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Pullback formulas for arithmetic cycles on orthogonal Shimura varieties

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On an orthogonal Shimura variety, one has a collection of arithmetic special cycles in the Gillet–Soulé arithmetic Chow group. We describe how these cycles behave under pullback to an embedded orthogonal Shimura variety of lower dimension. The bulk of the paper is devoted to cases in which the special cycles intersect the embedded Shimura variety improperly, which requires that we analyze logarithmic expansions of Green currents on the deformation to the normal bundle of the embedding.

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1. Introduction

On an orthogonal Shimura variety M, one has a systematic supply of special cycles coming from embeddings of smaller orthogonal Shimura varieties. These cycles are the subject of a series of conjectures of Kudla [2004], whose central theme is that they should be geometric analogues of the coefficients of Siegel theta functions.

In order to do arithmetic intersection theory with these cycles, one must endow them with Green currents. One construction of such Green currents was done by Garcia and Sankaran [2019], using ideas of Bismut [1990] and Bismut, Gillet and Soulé [Bismut et al. 1990a]. The special cycles endowed with these currents define arithmetic special cycles in the Gillet–Soulé arithmetic Chow group of M. The goal of this paper is to show that these arithmetic special cycles behave nicely under pullbacks via embeddings $M_0 \to M$ of smaller orthogonal Shimura varieties, in the sense that the pullback of an arithmetic special cycle on M is a prescribed linear combination of arithmetic special cycles on M_0.

When an arithmetic cycle intersects M_0 properly, its pullback to can be defined in a naive way, and is easy to compute directly from the definitions. Unfortunately, the intersections that arise in our setting

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are very rarely proper. For improper intersections, Gillet and Soulé define pullbacks using the moving lemma, which is poorly suited to any kind of explicit calculation. One doesn't have any natural choice of rationally trivial cycle with which to move the special cycle, and even if one did, replacing an arithmetic special cycle by a rationally equivalent one would destroy all the nice properties of the special cycle and Green current that one started with.

Our approach to treating improper intersections is to use [Hu 1999], which gives an alternative definition of arithmetic pullbacks based on Fulton's deformation to the normal cone approach to intersection theory. One can always specialize a cycle on M to a cycle on the normal cone to $M_0 \subset M$. As M_0 is smooth, the normal cone is canonically identified with (the total space of) the normal bundle $N_{M_0/M} \to M_0$. Hu showed that there is an analogous specialization of Green currents. The core of this paper is the calculation of the specializations of Garcia–Sankaran Green currents to $N_{M_0/M}$, or at least the calculation of enough of them to deduce the pullback formula.

1.1. ***Statement of the main result.*** Fix a quadratic space V of dimension $n+2 \geq 3$ over a totally real number field F. Assume that V has signature $(n, 2)$ at one embedding $\sigma : F \to \mathbb{R}$, but is positive definite at every other embedding.

From V one can construct a Shimura datum $(G, \mathcal{D})$ in which G is the restriction of scalars to $\mathbb{Q}$ of either $\mathrm{SO}(V)$ or $\mathrm{GSpin}(V)$, and $\mathcal{D}$ is a hermitian symmetric domain of dimension n. Fixing a sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f)$, we obtain a quasiprojective Shimura variety M over the reflex field $\sigma(F) \subset \mathbb{C}$ with complex points

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K.$$

For the rest of the Introduction we assume that V is anisotropic, so that M is projective.

Fix a positive integer $d \leq n+1$. Given the data of a nonsingular symmetric matrix $T \in \mathrm{Sym}_d(F)$ and a K-fixed $\mathbb{Z}$-valued Schwartz function

$$\varphi \in S(\widehat{V}^d),$$

one can define a special cycle $Z(T, \varphi)$ on M of codimension d, as in [Kudla 2004]. After fixing a positive definite $v \in \mathrm{Sym}_d(\mathbb{R})$, Garcia and Sankaran [2019] constructed a smooth form $\mathfrak{g}^\circ(T, v, \varphi)$ of type $(d-1, d-1)$ on the complex fiber of $M \smallsetminus Z(T, \varphi)$. This form is locally integrable on $M(\mathbb{C})$, and its associated current satisfies the Green equation

$$dd^c[\mathfrak{g}^\circ(T, v, \varphi)] + \delta_{Z(T,\varphi)} = [\omega^\circ(T, v, \varphi)]$$

for a smooth form $\omega^\circ(T, v, \varphi)$. In particular, it determines an arithmetic cycle class

$$\widehat{C}_M(T, v, \varphi) = (Z(T, \varphi), \mathfrak{g}^\circ(T, v, \varphi)) \in \widehat{\mathrm{CH}}^d(M) \tag{1.1.1}$$

in the Gillet–Soulé arithmetic Chow group.

Because our main focus is on the Green currents, in this paper we work exclusively with the arithmetic Chow group of the canonical model over the reflex field. No integral models will appear.

Following [Garcia and Sankaran 2019], in Section 4.3 we extend the definition of (1.1.1) to all $T \in \mathrm{Sym}_d(F)$, including the case $\det(T) = 0$. For the purposes of this Introduction, we say only that this extension makes use of a special hermitian line bundle

$$\hat{\omega} \in \widehat{\mathrm{Pic}}(M) \cong \widehat{\mathrm{CH}}^1(M).$$

For example, in the degenerate case of the zero matrix $0_d \in \mathrm{Sym}_d(F)$, the definition is

$$\widehat{C}(0_d, v, \varphi) = \varphi(0) \cdot \big(\underbrace{\hat{\omega}^{-1} \cdots \hat{\omega}^{-1}}_{d} + (0, -\log(\det(v)) \cdot \Omega^{d-1})\big),$$

where the $\cdots$ on the right-hand side is iterated arithmetic intersection, and Ω^{d-1} is the $d-1$ exterior power of the Chern form of $\hat{\omega}^{-1}$.

Now suppose that our quadratic space is presented as an orthogonal direct sum $V = V_0 \oplus W$, with W totally positive definite and $\dim(V_0) \geq 3$. In particular, V_0 has signature $(n_0, 2)$ at the real embedding $\sigma : F \to \mathbb{R}$ and is positive definite at all other embeddings. As such, it has its own Shimura datum $(G_0, \mathcal{D}_0)$, and a choice of compact open $K_0 \subset G_0(\mathbb{A}_f)$ determines a Shimura variety M_0 over $\sigma(F) \subset \mathbb{C}$ with its own family of arithmetic special cycles

$$\widehat{C}_{M_0}(T, v, \varphi_0) \in \widehat{\mathrm{CH}}^d(M_0).$$

The inclusion $V_0 \to V$ induces an embedding of Shimura data $(G_0, \mathcal{D}_0) \to (G, \mathcal{D})$. We choose K_0 and K in such a way that $K_0 \subset G_0(\mathbb{A}_f) \cap K$, and so that the induced $i_0 : M_0 \to M$ is a closed immersion. Our main result, stated in the text as Corollary 4.4.3, is the following pullback formula for arithmetic special cycles.

Theorem A. *Fix a K-fixed Schwartz function*

$$\varphi = \varphi_0 \otimes \psi \in S(\widehat{V}_0^d) \otimes S(\widehat{W}^d) \subset S(\widehat{V}^d),$$

with φ_0 fixed by K_0, and both φ_0 and ψ valued in $\mathbb{Z}$. The pullback

$$i_0^* : \widehat{\mathrm{CH}}^d(M) \to \widehat{\mathrm{CH}}^d(M_0)$$

satisfies

$$i_0^* \widehat{C}_M(T, v, \varphi) = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\ T_0 + T(y) = T}} \psi(y) \cdot \widehat{C}_{M_0}(T_0, v, \varphi_0)$$

for all $T \in \mathrm{Sym}_d(F)$ and all positive definite $v \in \mathrm{Sym}_d(\mathbb{R})$. Here $T(y) \in \mathrm{Sym}_d(F)$ is the moment matrix of the tuple $y \in W^d$, as in (4.3.2).

Remark 1.1.1. If one forgets Green currents and works with the usual Chow groups of M and M_0, the above pullback formula appears [Kudla 2021].

Remark 1.1.2. The constructions of Green currents in [Garcia and Sankaran 2019] are carried out on the Shimura varieties for special orthogonal groups of signature $((n, 2), (n+2, 0), \ldots, (n+2, 0))$ as above, and also on the Shimura varieties for unitary groups of signature $((n, 1), (n+1, 0), \ldots, (n+1, 0))$. There is no difficulty at all in proving the analogue of Theorem A also in the unitary case, using exactly the same argument. We have restricted attention to the orthogonal case only for concreteness, and to avoid excessively cluttering the exposition.

We have not attempted to exhaust the methods, which can presumably be pushed farther. For example, one would like a similar statement for noncompact Shimura varieties and integral models, as well as a formula expressing the intersection of two arithmetic special cycles as a linear combination of other arithmetic special cycles. There should be similar results for the Shimura varieties associated to quadratic spaces with signature $(n, 2)$ at several archimedean places. One could also try to prove similar formulas for other Green currents, such as those of [Funke and Hofmann 2021]. The author hopes to address some of these questions in future work.

1.2. ***Outline of the paper.*** In Section 2 we recall what we need from Hu's thesis [1999]. Suppose $X_0 \to X$ is a closed immersion of complex manifolds. If G is a Green current for an analytic cycle Z on X, one would like to construct a Green current $\sigma_{X_0/X}(G)$ for the specialization $\sigma_{X_0/X}(Z)$ of Z to the normal bundle $N_{X_0/X}$.

To see how this works, denote by $\widetilde{X}$ the deformation to the normal bundle $N_{X_0/X}$. It comes with a holomorphic function $\tau : \widetilde{X} \to \mathbb{C}$ whose fiber over $t \in \mathbb{C}$ we denote by $\widetilde{X}_t$. The fiber at $t = 0$ is $\widetilde{X}_0 = N_{X_0/X}$. If $t \neq 0$ there is a canonical identification $X \cong \widetilde{X}_t$, and hence a closed immersion $j_t : X \cong \widetilde{X}_t \hookrightarrow \widetilde{X}$. In this way we obtain a current $j_{t*}G$ on $\widetilde{X}$. Hu's idea is to look for a logarithmic expansion

$$j_{t*}G = \sum_{i \geq 0} G_i(t) \cdot (\log |t|)^i$$

whose coefficients $G_i(t)$ are currents on $\widetilde{X}$ with the property that each function $t \mapsto G_i(t)$ extends continuously to $t = 0$, and define $\sigma_{X_0/X}(G)$ in terms of the current $G_0(0)$. In this generality such a logarithmic expansion need not exist. If it does exist the logarithmic expansion will not be unique, but $\sigma_{X_0/X}(G)$ is independent of the choice.

In Theorems 2.2.5 and 2.3.1 we quote two results of Hu. The first guarantees the existence of logarithmic expansions (and hence specializations to the normal bundle) for certain currents on X. The second shows that if X is a projective variety over a number field, one can use the specialization of cycles and Green currents to define a morphism from the arithmetic Chow group of X to the arithmetic Chow group of $N_{X_0/X}$. This morphism of arithmetic Chow groups agrees with the one induced by pullback through the composition $N_{X_0/X} \to X_0 \to X$.

Now suppose L is a hermitian line bundle on X. In Section 3 we recall from [Garcia and Sankaran 2019] a construction that takes a tuple $s = (s_1, \dots, s_d)$ of global sections of $L^\vee$ and produces a Green current $\mathfrak{g}^\circ(s)$ for the analytic cycle $Z(s)$ defined by $s_1 = \cdots = s_d = 0$. The central problem is to understand the specialization $\sigma_{X_0/X}(\mathfrak{g}^\circ(s))$. For our applications it is enough to assume that $s = (p, q)$ is the concatenation of tuples $p = (p_1, \dots, p_k)$ and $q = (q_1, \dots, q_\ell)$, arranged so that the cycle $Z(p)$ meets X_0 properly, while $X_0 \subset Z(q)$.

In essence, our strategy is to show that the star product formula

$$\mathfrak{g}^\circ(s) = \mathfrak{g}^\circ(p) \star \mathfrak{g}^\circ(q_1) \star \cdots \star \mathfrak{g}^\circ(q_\ell) \tag{1.2.1}$$

of Garcia and Sankaran implies the analogous star product formula

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(s)) = \sigma_{X_0/X}(\mathfrak{g}^\circ(p)) \star \sigma_{X_0/X}(\mathfrak{g}^\circ(q_1)) \star \cdots \star \sigma_{X_0/X}(\mathfrak{g}^\circ(q_\ell)) \tag{1.2.2}$$

for specializations, and then compute each specialization on the right individually. As $Z(p)$ intersects X_0 properly, the specialization $\sigma_{X_0/X}(\mathfrak{g}^\circ(p))$ is easy to compute. To compute the specialization of $\mathfrak{g}^\circ(q_i)$, one must do more work, but the idea is imitate the construction of the current with X replaced by the deformation to the normal bundle $\widetilde{X}$, and use the resulting current on $\widetilde{X}$ to find an explicit logarithmic expansion for $\mathfrak{g}^\circ(q_i)$

It is not obvious to the author that Hu's specialization to the normal bundle is compatible with $\star$ products in general; that is to say, deducing (1.2.2) from (1.2.1) seems to require using particular properties of the Green currents $\mathfrak{g}^\circ(s)$. *After* passing to arithmetic Chow groups the compatibility of specialization with star products follows from Theorem 2.3.1, but on the level of arithmetic cycles (that is, before passing to their rational equivalence classes) things are more complicated. When we apply the calculations described above to the case of orthogonal Shimura varieties, the complex manifold X is not the Shimura variety $M(\mathbb{C})$, but rather the hermitian symmetric domain $\mathcal{D}$ that uniformizes it. As $\mathcal{D}$ does not have an arithmetic Chow group in any useful sense, our calculations must be carried out before passing to rational equivalence classes of arithmetic cycles.

To prove the compatibility of specializations with star products we need to show that the Green currents in question admit logarithmic expansions of an especially nice form; this is essentially Lemma 3.3.3, which is the core of the proof of Proposition 3.3.7. While Hu's proof of Theorem 2.3.1 provides a general construction of logarithmic expansions, the expansions one gets from this method are quite unpleasant. For example, if one starts with a current G that is represented by a locally integrable form, the currents $G_i(t)$ produced by Hu's construction will typically not have this form (Hu's construction of logarithmic expansions uses an inductive process, and each step of the induction introduces δ-currents that are not represented by smooth forms). It is essential to our method that we find logarithmic expansions for $\mathfrak{g}^\circ(s)$ that are better behaved than those obtained by tracing through Hu's proof of Theorem 2.3.1.

We emphasize that all the calculations in Section 3 are carried out in the setting of an arbitrary complex manifold X, and don't involve orthogonal Shimura varieties (or any Shimura varieties) at all.

Finally, in Section 4 we define the precise arithmetic cycle classes $\widehat{C}(T, v, \varphi)$ appearing in Theorem A, and apply the general constructions of the preceding sections to the case of orthogonal Shimura varieties. The strategy for proving Theorem A is to use the explicit calculation of specializations of cycles and Green currents to show that the arithmetic cycles appearing in the equality of that theorem become equal after pullback via the bundle map $N_{M_0/M} \to M_0$. By Proposition 2.3.3, they must have been equal before the pullback as well.

Something like specializations to the normal bundle of Green currents were computed in [Andreatta et al. 2017], but for the Green functions for special divisors in [Bruinier 2002]. When working only with arithmetic divisors, the situation is much simpler, and one doesn't really need specialization to the normal bundle at all. The codimension-1 arithmetic Chow group can be identified with the arithmetic Picard group, and pullback then agrees with the naive notion of pullback of hermitian line bundles. One can compute arithmetic pullbacks (even in cases of improper intersection) more directly using this interpretation. This is the approach taken in [Bruinier et al. 2015], which is the unitary Shimura variety analogue of [Andreatta et al. 2017]. To compute pullbacks for higher-codimension arithmetic Chow groups, the author knows of no method other than the specialization to the normal bundle approach developed here.

As a final remark, we note that the proof of Theorem 4.13 of [Bismut et al. 1990b], whose statement involves pullbacks of arithmetic cycle classes via closed immersions, also makes use of the deformation to the normal bundle. The connection between the methods used in [loc. cit.] and the logarithmic expansions of [Hu 1999] are not obvious to the author.

2. Arithmetic specialization to the normal bundle

In this section we recall some results from Hu's thesis [1999], and restate them in the precise form they will be needed later.

2.1. ***Logarithmic differential forms.*** We recall some definitions and results from [Burgos 1994]. Let X be a complex manifold of dimension $n = \dim(X)$.

Definition 2.1.1. If $Z \subset X$ is any analytic subset (i.e., a reduced closed analytic subspace), a *resolution of singularities*

$$r : (X', Z') \to (X, Z) \tag{2.1.1}$$

is a complex manifold X' together with a proper surjection $r : X' \to X$ such that $Z' = r^{-1}(Z)$ is a divisor with normal crossings and r restricts to an isomorphism $X' \smallsetminus Z' \cong X \smallsetminus Z$.

Remark 2.1.2. A resolution of singularities always exists by Theorem 3.3.5 of [Kollár 2007], extended to analytic spaces as in Section 3.4.4 of that work. See also [Włodarczyk 2009]. For quasiprojective varieties, this is Hironaka's theorem.

Denote by

$$E_X^\bullet = \bigoplus_{k \geq 0} E_X^k$$

the graded $\mathbb{C}$-algebra of smooth differential forms on X. For $\phi \in E_X^\bullet$, let

$$\phi_{[k]} \in E_X^k$$

be its component in degree k. Let ${}_cE_X^\bullet \subset E_X^\bullet$ be the graded subspace of compactly supported forms, and denote by D_X^k be the space of currents dual to ${}_cE_X^{2n-k}$. There is a canonical injection $E_X^k \to D_X^k$, denoted by $g \mapsto [g]$, defined by

$$[g](\phi) = \int_X g \wedge \phi. \tag{2.1.2}$$

When no confusion can arise, we sometimes omit the brackets, and write g both for the form and its associated current.

Given a divisor with normal crossings $Z \subset X$, let

$$E^\bullet_X(\log Z) \subset E^\bullet_{X \smallsetminus Z} \tag{2.1.3}$$

be the graded subalgebra of forms with logarithmic growth along Z as in Section 1.2 of [Burgos 1994]: in local coordinates on X such that Z is given by the equation $z_1 \cdots z_m = 0$, the forms of logarithmic growth are generated, as an algebra over the ring of smooth forms, by

$$\log|z_i|, \frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i} \quad \text{for } 1 \le i \le m.$$

Now let $Z \subset X$ be any analytic subset of codimension $d > 0$. A choice of resolution of singularities (2.1.1) determines a subspace

$$E^\bullet_X(\log Z) \subset E^\bullet_{X \smallsetminus Z} \tag{2.1.4}$$

consisting of those forms whose pullback to $E^\bullet_{X' \smallsetminus Z'}$ has logarithmic growth along the normal crossing divisor Z'. Although the notation does not indicate it, this subspace genuinely depends on the choice of resolution of singularities.

Denote by

$$E^\bullet_X(\operatorname{null} Z) \subset E^\bullet_X$$

the graded subspace of forms whose pullback to (the smooth locus of) Z vanishes, and let ${}_cE^\bullet_X(\operatorname{null} Z)$ be the graded subspace of compactly supported such forms. The inclusion ${}_cE^\bullet_X(\operatorname{null} Z) \subset {}_cE^\bullet_X$ induces a canonical surjection

$$D^\bullet_X \to D^\bullet_{X/Z},$$

where $D^k_{X/Z}$ is the space of currents dual to ${}_cE^{2n-k}_X(\operatorname{null} Z)$.

Proposition 2.1.3 (Burgos). *For any $g \in E^\bullet_X(\log Z)$ and $\phi \in E^\bullet_X(\operatorname{null} Z)$, the form $g \wedge \phi$ is locally integrable on X. The integral* (2.1.2) *defines an injection*

$$E^\bullet_X(\log Z) \xrightarrow{g \mapsto [g]} D^\bullet_{X/Z}$$

satisfying $\partial[g] = [\partial g]$, and similarly for $\bar\partial$.

Proof. See Corollaries 3.7 and 3.8 in [Burgos 1994]. □

Remark 2.1.4. If $k < 2d$ then any form in E^{2n-k}_X has trivial pullback to Z, and hence $D^k_{X/Z} = D^k_X$. In particular, we obtain an injection

$$E^k_X(\log Z) \xrightarrow{g \mapsto [g]} D^k_X.$$

For $g \in E^k_X(\log Z)$ with $k < 2d - 1$ we have $\partial[g] = [\partial g]$ in D^{k+1}_X, and similarly for $\bar\partial$.

Definition 2.1.5. Suppose that U is a smooth quasiprojective complex variety. By a *smooth compactification* of U we mean a smooth projective variety U^*, a divisor with normal crossings $\partial U^* \subset U^*$, and an isomorphism $i : U \cong U^* \smallsetminus \partial U^*$. The smooth compactifications of U form a cofiltered category in a natural way, allowing us to form the graded subalgebra

$$E^\bullet_{\log}(U) = \varinjlim_{(U^*,\partial U^*,i)} E^\bullet_{U^*}(\log \partial U^*) \subset E^\bullet_U$$

of *forms with logarithmic singularities along* ∞; see Definition 1.2 of [Burgos 1997] and the discussion surrounding it.

Remark 2.1.6. Of special interest is the case in which X is a smooth quasiprojective complex variety, $Z \subset X$ is a closed subvariety of codimension d, and we take $U = X \smallsetminus Z$. In this case, for any

$$g \in E^k_{\log}(X \smallsetminus Z)$$

there is a resolution of singularities (2.1.1) such that $g \in E^k_X(\log Z)$. When $k < 2d$, Remark 2.1.4 therefore provides us with an injection

$$E^k_{\log}(X \smallsetminus Z) \xrightarrow{g \mapsto [g]} D^k_X.$$

In the usual way, the complex structure on X induces bigradings

$$E^k_X = \bigoplus_{p+q=k} E^{p,q}_X \quad \text{and} \quad D^k_X = \bigoplus_{p+q=k} D^{p,q}_X,$$

and similarly for the other spaces of forms and currents appearing above.

2.2. ***Specialization to the normal bundle.*** For a closed immersion of schemes $X_0 \subset X$ one has the normal cone $C_{X_0/X} \to X_0$. If $X_0 \subset X$ is a regular immersion, the normal cone agrees with (the total space of) the normal bundle $N_{X_0/X} \to X_0$. These constructions, as well as the deformation to the normal cone, generalize in an obvious way to a closed immersion of complex analytic spaces; the necessary technical details are in [Axelsson and Magnússon 1986].

Now suppose that $X_0 \subset X$ is a closed immersion of complex manifolds. Denote by $\widetilde{X}$ the deformation to $N_{X_0/X} = C_{X_0/X}$. By construction, it comes with morphisms

$$X \xleftarrow{\pi} \widetilde{X} \xrightarrow{\tau} \mathbb{C} \tag{2.2.1}$$

such that π identifies every fiber $\widetilde{X}_t = \tau^{-1}(t)$ with

$$\widetilde{X}_t \cong \begin{cases} X & \text{if } t \neq 0, \\ N_{X_0/X} & \text{if } t = 0. \end{cases}$$

When $t \neq 0$, we denote by

$$j_t : X \cong \widetilde{X}_t \hookrightarrow \widetilde{X} \tag{2.2.2}$$

the inclusion, and similarly for $j_0 : N_{X_0/X} \cong \widetilde{X}_0 \hookrightarrow \widetilde{X}$.

Let $Z \subset X$ be any equidimensional analytic subset, endowed with its reduced complex analytic structure. The *strict transform*

$$\widetilde{Z} \subset \widetilde{X} \tag{2.2.3}$$

of Z is defined as the deformation to the normal cone $C_{(Z\times_X X_0)/Z}$. It is again reduced (although $Z\times_X X_0$ and the normal cone $C_{(Z\times_X X_0)/Z}$ need not be), and can be characterized as the union of all irreducible components of $\pi^{-1}(Z)$ not contained in $N_{X_0/X}$. Equivalently, (2.2.3) is the closure of $\pi^{-1}(Z)\smallsetminus\widetilde{X}_0$ in $\widetilde{X}$.

By an *analytic cycle* on a complex manifold we mean a locally finite formal $\mathbb{Z}$-linear combination of irreducible analytic subsets, all of the same codimension. Being reduced and equidimensional, we may view $\widetilde{Z}$ as an analytic cycle on $\widetilde{X}$ and form, for every $t\in\mathbb{C}$, the analytic cycle

$$\widetilde{Z}_t=\widetilde{Z}\cdot\widetilde{X}_t \tag{2.2.4}$$

on $\widetilde{X}$ supported on the fiber $\widetilde{X}_t$. Here the proper intersection of analytic cycles on the right is taken in the sense of [Draper 1969]. Of special interest is the analytic cycle (2.2.4) at $t=0$.

Definition 2.2.1. The analytic cycle

$$\sigma_{X_0/X}(Z)\stackrel{\text{def}}{=}\widetilde{Z}_0 \tag{2.2.5}$$

on $N_{X_0/X}=\widetilde{X}_0$ is the *specialization of Z to the normal bundle $N_{X_0/X}$*.

Having defined $\widetilde{Z}$, $\widetilde{Z}_t$, and $\sigma_{X_0/X}(Z)$ for a reduced analytic subset $Z\subset X$, extend the definitions linearly to all analytic cycles Z on X.

Remark 2.2.2. If $t\neq 0$ then $\widetilde{Z}_t$ is simply the pushforward of Z under the inclusion (2.2.2). The cycle $\widetilde{Z}_0$, which may be nonreduced, is then uniquely determined by the continuity at $t=0$ of the function

$$t\mapsto\delta_{\widetilde{Z}_t}(\phi)\stackrel{\text{def}}{=}\int_{\widetilde{Z}_t}\phi \tag{2.2.6}$$

for every $\phi\in{}_cE^{2\dim(Z)}_{\widetilde{X}}$. Moreover, if we temporarily denote by I_ϕ the continuous compactly supported function on $\mathbb{C}$ defined by (2.2.6), one has the Fubini-style integration formula

$$\int_{\widetilde{Z}}\phi\wedge\tau^*\omega=\int_{\mathbb{C}}I_\phi\wedge\omega$$

for any smooth 2-form ω on $\mathbb{C}$.

The continuity of (2.2.6) and the Fubini formula (2.2.6) are due to King [1971]. More precisely, Theorem 3.3.2 of [loc. cit.] constructs a family of analytic cycles $t\mapsto\widetilde{Z}_t$ on $\widetilde{X}$ for which these properties hold; the equality of this family of cycles with (2.2.4) is then a consequence of the results of Section 4.1 of [loc. cit.], especially Proposition 4.1.6 and the remarks that follow it.

Remark 2.2.3. In the case where X and Z are the complex analytic spaces associated to finite type schemes over $\mathbb{C}$, Draper's analytic intersection (2.2.4) agrees with the proper intersection of cycles in the algebraic sense of [Fulton 1984; Soulé 1992], and the cycle (2.2.5) agrees with the specialization to the normal cone in the sense of [Fulton 1984].

Definition 2.2.4. Fix a $G\in D^k_X$, and note that every $t\in\mathbb{C}\smallsetminus\{0\}$ determines a current

$$j_{t*}G\in D^{k+2}_{\widetilde{X}}.$$

We say that *G admits a logarithmic expansion along X_0* if there is a sequence of functions $G_0,G_1,G_2,\dots:\mathbb{C}\to D^{k+2}_{\widetilde{X}}$ with the following properties:

(1) For all $t \in \mathbb{C} \smallsetminus \{0\}$ we have

$$j_{t*}G = \sum_{i \geq 0} G_i(t) \cdot (\log|t|)^i,$$

and the sum is locally finite: for every compact subset $K \subset \widetilde{X}$ there is an integer M_K such that $G_i(t)(\phi) = 0$ for all $i > M_K$, all $t \in \mathbb{C} \smallsetminus \{0\}$, and all $\phi \in E^{2n-k}_{\widetilde{X}}$ with support contained in K.

(2) For every $i \geq 0$ and every $\phi \in {}_cE^{2n-k}_{\widetilde{X}}$, the function $t \mapsto G_i(t)(\phi)$ is continuous at $t = 0$, and is Hölder continuous at $t = 0$ if $i > 0$.

(3) Each $G_i(0)$ lies in the image of $j_{0*} : D^k_{\widetilde{X}_0} \to D^{k+2}_{\widetilde{X}}$.

The following result is slightly weaker than Theorem 3.2.2 of [Hu 1999]; see Remark 2.2.11 below. It provides a general criterion for the existence of logarithmic expansions.

Theorem 2.2.5 (Hu). *Suppose we have a form*

$$g \in E^k_X(\log Z)$$

in the subspace (2.1.4) *for some equidimensional analytic subset $Z \subset X$ of positive codimension, and some choice of resolution of singularities. If g is locally integrable on X, then the associated current $[g] \in D^k_X$ admits a logarithmic expansion along X_0. Moreover, if X is compact, there exists a logarithmic expansion with $G_i = 0$ for $i \gg 0$.*

Remark 2.2.6. Hu works on smooth quasiprojective complex varieties, but the same proof works for complex manifolds. The only difference is that in the quasiprojective case one can use the existence of smooth compactifications of X and $\widetilde{X}$ to prove the existence of finite (not just locally finite) logarithmic expansions. See Remark 2.2.11 below.

Remark 2.2.7. Suppose we are given functions $f_0, \dots, f_m : \mathbb{C} \to \mathbb{C}$ with f_0 continuous at 0, and $f_1, \dots, f_m$ Hölder continuous at 0. An easy induction on m, as in Lemma 3.1.5 of [Hu 1999], shows that if

$$\lim_{t \to 0} \sum_{i=0}^{m} f_i(t) \cdot (\log|t|)^i = 0,$$

then $f_i(0) = 0$ for all i.

The functions G_i in Definition 2.2.4, when they exist, are not uniquely determined. However, it follows from Remark 2.2.7 that the currents $G_i(0)$ are independent of the choice of logarithmic expansion. This allows us to make the following definition.

Definition 2.2.8. If $G \in D^k_X$ admits a logarithmic expansion along X_0, its *specialization to the normal bundle* is the current

$$\sigma_{X_0/X}(G) \in D^k_{N_{X_0/X}}$$

on the normal bundle $N_{X_0/X}$ characterized by $j_{0*}\sigma_{X_0/X}(G) = G_0(0)$.

As a trivial example, if $g \in E^k_X$ then $\pi^* g$ is a smooth form on $\widetilde{X}$, and

$$j_{t*}[g](\phi) = \int_{\widetilde{X}_t} \pi^* g \wedge \phi$$

for all $t \neq 0$. By Remark 2.2.2, the right-hand side is a continuous function of $t \in \mathbb{C}$, and we obtain a logarithmic expansion of $[g]$ by setting $G_0(t) = \pi^* g \wedge \delta_{\widetilde{X}_t}$ and $G_i(t) = 0$ for $i > 0$. In particular,

$$\sigma_{X_0/X}(g) = [j_0^* \pi^* g] = [\pi_0^* i_0^* g],$$

where $\pi_0 : N_{X_0/X} \to X_0$ is the bundle map and $i_0 : X_0 \to X$ is the inclusion.

Remark 2.2.9. If $G \in D^k_X$ admits a logarithmic expansion along X_0, then so does ∂G, and

$$\partial \sigma_{X_0/X}(G) = \sigma_{X_0/X}(\partial G).$$

The same holds with ∂ replaced by $\bar{\partial}$. This is a formal consequence of the definitions; see Theorem 3.1.6 of [Hu 1999].

The following proposition connects Definitions 2.2.1 and 2.2.8. The proof is extracted from the second proof of Theorem 3.2.3 in [Hu 1999].

Proposition 2.2.10. *Suppose $X_0 \subset X$ is a closed complex submanifold, Z is a codimension-d analytic cycle on X, and $G \in D^{d-1,d-1}_X$ satisfies the Green equation*

$$dd^c G + \delta_Z = [\omega]$$

for some $\omega \in E^{d,d}_X$. If G admits a logarithmic expansion along X_0, then its specialization to the normal bundle satisfies the Green equation

$$dd^c \sigma_{X_0/X}(G) + \delta_{\sigma_{X_0/X}(Z)} = [\pi_0^* i_0^* \omega].$$

Here $\pi_0 : N_{X_0/X} \to X_0$ is the bundle map and $i_0 : X_0 \to X$ is the inclusion.

Proof. When $t \neq 0$, we may push forward the Green equation for G via $j_t : X \to \widetilde{X}$. This yields the equality

$$dd^c j_{t*} G + \delta_{\widetilde{Z}_t} = \pi^* \omega \wedge \delta_{\widetilde{X}_t}$$

of currents on $\widetilde{X}$. Replacing $j_{t*}G$ by a logarithmic expansion results in

$$(dd^c G_0(t) + \delta_{\widetilde{Z}_t} - \pi^* \omega \wedge \delta_{\widetilde{X}_t}) + \sum_{i>0} dd^c G_i(t) \cdot (\log |t|)^i = 0,$$

and it follows from Remarks 2.2.2 and 2.2.7 that

$$dd^c G_0(0) + \delta_{\widetilde{Z}_0} - \pi^* \omega \wedge \delta_{\widetilde{X}_0} = 0.$$

The claim now follows using $\pi^* \omega \wedge \delta_{\widetilde{X}_0} = j_{0*}[j_0^* \pi^* \omega] = j_{0*}[\pi_0^* i_0^* \omega]$. □

Remark 2.2.11. In Hu's version of Theorem 2.2.5 it is assumed that X is a quasiprojective variety, and that $g \in E^k_{\log}(X \smallsetminus Z)$. These extra assumptions are not used in the proof in any essential way. However, the first guarantees the existence of a smooth compactification of X. Using this, Hu proves a stronger result than what we have stated.

After choosing a smooth compactification $X \subset X^*$, Hu constructs a smooth compactification $\widetilde{X} \subset \widetilde{X}^*$ of the deformation to the normal bundle, a diagram

$$X^* \xleftarrow{\pi} \widetilde{X}^* \xrightarrow{\tau} \mathbb{C}$$

extending (2.2.1), and a finite expansion of currents

$$\pi^* g \wedge \delta_{\widetilde{X}^*} = \sum_{i=0}^{M} G_i(t) \cdot (\log|t|)^i \tag{2.2.7}$$

in the space $D^{k+2}_{\widetilde{X}^*/\partial\widetilde{X}^*}$. The inclusion ${}_cE^\bullet_{\widetilde{X}} \to {}_cE^\bullet_{\widetilde{X}^*}(\text{null}\,\partial\widetilde{X}^*)$ induces a surjection

$$D^\bullet_{\widetilde{X}^*/\partial\widetilde{X}^*} \to D^\bullet_{\widetilde{X}},$$

and applying this map to both sides of (2.2.7) yields a finite logarithmic expansion of $[g]$. The refined logarithmic expansion (2.2.7) contains more information than a logarithmic expansion in our sense. Using it, Hu is able to construct a smooth compactification $N_{X_0/X} \subset N^*_{X_0/X}$ of the normal bundle, and a distinguished lift of $\sigma_{X_0/X}(g)$ under the surjection

$$D^\bullet_{N^*_{X_0/X}/\partial N^*_{X_0/X}} \to D^\bullet_{N_{X_0/X}}.$$

Although we will not need such a lift, the benefits of having one are explained in Remark 2.3.2.

2.3. ***Arithmetic Chow groups.*** We will use the arithmetic Chow groups defined in Section 3 of [Gillet and Soulé 1990], but only in the simple case of varieties over a field F with a chosen real embedding $\sigma : F \to \mathbb{R}$. If we let $c \in \mathrm{Aut}(\mathbb{C}/\mathbb{R})$ be complex conjugation, the triple $(F, \{\sigma\}, c)$ is an arithmetic ring, and any smooth quasiprojective variety X over F is an arithmetic variety over $(F, \{\sigma\}, c)$ in the sense of [loc. cit.].

Let $X_\mathbb{R} = X \otimes_{F,\sigma} \mathbb{R}$ be the base change of X to $\mathbb{R}$, and regard $X(\mathbb{C}) = X_\mathbb{R}(\mathbb{C})$ as a complex manifold. Define a real vector space

$$E^{d,d}_X = \{\omega \in E^{d,d}_{X(\mathbb{C})} : \omega \text{ is real and } c^*\omega = (-1)^d\omega\},$$

where now $c : X(\mathbb{C}) \to X(\mathbb{C})$ is complex conjugation, and similarly

$$D^{d,d}_X = \{G \in D^{d,d}_{X(\mathbb{C})} : G \text{ is real and } c^*G = (-1)^d G\}.$$

A *codimension-d arithmetic cycle* on X is a pair (Z, G) in which Z is a codimension-d cycle in the usual sense, and

$$G \in \widetilde{D}^{d-1,d-1}_X \stackrel{\text{def}}{=} \frac{D^{d-1,d-1}_X}{\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})}$$

satisfies the Green equation $dd^cG + \delta_{Z(\mathbb{C})} = [\omega]$ for some $\omega \in E^{d,d}_X$. Denote by $\widehat{Z}^d(X)$ the abelian group of all such pairs. The arithmetic Chow group is the quotient

$$\widehat{\mathrm{CH}}^d(X) = \widehat{Z}^d(X)/(\text{rational equivalence}).$$

Now assume that X is projective, and that $X_0 \subset X$ is a smooth closed subvariety. Let (Z, G) be any codimension-d arithmetic cycle, and set $U = X \smallsetminus Z$. Recalling Definition 2.1.5, define a real vector space

$$E^{d,d}_{\log}(U) = \{g \in E^{d,d}_{\log}(U(\mathbb{C})) : g \text{ is real and } F^*_\infty g = (-1)^d g\}.$$

By Remark 2.1.6 there is a canonical map

$$E^{d-1,d-1}_{\log}(U) \xrightarrow{g \mapsto [g]} D^{d-1,d-1}_X,$$

and Theorem 4.4 of [Burgos 1994] implies the existence of a unique lift of G to

$$g \in \widetilde{E}^{d-1,d-1}_{\log}(U) \stackrel{\text{def}}{=} \frac{E^{d-1,d-1}_{\log}(U)}{\operatorname{Im}(\partial) + \operatorname{Im}(\bar\partial)}.$$

Theorem 2.2.5 therefore implies that the current $G = [g]$ admits a logarithmic expansion along X_0. Combining this with Remark 2.2.9 and Proposition 2.2.10, we obtain an arithmetic cycle

$$(\sigma_{X_0/X}(Z), \sigma_{X_0/X}(G)) \in \widehat{Z}^d(N_{X_0/X}).$$

This defines a homomorphism

$$\widehat{Z}^d(X) \to \widehat{Z}^d(N_{X_0/X}). \tag{2.3.1}$$

The following is slightly weaker than what is proved in Section 4.1 of [Hu 1999]; see Remark 2.3.2 below.

Theorem 2.3.1 (Hu). *Still assuming that X is projective, the homomorphism* (2.3.1) *descends to*

$$\widehat{\mathrm{CH}}^d(X) \to \widehat{\mathrm{CH}}^d(N_{X_0/X}),$$

and this map agrees with the composition

$$\widehat{\mathrm{CH}}^d(X) \xrightarrow{i_0^*} \widehat{\mathrm{CH}}^d(X_0) \xrightarrow{\pi_0^*} \widehat{\mathrm{CH}}^d(N_{X_0/X}).$$

Here $i_0 : X_0 \to X$ is the inclusion, $\pi_0 : N_{X_0/X} \to X_0$ is the bundle map, and i_0^* and π_0^* are the induced pullbacks on arithmetic Chow groups.

Remark 2.3.2. Assuming only that X is quasiprojective, there are canonical maps

$$\widehat{Z}^d(X, \mathcal{D}_{\log}) \to \widehat{Z}^d(X) \quad \text{and} \quad \widehat{\mathrm{CH}}^d(X, \mathcal{D}_{\log}) \to \widehat{\mathrm{CH}}^d(X),$$

where the domains are the $\mathcal{D}_{\log}$ arithmetic cycles and Chow groups of [Burgos Gil et al. 2007]. These agree with those of [Burgos 1997], and both maps are isomorphisms if X is projective. Hu proves the existence of a distinguished lift of (2.3.1) to

$$\widehat{Z}^d(X, \mathcal{D}_{\log}) \to \widehat{Z}^d(N_{X_0/X}, \mathcal{D}_{\log}), \tag{2.3.2}$$

which then descends to a map on $\mathcal{D}_{\log}$ arithmetic Chow groups. This descent agrees with the composition

$$\widehat{\mathrm{CH}}^d(X, \mathcal{D}_{\log}) \xrightarrow{i_0^*} \widehat{\mathrm{CH}}^d(X_0, \mathcal{D}_{\log}) \xrightarrow{\pi_0^*} \widehat{\mathrm{CH}}^d(N_{X_0/X}, \mathcal{D}_{\log}).$$

Even when X is projective, this is stronger than Theorem 2.3.1 (because $N_{X_0/X}$ is not projective). The construction of the lift (2.3.2) is subtle, but the key ingredient is the lift of $\sigma_{X_0/X}(g)$ mentioned at the end of Remark 2.2.11.

Proposition 2.3.3. *The pullback π_0^* in Theorem 2.3.1 is injective.*

Proof. A similar statement is found in [Burgos 1997], but for the $\mathcal{D}_{\log}$ arithmetic Chow groups of Remark 2.3.2. The proof for Gillet–Soulé arithmetic Chow groups is essentially the same: By Theorem 3.3.5 of [Gillet and Soulé 1990] there is commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{CH}^{d,d-1}(X_0) & \longrightarrow & \widetilde{E}^{d-1,d-1}_{X_0} & \longrightarrow & \widehat{\mathrm{CH}}^d(X_0) & \longrightarrow & \mathrm{CH}^d(X_0) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_0^*} & & \downarrow \\
\mathrm{CH}^{d,d-1}(N_{X_0/X}) & \longrightarrow & \widetilde{E}^{d-1,d-1}_{N_{X_0/X}} & \longrightarrow & \widehat{\mathrm{CH}}^d(N_{X_0/X}) & \longrightarrow & \mathrm{CH}^d(N_{X_0/X})
\end{array}$$

The first and last vertical arrows are isomorphisms by Theorem 8.3 of [Gillet 1981]. The second vertical arrow is injective, and hence the third is as well. □

3. Green currents of Garcia and Sankaran

Given a closed immersion of complex manifolds $X_0 \subset X$, the constructions of Garcia and Sankaran [2019], Bismut [1990], and Bismut, Gillet and Soulé [Bismut et al. 1990a] provide a systematic way to produce Green currents for certain cycles on X. Theorem 2.2.5 can be applied to these currents to prove the existence of logarithmic expansions, but this abstract existence theorem is not sharp enough for our purposes.

The goal of this section is to construct explicit logarithmic expansions for these currents, and so effectively compute their specializations to the normal bundle $N_{X_0/X}$.

3.1. ***Construction of Green forms.*** Let X be a complex manifold, and let L be a holomorphic line bundle on X. We use the same symbol for both the total space $L \to X$, viewed as a complex manifold fibered over X, and for its sheaf of holomorphic sections.

Let $h(-,-)$ be a hermitian metric on L. If s is any local holomorphic section of L, abbreviate $h(s) = h(s,s)$. The *Chern form* of L is the $(1,1)$-form defined locally by

$$\mathrm{ch}(L) = \frac{1}{2\pi i}\partial\bar{\partial}\log h(s).$$

We denote again by h the induced metric on the dual bundle $L^\vee$.

Fix an integer $1 \le d \le \dim(X)$ and a tuple $s = (s_1, \dots, s_d)$ with $s_i \in H^0(X, L^\vee)$, and abbreviate

$$h(s) = h(s_1) + \cdots + h(s_d).$$

Denote by $Z(s) \subset X$ the (possibly nonreduced) analytic subspace defined by $s_1 = \cdots = s_d = 0$.

Definition 3.1.1. Fix a point $x \in Z(s)$, trivialize L in a neighborhood of x, and use this to view $s_{1,x}, \dots, s_{d,x} \in \mathcal{O}_{X,x}$ as germs of holomorphic functions at x. We say that s is

- *regular at x* if $s_{1,x}, \dots, s_{d,x} \in \mathcal{O}_{X,x}$ is a regular sequence in the sense of commutative algebra;
- *smooth at x* if $s_{1,x}, \dots, s_{d,x}$ are linearly independent in $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$, where $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ is the maximal ideal.

The tuple $s = (s_1, \dots, s_d)$ is *regular* or *smooth* if it has this property at every point of $Z(s)$.

Remark 3.1.2. Regularity of s at x is equivalent to all irreducible components of $Z(s)$ passing through x having codimension d in X, and both are equivalent to $\mathcal{O}_{Z(s),x}$ being Cohen–Macaulay of dimension $\dim(X) - d$.

Remark 3.1.3. Smoothness of s at x is equivalent to $Z(s)$ being nonsingular (that is, a complex manifold) of codimension d in some open neighborhood of x, as both are equivalent to $\mathcal{O}_{Z(s),x}$ being regular of dimension $\dim(X) - d$.

Remark 3.1.4. If $s = (s_1, \dots, s_d)$ is smooth, we have the equality of cycles

$$Z(s) = Z(s_1) \cdots Z(s_d)$$

on X, where the intersection on the right is the proper analytic intersection of [Draper 1969]. In other words, in the smooth case the intersection $\mathrm{div}(s_1) \cdots \mathrm{div}(s_d)$ in Draper's sense is (of course) simply the reduced analytic subspace defined by $s_1 = \cdots = s_d = 0$.

Remark 3.1.5. If $s = (s_1, \dots, s_d)$ is regular or smooth at a point x, the same is true of all tuples obtained by reordering the components of s, and of all tuples $(s_1, \dots, s_r)$ with $1 \le r \le d$.

The claims of Remarks 3.1.2, 3.1.3, and 3.1.5, all follow from basic properties of regular sequences and complex analytic spaces, as found in [Matsumura 1989; Fischer 1976]. Similarly, it is elementary to check that regularity of s is equivalent to the corresponding morphism of vector bundles $s : L^{\oplus d} \to \mathcal{O}_X$ being regular in the sense of Section 2.1.1 of [Garcia and Sankaran 2019]. Therefore, if s is regular, the constructions (2.5) and (2.12) of [loc. cit.] define forms

$$\varphi^\circ(s) \in E_X^\bullet \quad \text{and} \quad \nu^\circ(s) \in E_X^\bullet.$$

Both have trivial components in odd degree, and their components in even degree $2p$ have type (p, p). Abbreviate

$$\omega^\circ(s) = (-2\pi i)^{-d} \cdot \varphi^\circ(s)_{[2d]} \in E_X^{d,d}.$$

We will not recall the detailed construction of the forms above, as we only need the degree-$2d$ component of $\varphi^\circ(s)$ and the degree-$(2d-2)$ component of $\nu^\circ(s)$. Explicit formulas for these can be found in [Garcia and Sankaran 2019; Garcia 2018]. If $d = 1$ then

$$\varphi^\circ(s)_{[2]} = 2\pi i e^{-2\pi h(s)} \left(\mathrm{ch}(L) - i \frac{\partial h(s) \wedge \bar{\partial} h(s)}{h(s)} \right) \tag{3.1.1}$$

and

$$\nu^\circ(s)_{[0]} = e^{-2\pi h(s)}. \tag{3.1.2}$$

If $d > 1$ then

$$\varphi^\circ(s_1, \dots, s_d)_{[2d]} = \varphi^\circ(s_1)_{[2]} \wedge \cdots \wedge \varphi^\circ(s_d)_{[2]}, \tag{3.1.3}$$

$$\nu^\circ(s_1, \dots, s_d)_{[2d-2]} = \sum_{j=1}^{d} \nu^\circ(s_i)_{[0]} \wedge \varphi^\circ(s_1, \dots, \hat{s}_j, \dots, s_d)_{[2d-2]}. \tag{3.1.4}$$

Strictly speaking, the above formulas are given in [Garcia and Sankaran 2019; Garcia 2018] only for specific hermitian line bundles on hermitian symmetric domains associated to orthogonal and unitary groups, but the derivations of these formulas hold verbatim in our more general setting.

As explained in [Garcia and Sankaran 2019], results of [Bismut 1990; Bismut et al. 1990a] can be used to produce Green currents for the cycles $Z(s) \subset X$ defined above. We need a slight strengthening of those results.

Proposition 3.1.6. *If s is regular, the integral*

$$\mathfrak{g}^\circ(s) = \Big(-\frac{1}{2\pi i}\Big)^{d-1} \int_1^\infty \nu^\circ(\sqrt{u} \cdot s)_{[2d-2]} \frac{du}{u} \tag{3.1.5}$$

defines a smooth form on $X \smallsetminus Z(s)$ with

$$\mathfrak{g}^\circ(s) = \frac{a(s)}{h(s)^{d-1}} + b(s) \cdot \log(h(s)) \tag{3.1.6}$$

for some $a(s), b(s) \in E_X^{d-1,d-1}$. If s is smooth, then

$$\mathfrak{g}^\circ(s) \in E_X^{d-1,d-1}(\log Z(s)) \tag{3.1.7}$$

with respect to the resolution of singularities of $(X, Z(s))$ obtained by blowing up along $Z(s) \subset X$, and the associated current (Remark 2.1.4) satisfies the Green equation

$$dd^c[\mathfrak{g}^\circ(s)] + \delta_{Z(s)} = [\omega^\circ(s)].$$

Proof. First assume $d = 1$, so that s is a nonzero section of $L^\vee$. Plugging (3.1.2) into (3.1.5) yields

$$\mathfrak{g}^\circ(s) = \int_1^\infty e^{-2\pi u h(s)} \frac{du}{u} = E_1(2\pi h(s)), \tag{3.1.8}$$

where

$$E_1(x) = \int_1^\infty e^{-xu} \frac{du}{u} = -\log(x) - \gamma - \sum_{k=1}^\infty \frac{(-x)^k}{k \cdot k!}. \tag{3.1.9}$$

This gives a more precise version of (3.1.6), which will be essential later.

Now suppose $d > 1$. For each $1 \le j \le d$ abbreviate

$$\eta(s_j) = -i \cdot \frac{\partial h(s_j) \wedge \bar{\partial} h(s_j)}{h(s_j)} \in E_X^{1,1}, \tag{3.1.10}$$

so that (3.1.1) becomes

$$\varphi^\circ(s_j)_{[2]} = 2\pi i e^{-2\pi h(s_j)} (\mathrm{ch}(L) + \eta(s_j)),$$

and (3.1.3) and (3.1.4) imply

$$\nu^\circ(s)_{[2d-2]} = (2\pi i)^{d-1} e^{-2\pi h(s)} \sum_{j=1}^{d} \underbrace{(\mathrm{ch}(L)+\eta(s_1)) \wedge \cdots \wedge (\mathrm{ch}(L)+\eta(s_d))}_{\text{omit } j\text{-th factor}}.$$

Expanding out the wedge products in each term, we rewrite this as

$$\nu^\circ(s)_{[2d-2]} = e^{-2\pi h(s)} \sum_{k=0}^{d-1} \eta_k(s), \tag{3.1.11}$$

in which each $\eta_k(s) \in E_X^{d-1,d-1}$ is (up to multiplication by a constant) the wedge product of $\mathrm{ch}(L)^{d-k-1}$ with a sum of k-fold wedges of $\eta(s_1), \dots, \eta(s_d)$. For any $t \in \mathbb{C}$ we have $\eta(ts_j) = |t|^2 \eta(s_j)$, and hence

$$\eta_k(ts) = |t|^{2k} \eta_k(s). \tag{3.1.12}$$

Plugging (3.1.11) into (3.1.5) results in

$$(-2\pi i)^{d-1} \mathfrak{g}^\circ(s) = \sum_{k=0}^{d-1} \eta_k(s) \int_1^\infty u^k e^{-2\pi u h(s)} \, \frac{du}{u}.$$

If $k > 0$, a calculus exercise shows that

$$\int_1^\infty u^k e^{-ux} \, \frac{du}{u} = \frac{e^{-x} \cdot (k-1)!}{x^k} \cdot \sum_{i=0}^{k-1} \frac{x^i}{i!}. \tag{3.1.13}$$

Rewriting this as $e^{-x} x^{-k} P_k(x)$ for some polynomial $P_k(x)$, we obtain

$$(-2\pi i)^{d-1} \mathfrak{g}^\circ(s) = \eta_0(s) \cdot E_1(2\pi h(s)) + e^{-2\pi h(s)} \sum_{k=1}^{d-1} \frac{\eta_k(s)}{h(s)^k} \cdot P_k(h(s)). \tag{3.1.14}$$

The equality (3.1.6) follows immediately by putting all terms in the sum over the common denominator $h(s)^{d-1}$ and using (3.1.9).

Assuming now that s is smooth, we establish (3.1.7). Near any point $x \in Z(s)$ we may choose an open neighborhood U over which the line bundle $L^\vee$ admits a trivializing section σ. Each component of $s = (s_1, \dots, s_d)$ then has the form

$$s_i = z_i \cdot \sigma$$

for some holomorphic function z_i, and $h(s_i) = f \cdot |z_i|^2$, where $f = \|\sigma\|^2$ is a smooth function on U valued in the positive real numbers.

The smoothness of the section s implies that $z_1, \dots, z_d$ can be completed to a system of local coordinates $z_1, \dots, z_{\dim(X)}$ on (a possibly smaller) U. In these coordinates the cycle $Z(s) \cap U$ is defined by $z_1 = \cdots = z_d = 0$. Moreover,

$$h(s)|_U = f \cdot (|z_1|^2 + \cdots + |z_d|^2), \tag{3.1.15}$$

and the $(1, 1)$-form (3.1.10) can be expressed as

$$\eta(s_j)|_U = |z_j|^2 \wedge \text{smooth} + z_j \, d\bar{z}_j \wedge \text{smooth} + \bar{z}_j \, dz_j \wedge \text{smooth} + dz_j \wedge d\bar{z}_j \wedge \text{smooth}, \tag{3.1.16}$$

where each “smooth” is some smooth form of the appropriate bidegree.

Now consider the pullback of (3.1.14) to the blowup of U along $Z(s)\cap U$. This blowup is isomorphic to the submanifold

$$V \subset U \times \mathbb{P}^{d-1}$$

defined by $\dot{w}_j z_i = \dot{w}_i z_j$ for all $1 \le i, j \le d$, where $\dot{w}_1, \dots, \dot{w}_d$ are the homogeneous coordinates on $\mathbb{P}^{d-1}$. It is covered by open subsets $V_1, \dots, V_d$, with $V_i \subset V$ defined by the condition $\dot{w}_i \neq 0$.

For ease of notation, let's work on the open subset $V_1 \subset V$ where $\dot{w}_1 \neq 0$, and denote by $\pi_1 : V_1 \to U$ the projection. On V_1 we have coordinates

$$z_1, w_2, \dots, w_d, z_{d+1}, \dots, z_{\dim(X)},$$

and the functions $z_2, \dots, z_d$ are expressed in these coordinates as

$$z_j = z_1 w_j. \tag{3.1.17}$$

In particular, the preimage of $Z(s)\cap U$ under $\pi_1 : V_1 \to U$ is defined by the single equation $z_1 = 0$.

Plugging (3.1.17) into (3.1.15) and (3.1.16), we find that

$$\pi_1^* h(s) = \phi \cdot |z_1|^2$$

for ϕ a smooth function on V_1 valued in the positive real numbers, and

$$\pi_1^* \eta(s_j) = |z_1|^2 \wedge \text{smooth} + z_1\, d\bar{z}_1 \wedge \text{smooth} + \bar{z}_1\, dz_1 \wedge \text{smooth} + dz_1 \wedge d\bar{z}_1 \wedge \text{smooth}.$$

Recalling the discussion surrounding (2.1.3), it follows that the pullback of $\eta(s_j)h(s)^{-1}$ has logarithmic growth along $\pi_1^* Z(s) \subset V_1$ for every $1 \le j \le d$.

The pullback to V_1 of each $\eta_k(s)h(s)^{-k}$ appearing in (3.1.14) has logarithmic growth along $\pi_1^* Z(s)$, because each is a sum of wedge products of smooth forms and the $\eta(s_j)h(s)^{-1}$ just analyzed. Similarly, (3.1.9) implies that singularities of $\eta_0(s)E_1(2\pi h(s))$ are the same as those of $\log h(s)$, and so the pullback of this form also has logarithmic growth along $\pi_1^* Z(s)$.

Of course the same analysis applies on each of the open subsets $V_i \subset V$, proving that the pullback of (3.1.14) via the blowup morphism $V \to U$ has logarithmic singularities along the preimage of $Z(s)\cap U$. This completes the proof of (3.1.7).

For the Green equation, see Proposition 2.2 of [Garcia and Sankaran 2019]. □

3.2. ***The star product formula.*** Suppose G_1 and G_2 are currents on X satisfying the Green equations

$$dd^c G_i + \delta_{Z_i} = [\omega_i]$$

for analytic cycles Z_1 and Z_2 of codimensions d_1 and d_2 intersecting properly. Suppose also that $G_2 = [g_2]$ is the current defined by a smooth form g_2 on $X \smallsetminus Z_2$, locally integrable on X. The form g_2 is then uniquely determined by G_2, and we define

$$G_1 \star G_2 = \delta_{Z_1} \wedge G_2 + G_1 \wedge \omega_2 \in D_X^{d_1+d_2-1, d_1+d_2-1},$$

provided that the integral

$$(\delta_{Z_1} \wedge G_2)(\phi) = \int_{Z_1} g_2 \wedge \phi$$

converges for all $\phi \in {}_c E^\bullet_X$ of the appropriate degree.

Remark 3.2.1. Note that we understand the star product to be a current on X, not an element of the space of currents modulo currents of the form $\partial a + \bar\partial b$. Because of this, the star product is neither commutative nor associative, and in fact it may be that $G_1 \star G_2$ is defined while $G_2 \star G_1$ is not.

Remark 3.2.2. Keeping the previous remark in mind, we caution the reader that we are using the convention for star products opposite to [Soulé 1992; Garcia and Sankaran 2019]: our $G_1 \star G_2$ is their $G_2 \star G_1$.

Remark 3.2.3. The expression $G_1 \star (G_2 \star G_3)$ does not make sense, as $G_2 \star G_3$ is not represented by a locally integrable form (even if G_2 and G_3 are). We therefore understand

$$\begin{gathered} G_1 \star G_2 \star G_3 = (G_1 \star G_2) \star G_3, \\ G_1 \star G_2 \star G_3 \star G_4 = ((G_1 \star G_2) \star G_3) \star G_4, \\ \vdots \end{gathered}$$

provided that each star product on the right is defined.

Fix a smooth tuple $s = (s_1, \dots, s_d)$ with $s_i \in H^0(X, L^\vee)$. If we write $d = k + \ell$ with $k, \ell > 0$, and express $s = (p, q)$ as the concatenation of the smooth tuples

$$p = (s_1, \dots, s_k) \quad \text{and} \quad q = (s_{k+1}, \dots, s_d),$$

then $Z(s) = Z(p) \times_X Z(q)$ as analytic spaces.

Lemma 3.2.4. *If $G(p) \in D_X^{k-1,k-1}$ is any Green current for $Z(p)$, the star product $G(p) \star \mathfrak{g}^\circ(q)$ is defined.*

Proof. The pullback of $\mathfrak{g}^\circ(q)$ to $Z(p)$ is the form $\mathfrak{g}^\circ(q|_{Z(p)})$ obtained by applying the construction of Proposition 3.1.6 to the smooth ℓ-tuple

$$q|_{Z(p)} = (s_{k+1}|_{Z(p)}, \dots, s_d|_{Z(p)})$$

of sections of $L^\vee|_{Z(p)}$ on the complex manifold $Z(p)$. In particular, this pullback is locally integrable on $Z(p)$. □

In particular, the lemma implies that the star product in the following theorem is defined.

Theorem 3.2.5 (Garcia–Sankaran). *We have the equality of currents*

$$\mathfrak{g}^\circ(s) = \mathfrak{g}^\circ(p) \star \mathfrak{g}^\circ(q) - \partial[A(p;q)] - \bar\partial[B(p;q)]$$

on X, where

$$A(p;q) = \left(-\frac{1}{2\pi i}\right)^{d-1} \int_{1<v<u<\infty} \bar\partial(\nu^\circ(\sqrt{u}\,p)_{[2k-2]}) \wedge \nu^\circ(\sqrt{v}\,q)_{[2\ell-2]} \frac{du}{u}\frac{dv}{v},$$

$$B(p;q) = \left(-\frac{1}{2\pi i}\right)^{d-1} \int_{1<v<u<\infty} \nu^\circ(\sqrt{u}\,p)_{[2k-2]} \wedge \partial(\nu^\circ(\sqrt{v}\,q)_{[2\ell-2]}) \frac{du}{u}\frac{dv}{v}$$

are smooth forms on $X \smallsetminus (Z(p) \cup Z(q))$, *locally integrable on* X. *Moreover*, *there is a resolution of singularities of*

$$Z(p) \cup Z(q) \subset X$$

for which

$$A(p; q), B(p; q) \in E^{\bullet}_{X}(\log Z(p) \cup Z(q)). \tag{3.2.1}$$

Proof. Except for the final claim, this is Theorem 2.16 of [Garcia and Sankaran 2019], modified as per Remark 3.2.2. For those authors X is a particular hermitian symmetric domain, but the same argument works on any complex manifold.

It remains to prove (3.2.1). Construct resolutions of singularities

$$(X', D') \to (Y', E') \xrightarrow{r'} (X, Z(p))$$

and

$$(X'', D'') \to (Y'', E'') \xrightarrow{r''} (X, Z(q))$$

by taking Y' and Y'' to be the blowups of X along $Z(p)$ and $Z(q)$, respectively. Then let X' and X'' be the blowups of Y' and Y'' along the preimages of $Z(s) = Z(p) \cap Z(q)$ under r' and r''.

Now fix a resolution of singularities $(X^{\dagger}, D^{\dagger})$ of the analytic subspace

$$D' \times_X D'' \subset X' \times_X X''.$$

The natural map $X^{\dagger} \to X$ is then a resolution of singularities

$$(X^{\dagger}, D^{\dagger}) \to (X, Z(p) \cup Z(q)),$$

and we claim that (3.2.1) is satisfied for any such choice. The proof will require the following elementary lemma.

Lemma 3.2.6. *The pullback of* $h(p)/h(s)$ *to*

$$X' \smallsetminus D' \cong X \smallsetminus Z(p)$$

extends smoothly to X', *and the pullback of* $h(q)/h(s)$

$$X'' \smallsetminus D'' \cong X \smallsetminus Z(q)$$

extends smoothly to X''. *In particular*, *both pullbacks to*

$$X^{\dagger} \smallsetminus D^{\dagger} \cong X \smallsetminus (Z(p) \cup Z(q))$$

extend smoothly to $X^{\dagger}$.

Proof. The function $h(p)/h(s)$ is smooth on the open complement of $Z(s) \subset X$, so it suffices to analyze its singularities on an open neighborhood of a point of $Z(s)$.

As in the proof of Proposition 3.1.6, we use the smooth tuple $s = (p, q)$ to choose local coordinates $z_1, \dots, z_{\dim(X)}$ in such a way that

$$h(p) = f \cdot (|z_1|^2 + \cdots + |z_k|^2),$$
$$h(q) = f \cdot (|z_{k+1}|^2 + \cdots + |z_d|^2),$$

where f is a smooth function valued in the positive real numbers. In particular

$$\frac{h(p)}{h(s)} = \frac{|z_1|^2 + \cdots + |z_k|^2}{|z_1|^2 + \cdots + |z_k|^2 + |z_{k+1}|^2 + \cdots + |z_d|^2}. \tag{3.2.2}$$

Using the explicit description of blowups in coordinates, as in the proof of Proposition 3.1.6, it is easy to see that if one first blows up along the cycle $Z(p)$ defined by $z_1 = \cdots = z_k = 0$, and then blows up along the preimage of the cycle $Z(s)$ defined by $z_1 = \cdots = z_d = 0$, the pullback of (3.2.2) to this double blowup has no singularities. This proves the first claim of the lemma.

The proof of the second is identical, and the third claim follows from the first two, as the map $X^\dagger \to X$ factors through both X' and X''. □

Continuing with the proof of Theorem 3.2.5, abbreviate $\hbar = 2\pi h$, and expand

$$\nu^\circ(p)_{[2k-2]} = e^{-\hbar(p)} \sum_{a=0}^{k-1} \eta_a(p) \quad \text{and} \quad \nu^\circ(q)_{[2\ell-2]} = e^{-\hbar(q)} \sum_{b=0}^{\ell-1} \eta_b(q)$$

as in (3.1.11). Plugging this expansion into the definitions of $A(p;q)$ and $B(p;q)$, and noting that $\eta_0(p)$ and $\eta_0(q)$ are closed, we find that

$$\begin{aligned}(-2\pi i)^{d-1} A(p;q) = \sum_{\substack{0<a<k \\ 0\le b<\ell}} & F_{a,b}(\hbar(p),\hbar(q)) \cdot \bar\partial \eta_a(p) \wedge \eta_b(q) \\ & - \sum_{\substack{0\le a<k \\ 0\le b<\ell}} F_{a+1,b}(\hbar(p),\hbar(q)) \cdot \bar\partial \hbar(p) \wedge \eta_a(p) \wedge \eta_b(q),\end{aligned} \tag{3.2.3}$$

$$\begin{aligned}(-2\pi i)^{d-1} B(p;q) = \sum_{\substack{0\le a<k \\ 0<b<\ell}} & F_{a,b}(\hbar(p),\hbar(q)) \cdot \eta_a(p) \wedge \partial \eta_b(q) \\ & - \sum_{\substack{0\le a<k \\ 0\le b<\ell}} F_{a,b+1}(\hbar(p),\hbar(q)) \cdot \eta_a(p) \wedge \partial \hbar(q) \wedge \eta_b(q),\end{aligned} \tag{3.2.4}$$

in which we have set

$$F_{a,b}(x,y) = \int_{1<v<u<\infty} u^a v^b e^{-ux} e^{-vy} \frac{du}{u}\frac{dv}{v} = \int_1^\infty v^{a+b} e^{-vy} \left(\int_1^\infty u^a e^{-uvx} \frac{du}{u} \right) \frac{dv}{v}.$$

If $a, b > 0$, then (3.1.13) applies to the inner integral, leaving

$$F_{a,b}(x,y) = \sum_{i=0}^{a-1} \frac{(a-1)!}{x^{a-i} \cdot i!} \int_1^\infty v^{b+i} e^{-v(x+y)} \frac{dv}{v}. \tag{3.2.5}$$

Applying (3.1.13) once again leaves

$$F_{a,b}(x,y) = e^{-x-y} \sum_{i=0}^{a-1} \frac{\mathrm{poly}(x,y)}{x^{a-i} \cdot (x+y)^{b+i}}, \tag{3.2.6}$$

where in each term poly(x, y) is some polynomial (depending on i) in x and y whose exact value is irrelevant to us. If $a > 0$ and $b = 0$ one argues in the same way, except that the integral appearing in the

$i=0$ term of (3.2.5) is $E_1(x+y)$. Thus

$$F_{a,0}(x,y)=\frac{E_1(x+y)\cdot(a-1)!}{x^a}+e^{-x-y}\sum_{i=1}^{a-1}\frac{\mathrm{poly}(x,y)}{x^{a-i}\cdot(x+y)^i}. \tag{3.2.7}$$

If $a=0$ and $b>0$ then, again using (3.1.13), rewrite $F_{0,b}(x,y)$ as

$$\int_1^\infty\left(\int_1^\infty v^b e^{-v(y+ux)}\frac{dv}{v}\right)\frac{du}{u}=\sum_{i=0}^{b-1}\frac{(b-1)!}{i!}\int_1^\infty\frac{e^{-(y+ux)}}{(y+ux)^{b-i}}\frac{du}{u}. \tag{3.2.8}$$

The integral on the right can again be evaluated using elementary methods: for any $r\geq 1$ we have

$$\int_1^\infty\frac{e^{-(y+ux)}}{(y+ux)^r}\frac{du}{u}=\frac{e^{-y}}{y^r}E_1(x)+\sum_{j=1}^{r}\frac{(-1)^j}{(j-1)!}\frac{E_1(x+y)}{y^{b-i-j+1}}+\sum_{j=2}^{r}\frac{e^{-(x+y)}\cdot\mathrm{poly}(x,y)}{y^{r-j+1}(x+y)^{j-1}}.$$

Using this, one sees that (3.2.8) has the form

$$F_{0,b}(x,y)=e^{-y}E_1(x)\cdot\frac{\mathrm{poly}(y)}{y^b}+E_1(x+y)\cdot\frac{\mathrm{poly}(y)}{y^b}+\sum_{j=1}^{b-1}\frac{e^{-(x+y)}\cdot\mathrm{poly}(x,y)}{y^{b-j}(x+y)^j}.$$

With these explicit formulas for the $F_{a,b}$ in hand, let us consider the behavior singularities of (3.2.3) after pullback via $X^\dagger\to X$.

For the first sum of (3.2.3), one can use (3.2.6) and (3.2.7) to write each term in the form

$$F_{a,b}(\hbar(p),\hbar(q))\cdot\bar\partial\eta_a(p)\wedge\eta_b(q)=\frac{\bar\partial\eta_a(p)}{\hbar(p)^a}\wedge\frac{\eta_b(q)}{\hbar(q)^b}\wedge\left(\frac{\hbar(q)^b}{\hbar(s)^b}\sum_{i=0}^{a-1}\phi_i\frac{\hbar(p)^i}{\hbar(s)^i}\right)+E_1(\hbar(s))\wedge\frac{\bar\partial\eta_a(p)}{\hbar(p)^a}\wedge\psi.$$

Here each ϕ_i is a smooth function on X, and ψ is a smooth form (in fact, $\psi=0$ except when $b=0$). The singularities of every form appearing here are understood:

- The function in parentheses pulls back to a smooth function on $X^\dagger$, by Lemma 3.2.6.
- By the analysis of singularities in the proof of Proposition 3.1.6, the pullback of $\bar\partial\eta_a(p)/\hbar(p)^a$ to the blowup along $Z(p)\subset X$ has logarithmic growth along the preimage of $Z(p)$; hence its pullback to X' has logarithmic growth along D'.
- Again by the proof of Proposition 3.1.6, the pullback of $\eta_b(q)/\hbar(q)^b$ to the blowup along $Z(q)\subset X$ has logarithmic growth along the preimage of $Z(q)$; hence its pullback to X'' has logarithmic growth along D''.
- By (3.1.9), the function $E_1(\hbar(s))$ differs from $-\log\hbar(s)$ by a smooth function. Using the coordinates from the proof of Lemma 3.2.6, one sees that $-\log\hbar(s)$ pulls back to a function on X' with logarithmic growth along D', and also to a function on X'' with logarithmic growth along D''.

It follows that every term in the first summation in (3.2.3) pulls back to a form on $X^\dagger$ with logarithmic growth along $D^\dagger$.

For the second sum of (3.2.3), one similarly uses (3.2.6) and (3.2.7) to write each term as

$$F_{a+1,b}(\hbar(p),\hbar(q))\cdot\bar\partial\hbar(p)\wedge\eta_a(p)\wedge\eta_b(q)$$
$$=\frac{\bar\partial\hbar(p)}{\hbar(p)}\wedge\frac{\eta_a(p)}{\hbar(p)^a}\wedge\frac{\eta_b(q)}{\hbar(q)^b}\wedge\left(\frac{\hbar(q)^b}{\hbar(s)^b}\sum_{i=0}^{a}\phi_i\frac{\hbar(p)^i}{\hbar(s)^i}\right)+E_1(\hbar(s))\wedge\frac{\bar\partial\hbar(p)}{\hbar(p)}\wedge\frac{\eta_a(p)}{\hbar(p)^a}\wedge\psi.$$

The only new expression appearing here is $\bar\partial\hbar(p)/\hbar(p)$. As in the proof of Proposition 3.1.6, one can find local coordinates $z_1,\dots,z_{\dim(X)}$ near a point of $Z(p)\subset X$ such that

$$h(p)=f\cdot(|z_1|^2+\cdots+|z_k|^2)$$

for some smooth function f. In these coordinates

$$\frac{\bar\partial\hbar(p)}{\hbar(p)}=\bar\partial f+f\wedge\frac{z_1d\bar z_1+\cdots+z_kd\bar z_k}{|z_1|^2+\cdots+|z_k|^2}.$$

The pullback of this form to the blowup along $Z(p)\subset X$, which is defined by $z_1=\cdots=z_k=0$, has logarithmic growth along the preimage of $Z(p)$, as one immediately sees from the explicit coordinates on the blowup given in the proof of Proposition 3.1.6. Hence the pullback of $\bar\partial\hbar(p)/\hbar(p)$ to X' has logarithmic growth along D'; hence all terms in the second sum in (3.2.3) pull back to forms on $X^\dagger$ with logarithmic growth along $D^\dagger$.

This proves that (3.2.3) satisfies (3.2.1), and the argument for (3.2.4) is entirely similar. □

As a special case of Theorem 3.2.5,

$$\mathfrak{g}^\circ(s_1,\dots,s_d)=\mathfrak{g}^\circ(s_1,\dots,s_{d-1})\star\mathfrak{g}^\circ(s_d)-\partial[A(s_1,\dots,s_{d-1};s_d)]-\bar\partial[B(s_1,\dots,s_{d-1};s_d)].$$

Repeated application of this results in

$$\mathfrak{g}^\circ(s)=\mathfrak{g}^\circ(s_1)\star\cdots\star\mathfrak{g}^\circ(s_d)-\partial[\mathfrak{a}(s)]-\bar\partial[\mathfrak{b}(s)]\tag{3.2.9}$$

for locally integrable forms

$$\mathfrak{a}(s)=\sum_{r=2}^{d}A(s_1,\dots,s_{r-1};s_r)\wedge\omega^\circ(s_{r+1})\wedge\cdots\wedge\omega^\circ(s_d),$$
$$\mathfrak{b}(s)=\sum_{r=2}^{d}B(s_1,\dots,s_{r-1};s_r)\wedge\omega^\circ(s_{r+1})\wedge\cdots\wedge\omega^\circ(s_d).$$

3.3. ***Explicit logarithmic expansions.*** We now return to the setting of Section 2.2, so that $X_0\subset X$ is a closed complex submanifold, but now assume that X_0 is presented to us in a particular way: there is a holomorphic vector bundle $N\to X$ of dimension $\dim(X)-\dim(X_0)$ and a section

$$u\in H^0(X,N)$$

such that $X_0\subset X$ is defined (as an analytic space) by the equation $u=0$.

This presentation of $X_0\subset X$ identifies

$$N_{X_0/X}\cong N|_{X_0}.\tag{3.3.1}$$

Indeed, if we denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of holomorphic functions vanishing along X_0, then evaluation at u defines an isomorphism $N^\vee \cong \mathcal{I}$. Restricting this to X_0 yields an isomorphism $N|_{X_0}^\vee \cong \mathcal{I}/\mathcal{I}^2$ of vector bundles on X_0, and the normal bundle to $X_0 \subset X$ is (by definition) the dual of the right-hand side.

Viewing points of the total space $N \to X$ as pairs (x, v_x) consisting of a point $x \in X$ and a vector $v_x \in N_x$ in the fiber at x, the deformation to the normal bundle of $X_0 \subset X$ can be identified with the subset

$$\widetilde{X} \subset N \times \mathbb{C}$$

of triples (x, v_x, t) consisting of a point $(x, v_x) \in N$, and a scalar $t \in \mathbb{C}$ satisfying $t \cdot v_x = u_x$. The morphisms

$$X \xleftarrow{\pi} \widetilde{X} \xrightarrow{\tau} \mathbb{C}$$

of (2.2.1) are given by $\pi(x, v_x, t) = x$ and $\tau(x, v_x, t) = t$. This is essentially McPherson's description of the deformation to the normal bundle, as in Remark 5.1.1 of [Fulton 1984].

As in Section 3.1, fix a line bundle $L \to X$ with a hermitian metric h. Any morphism of holomorphic vector bundles $y : N \to L^\vee$ determines a section

$$q = y(u) \in H^0(X, L^\vee) \tag{3.3.2}$$

vanishing along X_0. We call this the *degenerating section* determined by y. Like any vector bundle, $\pi_0 : N_{X_0/X} \to X_0$ acquires a tautological section

$$v_0 \in H^0(N_{X_0/X}, \pi_0^* N_{X_0/X}) \tag{3.3.3}$$

after pullback via its own bundle map. Setting $L_0 = L|_{X_0}$, we may restrict $y : N \to L^\vee$ to a morphism

$$N_{X_0/X} \overset{(3.3.1)}{=} N|_{X_0} \xrightarrow{y} L^\vee|_{X_0} = L_0^\vee$$

of vector bundles on X_0, and then pull back by $\pi_0 : N_{X_0/X} \to X_0$. Applying this pullback to the tautological section (3.3.3) defines the *specialization to the normal bundle* of the degenerating section (3.3.2), denoted by

$$\sigma_{X_0/X}(q) = (\pi_0^* y)(v_0) \in H^0(N_{X_0/X}, \pi_0^* L_0^\vee). \tag{3.3.4}$$

The degenerating section (3.3.2) and its specialization (3.3.4) satisfy the informal relation

$$\sigma_{X_0/X}(q) = \frac{\pi^* q}{\tau}\Big|_{\tau=0},$$

which we formulate more precisely as the following lemma.

Lemma 3.3.1. *For any $q = y(u)$ as above, there is a unique section*

$$\tilde{q} \in H^0(\widetilde{X}, \pi^* L^\vee)$$

satisfying $\tau \cdot \tilde{q} = \pi^* q$. The pullback of $\tilde{q}$ to $N_{X_0/X} = \widetilde{X}_0$ is $\sigma_{X_0/X}(q)$.

Proof. There is a tautological section $v \in H^0(\widetilde{X}, \pi^*N)$ whose fiber at a point $(x, v_x, t) \in \widetilde{X}$ is v_x. This section satisfies $\tau \cdot v = \pi^* u$, and its restriction to $N_{X_0/X}$ is (3.3.3). The image of v under the map

$$H^0(\widetilde{X}, \pi^*N) \xrightarrow{\pi^* y} H^0(\widetilde{X}, \pi^* L^\vee)$$

is a section $\tilde{q}$ with the desired properties. □

Now fix a smooth tuple $s = (s_1, \dots, s_d)$ with $s_i \in H^0(X, L^\vee)$ and assume $s = (p, q)$ is the concatenation of

$$p = (p_1, \dots, p_k) \quad \text{and} \quad q = (q_1, \dots, q_\ell)$$

satisfying the following properties:

(1) The tuple $p|_{X_0}$ formed from the restrictions

$$p_1|_{X_0}, \dots, p_k|_{X_0} \in H^0(X_0, L_0^\vee)$$

is again smooth; equivalently, the analytic subspace

$$Z(p|_{X_0}) = Z(p) \times_X X_0 \subset X_0$$

is smooth of codimension k.

(2) The sections $q_1, \dots, q_\ell \in H^0(X, L^\vee)$ are the degenerating sections determined by morphisms $y_1, \dots, y_\ell : N \to L^\vee$ as above. In what follows, we denote by

$$\sigma_{X_0/X}(q_i) \in H^0(N_{X_0/X}, \pi_0^* L_0^\vee)$$

the section associated to $q_i = y_i(u)$ by (3.3.4), and by

$$\tilde{q}_i \in H^0(\widetilde{X}, \pi^* L^\vee)$$

the section associated to $q_i = y_i(u)$ by Lemma 3.3.1.

Our assumptions imply that $Z(p)$ intersects X_0 transversely, while $X_0 \subset Z(q)$. We allow the possibility that $s = p$ or $s = q$. Note that the tuples p and q are again smooth, by Remark 3.1.5. We consider the specializations of $Z(s)$, $Z(p)$, and $Z(q)$ to $N_{X_0/X}$.

Proposition 3.3.2. *For $s = (p, q)$ as above, the following properties hold*:

(1) *We have the equalities $\sigma_{X_0/X}(Z(p)) = \pi_0^* Z(p|_{X_0})$ and*

$$\sigma_{X_0/X}(Z(s)) = \sigma_{X_0/X}(Z(p)) \cdot \sigma_{X_0/X}(Z(q))$$

of cycles on $N_{X_0/X}$.

(2) *The tuple $\tilde{q} = (\tilde{q}_1, \dots, \tilde{q}_\ell)$ is smooth, and the cycle*

$$Z(\tilde{q}) \subset \widetilde{X}$$

defined by the vanishing of its components satisfies the equality

$$\sigma_{X_0/X}(Z(q)) = Z(\tilde{q}) \cdot N_{X_0/X}$$

of cycles on $N_{X_0/X}$.

(3) *The tuple* $\sigma_{X_0/X}(q) = (\sigma_{X_0/X}(q_1), \dots, \sigma_{X_0/X}(q_\ell))$ *is smooth, and the analytic cycle on* $N_{X_0/X}$ *defined by the vanishing of its components is equal to* $\sigma_{X_0/X}(Z(q))$.

All intersections above are understood in the sense of [Draper 1969].

Proof. Let $\Delta \subset \mathbb{C}$ be the open unit disk, and abbreviate $n = \dim(X)$ and $m = \dim(X) - \dim(X_0)$. The smoothness of $s = (p, q)$ implies that we may find a coordinate neighborhood in X near a point of X_0 of the form

$$U \cong \Delta^n = \{(z_1, \dots, z_n) : z_i \in \Delta\}$$

in such a way that

- the line bundle L is trivial on U,
- $U_0 = X_0 \cap U$ is defined by the vanishing of $z_1, \dots, z_m$,
- $p_1 = z_{m+1}, \dots, p_k = z_{m+k}$,
- $q_1 = z_1, \dots, q_\ell = z_\ell$.

The deformation to the normal bundle of $U_0 \subset U$ is identified with

$$\widetilde{U} = \{(z_1, \dots, z_n, w_1, \dots, w_m, t) \in \Delta^n \times \mathbb{C}^m \times \mathbb{C} : z_i = t w_i \text{ for all } 1 \le i \le m\},$$

and $\tilde{q}_i = w_i$ for all $1 \le i \le \ell$. The normal bundle itself is identified with

$$N_{U_0/U} = \{(0, \dots, 0, z_{m+1}, \dots, z_n, w_1, \dots, w_m, 0) \in \Delta^n \times \mathbb{C}^m \times \mathbb{C}\},$$

and $\sigma_{X_0/X}(q_i) = w_i$ for all $1 \le i \le \ell$. The strict transforms of $Z(p)$ and $Z(q)$ are defined by (respectively) the vanishing of $z_{m+1}, \dots, z_{m+k}$ and the vanishing of $w_1, \dots, w_\ell$. Their specializations to $N_{U_0/U}$ are defined by the same equations. All parts of the proposition follow immediately from computations in these local coordinates. ☐

Now we turn to the Green current

$$\mathfrak{g}^\circ(s) \in E_X^{d-1,d-1}(\log Z(s))$$

of Proposition 3.1.6, and the similar currents $\mathfrak{g}^\circ(p)$ and $\mathfrak{g}^\circ(q)$. The following lemmas are the key to understanding their logarithmic expansions along X_0, and hence their specializations to $N_{X_0/X}$.

Lemma 3.3.3. *There are forms* $a, b, c \in E_{\widetilde{X}}^{\ell-1,\ell-1}$ *such that*

$$\mathfrak{g}^\circ(q) = j_t^*\left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q}) + c \cdot \log|\tau|\right) \tag{3.3.5}$$

for all $t \in \mathbb{C} \smallsetminus \{0\}$. *If we define currents*

$$G_0(t) = \left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q})\right) \wedge \delta_{\widetilde{X}_t}$$

and $G_1(t) = c \wedge \delta_{\widetilde{X}_t}$ *on* $\widetilde{X}$, *then*

$$j_{t*}[\mathfrak{g}^\circ(q)] = G_0(t) + G_1(t) \log|t|$$

is a logarithmic expansion of $\mathfrak{g}^\circ(q)$ *along* X_0.

Proof. The smoothness of $\tilde{q}$ allows us to apply the constructions of Section 3.1 to obtain Green forms $\mathfrak{g}^\circ(\tilde{q})$ and $\mathfrak{g}^\circ(\tau\tilde{q})$ for the cycles $Z(\tilde{q}) \subset \widetilde{X}$ and

$$Z(\tilde{q}) \smallsetminus N_{X_0/X} \subset \widetilde{X} \smallsetminus N_{X_0/X},$$

respectively. Recalling that $\tau\tilde{q} = \pi^* q$, for $t \neq 0$ these are related by

$$\mathfrak{g}^\circ(q) = j_t^* \pi^* \mathfrak{g}^\circ(q) = j_t^* \mathfrak{g}^\circ(\tau\tilde{q}).$$

As in the proof Proposition 3.1.6, we may write

$$(-2\pi i)^{\ell-1}\mathfrak{g}^\circ(\tilde{q}) = \eta_0(\tilde{q}) \cdot E_1(2\pi h(\tilde{q})) + e^{-2\pi h(\tilde{q})} \sum_{j=1}^{\ell-1} \frac{\eta_j(\tilde{q})}{h(\tilde{q})^j} \cdot P_j(h(\tilde{q})), \tag{3.3.6}$$

where P_j is a polynomial and $\eta_j(\tilde{q})$ is a smooth form on $\widetilde{X}$ satisfying the homogeneity property (3.1.12). If we replace $\tilde{q}$ by $\tau\tilde{q}$ in (3.3.6), pull back by $j_t : X \to \widetilde{X}$, and use

$$j_t^* \left(\frac{\eta_j(\tau\tilde{q})}{h(\tau\tilde{q})^j} \right) = j_t^* \left(\frac{\eta_j(\tilde{q})}{h(\tilde{q})^j} \right),$$

we find that $\mathfrak{g}^\circ(q) = j_t^* \mathfrak{g}^\circ(\tau\tilde{q}) = (-2\pi i)^{1-\ell} j_t^* \Psi$, where

$$\Psi = \eta_0(\tilde{q}) \cdot E_1(2\pi|\tau|^2 h(\tilde{q})) + e^{-2\pi|\tau|^2 h(\tilde{q})} \sum_{j=1}^{\ell-1} \frac{\eta_j(\tilde{q})}{h(\tilde{q})^j} \cdot P_j(h(\tau\tilde{q})).$$

The equality (3.3.5) follows easily from this and (3.1.9).

Applying j_{t*} to both sides of (3.3.5) yields

$$j_{t*}[\mathfrak{g}^\circ(q)] = \left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q}) + c \cdot \log|t| \right) \wedge \delta_{\widetilde{X}_t}.$$

To show that this is a logarithmic expansion, one must verify the continuity and Hölder continuity at $t = 0$ of $G_0(t)(\varphi)$ and $G_1(t)(\varphi)$, respectively, for any smooth compactly supported form φ on $\widetilde{X}$. Using a partition of unity argument, we may reduce to the case in which the support of φ is contained in a coordinate neighborhood

$$\begin{aligned} \widetilde{U} &= \{(z_1, \dots, z_n, w_1, \dots, w_m, t) \in \Delta^n \times \mathbb{C}^m \times \mathbb{C} : z_i = t w_i \text{ for all } 1 \le i \le m\} \\ &\subset \{(z_{m+1}, \dots, z_n, w_1, \dots, w_m, t) \in \mathbb{C}^{n-m} \times \mathbb{C}^m \times \mathbb{C}\} \end{aligned}$$

chosen as in the proof of Proposition 3.3.2. In particular, $\tilde{q}_i = w_i$ for all $1 \le i \le \ell$, and the function $h(\tilde{q})$ has the form

$$H(z, w, t) = f_1(z, w, t) \cdot |w_1|^2 + \dots + f_\ell(z, w, t) \cdot |w_\ell|^2$$

for smooth compactly supported $f_1, \dots, f_\ell : \mathbb{C}^{n-m} \times \mathbb{C}^m \times \mathbb{C} \to \mathbb{R}^{>0}$.

The continuity of $G_0(t)(\varphi)$ now amounts to the continuity in t of

$$\int_{\mathbb{C}^{n-m} \times \mathbb{C}^m} \frac{g(z, w, t)}{H(z, w, t)^{\ell-1}} \cdot \mu \quad \text{and} \quad \int_{\mathbb{C}^{n-m} \times \mathbb{C}^m} g(z, w, t) \cdot \log H(z, w, t) \cdot \mu$$

for any smooth compactly supported function $g(z,w,t)$ on $\mathbb{C}^{n-m}\times\mathbb{C}^m\times\mathbb{C}$, where

$$\mu = dz_{m+1}\wedge d\bar{z}_{m+1}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n\wedge dw_1\wedge d\bar{w}_1\wedge\cdots\wedge dw_m\wedge d\bar{w}_m.$$

The smoothness (hence Hölder continuity) of $G_1(t)(\varphi)$ amounts to the smoothness in t of

$$\int_{\mathbb{C}^{n-m}\times\mathbb{C}^m} g(z,w,t)\cdot\mu.$$

These are routine calculus exercises, left to the reader. □

For the Green currents $\mathfrak{g}^\circ(q_i)$ associated to the individual components of $q=(q_1,\dots,q_\ell)$, one has a more precise version of Lemma 3.3.3.

Lemma 3.3.4. *For $1\le i\le\ell$ there is a smooth function f_i on $\widetilde{X}$ such that*

$$\mathfrak{g}^\circ(q_i) = j_t^*\big(-\log(2\pi e^\gamma h(\tilde{q}_i)) + |\tau|^2 f_i - 2\log|\tau|\big)$$

for all $t\in\mathbb{C}\smallsetminus\{0\}$. If we define currents

$$G_0 = \big(-\log(2\pi e^\gamma h(\tilde{q}_i)) + |\tau|^2 f_i\big)\wedge\delta_{\widetilde{X}_t}$$

and $G_1=-2\delta_{\widetilde{X}_t}$ on $N_{X_0/X}$, then

$$j_{t*}[\mathfrak{g}^\circ(q_i)] = G_0 + G_1\cdot\log|t|$$

is a logarithmic expansion of $\mathfrak{g}^\circ(q_i)$ along X_0.

Proof. The proof is the same as that of Lemma 3.3.3, except that one replaces (3.3.6) with the simpler equality $\mathfrak{g}^\circ(\tilde{q}_i)=E_1(2\pi h(\tilde{q}_i))$ of (3.1.8). □

Proposition 3.3.5. *We have the equality of currents*

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(p)) = [\pi_0^*\mathfrak{g}^\circ(p|_{X_0})]\in D^{k-1,k-1}_{N_{X_0/X}},$$

where $\pi_0: N_{X_0/X}\to X_0$ is the bundle map, and there are smooth forms a_0 and b_0 on $N_{X_0/X}$ such that

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(q)) = \frac{a_0}{h(\sigma_{X_0/X}(q))^{\ell-1}} + b_0\log h(\sigma_{X_0/X}(q))\in D^{\ell-1,\ell-1}_{N_{X_0/X}}.$$

For the individual components of $q=(q_1,\dots,q_\ell)$ we have the exact formula

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(q_i)) = -\log h(\sigma_{X_0/X}(q_i)) - \log(2\pi e^\gamma)\in D^{0,0}_{N_{X_0/X}}.$$

Proof. For $\mathfrak{g}^\circ(p)$, note that the tuple p remains smooth (as one can check in the local coordinates of Proposition 3.3.2) after pullback via any arrow in

$$\begin{array}{ccc} N_{X_0/X} & \xrightarrow{j_0} & \widetilde{X} \\ {\scriptstyle\pi_0}\downarrow & & \downarrow{\scriptstyle\pi} \\ X_0 & \xrightarrow[i_0]{} & X. \end{array}$$

Each of these pullbacks has its own Green form $\mathfrak{g}^\circ(\,\cdot\,)$ associated to it, and these satisfy obvious functorial properties, e.g., $\pi^*\mathfrak{g}^\circ(p)=\mathfrak{g}^\circ(\pi^*p)$. For any $t\neq 0$ we have the (particularly simple) logarithmic expansion

$$j_{t*}[\mathfrak{g}^\circ(p)]=\mathfrak{g}^\circ(\pi^*p)\wedge\delta_{\widetilde{X}_t}$$

of $\mathfrak{g}^\circ(p)$ along $X_0\subset X$. Of course one must check that the family of currents on the right-hand side is defined at $t=0$ and satisfies the continuity condition of Definition 2.2.4; using the analysis of singularities of $\mathfrak{g}^\circ(\pi^*p)$ from (3.1.6), this is an easy calculation in local coordinates as in the proof of Lemma 3.3.3. The constant term at $t=0$ of this expansion is

$$\mathfrak{g}^\circ(\pi^*p)\wedge\delta_{N_{X_0/X}}=j_{0*}[\mathfrak{g}^\circ(\pi^*p)|_{N_{X_0/X}}]=j_{0*}[\pi_0^*\mathfrak{g}^\circ(p|_{X_0})],$$

proving the first claim.

The claims about $\mathfrak{g}^\circ(q)$ and $\mathfrak{g}^\circ(q_i)$ follow by taking $t=0$ in the logarithmic expansions of Lemmas 3.3.3 and 3.3.4, and recalling from Lemma 3.3.1 that the restriction of $\tilde{q}$ to the fiber $N_{X_0/X}=\widetilde{X}_0$ is $\sigma_{X_0/X}(q)$. □

Remark 3.3.6. Using Proposition 3.1.6 and (3.1.8), each section $\sigma_{X_0/X}(q_i)$ of the hermitian line bundle $\pi_0^*L_0^\vee$ on $N_{X_0/X}$ determines a Green function

$$\mathfrak{g}^\circ(\sigma_{X_0/X}(q_i))=E_1(2\pi h(\sigma_{X_0/X}(q_i)))$$

for the divisor $\sigma_{X_0/X}(q_i)=0$ on $N_{X_0/X}$. By the third claim of Proposition 3.3.2, this divisor is none other than the specialization of $Z(q_i)\subset X$ to the normal bundle, which also admits the Green function $\sigma_{X_0/X}(\mathfrak{g}^\circ(q_i))$ obtained by specializing $\mathfrak{g}^\circ(q_i)$. Proposition 3.3.5 shows that

$$\mathfrak{g}^\circ(\sigma_{X_0/X}(q_i))\neq\sigma_{X_0/X}(\mathfrak{g}^\circ(q_i)).$$

This should not cause confusion, as the Green function on the left-hand side plays no role in our arguments, and will never appear again.

Proposition 3.3.7. *The specializations of $\mathfrak{g}^\circ(s)$, $\mathfrak{g}^\circ(p)$, and $\mathfrak{g}^\circ(q)$ to $N_{X_0/X}$ are related by*

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(s))=\sigma_{X_0/X}(\mathfrak{g}^\circ(p))\star\sigma_{X_0/X}(\mathfrak{g}^\circ(q))-\partial\sigma_{X_0/X}(A(p;q))-\bar{\partial}\sigma_{X_0/X}(B(p;q)),$$

where $A(p;q)$ and $B(p;q)$ are the currents of Theorem 3.2.5. Moreover,

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(q))=\sigma_{X_0/X}(\mathfrak{g}^\circ(q_1))\star\cdots\star\sigma_{X_0/X}(\mathfrak{g}^\circ(q_\ell))-\partial\sigma_{X_0/X}(\mathfrak{a}(q))-\bar{\partial}\sigma_{X_0/X}(\mathfrak{b}(q)),$$

where $\mathfrak{a}(q)$ and $\mathfrak{b}(q)$ are the currents of (3.2.9). *In particular, all currents on X appearing in these formulas admit logarithmic expansions along X_0, and the star products in both formulas are defined.*

Proof. The core of the proof is the following lemma.

Lemma 3.3.8. *Suppose $G\in D_X^{k-1,k-1}$ is any Green current for $Z(p)$. If G admits a logarithmic expansion along $X_0\subset X$, then so does $G\star\mathfrak{g}^\circ(q)$, and its specialization to the normal bundle satisfies*

$$\sigma_{X_0/X}(G\star\mathfrak{g}^\circ(q))=\sigma_{X_0/X}(G)\star\sigma_{X_0/X}(\mathfrak{g}^\circ(q)).$$

In particular, the star product on the right is defined.

Proof. Abbreviate $Z = Z(p)$, and recall from Lemma 3.2.4 that the star product

$$G \star \mathfrak{g}^\circ(q) = \delta_Z \wedge \mathfrak{g}^\circ(q) + G \wedge \omega^\circ(q)$$

is defined. Applying j_{t*} to both sides results in

$$j_{t*}[G \star \mathfrak{g}^\circ(q)](\varphi) = \int_Z \mathfrak{g}^\circ(q) \wedge j_t^* \varphi + (j_{t*}G)(\pi^* \omega^\circ(q) \wedge \varphi)$$

for any smooth compactly supported form φ on $\widetilde{X}$, and any $t \neq 0$.

Using $j_t : Z \cong \widetilde{Z}_t$ and the equality

$$\mathfrak{g}^\circ(q) = j_t^* \left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q}) + c \cdot \log |\tau| \right)$$

of Lemma 3.3.3, the integral on the right becomes

$$\int_Z \mathfrak{g}^\circ(q) \wedge j_t^* \varphi = \int_{\widetilde{Z}_t} \left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q}) + c \cdot \log |\tau| \right) \wedge \varphi.$$

Fixing a logarithmic expansion $j_{t*}G = \sum_{i \geq 0} G_i(t)(\log |t|)^i$, we obtain

$$j_{t*}[G \star \mathfrak{g}^\circ(q)] = \sum_{i \geq 0} C_i(t) \cdot (\log |t|)^i,$$

in which

$$\begin{aligned} C_0(t) &= \delta_{\widetilde{Z}_t} \wedge \left(\frac{a}{h(\tilde{q})^{\ell-1}} + b \log h(\tilde{q}) \right) + G_0(t) \wedge \pi^* \omega^\circ(q), \\ C_1(t) &= c \wedge \delta_{\widetilde{Z}_t} + G_1(t) \wedge \pi^* \omega^\circ(q), \\ C_i(t) &= G_i(t) \wedge \pi^* \omega^\circ(q) \quad \text{for } i > 1. \end{aligned}$$

To see that this is a logarithmic expansion of $G \star \mathfrak{g}^\circ(q)$, one must check that the terms involving $\delta_{\widetilde{Z}_t}$ are well-defined currents (including at $t = 0$) that satisfy the continuity conditions of Definition 2.2.4; this is easily verified in the local coordinates of the proof of Proposition 3.3.2.

The current $C_0(0)$ is the pushforward via $j_0 : N_{X_0/X} \to \widetilde{X}$ of

$$\delta_{\sigma_{X_0/X}(Z)} \wedge \sigma_{X_0/X}(\mathfrak{g}^\circ(q)) + \sigma_{X_0/X}(G) \wedge \pi_0^* i_0^* \omega^\circ(q),$$

which agrees with $\sigma_{X_0/X}(G) \star \sigma_{X_0/X}(\mathfrak{g}^\circ(q))$ by Proposition 2.2.10. □

Recall the equality

$$\mathfrak{g}^\circ(s) = \mathfrak{g}^\circ(p) \star \mathfrak{g}^\circ(q) - \partial[A(p;q)] - \bar{\partial}[B(p;q)]$$

of Theorem 3.2.5. The currents $A(p,q)$ and $B(p,q)$ admit logarithmic expansions along X_0 by Theorem 2.2.5 and the final claim of Theorem 3.2.5. The star product admits a logarithmic expansion by Lemma 3.3.8. The Green current on the left admits a logarithmic expansion by Theorem 2.2.5 and (3.1.7), and also because the right-hand side does. Specializing both sides to $N_{X_0/X}$ and using Remark 2.2.9 and Lemma 3.3.8 proves the first claim of Proposition 3.3.7.

For the second claim we use the following lemma.

Lemma 3.3.9. *Fix $1 \le r < \ell$, and let $G \in D_X^{r-1,r-1}$ be any Green current for $Z(q_1, \dots, q_r)$. If G admits a logarithmic expansion along $X_0 \subset X$, then so does $G \star \mathfrak{g}^\circ(q_{r+1})$, and*

$$\sigma_{X_0/X}(G \star \mathfrak{g}^\circ(q_{r+1})) = \sigma_{X_0/X}(G) \star \sigma_{X_0/X}(\mathfrak{g}^\circ(q_{r+1})).$$

In particular, the star product on the right is defined.

Proof. The proof is virtually identical to that of Lemma 3.3.8, using Lemma 3.3.4 instead of Lemma 3.3.3. □

To complete the proof of the second claim of Proposition 3.3.7, we begin with the equality

$$\mathfrak{g}^\circ(q) = \mathfrak{g}^\circ(q_1) \star \cdots \star \mathfrak{g}^\circ(q_\ell) - \partial[\mathfrak{a}(q)] - \bar{\partial}[\mathfrak{b}(q)]$$

of (3.2.9). Applying Lemma 3.3.9 inductively allows us to specialize both sides to $N_{X_0/X}$ and also shows that

$$\sigma_{X_0/X}(\mathfrak{g}^\circ(q_1) \star \cdots \star \mathfrak{g}^\circ(q_r)) = \sigma_{X_0/X}(\mathfrak{g}^\circ(q_1)) \star \cdots \star \sigma_{X_0/X}(\mathfrak{g}^\circ(q_r)).$$

Recalling Remark 2.2.9, we obtain the desired formula. □

4. Orthogonal Shimura varieties

We now apply the general theory of the previous subsections to the special case in which X is either the hermitian symmetric domain $\mathcal{D}$ associated to an orthogonal group over a totally real field, or the complex Shimura variety $M(\mathbb{C})$ determined by such a group. This allows us to prove our main result: a description of the behavior of special arithmetic cycles on the canonical model M under pullback via the inclusion $M_0 \to M$ of a smaller orthogonal Shimura variety.

4.1. ***The hermitian symmetric domain.*** Let (V, Q) be a quadratic space of dimension $n+2 \ge 3$ over a totally real number field F. Assume there is one embedding $\sigma : F \to \mathbb{R}$ for which the real quadratic space

$$V_\sigma = V \otimes_{F,\sigma} \mathbb{R}$$

has signature $(n, 2)$, while $V_\tau = V \otimes_{F,\tau} \mathbb{R}$ is positive definite for all embeddings $\tau \ne \sigma$. Denote by

$$[x, y] = Q(x+y) - Q(x) - Q(y) \tag{4.1.1}$$

the associated F-bilinear form on V. Extend it $\mathbb{R}$-bilinearly to V_σ, and $\mathbb{C}$-bilinearly to $V_\sigma \otimes_{\mathbb{R}} \mathbb{C}$.

The data (V, Q) determines a hermitian symmetric domain

$$\mathcal{D} = \{z \in V_\sigma \otimes_{\mathbb{R}} \mathbb{C} : [z, z] = 0,\ [z, \bar{z}] < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V_\sigma \otimes_{\mathbb{R}} \mathbb{C}).$$

Denote by

$$V_{\mathcal{D}} = V_\sigma \otimes_{\mathbb{R}} \mathcal{O}_{\mathcal{D}}$$

the constant vector bundle on $\mathcal{D}$ whose fiber at every point is V_σ. It comes equipped with a symmetric bilinear pairing

$$[\,\cdot\,,\cdot\,] : V_{\mathcal{D}} \times V_{\mathcal{D}} \to \mathcal{O}_{\mathcal{D}}, \tag{4.1.2}$$

which on fibers is just the $\mathbb{C}$-bilinear pairing induced by (4.1.1).

The vector bundle $V_{\mathcal{D}}$ is equipped with a filtration by $\mathcal{O}_{\mathcal{D}}$-module local direct summands

$$L_{\mathcal{D}} \subset L_{\mathcal{D}}^{\perp} \subset V_{\mathcal{D}}, \tag{4.1.3}$$

whose fibers at any point $z \in \mathcal{D}$ are identified with the subspaces

$$\mathbb{C}z \subset (\mathbb{C}z)^{\perp} \subset V_{\sigma} \otimes_{\mathbb{R}} \mathbb{C}.$$

In particular $L_{\mathcal{D}}$ is isotropic under the pairing (4.1.2), which induces an isomorphism

$$V_{\mathcal{D}}/L_{\mathcal{D}}^{\perp} \cong L_{\mathcal{D}}^{\vee}. \tag{4.1.4}$$

At each $z \in \mathcal{D}$ we endow the isotropic line

$$L_{\mathcal{D},z} = \mathbb{C}z \subset V_{\sigma} \otimes_{\mathbb{R}} \mathbb{C}$$

with the positive definite hermitian form h determined by

$$h(z, z) = -\frac{[z, \bar{z}]}{2}. \tag{4.1.5}$$

This makes $L_{\mathcal{D}}$ into a hermitian line bundle.

Using (4.1.4), any $x \in V_{\sigma}$ determines first a global section of $V_{\mathcal{D}}$, and then a global section

$$s(x) \in H^0(\mathcal{D}, L_{\mathcal{D}}^{\vee}), \tag{4.1.6}$$

with zero locus the smooth analytic divisor

$$Z_{\mathcal{D}}(x) = \{z \in \mathcal{D} : [z, x] = 0\}.$$

More generally, any tuple $x = (x_1, \dots, x_d) \in V_{\sigma}^d$ determines a tuple $s(x) = (s(x_1), \dots, s(x_d))$ of sections, and we denote by

$$Z_{\mathcal{D}}(x) \subset \mathcal{D}$$

the analytic subspace defined by the vanishing of all components. In other words, $Z_{\mathcal{D}}(x)$ is those lines $\mathbb{C}z \subset \mathcal{D}$ such that $[z, x_i] = 0$ for all $1 \le i \le d$. This is a complex submanifold which depends only on $\mathrm{Span}_{\mathbb{R}}\{x_1, \dots, x_d\} \subset V_{\sigma}$. It is nonempty precisely when this subspace is positive definite, in which case it has codimension $\dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}\{x_1, \dots, x_d\}$. Recalling the notation and terminology of Section 3.1, the smoothness and regularity of the tuple $s(x)$ are both equivalent to the linear independence of the vectors $x_1, \dots, x_d$.

Now fix a positive definite $v \in \mathrm{Sym}_d(\mathbb{R})$ and an $\alpha \in \mathrm{GL}_d(\mathbb{R})$ with positive determinant such that

$$v = \alpha \cdot {}^t\alpha.$$

If $x \in V_{\sigma}^d$ is a tuple with linearly independent components, we may form a new d-tuple $x\alpha \in V_{\sigma}^d$, and hence a corresponding smooth tuple $s(x\alpha)$ of sections of $L_{\mathcal{D}}^{\vee}$. Applying the constructions of Section 3.1 to this tuple of sections determines forms

$$\begin{aligned} \mathfrak{g}_{\mathcal{D}}^{\circ}(x, v) &= \mathfrak{g}^{\circ}(s(x\alpha)) \in E_{\mathcal{D} \smallsetminus Z(x)}^{d-1,d-1}, \\ \omega_{\mathcal{D}}^{\circ}(x, v) &= \omega^{\circ}(s(x\alpha)) \in E_{\mathcal{D}}^{d,d}, \end{aligned} \tag{4.1.7}$$

related by the Green equation

$$dd^c[\mathfrak{g}^\circ_{\mathcal{D}}(x,v)]+\delta_{Z_{\mathcal{D}}(x)}=[\omega^\circ_{\mathcal{D}}(x,v)].$$

These forms are independent of α by Proposition 2.6(d) of [Garcia and Sankaran 2019].

4.2. ***Canonical models.*** The quadratic space (V, Q) determines a short exact sequence

$$1\to\mathbb{G}_{m/F}\to \mathrm{GSpin}(V)\to \mathrm{SO}(V)\to 1$$

of reductive groups over F. From now on we denote by G either

$$\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(V)\quad\text{or}\quad \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SO}(V).$$

For our purposes these two groups are interchangeable. The group $G(\mathbb{R})$ acts on $\mathcal{D}$ via the projection

$$G(\mathbb{R})\to\prod_{\tau:F\to\mathbb{R}}\mathrm{SO}(V_\tau)\to \mathrm{SO}(V_\sigma),$$

and the pair $(G,\mathcal{D})$ is a Shimura datum. A choice of sufficiently small compact open subgroup $K\subset G(\mathbb{A}_f)$ determines a smooth quasiprojective variety M over the reflex field $F\cong\sigma(F)\subset\mathbb{C}$ with $\mathbb{C}$-points

$$M(\mathbb{C})=G(\mathbb{Q})\backslash\mathcal{D}\times G(\mathbb{A}_f)/K.$$

It is projective if and only if V is anisotropic. Any $g\in G(\mathbb{A}_f)$ determines an open and closed submanifold

$$(gKg^{-1}\cap G(\mathbb{Q}))\backslash\mathcal{D}\subset M(\mathbb{C}),\tag{4.2.1}$$

where the inclusion is $z\mapsto(z,g)$.

For a point $z\in\mathcal{D}$, the action of any $\gamma\in G(\mathbb{R})$ on $V_\sigma\otimes_{\mathbb{R}}\mathbb{C}$ identifies the fibers of (4.1.3) at z and γz. This allows us to descend the filtered vector bundle (4.1.3) from $\mathcal{D}$ to every quotient (4.2.1). By the theory of canonical models of automorphic vector bundles, there is a canonical filtered vector bundle

$$L_M\subset L_M^\perp\subset V_M$$

on M whose restriction to (4.2.1) agrees with this descent.

In a similar way, the pairing (4.1.2) descends to an $\mathcal{O}_M$-bilinear pairing

$$[\,\cdot\,,\cdot\,]:V_M\times V_M\to\mathcal{O}_M,\tag{4.2.2}$$

under which L_M and $L_M^\perp$ are the orthogonal subbundles to one another, and this pairing induces a canonical isomorphism

$$V_M/L_M^\perp\cong L_M^\vee\tag{4.2.3}$$

that agrees with (4.1.4) under the complex uniformization.

The vector bundle V_M is nonconstant, but it is *infinitesimally constant* in the sense that it carries a flat connection

$$\nabla:V_M\to V_M\otimes_{\mathcal{O}_M}\Omega^1_M\tag{4.2.4}$$

characterized by the property that its pullback to $\mathcal{D}$ via the uniformization (4.2.1) agrees with the constant connection $\mathrm{id}\otimes d$ on $V_{\mathcal{D}} = V_\sigma \otimes_{\mathbb{R}} \mathcal{O}_{\mathcal{D}}$. This allows us to perform parallel transport through square-zero thickenings:

Proposition 4.2.1. *Suppose $M_0 \subset M$ is a closed subscheme, smooth over F. A flat section of $V_M|_{M_0}$ extends uniquely to a flat section of V_M over the first-order infinitesimal neighborhood of M_0 in M.*

Proof. This can be extracted from the arguments of Section 2 of [Berthelot and Ogus 1978]. In fact, as we are working with smooth schemes in characteristic 0, the results of [loc. cit.] can be used to show that a flat section defined over M_0 extends uniquely to a flat section over the entire formal completion along $M_0 \subset M$. We instead sketch a more direct argument working only over the first-order infinitesimal neighborhood $M_0^\square \subset M$. Thus $M_0^\square$ is the closed subscheme defined by the square $I^2 \subset \mathcal{O}_M$ of the ideal sheaf $I \subset \mathcal{O}_M$ defining $M_0 \subset M$.

Denote by $U \subset M \times_F M$ the first-order infinitesimal neighborhood of the diagonal $M \subset M \times_F M$, and let $p_1, p_2 : U \to M$ be the projection maps. By Proposition 2.9 of [loc. cit.], the connection ∇ determines an isomorphism

$$p_1^* V_M \cong p_2^* V_M$$

of vector bundles on U satisfying a cocycle relation encoding the flatness of the connection. This cocycle condition implies that the above isomorphism is an isomorphism of vector bundles *with connections*, where the left- and right-hand sides are endowed with the pullbacks of ∇ through p_1 and p_2, respectively.

The smoothness of M_0 implies that, Zariski locally on $M_0^\square$, one can find a retraction $\rho : M_0^\square \to M_0$. Denoting by i and $i^\square$ the inclusions of M_0 and $M_0^\square$ into M, the product morphism

$$(i^\square, i \circ \rho) : M_0^\square \to M \times_F M$$

factors through U, and hence the pullbacks of V_M by $i^\square$ and $i \circ \rho$ are isomorphic as vector bundles with connections. The resulting isomorphism

$$V_M|_{M_0^\square} \cong \rho^*(V_M|_{M_0})$$

induces a homomorphism

$$H^0(M_0, V_M|_{M_0})^{\nabla=0} \xrightarrow{\rho^*} H^0(M_0^\square, \rho^*(V_M|_{M_0}))^{\nabla=0} \cong H^0(M_0^\square, V_M|_{M_0^\square})^{\nabla=0}$$

of spaces of flat sections, and it is not difficult to check that the first arrow is an isomorphism. Note that the composition does not depend on the choice of retraction ρ, because this is true of its inverse "restrict to M_0".

This proves the existence and uniqueness of flat extensions of flat sections over open subsets small enough that the required retractions exist, and the uniqueness allows us to glue the sections together over an open cover. □

4.3. ***Arithmetic cycle classes.*** Fix an integer d with $1 \le d \le n+1$. The group $G(\mathbb{A}_f)$ acts on

$$\widehat{V} = V \otimes_{\mathbb{Q}} \mathbb{A}_f,$$

and we fix a K-invariant $\mathbb{Z}$-valued Schwartz function $\varphi \in S(\widehat{V}^d)$.

Any $g \in G(\mathbb{A}_f)$ and $T \in \mathrm{Sym}_d(F)$ determine an analytic cycle

$$Z_{\mathcal{D}}(T,\varphi)_g = \sum_{\substack{x \in V^d \\ T(x)=T}} \varphi(g^{-1}x) \cdot Z_{\mathcal{D}}(x) \tag{4.3.1}$$

on $\mathcal{D}$. Here we denote by

$$T(x) = \left(\tfrac{1}{2}[x_i, x_j]\right) \in \mathrm{Sym}_d(F) \tag{4.3.2}$$

the moment matrix of a tuple $x \in V^d$. The cycle (4.3.1) descends to the quotient (4.2.1), and varying g yields an analytic cycle $Z_M(T,\varphi)(\mathbb{C})$ on $M(\mathbb{C})$. Being expressible as a union of smaller Shimura varieties constructed in the same way as M, this cycle is the complexification of an algebraic cycle $Z_M(T,\varphi)$ of codimension $\mathrm{rank}(T)$ on the canonical model M.

Fix a positive definite $v \in \mathrm{Sym}_d(\mathbb{R})$ and assume $\det(T) \neq 0$. As in Section 4.3 of [Garcia and Sankaran 2019], the sums

$$\begin{aligned} \mathfrak{g}^{\circ}_{\mathcal{D}}(T,v,\varphi)_g &= \sum_{\substack{x \in V^d \\ T(x)=T}} \varphi(g^{-1}x) \cdot \mathfrak{g}^{\circ}_{\mathcal{D}}(x,v) \in E^{d-1,d-1}_{\mathcal{D} \smallsetminus Z_{\mathcal{D}}(T,\varphi)(\mathbb{C})_g}, \\ \omega^{\circ}_{\mathcal{D}}(T,v,\varphi)_g &= \sum_{\substack{x \in V^d \\ T(x)=T}} \varphi(g^{-1}x) \cdot \omega^{\circ}_{\mathcal{D}}(x,v) \in E^{d,d}_{\mathcal{D}} \end{aligned} \tag{4.3.3}$$

also descend to the quotient (4.2.1). Again by varying g, we obtain forms

$$\mathfrak{g}^{\circ}_M(T,v,\varphi) \in E^{d-1,d-1}_{M(\mathbb{C}) \smallsetminus Z_M(T,\varphi)(\mathbb{C})} \quad \text{and} \quad \omega^{\circ}_M(T,v,\varphi) \in E^{d,d}_{M(\mathbb{C})}$$

related by the Green equation

$$dd^c[\mathfrak{g}^{\circ}_M(T,v,\varphi)] + \delta_{Z_M(T,\varphi)} = [\omega^{\circ}_M(T,v,\varphi)],$$

and an arithmetic cycle class

$$\widehat{Z}_M(T,v,\varphi) = (Z_M(T,\varphi), \mathfrak{g}^{\circ}_M(T,v,\varphi)) \in \widehat{\mathrm{CH}}^d(M). \tag{4.3.4}$$

We would like to extend the definition to include singular T.

Recall that we have endowed the tautological line bundle $L_{\mathcal{D}}$ on $\mathcal{D}$ with the hermitian metric h of (4.1.5) and have endowed $L^{\vee}_{\mathcal{D}}$ with the dual metric. These induce metrics on the canonical models L_M and $L^{\vee}_M$, and so determine arithmetic cycle classes

$$L_M, L^{\vee}_M \in \widehat{\mathrm{CH}}^1(M)$$

using the arithmetic Chern class map from Section III.4.2 of [Soulé 1992]. Of course $L^{\vee}_M = -L_M$. A distinguished role is played by

$$\hat{\omega}^{-1} = L^{\vee}_M + (0, -\log(2\pi e^{\gamma})) \in \widehat{\mathrm{CH}}^1(M). \tag{4.3.5}$$

In other words, if we endow L_M with the rescaled metric $(2\pi e^{\gamma})^{-1}h$, then (4.3.5) is the image of its dual under the arithmetic Chern class map. Write

$$\Omega = \mathrm{ch}(L^{\vee}_M) \in E^{1,1}_{M(\mathbb{C})}$$

for the Chern form of the dual of (L_M, h), and note that Ω is also the Chern form of (4.3.5).

Remark 4.3.1. Our L_M agrees with the $\mathcal{E}$ in (5.160) of [Garcia and Sankaran 2019], but our $\hat{\omega}$ differs from theirs by an inverse and a rescaling of metrics.

Remark 4.3.2. The factor of $2\pi e^{\gamma}$ in (4.3.5) is needed to make the arithmetic cycle classes defined below satisfy the pullback formula of Theorem A. More precisely, in the proof of Proposition 4.5.2 this factor will match up with the similar factor appearing in the logarithmic expansions of Lemma 3.3.4 and the specializations to the normal bundle of Proposition 3.3.5. There are other reasons why the particular normalization in (4.3.5) is a natural choice, as explained in the Introduction of [Kudla et al. 2004].

Theorem 4.3.3 (Garcia–Sankaran). *Assume that V is anisotropic. There are arithmetic cycle classes*

$$\widehat{Z}_M(T, v, \varphi) \in \widehat{\mathrm{CH}}^d(M)$$

indexed by $T \in \mathrm{Sym}_d(F)$, positive definite $v \in \mathrm{Sym}_d(\mathbb{R})$, and K-fixed $\mathbb{Z}$-valued $\varphi \in S(\widehat{V})$ satisfying the following properties:

(1) *For fixed T and v, the formation of $\widehat{Z}_M(T, v, \varphi)$ is linear in φ.*

(2) *If* $\det(T) \neq 0$ *then $\widehat{Z}_M(T, v, \varphi)$ agrees with* (4.3.4).

(3) *If $0_d \in \mathrm{Sym}_d(F)$ denotes the zero matrix, then*

$$\widehat{Z}_M(0_d, v, \varphi) = \varphi(0) \cdot \underbrace{\hat{\omega}^{-1} \cdots \hat{\omega}^{-1}}_{d \text{ times}}.$$

(4) *Assume that T and v have the form*

$$T = \begin{pmatrix} T_0 & \\ & 0_{d-r} \end{pmatrix} \quad \text{and} \quad v = {}^t\theta \cdot \begin{pmatrix} v_0 & \\ & w \end{pmatrix} \cdot \theta,$$

with $T_0 \in \mathrm{Sym}_r(F)$ nonsingular, $v_0 \in \mathrm{Sym}_r(\mathbb{R})$ and $w \in \mathrm{Sym}_{d-r}(\mathbb{R})$ of positive determinant, and

$$\theta = \begin{pmatrix} 1_r & * \\ & 1_{d-r} \end{pmatrix} \in \mathrm{GL}_d(\mathbb{R}).$$

If $\varphi = \varphi^{(r)} \otimes \varphi^{(d-r)} \in S(\widehat{V}^r) \otimes S(\widehat{V}^{d-r})$ is a product of $\mathbb{Z}$-valued K-fixed Schwartz functions, then

$$\widehat{Z}_M(T, v, \varphi) = \widehat{Z}_M(T_0, v_0, \varphi^{(r)}) \cdot \widehat{Z}_M(0_{d-r}, w, \varphi^{(d-r)}).$$

(5) *For any $a \in \mathrm{GL}_d(F)$ we have*

$$\widehat{Z}_M(T, v, \varphi) = \widehat{Z}_M({}^aT, {}^av, {}^a\varphi),$$

where

$${}^aT = {}^ta T a, \quad {}^av = \sigma(a^{-1}) v \sigma({}^ta^{-1}), \quad {}^a\varphi(x) = \varphi(xa^{-1}).$$

Proof. This is a minor modification of the construction of Section 5.4 of [Garcia and Sankaran 2019].

We have defined the forms (4.3.3) only when $\det(T) \neq 0$. If we drop this assumption, the construction of $\omega^{\circ}_M(T, v, \varphi)$ still makes sense word-for-word. The construction of $\mathfrak{g}^{\circ}_M(T, v, \varphi)$ does not, because Proposition 3.1.6 only applies to regular tuples, and the tuple $s(x\alpha)$ appearing in the definition (4.1.7) is not regular if $T(x)$ is a singular matrix. Nevertheless, Propositions 4.3 and 4.4 of [Garcia and Sankaran 2019] provide the construction of a current $\mathfrak{g}^\circ_M(T, v, \varphi)$ on $M(\mathbb{C})$ of type $(d-1, d-1)$ satisfying the generalized Green equation

$$dd^c \mathfrak{g}^\circ_M(T, v, \varphi) + \delta_{Z_M(T,\varphi)} \wedge \Omega^{d-\mathrm{rank}(T)} = \omega^\circ_M(T, v, \varphi).$$

We remark that when $\mathrm{rank}(T) < d$ the current $\mathfrak{g}^\circ_M(T, v, \varphi)$ is not represented by a locally integrable form on $M(\mathbb{C})$.

Now let $\mathfrak{g}$ be any choice of Green current for the cycle $Z_M(T, \varphi)$ of codimension $r = \mathrm{rank}(T)$. The arithmetic cycle class

$$\widehat{Z}_M(T, v, \varphi) = (Z_M(T, \varphi), \mathfrak{g}) \cdot (\hat{\omega}^{-1})^{d-r} + (0, \mathfrak{g}^\circ_M(T, v, \varphi) - \mathfrak{g} \wedge \Omega^{d-r})$$

is easily seen to be independent of $\mathfrak{g}$. This is the same definition as (5.158) of [Garcia and Sankaran 2019], except that we have used the class (4.3.5) in place of $L^\vee_M$, and have used the current $\mathfrak{g}^\circ_M(T, v, \varphi)$ instead of the modified version of Definition 4.7 of [loc. cit.].

Properties (1) and (2) are immediate from the definitions. Property (3) follows from $Z_M(0_d, \varphi) = \varphi(0)M$ and $\mathfrak{g}^\circ_M(0_d, v, \varphi) = 0$, as in (4.43) of [loc. cit.]. Property (4) is a consequence of the relations

$$Z_M(T, \varphi) = \varphi^{(d-r)}(0) \cdot Z_M(T_0, \varphi^{(r)}),$$
$$\mathfrak{g}^\circ_M(T, v, \varphi) = \varphi^{(d-r)}(0) \cdot \mathfrak{g}^\circ_M(T_0, v_0, \varphi^{(r)}) \wedge \Omega^{d-r} + \partial A + \bar{\partial} B$$

for currents A and B on $M(\mathbb{C})$, as in Examples 2.14 and 4.8 of [loc. cit.]. Property (5) follows from

$$Z_M(T, \varphi) = Z_M({}^aT, {}^a\varphi),$$
$$\mathfrak{g}^\circ_M(T, v, \varphi) = \mathfrak{g}^\circ_M({}^aT, {}^av, {}^a\varphi),$$

as in Remark 4.9 of [loc. cit.]. □

Remark 4.3.4. The arithmetic cycle classes of Theorem 4.3.3 are uniquely determined by the properties listed there. The key point is that for any T and v one may find an $a \in \mathrm{GL}_d(F)$ such that the matrices ${}^aT$ and ${}^av$ appearing in (5) have the form described in (4). The classes determined by such matrices are obviously determined by properties (1)–(4).

We now modify the arithmetic cycle classes of Theorem 4.3.3. Given data (T, v, φ) as in that theorem, choose $a \in \mathrm{GL}_d(F)$ in such a way that

$${}^aT = \begin{pmatrix} T_0 & \\ & 0_{d-\mathrm{rank}(T)} \end{pmatrix} \quad \text{and} \quad {}^av = {}^t\theta \cdot \begin{pmatrix} v_0 & \\ & w \end{pmatrix} \cdot \theta$$

have the form described in part (4), and define

$$\widehat{C}_M(T, v, \varphi) = \widehat{Z}_M({}^aT, {}^av, {}^a\varphi) + (0, -\log(\det(w)) \cdot \delta_{Z_M(T,\varphi)} \wedge \Omega^{d-\mathrm{rank}(T)-1}). \tag{4.3.6}$$

Note that if T is not totally positive semidefinite, then $Z_M(T, \varphi) = 0$ and the correction term disappears. If $\det(T) \neq 0$ we understand $\det(w) = 1$, so that the correction term again vanishes, leaving

$$\widehat{C}_M(T, v, \varphi) = \widehat{Z}_M({}^aT, {}^av, {}^a\varphi) = \widehat{Z}_M(T, v, \varphi).$$

Proposition 4.3.5. *The arithmetic cycle class $\widehat{C}_M(T, v, \varphi)$ does not depend on the choice of $a \in \mathrm{GL}_d(F)$ used in its construction. It satisfies all the properties listed in Theorem 4.3.3, except that now*

$$\widehat{C}_M(0_d, v, \varphi) = \varphi(0) \cdot [\underbrace{\hat{\omega}^{-1} \cdots \hat{\omega}^{-1}}_{d} + (0, -\log(\det(v)) \cdot \Omega^{d-1})].$$

In particular, if $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_d$ is a pure tensor then

$$\widehat{C}_M(0_d, v, \varphi) = \widehat{C}_M(0, v_1, \varphi_1) \cdots \widehat{C}_M(0, v_d, \varphi_d),$$

where $v_1, \dots, v_d$ are the eigenvalues of v, and

$$\widehat{C}_M(0, v_i, \varphi_i) = \varphi_i(0) \cdot [\hat{\omega}^{-1} + (0, -\log(v_i))] \in \widehat{\mathrm{CH}}^1(M).$$

Proof. For the independence of the choice of a, a linear algebra exercise shows that choosing a different a has the effect of multiplying both $\det(v_0)$ and $\det(w)$ by nonzero squares in $\sigma(F)$. Thus it suffices to show that the arithmetic cycle class

$$(0, -\log \sigma(\xi^2) \cdot \delta_{Z_M(T,\varphi)} \wedge \Omega^{d-\mathrm{rank}(T)-1}) \in \widehat{\mathrm{CH}}^d(M) \tag{4.3.7}$$

is trivial for any $\xi \in F^\times$. If we view ξ as a (constant) rational function on $Z_M(T, \varphi)$, it determines an arithmetic cycle

$$(i_* \operatorname{div}(\xi), i_*[-\log \sigma(\xi^2)]) = (0, -\log \sigma(\xi^2) \wedge \delta_{Z(T,\varphi)}) \in \widehat{Z}^{\mathrm{rank}(T)+1}(M),$$

where $i : Z_M(T, \varphi) \to M$ is the inclusion. As in the discussion leading to Definition 1 in Section III.1.1 of [Soulé 1992], this arithmetic cycle is trivial in the arithmetic Chow group. On the other hand, its arithmetic intersection with $d - \mathrm{rank}(T) - 1$ copies of $\hat{\omega}^{-1}$ is (4.3.7), which is therefore also trivial.

The remaining claims follow from Theorem 4.3.3 and the definitions. □

Remark 4.3.6. If there is no $x \in V^d$ such that $T(x) = T$, then

$$\widehat{C}_M(T, v, \varphi) = \widehat{Z}_M(T, v, \varphi) = 0.$$

If T is nonsingular, this is clear from the definitions. The general case can be reduced to the nonsingular case using Remark 4.3.4.

Remark 4.3.7. Our classes (4.3.6) agree with those of (5.158) of [Garcia and Sankaran 2019] when $\det(T) \neq 0$. For singular matrices they do not quite agree. As remarked in the proof of Theorem 4.3.3, the classes $\widehat{Z}_M(T, v, \varphi)$ differ from the Garcia–Sankaran classes in two ways: the extra factor of $-\log(2\pi e^\gamma)$ in (4.3.5), and the use of the current $\mathfrak{g}^\circ_M(T, v, \varphi)$ instead of the modified current of Definition 4.7 of [Garcia and Sankaran 2019]. Using the vanishing of (4.3.7), one can see that adding the correction term in (4.3.6) eliminates the second of these two differences. Thus the only difference between our $\widehat{C}_M(T, v, \varphi)$ and the classes of Garcia–Sankaran is the shifted metric in (4.3.5).

4.4. ***The pullback formula.*** We now state our main result. The proof will occupy the rest of the paper.

Suppose we are given an orthogonal decomposition

$$V = V_0 \oplus W$$

with V_0 and W of dimensions $n_0+2 \geq 3$ and m, respectively. Assume moreover that $W_\tau = W \otimes_{F,\tau} \mathbb{R}$ is positive definite for every $\tau : F \to \mathbb{R}$. The assumptions on V imposed in Section 4.1 imply that $V_{0,\tau} = V_0 \otimes_{F,\tau} \mathbb{R}$ has signature

$$\operatorname{sig}(V_{0,\tau}) = \begin{cases} (n_0, 2) & \text{if } \tau = \sigma, \\ (n_0+2, 0) & \text{if } \tau \neq \sigma. \end{cases}$$

The quadratic space V_0 therefore has its own Shimura datum $(G_0, \mathcal{D}_0)$, and the inclusion $V_0 \subset V$ induces a injection of Shimura data

$$i_0 : (G_0, \mathcal{D}_0) \to (G, \mathcal{D})$$

realizing $\mathcal{D}_0 \subset \mathcal{D}$ as a codimension-m submanifold. Fix a compact open subgroup $K_0 \subset G_0(\mathbb{A}_f) \cap K$, and let M_0 be the associated Shimura variety over $F = \sigma(F)$ with complex points

$$M_0(\mathbb{C}) = G_0(\mathbb{Q})\backslash \mathcal{D}_0 \times G_0(\mathbb{A}_f)/K_0.$$

The induced map $i_0 : M_0 \to M$ is finite and unramified. The Shimura variety M_0 has its own hermitian line bundle L_{M_0}, related to the one on M by a canonical isomorphism

$$L_{M_0} \cong i_0^* L_M.$$

Hypothesis 4.4.1. *We assume throughout that the compact open subgroups $K_0 \subset G_0(\mathbb{A}_f)$ and $K \subset G(\mathbb{A}_f)$ have been chosen so that*

$$i_0 : M_0 \to M$$

is a closed immersion. This is always possible, by Proposition 1.15 *of* [Deligne 1971].

Theorem 4.4.2. *Assume that V is anisotropic. Fix an integer $1 \leq d \leq n_0+1$ and a K-fixed Schwartz function*

$$\varphi = \varphi_0 \otimes \psi \in S(\widehat{V}_0^d)^{K_0} \otimes S(\widehat{W}^d) \subset S(\widehat{V}^d),$$

with both factors φ_0 and ψ valued in $\mathbb{Z}$. Recall that Section 4.3 associates to any $T \in \operatorname{Sym}_d(F)$ and any positive definite $v \in \operatorname{Sym}_d(\mathbb{R})$ arithmetic cycle classes

$$\widehat{C}_M(T, v, \varphi) \in \widehat{\operatorname{CH}}^d(M) \quad \textit{and} \quad \widehat{C}_{M_0}(T, v, \varphi_0) \in \widehat{\operatorname{CH}}^d(M_0). \tag{4.4.1}$$

The specialization to the normal bundle

$$\sigma_{M_0/M} : \widehat{\operatorname{CH}}^d(M) \to \widehat{\operatorname{CH}}^d(N_{M_0/M})$$

of Theorem 2.3.1 satisfies

$$\sigma_{M_0/M}(\widehat{C}_M(T, v, \varphi)) = \sum_{\substack{T_0 \in \operatorname{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \cdot \pi_0^* \widehat{C}_{M_0}(T_0, v, \varphi_0),$$

where $\pi_0 : N_{M_0/M} \to M_0$ is the bundle map.

Theorem 4.4.2 will be proved below. First, we record a corollary explaining the precise connection between the classes of (4.4.1).

Corollary 4.4.3. *Keeping the notation and assumptions of Theorem 4.4.2, the pullback*

$$i_0^* : \widehat{\mathrm{CH}}^d(M) \to \widehat{\mathrm{CH}}^d(M_0)$$

satisfies

$$i_0^* \widehat{C}_M(T, v, \varphi) = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \cdot \widehat{C}_{M_0}(T_0, v, \varphi_0).$$

Proof. This is immediate from Theorems 2.3.1 and 4.4.2, along with the injectivity of π_0^* proved in Proposition 2.3.3. □

4.5. ***Specialization of degenerate cycles.*** We now state and prove the key ingredient in the proof of Theorem 4.4.2. This is Proposition 4.5.2 below, which allows us to compute the specializations to the normal bundle $N_{M_0/M}$ of those arithmetic cycles on M that intersect M_0 improperly.

The action of $G_0(\mathbb{R})$ on the pair $\mathcal{D}_0 \subset \mathcal{D}$ induces an action on $N_{\mathcal{D}_0/\mathcal{D}}$, and the normal bundle to $M_0 \to M$ has complex points

$$N_{M_0/M}(\mathbb{C}) = G_0(\mathbb{Q}) \backslash N_{\mathcal{D}_0/\mathcal{D}} \times G_0(\mathbb{A}_f)/K_0.$$

As in (4.2.1), every $g \in G_0(\mathbb{A}_f)$ determines a commutative diagram

$$\begin{array}{ccc} \Gamma_g \backslash N_{\mathcal{D}_0/\mathcal{D}} & \xrightarrow{z \mapsto (z,g)} & N_{M_0/M}(\mathbb{C}) \\ \downarrow & & \downarrow \pi_0 \\ \Gamma_g \backslash \mathcal{D}_0 & \xrightarrow{z \mapsto (z,g)} & M_0(\mathbb{C}) \end{array} \tag{4.5.1}$$

in which $\Gamma_g = gK_0g^{-1} \cap G_0(\mathbb{Q})$, and the horizontal arrows are open and closed immersions.

Define complex manifolds

$$X_0 = \bigsqcup_g \Gamma_g \backslash \mathcal{D}_0 \quad \text{and} \quad X = \bigsqcup_g \Gamma_g \backslash \mathcal{D},$$

where both unions are taken over a set of representatives for the double quotient $G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}_f)/K_0$. This gives a diagram of complex manifolds

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \| & & \downarrow \\ M_0(\mathbb{C}) & \longrightarrow & M(\mathbb{C}), \end{array}$$

in which the horizontal arrows are closed immersions, and the right vertical arrow is a holomorphic covering of the union of all connected components of $M(\mathbb{C})$ having nonempty intersection with $M_0(\mathbb{C})$. There are canonical identifications

$$N_{M_0/M}(\mathbb{C}) \cong N_{X_0/X} \cong \bigsqcup_g \Gamma_g \backslash N_{\mathcal{D}_0/\mathcal{D}} \tag{4.5.2}$$

of holomorphic vector bundles on $X_0 = M_0(\mathbb{C})$. Note that X, unlike X_0, is not (in any obvious way) the complex points of an algebraic variety.

Fix a tuple $y = (y_1, \ldots, y_d) \in W^d$ with linearly independent components, and a positive definite $w \in \mathrm{Sym}_d(\mathbb{R})$. As explained in Section 4.1, this data determines a pair

$$(Z_{\mathcal{D}}(y), \mathfrak{g}^{\circ}_{\mathcal{D}}(y, w)) \tag{4.5.3}$$

consisting of an analytic cycle $Z_{\mathcal{D}}(y) \subset \mathcal{D}$ and a Green current for it, represented by a smooth form on $\mathcal{D} \smallsetminus Z_{\mathcal{D}}(y)$. Because the group $G_0(\mathbb{Q})$ acts trivially on the subspace $W \subset V$, and hence fixes y componentwise, this pair is invariant under the action of each Γ_g. Thus it descends to each quotient $\Gamma_g \backslash \mathcal{D}$, and by varying g we obtain a pair

$$(Z_X(y), \mathfrak{g}^{\circ}_X(y, w)) \tag{4.5.4}$$

consisting of an analytic cycle $Z_X(y) \subset X$ and a Green current for it, represented by a smooth form on $X \smallsetminus Z_X(y)$. Alternatively, rather than descending from $\mathcal{D}$, one could obtain this pair by simply repeating the construction of (4.5.3) with $\mathcal{D}$ replaced by X everywhere.

Using the constructions of Section 2.2, one can specialize (4.5.4) to a pair

$$(Z(\mathbb{C}), \mathfrak{g}) \stackrel{\text{def}}{=} \big(\sigma_{X_0/X}(Z_X(y)), \sigma_{X_0/X}(\mathfrak{g}^{\circ}_X(y, w))\big) \tag{4.5.5}$$

on the normal bundle (4.5.2). Equivalently, one could specialize (4.5.3) to obtain a $G(\mathbb{Q})$-invariant pair on the normal bundle $N_{\mathcal{D}_0/\mathcal{D}}$, pass to the quotient by each Γ_g in (4.5.2), and then vary g to obtain a pair on $N_{X_0/X}$.

Remark 4.5.1. In specializing $\mathfrak{g}^{\circ}_X(y, w)$ to $N_{X_0/X}$, we are using Theorem 2.2.5 and (3.1.7) to guarantee the existence of a logarithmic expansion of $\mathfrak{g}^{\circ}_X(y, w)$ along $X_0 \subset X$. Alternatively, we will soon see that $\mathfrak{g}^{\circ}_X(y, w)$ is a special example of the Green form obtained from a tuple of degenerating sections in the sense of Section 3.3, and so it has logarithmic expansion of the more concrete type described in Lemma 3.3.3.

The pair (4.5.5) is a subtle thing to understand, as the intersection of X_0 with $Z_X(y)$ is improper (in fact $X_0 \subset Z_X(y)$). It is not even obvious that the analytically defined cycle $Z(\mathbb{C})$ on (4.5.2) is algebraic, let alone that it is defined over the reflex field. Nevertheless, the following proposition gives us good control over it.

Proposition 4.5.2. *The analytic cycle $Z(\mathbb{C}) \subset N_{X_0/X}$ in* (4.5.5) *is the complexification of an algebraic cycle $Z \subset N_{M_0/M}$, and the equality*

$$(Z, \mathfrak{g}) = \pi_0^*(\underbrace{\hat{\omega}_0^{-1} \cdots \hat{\omega}_0^{-1}}_{d}) + \pi_0^*(0, -\log(\det(w)) \cdot \Omega_0^{d-1})$$

holds in the codimension-d arithmetic Chow group of $N_{M_0/M}$. Here $\pi_0 : N_{M_0/M} \to M_0$ is the bundle map, $\hat{\omega}_0^{-1}$ is the analogue of (4.3.5) *on M_0, and $\Omega_0 \in E^{1,1}_{M_0(\mathbb{C})}$ is its Chern form.*

The proof of Proposition 4.5.2, which occupies the remainder of this subsection, uses the degenerating sections of Section 3.3 in an essential way. The closed immersion $\mathcal{D}_0 \subset \mathcal{D}$ admits a presentation of the type considered in Section 3.3. More precisely, if we denote by $W_{\mathcal{D}} = W_\sigma \otimes_{\mathbb{R}} \mathcal{O}_{\mathcal{D}}$ the constant vector bundle on $\mathcal{D}$ with fibers $W_\sigma \otimes_{\mathbb{R}} \mathbb{C}$, so that $W_{\mathcal{D}} \subset V_{\mathcal{D}}$, the composition

$$W_{\mathcal{D}} \to V_{\mathcal{D}}/L_{\mathcal{D}}^{\perp} \xrightarrow{(4.1.4)} L_{\mathcal{D}}^{\vee} \tag{4.5.6}$$

defines a global section of

$$N_{\mathcal{D}} = \underline{\mathrm{Hom}}(W_{\mathcal{D}}, L_{\mathcal{D}}^{\vee}) \tag{4.5.7}$$

with vanishing locus $\mathcal{D}_0$. In particular,

$$N_{\mathcal{D}_0/\mathcal{D}} \overset{(3.3.1)}{\cong} \underline{\mathrm{Hom}}(W_{\mathcal{D}}, L_{\mathcal{D}}^{\vee})|_{\mathcal{D}_0} \cong \underline{\mathrm{Hom}}(W_{\mathcal{D}_0}, L_{\mathcal{D}_0}^{\vee}).$$

These isomorphisms are equivariant with respect the natural actions of $G_0(\mathbb{Q})$, and so, using (4.5.2), define an isomorphism

$$N_{X_0/X} \cong \underline{\mathrm{Hom}}(W_{X_0}, L_{X_0}^{\vee}) \tag{4.5.8}$$

of holomorphic vector bundles on $X_0 = M_0(\mathbb{C})$. Here W_{X_0} and $L_{X_0}^{\vee}$ have the obvious meanings: they are constructed from the vector bundles $W_{\mathcal{D}_0}$ and $L_{\mathcal{D}_0}^{\vee}$ using (4.5.2). We now explain how to algebraize (4.5.8).

Lemma 4.5.3. *Let $W_{M_0} = W \otimes_F \mathcal{O}_{M_0}$ be the constant vector bundle. There is an isomorphism*

$$N_{M_0/M} \cong \underline{\mathrm{Hom}}(W_{M_0}, L_{M_0}^{\vee})$$

of vector bundles on M_0 that agrees, using the first identification in (4.5.2), *with* (4.5.8) *on the complex fiber.*

Proof. The subspace $W \subset V$ is not stable under $G(\mathbb{Q})$, so it does not determine a subbundle of V_M. However, the decomposition $V = V_0 \oplus W$ is stable under $G_0(\mathbb{Q})$, which implies that the pullback of V_M via the inclusion $M_0 \to M$ acquires a canonical splitting

$$V_M|_{M_0} = V_{M_0} \oplus W_{M_0}.$$

This splitting is orthogonal with respect to the bilinear form (4.2.2), and the restriction to M_0 of the flat connection (4.2.4) is identified with the sum of the analogous connection on V_{M_0} and the constant connection on W_{M_0} (for which the constant sections $W \subset H^0(M_0, W_{M_0})$ are flat).

In particular, any vector $w \in W$ determines a flat section

$$f_w \in H^0(M_0, V_M|_{M_0}).$$

This section is orthogonal to the line $L_M|_{M_0} \cong L_{M_0} \subset V_{M_0}$, and so lies in the kernel of

$$V_M|_{M_0} \xrightarrow{(4.2.3)} L_M^{\vee}|_{M_0}.$$

By parallel transport (Proposition 4.2.1) the section f_w extends to a flat section

$$f_w^{\square} \in H^0(M_0^{\square}, V_M|_{M_0^{\square}})$$

over the first-order infinitesimal neighborhood $M_0^\square \subset M$ of M_0, whose ideal sheaf $I^2 \subset \mathcal{O}_M$ is the square of the ideal sheaf $I \subset \mathcal{O}_M$ defining M_0. The image of $f_w^\square$ under

$$V_M|_{M_0^\square} \xrightarrow{(4.2.3)} L_M^\vee|_{M_0^\square}$$

vanishes identically along $M_0 \subset M_0^\square$, so may be viewed as a section of the coherent $\mathcal{O}_M$-module

$$\frac{I L_M^\vee}{I^2 L_M^\vee} \cong L_M^\vee \otimes I/I^2.$$

The construction sending $w \in W$ to this last section defines a morphism of $\mathcal{O}_M$-modules

$$W \otimes_F \mathcal{O}_M \to L_M^\vee \otimes I/I^2.$$

Restricting to M_0 yields the morphism

$$W_{M_0} \to L_{M_0}^\vee \otimes N_{M_0/M}^\vee,$$

which we rewrite as

$$N_{M_0/M} \to \underline{\mathrm{Hom}}(W_{M_0}, L_{M_0}^\vee).$$

By direct comparison of the constructions, one can see that this agrees with (4.5.8) in the complex fiber, and hence is an isomorphism. □

Each component $y_i \in W$ of the tuple $y \in W^d$ determines a global section of the constant vector bundle W_{M_0} on M_0. Using Lemma 4.5.3, this section determines a morphism

$$y_i : N_{M_0/M} \to L_{M_0}^\vee,$$

which we pull back via the bundle map $\pi_0 : N_{M_0/M} \to M_0$ to a morphism

$$\pi_0^* y_i : \pi_0^* N_{M_0/M} \to \pi_0^* L_{M_0}^\vee.$$

Now apply this morphism to the tautological section

$$v_0 \in H^0(N_{M_0/M}, \pi_0^* N_{M_0/M}),$$

as in (3.3.3), to obtain a global section

$$Q_i = (\pi_0^* y_i)(v_0) \in H^0(N_{M_0/M}, \pi_0^* L_{M_0}^\vee).$$

The following lemma proves the first claim of Proposition 4.5.2.

Lemma 4.5.4. *The cycle* $Z(\mathbb{C}) \subset N_{X_0/X}$ *from* (4.5.5) *is the complexification of the codimension-d cycle*

$$Z = \mathrm{div}(Q_1) \cdots \mathrm{div}(Q_d) \subset N_{M_0/M}$$

obtained by iterated proper intersection.

Proof. It suffices to prove the stated equality after pullback via each of the uniformization maps $N_{\mathcal{D}_0/\mathcal{D}} \to N_{X_0/X}$ of (4.5.2), so we work over $N_{\mathcal{D}_0/\mathcal{D}}$.

Each $y_i \in W$ determines a global section of the constant bundle $W_{\mathcal{D}}$ on $\mathcal{D}$, and hence, by the definition (4.5.7), a morphism

$$y_i : N_{\mathcal{D}} \to L_{\mathcal{D}}^{\vee}.$$

As in (3.3.2), this morphism determines a degenerating section

$$q_i = y_i(u) \in H^0(\mathcal{D}, L_{\mathcal{D}}^{\vee}),$$

where u is the section of (4.5.7) determined by (4.5.6). On the other hand, directly comparing the constructions shows that

$$q_i = s(y_i), \tag{4.5.9}$$

where the right-hand side is the section of $L_{\mathcal{D}}^{\vee}$ defined by (4.1.6). Setting $q = (q_1, \dots, q_d)$ gives the equality

$$Z_{\mathcal{D}}(y) = Z_{\mathcal{D}}(q)$$

of analytic cycles on $\mathcal{D}$, and we have now shown that

$$Z(\mathbb{C}) = \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(q)), \tag{4.5.10}$$

where the left-hand side now denotes (by abuse of notation) the pullback of $Z(\mathbb{C})$ via $N_{\mathcal{D}_0/\mathcal{D}} \to N_{X_0/X}$.

The construction (3.3.4) associates to the degenerating section q_i a section

$$\sigma_{\mathcal{D}_0/\mathcal{D}}(q_i) \in H^0(N_{\mathcal{D}_0/\mathcal{D}}, \pi_0^* L_{\mathcal{D}_0}^{\vee}),$$

and by directly comparing the constructions we have

$$Q_i = \sigma_{\mathcal{D}_0/\mathcal{D}}(q_i), \tag{4.5.11}$$

where the left-hand side denotes (by similar abuse of notation) the pullback of the complexification of Q_i via $N_{\mathcal{D}_0/\mathcal{D}} \to N_{X_0/X} \cong N_{M_0/M}(\mathbb{C})$.

By the third claim of Proposition 3.3.2, the tuple

$$\sigma_{\mathcal{D}_0/\mathcal{D}}(q) = (\sigma_{\mathcal{D}_0/\mathcal{D}}(q_1), \dots, \sigma_{\mathcal{D}_0/\mathcal{D}}(q_d)) = (Q_1, \dots, Q_d)$$

is smooth, and (4.5.10) is defined by the vanishing of its components. Thus $Z(\mathbb{C})$ is defined by the equations $Q_1 = \cdots = Q_d = 0$, so it is equal to the intersection of the divisors of $Q_1, \dots, Q_d$ by Remark 3.1.4. □

As the cycle $Z \subset N_{M_0/M}$ of Lemma 4.5.4 is presented to us as the proper intersection of the divisors of sections $Q_i \in H^0(N_{M_0/M}, \pi^* L_{M_0}^{\vee})$, it is easy to construct a Green current for it. Each divisor $\mathrm{div}(Q_i)$ has a Green current $-\log(2\pi e^{\gamma} h(Q_i))$, and the iterated star product

$$G = [-\log(2\pi e^{\gamma} h(Q_1))] \star \cdots \star [-\log(2\pi e^{\gamma} h(Q_d))] \in D^{d-1,d-1}_{N_{M_0/M}(\mathbb{C})}$$

is a Green current for Z.

This construction can be generalized. For $\beta \in \mathrm{GL}_d(\mathbb{R})$, consider the tuple

$$(Q'_1, \dots, Q'_d) = (Q_1, \dots, Q_d) \cdot \beta \in H^0(N_{M_0/M}(\mathbb{C}), \pi_0^* L^\vee_{M_0(\mathbb{C})})^d$$

of sections defined over the complex fiber (of course they will not be defined over the reflex field $F = \sigma(F)$ unless β is). Because

$$\mathrm{div}(Q'_1) \cdots \mathrm{div}(Q'_d) = Z(\mathbb{C}),$$

the iterated star product

$$G(\beta) = [-\log(2\pi e^\gamma h(Q'_1))] \star \cdots \star [-\log(2\pi e^\gamma h(Q'_d))] \in D^{d-1,d-1}_{N_{M_0/M}(\mathbb{C})}$$

is also a Green current for Z.

Lemma 4.5.5. *For any $\beta \in \mathrm{GL}_d(\mathbb{R})$, the pullback of*

$$\underbrace{\hat{\omega}_0^{-1} \cdots \hat{\omega}_0^{-1}}_{d\ times} + (0, -\log|\det(\beta)|^2 \cdot \Omega_0^{d-1}) \in \widehat{\mathrm{CH}}^d(M_0)$$

via the bundle map $\pi_0 : N_{M_0/M} \to M_0$ is represented by the arithmetic cycle

$$(Z, G(\beta)) \in \widehat{Z}^d(N_{M_0/M}).$$

Proof. By construction, (Z, G) is the arithmetic intersection of the

$$(\mathrm{div}(Q_i), -\log(h(Q_i))) + (0, -\log(2\pi e^\gamma)) \in \widehat{\mathrm{CH}}^1(N_{M_0/M})$$

as i varies over $1 \le i \le d$. Each Q_i is a section of $\pi_0^* L^\vee_{M_0}$, and so, recalling (4.3.5), each of these arithmetic divisors represents

$$\pi_0^* L^\vee_{M_0} + (0, -\log(2\pi e^\gamma)) = \pi_0^* \hat{\omega}_0^{-1}.$$

Thus

$$(Z, G) = \underbrace{\pi_0^* \hat{\omega}_0^{-1} \cdots \pi_0^* \hat{\omega}_0^{-1}}_{d \text{ times}} \in \widehat{\mathrm{CH}}^d(N_{M_0/M}),$$

and the claim is true when β is the identity matrix.

It now suffices to show that, for any $\alpha, \beta \in \mathrm{GL}_d(\mathbb{R})$, we have

$$(Z, G(\alpha\beta)) = (Z, G(\alpha)) + (0, -\log|\det(\beta)|^2 \cdot \Omega_0^{d-1})$$

in the arithmetic Chow group of $N_{M_0/M}$. If β is a permutation matrix, this follows from the usual associativity and commutativity of the star product (modulo currents of the form $\partial a + \bar{\partial} b$). The cases

$$\beta = \begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & I_{d-2} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \lambda & \\ & I_{d-1} \end{pmatrix}$$

with $\lambda \in \mathbb{R}^\times$ follow immediately from the definition of the star product. The general case follows by writing β as a product of such matrices. □

Proof of Proposition 4.5.2. The first claim of Proposition 4.5.2 follows from Lemma 4.5.4. For the second claim, factor $w = \beta \cdot {}^t\beta$ with $\beta \in \mathrm{GL}_d(\mathbb{R})$ of positive determinant. By Lemma 4.5.5, it suffices to prove the equality

$$(Z, \mathfrak{g}) = (Z, G(\beta))$$

in the arithmetic Chow group of $N_{M_0/M}$. Thus we seek currents a and b on $N_{X_0/X} = N_{M_0/M}(\mathbb{C})$ satisfying

$$\mathfrak{g} + \partial a + \bar\partial b = G(\beta).$$

To this end, we work with the pullbacks of $\mathfrak{g}$ and $G(\beta)$ via

$$N_{\mathcal{D}_0/\mathcal{D}} \to \Gamma_g \backslash N_{\mathcal{D}_0/\mathcal{D}} \xrightarrow{z \mapsto (z,g)} N_{X_0/X}$$

for a fixed $g \in G_0(\mathbb{A}_f)$, as in (4.5.1). Recall from (4.5.11) the equality

$$Q_i = \sigma_{\mathcal{D}_0/\mathcal{D}}(q_i) \in H^0(N_{\mathcal{D}_0/\mathcal{D}}, \pi_0^* L^\vee_{\mathcal{D}_0}).$$

The final claim of Proposition 3.3.5 implies that the pullback of $G(\beta)$ to $N_{\mathcal{D}_0/\mathcal{D}}$ is equal to

$$[-\log(2\pi h(\sigma_{\mathcal{D}_0/\mathcal{D}}(q'_1))] \star \cdots \star [-\log(2\pi e^\gamma h(\sigma_{\mathcal{D}_0/\mathcal{D}}(q'_d)))] = \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q'_1)) \star \cdots \star \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q'_d)),$$

where the $q'_i \in H^0(\mathcal{D}, L^\vee_{\mathcal{D}})$ are the components of the tuple $q' = q\beta$.

It now follows from the second claim of Proposition 3.3.7 that

$$G(\beta) = \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q'_1)) \star \cdots \star \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q'_d)) = \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q')) + \partial a + \bar\partial b$$

for currents

$$a = \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{a}(q')) \quad \text{and} \quad b = \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{b}(q'))$$

on $N_{\mathcal{D}_0/\mathcal{D}}$. Here, by abuse of notation, the left-hand side is the pullback of $G(\beta)$ to $N_{\mathcal{D}_0/\mathcal{D}}$. As in (4.5.9), we have the equality

$$q' = q\beta = s(y\beta)$$

of tuples of sections of $L^\vee_{\mathcal{D}}$, which implies

$$\mathfrak{g}^\circ(q') = \mathfrak{g}^\circ(s(y\beta)) \overset{(4.1.7)}{=} \mathfrak{g}^\circ_{\mathcal{D}}(y, w).$$

Putting everything together, and recalling (4.5.5), we find

$$\begin{aligned} G(\beta) &= \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ(q')) + \partial a + \bar\partial b \\ &= \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ_{\mathcal{D}}(y, w)) + \partial a + \bar\partial b = \mathfrak{g} + \partial a + \bar\partial b \end{aligned}$$

as currents on $N_{\mathcal{D}_0/\mathcal{D}}$.

The only thing left to verify is that the currents a and b are $G_0(\mathbb{Q})$-invariant, so they descend to currents on $\Gamma_g \backslash N_{\mathcal{D}_0/\mathcal{D}} \subset N_{X_0/X}$. This follows directly from their construction (3.2.9), as the components of the tuple q' from which a and b are built are $G_0(\mathbb{Q})$-invariant sections of the line bundle $L^\vee_{\mathcal{D}}$ on $\mathcal{D}$ (alternatively, one could have carried out the entirety of the proof with $\mathcal{D}_0 \subset \mathcal{D}$ replaced by $X_0 \subset X$ everywhere). □

4.6. ***Proof of Theorem 4.4.2.*** Keep the notation and assumptions of Theorem 4.4.2.

Assume for the moment that $\det(T) \neq 0$. Using the orthogonal decomposition

$$V = V_0 \oplus W,$$

each $x \in V^d$ decomposes as $x = x_0 + y$, with $x_0 \in V_0^d$ and $y \in W^d$ satisfying

$$T(x_0) + T(y) = T(x).$$

For a fixed $g \in G_0(\mathbb{A}_f)$ we may decompose (4.3.1) and (4.3.3) as

$$Z_{\mathcal{D}}(T, \varphi)_g = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) Z_{\mathcal{D}}(x_0 + y), \tag{4.6.1}$$

$$\mathfrak{g}^{\circ}_{\mathcal{D}}(T, v, \varphi)_g = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) \mathfrak{g}^{\circ}_{\mathcal{D}}(x_0 + y, v). \tag{4.6.2}$$

To compute their specializations to $N_{\mathcal{D}_0/\mathcal{D}}$, it suffices to do so for the inner summations for fixed T_0, v, and y.

This is done by reduction to the following special case. Suppose that for some $1 \leq r \leq d$ we have

$$T_0 = \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Sym}_d(F),$$

with $S_0 \in \mathrm{Sym}_r(F)$ nonsingular, and

$$v = \begin{pmatrix} v_0 & 0 \\ 0 & w \end{pmatrix} \in \mathrm{Sym}_d(\mathbb{R}),$$

with $v_0 \in \mathrm{Sym}_r(\mathbb{R})$ and $w \in \mathrm{Sym}_{d-r}(\mathbb{R})$. Let $y \in W^d$ be any tuple such that

$$\mathrm{rank}(T_0 + T(y)) = d,$$

and write $y = (y', y'')$ as the concatenation of $y' \in W^r$ and $y'' \in W^{d-r}$.

Lemma 4.6.1. *Assume V_0 is anisotropic and that*

$$\varphi_0 = \varphi_0^{(r)} \otimes \varphi_0^{(d-r)} \in S(\widehat{V}_0^r) \otimes S(\widehat{V}_0^{d-r}),$$

with both factors in the tensor product $\mathbb{Z}$-valued and K_0-fixed. For any fixed $g \in G_0(\mathbb{Q})$, we have the equalities

$$\sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(x_0 + y)) = \varphi_0^{(d-r)}(0) \cdot \pi_0^* Z_{\mathcal{D}_0}(S_0, \varphi_0, \varphi_0^{(r)})_g \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(y'')),$$

$$\sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) \cdot \sigma_{\mathcal{D}_0/D}(\mathfrak{g}^{\circ}_{\mathcal{D}}(x_0 + y, v)) = \varphi_0^{(d-r)}(0) \cdot \pi_0^* \mathfrak{g}^{\circ}_{\mathcal{D}_0}(S_0, v_0, \varphi_0^{(r)})_g \star \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^{\circ}_{\mathcal{D}}(y'', w)) - \partial A_g - \bar{\partial} B_g$$

of cycles and currents on $N_{\mathcal{D}_0/\mathcal{D}}$ for some currents A_g and B_g invariant under the action of the subgroup $\Gamma_g \subset G_0(\mathbb{Q})$ from (4.5.1). *On the right-hand side*

$$\pi_0 : N_{\mathcal{D}_0/\mathcal{D}} \to \mathcal{D}_0$$

is the bundle map, and $Z_{\mathcal{D}_0}(S_0, \varphi_0^{(r)})(\mathbb{C})_g$ *and* $\mathfrak{g}^\circ_{\mathcal{D}_0}(S_0, v_0, \varphi_0^{(r)})_g$ *are the cycle and current on* $\mathcal{D}_0$ *defined in the same way as* (4.3.1) *and* (4.3.3), *but with the Shimura datum* $(G, \mathcal{D})$ *replaced by* $(G_0, \mathcal{D}_0)$.

Proof. Given $x \in V^d$, write $x' \in V^r$ and $x'' \in V^{d-r}$ for the tuples formed from the first r and final $d-r$ components of x.

For any $x_0 \in V_0^d$ satisfying $T(x_0) = T_0$ we have $T(x_0') = S_0$ and $T(x_0'') = 0$. Hence $x_0'' = 0$ by our assumption that V_0 is anisotropic, and $x_0 + y \in V^d$ is the concatenation of $x_0' + y' \in V^r$ and $y'' \in W^{d-r} \subset V^{d-r}$. As in the discussion surrounding (4.1.6), these tuples determine tuples of sections

$$p = s(x_0' + y') \in H^0(\mathcal{D}, L_{\mathcal{D}}^\vee)^r \quad \text{and} \quad q = s(y'') \in H^0(\mathcal{D}, L_{\mathcal{D}}^\vee)^{d-r},$$

whose concatenation is $(p, q) = s(x_0 + y)$. These satisfy the assumptions imposed in Section 3.3. More precisely:

(1) The restriction $p|_{\mathcal{D}_0} = s(x_0') \in H^0(\mathcal{D}_0, L_{\mathcal{D}_0}^\vee)^r$ is the tuple of sections formed from $x_0' \in V_0^r$. As $T(x_0') = S_0$ is nonsingular, this restriction is again smooth, and

$$Z_{\mathcal{D}_0}(p|_{\mathcal{D}_0}) = Z_{\mathcal{D}_0}(x_0').$$

(2) As explained in the discussion surrounding (4.5.9), the components of q are degenerating sections in the sense of Section 3.3.

Thus Proposition 3.3.2 applies, and shows that

$$\sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(x_0 + y)) = \pi_0^* Z_{\mathcal{D}_0}(x_0') \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(y'')).$$

Now sum both sides of this last equality over all $x_0 \in V_0^d$ with $T(x_0) = T_0$. As $x_0'' = 0$ for every such x_0, that sum can be replaced by the sum over all $x_0' \in V_0^r$ satisfying $T(x_0') = S_0$. The result is

$$\sum_{\substack{x_0 \in V_0^d \\ T(x_0)=T_0}} \varphi_0(g^{-1}x_0) \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(x_0 + y)) = \sum_{\substack{x_0' \in V_0^r \\ T(x_0)=S_0}} \varphi_0^{(r)}(g^{-1}x_0')\varphi_0^{(d-r)}(0) \cdot \pi_0^* Z_{\mathcal{D}_0}(x_0') \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(y'')),$$

proving the first claim of the proposition.

For the specialization of Green forms, choose the $\alpha \in \mathrm{GL}_d(\mathbb{R})$ of (4.1.7) in the block diagonal form

$$\alpha = \begin{pmatrix} \alpha_0 & \\ & \beta \end{pmatrix},$$

with $\alpha_0 \in \mathrm{Sym}_r(\mathbb{R})$ and $\beta \in \mathrm{Sym}_{d-r}(\mathbb{R})$ of positive determinant. By definition, $\mathfrak{g}^\circ_{\mathcal{D}}(x_0 + y, v)$ is the Green current associated to the tuple of sections

$$s(x_0\alpha + y\alpha) \in H^0(\mathcal{D}, L_{\mathcal{D}}^\vee)^d.$$

Writing this as the concatenation of $p\alpha_0$ and $q\beta$, Propositions 3.3.5 and 3.3.7 imply

$$\sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ_{\mathcal{D}}(x_0+y, v)) = \pi_0^* \mathfrak{g}^\circ_{\mathcal{D}_0}(x_0', v_0) \star \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ_{\mathcal{D}}(y'', w)) - \partial \sigma_{\mathcal{D}_0/\mathcal{D}}(A(p\alpha_0; q\beta)) - \bar{\partial} \sigma_{\mathcal{D}_0/\mathcal{D}}(B(p\alpha_0; q\beta)).$$

As above, summing both sides over all $x_0 \in V_0^d$, with $T(x_0) = T_0$, proves the second claim of the proposition. □

The proof of Theorem 4.4.2 will now proceed in two steps; first assuming $\det(T) \neq 0$, and then without this assumption.

Proof of Theorem 4.4.2: *the nonsingular case.* Assume $\det(T) \neq 0$, so that

$$\widehat{C}_M(T, v, \varphi) = \widehat{Z}_M(T, v, \varphi) = (Z_M(T, v, \varphi), \mathfrak{g}^\circ_M(T, v, \varphi)).$$

For a given $T_0 \in \mathrm{Sym}_d(F)$ and $y \in W^d$ satisfying $T_0 + T(y) = T$, abbreviate $r = \mathrm{rank}(T_0)$, and choose $a \in \mathrm{GL}_d(F)$ in such a way that the matrices

$$^aT_0 = {}^t a T_0 a, \quad {}^a v = \sigma(a^{-1}) v \sigma({}^t a^{-1})$$

have the form

$$^aT_0 = \begin{pmatrix} S_0 & \\ & 0_{d-r} \end{pmatrix}, \quad {}^a v = {}^t\theta \cdot \begin{pmatrix} v_0 & \\ & w \end{pmatrix} \cdot \theta \tag{4.6.3}$$

of part (4) of Theorem 4.3.3, with $\det(S_0) \neq 0$. Decompose

$$^a\varphi_0 = \sum_i \Phi_i^{(r)} \otimes \Phi_i^{(d-r)} \in S(\widehat{V}_0^r) \otimes S(\widehat{V}_0^{d-r})$$

as a sum of pure tensors, with all Schwartz functions appearing here $\mathbb{Z}$-valued and K_0-fixed.

Fix a $g \in G_0(\mathbb{A}_f)$, and let $\Gamma_g \subset G_0(\mathbb{Q})$ be the subgroup from (4.5.1). It follows from $T(x_0 a) = {}^aT(x_0)$ that

$$\sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) Z_{\mathcal{D}}(x_0 + y) = \sum_{\substack{x_0 \in V_0^d \\ T(x_0) = {}^aT_0}} {}^a\varphi_0(g^{-1}x_0) Z_{\mathcal{D}}(x_0 + ya)$$

as Γ_g-invariant cycles on $\mathcal{D}$. Specializing both sides to $N_{\mathcal{D}_0/\mathcal{D}}$ and using Lemma 4.6.1 yields the equality

$$\sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(x_0 + y)) = \sum_i \Phi_i^{(d-r)}(0) \pi_0^* Z_{\mathcal{D}_0}(S_0, \Phi_i^{(r)})_g \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(\bar{y})) \tag{4.6.4}$$

of Γ_g-invariant cycles on $N_{\mathcal{D}_0/\mathcal{D}}$, where

$$\bar{y} = (\bar{y}_1, \dots, \bar{y}_{d-r}) \in W^{d-r}$$

consists of the final $d - r$ components of ya. The same reasoning shows that

$$\begin{aligned} &\sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0) \sigma_{\mathcal{D}_0/D}(\mathfrak{g}^\circ_{\mathcal{D}}(x_0 + y, v)) \\ &\qquad = \sum_i \Phi_i^{(d-r)}(0) \pi_0^* \mathfrak{g}^\circ_{\mathcal{D}_0}(S_0, v_0, \Phi_i^{(r)}) \star \sigma_{\mathcal{D}_0/\mathcal{D}}(\mathfrak{g}^\circ_{\mathcal{D}}(\bar{y}, w)) - \partial A - \bar{\partial} B, \end{aligned} \tag{4.6.5}$$

where the Γ_g-invariant currents A and B on $N_{\mathcal{D}_0/\mathcal{D}}$ depend on T_0, y, and the choice of a.

Now sum both sides of the equality (4.6.4) over all $T_0 \in \mathrm{Sym}_d(F)$ and $y \in W^d$ for which $T_0 + T(y) = T$ to obtain

$$\sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(T,\varphi)_g) \overset{(4.6.1)}{=} \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_{\substack{x_0 \in V_0^d \\ T(x_0) = T_0}} \varphi_0(g^{-1}x_0)\sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(x_0 + y))$$
$$\overset{(4.6.4)}{=} \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_i \Phi_i^{(d-r)}(0)\pi_0^* Z_{\mathcal{D}_0}(S_0, \Phi_i^{(r)})_g \cdot \sigma_{\mathcal{D}_0/\mathcal{D}}(Z_{\mathcal{D}}(\bar{y})).$$

Note that in the inner sum the data S_0, $\bar{y}$, $r = \mathrm{rank}(T_0)$, and the Schwartz functions Φ_i all depend on T_0, y, and a choice of $a \in \mathrm{SL}_d(F)$ as in (4.6.3). These are equalities of Γ_g-invariant analytic cycles on $N_{\mathcal{D}_0/\mathcal{D}}$. By descending to $\Gamma_g \backslash N_{\mathcal{D}_0/\mathcal{D}} \subset N_{M_0/M}(\mathbb{C})$ and then varying $g \in G_0(\mathbb{A}_f)$, we deduce the analogous equality

$$\sigma_{M_0/M}(Z_M(T,\varphi)) = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_i \Phi_i^{(d-r)}(0)\pi_0^* Z_{M_0}(S_0, \Phi_i^{(r)}) \cdot \sigma_{X_0/X}(Z_X(\bar{y}))$$

of cycles on $N_{M_0/M}$. Here $\sigma_{X_0/X}(Z_X(\bar{y}))$ is the specialization to the normal bundle

$$N_{X_0/X} \cong N_{M_0/M}(\mathbb{C}) \tag{4.6.6}$$

of the analytic cycle $Z_X(\bar{y}) \subset X$ associated to $\bar{y} \in W^{d-r}$ as in (4.5.4). It is algebraic and defined over the reflex field by Proposition 4.5.2.

The same reasoning, using (4.6.2) and (4.6.5) in place of (4.6.1) and (4.6.4), gives the equality of currents

$$\sigma_{M_0/M}(\mathfrak{g}_M^\circ(T,v,\varphi)) = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_i \Phi_i^{(d-r)}(0)\pi_0^* \mathfrak{g}_{M_0}^\circ(S_0, v_0, \Phi_i^{(r)}) \star \sigma_{X_0/X}(\mathfrak{g}_X^\circ(\bar{y}, w))$$

on (4.6.6), modulo currents of the form ∂A and $\bar{\partial} B$. Here $\sigma_{X_0/X}(\mathfrak{g}_X^\circ(\bar{y}, w))$ is the specialization to (4.6.6) of the Green current $\mathfrak{g}_X^\circ(\bar{y}, w)$ associated to $\bar{y} \in W^{d-r}$ and $w \in \mathrm{Sym}^{d-r}(\mathbb{R})$ as in (4.5.4). As in the previous paragraph, in the inner sum the data S_0, $\bar{y}$, $r = \mathrm{rank}(T_0)$, v_0, w, and the Schwartz functions Φ_i all depend on T_0, y, and a choice of $a \in \mathrm{SL}_d(F)$ as in (4.6.3).

Passing to the arithmetic Chow group of $N_{M_0/M}$, the above equalities show that

$$\sigma_{M_0/M}(\widehat{Z}_M(T,v,\varphi)) = \sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \sum_i \pi_0^* \widehat{Z}_{M_0}(S_0, v_0, \Phi_i^{(r)}) \cdot (Z_i, \mathfrak{g}_i), \tag{4.6.7}$$

where each arithmetic cycle

$$(Z_i, \mathfrak{g}_i) = \Phi_i^{(d-r)}(0)\big(\sigma_{X_0/X}(Z_X(\bar{y})), \sigma_{X_0/X}(\mathfrak{g}_X^\circ(\bar{y}, w))\big)$$

in the sum depends on T_0, y, and a choice of $a \in \mathrm{SL}_d(F)$ as in (4.6.3).

Loosely speaking, the above decomposition (4.6.7) separates the parts of the specialization to the normal bundle that arise from proper intersection between $Z_M(T,\varphi)$ and M_0 from those parts that arise from improper intersection, with the improper parts corresponding to the various $(Z_i, \mathfrak{g}_i)$.

We now come to the central point of the proof: Proposition 4.5.2 tells us that each $(Z_i, \mathfrak{g}_i)$ is equal to the pullback via $\pi_0 : N_{M_0/M} \to M_0$ of the arithmetic cycle class

$$\widehat{C}_{M_0}(0_{d-r}, w, \Phi_i^{(d-r)}) = \Phi_i^{(d-r)}(0) \cdot [\underbrace{\hat{\omega}_0^{-1} \cdots \hat{\omega}_0^{-1}}_{d-r \text{ times}} + (0, -\log(\det(w)) \cdot \Omega_0^{d-r-1})]$$

of Proposition 4.3.5, where $r = \operatorname{rank}(T_0)$. Hence the inner sum in (4.6.7) simplifies to

$$\begin{aligned}\sum_i \pi_0^* \widehat{Z}_{M_0}(S_0, v_0, \Phi_i^{(r)}) \cdot (Z_i, \mathfrak{g}_i) &= \sum_i \pi_0^* \widehat{C}_{M_0}(S_0, v_0, \Phi_i^{(r)}) \cdot \pi_0^* \widehat{C}_{M_0}(0_{d-r}, w, \Phi_i^{(d-r)}) \\ &= \pi_0^* \widehat{C}_{M_0}({}^a T_0, {}^a v, {}^a \varphi_0) = \pi_0^* \widehat{C}_{M_0}(T_0, v, \varphi_0).\end{aligned}$$

Plugging this back into (4.6.7) completes the proof of Theorem 4.4.2 when $\det(T) \neq 0$. □

Proof of Theorem 4.4.2: the general case. Now let $T \in \operatorname{Sym}_d(F)$ be arbitrary, and set $r = \operatorname{rank}(T)$. Using Remark 4.3.4 and Proposition 4.3.5, one immediately reduces to the case in which

$$T = \begin{pmatrix} S & \\ & 0_{d-r} \end{pmatrix} \quad \text{and} \quad v = {}^t\theta \cdot \begin{pmatrix} v_0 & \\ & w \end{pmatrix} \cdot \theta$$

as in part (4) of Theorem 4.3.3, with $S \in \operatorname{Sym}_r(F)$ nonsingular. We may also assume that the factors in $\varphi = \varphi_0 \otimes \psi$ admit further factorizations

$$\begin{aligned}\varphi_0 &= \varphi_0^{(r)} \otimes \varphi_0^{(d-r)} \in S(\widehat{V}_0^r) \otimes S(\widehat{V}_0^{d-r}), \\ \psi &= \psi^{(r)} \otimes \psi^{(d-r)} \in S(\widehat{W}^r) \otimes S(\widehat{W}^{d-r}),\end{aligned}$$

so that Proposition 4.3.5 implies

$$\widehat{C}_M(T, v, \varphi) = \widehat{C}_M(S, v_0, \varphi^{(r)}) \cdot \widehat{C}_M(0_{d-r}, w, \varphi^{(d-r)}), \tag{4.6.8}$$

with $\varphi^{(r)} = \varphi_0^{(r)} \otimes \psi^{(r)}$, and similarly with r replaced by $d-r$.

It is clear from the definitions that pullback via $i_0 : M_0 \to M$ satisfies

$$i_0^* \hat{\omega}^{-1} = \hat{\omega}_0^{-1},$$

and so Proposition 4.3.5 and Theorem 2.3.1 imply

$$\sigma_{M_0/M}(\widehat{C}_M(0_{d-r}, w, \varphi^{(d-r)})) = \psi^{(d-r)}(0) \cdot \pi_0^* \widehat{C}_{M_0}(0_{d-r}, r, \varphi_0^{(d-r)}).$$

We have already proved Theorem 4.4.2 for the nonsingular matrix S, so

$$\sigma_{M_0/M}(\widehat{C}_M(S, v_0, \varphi^{(r)})) = \sum_{\substack{S_0 \in \operatorname{Sym}_r(F) \\ y \in W^r,\, S_0 + T(y) = S}} \psi^{(r)}(y) \cdot \pi_0^* \widehat{C}_{M_0}(S_0, v_0, \varphi_0^{(r)}).$$

Specialization to the normal bundle commutes with arithmetic intersection (this is immediate from Theorem 2.3.1 and the fact that pullbacks commute with arithmetic intersection), and so the specialization of (4.6.8) is equal to the pullback via $\pi_0 : N_{M_0/M} \to M_0$ of

$$\sum_{\substack{S_0 \in \mathrm{Sym}_r(F) \\ y \in W^r,\, S_0 + T(y) = S}} \psi^{(r)}(y) \cdot \psi^{(d-r)}(0) \cdot \widehat{C}_{M_0}(S_0, v_0, \varphi_0^{(r)}) \cdot \widehat{C}_{M_0}(0_{d-r}, w, \varphi_0^{(d-r)}). \tag{4.6.9}$$

To complete the proof, we must show that (4.6.9) is equal to

$$\sum_{\substack{T_0 \in \mathrm{Sym}_d(F) \\ y \in W^d,\, T_0 + T(y) = T}} \psi(y) \cdot \widehat{C}_{M_0}(T_0, v, \varphi_0). \tag{4.6.10}$$

If the (T_0, y)-term in (4.6.10) is nonzero then, by Remark 4.3.6, there is an $x \in V_0^d$ such that $T(x) = T_0$. The tuple $x + y \in V^d$ then satisfies $T(x + y) = T$, and so its i-th component is isotropic for $r < i \le d$. As we have assumed that V is anisotropic, we deduce that y has the form

$$y = (y_1, \ldots, y_r, 0, \ldots, 0).$$

It then follows from $T_0 + T(y) = T$ that

$$T_0 = \begin{pmatrix} S_0 & \\ & 0_{d-r} \end{pmatrix}$$

for some $S_0 \in \mathrm{Sym}_r(F)$, and we know from Proposition 4.3.5 that

$$\widehat{C}_{M_0}(T_0, v, \varphi_0) = \widehat{C}_{M_0}(S_0, v_0, \varphi_0^{(r)}) \cdot \widehat{C}_{M_0}(0_{d-r}, w, \varphi_0^{(d-r)})).$$

Thus in (4.6.10) we may replace the sum over $T_0 \in \mathrm{Sym}_d(F)$ with a sum over $S_0 \in \mathrm{Sym}_r(F)$, replace the sum over $y \in W^d$ with a sum over $y \in W^r$, and the result is (4.6.9). □

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