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The mystery of plethysm coefficients

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Abstract. Composing two representations of the general linear groups gives rise to Littlewood's (outer) plethysm. On the level of characters, this poses the question of finding the Schur expansion of the plethysm of two Schur functions. A combinatorial interpretation for the Schur expansion coefficients of the plethysm of two Schur functions is, in general, still an open problem. We identify a proof technique of combinatorial representation theory, which we call the "s-perp trick", and point out several examples in the literature where this idea is used. We use the s-perp trick to give algorithms for computing monomial and Schur expansions of symmetric functions. In several special cases, these algorithms are more efficient than those currently implemented in SageMath.

1. Introduction

The isomorphism classes of complex irreducible polynomial representations of $GL_n := GL_n(\mathbb{C})$ are indexed by integer partitions λ with at most n parts. We denote such a representation by ρ^λ. Its character is identified with the *Schur polynomial* (see [**39**])

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathsf{SSYT}(\lambda)} x^{\mathrm{weight}(T)},$$

where $\mathsf{SSYT}(\lambda)$ is the set of all semistandard Young tableaux of shape λ over the alphabet $\{1, 2, \ldots, n\}$ and $\mathrm{weight}(T)$ is an n-dimensional vector, where the i-th entry contains the number of occurrences of the letter i in T, and where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for any n-dimensional vector α. The composition of two such representations, say $\rho^\lambda : GL_n \to GL_m$ and $\rho^\mu : GL_m \to GL_\ell$, is also a polynomial representation of GL_n, and its character is denoted by $s_\lambda[s_\mu]$. This operation can be viewed as an operation on symmetric polynomials, which was named *(outer) plethysm* by Littlewood [**29**].

The main objective of this paper is to discuss the following open problem.

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PROBLEM 1.1. Since $s_\lambda[s_\mu]$ is the character of a GL_n-representation, it is an $\mathbb{N}$-linear combination of Schur polynomials. Find a combinatorial interpretation of the coefficients $a^\nu_{\lambda,\mu} \in \mathbb{N}$ in the expansion

$$s_\lambda[s_\mu] = \sum_\nu a^\nu_{\lambda,\mu} s_\nu.$$

In the last century, the problem of understanding the coefficients $a^\nu_{\lambda,\mu}$ has stood as a measure of progress in the field of GL_n-representation theory (see for instance Problem 9 of [**40**]). Here we identify the *s-perp trick* as a possible way to approach this problem by proving known cases in a simple way and finding some new combinatorial descriptions of the $a^\nu_{\lambda,\mu}$. In addition, we demonstrate that the s-perp trick gives an efficient way to compute plethysm coefficients.

The simplest form of the problem occurs when the partitions λ and μ are both of row shape, however even this case is notoriously difficult and explicit formulae are known only in very special circumstances. The following table presents a (non-exhaustive) list of some of the known results in this direction.

Formulae for plethysms of the form $s_m[s_n]$	References
$s_2[s_n]$ in terms of Schur functions	[**28**]
$s_3[s_n]$ in terms of Schur functions	[**43**]
$s_4[s_n]$ in terms of Schur functions	[**16**, **21**]
$s_2[s_\lambda]$ in terms of Schur functions	[**8**, **45**]
$s_\lambda[s_\mu]$ in terms of fundamental quasisymmetric functions	[**30**]
$s_2[s_b[s_a]]$ and $s_c[s_2[s_a]]$ in terms of Schur functions	[**19**]

Since general formulae for the coefficients $a^\nu_{\lambda,\mu}$ have been elusive, various methods for computing plethysm have been developed [**6**, **7**, **10**, **14**, **15**, **24**, **27**, **36**, **44**, **46**, **47**] as well as representation-theoretic approaches [**4**, **5**, **12**, **21**, **33**, **34**, **38**].

The paper is organized as follows. In Section 2, we set up notation in the framework of symmetric functions and recall the definition of plethysm. In Section 3, we describe the s-perp trick to prove symmetric function identities and identify places in the literature where this trick has been used. In Section 4, we state an algorithm for computing the Schur expansion for a symmetric function f given the Schur expansions of $s_r^\perp f$. In Section 5, we apply this algorithm to plethysm expressions to show how it is used to speed up calculations of this type and to prove/derive new combinatorial formulas for the coefficients $a^\nu_{\lambda,\mu}$ in Problem 1.1.

2. Symmetric functions and plethysm

We begin by reviewing some notation in the framework of the ring of symmetric functions. We refer the reader to references like [**32**, **37**, **39**] for more details.

A *partition* of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ whose sum $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ is n. We use the notation $\lambda \vdash n$ to indicate that λ is a partition of $n \in \mathbb{N}$. The *length* of λ is denoted $\ell(\lambda) := r$. We assume that the empty partition, $\lambda = ()$, is the only partition of $n = 0$ and its length is 0. Also, we use the notation $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ and λ' to denote conjugate partition.

Let $\mathbb{K}$ be a ring containing $\mathbb{Q}$ as a subfield, such as $\mathbb{C}$ or $\mathbb{Q}(q,t)$. The ring of symmetric functions is defined as

$$\Lambda := \mathbb{K}[p_1, p_2, p_3, \ldots],$$

where the generators p_r are known as the *power sums* and the degree of p_r is equal to r. The subspace of symmetric functions of degree n is denoted $\Lambda_{=n}$ and is spanned by the products $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$, where $\lambda \vdash n$.

There are five distinguished bases for $\Lambda_{=n}$ that we will refer to in this exposition: the *power sum* $\{p_\lambda\}_{\lambda\vdash n}$, *complete* (or homogeneous) $\{h_\lambda\}_{\lambda\vdash n}$, *elementary* $\{e_\lambda\}_{\lambda\vdash n}$, *monomial* $\{m_\lambda\}_{\lambda\vdash n}$ and *Schur* $\{s_\lambda\}_{\lambda\vdash n}$ bases. (See [**32**, **37**, **39**] for the definition of these bases and the basic relations between them.)

The ring of symmetric functions is endowed with a *standard involution* defined by setting $\omega(h_\lambda) = e_\lambda$, $\omega(p_\lambda) = (-1)^{|\lambda|+\ell(\lambda)} p_\lambda$, or $\omega(s_\lambda) = s_{\lambda'}$. It also has a scalar product, known as the *Hall inner product*, which is defined by

$$\langle s_\lambda, s_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

If multiplication by a symmetric function f is thought of as a linear operator, then the linear operator which is adjoint to multiplication by f with respect to this scalar product is denote $f^\perp$. It satisfies $\langle f^\perp(g), h \rangle = \langle g, fh \rangle$ for all symmetric functions $f, g, h \in \Lambda$ and can be calculated via the formula

$$f^\perp(g) = \sum_\mu \langle g, f s_\mu \rangle \, s_\mu. \tag{2.1}$$

Then for $f \in \Lambda$, we can view $f^\perp$ as a (bi-)linear operator since $(f+g)^\perp = f^\perp + g^\perp$ and $f^\perp(g+h) = f^\perp(g) + f^\perp(h)$. For the Schur basis, $s_\lambda^\perp(s_\mu) = \sum_\nu c^\mu_{\lambda\nu} s_\nu$, where $c^\mu_{\lambda\nu}$ is the Littlewood–Richardson coefficient.

As one of the applications of the s-perp trick we will need the notion of composition of symmetric functions, or *plethysm*, and extend its use to *plethystic notation* on symmetric functions. In [**32**], the operation is denoted $f \circ g$ but it is convenient to express composition using square brackets. If $g \in \Lambda$ with $g = \sum_\lambda c_\lambda p_\lambda$ for some coefficients $c_\lambda \in \mathbb{Q}$,[1] then

$$p_r[g] = \sum_\lambda c_\lambda p_{r\lambda_1} p_{r\lambda_2} \cdots p_{r\lambda_{\ell(\lambda)}}$$

for all positive integers r, and $p_\mu[g] = p_{\mu_1}[g] p_{\mu_2}[g] \cdots p_{\mu_{\ell(\mu)}}[g]$ for all partitions μ. The plethysm of f and g, for $f \in \Lambda$ with $f = \sum_\mu c'_\mu p_\mu$ with $c'_\mu \in \mathbb{K}$, is defined as

$$f[g] = \sum_\mu c'_\mu p_\mu[g].$$

Next, *plethystic notation* extends the operation of plethysm to expressions containing variables from the base ring $\mathbb{K}$. For $E := E(x_1, x_2, x_3, \ldots) \in \mathbb{K}$, we have

$$p_r[E(x_1, x_2, x_3, \ldots)] = E(x_1^r, x_2^r, x_3^r, \ldots)$$

[1]The condition that the coefficients c_λ are in $\mathbb{Q}$ is important because using plethystic notation, if the base ring contains variables (for example, if $\mathbb{K} = \mathbb{Q}(q,t)$), then the variables must be treated differently.

and for $f \in \Lambda$, where $f = \sum_\lambda c_\lambda p_\lambda$ (with $c_\lambda \in \mathbb{K}$),

$$f[E] = \sum_\lambda c_\lambda p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E] \, .$$

We will use the symbol X to stand for an arbitrary alphabet of variables $X := x_1 + x_2 + x_3 + \cdots$, but as that expression is sufficiently general it may be replaced with any element of Λ and the identity will hold true. In particular, $f[X]$ is an expression equivalent to f since if $X = p_1$, then by definition, $f[p_1] = f$.

For our purposes, we require the following identities which can be derived from this definition (see [**32**]):

$$f[X+t] = \sum_{r \geqslant 0} (s_r^\perp f)[X] t^r, \qquad f[X-t] = \sum_{r \geqslant 0} (s_{1^r}^\perp f)[X](-t)^r, \tag{2.2}$$

$$f[-X] = (-1)^{\mathsf{degree}(f)} (\omega f)[X], \qquad f[tX] = t^{\mathsf{degree}(f)} f[X], \tag{2.3}$$

where $f \in \Lambda$ is of homogeneous degree equal to $\mathsf{degree}(f)$ and t is a variable in the base ring $\mathbb{K}$. Then for expressions $A_1, A_2, \ldots, A_k \in \Lambda$,

$$f[A_1 + A_2 + \cdots + A_k] = \sum_{\nu^{(*)}} s_{\nu^{(1)}}[A_1] s_{\nu^{(2)}}[A_2] \cdots s_{\nu^{(k)}}[A_k] \left\langle f, s_{\nu^{(1)}} s_{\nu^{(2)}} \cdots s_{\nu^{(k)}} \right\rangle , \tag{2.4}$$

where the sum is over all sequences of partitions $\nu^{(*)} = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(k)})$ with $\sum_{i=1}^k |\nu^{(i)}| = \mathsf{degree}(f)$.

Given two symmetric functions $f, g \in \Lambda$ with known monomial expansion $f = \sum_{i \geqslant 1} x^{a^i}$, where $a^1, a^2, \ldots$ are vectors, the plethysm is also given by

$$g[f] = g(x^{a^1}, x^{a^2}, \ldots).$$

Example 2.1. Since $s_1 = x_1 + x_2 + \cdots$, it hence immediately follows that

$$g[s_1] = g(x_1, x_2, \ldots) = g$$

and since $p_n = x_1^n + x_2^n + \cdots$, it follows that if $f = \sum_{i \geqslant 1} x^{a^i}$, then

$$f[p_n] = f(x_1^n, x_2^n, \ldots) = \sum_{i \geqslant 1} x^{a^i n} = p_n[f].$$

Example 2.2. For a slightly more advanced example, consider

$$s_2[x_1, x_2] \quad = \quad \underset{\boxed{1}\boxed{1}}{x_1^2} \quad + \quad \underset{\boxed{1}\boxed{2}}{x_1 x_2} \quad + \quad \underset{\boxed{2}\boxed{2}}{x_2^2} \tag{2.5}$$

where we indicated the semistandard Young tableau which contributes to each term. Then the plethysm

$$\begin{aligned} s_2[s_2[x_1, x_2]] &= s_2[x_1^2, x_1 x_2, x_2^2] \\ &= \underset{\boxed{1}\boxed{1}}{x_1^4} + \underset{\boxed{1}\boxed{2}}{x_1^3 x_2} + \underset{\boxed{1}\boxed{3}}{x_1^2 x_2^2} \\ &\quad + \underset{\boxed{1}\boxed{1}\boxed{1}\boxed{1}}{x_1^2 x_2^2} + \underset{\boxed{1}\boxed{1}\boxed{1}\boxed{2}}{x_1 x_2^3} + \underset{\boxed{1}\boxed{1}\boxed{2}\boxed{2}}{x_2^4} \\ &\quad + \underset{\boxed{2}\boxed{2}}{x_1^2 x_2^2} + \underset{\boxed{2}\boxed{3}}{x_1 x_2^3} + \underset{\boxed{3}\boxed{3}}{x_2^4} \\ &\quad + \underset{\boxed{1}\boxed{2}\boxed{1}\boxed{2}}{x_1^2 x_2^2} + \underset{\boxed{1}\boxed{2}\boxed{2}\boxed{2}}{x_1 x_2^3} + \underset{\boxed{2}\boxed{2}\boxed{2}\boxed{2}}{x_2^4} \\ &= s_4[x_1, x_2] + s_{2,2}[x_1, x_2]. \end{aligned}$$

Note that the first row of semistandard Young tableaux consists of the usual tableaux of shape (2) in the alphabet $\{1,2,3\}$ since the Schur polynomial in the example depends on three variables. The second row of tableaux takes into account that by (2.5) each variable itself corresponds to a tableau already, yielding tableaux of tableaux.

3. The s-perp trick

This section identifies a technique in symmetric function theory, which we coin the "*s-perp trick*". It has been used repeatedly in the literature to establish results in combinatorial representation theory. Here we apply it to Macdonald symmetric functions and plethysm to compute the monomial and Schur expansions of these symmetric functions. There are various other algorithms for the computation of plethysm that have appeared in the literature [**1**, **9**, **10**, **27**, **30**, **47**, **48**]. Surprisingly, this method is in many cases more efficient than the current implementation in SageMath; in particular, the computation of the Schur expansion of a single Schur plethysm of the form $s_\lambda[s_m]$ or $s_\lambda[s_{1^m}]$ is usually faster using the s-perp trick.

To explain the s-perp trick, we consider the ring of symmetric functions Λ spanned by the Schur basis s_λ, where λ is a partition of a non-negative integer. The degree of the Schur function indexed by the partition λ is given by the size of the partition. A function $f \in \Lambda$ is of homogeneous degree d if its expansion in the Schur basis only involves Schur functions indexed by partitions of d. Recall that the ring Λ is endowed with a scalar product, where the Schur basis is orthonormal. Also from (2.1), recall the action of the *s-perp operator*.

Definition 3.1. Let λ be a partition and $f \in \Lambda$. The *action of $s_\lambda^\perp$* on f is defined by

$$s_\lambda^\perp f = \sum_\mu \langle f, s_\lambda s_\mu \rangle \, s_\mu \ .$$

Now, we are ready to present the technique studied in this paper.

Proposition 3.2. (*The s-perp trick*) *Let f and g be two symmetric functions of homogeneous degree d. If*

$$s_r^\perp f = s_r^\perp g \qquad \text{for all } 1 \leqslant r \leqslant d,$$

then $f = g$. The same statement is true if $s_r^\perp$ is replaced by $s_{1^r}^\perp$, mutatis mutandi.

This statement is [**20**, Proposition 6.20.1], where the idea is used to compute the monomial expansion of $\nabla(e_n)$ in the Shuffle Conjecture. This technique occurs relatively frequently in the computation of combinatorial formulas for q,t-symmetric functions with labeled lattice paths (e.g. see the proof of [**23**, Theorem 5.2] in the recent paper by A. Iraci and A. Vanden Wyngaerd).

Proposition 3.2 can be interpreted as a recursive method to determine a symmetric function f by knowing the expressions $s_r^\perp f$ for each r between 1 and the degree of f. In fact, the proof of this proposition follows from the algorithm presented in Section 4, which provides a method for recovering the monomial (Equation (3.1)) and Schur expansions of f from the symmetric functions $s_r^\perp f$.

As we mentioned at the beginning of this section, this technique has been used repeatedly in the literature. For instance, one manifestation of the s-perp trick is

to give a monomial expansion of a symmetric polynomial using the recurrence

$$f(x_1, x_2, \ldots, x_n) = \sum_{r \geqslant 0} (s_r^{\perp} f)(x_1, x_2, \ldots, x_{n-1}) x_n^r \ .$$

This is the method for computing the monomial expansion of the modified Macdonald symmetric functions $\tilde{H}_\lambda[X; q, t]$ (see [**17**]) given by F. Bergeron and M. Haiman in [**3**, Proposition 5]. They provide an explicit formula for the coefficient $d_{\lambda\nu}^{(r)}$ in the expression defined by

$$s_r^{\perp} \tilde{H}_\lambda[X; q, t] = \sum_{\nu} d_{\lambda\nu}^{(r)} \tilde{H}_\nu[X; q, t] \ .$$

Denoting $\overline{\mu} = (\mu_2, \mu_3, \ldots, \mu_{\ell(\mu)})$, it follows that if we denote the coefficient of m_μ in $\tilde{H}_\lambda[X; q, t]$ by $L_{\lambda\mu}$, then it can be computed with the recursive formula[2]

$$L_{\lambda\mu} = \sum_{\beta} d_{\lambda\beta}^{(\mu_1)} L_{\beta\overline{\mu}} \ . \tag{3.1}$$

Another example of the s-perp trick used in combinatorial representation theory appears in a paper by A. Garsia and C. Procesi [**18**], where the authors show that the Frobenius characteristic of a certain quotient module is equal to the Hall–Littlewood symmetric function. Let $H_\lambda[X; q] = \sum_\mu K_{\lambda\mu}(q) s_\mu$ be the Hall–Littlewood symmetric function with $K_{\lambda\mu}(q)$ representing the q-Kostka coefficient (see [**32**, Section III.6]) and let $C_\lambda[X; q]$ be the Frobenius characteristic of a certain module. It is shown in [**18**] that since $s_{1^r}^{\perp} H_\lambda[X; q]$ and $s_{1^r}^{\perp} C_\lambda[X; q]$ satisfy exactly the same recursive expression, the identity $H_\lambda[X; q] = C_\lambda[X; q]$ holds.

Iraci, Rhoades and Romero use an almost identical method [**22**, Lemma 3.1] to prove that the diagonal fermionic coinvariants $\bigwedge\{\theta_n, \xi_n\}/I^+$ have Frobenius image of a special case of the Theta conjecture [**11**, Conjecture 9.1].

We will apply this technique to the application of computing the plethysm of symmetric functions.

4. Application: Schur expansions

In the previous section, we saw that the monomial expansion of a symmetric function f can be computed by recursively computing the monomial expansion of $s_r^{\perp} f$. In this section, we state an algorithm for computing the Schur expansion of a symmetric function f by recursively computing the Schur expansion of $s_r^{\perp} f$ (respectively $s_{1^r}^{\perp} f$).

Let $\mathsf{addrow}(s\_expansion, r)$ be a function which takes as input a Schur expansion of an element of Λ and a positive integer r. If each partition indexing the terms in the Schur expansion of f has first part less than or equal to r, then return the expression with each indexing partition having a row of size r appended to it, or 0 otherwise.

REMARK 4.1. The idea for this algorithm occurred to us because we had a symmetric group module and we could combinatorially describe $s_1^{\perp} f$, where f is the Frobenius image of this module. From this we wanted to deduce a combinatorial formula for f. In general, $s_1^{\perp} f$ does not characterize f (e.g. consider the symmetric

[2]This algorithm was implemented in 2015 as the default method for computing the $\tilde{H}_\mu[X; q, t]$ symmetric functions in the computer algebra system SAGEMATH [**42**] and there was roughly a $2\times$ speedup against the previous most efficient method for computing these functions.

functions $s_{22}+s_4$ and s_{31} which both satisfy $s_1^\perp(s_{22}+s_4)=s_{21}+s_3=s_1^\perp(s_{31})$). However the idea seemed quite close. Then we asked what additional information would we need to recover f and realized that there is a general technique that is regularly used in symmetric function theory that could be identified.

Algorithm 1: An algorithm for finding the Schur expansion of f from the Schur expansion of $s_r^\perp f$ for $1 \leqslant r \leqslant \mathsf{degree}(f)$

```
1 function ExpandSchur A;
  Input  : A - array with A[r] the Schur expansion of s_r^⊥ f for
           1 ⩽ r ⩽ degree(f)
  Output: out - a Schur expansion of f
2 begin
3 |  out ⟵ 0;
4 |  for r = length(A) downto 1 do
5 |  |  out ⟵ out + addrow(A[r] − s_r^⊥(out), r);
6 |  end
7 |  return out;
8 end
```

Algorithm 1 works because when $r=k$ at line 4, *out* is equal to the sum of the terms of the Schur expansion of f which have first part of the indexing partition greater than or equal to k and so $f-out$ is in the linear span of Schur functions with parts less than or equal to k. Hence

$$\mathsf{addrow}(A[k]-s_k^\perp(out),k)=\mathsf{addrow}(s_k^\perp(f-out),k)$$

is equal to the sum of terms in the Schur expansion of f with first part of the indexing partition equal to k. When the *for* loop completes, *out* will equal the Schur expansion of f.

This algorithm can be modified to deduce the Schur expansion of f from the Schur expansion of $s_{1^r}^\perp f$, for $1 \leqslant r \leqslant \mathsf{degree}(f)$, by replacing the function addrow with addcol, which adds a column on the partitions indexing the expansions of the Schur functions.

5. Application to plethysm

5.1. s-perp formulae for plethysm. Equation (2.2) can be used to compute $s_r^\perp$ or $s_{1^r}^\perp$ acting on plethysms. The expression that we will specialize for our application is the following identity.

PROPOSITION 5.1. *For partitions λ and μ and a positive integer r,*

(5.1)
$$s_r^\perp s_\lambda[s_\mu]=\sum_{\nu^{(i)}} s_{\nu^{(0)}}[s_\mu]s_{\nu^{(1)}}[s_1^\perp s_\mu]s_{\nu^{(2)}}[s_2^\perp s_\mu]\cdots s_{\nu^{(r')}}[s_{r'}^\perp s_\mu]\left\langle s_\lambda, s_{\nu^{(0)}}s_{\nu^{(1)}}\cdots s_{\nu^{(r')}}\right\rangle$$

where $r'=\min(r,\mu_1)$ and the sum is over all sequences of partitions $(\nu^{(0)},\nu^{(1)},\ldots,\nu^{(r')})$ with $\sum_{i=0}^r|\nu^{(i)}|=|\lambda|$ and $\sum_{i=0}^r i|\nu^{(i)}|=r$.

PROOF. If we let $r'=\min(r,\mu_1)$, then by Equation (2.2) we have

$$s_\mu[X+t]=s_\mu+ts_1^\perp s_\mu+t^2s_2^\perp s_\mu+\cdots+t^{r'}s_{r'}^\perp s_\mu.$$

Now Equation (5.1) follows from an application of Equation (2.4), factoring out a power of t using the right expression of Equation (2.3) and then taking a coefficient of t^r on both sides of the equation. □

A nearly identical proof will derive a similar identity for $s_{1^r}^{\perp} s_\lambda[s_\mu]$ by applying Equation (2.2) and in addition using Equation (2.3) so that in the product we have terms of the form $(\omega s_{\nu^{(k)}})[s_{1^k}^{\perp} s_\mu] = s_{(\nu^{(k)})'}[s_{1^k}^{\perp} s_\mu]$ if k is odd and $s_{\nu^{(k)}}[s_{1^k}^{\perp} s_\mu]$ if k is even.

PROPOSITION 5.2. *For partitions λ and μ and a positive integer r, $s_{1^r}^{\perp} s_\lambda[s_\mu]$ is equal to*

(5.2)
$$\sum_{\nu^{(i)}} s_{\nu^{(0)}}[s_\mu](\omega s_{\nu^{(1)}})[s_1^{\perp} s_\mu] s_{\nu^{(2)}}[s_{1^2}^{\perp} s_\mu] \cdots (\omega^{r'} s_{\nu^{(r')}})[s_{1^{r'}}^{\perp} s_\mu] \left\langle s_\lambda, s_{\nu^{(0)}} s_{\nu^{(1)}} \cdots s_{\nu^{(r')}} \right\rangle$$

where $r' = \min(r, \ell(\mu))$ and the sum is over all sequences of partitions $(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(r')})$ with $\sum_{i=0}^{r} |\nu^{(i)}| = |\lambda|$ and $\sum_{i=0}^{r} i|\nu^{(i)}| = r$.

Applying Algorithm 1 to either Proposition 5.1 or 5.2 will have too many terms, and so it cannot be considered an efficient method for computing plethysms in general. However, special cases of these identities can be used to recursively compute classes of plethysms much more efficiently than the standard algorithms.

Special cases of Equations (5.1) and (5.2) include

$$s_r^{\perp} s_\lambda[s_{1^w}] = \sum_{\mu,\gamma} c_{\mu\gamma}^{\lambda} s_\mu[s_{1^w}] s_\gamma[s_{1^{w-1}}] \qquad s_{1^r}^{\perp} s_\lambda[s_w] = \sum_{\mu,\gamma} c_{\mu\gamma'}^{\lambda} s_\mu[s_w] s_\gamma[s_{w-1}],$$

where the sums are over partitions μ and γ with $|\mu| = |\lambda| - r$ and $|\gamma| = r$ and $c_{\mu\gamma}^{\lambda} := \langle s_\lambda, s_\mu s_\gamma \rangle$ are the Littlewood–Richardson coefficients.

While we saw improvements in recursive computations with these formulas, even smaller classes of functions are the special cases:

$$s_r^{\perp} s_{1^h}[s_{1^w}] = s_{1^{h-r}}[s_{1^w}] s_{1^r}[s_{1^{w-1}}] \qquad s_{1^r}^{\perp} s_h[s_w] = s_{h-r}[s_w] s_{1^r}[s_{w-1}] \tag{5.3}$$

$$s_r^{\perp} s_h[s_{1^w}] = s_{h-r}[s_{1^w}] s_r[s_{1^{w-1}}] \qquad s_{1^r}^{\perp} s_{1^h}[s_w] = s_{1^{h-r}}[s_w] s_r[s_{w-1}] \ . \tag{5.4}$$

We compared this method with the current method used by the SAGEMATH computer algebra system [**42**] and found it to be significantly faster for computing the plethysms $s_{1^h}[s_{1^w}]$. The current implementation in SAGEMATH uses the definition stated in Section 2, which requires a change of basis from the Schur functions to power sums and back again. An implementation of our method in SAGEMATH was able to compute $s_{1^4}[s_{1^6}]$ in less than a second compared to 36 seconds with the current implementation in SAGEMATH; and to compute $s_{1^6}[s_{1^6}]$ it took 21 seconds compared to well over an hour.

5.2. A formula for $s_\lambda[s_k]$ for $|\lambda| \leqslant 3$. Quasi-polynomial and integer polytope expressions for the coefficients of s_μ in $s_\lambda[s_k]$ with $|\lambda| \leqslant 3$ and k a positive integer have been extensively studied [**2**, **10**, **13**, **21**, **24**, **35**, **41**, **43**]. The quasi-polynomial expressions are probably the most efficient means of computation of a single coefficient in this expression. In [**25**], the authors use the expressions for $s_3[s_k]$ and $s_k[s_3]$ to conclude that certain families of these coefficients do not need to be given by Ehrhart functions of rational polytopes.

In the following theorem we use the s-perp trick and some of the properties and analysis that others have derived about these formulae to give a combinatorial

interpretation in terms of semi-standard Young tableaux. This formulation gives an idea of what we expect a combinatorial interpretation for the plethysm coefficients should look like in general. The result was obtained independently using different methods by Florence Maas-Gariépy and Étienne Tétreault [**31**], and our proof makes use of one of their earlier results [**41**]. Let $\mathsf{SSYT}_{(k,k,k)}$ represent the set of semi-standard Young tableaux with k 1s, k 2s and k 3s and denote the shape of a tableau by $\mathsf{shape}(S)$.

THEOREM 5.3. *Let* $T \in \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 3\\\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 2\\\hline 1&3\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \right\}$ *be a standard tableau with shape a partition of 3. Then for any* $k \geqslant 1$,

$$s_{\mathsf{shape}(T)}[s_k] = \sum_{\substack{S \in \mathsf{SSYT}_{(k,k,k)} \\ \mathsf{type}(S)=T}} s_{\mathsf{shape}(S)}, \tag{5.5}$$

where $\mathsf{type}(S)$ *for* $S \in \mathsf{SSYT}_{(k,k,k)}$ *is given in Definition* 5.4 *below.*

DEFINITION 5.4. For $S \in \mathsf{SSYT}_{(k,k,k)}$, let N_r represent the number of cells with the label r in the second row of S for $1 \leqslant r \leqslant 3$.

- If S is standard and $k = 1$, then let $\mathsf{type}(S) = S$.
- If S has one or two rows, then we define
  (a) $\mathsf{type}(S) = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ if N_2 is even, $N_3 \geqslant 2N_2$, but $N_3 \neq 2N_2 + 1$.
  (b) $\mathsf{type}(S) = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$ if N_2 is odd, $N_3 \geqslant 2N_2$, but $N_3 \neq 2N_2 + 1$.
  (c) $\mathsf{type}(S) = \begin{array}{|c|c|}\hline 3\\\hline 1&2\\\hline\end{array}$ if N_2 is even, and $N_3 < 2N_2$ or $N_3 = 2N_2 + 1$.
  (d) $\mathsf{type}(S) = \begin{array}{|c|c|}\hline 2\\\hline 1&3\\\hline\end{array}$ if N_2 is odd, and $N_3 < 2N_2$ or $N_3 = 2N_2 + 1$.
- If S has three rows and $\overline{S}$ is S with the first column removed, then $\mathsf{type}(S) = \mathsf{type}(\overline{S})^t$.

For T a standard tableau of size 3, let

$$\mathsf{Tab}_{T,k} = \{S \in \mathsf{SSYT}_{(k,k,k)} \mid \mathsf{type}(S) = T\}.$$

Define a linear operator on symmetric functions as

$$s_\mu \downarrow_k = \begin{cases} s_\mu & \text{if } \ell(\mu) \leqslant k, \\ 0 & \text{else.} \end{cases}$$

Also define the following bilinear (and commutative) operator by

$$s_\mu \odot s_\lambda = s_{\mu+\lambda},$$

where the sum of the two partitions is done componentwise, adding zeros if necessary.

LEMMA 5.5. *For* $k \geqslant 1$,

$$s_3[s_k] \downarrow_2 = \sum_S s_{\mathsf{shape}(S)},$$

where the sum is over all $S \in \mathsf{Tab}_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},k}$ *with* $\ell(S) \leqslant 2$.

PROOF. A result of [**41**, Theorem 2.1] says that

$$s_3[s_k]\downarrow_2 = s_{66}\odot s_3[s_{k-4}]\downarrow_2 + \sum_{r=2}^{k} s_{3k-r,r} + s_{3k}.$$

The tableaux $S \in \mathsf{Tab}_{\boxed{1|2|3},k}$ and $\ell(S)\leqslant 2$ also satisfies this recurrence since they either contain $\boxed{2|2|3|3|3|3}$ in the top row and $\boxed{1|1|1|1|2|2}$ in the bottom row or not. Those that do are (by induction) enumerated by the expression $s_{66}\odot s_3[s_{k-4}]\downarrow_2$. Those tableaux that do not contain $\boxed{2|2|3|3|3|3}$ and $\boxed{1|1|1|1|2|2}$ of type $\boxed{1|2|3}$ are either a single row or have a second row consisting of exactly r 3's for $2\leqslant r\leqslant k$. □

LEMMA 5.6. *For $k\geqslant 1$,*

$$\sum_{S\in\mathsf{Tab}_{\substack{\boxed{3}\\ \boxed{1|2}},k}} s_{\mathsf{shape}(S)} = \sum_{S\in\mathsf{Tab}_{\substack{\boxed{2}\\ \boxed{1|3}},k}} s_{\mathsf{shape}(S)}\ .$$

PROOF. We define a map which we denote

$$\phi\colon \mathsf{Tab}_{\substack{\boxed{3}\\ \boxed{1|2}},k} \to \mathsf{Tab}_{\substack{\boxed{2}\\ \boxed{1|3}},k}\ .$$

CASE 1. If the length of S is 3, then let $\overline{S}\in \mathsf{SSYT}_{(k,k,k)}$ be the tableau S with the first column removed. Define $\phi(S)$ to be a column of length 3 added to $\phi^{-1}(\overline{S})$.

CASE 2. If the length of the second row of S is odd, then let S' be the tableau where one takes a 3 from the second row of S and exchanges it with a 2 in the first row.

CASE 3. If the length of the second row of S is even, then let S' be the tableau such that one takes a 3 from the first row of S and exchanges it with a 2 in the second row.

We leave to the reader to check the details that this is a bijection. □

PROOF OF THEOREM 5.3. Assume by induction that (5.5) holds for some fixed k and all $T\in\left\{\boxed{1|2|3},\ \boxed{3}\boxed{1|2},\ \boxed{2}\boxed{1|3},\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right\}$. By Lemma 5.5 the terms of length less than or equal to 2 in $s_3[s_{k+1}]$ are given by the $S\in\mathsf{Tab}_{\boxed{1|2|3},k+1}$ such that $\ell(S)\leqslant 2$ and by (5.3) we have $s_{111}^{\perp}s_3[s_{k+1}] = s_{111}[s_k]$. Then, it follows by induction that the terms of length 3 are given by those $S\in\mathsf{Tab}_{\boxed{1|2|3},k+1}$ with $\ell(S)=3$. Therefore,

$$s_3[s_{k+1}] = \sum_{S\in\mathsf{Tab}_{\boxed{1|2|3},k+1}} s_{\mathsf{shape}(S)}\ . \tag{5.6}$$

From [**32**, Section I.8 Example 9],

$$s_2[s_{k+1}] = \sum_S s_{\mathsf{shape}(S)},$$

where the sum is over $S \in \mathsf{SSYT}_{(k+1,k+1)}$ with N_2 is even. Therefore,

$$s_{21}[s_{k+1}]+s_3[s_{k+1}] = s_2[s_{k+1}]s_{k+1} = \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)} + \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 3\\\hline 1&2\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)},$$

where the last equality holds because $\mathsf{Tab}_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,k+1} \cup \mathsf{Tab}_{\begin{array}{|c|}\hline 3\\\hline 1&2\\\hline\end{array}\,,k+1}$ are all $S \in \mathsf{SSYT}_{(k+1,k+1,k+1)}$ such that N_2 is even. We conclude by subtracting Equation (5.6) from both sides of the equation and Lemma 5.6 that

$$s_{21}[s_{k+1}] = \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 3\\\hline 1&2\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)} = \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 2\\\hline 1&3\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)}\ . \tag{5.7}$$

We again use [**32**, Section I.8 Example 9], which says that

$$s_{11}[s_{k+1}] = \sum_S s_{\mathsf{shape}(S)},$$

where the sum is over $S \in \mathsf{SSYT}_{(k+1,k+1)}$ with N_2 is odd so that

$$s_{21}[s_{k+1}] + s_{111}[s_{k+1}] = s_{11}[s_{k+1}]s_{k+1} = \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)} + \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 2\\\hline 1&3\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)}.$$

The last equality holds because $\mathsf{Tab}_{\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\,,k+1} \cup \mathsf{Tab}_{\begin{array}{|c|c|}\hline 2\\\hline 1&3\\\hline\end{array}\,,k+1}$ are all $S \in \mathsf{SSYT}_{(k+1,k+1,k+1)}$ such that N_2 is odd. It then follows by subtracting Equation (5.7) from both sides of the equation that

$$s_{111}[s_{k+1}] = \sum_{S\in\mathsf{Tab}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\,,k+1} s_{\mathsf{shape}(S)}\ .$$

This concludes the proof by induction since Equation (5.5) holds for $k \to k+1$ and all $T \in \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 2\\\hline 1&3\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \right\}$. □

5.3. Explicit formulas for $s_\lambda[s_2]$ and $s_\lambda[s_{1^2}]$ in special cases. A partition λ is called *even* if all columns have even length. A partition λ is called *threshold* if $\lambda'_i = \lambda_i + 1$ for all $1 \leqslant i \leqslant d(\lambda)$ where $d(\lambda)$ is the maximal i such that $(i,i) \in \lambda$.

THEOREM 5.7. *We have*

$$s_h[s_2] = \sum_{\substack{\lambda\vdash 2h\\ \lambda\ even}} s_{\lambda'}, \qquad s_h[s_{1^2}] = \sum_{\substack{\lambda\vdash 2h\\ \lambda\ even}} s_\lambda, \tag{5.8}$$

$$s_{1^h}[s_2] = \sum_{\substack{\lambda\vdash 2h\\ \lambda\ threshold}} s_{\lambda'}, \qquad s_{1^h}[s_{1^2}] = \sum_{\substack{\lambda\vdash 2h\\ \lambda\ threshold}} s_\lambda. \tag{5.9}$$

REMARK 5.8. The second equation in (5.9) has appeared in [**26**, Theorem 5.2]. The formulas of Theorem 5.7 also appear in [**32**, Page 138] and were originally proven in [**28**, Equations (11.9; 1) through (11.9; 4)], however these proofs are different from the ones we present here.

A *cell* of a partition λ is a pair (r, c), where $1 \leqslant r \leqslant \ell(\lambda)$ and $1 \leqslant c \leqslant \lambda_r$. A *corner* of a partition is a cell of the partition (r, c) such that $(r+1, c)$ and $(r, c+1)$ are not cells of the partition. For the proof of Theorem 5.7, we need the notion of *opposite cell* for each cell (s, t) in a threshold partition λ. The opposite cell $\mathsf{op}(s, t)$ of (s, t) is defined to be $(t+1, s)$ if $s \leqslant t$ and $(t, s-1)$ otherwise.

PROOF OF THEOREM 5.7. We prove the second equation in (5.8) by checking that the right hand side satisfies

$$s_r^{\perp} s_h[s_{1^2}] = s_{h-r}[s_{1^2}] s_r \tag{5.10}$$

from (5.4) holds for all r. We have

$$s_r^{\perp} s_\lambda = \sum_{\nu} s_\nu,$$

where the sum is over all ν such that λ/ν has at most one cell per column and $|\lambda/\nu| = r$. In the columns, where there is a cell in λ/ν, the length of the columns of ν will be odd. This implies that ν contains an even partition μ with ν/μ containing the cells which make those columns odd and so will also have at most one cell per column. Therefore,

$$s_r^{\perp} \sum_{\substack{\lambda \vdash 2h \\ \lambda \text{ even}}} s_\lambda = \sum_{\substack{\nu \vdash 2h-r \\ \nu/\mu \text{ horizontal } r\text{-strip} \\ \text{for some } \mu \vdash 2h-2r \text{ even}}} s_\nu = \sum_{\substack{\mu \vdash 2h-2r \\ \mu \text{ even}}} s_\mu s_r = \Big(\sum_{\substack{\mu \vdash 2h-2r \\ \mu \text{ even}}} s_\mu \Big) s_r$$

where the second to last equality follows from the Pieri rule. This confirms that the right hand side of the second equation in (5.8) satisfies (5.10).

Similarly, we prove the second equation in (5.9) by checking that the right hand side satisfies

$$s_r^{\perp} s_{1^h}[s_{1^2}] = s_{1^{h-r}}[s_{1^2}] s_{1^r}$$

from (5.3) holds for all r. We start with

$$s_r^{\perp} \sum_{\substack{\lambda \vdash 2h \\ \lambda \text{ threshold}}} s_\lambda = \sum_{\substack{\nu \vdash 2h-r \\ \lambda/\nu \text{ vertical } r\text{-strip} \\ \text{for some } \lambda \vdash 2h \text{ threshold}}} s_\nu.$$

For each cell (s, t) in λ/ν in the last sum from rightmost to leftmost (that is largest t to smallest t), check whether $\mathsf{op}(s, t)$ is in ν. If it is, remove $\mathsf{op}(s, t)$ from ν, otherwise leave the partition unchanged. Call the partition with all (possible) opposite cells removed τ. Note that ν/τ is a vertical strip since the cells in λ/ν form a horizontal strip and $\mathsf{op}(s, t)$ involves transposition. For each cell (s, t) in λ/ν from largest to smallest t with $t \geqslant s$, for which $(s', t') = \mathsf{op}(s, t)$ is not in ν, find the cell (x, y) in τ with smallest $y \geqslant t'$ such that $\nu/(\tau \setminus \{(x, y)\})$ is a vertical strip. Remove (x, y) and $\mathsf{op}(x, y)$ from τ. Call the resulting partition μ. Note that μ is threshold and

furthermore, ν/μ is a vertical r-strip. Hence

$$s_r^{\perp} \sum_{\substack{\lambda\vdash 2h \\ \lambda \text{ threshold}}} s_\lambda = \sum_{\substack{\nu\vdash 2h-r \\ \nu/\mu \text{ vertical } r\text{-strip} \\ \text{for some } \mu\vdash 2h-2r \text{ threshold}}} s_\nu = \Big(\sum_{\substack{\mu\vdash 2h-2r \\ \mu \text{ threshold}}} s_\mu\Big) s_{1^r}.$$

The other two formulas follow from the identity

$$s_\lambda[s_{\mu'}] = \sum_\gamma \langle s_\lambda[s_\mu], s_\gamma\rangle\, s_{\gamma'} \tag{5.11}$$

for $|\mu|$ even (see Equation (2.3)). □

For $h \geqslant 2$ and $1 \leqslant r < h$, we have

$$s_{h-r}[s_{1^2}]s_{1^r}[s_{1^2}] = s_{(h-r,1^r)}[s_{1^2}] + s_{(h-r+1,1^{r-1})}[s_{1^2}]\ , \tag{5.12}$$

since by the dual Pieri rule $s_{h-r}s_{1^r} = s_{(h-r,1^r)} + s_{(h-r+1,1^{r-1})}$.

The next result is new.

Corollary 5.9. *We have*

$$s_{(h-1,1)}[s_{1^2}] = \sum_{\mu\in\mathcal{P}_{2h}} s_\mu + \sum_{\nu\vdash 2h \text{ even}} (b_\nu - 1)s_\nu, \tag{5.13}$$

$$s_{(h-1,1)}[s_2] = \sum_{\mu\in\mathcal{P}_{2h}} s_{\mu'} + \sum_{\nu\vdash 2h \text{ even}} (b_\nu - 1)s_{\nu'}, \tag{5.14}$$

where $\mathcal{P}_{2h}$ is the set of all partitions of $2h$ with columns of even length except two columns of distinct odd length, and b_ν is the number of corners of ν.

Proof. By (5.12) with $r = 1$, we have $s_{(h-1,1)}[s_{1^2}] = s_{h-1}[s_{1^2}]s_{1^2} - s_h[s_{1^2}]$. On the other hand by Theorem 5.7, the Schur expansion of $s_{h-1}[s_{1^2}]$ contains the sum over s_ν where $\nu \vdash 2h-2$ is even. Multiplication by s_{1^2} adds a vertical strip of size 2 to the partitions in the Schur expansion by the Pieri rule, so that

$$s_{h-1}[s_{1^2}]s_{1^2} = \sum_{\mu\in\mathcal{P}_{2h}} s_\mu + \sum_{\nu\vdash 2h \text{ even}} b_\nu s_\nu.$$

This follows from the fact that adding a vertical strip of length 2 to an even partition gives a partition with two different odd and otherwise even length columns (if the two boxes are added to different columns) or an even partition (if the two boxes are added to the same column). An even partition ν of size $2h$ can be obtained in b_ν ways from an even partition of size $2h-2$ by adding two boxes to a column. Noting that, by Theorem 5.7, the Schur expansion of $s_h[s_{1^2}]$ contains all even partitions of size $2h$ proves (5.13).

Equation (5.14) follows from (5.11). □

Remark 5.10. Note that there is an involution on the partitions in $\mathcal{S}_{2h} = \mathcal{P}_{2h} \cup \{\nu \vdash 2h \mid \nu \text{ even}\}$ appearing in the expansion in Corollary 5.9. Namely, map the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ with ℓ even (with possibly $\nu_\ell = 0$) to

$$w\colon \nu \mapsto (\nu_1 + \nu_2, \nu_3 + \nu_4, \ldots, \nu_{\ell-1} + \nu_\ell)'.$$

Note that $w(\nu) \in \mathcal{P}_{2h}$ if $\nu \in \mathcal{P}_{2h}$ and $w(\nu)$ is even if ν is even. Also, it is not hard to see that $w^2(\nu) = \nu$ and $b_{w(\nu)} = b_\nu$. The involution w imposes a symmetry on the Schur expansion of $s_{(h-1,1)}[s_{1^2}]$ and $s_h[s_{1^2}]$.

EXAMPLE 5.11. We have

$$s_{(3,1)}[s_{1^2}] = s_{(21^6)} + s_{(2^21^4)} + s_{(2^31^2)} + s_{(321^3)} + s_{(32^21)} + s_{(3^21^2)} + s_{(431)}.$$

Note that $w(431) = (21^6)$, $w(3^21^2) = (2^21^4)$, $w(32^21) = (2^31^2)$, and $w(321^3) = (321^3)$ and indeed the coefficients of s_γ and $s_{w(\gamma)}$ match.

COROLLARY 5.12. *We have*

$$s_{(2,1^{h-2})}[s_{1^2}] = \sum_{\mu\in\mathcal{T}_{2h}} s_\mu + \sum_{\nu\vdash 2h \text{ threshold}} \left\lfloor \frac{b_\nu - 1}{2} \right\rfloor s_\nu, \tag{5.15}$$

$$s_{(2,1^{h-2})}[s_2] = \sum_{\mu\in\mathcal{T}_{2h}} s_{\mu'} + \sum_{\nu\vdash 2h \text{ threshold}} \left\lfloor \frac{b_\nu - 1}{2} \right\rfloor s_{\nu'}, \tag{5.16}$$

where $\mathcal{T}_{2h}$ is the set of all partitions λ of $2h$ with $\lambda'_i = \lambda_i + 1$ for all i except either

(i) *two distinct i in the range $1 \leqslant i \leqslant d(\lambda)$ for which either $\lambda'_i = \lambda_i + 2$ and the cell (λ'_i, i) is a corner or $\lambda'_i = \lambda_i$; or*
(ii) *one i in the range $1 \leqslant i \leqslant d(\lambda)$ with $\lambda'_i = \lambda_i + 3$.*

PROOF. The proof follows the same outline as the proof of Corollary 5.9 using (5.12) with $r = h - 1$ (instead of $r = 1$) so that $s_{(2,1^{h-2})}[s_{1^2}] = s_{1^2}s_{1^{h-1}}[s_{1^2}] - s_{1^h}[s_{1^2}]$. □

COROLLARY 5.13. *We have for $h \geqslant 3$*

$$s_{(h-2,1,1)}[s_{1^2}] = \sum_{\mu\vdash 2h} a_\mu s_\mu,$$

$$s_{(h-2,1,1)}[s_2] = \sum_{\mu\vdash 2h} a_\mu s_{\mu'},$$

where

$$a_\mu = \begin{cases} \binom{b_\mu - 1}{2} & \text{if } \mu \text{ is even,} \\ \left(\sum_{\nu\vdash 2(h-2) \text{ even}} c^\mu_{\nu(2,1,1)}\right) - 1 & \text{if } \mu \in \mathcal{P}_{2h}, \\ \sum_{\nu\vdash 2(h-2) \text{ even}} c^\mu_{\nu(2,1,1)} & \text{otherwise.} \end{cases}$$

PROOF. From Equation (5.12) with $r = 2$ we know that $s_{(h-2,1,1)}[s_{1^2}] = s_{h-2}[s_{1^2}]s_{1^2}[s_{1^2}] - s_{(h-1,1)}[s_{1^2}]$. We use the fact that $s_{1^2}[s_{1^2}] = s_{(2,1,1)}$ and the expression for $s_{(h-1,1)}[s_{1^2}]$ from Corollary 5.9. In the Schur expansion of $s_{h-2}[s_{1^2}]$ only even partitions appear. The partitions μ indexing the Schur functions in the product $s_{h-2}[s_{1^2}]s_{1^2}[s_{1^2}] = s_{h-2}[s_{1^2}]s_{(2,1,1)}$ fall into four cases:

- Case 1: μ has exactly 4 odd columns
- Case 2: μ has exactly two equal odd columns
- Case 3: μ has exactly two distinct odd columns
- Case 4: μ is an even partition.

Subtracting the combinatorial formula for $s_{(h-1,1)}[s_{1^2}]$ given by Corollary 5.9 does not subtract anything in Cases 1 and 2, subtracts 1 in Case 3, and $b_\mu - 1$ in Case 4. Using that the Littlewood–Richardson coefficient $c^\mu_{\nu(2,1,1)}$ for all $\nu \subseteq \mu$ with $\nu \vdash 2(h-2)$ even count the coefficients of s_μ in $s_{h-2}[s_{1^2}]s_{(2,1,1)}$ hence proves the last two cases in the formula for a_μ. To prove the first case in a_μ, note that $\sum_{\nu\vdash 2(h-2) \text{ even}} c^\mu_{\nu(2,1,1)}$ for μ even is equal to $\binom{b_\mu}{2}$ since this number is counted by Littlewood–Richardson tableaux of shape μ/ν and weight $(2,1,1)$. Recall that a

Littlewood–Richardson tableau is a semistandard Young tableau such that the row reading of the tableau is a reverse lattice word. For μ and ν even these tableaux have to have 21 in the rightmost column of μ/ν and 31 in the leftmost column of μ/ν. This means we need to pick two corners of μ, which can be done in $\binom{b_\mu}{2}$ ways. Noting $\binom{b_\mu}{2} - (b_\mu - 1) = \binom{b_\mu - 1}{2}$ proves the first equation in the corollary.

The second equation follows again from (5.11). □

For general hooks, we have the following result.

COROLLARY 5.14. *For $h > k \geqslant 0$, we have*

$$s_{(h-k,1^k)}[s_{1^2}] = \sum_{i=0}^{k} \sum_{\substack{\mu\vdash 2h \\ \nu\vdash 2(h-k+i)\ even \\ \rho\vdash 2(k-i)\ threshold}} (-1)^i c^\mu_{\nu\rho} s_\mu. \tag{5.17}$$

PROOF. We prove this result by induction on k. For $k = 0$, Equation (5.17) reads

$$s_h[s_{1^2}] = \sum_{\substack{\mu\vdash 2h \\ \nu\vdash 2h \text{ even} \\ \rho\vdash 0 \text{ threshold}}} c^\mu_{\nu\rho} s_\mu = \sum_{\mu\vdash 2h \text{ even}} s_\mu,$$

which is true by (5.8).

Now assume by induction that (5.17) holds for $k-1$. Note that by (5.12)

$$s_{(h-k,1^k)}[s_{1^2}] = s_{h-k}[s_{1^2}]s_{1^k}[s_{1^2}] - s_{(h-k+1,1^{k-1})}[s_{1^2}].$$

By Theorem 5.7 and the Littlewood–Richardson rule, the term $s_{h-k}[s_{1^2}]s_{1^k}[s_{1^2}]$ is the term $i = 0$ in (5.17). By induction, the second term equals

$$\begin{aligned} s_{(h-k+1,1^{k-1})}[s_{1^2}] &= \sum_{i=0}^{k-1} \sum_{\substack{\mu\vdash 2h \\ \nu\vdash 2(h-k+1+i) \text{ even} \\ \rho\vdash 2(k-1-i) \text{ threshold}}} (-1)^i c^\mu_{\nu\rho} s_\mu \\ &= -\sum_{i=1}^{k} \sum_{\substack{\mu\vdash 2h \\ \nu\vdash 2(h-k+i) \text{ even} \\ \rho\vdash 2(k-i) \text{ threshold}}} (-1)^i c^\mu_{\nu\rho} s_\mu, \end{aligned}$$

which are the remaining terms in (5.17), proving the claim. □

REMARK 5.15. It can be deduced from Corollary 5.14 that the coefficient of s_μ with μ even in $s_{(h-3,1,1,1)}[s_{1^2}]$ is $(b_\mu - 1)(b_\mu - 2)(2b_\mu - 3)/6$. The coefficient of s_μ with μ even in $s_{(h-4,1,1,1,1)}[s_{1^2}]$ is not dependent solely on b_μ since the coefficient of $s_{(3^2 2^2 1^4)}$ in $s_{(3,1,1,1,1)}[s_{1^2}]$ is 1 while the coefficient of $s_{(4^2 2^2 1^2)}$ is 2.

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