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Quantitative operators as an analytical tool for explaining differential equation students' construction of new quantities during modeling

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ABSTRACT

Theories of quantitative reasoning have taken precedence as an analytical tool to interpret and describe students' mathematical reasonings, especially as students engage in mathematical modeling tasks. These theories are particularly useful to describe how students construct new quantities as they model. However, while using this lens to analyze Differential Equations students' construction of mathematical models of dynamic situations, we found cases of quantity construction that were not fully characterized by extant concepts. In this theory-building paper, we present five examples of such cases. Additionally, we introduce a new construct—quantitative operators—as an extended analytical tool to characterize those cases. Our findings suggest that quantitative operators may be viewed as an extension for theories of quantity construction and complementary to symbolic forms, when localizing theories of quantity construction *for* mathematical modeling, especially at the undergraduate differential equation level.

1. Introduction

Mathematical modeling is important globally for STEM majors (OECD, 2016). However, across grade bands, mathematical modeling (hereafter: modeling) presents many challenges to learners (e.g., Jankvist & Niss, 2020; Lyon & Magana, 2020). At the same time, cultivating students' modeling skills presents challenges to educators (Blum & Borromeo-Ferri, 2009; Manouchehri, 2017). At its core, modeling is a representational activity—a mathematical model *represents* a real-world scenario mathematically to the modeler. For these reasons, the prevailing approach to studying mathematical modeling has been the cognitive perspective due to its focus on the cognitive processes involved in model construction and its promise of intervening in students' model construction (Kaiser, 2017; Kaiser & Sriraman, 2006). However, as it stands, the cognitive perspective on modeling does not afford the detailed analysis of the operations involved in constructing quantitative relationships, which we believe is central to model construction.

Existing research suggests that operationalizing modeling through theories of quantitative reasoning (Thompson, 2011) has productive implications for research (e.g., Czocher et al., 2022) and education (e.g., Carlson et al., 2015). Several studies have illustrated that students' construction of robust quantitative relationships supports the construction of meaningful mathematical expressions,

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with deep mathematical understandings rooted to real-world scenarios (e.g., Castillo-Garsow, 2010; Ellis, 2007,2011; Kafetzopoulos & Psycharis, 2022; Moore, 2014; Moore & Carlson, 2012). Conceiving modeling as constructing a network of quantitative relationships (Thompson, 2011) has the power to keep the students on track through goal setting and encourages them to reason about how the quantities vary (or not) with each other in order to determine mathematical structures for real world situations (e.g., Basu & Panorkou, 2019; Castillo-Garsow, 2010).

Following this trend, as part of a larger project, our research team studied scaffolding moves that could successfully guide Differential Equations (DE) students toward constructing models of dynamical systems. To accomplish this major research goal, we needed to catalog the scaffolds that supported the modelers to make progress in their model construction process, which we operationalized in terms of constructing and manipulating quantities. Thus, we set a sub-goal to analyze and document the ways in which DE students constructed new quantities and quantitative relationships as they worked on modeling tasks. During our analysis, we found that there were cases where participants constructed quantities for which we were not able to use existing theories on quantity construction for characterizing them. As a tool, the theories on quantity construction we referenced above were limited in accounting for the observations we made, in a new setting with a new purpose. In such cases, it is recommended to take theory as an object of inquiry with the intention of adapting it and expanding its domain of validity to new contexts (Assude et al., 2008; Cobb, 2007; Lester, 2005).

With these considerations in mind, in this theory-building paper, we showcase our extension of existing theories on quantity construction to examine a novel phenomenon, distinct from the context (setting, participants, and mathematical concepts) and focus (research goal and phenomenon of study) in which those theories of quantity construction were developed. We present a set of cases of modelers' construction of quantities for which existing theories on quantity construction were limited in their capacity to analyze. When we say existing theories were limited, we mean we encountered instances of modelers' construction of quantities where there was no clear evidence that modelers' activities were compatible with the already established theories *or* that the theoretical lens was limited in describing modelers' *mathematizing* activities. We propose a new construct—*quantitative operators*—as an analytical tool to explain DE students' construction of new quantities during modeling. We make the argument that quantitative operators can be viewed as an extension to existing theories on quantity construction *for* analyzing the mathematics of modeler. In particular, to explain cases of DE students' construction of new quantities by operating on existing quantities, during modeling. In light of this argument, we first summarize the existing theories on quantity construction. Next, we explain our perspective on modeling. Next, we discuss our reasons for adopting a quantitative reasoning perspective to study modeling and its prospective limitations. Then, we discuss how our approach overcame those limitations in the methods section. Next, we present our findings as results of the methods we undertook to overcome the theoretical limitations. Finally, we end with connections to other parallel theories and implications of our findings.

2. Theories on the construction of quantities

Quantitative reasoning entails the mental operations involved in conceiving a real-world situation consisting of quantities and relationships among conceived quantities (Thompson, 2011). Within this perspective, quantities are mental constructs of measurable attributes that consists of three inter dependent entities: object, attribute, and a conceived measurement process (Thompson, 2011). Hence, quantification of a measurable attribute entails "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (Thompson, 2011, p.37). For example, consider the quantity *the number of distinct birds that visited the backyard on a Saturday morning, from 9am to 1pm.* Here, the object is the backyard habitat, and the attribute is the number of birds during the given time. We propose two measurement processes to highlight the distinctions in the ways in which this quantity may be conceived. First, a modeler may stay out in the backyard and count the distinct number of birds, one-by-one, that visited the backyard. Second, a modeler may count the number of birds that visited the backyard during the first hour, and then *instantiate a rate* between the number of birds that visited the backyard during an hour and the total time period (e.g., 7 birds an hour for 4 hours). In the first method we proposed, the quantification was directly measurable; in the second method quantification required *operation on existing quantities*, which this study focuses on.

Thompson (1994) defined *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.9). Other examples of quantitative operations include: combining quantities additively, combining quantities multiplicatively, comparing quantities additively, comparing quantities multiplicatively, and generalizing a ratio (see Table 1). As a result of a quantitative operation a *quantitative relationship* is created among the quantities operated, the quantitative operation, and the result of operating—the new quantity (Thompson, 1990). In other words, a quantitative relationship is

Table 1
Quantitative structures and their arithmetic operations (Thompson, 1990, p.26).

Quantitative Structure	Arithmetic operation to evaluate the resultant quantity
A quantity is the result of an additive combination of two quantities	Addition
A quantity is the result of an additive comparison of two quantities	Subtraction
A quantity is the result of a multiplicative combination of two quantities	Multiplication
A quantity is the result of a multiplicative comparison of two quantities	Division
A quantity is the result of an instantiation of a rate	Multiplication
A quantity is the result of a composition of ratios	Multiplication

the "conception of three quantities, two of which determine the third by a quantitative operation (Thompson, 1990, p. 12)." A network of such quantitative relationships is known as a quantitative structure (Thompson, 1990).

New quantities may also be constructed through reasoning about how existing quantities vary in relation to each other, known as covariational reasoning. Covariational reasoning (Carlson et al., 2002) means "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). For example, a modeler who is modeling a backyard habitat where cats prey on birds, might reason that "as the cat population increases, the bird population decreases." This kind of reasoning is an example of gross coordination of values (Thompson & Carlson, 2017). In contrast, if a modeler coordinates values of the cat population with values of bird population (for example, evidenced through drawing a graph, making a table), she is engaging in the coordination of values. Thompson and Carlson (2017) and Carlson et al. (2002) have proposed frameworks for the different mental operations involved in covariational reasoning. Scholars have built on frameworks of covariational reasoning to describe how students construct new quantities. For example, Johnson (2015) investigated how students quantified *rate* through unifying theories from quantitative reasoning (Thompson, 2011) and covariational reasoning (Carlson et al., 2002).

According to Thompson (1990), quantitative operations and arithmetic operations are distinct in the following way: while quantitative operations are the mental operations involved in constructing quantitative relationships, an arithmetic operation is a numerical operation that is used to determine the value of the quantity. Table 1 illustrates the distinct quantitative structures and the canonical arithmetic operations that are used to evaluate the resultant quantity. In our study, we used the quantitative structures illustrated in Table 1 as an analytical tool to make sense of modelers' construction of quantitative structures during modeling. We refer to this table again in the discussion section.

In some theories of quantitative reasoning, importance is placed on values of quantities and arithmetic operations for combining them (e.g., Schwartz, 1996). In contrast, Thompson (2011) proposes quantity as an object that is constructed by an individual and emphasizes the amount-ness (Moore et al., 2022)of the quantities and the mental operations that are involved in their construction. This view on quantities enables us to explain the evolution of a modeler's model in terms of the quantities the modeler conceived and the mental operations she performed. Therefore, in our study, we adopted Thompson's (2011) and his predecessors' view on quantity construction *for* modeling. We refer to Thompson's notion of constructing quantities through operating on existing quantities as *Theories of Quantity Construction* (TQC) through the rest of this paper. Additionally, we provide our perspective on modeling through a cognitive constructivist lens and further justification for adopting theories of quantitative reasoning (referred to as QRT), the umbrella of TQC, to analyze modelers' mathematics[1] in the sections that follow.

3. Our perspective on modeling

Within modeling research, our work lies within the context of the cognitive perspective of modeling (Kaiser, 2017), which entails the analysis and understanding of the cognitive processes involved in model construction (Kaiser & Sriraman, 2006). Within this perspective, modeling is typically seen as a non-linear process consisting of a set of competencies (and sub-competencies) that are unique to organizing a real-world situation into a mathematical representation and are depicted using mathematical modeling cycles (MMC). While pluralities in the MMCs adopted by scholars exist (e.g., Blomhoj & Hojgaard-Jensen, 2003; Blum & Leiss, 2007), many of them share the following modeling competencies: *understanding* of the messy real-world situation, *simplifying* the messy real-world situation into a problem that can be solved using mathematics, *mathematizing* the problematized situation, *performing mathematical analysis* to obtain mathematical results, *interpreting* the mathematical results, and *validating* the results against real-world constraints.

Within the cognitive perspective of modeling, the primary aim of research has been to provide description of students' cognitive processes involved in developing mathematical models of real-world situations and to document the challenges modelers experience (from a researcher's perspective) while working on modeling tasks. For example, researchers have re-constructed students' individual modeling routes through examining modeling sub-competencies (e.g., Borromeo-Ferri, 2007), examined how modelers simplify messy real-world situations (e.g., Jablonski, 2023), examined how learners organize these parameters and variables into mathematical representations (e.g., Murata & Kattubadi, 2012; Stillman & Brown, 2014), reported the ways in which students validate their models (e.g., Czocher, 2018), and catalogued the cognitive blockages students experience during modeling (e.g., Jankvist & Niss, 2020; Stillman et al., 2010;).

The cognitive perspective of modeling, with its focus on competencies that contribute to (or hinder) students' progress in a modeling task, has been effective in describing and interpreting modelers' engagement with modeling tasks, while also framing the learning outcomes educators can expect from that engagement. For example, scholars have proffered ways for improving the learning of modeling through supporting modeling competencies directly and designing the appropriate learning environments to deliver that support (e.g., Anhalt, et al., 2018; Lesh et al., 2003; Stillman, 2011; Stillman et al., 2010). However, challenges in cultivating students' modeling skills stem from that same focus on describing *macroscopic phases*[2] of modeling, which do not provide fine grained analysis (microscopic view) of how the models themselves *evolve* as modelers construct them. Without an understanding of how modelers' models evolve, educators are constrained in articulating interventions to modelers' model construction in ways that uphold modelers'

[2] Following Steffe & Thompson (2000), we use the term modelers' mathematics to refer to modelers' mathematical realities, which are distinct from the modeling-educators' mathematics.

[3] We refer to macroscopic phases of modeling to mean the phases included in a typical MMC, such as *mathematizing* or *interpreting*, and microscopic view as the immediate next step a modeler makes in her model construction process, such as coordinating change in a population with change in time.

autonomy while generating meaningful (to modelers) representations. Therefore, bringing in novel theoretical approaches to viewing cognitive modeling may benefit the field (Cevikbas et al., 2021).

As a result, we synthesize a *cognitive constructivist* view on modeling by integrating constructivist TQC to analyze DE students' mathematical model construction. From our perspective, modeling (and in particular *mathematizing*) can be viewed as analyzing a real-world situation into a quantitative structure (Thomson, 1990). Our view of modeling also goes beyond the mental operations involved with quantities to also include other mathematical operations that aid in representing a real-world situation as a mathematical object. From this perspective, a mathematical model can be viewed as a quantitative structure that represents how the quantities of a real-world situation are related to each other, where the quantities act as the building blocks that make up the model (Larson, 2013).

4. Prospective benefits and existing limitations of adopting and integrating QRT to strengthen the cognitive perspective of model construction

There is value to theorizing mathematical model construction through the lens of quantitative reasoning (Czocher et al., 2022; Moore & Carlson, 2012; Thompson, 2011). First, many studies investigate students' quantitative reasoning while treating mathematical modeling as a *means* to learning mathematical concepts. Curricular materials developed in this way have proven successful for the teaching and learning of mathematical concepts such as algebra (e.g., Smith & Thompson, 2007), calculus (e.g., Carlson et al., 2015), and functions (e.g., Ellis, 2011). Elaborating points of contact between QRTs and the cognitive perspectives on modeling would enable researchers in both traditions to design curricular materials that have the potential to advance students' modeling skills as an *end* to learning.

Second, and more importantly to our argument, there is a natural connection between modeling (creating a mathematical model) and quantitative reasoning (constructing a quantitative structure). We can understand mathematical modeling as the organization of real-world situations using mathematical representations that illustrate how quantities within the real-world situation are in relation to each other. Using QRT as an analytic tool is a powerful way to analyze students' model construction process in terms of the quantities they impute to the situation, the reasonings they make on how the values of the quantities vary in relation to each other, the new quantities they construct through operating on existing quantities, and the relationships they form between the quantities. In this way, QRT can be used to describe students' construction of new quantities, hence the evolution of their models (Czocher & Hardison, 2021). This approach to analyzing modelers' modeling activities formulates the immediate next step in the modelers' model construction process. It thus organizes students' modeling activity, at a fine-grained level, from the perspective of an observer and therefore sheds light on developing instructional moves that meet the students where they are.

Amending the cognitive perspective on modeling with QRTs suggests that construction of new quantities by operating on existing quantities is responsible for the evolution of a mathematical model. For research programs whose focus is not on leveraging students' quantitative reasoning to promote mathematical conceptual development—but instead on development of a mathematical model—we found an opportunity to extend and complement TQC through examining instances of modelers' mathematics that (a) were contributing to model evolution *and* (b) were borderline cases within TQC. These borderline instances may be easily discounted as a construction of a new quantity through operating on existing quantity(s) but could provide valuable insights to modelers' model construction process. We motivate our work by presenting a simple example of this case.

One of our participants, Pai (a senior, Economics major), was working on constructing a model for the rate of growth of a simple interest bank account. Pai first constructed an expression for the amount of money present in the bank account in terms of time, t. He next took the derivative with respect to time to construct the rate at which the account grows. While Pai's activities may signal typical computations that use calculus-based operations, we provide an alternative interpretation in terms of quantity construction. Pai's construction of the bank account's rate of growth followed the following order of operations: (a) Pai first constructed a quantity that measured an amount in relation to other quantities (i.e., time and initial deposit) and (b) Pai operated on the quantity that measured an *amount* by formulating a derivative to construct a quantity that measured a *rate*. In essence, Pai transformed an amount into a rate through a derivative. We found that we were not able to use TQC to describe Pai' modeling activities. However, we observed from our data that taking the derivative to construct a rate of change is an operation that is common among STEM majors and often is a fruitful endeavor in modeling canonical scenarios from across DE curricula.

Cases like Pai's may pose challenges for analyzing modelers' mathematics through TQCs for the following reasons. First, TQC is concerned with the mental operations involved in organizing the world around us into a quantitative structure. The focus is primarily on *mental operations* and *amount-ness* of the quantities. Within modeling, representing the quantitative relationships into mathematical representations that incorporate mathematical symbols and testing the model against real-world conditions are processes that are indispensable. This disparity between the two theoretical perspectives may pose limitations in using TQC to explain modelers' mathematics (Czocher et al., 2022). Second, TQC were developed through investigating students' thinking up to the calculus level. Therefore, using TQC for analyzing students' construction of new quantities with students who have already taken differential equations, may pose limitations. Third, existing TQC were developed to meet students' intellectual need (Harel, 2013) for constructing quantitative relationships (Johnson, 2023). Students' intellectual need for quantitative relationships entails the students' need to mathematize a problematized situation through constructing quantitative relationships through *operating among constituent quantities* (Johnson, 2023). However, we hypothesize that existing TQC may not attend to students' intellectual need to construct quantitative relationships through *transforming* a single quantity using mathematical operators that enable that transformation, like Pai did. We address such cases in this paper.

Finally, although Thompson (1994) defined quantitative operations as the mental operations on "one or more already-conceived quantities" to construct a new quantity, the definition of a quantitative relationship (Thompson, 1990) and the examples given for quantitative operations (Thompson, 1994) emphasize operations on two quantities to conceive a third new quantity. Therefore, it is not clear whether operations performed on one quantity to construct a new quantity fall within the scope of quantitative operations (or what they would even look like). In addition, through our analysis, we observed that students may engage in operations on quantities without clear evidence of the operations having a situationally relevant quantitative meaning, but the resultant quantity has a quantitative meaning for the student. For example, in Pai's construction of rate of growth of the bank account, there was no discernible evidence that taking the derivative of the amount constituted the multiplicative comparison of change in amount and change in time. However, the result of the operation carried situationally relevant meaning—a measurable attribute of the bank account.

To address these nuanced challenges—where a modelers' operations might appear simplistic at first glance but are crucial for advancing their model construction process—we introduce a tool that captures the borderline cases. We define *quantitative operators* as machines that take in a singular quantity and output a new quantity, depicted in Fig. 1. We see the purpose of a quantitative operator as *transforming* an existing quantity into a new quantity, using mathematical operators that enable the modeler's intended transformation. We distinguish quantitative operators from Thompson's quantitative operations as follows: Thomspon's quantitative operations emphasize that both the operation (enactment) and the resultant quantities (product) have situationally relevant quantitative meanings. In contrast, quantitative operators do not require this situational relevance during modelers' enactment of them, but the resultant quantity still holds situationally relevant quantitative meaning to the modeler. Additionally, quantitative operators diverge from TQC by emphasizing the transformation of a singular quantity rather than operation among multiple quantities. In the study we describe below, we address the following questions: *What are some cases of modelers' construction of new quantities, through operating on existing quantities, that cannot be characterized by extant TQC? How can quantitative operators be used as an analytical tool to describe such cases?*

5. Methods

5.1. Setting and events

The data reported in this paper were drawn from a larger project that studied how to effectively scaffold DE students to construct mathematical models for dynamic situations. In the larger study, data was collected through a combination of teaching experiments (Steffe & Thompson, 2000) and clinical interviews (Goldin, 1997) from 30 STEM undergraduates at a large public university in the southwestern United States. In this paper, we focus on a subset of the data we identify as events, arising from five participants' work on modeling tasks from the larger project. These events were purposefully streamlined from the larger data set due to the presence of quantity construction through *transforming* existing quantities, that existing TQC were limited in characterizing. The five participants considered here were undergraduate STEM majors (economics, computer science, physics, civil engineering, and mathematics), who were either in their 3rd or 4th year of college. All five participants had already taken a course in differential equations prior to the interviews.

Each participant engaged in modeling tasks[3] during interview sessions that lasted for around 60–90 minutes. No constraints were imposed onto the participants in conceiving the situation given to model. Each interview session was accompanied by a lead-interviewer and a witness-interviewer. The lead interviewer's primary goal was to unpack how the participants conceived the real-world scenario and provide support through questioning, rather than delivering information, to help the students advance in their modeling process.

Our five events consisted of (1) Pai's work on the Cats and Birds task, (2) Winnow's work on the Tuberculosis task, (3) Ivory's work on the Tropical Fish task, (4) Pattern's work on the Cats and Birds task, and (5) Szeth's work on the Pruning task (see Table 2 for task statements).

5.2. Data analysis and constitution of cases

We adopt Ragin's (1992) view of cases as empirically observable phenomena. Constitution of cases emerged through analysis, as we encountered instances of participants constructing new quantities through operating on existing quantities, that TQC were limited in accounting for. Once these instances were identified, we developed rich, interpretive accounts of the cases (see Geertz, 1973). The identification and development of the cases comprised of the following steps: First, we watched the videos and identified instances where participants constructed new quantities by operating on existing quantities. Next, we summarized what the participants did in each of those instances. Next, we leveraged criteria for quantification (Czocher & Hardison, 2021), constructs from quantitative operations (Thompson, 1990; 2011), and the mental operations involved in covariational reasoning (Thompson & Carlson, 2017) to interpret how the participants constructed new quantities by fine-tuning our summaries from the previous step. At this level, we also carefully sorted borderline instances where the participants did not show clear evidence of having situationally relevant quantitative meanings for the operations they performed on existing quantities to construct new quantities *or* instances where the existing TQC were limited for describing what the participants did in order to construct new quantities. Once we had a catalogue of the cases where participants' construction of new quantities could not be explained by existing TQC, we asked: how did the participant *transform* an existing quantity (or quantities) to conceive a new quantity? Answering this question engendered the analytical tool *quantitative*

[4] We take an expansive view on modeling tasks in the sense that the characteristics of a modeling task are considered from the perspective of the modeler actively working on them (rather than relying solely on the information provided in the task).

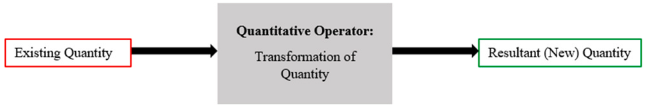


Fig. 1. Quantitative operators as a machine.

Table 2
Tasks (abridged versions) giving rise to the events.

Pruning	Imagine you have a hedge in your garden of some size, *S*, and you want it to increase its size even more. Your gardener advises you that the overall rate of growth of a plant will depend both on the extent of pruning and on the regrowth rate, which is particular to the plant species and environmental conditions. Both rates can be measured as a percentage of the size of the plant. The pruning rate can be adjusted to result in a target overall growth rate. Can you derive a model for the rate of change of the size of the plant?
Cats and Birds	Consider a backyard habitat, where cats are the natural predators of birds. Model the rate of decrease of the population of birds due to predation by cats.
Tropical Fish	The strength of a buffering solution entering the tank varies according to $1 - e^{-t/20}$ grams per liter. The buffering solution enters the tank at a rate of 5 liters per minute. Create an expression that models how quickly the amount of buffering agent in the tank is changing at any moment in time
Tuberculosis	Tuberculosis (TB) is a serious infectious disease that can cause death. Imagine a community where sick and well members move about freely among one another. Create a mathematical model for the rate that the disease will spread through the community (Boyce & DiPrima, 2012).

operators and nominalized the five cases, reported as findings. We nominalized each case to exhibit the modeler's intended quantitative structures, despite the canonical meanings each nominalization may carry. Participants' intended quantitative structures were inferred through localized goals participants' set to resolve a problematized situation (Simon et al., 2004; Johnson, 2023). A sample analytical process, using Pai's construction of the rate of growth of a simple interest bank account, is illustrated in Table 3.

6. Results

We report five cases of participants' construction of new quantities that extant TQC were limited in characterizing, but the analytical tool *quantitative operators* could. For each case, we situate the participant's work within the task setting, describe how each case is difficult to characterize using TQC, and the theoretical need for placing the participant's work under the new construct—quantitative operators.

6.1. The derivative operator: transforming an amount to a rate

The derivative operator arises from transforming a quantity that measures the amount of an object to a quantity that measures the rate at which the amount changes. To illustrate this case, we present Pai's work from The Cats and Birds Task. Pai constructed Expression 1 to represent "the number of bird-cat interactions that resulted in a kill" (which for Pai, was also the "number of birds dead at time *t*").

$$f_k(t) = [\alpha \cdot B(t) \cdot C(t)]\beta \qquad \text{(Expression 1)}$$

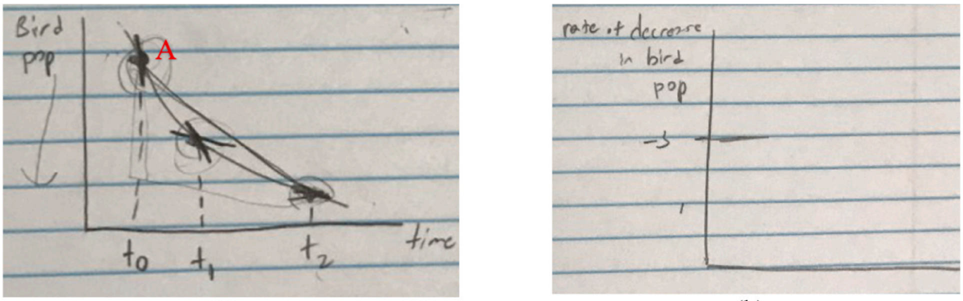
Above, $B(t)$ and $C(t)$ represented the bird and cat population at time t and α was given as the percentage of potential cat-bird interactions that was realized. Pai defined β as the percentage of the realized cat-bird interactions that resulted in a bird's deaths. The interviewer asked Pai to construct an expression for the rate of decrease of the bird population due to cat predation with respect to time. Pai's initial conception of this prompt was to draw a graph of bird population vs time, where he coordinated the values of the bird population and time (Fig. 2(a)).

After drawing the graph in Fig. 2(a), Pai explained that the slope of the tangent gives the rate of decrease of bird population:

Pai: I think this question is trying to plot the function for the derivative of my model, because it wants the rate of decrease. I think we're asking for decrease in bird pop would be the y axis. Over time *t* [*draws the x axis in* Fig. 2*(b)*]. The rate would be whatever this slope is [*pointing at A in* Fig. 2*(a)],* called, I don't know. Negative three. It would be like that [*draws the graph in* Fig. 2*(b)]*

Table 3
Sample analytical process.

Resultant Quantity	Quantity(ies) Operated Upon	Description of Construction Process	Can TQC be used to explain the quantification process?
Amount of money in the bank after *t* years	Initial deposit amount, interest gained during *t* years	Amount of money in the bank after *t* years was constructed through *additively combining* two quantities: initial deposit amount, interest gained during *t* years.	Yes, additive combination is a quantitative operation that combines two existing quantities to construct a new quantity.
Rate of growth of the bank account	Amount of money in the bank after *t* years	To construct rate of growth, Pai set a subgoal to construct the amount of money in the account. He then took the derivative of the Amount of money in the bank after *t* years	It is unclear if taking the derivative qualifies as a quantitative operation without additional evidence.



(a) (b)

Fig. 2. Pai's graph for (a) the bird population varying with time (b) the rate of decrease of bird population with respect to time varying with time.

We interpret that Pai constructed an image of the slope of the tangent line to the curve in his graph in Fig. 2(a), at different points in time, in order to derive a model for the rate of change of the bird population due to cat predation with respect to time. Pai stopped drawing the graph he initiated in Fig. 2(b) and decided to take the derivative both sides of Expression 1 with respect to time. Pai justified his decision as below.

Pai: We're plotting the derivative, the rate of decrease at these times [*referring to* Fig. 2*(a)*]. I need a model for the derivative. I think it would be $f_k{}'(t)$…If we do the product rule, $f_k{}'(t)$ we get something like that [*Referring to the result of applying the product rule]*, where B' and $C'(t)$ respectively are the change in the bird [and cat] population.

Pai used the product rule to obtain Expression 2 as the rate at which the bird population decreases due to predation by cats with respect to time. Upon constructing Expression 2 and reasoning through Expression 2, Pai clarified that $f_k'(t)$ represented the rate of decrease of bird population due to cat predation with respect to time and $B'(t)$ represented the rate of change in bird population due to all causes with respect to time.

$$f_k{}'(t) = \beta \cdot \alpha[B'(t) \cdot C(t) - B(t) \cdot C'(t)] \qquad \text{(Expression 2)}$$

In this example, we described how Pai operated on existing quantities to derive an expression for the rate of decrease of bird population due to predation by cats with respect to time. First, Pai constructed the slope of the tangent line at different points on the curve for the graph he drew for how the bird population varies with time. Soon after, Pai took the derivative of Expression 1 (to Pai Exepression 1 represented the "the number of bird-cat interactions that resulted in a kill") with respect to time. These modeling activities, especially Pai's construction of the tangent line, may be seen as Pai enacting some sort of coordination between two quantities. In particular, the *multiplicative comparison* between change in bird population and change in time. However, there was not enough evidence to support this claim. We infer Pai's actions of drawing the tangent lines to find the slope was largely in service of finding the derivative in order to construct the rate of change. This is because (a) Pai did not complete the graph he initiated in Fig. 2(b) and (b) decided to take the derivative of $f_k(t)$ to construct the rate of decrease of the bird population due to cat predation with respect to time. We conjecture that, for Pai, the phrase "rate of change" acted as a cue to take the derivative of a function (e.g., Jones, 2017), dropping the situational attribute that function was measuring. Regardless, through taking the derivative, Pai constructed a quantity that was meaningful for him. If Pai had explicitly indicated that through constructing the tangent line he was intending to multiplicatively compare the change in bird population due to cat predation with change in time, we may have credited Pai as having engaged in quantitative operations as described in TQC. In turn, Pai proceeded with taking the derivative of "the number of birds dead at time t" with respect to time. In both these operations on quantities, Pai's goal was to *transform* an existing quantity that measured an amount, through formulating a derivative, into a quantity that measured the rate of change.

6.2. The negation operator: transforming an inflow to an outflow

The negation operator arises from transforming a quantity to a new quantity that measures the opposite flow of the extant quantity. To illustrate this case, we present Winnow's work from The Tuberculosis Task. Winnow constructed Expression 3 as a model for the rate of change of the sick people with respect to time.

$$\frac{dS}{dt} = \frac{m}{S(t) \times H(t)} \cdot H(t) \cdot r \qquad \text{(Expression 3)}$$

In Expression 3, Winnow defined $H(t)$ as the number of healthy people at time t, $S(t)$ as the number of sick people at time t, m as the number of contacts between healthy and sick people that actually occur, and r as the transmission rate. For Winnow, $\frac{m}{S(t)\times H(t)}$ represented the percentage of the healthy people encountering sick people per unit time. Through interviewer support, Winnow agreed that, $\frac{m}{S(t)\times H(t)}$ represented the exposure rate. He then multiplied $\frac{m}{S(t)\times H(t)}$, $H(t)$, and r to model the number of healthy people that get sick per unit time. When the interviewer asked for a model for the rate of change of the healthy people with respect to time, Winnow constructed Expression 4 as an initial model for the rate of change of healthy people with respect to time.

$$\frac{dH}{dt} = -\frac{m}{S(t) \times H(t)} \cdot S(t) \cdot r \qquad \text{(Expression 4)}$$

Winnow validated his model by checking against specific conditions, $S(t) = 10$ and $H(t) = 10$. He was satisfied that substituting for $S(t) = 10$ and $H(t) = 10$ in Expression 3 and Expression 4 yielded the same value, but opposite in signs. He asserted that the rate of change with respect to time for sick people would have the same value as the rate of change with respect to time for healthy people. To perturb Winnow, the Interviewer asked him to validate his model against the values $S(t) = 2$ and $H(t) = 10$. Realizing that the output values from his expressions for $\frac{dS}{dt}$ and $\frac{dH}{dt}$ would not be equal, Winnow modified his model for the rate of change of healthy people with respect to time to be $\frac{dH}{dt} = -\frac{dS}{dt}$. Winnow justified his model, asserting that "the number of new sick people and the decrease in healthy people should be the same." We take Winnow's actions as evidence that he had conceived that the rate of change of healthy people with respect to time should have the same value, but opposite in flow, as the rate of change of sick people with respect to time.

For Winnow, the " $-$ " in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented only a gross covariation between the number of healthy people and number of sick people. We make this inference due to the justification he gave to his model as: "[the rate of change of healthy people with respect to time] would be a negative number because…the number of healthy people would be decreasing [as the number of sick people increase]." However, there was no clear evidence if the $-$ was measuring a situational attribute. If Winnow had indicated that the $-$ in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented $\frac{-1 \text{ healthy person}}{1 \text{ sick person}}$ (the number of healthy people decrease by one for each person that gets sick)—a measurable attribute of the system of healthy and sick people—then we would have credited Winnow to have engaged in the *multiplicative combination* of two quantities (number of people removed from the healthy population for each person getting sick and the rate of change of the sick people with respect to time), making it a quantitative operation as defined in TQC. However, there was no clear evidence whether the " $-$ " in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented a quantity, let alone -1, for Winnow. Through Winnow's modeling actions we infer that Winnow negated the rate of change of sick people with respect to time to construct the rate of change of healthy people with respect to time, because for Winnow, the number of *new* sick people, directly corresponded to the *decrease* in the number of healthy people. Therefore, the " $-$ " in $\frac{dH}{dt} = -\frac{dS}{dt}$ acted as a quantitative operator to transform a quantity that measured a flow into a quantity that measured the opposing flow, that was afforded through allowing gross variation of those quantities.

6.3. *The anti-derivative operator: transforming a rate into an amount*

The anti-derivative operator arises from transforming a quantity that measures a *rate* at which the amount of a quantity changes to a quantity that measures the *amount* of the quantity. To illustrate this case, we present Ivory's work from The Tropical Fish Task. Ivory was working towards constructing a model for the amount of buffering agent in the tank at time *t*. To accomplish this goal, Ivory first constructed Expression 5 to represent the rate at which the amount of buffering agent enters the tank.

$$m_E(t) = 5 \cdot \left(1 - e^{\frac{-t}{20}}\right) \qquad \text{(Expression 5)}$$

Ivory defined $m_E(t)$ as the rate at which the amount of buffering agent enters the tank at time *t*. After constructing Expression 5, she stated that she would take the integral of $m_E(t)$ in order to construct an expression for the total amount of buffering agent in the tank at time *t*. Following this reasoning, Ivory constructed expression 6 to represent the amount of buffering agent inside the tank at time *t*.

$$M(t) = \int m_E(t) \qquad \text{(Expression 6)}$$

After the interviewer reflected that Expression 6, as written, does not take into account the amount of buffering agent that leaves the tank, Ivory modified expression 6 to the one shown in Fig. 3 to account for the amount of buffering agent that had also exited the tank. She confidently voiced that the expression in Fig. 3 "would work," because she justified "the total things that have entered minus the total things that have left." In Fig. 3, Ivory defined $m_L(t)$ as the rate at which the amount of buffering agent leaves the tank at time *t*.

Through Ivory's modeling activities, we infer that Ivory conceptualized that the integral of the rates at which the amount of buffering agent enters and leaves the tank would yield the total amount of buffering agent that enters the tank and the total amount of buffering agent that leaves the tank, respectively. Even though Ivory constructed a quantity that had situationally relevant meaning to her (amount of buffering agent that enters (leaves) the tank at time *t*), we were not able to infer if the operation (taking the anti-derivative) on the singular quantity (rate at which the amount of buffering agent enters (leaves) the tank) had clear evidence of a

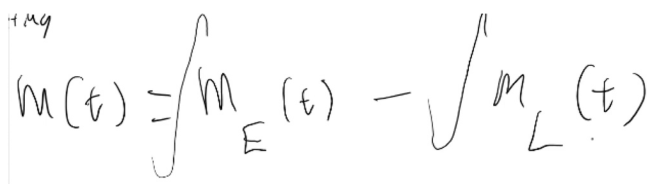


Fig. 3. Ivory's model for the amount of buffering agent in the tank at time t.

situationally relevant quantitative meaning. That is, it was not clear to us, through taking the integral, if Ivory was envisioning additively combining all the little changes in amount of buffering agent (i.e., $m_E(t) \cdot \Delta t$) that had entered (left) the tank since the time of interest to the present time, while also simultaneously coordinating the values of t, $m_E(t)$, and $\int_0^t m_E(t) \cdot dt$ (Thompson & Silverman, 2008). However, we wanted to document Ivory's construction of the amount of buffering agent in the tank at time t, through her transformation of the quantities $m_E(t)$ and $m_L(t)$. In this particular instance, we infer that for Ivory, the anti-derivative acted as an operator to transform a quantity that measured a *rate* to quantity that measured *amount*.

6.4. *The sub-setting operator: transforming an amount to a smaller amount*

The sub-setting operator arises from transforming a quantity that measures an *amount* to a quantity that measures a *part of the whole amount*. We present Pattern's work from The Cats and Birds Task to illustrate the sub-setting operator. Pattern was working towards constructing a model for the number of cat-bird encounters that result in a bird's death. To accomplish this goal, Pattern first constructed an expression to calculate the potential number of cat-bird encounters at time t (Fig. 4(a)). Pattern was then asked to consider how he might modify his expression to account for the fact that only some percentage, α, of the potential encounters are realized. In response to this, Pattern elected to multiply the potential number of cat-bird encounters at time t by α as shown in Fig. 4(b). Pattern explained his reasoning for multiplying by α as follows.

Pattern: So, this [*referring to the expression in* Fig. 4*(a)*] is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total here [*referring to* Fig. 4*(b)*].

Later, Pattern was asked to adapt his previous equations to model the number of birds that died due to a cat encounter accounting for the fact that sometimes a bird might escape. In response, Pattern decided to multiply the potential number of cat-bird encounters at time t by β as shown in Fig. 4(c). Pattern verbally expressed that β is "some kind of rate that gives me the percentage of bird died." When the interviewer asked how he was distinguishing between α and β, Pattern gave the reasoning below.

Pattern: They [α and β] basically function the same way it's just they are solving for different things. Because what I did here was I made it really easy for myself by creating this baseline [*referring to* Fig. 4*(a)]* And from here, you can… I can add whatever I want. So that gives me the freedom of being like, well, since you want to know how many birds die, we can just create this percentage [*referring to* $(C(t) \cdot B(t)) \cdot \beta$, Fig. 4*(c)*].

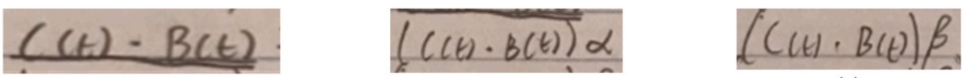
In the first instance, Pattern constructed the number of encounters that actually happened by envisioning a proportion of the potential number of cat-bird encounters at time t. For Pattern, α acted as a multiplicative scaling factor to quantify the proportion of cat-bird interaction that were realized. Next, Pattern constructed the number of encounters that result in a bird's death by taking a different proportion of the potential number of cat-bird encounters at time t. There was no clear evidence that α or β had a situationally relevant quantitative meaning for Pattern. That is, we were not able to infer a situational referent (an object) that the attribute represented by, say for example β, was measuring. We would have credited Pattern to have engaged in a quantitative operation —*instantiating a rate*—between β and $C(t) \cdot B(t)$, if he had shown clear evidence that β measured the number of cat-bird interactions that resulted in a bird death for every 100 potential cat-bird interactions, an attribute of the system of cats and birds. This way β would have been credited as a scalar quantity (Thompson, 1990). However, for Pattern, β acted as an operator that shrank the size the total number of potential cat-bird interaction to represent a subset of that amount. This interpretation was evidenced when Pattern stated that "α and β basically function the same way" and that $C(t) \cdot B(t)$ acted as a "baseline" to consider different proportions of the number of potential cat-bird interactions—number of cat-bird interactions that realized and number of cat-bird interactions that resulted in a dead bird. Pattern accomplished this by using multiplicative scaling factors (α% and β%, respectively) that *transformed* the size of the whole—potential number of cat-bird interactions—through reducing it. In both of these instances, Pattern constructed new quantities by considering a proportion of the "baseline" via using multiplicative scaling factors. However, it was difficult to determine the situational referent that Pattern was associating α% and β% with.

6.5. *The percent-taking operator: transforming an amount to a percentage of the amount*

To illustrate this operator, we present Szeth's work from The Pruning Task. Szeth first constructed Expression 7 where Szeth defined R' as the rate at which the plant would be growing, P as the "pruning," G' as the "regrowth rate," and E as "environmental conditions."

$$R' = P + G' + E \qquad \text{(Expression 7)}$$

After Szeth constructed Expression 7, he mathematized R' and G' as $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ because "both rates can be measured as a percentage of the size of the plant." We interpret that when Szeth read the task "the overall rate of growth [of the plant] will depend



(a) (b) (c)

Fig. 4. Pattern's mathematical models for the Cats and Birds task.

both on the extent of pruning and on the regrowth rate…Both rates can be measured as a percentage of the size of the plant," he interpreted "both rates" to be the rate at which the plant is growing (R') and the regrowth rate (G'), as oppose to regrowth rate and the pruning rate. Nevertheless, Szeth constructed R' and G' as a percentage of the size of the plant, S, through considering $\frac{1}{100}^{th}$ of the size of the whole plant S. In this instance, we were not able to discern a situational object or attribute that $\frac{1}{100}$ was measuring.

Szeth said that he would substitute $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ in Expression 7. Following this, the conversation below was exchanged among us.

Interviewer: You have $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$. So, does that say that R' and G' are both equal to each other, or can they be different percentages?

Szeth: I guess it does say they're equal. I wouldn't take them to be equal. In real life perspective, they are supposed to be different things.

In the above excerpt, when the interviewer asked Szeth if R' and G' were equal, Szeth responded that "in real life…they are supposed to be different things." By that, we interpret that he meant R' and G' measure different qualities of the plant, that may (or may not) have different values. We take this as evidence that for Szeth considering a fraction of the size of the plant—in particular considering $\frac{1}{100}^{th}$ of S—was merely an operation to transform the size of the plant to a percentage of the size of the plant.

In this case, we illustrated how Szeth constructed the quantity G' — "regrowth rate"—through operating on the quantity S—"the size of the plant"—by considering $\frac{1}{100}^{th}$ of S. If Szeth had shown evidence that $\frac{1}{100}$ to him represented the ratio between the size of the plant that grew due to pruning and the size of the plant that was pruned, we would have described his construction of quantity through TQC. However, for Szeth, multiplying by $\frac{1}{100}$ acted as an operator to transform a quantity that measured amount to a percentage of that amount.

7. Discussion

The purpose of this paper was to demonstrate our adaptation of existing TQC to a modeling context at the undergraduate, differential equations level. We presented five cases of DE students' construction of new quantities where we were not able to characterize them use existing TQC. We argued how each case fits the definition of our new construct—quantitative operators. In particular, we presented cases of constructing quantities through transforming (1) an amount to a rate, (2) an inflow to an outflow, (3) a rate to an amount, (4) an amount to a smaller amount, and (5) an amount to a percentage of the amount. To characterize these cases, we introduced the derivative operator, negation operator, anti-derivative operator, sub-setting operator, and the percent-taking operator, respectively. In this section, we discuss its contribution to theory, implications for research and teaching, and limitations.

7.1. *Contribution to theory*

7.1.1. *An extended framework of TQC for modeling at the DE level*

A significant implication of the findings reported in this paper is an extension of TQC for modeling. Thompson (2011) proposed a list of distinct structures involved in the construction of new quantities, through quantitative operations, and their respective arithmetic operations to evaluate the resultant quantity (See Table 1). Thompson acknowledged that the list he proposed "[while] sufficient for middle school mathematics, and most of high school, it can easily be extended" (p. 43). We view our findings as expanding the list of quantitative structures to accommodate the construction of new quantities that may arise while modeling dynamic systems in undergraduate STEM education. Table 4 presents the quantitative relationships and the canonical arithmetic/calculus operations, that were used by our participants, to evaluate the quantity that results from using a quantitative operator to transform a quantity.

Following Thompson, we distinguish between arithmetic/calculus-based operators and quantitative operators in the following way. An arithmetic/calculus-based operator is used when an individual is attempting to evaluate the measure of a quantity. While arithmetic operators are sufficient at the K-8 educational setting, calculus-based operators appear in secondary and tertiary educational levels, especially with students' who have experience with DE. In contrast, a quantitative operator is used when a modeler constructs a new quantity through transforming an existing quantity, using a mathematical operator that affords such transformation. For example, consider a modeler who quantifies rate by applying the derivative operator to transform an amount, like Pai did. This process leads to the construction of a new quantity. In contrast, the modeler may evaluate the *measure* of the quantity using the rules of derivatives. The use of such mathematical operators as *transformational tools* to *construct new quantities* (rather than for symbolic manipulations and calculating values) signal a mark of sophistication in modelers' conceptualization and the purposes of mathematical operators.

7.1.2. *Quantitative operators and symbolic forms are complementary views on model construction*

In this section, we discuss connections between *quantitative operators* and *symbolic forms* (Sherin, 2001). Symbolic forms encapsulate the different meanings students ascribe to equations (Sherin, 2001). Symbolic forms consist of a symbol template and a conceptual schema associated with that symbol template. The symbol template refers to the arrangements of symbols in an expression or equation and the conceptual schema refers to the semantics underlying the arrangement (Jones, 2013, p. 124). The findings presented in this paper can be examined through the lens of symbolic forms. Consider Pattern's model for the number of cat-bird encounters that are realized—$\alpha \cdot C(t) \cdot B(t)$. Pattern created a "baseline" for the potential number of cat-bird encounters at time t—$C(t) \cdot B(t)$—and multiplied this "baseline" by α to account for the fact that only a proportion of the potential number of cat-bird encounters are realized.

Table 4
Construction of a quantity through quantitative operators.

Quantitative Structure	Arithmetic/calculus-based operations to evaluate the resultant quantity
A quantity is the result of transforming an amount to a rate	Rules of derivatives
A quantity is the result of transforming a quantity to its opposite flow	Multiplying by -1
A quantity is the result of sub-setting	Multiplication
A quantity is the result of transforming a rate to an amount	Rules of Anti-derivative
A quantity is a result of transforming an amount to a percentage of that amount	Multiplication

In $\alpha \cdot [C(t) \cdot B(t)]$, Pattern used α as a scalar to control the size of the effect on $C(t) \cdot B(t)$—the maximum number of cat-bird encounters. This scenario can be interpreted as a case where the *scaling form* is activated during Pattern's modeling activity. The symbol template associated with this form is $\alpha \cdot \blacksquare$ and the conceptual schema involved is the "production of an entity of the same sort that is larger or smaller than the original" (Sherin, 2001, p.536).

On the surface, symbolic forms seem conceptually similar to quantitative operators, but our empirical work suggests that using quantitative operators to examine students' mathematical modeling may have distinct theoretical advantages. The mechanisms involved in the activation of the symbolic forms scheme is the *mapping* of the conceptual schemes to symbolic templates. In contrast, the activation of quantitative operator schemes is the *transformation* of an existing quantity into a new quantity. A modeler who is trying to construct an equation for an established quantitative relationship may be engaging with the symbolic forms scheme. In contrast, a modeler who is trying to construct a new quantity through transforming an existing quantity is engaging with the quantitative operators scheme. Symbolic forms and quantitative operators may also diverge methodologically in terms of the researchers' use of them as an analytical tool to interpret modelers' construction of two distinct mathematical objects—mathematical expressions and new quantities—principal to model construction. That is, while symbolic forms can be used to analyze how modelers construct mathematical expressions, quantitative operators can be used to analyze how modelers construct new quantities.

In this light, we discuss an alternative interpretation of our findings from the perspective of symbolic forms. While the function that quantitative operators serve is to transform a quantity into a new quantity, this function is achieved by operators that may be expressed by symbolic forms. For example, in order to construct the rate of change, for Pai, $\frac{d\blacksquare}{dt}$ acted as an operator to transform Q into the rate at which the amount of Q is changing with respect to time. At the same time, for Pattern, $\alpha \cdot \blacksquare$ acted as an operator to shrink the potential number of cat-bird encounters to a proportion of it. Through our findings we speculate five such operators that take the following symbolic forms: $\frac{d\blacksquare}{dt}$, $-\blacksquare$, $\int\blacksquare$, $\alpha \cdot \blacksquare$, and $\frac{1}{100} \times \blacksquare$,. In Table 5, we present the symbolic forms that are associated with each quantitative operator presented in our findings.

7.2. Implications to research and teaching

We view the notion of quantitative operators to also play the role of a placeholder to capture the genesis of quantities that (a) do not fit the theoretical definitions of quantitative operations or (b) not evidenced, thereof. Documenting these borderline instances has implications for research: it opens avenues for inquiry in terms of exploring the quantitative meanings behind each of these operations. For example, the field can make progress addressing questions such as *what kinds of analytical indicators serve as evidence of a quantitative operation when a student takes the derivative of a function?* or *what quantitative meanings does taking the anti-derivative bear for an individual?* These revelations will help in building theories of student cognition surrounding calculus concepts that foreground quantitative reasoning. This endeavor is especially important to study DE students' thinking about dynamic situations. In addition, quantitative operators can be used as an analytical tool by researchers to describe modelers' mathematics that otherwise cannot be described using theories of quantitative reasoning (and therefore missed or discarded), giving credence to modeler's mathematics. Documenting these instances and being able to interpret modelers' mathematics during these instances is critical to constructing exhaustive accounts of how modelers' models evolve.

The findings reported here have implications for the teaching and learning of modeling through quantitative reasoning. First, the findings suggest that attending to students' quantitative meanings behind certain mathematical operations, such as taking the derivative or integral, may be a fertile avenue to help modelers progress in their model construction process. At the same time, we also think that educators should not get too caught up comprehending modelers' quantitative meanings in the event that the modeler successfully[4] progresses in their modeling process (e.g., Pattern's use of the sub-setting operator). Second, we view the discourse between a modeling-educator and a modeler, during mathematical modeling instruction, to be turn-taking. Consequently, a modeling-educator's move has the potential to influence the modeler's next course of action, and thus the evolution of the modelers' models. We view our findings to be contributing towards a repertoire of how modelers think and construct new quantities. Modeling-educators and researchers can use this information to inform their moves during in-the-moment interactions with the modelers. Despite reasoning with quantities, constructing mathematical expressions that represent those quantitative relationships can still be challenging for modelers (Czocher et al., 2022)—a critical component of *mathematizing*. Our findings emphasize the importance of paying detailed attention to modelers' mathematization activities (rather than overlooking them), as it may offer valuable insights into the constraints

[5] By successful, we mean both mathematical success and the absence of any internal or external conflict that impedes the modelers' model construction.

Table 5
Symbolic forms associated with each quantitative operator.

Quantitative operator	Symbolic form	
	Conceptual schema	Symbol template
The derivative operator	Taking the derivative gives the rate of change	$\frac{d\blacksquare}{dt}$
The negation operator	Negating a magnitude gives the opposite flow	$-\blacksquare$
The anti-derivative operator	The anti-derivative of rate of change will yield the amount.	$\int\blacksquare$
The sub-setting operator	Scaling to smaller than the original size	$\alpha \bullet \blacksquare$
The percent-taking operator	Scaling to $\frac{1}{100}^{th}$ of the original size	$\frac{1}{100}\times\blacksquare$

modelers may encounter during mathematization. We believe that employing a quantity-construction oriented lens to modeling (rather than a modeling cycle-oriented lens), afforded the micro-level analysis of modelers' models.

7.3. Limitations

We acknowledge that there exist limitations to our findings. First, we do not claim that the quantitative operators we listed are exhaustive. Studying a set of participants at a different educational level working on a different set of modeling tasks may yield cases absent from the list provided in this report or may yield none at all. That is, we conjecture that the quantitative operators we reported are closely tied to the educational level of our participants and the nature of tasks given to the participants, which required first order differential equations to model the scenarios. In that sense, the generalizability of the theory proposed in this paper may be limited to only cases of modelers constructing quantities while modeling dynamic situations that require differential equations.

Second, the findings reported in this study emerged as a consequence of analyzing modelers' quantitative reasoning during a mathematical modeling instructional sequence, as part of a larger study that followed a design experiment methodology. Due to the adaptive nature of a design experiment, the need to attend to modelers' justifications of mathematical operations and their quantitative meanings consistently and systematically emerged as we conducted our ongoing analysis. Therefore, during the interview we may have missed some opportunities to attend to the justification students made for their mathematical operations or systematically seek the quantitative meanings they carry. Designing a study for the purposes of determining the quantitative meaning behind the operations modelers perform to construct new quantities may yield more complete outcomes. We see this as an avenue for future research.

Third, while we propose quantitative operators as an extended framework for TQC for the purposes of studying modeling, the study is not suited to answer whether modelers' use of quantitative operators can be considered as mental operations. Mental operations are operations of the mind that do not depend on specific sensory material and therefore are not observable (von. Glasersfeld, 1995,p. 86). In this view, whether an operation is mental or not, depends on how the user used it. For example, Pattern's use of the sub-setting operator, to construct the number of realized cat-bird encounters, was motivated by his intellectual need (i.e. a goal that he established) to construct a proportion of the "base line"—number of potential cat-bird encounters. This action may fit the definition of a mental operation if Pattern imagined constructing a subset of the whole either by *shrinking* the size of the whole or *extracting* a portion of the whole. On the other hand, Ivory's use of the anti-derivative operator stemmed from her conception that the anti-derivative of Q' would yield Q. As observers do not have direct access to modelers' minds, we can only make inferences about modelers' cognitive processes through their observable activities (Ginsburg, 1997). And sometimes there is no recognizable one-to-one correspondence between the new quantities, the equations, and other representations modelers construct. Distinguishing whether these cognitive processes were mental operations or not was beyond the scope of this study.

7.4. Concluding remarks

We join with other mathematics education scholars arguing that theories in mathematics education should be dynamic; theories should be applied to novel contexts, allowing for adaptation, modification, and expansion to align with the available data and the specific requirements of the phenomenon under investigation (e.g., Cobb, 2007). At the same time, it is advised that researchers should proceed with caution when attributing quantitative reasoning to students based on their mathematical operations (Boyce, 2024). Therefore, in this paper, we showcased our adaptation of existing TQC for analyzing DE students' modeling of dynamic situations. Our intention for presenting the notion of quantitative operators is twofold. First, we wanted to document instances where students evidently constructed quantities that could not be explained using existing TQC. Second, we intended to present an alternative way to analyze modelers' construction of new quantities that belong in these borderline instances. Even though taking the derivative, anti-derivative, using a multiplicative scaling factor, and negating in and of itself may not be credited as quantitative operations, they were operations that our participants performed on existing quantities to construct a new quantity, furthering their modeling activity. Our findings resonate with Boyce's (2024) statement that quantitative and operator-based reasonings "can be considered to be co-constructed via reasoning quantitatively[5]" (p. 421). Adhering to the definitions of quantitative operations, as defined in TQC,

[6] See Boyce (2024) for more information on how quantitative reasoning and reasoning quantitatively are distinguished.

instances such as the ones illustrated in this paper will be missed, despite their potential centrality to constructing models. Therefore, to investigate students' quantitative reasoning *for* modeling, we propose to extend the definition of *operations on quantities* to include operations on singular quantities to construct new quantities through the aid of *quantitative operators*. The inclusion of these *quantitative operators* allows us to make better sense of and to engage deeply in modelers' construction of new quantities at the undergraduate level.

CRediT authorship contribution statement

Elizabeth Roan: Writing – review & editing, Resources, Methodology, Investigation, Formal analysis, Data curation. **Jennifer Czocher:** Writing – review & editing, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Writing – original draft. **Sindura Subanemy Kularajan:** Writing – review & editing, Writing – original draft, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of Competing Interest

None

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