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REVISED IDENTIFICATION OF STRAIN GRADIENT ELASTIC PARAMETERS

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Abstract. The work reported in "Granular micromechanics-based identification of isotropic strain gradient parameters for elastic geometrically nonlinear deformations" misidentified key terms in the grain-pair objective relative displacement when accounting for the second gradient of placement. In this paper, we correct that oversight by deriving a revised expression for the grain-pair objective relative displacement within the granular micromechanics framework. The amended terms, which resemble Christoffel symbols expressed in terms of strain gradients, modify the contributions of both the normal and tangential components to the strain energy and, consequently, alter the identified strain gradient elastic parameters. Importantly, the identification of the standard (first gradient) elastic tensor remains unchanged. This brief paper presents the corrected derivation, the resulting stiffness tensors for anisotropic strain gradient elasticity, and updated analytical expressions for the material parameters in both 2D and 3D isotropic settings.

1. Introduction

In recent years, strain gradient elasticity theories have garnered significant attention for their ability to capture size-dependent mechanical behavior in nano-structured materials and metamaterials, where classical continuum models fail to provide accurate predictions [44, 45, 46, 47, 48]. These advanced models, particularly those incorporating higher-order or second strain gradient effects, have been successfully applied to investigate a wide range of phenomena, such as wave propagation in periodic nanostructures and graphene sheets [2, 3, 4], static and dynamic analyses of nanoplates and nanobeams [5, 6, 7, 8, 9, 10, 11, 12, 13], and hygro-thermal responses of functionally graded and magneto-electro-elastic nano-components [10, 11, 12, 13]. Among these, strain gradient elasticity—particularly its higher-order and second-gradient formulations—has proven effective in describing phenomena such as wave dispersion, localization, and boundary layer effects in nano-beams, plates, and metamaterials [14, 16, 17, 18, 19, 20, 21, 22]. Computational approaches leveraging meshless methods [5, 6, 7], finite element techniques [15, 19], and semi-analytical formulations [12] have further enhanced the applicability of strain gradient models to complex geometries and loading conditions. Moreover, recent developments in homogenization techniques and metamaterial modeling have emphasized the role of second gradient continua in capturing microscale interactions and deformation patterns in engineered structures [23, 24, 25, 31, 32, 33, 34]. Despite this growing body of work, the accurate identification of strain gradient elastic coefficients—crucial for both predictive modeling and material design—remains an open challenge. In this context, the present study aims to contribute to this field by proposing a robust framework for the identification of such coefficients, thereby advancing the reliability and effectiveness of strain gradient theories in engineering applications. In other words, a central challenge in deploying these models lies in the accurate identification of the additional constitutive parameters introduced by gradient elasticity. While most existing works focus on the implementation of gradient theories in numerical and analytical frameworks, including finite element and meshless methods [17, 18, 27, 28], only a limited number of studies address the direct physical interpretation and quantitative determination of the gradient moduli. Recent advances, particularly those based on homogenization [29, 30, 35, 39] and microstructure-informed modeling [37, 38, 40, 41, 42], have paved the way toward more physically grounded identification techniques. Further developments in this direction include the use of multiscale and stress-driven models [26, 36, 43, 44, 45], as well as numerical studies of lattice and pantographic structures [31, 32, 33, 34]. Building on these foundations, the present study introduces a granular micromechanical approach for the identification of strain gradient elastic moduli, grounded in the physical interpretation of microscale interactions and compatible

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with continuum models of second-gradient type. This contribution seeks to bridge the gap between micromechanical modeling and constitutive parameter identification, enabling more predictive and physically consistent applications of strain gradient elasticity.

2. A recap before the correction

In granular micromechanics, the density of the (internal) strain energy is, according to (15) of [1], the integral of the energy of each orientation $\hat{c}$ over the unit circle S^1 in the 2D case or over the unit sphere S^2 in the 3D case,

$$U = \int_{\mathcal{S}^{1,2}} \left(\frac{1}{2} k_\eta u_\eta^2 + \frac{1}{2} k_\tau u_\tau^2 \right) dS, \tag{1}$$

where k_η is the normal stiffness relative to the orientation $\hat{c}$, k_τ is the tangent stiffness relative to the orientation $\hat{c}$, and dS is the angular element over S^1 or S^2. In addition, u_η is the normal displacement, which is a scalar, and u_τ is the tangent displacement, which is a vector. The normal displacement is defined by equation (11) of [1],

$$u_\eta = \frac{1}{2} u^{np} \cdot \hat{c} = \frac{1}{2} u_i^{np} \hat{c}_i, \tag{2}$$

and the tangential displacement by the equation (12) of [1],

$$u_\tau = u^{np} - (u^{np} \cdot \hat{c})\, \hat{c} = u^{np} - 2u_\eta \hat{c}. \tag{3}$$

In (2-3) we use the definition of the objective relative displacement, u^{np}, which is defined below (see (4)).

3. The objective relative displacement and its projections

The objective relative displacement is, from the first equivalence of the equation (9) of [1],

$$u^{np} = 2G\hat{c}L + \frac{L^2}{2} \left(F^T \nabla F \right) \hat{c}\hat{c}, \tag{4}$$

where G is the Green-Saint-Venant strain tensor,

$$G = \frac{1}{2} \left(F^T F - I \right), \tag{5}$$

F is the deformation gradient,

$$F = \nabla \chi, \tag{6}$$

χ is the placement function, $\hat{c}$ is a unit vector, L is average distance between the neighboring grain-pairs and ∇ is the gradient operator. In index notation (4) yields

$$u_i^{np} = 2G_{ij}\hat{c}_j L + \frac{L^2}{2} F_{ia}^T \left(\nabla F \right)_{abc} \hat{c}_c \hat{c}_b = 2G_{ij}\hat{c}_j L + \frac{L^2}{2} H_{ibc} \hat{c}_c \hat{c}_b, \tag{7}$$

where the definition (5) has been taken into account.

$$H_{ibc} = F_{ai} F_{ab,c} = \chi_{a,i} \chi_{a,bc} = H_{icb}$$

is a third order tensor, symmetric with respect the last two indices. The error of equation (9) in [1] is due to a wrong identification of the last term of (7),

$$\frac{L^2}{2} H_{ibc} \hat{c}_c \hat{c}_b \to \frac{L^2}{2} G_{ib,c} \hat{c}_c \hat{c}_b,$$

which is simply an oversight. Indeed, it is possible to prove the following identity for the components of the tensor H, similar in form to Christoffel symbols:

$$H_{ibc} = G_{ib,c} + G_{ic,b} - G_{bc,i}. \tag{8}$$

Let us prove (8). From (5) we have

$$G_{ij} = \frac{1}{2} \left(F_{ki} F_{kj} - \delta_{ij} \right).$$

Its gradient is

$$G_{ij,h} = \frac{1}{2} \left(F_{ki,h} F_{kj} + F_{ki} F_{kj,h} \right). \tag{9}$$

Thus, by changing the names of the indices of (9) we have the following three expressions,

$$G_{ib,c} = \frac{1}{2}\left(F_{ki,c}F_{kb} + F_{ki}F_{kb,c}\right),$$
$$G_{ic,b} = \frac{1}{2}\left(F_{ki,b}F_{kc} + F_{ki}F_{kc,b}\right),$$
$$G_{bc,i} = \frac{1}{2}\left(F_{kb,i}F_{kc} + F_{kb}F_{kc,i}\right).$$

From the previous three equations, the combination expressed on th right-hand side of (8) yields

$$G_{ib,c} + G_{ic,b} - G_{bc,i} = \frac{1}{2}\left(F_{ki,c}F_{kb} + F_{ki}F_{kb,c} + F_{ki,b}F_{kc} + F_{ki}F_{kc,b} - F_{kb,i}F_{kc} - F_{kb}F_{kc,i}\right). \tag{10}$$

Because of the symmetry on the last two indices of the gradient of the deformation gradient in (6),

$$(\nabla F)_{abc} = F_{ab,c} = \chi_{a,bc} = \chi_{a,cb} = F_{ac,b} = (\nabla F)_{acb},$$

the last two terms of (10) cancel the first, and the third terms and the second term equals the fourth one. As a consequence, we have,

$$G_{ib,c} + G_{ic,b} - G_{bc,i} = F_{ki}F_{kb,c} = H_{ibc} = H_{icb},$$

proving (8).

Thus, the objective relative displacement is given by the insertion of (8) into (7)

$$u_i^{np} = 2G_{ij}\hat{c}_j L + \frac{L^2}{2}\left(G_{ib,c} + G_{ic,b} - G_{bc,i}\right)\hat{c}_c\hat{c}_b. \tag{11}$$

The normal displacement u_η is defined by equation (11) of [1]. The insertion of (11) into the definition yields,

$$u_\eta = \frac{1}{2}u_i^{np}\hat{c}_i = G_{ij}\hat{c}_i\hat{c}_j L + \frac{L^2}{4}\left(G_{ib,c}\hat{c}_i\hat{c}_c\hat{c}_b + G_{ic,b}\hat{c}_i\hat{c}_c\hat{c}_b - G_{bc,i}\hat{c}_i\hat{c}_c\hat{c}_b\right). \tag{12}$$

The first and the third terms in parenthesis are the same because of symmetry (exchanging the names of the indices i and b), so that the normal displacement (12) is

$$u_\eta = G_{ij}\hat{c}_i\hat{c}_j L + \frac{L^2}{4}G_{ij,h}\hat{c}_i\hat{c}_j\hat{c}_h,$$

the same expression as (A.2) of the [1]. Its square is

$$u_\eta^2 = \left(G_{ij}\hat{c}_i\hat{c}_j L + \frac{L^2}{4}G_{ij,h}\hat{c}_i\hat{c}_j\hat{c}_h\right)\left(G_{ab}\hat{c}_a\hat{c}_b L + \frac{L^2}{4}G_{ab,c}\hat{c}_a\hat{c}_b\hat{c}_c\right).$$

Expanding the square of the previous binomial

$$\begin{aligned} u_\eta^2 = &\left(G_{ij}\hat{c}_i\hat{c}_j L\right)\left(G_{ab}\hat{c}_a\hat{c}_b L\right) \\ &+ \left(G_{ij}\hat{c}_i\hat{c}_j L\right)\left(\frac{L^2}{4}G_{ab,c}\hat{c}_a\hat{c}_b\hat{c}_c\right) \\ &+ \left(\frac{L^2}{4}G_{ij,h}\hat{c}_i\hat{c}_j\hat{c}_h\right)\left(G_{ab}\hat{c}_a\hat{c}_b L\right) \\ &+ \left(\frac{L^2}{4}G_{ij,h}\hat{c}_i\hat{c}_j\hat{c}_h\right)\left(\frac{L^2}{4}G_{ab,c}\hat{c}_a\hat{c}_b\hat{c}_c\right), \end{aligned}$$

we have the equivalence between the second and the third lines of the previous equation. Thus,

$$u_\eta^2 = L^2 G_{ij}G_{ab}\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b + \frac{L^3}{2}G_{ij}G_{ab,c}\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\hat{c}_c + \frac{L^4}{16}G_{ij,h}G_{ab,c}\hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c. \tag{13}$$

The tangent displacement is a vector, that is defined in equation (12) of [1],

$$u_\tau = u^{np} - \left(u^{np}\cdot\hat{c}\right)\hat{c} = u^{np} - 2u_\eta\hat{c},$$

where we have used the definition (12). In index notation the previous expression is $u_{\tau i} = u_i^{np} - 2u_\eta\hat{c}_i$. The squared tangent displacement is the scalar

$$u_\tau^2 = \left(u_i^{np} - 2u_\eta\hat{c}_i\right)\left(u_i^{np} - 2u_\eta\hat{c}_i\right) = u_i^{np}u_i^{np} - 4u_\eta u_i^{np}\hat{c}_i + 4u_\eta^2 = u_i^{np}u_i^{np} - 4u_\eta^2. \tag{14}$$

Let us analyze the first term of (14), which is the norm of the objective relative displacement,

$$u_i^{np}u_i^{np} = \left[2G_{ib}\hat{c}_b L + \frac{L^2}{2}\left(G_{ib,c} + G_{ic,b} - G_{bc,i}\right)\hat{c}_c\hat{c}_b\right] \times \left[2G_{ij}\hat{c}_j L + \frac{L^2}{2}\left(G_{ij,h} + G_{ih,j} - G_{jh,i}\right)\hat{c}_h\hat{c}_j\right]. \tag{15}$$

Expanding the square of the previous binomial we have

$$\begin{aligned} u_i^{np}u_i^{np} = {} & \left[2G_{ib}\hat{c}_b L\right]\left[2G_{ij}\hat{c}_j L\right] && (16)\\ & + \left[2G_{ib}\hat{c}_b L\right]\left[\frac{L^2}{2}\left(G_{ij,h} + G_{ih,j} - G_{jh,i}\right)\hat{c}_h\hat{c}_j\right] && (17)\\ & + \left[\frac{L^2}{2}\left(G_{ib,c} + G_{ic,b} - G_{bc,i}\right)\hat{c}_c\hat{c}_b\right]\left[2G_{ij}\hat{c}_j L\right] && (18)\\ & + \left[\frac{L^2}{2}\left(G_{ib,c} + G_{ic,b} - G_{bc,i}\right)\hat{c}_c\hat{c}_b\right]\left[\frac{L^2}{2}\left(G_{ij,h} + G_{ih,j} - G_{jh,i}\right)\hat{c}_h\hat{c}_j\right]. && (19) \end{aligned}$$

The second and the third lines of the previous expression are the same. Thus the norm of the objective relative displacement (15) is,

$$\begin{aligned} u_i^{np}u_i^{np} = {} & 4L^2 G_{ib}G_{ij}\hat{c}_b\hat{c}_j \\ & + 2L^3 G_{ib}\left(G_{ij,h} + G_{ih,j} - G_{jh,i}\right)\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}\left(G_{ib,c} + G_{ic,b} - G_{bc,i}\right)\left(G_{ij,h} + G_{ih,j} - G_{jh,i}\right)\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j. \end{aligned}$$

Expanding each term of the previous expressions we have

$$\begin{aligned} u_i^{np}u_i^{np} = {} & 4L^2 G_{ib}G_{ij}\hat{c}_b\hat{c}_j \\ & + 2L^3 G_{ib}G_{ij,h}\hat{c}_b\hat{c}_h\hat{c}_j \\ & + 2L^3 G_{ib}G_{ih,j}\hat{c}_b\hat{c}_h\hat{c}_j \\ & - 2L^3 G_{ib}G_{jh,i}\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}G_{ib,c}G_{ij,h}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}G_{ib,c}G_{ih,j}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & - \frac{L^4}{4}G_{ib,c}G_{jh,i}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}G_{ic,b}G_{ij,h}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}G_{ic,b}G_{ih,j}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & - \frac{L^4}{4}G_{ic,b}G_{jh,i}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & - \frac{L^4}{4}G_{bc,i}G_{ij,h}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & - \frac{L^4}{4}G_{bc,i}G_{ih,j}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\ & + \frac{L^4}{4}G_{bc,i}G_{jh,i}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j. \end{aligned}$$

Changing the names of the indices over which we have an implicit sum in order to collect the same strain and strain gradient components, the previous expression is

$$
\begin{aligned}
u_i^{np}u_i^{np} = & \, 4L^2 G_{ab}G_{ij}\delta_{ia}\hat{c}_b\hat{c}_j \\
& + 2L^3 G_{ab}G_{ij,h}\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j \\
& + 2L^3 G_{ab}G_{ij,h}\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j \\
& - 2L^3 G_{ab}G_{ij,h}\delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i .
\end{aligned}
$$

Let us now collect the terms of the previous expression into three parts. The first with the squared strain components, the third with the square of the strain gradient components and the second with the mixed strain-strain gradient components,

$$
\begin{aligned}
u_i^{np}u_i^{np} = & \, 4L^2 G_{ab}G_{ij}\delta_{ia}\hat{c}_b\hat{c}_j \\
& + 2L^3 G_{ab}G_{ij,h}\left(2\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j - \delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(4\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j - 2\delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i - 2\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i\right) .
\end{aligned}
\tag{20}
$$

Thus, the squared tangent displacement is from the insertion of (13) and (20) into (14),

$$
\begin{aligned}
u_\tau^2 = u_i^{np}u_i^{np} - 4u_\eta^2 = & \, 4L^2 G_{ab}G_{ij}\left(\delta_{ia}\hat{c}_b\hat{c}_j - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \\
& + 2L^3 G_{ab}G_{ij,h}\left(2\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j - \delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\hat{c}_c\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(4\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j - 2\delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i - 2\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i - \hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c\right) .
\end{aligned}
$$

Because of the symmetry of both the strain tensor G and its gradient ∇G with respect to the first two indices, we symmetrize the coefficients of the strain and strain gradient components as follows,

$$\begin{aligned}
u_\tau^2 = u_i^{np}u_i^{np} - 4u_\eta^2 = {} & L^2 G_{ab}G_{ij}\left(\delta_{ia}\hat{c}_b\hat{c}_j + \delta_{ja}\hat{c}_b\hat{c}_i + \delta_{ib}\hat{c}_a\hat{c}_j + \delta_{jb}\hat{c}_a\hat{c}_i - 4\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \\
& + \frac{L^3}{2} G_{ab}G_{ij,h}\left(2\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j + 2\delta_{ja}\hat{c}_b\hat{c}_h\hat{c}_i + 2\delta_{ib}\hat{c}_a\hat{c}_h\hat{c}_j + 2\delta_{jb}\hat{c}_a\hat{c}_h\hat{c}_i\right) \\
& - \frac{L^3}{2} G_{ab}G_{ij,h}\left(2\delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i + 2\delta_{hb}\hat{c}_a\hat{c}_j\hat{c}_i + 4\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\hat{c}_c\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j + \delta_{ib}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{ja}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_i + \delta_{jb}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_i\right) \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i + \delta_{hb}\hat{c}_c\hat{c}_a\hat{c}_j\hat{c}_i\right) \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jc}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_i\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i\right) \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c\right).
\end{aligned}$$

Collecting the terms we have a new form of the squared tangent displacement,

$$\begin{aligned}
u_\tau^2 = {} & L^2 G_{ab}G_{ij}\left(\delta_{ia}\hat{c}_b\hat{c}_j + \delta_{ja}\hat{c}_b\hat{c}_i + \delta_{ib}\hat{c}_a\hat{c}_j + \delta_{jb}\hat{c}_a\hat{c}_i - 4\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \\
& + \frac{L^3}{2} G_{ab}G_{ij,h}\left(2\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j + 2\delta_{ja}\hat{c}_b\hat{c}_h\hat{c}_i - 2\delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i + 2\delta_{ib}\hat{c}_a\hat{c}_h\hat{c}_j\right) \\
& + \frac{L^3}{2} G_{ab}G_{ij,h}\left(2\delta_{jb}\hat{c}_a\hat{c}_h\hat{c}_i - 2\delta_{hb}\hat{c}_a\hat{c}_j\hat{c}_i - 4\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\hat{c}_c\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j + \delta_{ja}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_i - \delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i\right) \\
& + \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ib}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jb}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_i - \delta_{hb}\hat{c}_c\hat{c}_a\hat{c}_j\hat{c}_i\right) \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jc}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_i - \delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i\right) \\
& - \frac{L^4}{4} G_{ab,c}G_{ij,h}\left(\hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c\right).
\end{aligned} \tag{21}$$

4. The density of the strain energy

The density of the (internal) strain energy is, according to (15) of [1], the integral of the energy of each orientation $\hat{c}$ over the unit circle S^1 in the 2D case or over the unit sphere S^2 in the 3D case,

$$U = \int_{\mathcal{S}^{1,2}} \left(\frac{1}{2}k_\eta u_\eta^2 + \frac{1}{2}k_\tau u_\tau^2\right) dS, \tag{22}$$

where k_η is the normal stiffness relative to the orientation $\hat{c}$, k_τ is the tangent stiffness relative to the orientation $\hat{c}$, and dS is the angular element over S^1 or S^2. Insertion of (13) and (21) into (22) yields

$$
\begin{aligned}
U = \int_{\mathcal{S}^{1,2}} \{ & \frac{1}{2} k_\eta L^2 G_{ij} G_{ab} \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \\
& + \frac{1}{2} k_\eta \frac{L^3}{2} G_{ij} G_{ab,c} \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \hat{c}_c \\
& + \frac{1}{2} k_\eta \frac{L^4}{16} G_{ij,h} G_{ab,c} \hat{c}_i \hat{c}_j \hat{c}_h \hat{c}_a \hat{c}_b \hat{c}_c \\
& \frac{1}{2} k_\tau L^2 G_{ab} G_{ij} \left(\delta_{ia} \hat{c}_b \hat{c}_j + \delta_{ja} \hat{c}_b \hat{c}_i + \delta_{ib} \hat{c}_a \hat{c}_j + \delta_{jb} \hat{c}_a \hat{c}_i - 4 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \right) \\
& + \frac{1}{2} k_\tau \frac{L^3}{2} G_{ab} G_{ij,h} \left(2\delta_{ia} \hat{c}_b \hat{c}_h \hat{c}_j + 2\delta_{ja} \hat{c}_b \hat{c}_h \hat{c}_i - 2\delta_{ha} \hat{c}_b \hat{c}_j \hat{c}_i + 2\delta_{ib} \hat{c}_a \hat{c}_h \hat{c}_j \right) \\
& + \frac{1}{2} k_\tau \frac{L^3}{2} G_{ab} G_{ij,h} \left(2\delta_{jb} \hat{c}_a \hat{c}_h \hat{c}_i - 2\delta_{hb} \hat{c}_a \hat{c}_j \hat{c}_i - 4 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \hat{c}_c \right) \\
& + \frac{1}{2} k_\tau \frac{L^4}{4} G_{ab,c} G_{ij,h} \left(\delta_{ia} \hat{c}_c \hat{c}_b \hat{c}_h \hat{c}_j + \delta_{ja} \hat{c}_c \hat{c}_b \hat{c}_h \hat{c}_i - \delta_{ha} \hat{c}_c \hat{c}_b \hat{c}_j \hat{c}_i \right) \\
& + \frac{1}{2} k_\tau \frac{L^4}{4} G_{ab,c} G_{ij,h} \left(\delta_{ib} \hat{c}_c \hat{c}_a \hat{c}_h \hat{c}_j + \delta_{jb} \hat{c}_c \hat{c}_a \hat{c}_h \hat{c}_i - \delta_{hb} \hat{c}_c \hat{c}_a \hat{c}_j \hat{c}_i \right) \\
& - \frac{1}{2} k_\tau \frac{L^4}{4} G_{ab,c} G_{ij,h} \left(\delta_{ic} \hat{c}_b \hat{c}_a \hat{c}_h \hat{c}_j + \delta_{jc} \hat{c}_b \hat{c}_a \hat{c}_h \hat{c}_i - \delta_{hc} \hat{c}_b \hat{c}_a \hat{c}_j \hat{c}_i \right) \\
& - \frac{1}{2} k_\tau \frac{L^4}{4} G_{ab,c} G_{ij,h} \left(\hat{c}_i \hat{c}_j \hat{c}_h \hat{c}_a \hat{c}_b \hat{c}_c \right) \} dS.
\end{aligned}
$$

Collecting the terms we have an important form of the density of the strain energy,

$$
\begin{aligned}
U = & \frac{1}{2} G_{ab} G_{ij} L^2 \int_{\mathcal{S}^{1,2}} \{ \left(k_\eta - 4k_\tau \right) \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \\
& \qquad k_\tau \left(\delta_{ia} \hat{c}_b \hat{c}_j + \delta_{ja} \hat{c}_b \hat{c}_i + \delta_{ib} \hat{c}_a \hat{c}_j + \delta_{jb} \hat{c}_a \hat{c}_i \right) \} dS \\
& + G_{ab} G_{ij,h} \frac{L^3}{4} \int_{\mathcal{S}^{1,2}} \{ \left(k_\eta - 4k_\tau \right) \hat{c}_a \hat{c}_b \hat{c}_i \hat{c}_j \hat{c}_h \\
& \qquad + 2k_\tau \left(\delta_{ia} \hat{c}_b \hat{c}_h \hat{c}_j + \delta_{ja} \hat{c}_b \hat{c}_h \hat{c}_i - \delta_{ha} \hat{c}_b \hat{c}_j \hat{c}_i \right) \\
& \qquad + 2k_\tau \left(\delta_{ib} \hat{c}_a \hat{c}_h \hat{c}_j + \delta_{jb} \hat{c}_a \hat{c}_h \hat{c}_i - \delta_{hb} \hat{c}_a \hat{c}_j \hat{c}_i \right) \\
& \qquad \} dS \\
& + \frac{1}{2} G_{ij,h} G_{ab,c} \frac{L^4}{16} \int_{\mathcal{S}^{1,2}} \{ \left(k_\eta - 4k_\tau \right) \hat{c}_i \hat{c}_j \hat{c}_h \hat{c}_a \hat{c}_b \hat{c}_c \\
& \qquad + 4k_\tau \left(\delta_{ia} \hat{c}_c \hat{c}_b \hat{c}_h \hat{c}_j + \delta_{ja} \hat{c}_c \hat{c}_b \hat{c}_h \hat{c}_i - \delta_{ha} \hat{c}_c \hat{c}_b \hat{c}_j \hat{c}_i \right) \\
& \qquad + 4k_\tau \left(\delta_{ib} \hat{c}_c \hat{c}_a \hat{c}_h \hat{c}_j + \delta_{jb} \hat{c}_c \hat{c}_a \hat{c}_h \hat{c}_i - \delta_{hb} \hat{c}_c \hat{c}_a \hat{c}_j \hat{c}_i \right) \\
& \qquad - 4k_\tau \left(\delta_{ic} \hat{c}_b \hat{c}_a \hat{c}_h \hat{c}_j + \delta_{jc} \hat{c}_b \hat{c}_a \hat{c}_h \hat{c}_i - \delta_{hc} \hat{c}_b \hat{c}_a \hat{c}_j \hat{c}_i \right) \} dS,
\end{aligned}
$$

where we have taken into account the fact that the strain and strain tensor components do not depend on the orientation $\hat{c}$ and go out of the integral over the unit circle S^1 in the 2D case or over the unit sphere S^2 in the 3D case. If we write the previous expression in a standard form with the use of the stiffness tensors $\mathbb{C}$, $\mathbb{M}$ and $\mathbb{D}$,

$$
U = \frac{1}{2} \mathbb{C}_{abij} G_{ij} G_{ab} + \mathbb{M}_{abijh} G_{ab} G_{ij,h} + \frac{1}{2} \mathbb{D}_{abcijh} G_{ij,h} G_{ab,c}
$$

then we have the following identifications of the three stiffness tensors,

$$
\begin{aligned}
\mathbb{C}_{abij} = L^2 \int_{\mathcal{S}^{1,2}} \{ & \left(k_\eta - 4k_\tau \right) \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \\
& k_\tau \left(\delta_{ia} \hat{c}_b \hat{c}_j + \delta_{ja} \hat{c}_b \hat{c}_i + \delta_{ib} \hat{c}_a \hat{c}_j + \delta_{jb} \hat{c}_a \hat{c}_i \right) \} dS,
\end{aligned}
\tag{23}
$$

$$\mathbb{M}_{abijh} = \frac{L^3}{4}\int_{\mathcal{S}^{1,2}} \{ (k_\eta - 4k_\tau)\,\hat{c}_a\hat{c}_b\hat{c}_i\hat{c}_j\hat{c}_h \\ + 2k_\tau\,(\delta_{ia}\hat{c}_b\hat{c}_h\hat{c}_j + \delta_{ja}\hat{c}_b\hat{c}_h\hat{c}_i - \delta_{ha}\hat{c}_b\hat{c}_j\hat{c}_i) \\ + 2k_\tau\,(\delta_{ib}\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jb}\hat{c}_a\hat{c}_h\hat{c}_i - \delta_{hb}\hat{c}_a\hat{c}_j\hat{c}_i)\}dS, \tag{24}$$

$$\mathbb{D}_{abcijh} = \frac{L^4}{16}\int_{\mathcal{S}^{1,2}} (k_\eta - 4k_\tau)\,\hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c \\ + 4k_\tau\,(\delta_{ia}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_j + \delta_{ja}\hat{c}_c\hat{c}_b\hat{c}_h\hat{c}_i - \delta_{ha}\hat{c}_c\hat{c}_b\hat{c}_j\hat{c}_i) \\ + 4k_\tau\,(\delta_{ib}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jb}\hat{c}_c\hat{c}_a\hat{c}_h\hat{c}_i - \delta_{hb}\hat{c}_c\hat{c}_a\hat{c}_j\hat{c}_i) \\ - 4k_\tau\,(\delta_{ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_j + \delta_{jc}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_i - \delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_i)\}dS. \tag{25}$$

A more synthetic way to propose such an identification is the following

$$\mathbb{C}_{abij} = L^2\int_{\mathcal{S}^{1,2}} \left[(k_\eta - 4k_\tau)\,\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b + 4k_\tau\delta_{(i(a}\hat{c}_{b)}\hat{c}_{j)}\right] dS, \tag{26}$$

$$\mathbb{M}_{abijh} = \frac{L^3}{4}\int_{\mathcal{S}^{1,2}} \left[(k_\eta - 4k_\tau)\,\hat{c}_a\hat{c}_b\hat{c}_i\hat{c}_j\hat{c}_h + 4k_\tau\left(2\delta_{(i(a}\hat{c}_{b)}\hat{c}_h\hat{c}_{j)} - \delta_{h(a}\hat{c}_{b)}\hat{c}_j\hat{c}_{i)}\right)\right] dS, \tag{27}$$

$$\mathbb{D}_{abcijh} = \frac{L^4}{16}\int_{\mathcal{S}^{1,2}} \{(k_\eta - 4k_\tau)\,\hat{c}_i\hat{c}_j\hat{c}_h\hat{c}_a\hat{c}_b\hat{c}_c \\ +4k_\tau\left(4\delta_{(i(a}\hat{c}_c\hat{c}_{b)}\hat{c}_h\hat{c}_{j)} - \delta_{h(a}\hat{c}_c\hat{c}_{b)}\hat{c}_j\hat{c}_i - 2\delta_{(ic}\hat{c}_b\hat{c}_a\hat{c}_h\hat{c}_{j)} + \delta_{hc}\hat{c}_b\hat{c}_a\hat{c}_j\hat{c}_{i)}\right)\}dS, \tag{28}$$

where the parentheses on the index means the standard single symmetrization rule

$$A_{(ikl j)} = \frac{1}{2}\left(A_{iklj} + A_{jkli}\right),$$

and nested parentheses are symmetrized first,

$$A_{(a(bc)d)} = \frac{1}{2}\left(A_{(abcd)} + A_{(acbd)}\right) = \frac{1}{4}\left(A_{abcd} + A_{acbd} + A_{dbca} + A_{dcba}\right)$$

It is worth to note that the identification (23) is the same as equation (18) of [1]. However, the identifications (24-25) change with respect to equation (19) of [1]. This change is the aim of this paper.

5. Identification of the 2D isotropic case

In the general anisotropic case, normal k_η and tangent k_τ stiffnesses are functions of the orientation $\hat{c}$. However, in the isotropic case we consider no functional dependence. In the 2D case, where the integration of (23-24-25) or (26-27-28) is done over the unit circle S^1, the equation that follows equation (36) of [1] is,

$$k_\eta = \frac{\bar{k}_\eta}{2\pi}, \qquad k_\tau = \frac{\bar{k}_\tau}{2\pi}, \tag{29}$$

where $\bar{k}_\eta$ and $\bar{k}_\tau$ are the integrated stiffnesses over the set of possible orientations. An analytical calculation of the insertion of (29) into (26) yields the following identification of the non null components of the standard stiffness tensor. They are divided into 3 groups.

$$\mathbb{C}_{1111} = \mathbb{C}_{2222} = \frac{1}{8}L^2\left(3\bar{k}_\eta + 4\bar{k}_\tau\right),$$
$$\mathbb{C}_{1122} = \mathbb{C}_{2211} = \frac{1}{8}L^2\left(\bar{k}_\eta - 4\bar{k}_\tau\right),$$
$$\mathbb{C}_{1212} = \mathbb{C}_{1221} = \mathbb{C}_{2112} = \mathbb{C}_{2121} = \frac{1}{8}L^2\left(\bar{k}_\eta + 4\bar{k}_\tau\right).$$

The rest of the components are null. The identification of the Lamé parameters are

$$\lambda = \mathbb{C}_{1122} = \frac{1}{8}L^2\left(\bar{k}_\eta - 4\bar{k}_\tau\right),\ \mu = \mathbb{C}_{1212} = \frac{L^2}{8}\left(\bar{k}_\eta + 4\bar{k}_\tau\right) = \frac{1}{2}\left(\mathbb{C}_{1111} - \mathbb{C}_{1122}\right).$$

The Young's modulus Y_{2D} and Poisson's ratio ν_{2D} are given from the expressions (A.12) of [1],

$$Y_{2D} = 4\mu\frac{\lambda+\mu}{\lambda+2\mu} = \frac{L^2\bar{k}_\eta\left(\bar{k}_\eta+4\bar{k}_\tau\right)}{3\bar{k}_\eta+4\bar{k}_\tau}, \qquad \nu_{2D} = \frac{\lambda}{\lambda+2\mu} = \frac{\bar{k}_\eta-4\bar{k}_\tau}{3\bar{k}_\eta+4\bar{k}_\tau}. \tag{30}$$

The inversion of (30) gives the normal and tangent integrated stiffnesses in terms of both the Young's modulus Y_{2D} and the Poisson's ratio ν_{2D},

$$\bar{k}_\eta = -\frac{2Y_{2D}}{L^2\left(\nu_{2D}-1\right)}, \qquad \bar{k}_\tau = \frac{\left(3\nu_{2D}-1\right)Y_{2D}}{2L^2\left(\nu_{2D}^2-1\right)}. \tag{31}$$

An analytical calculation of the insertion of (29) into (25) yields the following identification of the non-null components of the strain gradient stiffness tensor. They are divided into 6 groups.

The first,

$$d_1 = \mathbb{D}_{111111} = \mathbb{D}_{222222},$$

that, in terms of integrated stiffnesses $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_1 = \frac{L^4}{256}\left(5\bar{k}_\eta+4\bar{k}_\tau\right) = \frac{L^2\left(3+\nu_{2D}\right)Y_{2D}}{64\left(1-\nu_{2D}^2\right)}.$$

The second,

$$d_2 = \mathbb{D}_{111122} = \mathbb{D}_{111212} = \mathbb{D}_{121222} = \mathbb{D}_{122111},$$
$$d_2 = \mathbb{D}_{211222} = \mathbb{D}_{212111} = \mathbb{D}_{222121} = \mathbb{D}_{222211},$$

that, in terms of integrated stiffnesses $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_2 = \frac{L^4}{256}\left(\bar{k}_\eta+4\bar{k}_\tau\right) = \frac{L^2Y_{2D}}{64\left(\nu_{2D}+1\right)}.$$

The third,

$$d_3 = \mathbb{D}_{111221} = \mathbb{D}_{112222} = \mathbb{D}_{221111} = \mathbb{D}_{222112},$$

that ,in terms of integrated stiffnesses $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_3 = \frac{L^4}{256}\left(\bar{k}_\eta-12\bar{k}_\tau\right) = \frac{L^2\left(1-5\nu_{2D}\right)Y_{2D}}{64\left(\nu_{2D}^2-1\right)}.$$

The fourth,

$$d_4 = \mathbb{D}_{112112} = \mathbb{D}_{221221},$$

that, in terms of integrated stiffnesses $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_4 = \frac{L^4}{256}\left(\bar{k}_\eta+52\bar{k}_\tau\right) = \frac{L^2\left(19\nu_{2D}-7\right)Y_{2D}}{64\left(\nu_{2D}^2-1\right)}.$$

The fifth,

$$d_5 = \mathbb{D}_{112121} = \mathbb{D}_{112211} = \mathbb{D}_{121112} = \mathbb{D}_{122221},$$
$$d_5 = \mathbb{D}_{211112} = \mathbb{D}_{212221} = \mathbb{D}_{221122} = \mathbb{D}_{221212},$$

that, in terms of integrated stiffness $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_5 = \frac{L^4}{256}\left(\bar{k}_\eta-28\bar{k}_\tau\right) = \frac{L^2\left(3-11\nu_{2D}\right)Y_{2D}}{64\left(\nu_{2D}^2-1\right)}.$$

The sixth,

$$d_6 = \mathbb{D}_{121121} = \mathbb{D}_{121211} = \mathbb{D}_{122122} = \mathbb{D}_{122212},$$
$$d_6 = \mathbb{D}_{211121} = \mathbb{D}_{211211} = \mathbb{D}_{212122} = \mathbb{D}_{212212},$$

that, in terms of integrated stiffness $\bar{k}_\eta$ and $\bar{k}_\tau$ or in terms of the Young's modulus Y_{2D} and Poisson's ratio ν_{2D}, is

$$d_6 = \frac{L^4}{256}\left(\bar{k}_\eta + 20\bar{k}_\tau\right) = \frac{L^2\left(7\nu_{2D}-3\right)Y_{2D}}{64\left(\nu_{2D}^2-1\right)}.$$

The rest of the components of the stiffness tensor are null.

6. Identification of the 3D isotropic case

In the general anisotropic case, normal k_η and tangent k_τ stiffnesses are functions of the orientation $\hat{c}$. However, in the isotropic case we consider no such dependence. In the 3D case, where the integration of (23-24-25) or (26-27-28) is done over the unit sphere S^2, the expression of the equation that follows equation (56) of [1] is,

$$k_\eta = \frac{\bar{k}_\eta}{4\pi}, \qquad k_\tau = \frac{\bar{k}_\tau}{4\pi}, \tag{32}$$

where $\bar{k}_\eta$ and $\bar{k}_\tau$ are the are integrated stiffnesses over the set of possible orientations. An analytical calculation of the insertion of (32) into (26) yields the following identification of the non-null components of the standard stiffness tensor. They are divided into 3 groups.

$$\mathbb{C}_{1111} = \mathbb{C}_{2222} = \mathbb{C}_{3333} = \frac{L^2}{15}\left(3\bar{k}_\eta + 8\bar{k}_\tau\right),$$

$$\mathbb{C}_{1122} = \mathbb{C}_{1133} = \mathbb{C}_{2211} = \mathbb{C}_{2233} = \mathbb{C}_{3311} = \mathbb{C}_{3322} = \frac{L^2}{15}\left(\bar{k}_\eta - 4\bar{k}_\tau\right),$$

$$\mathbb{C}_{1212} = \mathbb{C}_{1221} = \mathbb{C}_{1313} = \mathbb{C}_{1331} = \mathbb{C}_{2112} = \mathbb{C}_{2121} = \frac{L^2}{15}\left(\bar{k}_\eta + 6\bar{k}_\tau\right),$$
$$\mathbb{C}_{2323} = \mathbb{C}_{2332} = \mathbb{C}_{3113} = \mathbb{C}_{3131} = \mathbb{C}_{3223} = \mathbb{C}_{3232} = \frac{L^2}{15}\left(\bar{k}_\eta + 6\bar{k}_\tau\right).$$

The rest of the components are null. Identification of the Lamé parameters are

$$\lambda = \mathbb{C}_{1122} = \frac{1}{15}L^2\left(\bar{k}_\eta - 4\bar{k}_\tau\right),\ \mu = \mathbb{C}_{1212} = \frac{L^2}{8}\left(\bar{k}_\eta + 6\bar{k}_\tau\right) = \frac{1}{2}\left(\mathbb{C}_{1111} - \mathbb{C}_{1122}\right).$$

The Young's modulus Y_{3D} and Poisson's ratio ν_{3D} are given from the expressions (A.15) of [1],

$$Y_{3D} = \mu\frac{3\lambda+\mu}{\lambda+2\mu} = \frac{L^2\bar{k}_\eta\left(\bar{k}_\eta + 6\bar{k}_\tau\right)}{6\left(\bar{k}_\eta + \bar{k}_\tau\right)}, \qquad \nu_{3D} = \frac{\lambda}{2\left(\lambda+\mu\right)} = \frac{\bar{k}_\eta - 4\bar{k}_\tau}{4\left(\bar{k}_\eta + \bar{k}_\tau\right)}. \tag{33}$$

The inversion of (33) gives the normal and tangent integrated stiffnesses in terms of the Young's modulus Y_{3D} and the Poisson's ratio ν_{3D},

$$\bar{k}_\eta = \frac{3Y_{3D}}{L^2\left(1-2\nu_{3D}\right)}, \qquad \bar{k}_\tau = \frac{\left(4\nu_{3D}-1\right)3Y_{3D}}{4L^2\left(2\nu_{3D}^2+\nu_{3D}-1\right)}. \tag{34}$$

If we represent the strain gradient stiffness matrix with the following isotropic representation

$$\begin{aligned}
\mathbb{D}_{ijklmn} = {} & c_3\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}\right) \\
& + c_4\delta_{ij}\delta_{kn}\delta_{ml} \\
& + c_5\left(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn}\right) \\
& + c_6\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jl}\delta_{kn}\right) \\
& + c_7\left(\delta_{il}\delta_{jn}\delta_{mk} + \delta_{im}\delta_{jn}\delta_{lk} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right)
\end{aligned} \tag{35}$$

then, it is sufficient to know the following five components of $\mathbb{D}$,

$$\mathbb{D}_{111111} = 4c_3 + c_4 + 4c_5 + 2c_6 + 4c_7, \quad \mathbb{D}_{221221} = c_4 + 2c_6, \tag{36}$$

$$\mathbb{D}_{111221} = 2c_3 + c_4, \quad \mathbb{D}_{221122} = c_3 + 2c_7, \quad \mathbb{D}_{112233} = c_3, \tag{37}$$

to identify the five independent isotropic strain gradient coefficients as follows,

$$c_3 = \mathbb{D}_{112233}, \quad c_4 = \mathbb{D}_{111221} - 2\mathbb{D}_{112233}, \tag{38}$$

$$c_5 = \frac{1}{4}\left(\mathbb{D}_{111111} - 2\mathbb{D}_{112233} - 2\mathbb{D}_{221122} - \mathbb{D}_{221221}\right), \tag{39}$$

$$c_6 = \frac{1}{2}\left(-\mathbb{D}_{111221} + 2\mathbb{D}_{112233} + \mathbb{D}_{221221}\right), \quad c_7 = \frac{1}{2}\left(-\mathbb{D}_{112233} + \mathbb{D}_{221122}\right). \tag{40}$$

It is worth to note that the expressions (36-37) have been calculated from (35) and the identifications (38-39) is given by the inversion of (36-37). An analytical calculation of the insertion of (32) into (25) yields the identification of these 5 components of the strain gradient stiffness tensor,

$$\mathbb{D}_{111111} = \frac{L^4}{560}\left(5\bar{k}_\eta + 8\bar{k}_\tau\right) = \frac{3L^2\left(3\nu_{3D} - 7\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)}, \tag{41}$$

$$\mathbb{D}_{221221} = \frac{L^4\left(3\bar{k}_\eta + 184\bar{k}_\tau\right)}{1680} = \frac{L^2\left(181\nu_{3D} - 49\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)}, \tag{42}$$

$$\mathbb{D}_{111221} = \frac{L^4\left(3\bar{k}_\eta - 40\bar{k}_\tau\right)}{1680} = \frac{L^2\left(7 - 43\nu_{3D}\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)}, \tag{43}$$

$$\mathbb{D}_{221122} = \frac{L^4\left(\bar{k}_\eta - 32\bar{k}_\tau\right)}{560} = \frac{3L^2\left(7 - 33\nu_{3D}\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)}, \tag{44}$$

$$\mathbb{D}_{112233} = \frac{L^4\left(\bar{k}_\eta - 32\bar{k}_\tau\right)}{1680} = \frac{L^2\left(7 - 33\nu_{3D}\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)}, \tag{45}$$

where the second equivalence is achieved because of (34). Insertion of (41-45) into (38-39) yields

$$c_3 = \frac{L^4}{1680}\left(\bar{k}_\eta - 32\bar{k}_\tau\right) = \frac{L^2\left(7 - 33\nu_{3D}\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)},$$

$$c_4 = \frac{L^4}{1680}\left(\bar{k}_\eta + 24\bar{k}_\tau\right) = \frac{L^2\left(23\nu_{3D} - 7\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)},$$

$$c_5 = \frac{L^4}{1680}\left(\bar{k}_\eta + 24\bar{k}_\tau\right) = \frac{L^2\left(23\nu_{3D} - 7\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)},$$

$$c_6 = \frac{L^4}{1680}\left(\bar{k}_\eta + 80\bar{k}_\tau\right) = \frac{L^2\left(79\nu_{3D} - 21\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)},$$

$$c_7 = \frac{L^4}{1680}\left(\bar{k}_\eta - 32\bar{k}_\tau\right) = \frac{L^2\left(7 - 33\nu_{3D}\right)Y_{3D}}{560\left(2\nu_{3D}^2 + \nu_{3D} - 1\right)},$$

where the second equivalence is achieved because of (34).

The Mindlin coefficients a_1, a_2, a_3, a_4 and a_5 are then related to c_3, c_4, c_5, c_6 and c_7 via

$$a_1 = 2c_3, \quad a_2 = 2c_4, \quad a_3 = 2c_5, \quad a_4 = c_6, \quad a_5 = 2c_7.$$

7. Concluding Remarks

The main contribution of this paper is a corrected expression for the grain-pair objective relative displacement within the granular micromechanics framework that incorporates second gradients of placement. This correction amends the formulation presented in [1] and leads to modified strain energy contributions from both the normal and tangential components of grain-pair interactions. As a result, the elastic energy density functional in the second gradient granular micromechanics approach is updated, and explicit stiffness tensors applicable to anisotropic second gradient elasticity are derived. We also revise the analytical expressions for the material parameters of 2D and 3D isotropic second gradient elasticity previously reported in [1]. The standard (first gradient) elastic parameters remain unchanged but are included here for completeness.

Finally, we note that second gradient elasticity is increasingly used to explain observed mechanical responses exhibiting intrinsic length scale effects. The parameters provided in this work offer a practical starting

point for such applications, especially since the formulation reveals the emergence of multiple characteristic lengths under different deformation modes.

Appendix: Complete list of strain gradient stiffness components

In this appendix, we list the complete list of the strain gradient stiffness tensor components. They are divided into 7 groups.

The first group is

$$d_1 = \mathbb{D}_{111111} = \mathbb{D}_{222222} = \mathbb{D}_{333333} = \frac{L^4}{560}\left(5\bar{k}_\eta + 8\bar{k}_\tau\right).$$

The second group is

$$\begin{aligned}
d_2 = \quad & \mathbb{D}_{111122} = \mathbb{D}_{111133} = \mathbb{D}_{111212} = \mathbb{D}_{111313} = \mathbb{D}_{121222} = \mathbb{D}_{122111} = \mathbb{D}_{131333} = \mathbb{D}_{133111}, \\
d_2 = \quad & \mathbb{D}_{211222} = \mathbb{D}_{212222} = \mathbb{D}_{222121} = \mathbb{D}_{222211} = \mathbb{D}_{222233} = \mathbb{D}_{222323} = \mathbb{D}_{232333} = \mathbb{D}_{233222}, \\
d_2 = \quad & \mathbb{D}_{311333} = \mathbb{D}_{313111} = \mathbb{D}_{322333} = \mathbb{D}_{323222} = \mathbb{D}_{333131} = \mathbb{D}_{333232} = \mathbb{D}_{333311} = \mathbb{D}_{333322},
\end{aligned}$$

where

$$d_2 = \frac{L^4}{1680}\left(3\bar{k}_\eta + 16\bar{k}_\tau\right).$$

The third group is

$$\begin{aligned}
d_3 = \quad & \mathbb{D}_{111221} = \mathbb{D}_{111331} = \mathbb{D}_{112222} = \mathbb{D}_{113333}, \\
d_3 = \quad & \mathbb{D}_{221111} = \mathbb{D}_{222112} = \mathbb{D}_{222332} = \mathbb{D}_{223333}, \\
d_3 = \quad & \mathbb{D}_{331111} = \mathbb{D}_{332222} = \mathbb{D}_{333113} = \mathbb{D}_{333223},
\end{aligned}$$

where

$$d_3 = \frac{L^4}{1680}\left(3\bar{k}_\eta - 40\bar{k}_\tau\right).$$

The fourth group is

$$d_4 = \quad \mathbb{D}_{112112} = \mathbb{D}_{113113} = \mathbb{D}_{221221} = \mathbb{D}_{223223} = \mathbb{D}_{331331} = \mathbb{D}_{332332},$$

where

$$d_4 = \frac{L^4}{1680}\left(3\bar{k}_\eta + 184\bar{k}_\tau\right).$$

The fifth group is

$$\begin{aligned}
d_5 = \quad & \mathbb{D}_{112121} = \mathbb{D}_{112211} = \mathbb{D}_{113131} = \mathbb{D}_{113311} = \mathbb{D}_{121112} = \mathbb{D}_{122221} = \mathbb{D}_{131113} = \mathbb{D}_{133331}, \\
d_5 = \quad & \mathbb{D}_{211112} = \mathbb{D}_{212221} = \mathbb{D}_{221122} = \mathbb{D}_{221212} = \mathbb{D}_{223232} = \mathbb{D}_{223322} = \mathbb{D}_{232223} = \mathbb{D}_{233332}, \\
d_5 = \quad & \mathbb{D}_{311113} = \mathbb{D}_{313331} = \mathbb{D}_{322223} = \mathbb{D}_{323332} = \mathbb{D}_{331133} = \mathbb{D}_{331313} = \mathbb{D}_{332233} = \mathbb{D}_{332323}, \\
3d_5 = \quad & \mathbb{D}_{112233} = \mathbb{D}_{112323} = \mathbb{D}_{113232} = \mathbb{D}_{113322} = \mathbb{D}_{121332} = \mathbb{D}_{122331} = \mathbb{D}_{123132} = \mathbb{D}_{123231}, \\
3d_5 = \quad & \mathbb{D}_{123312} = \mathbb{D}_{123321} = \mathbb{D}_{131223} = \mathbb{D}_{132123} = \mathbb{D}_{132213} = \mathbb{D}_{132231} = \mathbb{D}_{132321} = \mathbb{D}_{133221}, \\
3d_5 = \quad & \mathbb{D}_{211332} = \mathbb{D}_{212331} = \mathbb{D}_{213132} = \mathbb{D}_{213231} = \mathbb{D}_{213312} = \mathbb{D}_{213321} = \mathbb{D}_{221133} = \mathbb{D}_{221313}, \\
3d_5 = \quad & \mathbb{D}_{223131} = \mathbb{D}_{223311} = \mathbb{D}_{231123} = \mathbb{D}_{231132} = \mathbb{D}_{231213} = \mathbb{D}_{231312} = \mathbb{D}_{232113} = \mathbb{D}_{233112}, \\
3d_5 = \quad & \mathbb{D}_{311223} = \mathbb{D}_{312123} = \mathbb{D}_{312213} = \mathbb{D}_{312231} = \mathbb{D}_{312321} = \mathbb{D}_{313221} = \mathbb{D}_{321123} = \mathbb{D}_{321132}, \\
3d_5 = \quad & \mathbb{D}_{321213} = \mathbb{D}_{321312} = \mathbb{D}_{322113} = \mathbb{D}_{323112} = \mathbb{D}_{331122} = \mathbb{D}_{331212} = \mathbb{D}_{332121} = \mathbb{D}_{332211},
\end{aligned}$$

where

$$d_5 = \frac{L^4}{560}\left(\bar{k}_\eta - 32\bar{k}_\tau\right).$$

The sixth group is

$$
\begin{aligned}
d_6 = \quad & \mathbb{D}_{112332} = \mathbb{D}_{113223} = \mathbb{D}_{121233} = \mathbb{D}_{121323} = \mathbb{D}_{122133}, \\
d_6 = \quad & \mathbb{D}_{122313} = \mathbb{D}_{131232} = \mathbb{D}_{131322} = \mathbb{D}_{133122} = \mathbb{D}_{133212}, \\
d_6 = \quad & \mathbb{D}_{211233} = \mathbb{D}_{211323} = \mathbb{D}_{212133} = \mathbb{D}_{212313} = \mathbb{D}_{221331}, \\
d_6 = \quad & \mathbb{D}_{223113} = \mathbb{D}_{232131} = \mathbb{D}_{232311} = \mathbb{D}_{233121} = \mathbb{D}_{233211}, \\
d_6 = \quad & \mathbb{D}_{311232} = \mathbb{D}_{311322} = \mathbb{D}_{313122} = \mathbb{D}_{313212} = \mathbb{D}_{322131}, \\
d_6 = \quad & \mathbb{D}_{322311} = \mathbb{D}_{323121} = \mathbb{D}_{323211} = \mathbb{D}_{331221} = \mathbb{D}_{332112}, \\
3d_6 = \quad & \mathbb{D}_{121121} = \mathbb{D}_{121211} = \mathbb{D}_{122122} = \mathbb{D}_{122212} = \mathbb{D}_{131131} = \mathbb{D}_{131311} = \mathbb{D}_{133133} = \mathbb{D}_{133313}, \\
3d_6 = \quad & \mathbb{D}_{211121} = \mathbb{D}_{211211} = \mathbb{D}_{212122} = \mathbb{D}_{212212} = \mathbb{D}_{232232} = \mathbb{D}_{232322} = \mathbb{D}_{233233} = \mathbb{D}_{233323}, \\
3d_6 = \quad & \mathbb{D}_{311131} = \mathbb{D}_{311311} = \mathbb{D}_{313133} = \mathbb{D}_{313313} = \mathbb{D}_{322232} = \mathbb{D}_{322322} = \mathbb{D}_{323233} = \mathbb{D}_{323323},
\end{aligned}
$$

where

$$
d_6 = \frac{L^4}{1680}\left(\bar{k}_\eta + 24\bar{k}_\tau\right).
$$

The seventh group is

$$
\begin{aligned}
d_7 = \quad & \mathbb{D}_{123123} = \mathbb{D}_{123213} = \mathbb{D}_{132132} = \mathbb{D}_{132312}, \\
d_7 = \quad & \mathbb{D}_{213123} = \mathbb{D}_{213213} = \mathbb{D}_{231231} = \mathbb{D}_{231321}, \\
d_7 = \quad & \mathbb{D}_{312132} = \mathbb{D}_{312312} = \mathbb{D}_{321231} = \mathbb{D}_{321321},
\end{aligned}
$$

where

$$
d_7 = \frac{L^4}{1680}\left(\bar{k}_\eta + 80\bar{k}_\tau\right).
$$

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