

RECEIVED: October 10, 2024

ACCEPTED: October 23, 2024

PUBLISHED: December 3, 2024

Gauge-fixing local H symmetry in supergravities

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ABSTRACT: We discuss known maximal D -dimensional supergravities of two types: type I with G/H coset spaces and type II derived by compactification from higher dimensions without dualization, these have less manifest symmetries. In $4D$ and $6D$ in type I models we perform explicit gauge-fixing of local H symmetries in unitary gauges: symmetric, Iwasawa and partial Iwasawa. In $4D$ supergravity I in symmetric gauge global H -invariance and nonlinearly realized G -symmetry are valid on shell, classically. The global H -symmetry and G -symmetry in Iwasawa-type gauges in type I and in type II supergravities are not manifest, if at all present. This fact raises the issue of the gauge equivalence of the S-matrix of various gauge-fixed D -dimensional supergravities and its relation to the ones computable using superamplitude methods.

KEYWORDS: Extended Supersymmetry, Global Symmetries, Supergravity Models

ARXIV EPRINT: [2409.18950](https://arxiv.org/abs/2409.18950)

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1 Introduction

Consider well known (ungauged) maximal supergravity theories in $4D$ and $6D$. They contain physical scalar fields which belong to the coset space G/H . Here G is a non compact group, while H is the maximal compact subgroup of G :

$$\frac{G}{H}\Big|_4 = \frac{E_{7(7)}}{\mathrm{SU}(8)}, \quad \frac{G}{H}\Big|_6 = \frac{E_{5(5)}}{\mathrm{USp}(4) \times \mathrm{USp}(4)}. \quad (1.1)$$

In the original versions of ungauged supergravities in [1–4] the number of scalars is defined by the fundamental representation of a group G , and there is also a local H symmetry, which is the R -symmetry group. We will refer to these supergravities as supergravities of type I. In all integer dimensions D these supergravities of type I have physical scalars in the coset space $(G/H)_D$ and they have the corresponding global G_D and local H_D symmetries.

D -dimensional supergravities of the type I have global U-duality symmetry G_D . These are groups $E_{11-D}(11-D)$ [5, 6], they are often called E_{11-D} groups. They are nicely explained by B. Julia [5] in terms of Dynkin diagrams as the process of Group Disintegration. Starting from $E_{8(8)}$ in $3D$ removing the right node of the Dynkin diagram, one by one, one gets $E_{7(7)}$ in $4D$, $E_{6(6)}$ in $5D$, $E_{5(5)}$ in $6D$, and all the way up.

The local H symmetry in [1–4] has unusual features. In standard local gauge symmetries there are propagating gauge fields, but in local H symmetries the role of the gauge field is played by the composite scalar dependent connection. Some of the advantages in keeping local H symmetry not gauge fixed is that the global duality symmetry G and the local H symmetry are independent, and fermions transform under H and are neutral in G .

The action with local H symmetry depends on scalars which are in the adjoint representation of G , for example 133 in maximal $4D$ supergravity and 45 in $6D$. These scalars parametrize a G -valued matrix $\mathcal{V}(x)$ which transforms by G from the left and by H from the right. When local H symmetry in supergravities is gauge-fixed, only physical scalars remain, their number is reduced to the number of coordinates in the coset space G/H , 70 in $4D$ and 25 in $6D$. In symmetric gauges fermions in gauge-fixed theory transform under G symmetry due to a compensating H symmetry transformation, preserving the choice of the gauge. In $4D$ case symmetric gauge in [1, 2] supergravity was studied in detail in [1, 2, 7, 8], where it was shown that the global H -symmetry is valid on shell.

Thus, it is convenient to refer to original versions of ungauged supergravities in D -dimensions, with global G and local H symmetries, as supergravities I. We will refer to supergravities derived by compactification from $(D+n)$ dimensional supergravities, *without dualization*, as supergravities II. These are less known, in general, however, these models have played an important role in studies of black hole attractors in maximal $4D$ supergravity, see for example [9, 11–14]. Specifically, $1/2, 1/2, 1/2$ extremal BPS black holes are associated with type I supergravity, but the non-BPS Kaluza-Klein extremal black holes are associated with type II supergravity [13, 14].

We will mostly focus on the case of $n=1$ when D -dimensional supergravity is derived from the $D+1$ supergravity compactified on a circle. In $4D$ these type II supergravity are models in [13, 14] and in $6D$ in [15].

Our study of gauge-fixing H -symmetry in supergravities is based on the original work in $4D$ and in $6D$ in [1–4] as well as on supergravity studies in diverse dimensions in [16–24]. We will define classes of gauges we are interested in various dimensions, and we will fill in some gaps in the existing literature, for example in gauge-fixing of $6D$ maximal supergravity, as well as with regard to Iwasawa-type gauges in various dimensions.

Unitary gauges in the context of quantum field theory have the property that the gauge fixing function depends on the fields and not on their derivatives. Therefore there are no Faddeev-Popov propagating ghost fields.

Unitary *symmetric gauges* are well known in $4D$, but not in $6D$. In maximal $4D$ supergravity after gauge-fixing local $H = \text{SU}(8)$ symmetry [1, 2], there is a remaining nonlinearly realized $G = E_{7(7)}$ and a field-dependent compensating $H = \text{SU}(8)$ symmetry. The physical scalars $\phi_{ijkl}, \bar{\phi}^{pqmn}$ transform in a linear representation of $\text{SU}(8)$. All dependence on physical scalars is non-polynomial. The 1-loop anomalies of global $\text{SU}(8)$ -symmetry in this gauge cancel [25].

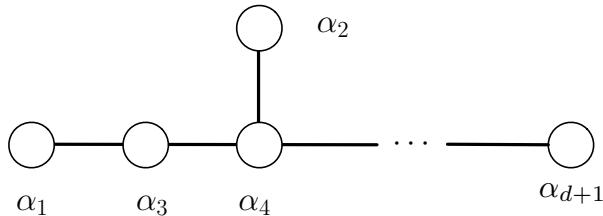


Figure 1. The Dynkin diagrams of U-duality groups $E_{d+1(d+1)}$ as given in [28], $0 \leq d \leq 7$, $D = 10 - d$. The groups $E_{d+1(d+1)}$ are often called E_{d+1} .

Unitary *Iwasawa gauges*, also called triangular gauges, in supergravities were described already in [1]. These gauges have a remarkable polynomiality in some of the scalars, which is absent in symmetric gauges. These are associated in case of maximal supergravities with dimensional reduction from $11D$ supergravity. In these gauges there is a maximal number of axionic scalars, which enter the action polynomially.

Unitary *partial Iwasawa gauges* we define here as the ones where D -dimensional theories are associated with dimensionally reduced maximal $(D+1)$ supergravities. They have the feature that the $D+1$ coset $(G/H)_{D+1}$ inherited in dimension D was gauge-fixed in a symmetric gauge in $D+1$. The number of axionic polynomial scalars in these gauges is always non-zero, but less than the one in Iwasawa triangular gauge.

Supergravity actions depend on scalars via the vielbein $\mathcal{V}(x)$. The vielbein transforms under global G symmetry and local H -symmetry

$$\mathcal{V}(x) \rightarrow \mathbf{g} \mathcal{V}(x) h^{-1}(x). \quad (1.2)$$

Before gauge-fixing the vielbein is in a fundamental representation of G , the number of scalars is $\dim[G]$. After gauge-fixing

$$\mathcal{V}(x) \rightarrow \mathcal{V}(x)_{\text{gf}}. \quad (1.3)$$

It is a matrix depending only on physical scalars, where the number of physical scalars n_{sc} is equal to $\dim[G] - \dim[H]$.

Different choices of coset representatives define different choices of gauges of local H symmetry. Various solutions of underlying mathematical problem to find G/H coset space representatives can be found in supergravity original papers [1–4] and reviews in [17–24] and in the textbooks like [26, 27].

Consider figure 1 here for Dynkin diagrams of U-duality groups as presented in [28]. Different gauges for local H -symmetry correspond to different ways of cutting some nodes in these Dynkin diagrams, thereby breaking global G -symmetry of the classical action in a class of Iwasawa type gauges.

The purpose of this paper is to provide a systematic gauge-fixing of local H -symmetries in D -dimensional supergravities in a class of gauges which either preserve global H -symmetry and G -symmetry, in symmetric gauges, or not (at least not manifestly), in Iwasawa-type gauges associated with $D+1$ supergravities. These various choices of gauge-fixing can be studied in the future to understand how the existence of different unitary gauges might affect

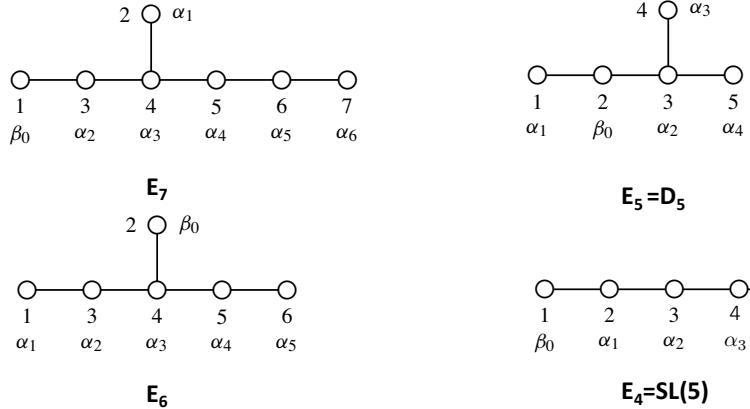


Figure 2. Left panel at the top shows Dynkin diagram for E_7 , the one on the bottom is E_6 . Right panel at the top shows Dynkin diagram for E_5 , the one on the bottom is $E_4 = \text{SL}(5)$. Figures taken from [29] where a detailed explanation can be found. Deleting the right node in E_7 we see E_6 and deleting the right node in $D_5 = E_{5(5)}$ we see $\text{SL}(5) = E_{4(4)}$.

the quantization of supergravities, the issue of local H -symmetry and global G -symmetry anomalies and UV divergences.

In particular, the 1-loop anomalies of global H -symmetry were computed in [25] in a symmetric gauge, where global H -symmetry is present in a classical action and where the vielbein is assumed to take a form

$$\mathcal{V}_{\text{gf}} = e^{\phi \cdot \mathbb{K}}. \quad (1.4)$$

Here \mathbb{K} are generators in a noncompact part of the algebra of G , which is a property of symmetric gauges. In other gauges the status of a global H -symmetry is not obvious, unless one can prove the on shell gauge independence of these theories. Therefore the issue of 1-loop anomalies in supergravity with account of symmetric and Iwasawa-type gauges requires an additional investigation.

2 D -dimensional supergravity I and supergravity II

Supergravities I were constructed in dimension D , the physical scalars are in $(G/H)_D$ coset space. These models have local H -symmetry and global on shell G symmetry. This local H symmetry can be gauge-fixed in various gauges. In symmetric gauges there is a non-polynomial dependence on all physical scalars and on shell $E_{11-D}(11-D)$ symmetry. Upon gauge-fixing the global G symmetry is still preserved but non-linearly realized. In Iwasawa and partial Iwasawa gauges there is a polynomial dependence on some of the physical scalars, often called axions, and non-polynomial dependence on other physical scalars, often called dilatons.

In Iwasawa type gauges the status of a global H_D symmetry and on shell G_D -symmetry is not known, a priori. The symmetries, like $E_{7(7)}$ in $4D$ and $E_{5(5)}$ symmetry in $6D$ are broken off shell, by construction. This is best explained using the Dynkin diagrams in figure 1 and figure 2.

We will use two of the parabolic subgroups of the U-duality groups described in [28]. The first one is the subgroup $P_{\alpha_{d+1}}$ obtained by removing the root α_{d+1} associated with the

most right last node $d + 1$ of the Dynkin diagram in figure 1. These gauges in D -dimensional supergravity are related to actions derived by compactification from $D + 1$ supergravity in accordance with the Group Disintegration [5]. The second subgroup P_{α_2} is obtained by removing the root α_2 associated to the node 2. These are triangular, Iwasawa unitary gauges for local H -symmetry. They were explored in [19] for D -dimensional supergravities derived by compactification of $11D$ supergravity on a torus T^{d+1} , $d = 10 - D$.

In figure 2 we show examples of Iwasawa gauges where right node was removed from the Dynkin diagram: in $4D$ case this describes the choice of the gauge in type I supergravity where $E_{7(7)}$ group is broken down to $E_{6(6)}$. In $6D$ the $E_{5(5)}$ duality is broken to $E_{4(4)}$.

Supergravities II in dimension D which we focus on are related to $(D + 1)$ -dimensional supergravity. They were derived from $D + 1$ supergravities by a reduction on a circle, or, sometimes were constructed directly in dimension D . These correspond to cases of parabolic subgroups $P_{\alpha_{d+1}}$ of $E_{d+1(d+1)}$ group with removal of the right node of a Dynkin diagram in figure 1. Their local symmetry is smaller, it is at most H_{D+1} -symmetry inherited from $D + 1$. This local H_{D+1} -symmetry can be gauge fixed already in $D + 1$, before dimensional reduction, or after, see for example [13, 15].

Supergravities II are also given by their actions in dimension D and they also have the same amount of maximal local supersymmetries as supergravities I. These models in general case were described in [18]. The physical scalars correspond to a decomposition of the $(G/H)_D$ coset space via the smaller coset space of $D + 1$ supergravity $\frac{G_{D+1}}{H_{D+1}}$, the radius of the circle, r_{D+1} and the scalars originating from the extra component of $D + 1$ vectors \mathbf{V}_r^{D+1} .

$$\frac{G_D}{H_D} \sim \left(\frac{G_{D+1}}{H_{D+1}}, r_{D+1}, \mathbf{V}_r^{D+1} \right). \quad (2.1)$$

It means that supergravities II always have some axions, as opposite to supergravities I where there is a class of symmetric gauges without axions. In supergravities II the minimal value of the axions is reached in a gauge where the local H_{D+1} symmetry is gauge-fixed in a symmetric gauge.

In supergravities I the number of axions depends on the gauge chosen to fix the local H -symmetry. It varies between zero and maximum in Iwasawa triangular gauge.

1. In symmetric gauge, there are no polynomial axions. All scalars are non-polynomial:

$$\#\text{axions}_{\text{sym}} = 0 \quad (2.2)$$

2. The Iwasawa triangular gauges are closely related to the supergravity actions derived by dimensional reduction from $11D$ supergravity. They involve a Borel subalgebra of the algebra of the group G , or they can also be described in terms of the solvable Lie group and using the solvable parametrization of the coset space G/H [22, 23]. The number of axions in these gauges in dimension D is defined by the properties of the U-duality group $G = E_{11-D}(11-D)$ in a given dimension D

$$\#\text{axions}_{\text{triangular}} = \frac{1}{2}(\dim G - \text{rank } G) \quad (2.3)$$

3. In gauges we call partial Iwasawa gauges related to $D + 1$ supergravities the scalars are defined by the decomposition of the coset space $(G/H)_D$ in eq. (2.1). In these

gauges of supergravity I, as in supergravities II, there are axions related to compactified vectors \mathbf{V}_r^{D+1} presenting the abelian ideal in D dimensions. The number of axions is the number of vector fields in one dimension above, it also has a group theoretic interpretation as a maximal abelian ideal [18]

$$\#\text{axions}_{\text{partial Iwasawa}} = \dim \mathcal{A}_D \quad (2.4)$$

This is the case when the relevant H -symmetry in $D+1$ theory was gauge-fixed in symmetric gauge. However, when the gauge in $D+1$ supergravity was an Iwasawa gauge, the number of axions is exceeding the ones in partial Iwasawa gauge.

To summarize, we find that in supergravity I there are gauges which have the same set of scalars as supergravity II, derived from $D+1$ supergravity by compactification on a circle. If in supergravity II a local H_{D+1} symmetry was gauge-fixed in Iwasawa gauge, the corresponding supergravity I needs to be gauge-fixed in the appropriate Iwasawa gauge. If in supergravity II a local H_{D+1} symmetry was gauge-fixed in a symmetric gauge, supergravity I has to be gauge-fixed in an appropriate partial Iwasawa gauge, to reproduce the same set of scalars. In general, the relation between vectors in supergravities I and II in various gauges is not as simple.

Thus, using different gauges in supergravities I we can study both supergravities I as well as supergravities II. The minimal number of axions in supergravities II is the same as in supergravities I in the partial Iwasawa gauge and always positive:

$$\#\text{axions}_{\text{partial Iwasawa}} = \#\text{axions}_{\text{supergravity II}} = \dim \mathcal{A}_D > 0. \quad (2.5)$$

This is opposite to symmetric gauges of supergravity I

$$\#\text{axions}_{\text{sym}} = \#\text{axions}_{\text{supergravity I}} = 0. \quad (2.6)$$

The mere existence of supergravities II in [13] and in [15] as well as the fact that there are gauges in supergravities I where scalars are the same as in supergravities II, suggest that these issues might be relevant to quantum properties of maximal supergravities.

In supergravity I the G -valued matrix, a vielbein $\mathcal{V}(x)$, transforms by G from the left and by H from the right. Before local H -symmetry is gauge-fixed, these are independent transformations

$$\mathcal{V}(x) \rightarrow \mathbf{g} \mathcal{V}(x), \quad \mathcal{V}(x) \rightarrow \mathcal{V}(x) h^{-1}(x), \quad \mathbf{g} \in G, \quad h(x) \in H. \quad (2.7)$$

The Lie algebra \mathfrak{g} of a group G can be decomposed into two orthogonal subspaces: the Lie algebra \mathfrak{h} of a group H and a coset space \mathfrak{k} . Here H is the maximal compact group in G :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}; \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}; \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \quad (2.8)$$

The supergravity reviews which we will use here have the following features. In [20–22] one starts with ungauged supergravity with a local H symmetry and global G symmetry, both independent and linearly realized. The action depends on scalars which form a fundamental representation of G . These scalars parametrize a G -valued matrix $\mathcal{V}(x)$ which transforms

by G and by H as shown in eq. (2.7). Gauge-fixing local H symmetry in unitary gauges requires to use all parameters of local H symmetry, like 63 in $H = \text{SU}(8)$ in $4D$, or 20 in $H = \text{SO}(5) \times \text{SO}(5)$ in $6D$, to bring the theory to the form where it depends only on physical scalars. Their number is $\dim[G] - \dim[H]$, i.e. in maximal $4D$ we have $133 - 63 = 70$ and in $6D$ $45 - 20 = 25$ physical scalars.

Our purpose is, in particular, to compare supergravity actions derived in a given dimension $D = 4, 6$ versus the ones obtained via dimensional reduction from $D + 1 = 5, 7$. For this purpose it is convenient to study all theories in dimension D at the stage where local H symmetry was not yet gauge-fixed. It means that we study supergravities I and gauge-fix them in various gauges.

In such case¹ we define these gauges in terms of the G -valued vielbein matrix $\mathcal{V}(x)$. The corresponding gauge-fixings which eliminate unphysical scalars correspond to certain choices of the vielbein \mathcal{V} : for example, one gauge is a symmetric one, the other is an Iwasawa gauge, or a partial Iwasawa gauge. All of these are unitary gauges, in a sense that they have only physical scalars and do not require propagating ghost fields.

3 Symmetric, Iwasawa and partial Iwasawa supergravity I unitary gauges

In D -dimensional supergravities with a local H symmetry, i.e. in supergravities I, one can make a choice of the vielbeins/of the coset representatives of a different kind. We discuss possible choices of unitary gauges starting with the well known *symmetric gauges* [1, 2, 30].

There are also gauges suitable for addressing the relation between D -dimensional supergravity with $(G/H)_D$ coset space with the one in D derived from higher dimensions, from $11d$ or from $D + 1$, for example. These gauges are, in general, different. But all of them have some axionic scalars interacting polynomially.

Iwasawa triangular gauges are also well known, they represent $11D$ supergravity dimensionally reduced to D -dimension, see [1, 19, 22]. There is also a class of gauges available in the literature, for example in [30], where D -dimensional supergravity was derived from a $D + 1$ -dimensional supergravity by compactification on a circle. If the relevant H_{D+1} -symmetry in $D + 1$ theory was gauge-fixed in a symmetric gauge, this theory can be related to D -dimensional supergravity in a class of gauges which we call *partially Iwasawa gauges*.

The interest to parabolic subgroups of the U-duality groups in string theory and supergravity was raised in the context of automorphism analysis in [31, 32]. It was shown that maximal parabolic subgroups of $E_{d(d)}$ (often called E_d) arising in string theory are of the form $\text{GL}(1) \times X_{d-1}$. For example, for $E_{7(7)}$ the X_6 is $E_{6(6)}$ and for $E_{5(5)}$ the X_5 is $\text{SL}(5)$. This is convenient to see in cases interesting for us using Dynkin diagrams in figure 2 here, which we took from [29]. In particular, after deleting the right node E_7 becomes E_6 and $D_5 = E_{5(5)}$ becomes $\text{SL}(5) = A_4 \sim E_4$, see also table 2 in [32].

¹In [17, 23] the stage of an ungauged supergravity with a local H symmetry is replaced from the beginning by various choices of coset space representatives $L(\phi^r)$. Namely, the coset representatives depend only on physical scalars ϕ^r , $r = 1, \dots, n_{\text{sc}}$. There is no local H symmetry independent on G symmetry, instead the action of G transforms the fields ϕ^r in a coset between themselves and might require a compensating field-dependent H transformation.

The reason for Iwasawa and partial Iwasawa gauges is the fact that in dimension $D + 1$, the scalars are in a smaller coset $(G/H)_{D+1}$, see eq. (2.1). These are gauges suitable for addressing the relation between D -dimensional supergravity with $(G/H)_D$ coset space with the one in D derived from higher dimensions, they are, in general, different. The ones in dimension D derived from $D + 1$ dimensions have some axionic type scalars where, according to eq. (2.1), the axions correspond to \mathbf{V}_r^{D+1} . The local symmetry in $D + 1$ is also smaller, it is H_{D+1} . One can gauge-fix H_{D+1} either in a the triangular, Iwasawa gauge or in a symmetric gauge. In dimension D the 1st choice will produce a triangular Iwasawa gauge, the second choice will produce partial Iwasawa gauge. The simplest example of this partial Iwasawa gauge one can see in the $4D$ maximal supergravity action in [13]. This is a limit of the action in [30] when all mass/gauging parameters $m_i \rightarrow 0$.

- *Symmetric gauge.*

The reason they are called symmetric is that they correspond to a generalization of the polar decomposition of a linear matrix into a product of the orthogonal and a symmetric matrix

$$\mathcal{V} = e^{\phi \cdot \Sigma} e^{\theta \cdot \Lambda}. \quad (3.1)$$

Here Λ are the generators of the H group and Σ are the coset generators. A symmetric gauge is a choice

$$\theta = 0. \quad (3.2)$$

In symmetric gauge

$$\mathcal{V}_{\text{sym}}(\phi^r) = e^{\phi^r K_r} \in \exp(\mathfrak{k}) \quad r = 1, \dots, n_{\text{sc}}. \quad (3.3)$$

where $\{K_r\}$ is a basis of the coset algebra \mathfrak{k} defined in (2.8). Note that the coset representative in the symmetric gauge is not in a subalgebra of G . *All scalars in the symmetric H -gauge occur in the gauge-fixed supergravity action non-polynomially.*

In symmetric gauge, the scalars ϕ^r transform in a linear representation of the maximal compact subgroup $H \subset G$, and global H -invariance of the Lagrangian is manifest. The 1-loop anomalies of global H -symmetry in this gauge were computed in [25].

- *Iwasawa triangular gauges.*

The early class of gauges relating D -dimensional supergravity to $11D$ supergravity was proposed and studied in [1, 19].

Iwasawa triangular gauges/coset space representatives discussed in general in [22, 23] are associated with the Iwasawa decomposition of G with respect to H and with a solvable parametrization so that $\mathcal{V}_{\text{Iwasawa}}(\phi^r)$ belongs to a solvable Lie group $G_S = \exp(\mathcal{S})$

$$\mathcal{V}_{\text{Iwasawa}}(\phi^r) = e^{\varphi^r T_r} \in \exp(\mathcal{S}). \quad (3.4)$$

Here $\{T_r\}$ is a basis of \mathcal{S} ($r = 1, \dots, n_{\text{sc}}$), it is also known as a Borel subalgebra of \mathfrak{g} . The algebra \mathcal{S} , parametrized by the scalar fields of the theory, has the following general structure:

$$\mathcal{S} = \mathcal{C} \oplus \mathcal{N}, \quad (3.5)$$

where \mathbf{C} is the Cartan subspace of the coset space \mathbf{k} and is defined as the maximal set of commuting semisimple generators, and \mathbf{N} is a nilpotent subalgebra. When the theory originates from dimensional reduction of a higher dimensional one, \mathbf{C} is parametrized by the *dilatonic moduli*. The space \mathbf{N} , is parametrized by *axionic fields*. Since \mathbf{N} is nilpotent, *the scalars parametrizing \mathbf{N} appear in the supergravity action polynomially, whereas the dilatonic scalars parametrizing the Cartan subalgebra \mathbf{C} occur non-polynomially, in particular exponentially.*

Iwasawa gauges in supergravity include cases where D -dimensional supergravity is derived from maximal $11D$, they are described by parabolic subgroups P_{α_2} in figure 1. These triangular gauges are discussed in [1, 22] and presented in details with examples in [19]. We discuss these triangular Iwasawa gauges in appendix D.

- *Partial Iwasawa gauge.*

These gauges arise from $D+1$ dimensional supergravity where part of the scalars are in G_{D+1}/H_{D+1} . The freedom to gauge-fix a local H_{D+1} symmetry includes cases where we can use a symmetric gauge for fixing local H_{D+1} symmetry. This choice will produce a partial Iwasawa gauge. In addition to known examples in $4D$ related to $5D$ in [13, 30], we will show analogous examples of partial Iwasawa gauges in $6D$ related to $7D$. In supergravity II in [15] the action still has a local $\text{SO}(5)$ symmetry associated with a $7D$ $(G/H)_7 = \frac{\text{SL}(5)}{\text{SO}(5)}$. We will gauge-fix the corresponding action of $6D$ supergravity II in Iwasawa as well as in partial Iwasawa gauges.

The first issue with regard to these choices of gauges is related to 1-loop anomalies, computed in symmetric gauges in [25]. For example, in maximal $4D$ supergravity the global $H = \text{SU}(8)$ anomalies cancel and in $6D$ the global $H = \text{SO}(5) \times \text{SO}(5)$ anomalies cancel. However, supergravities in Iwasawa type gauges do not have manifest global H_D symmetries, rather they have the properties related to theories in higher dimensions. For example, the parabolic groups P_D are shown in figure 2. In $4D$ P_7 subgroup of E_7 has the node 7 removed and we get E_6 , a symmetry of $D+1=5$ supergravity. In $6D$ P_5 subgroup of E_5 has the node 5 removed and we get A_4 , a symmetry of $D+1=7$ supergravity.

Therefore the relation between the quantum field theories in these different gauges, symmetric and Iwasawa, requires an investigation.

The gauge-fixing of maximal $4D$ supergravity is known in a symmetric gauge [1, 2], and in $\mathcal{N} \geq 5$ in $4D$ with $\frac{G}{H} = \frac{\text{SO}^*(12)}{\text{U}(6)}$ and $\frac{G}{H} = \frac{\text{SU}(1,5)}{\text{U}(5)}$ for $\mathcal{N} = 6, 5$ respectively, by truncation, or by using symplectic sections as in [17]. But less is known about Iwasawa type gauges, and no relation between these theories in different gauges was ever established. In $6D$ maximal supergravity no gauge-fixing of a local $\text{SO}(5) \times \text{SO}(5)$ was performed, neither in symmetric nor in a triangular gauge.

There is a certain relation between versions of supergravity constructed in various dimensions D with G/H coset space, and supergravities obtained by Scherk-Schwarz generalized dimensional reduction from higher D [33]. It was explained in [1, 19], it all boils down to gauge-fixing local H symmetry. The choice of the gauge, suggested in [1] is “up to the user”: one can have a symmetric gauge [1, 2] where the asymptotic fields transform in the linear representation of H and dependence on all scalars is non-polynomial. Or one can have Iwasawa

or partial Iwasawa gauges, where axionic scalars enter only polynomially. However, the local H -symmetry must be anomaly-free for the S-matrix to be independent of the user's choice!

4 Gauge-fixing $4D$ maximal supergravity I

4.1 Symmetric gauge

We start with a standard (ungaughed) Lagrangian in $4D$ [1, 2] with a manifest $\text{SL}(8, \mathbb{R})$ symmetry which is a subgroup of $E_{7(7)}$. The duality representation **56** of $E_{7(7)}$ and its adjoint representation **133** branch with respect to $\text{SL}(8, \mathbb{R})$ as follows:

$$\begin{aligned} \mathbf{56} &\rightarrow \mathbf{28} + \mathbf{28}', \\ \mathbf{133} &\rightarrow \mathbf{63} + \mathbf{70}. \end{aligned} \quad (4.1)$$

A general polar decomposition presented in eq. (3.1) and the corresponding choice of the symmetric gauge (3.2) in case of $\frac{G}{H} = \frac{E_{7(7)}}{\text{SU}(8)}$ is specified as follows. To perform gauge-fixing requires to put a restriction on the 56-bein \mathcal{V} depending on 133 scalars, so that it will depend only on 70 scalars, coset space coordinates. The restriction is for the 56-bein to take a form $\mathcal{V}_{\text{gf}} = \mathcal{V}_{\text{gf}}^\dagger$ [1, 2]

$$\mathcal{V} \rightarrow \mathcal{V}_{\text{gf}} = \mathcal{V}_{\text{gf}}^\dagger = e^X = \begin{pmatrix} \cosh \phi \bar{\phi} & \phi \frac{\sinh \bar{\phi} \phi}{\bar{\phi} \phi} \\ \bar{\phi} \frac{\sinh \phi \bar{\phi}}{\phi \bar{\phi}} & \cosh \bar{\phi} \phi \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \phi_{ijkl} \\ \bar{\phi}^{mnpq} & 0 \end{pmatrix}. \quad (4.2)$$

Here the self-dual scalars $\phi_{ijkl} = \pm \frac{1}{4!} \epsilon_{ijklpqmn} \bar{\phi}^{pqmn}$ transform in the 35-dimensional representation of $\text{SU}(8)$, and is considered as a 28×28 matrix in the antisymmetric combinations $[ij]$, $[kl]$. Indices are raised by complex conjugation. $\bar{\phi} \phi$ stands for the matrix $(\bar{\phi} \phi)^{ij}_{kl} = \frac{1}{2} \bar{\phi}^{ijmn} \phi_{mnkl}$. It is useful to use the inhomogeneous coordinates of $E_{7(7)}/\text{SU}(8)$

$$y_{ij,kl} = \left(\phi \frac{\tanh \sqrt{(\bar{\phi} \phi)}}{\sqrt{(\bar{\phi} \phi)}} \right)_{ij,kl} = \frac{1}{2} \phi_{ijmn} \left(\frac{\tanh \sqrt{(\bar{\phi} \phi)}}{\sqrt{(\bar{\phi} \phi)}} \right)^{mn}_{kl}, \quad (4.3)$$

such that

$$e^X = \begin{pmatrix} \frac{1}{\sqrt{1-yy}} & y \frac{1}{\sqrt{1-yy}} \\ \bar{y} \frac{1}{\sqrt{1-yy}} & \frac{1}{\sqrt{1-yy}} \end{pmatrix}. \quad (4.4)$$

In this way the 63 local parameters of $\text{SU}(8)$ are used to define the unitary symmetric gauge with 70 physical scalars, in agreement with eq. (3.3) above. Details showing that after gauge-fixing local $H = \text{SU}(8)$ symmetry, there is a remaining nonlinearly realized $E_{7(7)}$ and a field-dependent compensating $H = \text{SU}(8)$ symmetry, are given in [1, 2, 7, 8]. The scalars ϕ_{ijkl} , $\bar{\phi}^{pqmn}$ transform in a linear representation of $\text{SU}(8)$. The 1-loop anomalies of global $\text{SU}(8)$ -symmetry in this gauge cancel [25].

4.2 Iwasawa $D + 1$ gauge

In terms of figure 1 this is the parabolic group P_{α_7} when the α_7 node is removed. In figure 2 at the left panel we also see the Dynkin diagram of E_7 and the one for E_6 , once the right node was removed.

In attempt to understand the $5D/4D$ connection we can make a choice of the Iwasawa gauge for the action in [1, 2] based on the fact that in $5D$ scalars are in $\frac{E_{6(6)}}{\mathrm{USp}(8)}$ coset space. With respect to $E_{6(6)} \times \mathrm{SO}(1, 1)$ the **56** and the adjoint of $E_{7(7)}$ decompose instead of (4.1) as follows:

$$\begin{aligned} \mathbf{56} &\rightarrow \mathbf{1}_{-3} + \mathbf{27}'_{-1} + \mathbf{1}_{+3} + \mathbf{27}_{+1}, \\ \mathbf{133} &\rightarrow \mathbf{27}_{-2} + \mathbf{1}_0 + \mathbf{78}_0 + \mathbf{27}'_{+2}. \end{aligned} \quad (4.5)$$

Here **28** vectors split into $\mathbf{1}_{-3}$ from a $5D$ metric and 27 vector fields of $5D$ in $\mathbf{27}'_{-1}$ of the electric group. The **78** scalars are related to generators of $E_{6(6)}$, it is a part of $E_{7(7)}$ generators. The $\mathbf{27}'_{+2}$ scalars are 27 axionic scalars originating from the five-dimensional vector fields through the Kaluza-Klein reduction. The $\mathbf{1}_0$ is related to a radius of the 5th dimension. We can now choose a gauge-fixing of a 56-bein \mathcal{V} in the action of standard $4D$ supergravity in [1, 2] in the form depending on 70 scalars associated with $5D$ maximal supergravity.

The Lagrangian [1, 2] has 63 local H -symmetries. In Iwasawa gauge 36 of these can be associated with a local H -symmetry of $5D$ supergravity, $\mathrm{USp}(8)$, but there are 27 more which we can use to make a gauge-fixing related to compactified $5D$ supergravity:

$$\mathbf{63} = \mathbf{36} + \mathbf{27}. \quad (4.6)$$

Our $70 = 1 + 42 + 27$ scalars in notation of [23] consist of

$$1 + 42 + 27 : \quad \sigma, \hat{\phi}, a^\lambda. \quad (4.7)$$

Here in $\mathbf{133} \rightarrow \mathbf{27}_{-2} + \mathbf{1}_0 + \mathbf{78}_0 + \mathbf{27}'_{+2}$ we are using 36 local symmetries to get $78 - 36 = 42$ of $\hat{\phi}$, and we use 27 local symmetries to remove $\mathbf{27}_{-2}$ scalars. Thus there are 2 possible gauge-fixing condition on the 56-bein. The first one is the triangular Iwasawa gauge

$$\mathcal{V}_{\text{gf}} = e^{a^\lambda t_\lambda} e^{\hat{\phi}^r \hat{T}_r} e^{\sigma D} \quad r = 1, \dots, 42. \quad (4.8)$$

Here the expressions for the 56×56 matrices, operators D , t_λ are given in eq. (4.26) in [23] and 27 t_λ form an algebra $[t_\lambda, t_\delta] = 0$. The coset space $\frac{E_{6(6)}}{\mathrm{USp}(8)}$ representative for the 42 scalars in $e^{\hat{\phi}^r \hat{T}_r}$ is in the solvable parametrization, with solvable Lie algebra \mathcal{S} .

The choice (4.8) corresponds to describing the scalar manifold of the theory as isometric to the following manifold [23]

$$\mathcal{M}_{\text{scal}} \sim \left[\mathrm{O}(1, 1) \times \frac{E_{6(6)}}{\mathrm{USp}(8)} \right] \ltimes \exp(\mathbb{N}^{[\mathbf{27}'_{+2}]}) , \quad (4.9)$$

where \ltimes denotes the semi-direct product and $\mathbb{N}^{[\mathbf{27}'_{+2}]}$ is the 27-dimensional space spanned by t_λ . If we choose for $\frac{E_{6(6)}}{\mathrm{USp}(8)}$ coset the solvable parametrization, then (4.8) defines the solvable parametrization of $\mathcal{M}_{\text{scal}}$, with solvable Lie algebra \mathcal{S} . Using the property that the t_λ are commuting, one can verify that the scalars a^λ are covered by a derivative and the constant shifts $a^\lambda \rightarrow a^\lambda + c^\lambda$ are isometries. They are implemented by the $E_{7(7)}$ -transformation $\mathbf{g} = e^{c^\lambda t_\lambda}$:

$$\mathbf{g} \mathcal{V}_{\text{gf}}(a^\Lambda, \hat{\phi}, \sigma) = e^{(a^\lambda + c^\lambda)t_\lambda} e^{\hat{\phi}^r \hat{T}_r} e^{\sigma D}. \quad (4.10)$$

4.3 Partial Iwasawa $D + 1$ gauge

$$\mathcal{V}_{\text{gf}} = e^{a^\lambda t_\lambda} e^{\phi^{abcd} K_{abcd}} e^{\sigma D}. \quad (4.11)$$

Here the expressions for the 56×56 matrices, operators D , t_λ are given in eq. (4.26) in [23] and 27 t_λ form an algebra $[t_\lambda, t_\delta] = 0$. But the coset representative for the 42 scalars ϕ^{abcd} in $\frac{E_{6(6)}}{\text{USp}(8)}$ coset space is not taken in the solvable parametrization. Instead it is taken in a symmetric gauge in the form $e^{\phi^{abcd} K_{abcd}}$. It is the one which was chosen in [13], it corresponds to a symmetric gauge in $5D$ in [30] and leads to a partial Iwasawa gauge in $4D$ in [13].

5 Supergravity I and supergravity II in $4D$

The maximal supergravity I in $4D$ [1, 2] is rather well known, whereas supergravity II [13] is not well known. It was developed and applied in [14] in the context of extremal black holes and $\mathcal{N} = 8$ attractors [9, 10]. The non-BPS regular extremal black holes turned out to be solutions of supergravity II with its $E_{6(6)}$ basis. The $1/2$ BPS extremal black holes entropy is given by a quartic $E_{7(7)}$ invariant, which is also manifestly $\text{SU}(8)$ invariant. But the non-BPS black hole with the regular horizon has the entropy which is manifestly $E_{6(6)}$ invariant, it depends on a cubic $E_{6(6)}$ invariant. These black holes are in different orbits of the fundamental representations of the exceptional groups $E_{7(7)}$ and $E_{6(6)}$ [11, 12].

The structure of maximal supergravity II [13] in $4D$ was influenced by $5D$ supergravity, but it was constructed directly in $4D$ by decomposing $E_{7(7)}$ symmetry

$$\mathfrak{e}_{7,7} = \mathfrak{e}_{6,6} + \mathfrak{so}(1,1) + \mathfrak{p}, \quad \mathfrak{p} = \mathbf{27}_{-2} + \mathbf{27}'_{+2}, \quad (5.1)$$

where \mathfrak{p} carries the representations $\mathbf{27}_{-2}$ and $\mathbf{27}'_{+2}$ of $\mathfrak{e}_{6,6} + \mathfrak{so}(1,1)$. It was also shown to agree with the one derived from $5D$ by compactification in [30] in the limit of vanishing gaugings. An ungauged version of it was presented in [13] when all mass parameters/gauge couplings vanish.

The ungauged Lagrangian of supergravity in $4D$ in [13], which we call supergravity II, does not have local symmetries since the reduction in [30] started in $5D$ supergravity with $\frac{G}{H} = \frac{E_{6(6)}}{\text{USp}(8)}$ where the $5D$ local $\text{USp}(8)$ symmetry is gauge fixed in a symmetric gauge. It means that from 78 scalars of $E_{6(6)}$ the 36 are eliminated using the local $\text{USp}(8)$ symmetry, and only 42 are left. It also means that $E_{6(6)}$ symmetry is realized nonlinearly.

In addition to 42 scalars ϕ^{abcd} , coset coordinates of $\frac{E_{6(6)}}{\text{USp}(8)}$, the extra 28 scalars in $4D$ come from the size of the extra dimension circle ϕ and from the extra component of the 27 -dimensional vector in $5D$ $a^\Lambda = A_5^\Lambda$. Therefore the 70 scalars in [13] are

$$\phi, \quad a^\Lambda, \quad \phi^{abcd}. \quad (5.2)$$

It means that **70** of $\text{SU}(8)$ are decomposed as follows under $\text{USp}(8)$

$$\mathbf{70} \rightarrow \mathbf{1} + \mathbf{27} + \mathbf{42}. \quad (5.3)$$

This is to be compared with 133 scalars in standard $4D$ maximal supergravity [1, 2]

$$\mathbf{133} \rightarrow \mathbf{63} + \mathbf{70} \quad (5.4)$$

or in $E_{6(6)} \times SO(1, 1)$ decomposition

$$\mathbf{133} \rightarrow \mathbf{27}_{-2} + \mathbf{1}_0 + \mathbf{78}_0 + \mathbf{27}'_{+2} \quad (5.5)$$

in [13]. Now we need to consider gauge-fixing local $SU(8)$ in [1, 2] in a way to reproduce either 70 scalars transforming linearly under global $SU(8)$, or getting 70 by removing $\mathbf{27}_{-2}$ scalars as well as 36 out of $\mathbf{78}_0$. The first case is a standard unitary gauge called symmetric gauge in [1, 2]. The second case related to supergravity II in [13] is the choice we called a partial Iwasawa gauge.

It might be useful to bring up the scalar action in [13, 14] for a better understanding of the issues of gauge-fixing local symmetries in 4D supergravities, in particular the difference between symmetric and Iwasawa gauges. The scalar action in [13, 14] is

$$\frac{1}{e} \mathcal{L}_{5D4D}^{\text{sc}} = \frac{3}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-4\phi} \hat{\mathcal{N}}_{\Lambda\Sigma} \partial_\mu a^\Lambda \partial^\mu a^\Sigma + \frac{1}{4!} P_\mu^{abcd} P^\mu_{abcd}. \quad (5.6)$$

Here $\hat{\mathcal{N}}_{\Lambda\Sigma}$ is the 5D ($SO(1, 1)$ invariant) vector kinetic matrix, $\Lambda = 1, \dots, 27$, $a = 1, \dots, 8$ and 70 scalars are decomposed under $USp(8)$ as $\mathbf{70} \rightarrow \mathbf{1} + \mathbf{27} + \mathbf{42}$.

In the supergravity action [1, 2] the pure scalar part, before gauge-fixing local $SU(8)$ symmetry has the form

$$\frac{1}{e} \mathcal{L}_{4D}^{\text{sc}} = \frac{1}{4!} P_\mu^{ijkl} P^\mu_{ijkl} \quad (5.7)$$

where $(\mathcal{V}^{-1} D_\mu \mathcal{V})^{ijkl} = \mathcal{P}_\mu^{ijkl}$ is a local $SU(8)$ tensor in $\mathbf{70}$.

The 28 vectors in the action in [1, 2] in [13] are represented by 1+27 vectors in [13, 14]

$$B_\mu, \quad Z_\mu^\Lambda. \quad (5.8)$$

Both actions depend only on field strength's: 28 in $F_{\mu\nu}^{IJ} = \partial_\mu A_\nu^{IJ} - \partial_\nu A_\mu^{IJ}$ in [1, 2] and 1+27 $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $Z_{\mu\nu}^\Lambda = \partial_\mu B_\nu^\Lambda - \partial_\nu B_\mu^\Lambda$ in [13]. The scalar-vector Lagrangian is

$$\begin{aligned} \frac{1}{e} \mathcal{L}^{\text{vec}} = & \mathcal{I}_{00}(\phi) B_{\mu\nu} B^{\rho\sigma} + 2\mathcal{I}_{0\Lambda}(\phi) B_{\mu\nu} Z^{\Lambda\mu\nu} + \mathcal{I}_{\Lambda\Sigma}(\phi) Z_{\mu\nu}^\Lambda Z^{\Sigma\mu\nu} \\ & + \frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} [\mathcal{R}_{00}(\phi) B_{\mu\nu} B_{\rho\sigma} + 2\mathcal{R}_{0\Lambda}(\phi) B_{\mu\nu} Z_{\rho\sigma}^\Lambda + \mathcal{R}_{\Lambda\Sigma}(\phi) Z_{\mu\nu}^\Lambda Z_{\rho\sigma}^\Sigma] \end{aligned} \quad (5.9)$$

where \mathcal{I}_{IJ} and \mathcal{R}_{IJ} are given by 4 blocks with 28 split into 0 and 27 Λ 's

$$\mathcal{I}_{IJ} = \left(\begin{array}{c|c} \mathcal{I}_{00} & \mathcal{I}_{0\Sigma} \\ \hline \mathcal{I}_{\Lambda 0} & \mathcal{I}_{\Lambda\Sigma} \end{array} \right) = -e^{6\phi} \left(\begin{array}{c|c} 1 + e^{-4\phi} a_{\Lambda\Sigma} a^\Lambda a^\Sigma & -e^{-4\phi} a_{\Lambda\Sigma} a^\Lambda \\ \hline -e^{-4\phi} a_{\Lambda\Sigma} a^\Sigma & e^{-4\phi} a_{\Lambda\Sigma} \end{array} \right); \quad (5.10)$$

and

$$\mathcal{R}_{IJ} = \left(\begin{array}{c|c} \mathcal{R}_{00} & \mathcal{R}_{0\Sigma} \\ \hline \mathcal{R}_{\Lambda 0} & \mathcal{R}_{\Lambda\Sigma} \end{array} \right) = \left(\begin{array}{c|c} \frac{1}{3} d_{\Lambda\Sigma\Gamma} a^\Lambda a^\Sigma a^\Gamma & -\frac{1}{2} d_{\Sigma\Lambda\Gamma} a^\Lambda a^\Gamma \\ \hline -\frac{1}{2} d_{\Lambda\Sigma\Gamma} a^\Sigma a^\Gamma & d_{\Lambda\Sigma\Gamma} a^\Gamma \end{array} \right). \quad (5.11)$$

Here $d_{\Lambda\Sigma}$ is the symmetric invariant tensor of the representation **27** of $E_{6(6)}$ and $a_{\Lambda\Sigma}$ is the five dimensional (SO(1, 1) invariant) vector kinetic matrix. Note that the scalar-dependent kinetic terms of vectors are polynomial in axions a^Λ .

The scalar-vector Lagrangian in a symmetric gauge in 4D is

$$\frac{1}{e} \mathcal{L}^{\text{vec}} = \frac{1}{4} \mathcal{I}_{IJ}(\phi) F_{\mu\nu}^I F^{J\mu\nu} + \frac{1}{8e} \mathcal{R}_{IJ}(\phi) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^J. \quad (5.12)$$

Here $I, J = 1, \dots, 28$. See for example, eq. (2.1) in [23].

Both actions 4D, supergravity I and II, when supplemented with fermions have maximal local supersymmetry. Supergravity II has inherited local supersymmetry via dimensional reduction. The gauged supergravity action in [30] has spontaneously broken supersymmetry, however, it is restored in the ungauged supergravity with $m_i \rightarrow 0$.

6 Gauge-fixing 6D maximal supergravity I

Explicit gauge-fixing of 6D maximal supergravity is interesting, in particular, since Marcus anomaly cancellation is valid for a symmetric gauge in 6D [25]. But the gauge-fixing of local $\text{SO}(5) \times \text{SO}(5)$ symmetry in 6D supergravity was not done before.

There is a technical complications due to presence in the action of two kinds of vielbeins, 10×10 and 16×16 [3, 4] vielbeins, related to each other. It is therefore necessary to find coset representatives for both of these, and check that the relations between them are valid upon gauge-fixing. This is different from 4D where there is only 56×56 vielbein [1, 2].

We consider 6D supergravity [3, 4] and try to gauge-fix local $\text{SO}(5) \times \text{SO}(5)$ symmetry. Before gauge-fixing scalars form an $\text{SO}(5, 5)$ valued 16×16 vielbein matrix $V_{\mu\dot{\mu}}{}^{\alpha\dot{\alpha}}$ with 45 independent entries $\mu, \dot{\mu} = 1, 2, 3, 4, \alpha, \dot{\alpha} = 1, 2, 3, 4$. When two local $\text{SO}(5) \times \text{SO}(5)$ symmetries are gauged-fixed, the 25 scalars which are left are representative of the coset space $\frac{\mathcal{G}}{\mathcal{H}} = \frac{\text{E}_{5(5)}}{\text{USp}(4) \times \text{USp}(4)} \sim \frac{\text{SO}(5, 5)}{\text{SO}(5) \times \text{SO}(5)}$.

The theory has a symmetry in which the first $\text{SO}(5)$ is flipped into the second $\text{SO}(5)$ and chirality flips. For example, spin 1/2 fields are singlets of $\text{E}_{5(5)}$ and are either in (5, 4) or (4, 5) of $\text{SO}(5) \times \text{SO}(5)$, depending on their chirality.

6.1 Symmetric gauge

Now we consider 6D supergravity [3, 4] and try to gauge-fix local $\text{SO}(5) \times \text{SO}(5)$ symmetry.² The 10×10 coset representative of $\frac{\text{SO}(5, 5)}{\text{SO}(5) \times \text{SO}(5)}$ is available in explicit form in [26].

10 × 10 representation of the vielbein. The 10×10 $\text{SO}(5, 5)$ matrix U in eq. (8) in [3] satisfies the constraint

$$U^T \eta_d U = \eta_d \quad \eta_d = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (6.1)$$

For

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (6.2)$$

²We have repeated and extended their notations in appendix A.

eq. (6.1) means that

$$A^T A - C^T C = 1, \quad B^T B - D^T D = -1, \quad A^T B - C^T D = B^T A - D^T C = 0. \quad (6.3)$$

Under $\text{SO}(5, 5)$ matrix g and $\text{SO}(5) \times \text{SO}(5)$ matrix $h(x)$ it transforms as

$$U' = g U h^{-1}(x). \quad (6.4)$$

It has 45 independent entries and g and h are linearly realized and independent before local $\text{SO}(5) \times \text{SO}(5)$ is gauge-fixed. We can realize this unitary gauge where only 25 scalars are left, after local 10+10 parameters of $\text{SO}(5) \times \text{SO}(5)$ are used. Namely, we impose the condition on U to gauge-fix local $\text{SO}(5) \times \text{SO}(5)$ symmetry in the form³

$$U_{\text{gf}} = U_{\text{gf}}^T \quad (6.5)$$

in agreement with the coset space construction for $\frac{\text{SO}(r,s)}{\text{SO}(r) \times \text{SO}(s)}$ in [26]. In such case we get

$$U_{\text{gf}} = \exp \begin{pmatrix} 0 & \phi \\ \phi^T & 0 \end{pmatrix} = \begin{pmatrix} \cosh \sqrt{(\phi \phi^T)} & \phi \frac{\sinh \sqrt{(\phi \phi^T)}}{\sqrt{(\phi \phi^T)}} \\ \phi^T \frac{\sinh \sqrt{(\phi \phi^T)}}{\sqrt{(\phi \phi^T)}} & \cosh \sqrt{(\phi \phi^T)} \end{pmatrix}. \quad (6.6)$$

Here the asymptotic field $\phi_{a\dot{a}}$ is in $(5, 5)$ of $\text{SO}(5) \times \text{SO}(5)$, $a, \dot{a} = 1, \dots, 5$. Note that the matrices $\phi^T \phi$ and $\phi \phi^T$ (with ϕ^T written as $(\phi^T)_{\dot{a}a} = \delta_{\dot{a}\dot{b}} \phi_{b\dot{b}} \delta^{ba}$) are hermitian with nonnegative values and have nonzero eigenvalues identical. The square roots do not appear in the final expressions, since the entries in (6.6) are power series with even powers of $\sqrt{\phi \phi^T}$.

As in [1, 2, 7, 8] where the 4D supergravity local $\text{SU}(8)$ symmetry was gauge-fixed, we introduce a set of inhomogeneous coset coordinates

$$\begin{aligned} y_a{}^{\dot{a}} &= \phi_a{}^{\dot{b}} \left(\frac{\tanh \sqrt{(\phi \phi^T)}}{\sqrt{(\phi \phi^T)}} \right)_b{}^{\dot{a}} = \left(\frac{\tanh \sqrt{(\phi \phi^T)}}{\sqrt{(\phi \phi^T)}} \right)_a{}^{\dot{b}} \phi_b{}^{\dot{a}}, \quad (y^T)_{\dot{a}}{}^a = \delta_{\dot{a}\dot{b}} \phi_b{}^{\dot{b}} \delta^{ba}, \\ (U_{\text{gf}}^{\text{sym}})_A{}^B &= \begin{pmatrix} (U_{\text{gf}})_a{}^b & (U_{\text{gf}})_a{}^{\dot{b}} \\ (U_{\text{gf}})_{\dot{a}}{}^b & (U_{\text{gf}})_{\dot{a}}{}^{\dot{b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-yy^T}} & y \frac{1}{\sqrt{1-y^T y}} \\ y^T \frac{1}{\sqrt{1-yy^T}} & \frac{1}{\sqrt{1-y^T y}} \end{pmatrix}. \end{aligned} \quad (6.7)$$

16 \times 16 representation of the vielbein. The 45 generators of $\text{SO}(5, 5)$ in spinorial representation can be given in 32 \times 32 form, and then split in chiral and antichiral 16 \times 16 matrices:

$$\begin{aligned} \text{SO}(5, 5) \text{ representation } \frac{1}{2} \tilde{\Gamma}_{\underline{A}\underline{B}} &= \frac{1}{2} \begin{pmatrix} \Gamma_{\underline{A}\underline{B}} & 0 \\ 0 & \Gamma'_{\underline{A}\underline{B}} \end{pmatrix}, \\ 10: \quad \tilde{\Lambda}_{[ab]} &= \tilde{\Gamma}_{ab}, & \Lambda_{[ab]} &= \Gamma_{ab} = \Gamma'_{ab} = \mathbb{1}_4 \times \gamma_{ab}, \\ 10: \quad \tilde{\Lambda}_{[\dot{a}\dot{b}]} &= \tilde{\Gamma}_{\dot{a}\dot{b}}, & \Lambda_{[\dot{a}\dot{b}]} &= \Gamma_{\dot{a}\dot{b}} = \Gamma'_{\dot{a}\dot{b}} = -\gamma_{\dot{a}\dot{b}} \times \mathbb{1}_4, \\ 15: \quad \tilde{\Sigma}_{a\dot{a}} &= \tilde{\Gamma}_{a\dot{a}}, & \Sigma_{a\dot{a}} &= \Gamma_{a\dot{a}} = -\Gamma'_{a\dot{a}} = -\gamma_{\dot{a}} \times \gamma_a. \end{aligned} \quad (6.8)$$

³Note also that in 4D the related choice of a symmetric gauge in [1, 2] is for the 56 \times 56-bein $\mathcal{V}_{\text{gf}} = \mathcal{V}_{\text{gf}}^\dagger$.

Before gauge-fixing there are 32×32 matrices, block diagonal

$$\begin{aligned} W &= \exp \left[\frac{1}{4} \phi^{\underline{A}\underline{B}} (\tilde{\Gamma}_{\underline{A}\underline{B}})_{\underline{B}}^{\underline{A}} \right] \\ &= \exp \left[\frac{1}{2} (\phi_a{}^{\dot{a}} \tilde{\Sigma}^a{}_{\dot{a}}) + \frac{1}{4} \left(\theta_{ab} \tilde{\Lambda}^{[ab]} + \frac{1}{4} \theta_{ab} \tilde{\Lambda}^{[\dot{a}\dot{b}]} \right) \right]. \end{aligned} \quad (6.9)$$

We can split W into 16×16 matrices, chiral and antichiral parts

$$W_{\underline{A}}{}^{\underline{B}} = \begin{pmatrix} \bar{V}_{\underline{A}}{}^{\underline{B}} & 0 \\ 0 & V_{\underline{A}}{}^{\underline{B}} \end{pmatrix}, \quad \text{with} \quad \bar{V}^{\underline{B}}{}_{\underline{A}} = C^{\underline{B}\underline{C}} \bar{V}_{\underline{C}}{}^{\underline{D}} C_{\underline{D}\underline{A}} = (V^{-1})_{\underline{A}}{}^{\underline{B}}. \quad (6.10)$$

Thus \bar{V} is the transpose of the inverse of V , i.e.

$$V_{\underline{A}}{}^{\underline{C}} \bar{V}^{\underline{B}}{}_{\underline{C}} = \delta_{\underline{A}}^{\underline{B}}, \quad V_{\underline{A}}{}^{\underline{C}} \bar{V}^{\underline{A}}{}_{\underline{D}} = \delta_{\underline{D}}^{\underline{C}}. \quad (6.11)$$

These matrices are related to $U_{\underline{A}}{}^{\underline{B}}$ in (6.2), (6.3) by the requirement of invariance of $(\Gamma_{\underline{A}})^{\underline{A}\underline{B}}$ and $(\Gamma'_{\underline{A}})_{\underline{A}\underline{B}}$ (the left-right and right-left chiral parts of $\tilde{\Gamma}_{\underline{A}}^{\underline{A}\underline{B}}$) under simultaneous $\text{SO}(5,5)$ action on all their indices: [3, 4]⁴

$$\begin{aligned} V_{\underline{A}}{}^{\underline{C}} (\Gamma_{\underline{B}})^{\underline{A}\underline{B}} V_{\underline{B}}{}^{\underline{D}} &= U_{\underline{B}}{}^{\underline{A}} (\Gamma_{\underline{A}})_{\underline{C}\underline{D}}, \\ \bar{V}^{\underline{A}}{}_{\underline{C}} (\Gamma'_{\underline{B}})_{\underline{A}\underline{B}} \bar{V}^{\underline{B}}{}_{\underline{D}} &= U_{\underline{B}}{}^{\underline{A}} (\Gamma'_{\underline{A}})_{\underline{C}\underline{D}} \end{aligned} \quad (6.12)$$

or

$$\begin{aligned} U_{\underline{B}}{}^{\underline{A}} &= \frac{1}{16} V_{\underline{A}}{}^{\underline{C}} (\Gamma_{\underline{B}})^{\underline{A}\underline{B}} V_{\underline{B}}{}^{\underline{D}} (\Gamma'_{\underline{C}})_{\underline{D}\underline{C}}, \\ &= \frac{1}{16} \bar{V}^{\underline{A}}{}_{\underline{C}} (\Gamma'_{\underline{B}})_{\underline{A}\underline{B}} \bar{V}^{\underline{B}}{}_{\underline{D}} (\Gamma'_{\underline{C}})^{\underline{D}\underline{C}}. \end{aligned} \quad (6.13)$$

Note first that taking $V = \bar{V} = \mathbb{1}_{16}$, one obtains $U = \mathbb{1}_{10}$ (the value for $\phi = 0$ in the equations in [26]), see (6.6).

Secondly, these expressions lead to a matrix U that is in $\text{SO}(5,5)$, i.e.

$$U_{\underline{A}}{}^{\underline{C}} U_{\underline{B}\underline{C}} = (\eta_d)_{\underline{A}\underline{B}}. \quad (6.14)$$

In fact, inserting the first line of (6.13) in the second factor U of (6.14), we get

$$\begin{aligned} U_{\underline{A}}{}^{\underline{C}} U_{\underline{B}\underline{C}} &= U_{\underline{A}}{}^{\underline{C}} \frac{1}{16} V_{\underline{A}}{}^{\underline{C}} (\Gamma_{\underline{B}})^{\underline{A}\underline{B}} V_{\underline{B}}{}^{\underline{D}} (\Gamma'_{\underline{C}})_{\underline{D}\underline{C}} \\ &= \frac{1}{16} V_{\underline{A}}{}^{\underline{C}} (\Gamma_{\underline{B}})^{\underline{A}\underline{B}} V_{\underline{B}}{}^{\underline{D}} \bar{V}^{\underline{E}}{}_{\underline{D}} (\Gamma'_{\underline{A}})_{\underline{E}\underline{F}} \bar{V}^{\underline{F}}_{\underline{C}} \\ &= \frac{1}{16} (\Gamma_{\underline{B}})^{\underline{A}\underline{B}} (\Gamma'_{\underline{A}})_{\underline{B}\underline{A}} \\ &= (\eta_d)_{\underline{A}\underline{B}}, \end{aligned} \quad (6.15)$$

⁴Most equations in these references are written in a covariant basis, useful to write down the action (for the split electric/magnetic fields). For the group theory, the \underline{A} basis is more transparent. The equations are equivalent after multiplication with the matrix M in [4, (2.29)].

where for the second line the second line of (6.12) is used, for the third line (6.11) and finally (A.11) with an index lowered.

The 16×16 matrices are

$$\bar{V} = e^{\frac{1}{4}\phi^{\underline{A}\underline{B}}\Gamma_{\underline{A}\underline{B}}}, \quad V = e^{\frac{1}{4}\phi^{\underline{A}\underline{B}}\Gamma'_{\underline{A}\underline{B}}}. \quad (6.16)$$

Gauge fixing using 10+10 local $\text{SO}(5) \times \text{SO}(5)$

$$\theta_{ab} = \theta_{\dot{a}\dot{b}} = 0 \quad (6.17)$$

and thus gauge fixed W is

$$W_{\text{gf}} = \exp\left(\frac{1}{2}(\phi_a{}^{\dot{a}}\tilde{\Sigma}^a{}_{\dot{a}})\right), \quad W_{\mathcal{B}}{}^{\mathcal{A}} = -(W^{-1})^{\mathcal{A}}{}_{\mathcal{B}} = -\mathbb{C}^{\mathcal{A}\mathcal{C}}(W^{-1})_{\mathcal{C}}{}^{\mathcal{D}}\mathbb{C}_{\mathcal{D}\mathcal{B}}. \quad (6.18)$$

The exponential satisfies the latter equation, since

$$(\tilde{\Gamma}_{\underline{A}\underline{B}})_{\mathcal{B}}{}^{\mathcal{A}} = -(\tilde{\Gamma}_{\underline{A}\underline{B}})_{\mathcal{B}}{}^{\mathcal{A}}, \quad (6.19)$$

(see (A.5)) and for $\underline{A} = a$ and $\underline{B} = \dot{a}$, and the minus sign in (6.18) leads from W to W^{-1} .

Gauge fixed \bar{V}, V are

$$\bar{V}_{\text{gf}} = \exp\left(\frac{1}{2}\phi_a{}^{\dot{a}}\Sigma^a{}_{\dot{a}}\right), \quad V_{\text{gf}} = \exp\left(-\frac{1}{2}\phi_a{}^{\dot{a}}\Sigma^a{}_{\dot{a}}\right). \quad (6.20)$$

Finally, one can show that the linear part in ϕ of (6.18) leads with (6.13) to the linear part of (6.6). For that it is convenient to write the two equations (6.13) as (remembering that the $\Gamma_{\underline{A}}$ and $\Gamma'_{\underline{A}}$ are symmetric)

$$\begin{aligned} U_{\underline{B}}{}^{\underline{A}} &= \frac{1}{16}\text{Tr}\left[V\Gamma'^{\underline{A}}V^T\Gamma_{\underline{B}}\right] = \frac{1}{16}\text{Tr}\left[V\Gamma'^{\underline{A}}\bar{V}^{-1}\Gamma_{\underline{B}}\right] \\ &= \frac{1}{16}\text{Tr}\left[\bar{V}\Gamma^{\underline{A}}\bar{V}^T\Gamma'_{\underline{B}}\right] = \frac{1}{16}\text{Tr}\left[\bar{V}\Gamma^{\underline{A}}V^{-1}\Gamma'_{\underline{B}}\right], \end{aligned} \quad (6.21)$$

where we used that $V^T = \bar{V}^{-1}$. These are traces over 16×16 matrices. We can rewrite this expression in 32×32 language, which makes it easier to handle Γ matrices. Using (6.18), the equations (6.21) are the block diagonal parts of

$$\begin{aligned} U_{\underline{B}}{}^{\underline{A}} &= \frac{1}{32}\text{Tr}\left[W\tilde{\Gamma}^{\underline{A}}W^{-1}\tilde{\Gamma}_{\underline{B}}\right] \\ &= \frac{1}{32}\text{Tr}\left[\begin{pmatrix}\bar{V} & 0 \\ 0 & V\end{pmatrix}\begin{pmatrix}0 & \Gamma^{\underline{A}} \\ \Gamma'^{\underline{A}} & 0\end{pmatrix}\begin{pmatrix}\bar{V}^{-1} & 0 \\ 0 & V^{-1}\end{pmatrix}\begin{pmatrix}0 & \Gamma_{\underline{B}} \\ \Gamma'_{\underline{B}} & 0\end{pmatrix}\right]. \end{aligned} \quad (6.22)$$

Now we can take standard γ -algebra, and use that traces of $\tilde{\Gamma}_{\underline{A}\underline{B}}$ and $\tilde{\Gamma}_{\underline{A}\underline{B}\underline{C}\underline{D}}$ vanish. If we take the linear part of (6.9)

$$W = 1_{32} = \frac{1}{4}\phi^{\underline{A}\underline{B}}\tilde{\Gamma}_{\underline{A}\underline{B}} + \mathcal{O}(\phi^2), \quad (6.23)$$

we get

$$U_{\underline{B}}{}^{\underline{A}} = \delta_{\underline{B}}{}^{\underline{A}} + \phi_{\underline{B}}{}^{\underline{A}} + \mathcal{O}(\phi^2). \quad (6.24)$$

After gauge fixing this is

$$W = \mathbb{1}_{32} + \frac{1}{2} \phi_a{}^{\dot{a}} \tilde{\Gamma}^a \tilde{\Gamma}_{\dot{a}} + \mathcal{O}(\phi^2), \quad (6.25)$$

and we get

$$\begin{aligned} U_{\underline{B}}{}^A &= \delta_{\underline{B}}{}^A + \phi_a{}^{\dot{a}} \left(\delta_{\underline{B}}{}^a \delta_{\dot{a}}^A - (\eta_d)^{aA} (\eta_d)_{\dot{a}\underline{B}} \right) + \mathcal{O}(\phi^2) \\ &= \delta_{\underline{B}}{}^A + \phi_a{}^{\dot{a}} \left(\delta_{\underline{B}}^a \delta_{\dot{a}}^A + \delta^{aA} \delta_{\dot{a}\underline{B}} \right) + \mathcal{O}(\phi^2), \end{aligned} \quad (6.26)$$

which agrees with the linear part in (6.6), and is the linear part of the exponential of the fields times the $\text{SO}(5, 5)$ representation matrices.

This then implies for the 16×16 matrices

$$V = \mathbb{1}_{16} + \frac{1}{2} \phi^{a\dot{a}} \Gamma'_{a\dot{a}} + \mathcal{O}(\phi^2), \quad \bar{V} = \mathbb{1}_{16} + \frac{1}{2} \phi^{a\dot{a}} \Gamma_{a\dot{a}} + \mathcal{O}(\phi^2), \quad (6.27)$$

where the Γ are defined in (A.13).

6.2 Iwasawa $D + 1$ gauge

In terms of figure 1 this is the parabolic group P_{α_5} when the α_5 node is removed. This gauge is related to the field content originating from $7D$ supergravity.

We are looking at the ungauged supergravity in [3, 4] trying to find a gauge in which the scalar part becomes the one derived from $7D$ in [15]. It was derived from $7D$ gauged maximal supergravity in [34] by a reduction to $6D$ and taking the limit to $g \rightarrow 0$. The $6D$ ungauged supergravity action in [15] still has a local $\text{SO}(5)$ symmetry and $35 = 10 + 1 + 24$ scalars are:

$$B^{IJ}, \quad \sigma, \quad \Pi_I{}^j. \quad (6.28)$$

Here $I = 1, \dots, 5$, $i = 1, \dots, 5$. We will present the scalar part of the action in eqs. (7.2), (7.3). The action in [15] has $\text{SL}(5, \mathbb{R})$ symmetry with 24 generators, inherited from $7D$. The 10 axions B^{IJ} originate from $7D$ vector $A_7^{IJ} = B^{IJ}$, σ is related to an extra circle, and 24 scalars $\Pi_I{}^j$ represent a $7D$ $\text{SL}(5, \mathbb{R})/\text{SO}(5)$ vielbein.

The local $\text{SO}(5)$ symmetry can be gauge-fixed, so that only 25 physical scalars remain, these are B^{IJ} , σ as well as 14 representatives of the coset space $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5)}$. An Iwasawa triangular gauge in $6D$ supergravity I can be given in the form (3.4) where 25 generators of $\text{SO}(5, 5)$ form a solvable Lie algebra. The representative of the coset $\text{SO}(5, 5)/(\text{SO}(5) \times \text{SO}(5))$ is then

$$V_{\mu\dot{\mu}}{}^{\alpha\dot{\alpha}}(x) \quad \Rightarrow \quad V_{\text{gf}}(x) = \exp\{B^{IJ} N_{IJ}\} \Pi^{\text{Iwa}} \exp\{\sigma h_0\} \quad (6.29)$$

where

$$\Pi^{\text{Iwa}} = \exp\{\phi^{ij} t_{ij} + \phi^\lambda h_\lambda\}. \quad (6.30)$$

Here we have a set of 10 nilpotent generators, N_{IJ} , associated with axions B^{IJ} and a set of 14 generators in (6.30), representative of a coset space $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$ gauge-fixed in the Iwasawa gauge in Π^{Iwa} . These include 10 nilpotent generators in $\text{SL}(5, \mathbb{R})$ t_{ij} , $i, j = 1, \dots, 5$, $i < j$, and 4 non-compact Cartan generators of $\text{SL}(5, \mathbb{R})$, h_λ , $\lambda = 1, \dots, 4$, so this is the Borel subalgebra. There is also h_0 , the generator in $\text{SO}(5, 5)$ which commutes with $\text{SL}(5, \mathbb{R})$. Using this gauge allows to make a $7D$ origin of the $6D$ maximal supergravity transparent:

$$V_{\mu\dot{\mu}}{}^{\alpha\dot{\alpha}}(x) \quad \Rightarrow \quad V_{\text{gf}}(x) = \exp\{B^{IJ} N_{IJ}\} \exp\{\phi^{ij} t_{ij} + \phi^\lambda h_\lambda\} \exp\{\sigma h_0\}. \quad (6.31)$$

6.3 Partial Iwasawa $D + 1$ gauge

This gauge also has a $7D$ origin. To gauge-fix the local $\text{SO}(5)$ symmetry of the $6D$ maximal supergravity related to [15] we proceed following the strategy in $4D$ maximal supergravity which we already applied in the partial Iwasawa gauge above. We have to replace the $\text{SL}(5, \mathbb{R})$ matrix Π_I^j with 24-entries by a coset space $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$ representative in a symmetric gauge, so that with account of gauge-fixed local $\text{SO}(5)$ only 14 entries are left. These are given in [26].

The coset we need is described by the exponential of a hermitian symmetric (i.e. real) matrices e^H such that

$$H^\dagger = H, \quad H^T = H, \quad \text{Tr } H = 0 \quad (6.32)$$

and

$$(e^H)^\dagger = e^{H^\dagger} = e^H \quad (e^H)^T = e^{H^T} = e^H. \quad (6.33)$$

These matrices are unimodular

$$\|e^H\| = e^{\text{Tr } H} = 1. \quad (6.34)$$

Thus, the coset representative of $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$ consist of all real symmetric unimodular matrices. Once we constrain Π as in eqs. (6.33), (6.34)

$$\Pi \quad \rightarrow \quad \Pi = \Pi^\dagger = \Pi^T, \quad \|\Pi\| = 1 \quad (6.35)$$

we get a gauge-fixed unitary action with 25 scalars ($14 + 10 + 1$) and the remaining local $\text{SO}(5)$ symmetry inherited from $7D$ is fixed.

We can impose a symmetric gauge associated with 14 scalars in $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$

$$\Pi^{\text{sym}} = \exp\{\phi^{ab} K_{ab}\} \quad (6.36)$$

where K_{ab} are the coset generators and $\text{Tr } \phi = 0$. Note that in the Iwasawa gauge (6.29) we had instead Π^{Iwa} in (6.30). Thus the total vielbein in partial Iwasawa $D + 1$ gauge is

$$V_{\mu\dot{\mu}}^{\alpha\dot{\alpha}}(x) \quad \Rightarrow \quad V_{\text{gf}}(x) = \exp\{B^{IJ} N_{IJ}\} \exp\{\phi^{ab} K_{ab}\} \exp\{\sigma h_0\} \quad (6.37)$$

and we get a gauge-fixed unitary gauge with 25 scalars ($14 + 10 + 1$) and the remaining local $\text{SO}(5)$ symmetry fixed. Using this gauge allows to make a $7D$ origin of the $6D$ maximal supergravity transparent. This is to be compared with Iwasawa $D + 1$ gauge in (6.31).

$\text{SL}(5, \mathbb{R})$ acts by left multiplication on the vielbein (6.37) together with a compensating $\text{SO}(5)$ rotation from the right. So it rotates the B^{IJ} (as it does in $7D$), and acts non-linearly on the ϕ^{ab} via the standard coset action.

$E_{5(5)}$ is not manifest, but still present on the equations of motion (not the action), it acts by left multiplication on the vielbein (6.37) together with a compensating $\text{SO}(5) \times \text{SO}(5)$ rotation from the right. Such that this action is non-linear and in general mixes the B^{IJ} with the ϕ^{ab} .

7 Supergravity I and supergravity II in 6D

The standard 6D ungauged maximal (2, 2) supergravity with $\frac{\mathcal{G}}{\mathcal{H}} = \frac{E_{5(5)}}{\mathrm{USp}(4) \times \mathrm{USp}(4)} \sim \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5) \times \mathrm{SO}(5)}$ coset space was constructed in [3, 4]. It is presented in the form with local $\mathcal{H} = \mathrm{SO}(5) \times \mathrm{SO}(5)$ symmetry. Duality symmetry of Gaillard-Zumino type is realized acting on a 3-forms and their magnetic duals, they combine into the vector representation 10 of $\mathrm{SO}(5,5) \sim E_{5(5)}$. A detailed version of this theory, in slightly different notation, as well as a gauged version of it was presented in [4]. In this version a $\mathrm{GL}(5)$ electric subgroup of $E_{5(5)}$ is realized off-shell as a symmetry of the action.

Another version of 6D maximal supergravity was derived in [15] from 7D gauged maximal supergravity in [34]. First, in [15] the gauged theory was reduced to 6D. The gauged version has $\mathrm{SO}(5)_c \times \mathrm{SO}(5)_g$ local symmetry. Here the local $\mathrm{SO}(5)_c$ originates from the 11D Lorentz symmetry whereas $\mathrm{SO}(5)_g$ is a Yang-Mills local symmetry due to gauging. In the limit to ungauged 6D supergravity in [15] the only local symmetry left is $\mathrm{SO}(5)_c$ since the Yang-Mills $\mathrm{SO}(5)_g$ disappears in the ungauged theory.

The model with maximal local supersymmetry in 6D presented in eqs. (C.1)–(C.14) in [15] has the following properties originating from 7D theory in [34], which has scalars associated with the coset space $\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(5)}$. In 7D the limit $g \rightarrow 0$ removing the $\mathrm{SO}(5)_g$ gauge symmetry is singular. Meanwhile in 6D the limit of removing the $\mathrm{SO}(5)_g$ gauge symmetry is regular [15]. It has manifest linearly realized global $\mathrm{SL}(5,\mathbb{R})$ symmetry, it has an independent local $\mathrm{SO}(5)_c$ symmetry, both originating from D7 theory in [34] after reduction on a circle and after gauge coupling g is sent to zero.

The field content of the theory in eq. (C.2) in [15] is: 1 graviton $g_{\mu\nu}$, 5 two-index antisymmetric tensor potentials $C_{\mu\nu I}$, $(10 + 5 + 1)$ vectors $(B_{\mu I}{}^J, S_{\mu I}, A_\mu)$, 4 gravitini ψ_μ and $(16 + 4)$ spin 1/2 fermions (λ_i, χ) . All fermions ψ_μ , λ_i and χ are all $D = 6$ $\mathrm{USp}(4)$ symplectic Majorana spinors. The scalars are

$$\Pi_I{}^j, \quad B_{0I}{}^J, \quad \sigma \quad (7.1)$$

originating from a reduction of D7 theory [34] on a S^1 circle. The 10 axions B^{IJ} originate from 7D vector $A_7^{IJ} = B^{IJ}$, σ is related to an extra circle, and 24 scalars $\Pi_I{}^j$ represent a 7D $\mathrm{SL}(5,\mathbb{R})$ vielbein.

$\Pi_I{}^j$ is an $\mathrm{SL}(5,\mathbb{R})$ matrix and has 24 entries. Here $i, j = 1, \dots, 5$ are vector indices of $\mathrm{SO}(5)_c$, which is still a local symmetry with 10 local parameters in the Lagrangian (C.2) of [15]. The indices $I, J = 1, \dots, 5$ originate from a $\mathrm{SO}(5)_g$ which is left after the limit of the Yang-Mills gauge coupling was taken. Thus the total amount of scalars, physical and unphysical in 6D maximal supergravity in eq. (C.2) of [15] is $35 = 24 + 10 + 1$ and there are still 10 local parameters of $\mathrm{SO}(5)_c$ symmetry which can be used to remove the unphysical scalars to get the 25 physical $(14 + 10 + 1)$ scalars. A direct inspection of the 6D maximal supergravity in eq. (C.2) in [15] shows the following facts.

The scalar part of the action is

$$\frac{1}{e} \mathcal{L}_{7D6D}^{\mathrm{sc}} = -\frac{1}{2} e^{\frac{4\sigma}{\sqrt{10}}} (\Pi_I{}^i \Pi_J{}^j \partial_\mu B^{IJ})^2 - \frac{1}{2} (\partial_\mu \sigma)^2 - P_{\mu ij} P^{\mu ij} \quad (7.2)$$

where

$$P_{\mu ij} = \Pi^{-1}{}^I_{(i} \partial_\mu \Pi_{Ij)} . \quad (7.3)$$

This action is obtained from the scalar action of [3, 4] by partially gauge fixing the $\text{SO}(5, 5)$ vielbein as

$$V_{\text{gf}}(x) = \exp\{B^{IJ}N_{IJ}\} \Pi \exp\{\sigma h_0\}, \quad (7.4)$$

which upon further gauge fixing of the remaining $\text{SO}(5)$ yields (6.31) or (6.37).

To gauge-fix the local $\text{SO}(5)_c$ symmetry of the $6D$ maximal supergravity in [15] we proceed following the strategy in $4D$ maximal supergravity. As we already discussed in the context of the Iwasawa gauge for the action in [3] in section 6.2 and partial Iwasawa gauge in section 6.3 we have to replace the $\text{SL}(5, \mathbb{R})$ matrix Π_I^j with 24-entries by a coset space $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$ representative, so that with account of gauge-fixed local $\text{SO}(5)_c$ only 14 entries are left.

One choice for Π_I^j can be as in (6.30), the other as in (6.36). Once we constrain Π as shown above we get a gauge-fixed unitary action with 25 scalars (14+10+1) and the remaining local $\text{SO}(5)_c$ symmetry of the action in [15] is gauge-fixed. It is likely that, $\text{SL}(5, \mathbb{R})$ symmetry is still present but nonlinearly realized. But this requires an additional investigation. $\text{SL}(5, \mathbb{R})$ is still present after gauge fixing. With regard to $E_{5(5)}$ it is not manifest and has no obvious reason to be present in $6D$ maximal supergravity II as a hidden symmetry, either before or after local $\text{SO}(5)_c$ gauge-fixing. However, $E_{5(5)}$ might be realized on the equations of motion, not on the action.

There are 45 scalars in ungauged maximal $6D$ supergravity in [3, 4] in the vielbein in 16×16 representation V_{16} or in 10×10 representation in \mathcal{V}_{10} of $\text{SO}(5, 5)$. In notation of [4] the 16×16 vielbein is $V_M^{\alpha\dot{\alpha}}(x)$, and an $\text{SO}(5) \times \text{SO}(5)$ covariant 1-form is

$$P^{a\dot{a}} = \frac{1}{4}\bar{V}\gamma^a\gamma^{\dot{a}}dV \quad (7.5)$$

and the scalar Lagrangian has a local $\text{SO}(5) \times \text{SO}(5)$ symmetry and a global $E_{5(5)}$ symmetry

$$\frac{1}{e}\mathcal{L}_{6D}^{\text{sc}} = -\frac{1}{16}P_\mu^{a\dot{a}}P^\mu_{a\dot{a}}. \quad (7.6)$$

When the local $\text{SO}(5) \times \text{SO}(5)$ symmetry is gauge-fixed, the 20 local parameters can be used to eliminate 20 scalars so that only 25 physical scalars remain.

There are 5 two-index antisymmetric tensor potentials $C_{\mu\nu I}$ and $16 = 10 + 5 + 1$ vector fields $B_{\mu I}^J$, $S_{\mu I}$, A_μ in $6D$ ungauged supergravity in [15] with kinetic terms depending on 35 scalars, or on 25 after the local $\text{SO}(5)_c$ gauge symmetry is gauge-fixed as in (6.30) or (6.36).

The maximal $6D$ supergravity action in eq. (C.2) in [15] has the following action for the tensors and vectors

$$\begin{aligned} e^{-1}\mathcal{L}_{7D6D}^{\text{ten,vec}} = & -\frac{1}{4}e^{-\frac{5\sigma}{\sqrt{10}}}(f_{\mu\nu})^2 - \frac{1}{12}e^{-\frac{2\sigma}{\sqrt{10}}}(\Pi^{-1}_i{}^I H_{\mu\nu\rho I})^2 - \frac{1}{4}e^{-\frac{\sigma}{\sqrt{10}}}(\Pi_I{}^i\Pi_J{}^j F_{\mu\nu}^{IJ})^2 \\ & -\frac{1}{4}e^{\frac{3\sigma}{\sqrt{10}}}(\Pi^{-1}_i{}^I G_{\mu\nu I})^2 \\ & -\frac{e^{-1}}{36\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\lambda\tau}B_0{}^{IJ}H_{\mu\nu\rho I}H_{\sigma\lambda\tau J} - \frac{e^{-1}}{6\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\lambda\tau}H_{\mu\nu\rho I}B_\sigma{}^{IJ}G_{\lambda\tau J}. \end{aligned} \quad (7.7)$$

Here

$$\begin{aligned}
f_{\mu\nu} &\rightarrow f_2 = dA & G_{\mu\nu I} &\rightarrow G_{2I} = dS_{1I} \\
F_{\mu\nu}^{IJ} &\rightarrow F_{2I}^J = dB_{1I}^J + B_{0I}^J dA \\
H_{\mu\nu\rho I} &= 3 \left(\partial_{[\mu} C_{\nu\rho]I} + \frac{1}{2} G_{[\mu\nu I} A_{\rho]} \right). \tag{7.8}
\end{aligned}$$

By comparison, the tensor-vector part of the $6D$ action in [3, 4] is

$$\frac{1}{e} \mathcal{L}_{6D}^{\text{ten,vec}} = -\frac{1}{12} H_m \cdot K^{mn} H_n - \frac{1}{4} \mathcal{M}_{AB} F_{\mu\nu}^A F^{\mu\nu B} + H_m \cdot j\omega^m - \frac{1}{12} \omega_m \cdot j\omega^m. \tag{7.9}$$

The action in [3] is the same as the $g \rightarrow 0$ in [4], in slightly different notations. It has $\text{SO}(5) \times \text{SO}(5)$ local symmetry and an on shell $E_{5(5)}$ global symmetry. We refer to notations and details in [4].

Thus the unitary gauge of the action in (7.7) is established. It is hard to see where is the global $\text{SO}(5) \times \text{SO}(5)$ and reflection symmetries in this case. Therefore the absence of 1-loop anomalies for these symmetries found in [25] does not make sense here, unless one can prove that this unitary gauge is equivalent to a symmetric gauge in [3]. It is also not clear how this $6D$ supergravity theory II can be protected from UV divergences and anomalies.

8 Discussion

The purpose of this work was to study gauge-fixing of local H symmetries in G/H maximal D -dimensional supergravities. The existence of different unitary gauges in G/H maximal D -dimensional supergravities has a deep origin in existence of different versions of D -dimensional supergravities, some of which are associated with $D + n$ -dimensional supergravities. We studied mostly the case $n = 1$, i.e. $D + 1$ supergravities compactified on a circle. In certain gauges, like symmetric gauges, all scalars enter the action non-polynomially. In other gauges, like Iwasawa gauges, associated with $D + 1$ -dimensional supergravities, which we studied here, some of the scalars enter the action polynomially.

The reason for this difference has to do with the properties of the Lie algebra of the relevant duality $G_U = E_{11-D}(11-D)$ symmetries of D -dimensional maximal supergravities [18]. There is a *Cartan decomposition* of a duality symmetry Lie algebra

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K}. \tag{8.1}$$

Here \mathbb{H} is a compact subalgebra of \mathbb{G} and \mathbb{K} includes generators of the coset space G/H . For example, in $\mathcal{N} = 8$ in $4D$ there are 63 generators in $H = \text{SU}(8)$ and 70 in $\frac{E_{7(7)}}{\text{SU}(8)}$. In symmetric gauges global H -symmetry is preserved and supergravity vielbeins take the form

$$\mathcal{V}_{\text{sym}} = e^{\phi_{\text{sym}} \cdot \mathbb{K}}. \tag{8.2}$$

For symmetric gauge to be preserved under G -symmetry transformation a compensating field-dependent H -symmetry transformation is required. As the result, the global G -symmetry remains a symmetry of the theory and it is non-linearly realized. Therefore the studies of the vanishing soft scalar limits are efficient and useful in amplitude computations.

There is an *Iwasawa decomposition* of a duality symmetry Lie algebra

$$\mathbb{G} = \mathbb{H} \oplus \mathcal{S}. \quad (8.3)$$

Here \mathcal{S} is a solvable subgroup of the Lie algebra,

$$\mathcal{S} = C \oplus N, \quad (8.4)$$

C is the Cartan subspace of the coset space and N is a nilpotent subalgebra. Since N is nilpotent, *the axionic scalars parametrizing N appear in the supergravity action polynomially*. In Iwasawa-type gauges global H -symmetry as well as G -symmetry are either broken or, at least, not manifest

$$\mathcal{V}_{\text{Iwa}} = e^{\phi_{\text{Iwa}} \cdot \mathcal{S}}. \quad (8.5)$$

In Iwasawa gauges the scalars are the same as in supergravity theories II which were derived by a compactification from $D+1$ dimension where scalars are in a coset space $(G/H)_{D+1}$. In these versions of D -dimensional supergravity there is a smaller local symmetry $H_{D+1} < H_D$ as well as a smaller global symmetry $G_{D+1} < G_D$.

Thus, the existence of different D -dimensional supergravities, I and II, and different gauges for a local H -symmetry in supergravities I is universal, and we studied all of this in various examples, particularly in $4D$ and $6D$. The next question raised by our investigation here is the relation between different local H -symmetry gauges in supergravities I, i.e. the issue of local H -symmetry anomalies. Another question is the relation between different versions of supergravities, I and II, which involves the issue of G -symmetry anomalies since in supergravities II G_D -symmetry is broken down to G_{D+1} by the Group Disintegration process [5].

In $4D$ it was established in a set of papers in [35–37] that using an $\text{Sp}(56, \mathbb{R})$ Gaillard-Zumino duality symmetry one can show, for example, that starting with supergravity I in [1, 2] one can change a symplectic frame from the $\text{SL}(8, \mathbb{R})$ -basis to the $E_{6(6)}$ -basis and gauge-fix local symmetries in a way which allows to reproduce the results of supergravity II in [13] for physical observables. This suggests that the extra symmetries beyond U-duality may help to understand the relation between different version of D -dimensional supergravities and different gauges, which in turn might be helpful for understanding UV divergences in D -dimensional supergravities [38].

Acknowledgments

We had helpful suggestions from P. Aspinwall, E. Bergshoeff, G. Moore and M. Günaydin concerning the issue of the 16×16 representative of the coset space of $\frac{\text{SO}(5,5)}{\text{SO}(5) \times \text{SO}(5)}$ which is not standard, as opposite to the 10×10 representation available in the textbooks. We are grateful to participants of the Amplitudes 2024 conference for the interest to this work, especially to N. Arkani Hamed, L. Dixon, A. Edison, H. Johansson, J. Parra-Martinez and R. Roiban. The discussion with them suggested that the current work poses the following questions. Which of these various D -dimensional supergravities, quantized in different gauges, are described by the loop computations using the amplitude double-copy methods? How

to probe $\text{Sp}(2n)$ symmetry in amplitudes? We also thank C. Hull, M. Henneaux and the organizers of the Mons workshop ‘Gauge invariance: quantization and geometry- in memory of Igor Batalin’ for discussions.

The work of RK is supported by SITP and by the U.S. National Science Foundation grant PHY-2310429.

A $6D$ notation

Our notations are consistent with [39], and mostly with [3, 4, 40]. We start with a table of the indices:

$$\begin{aligned} a, \dot{a} &= 1, \dots, 5, & \underline{A} &= \{a, \dot{a}\} = 1, \dots, 10, \\ m &= 1, \dots, 5, & M &= 1, \dots, 10, \\ \alpha, \dot{\alpha} &= 1, \dots, 4, & A &= (\alpha \dot{\alpha}) = 1, \dots, 16, \\ i &= 1, 2, & \mathcal{A} &= (\dot{\alpha} \alpha i) = (Ai) = 1, \dots, 32. \end{aligned} \quad (\text{A.1})$$

The full $\text{SO}(5, 5)$ gamma matrices are written as

$$(\tilde{\Gamma}_a)_{\mathcal{A}}^{\mathcal{B}} = (\gamma_a)_a^{\beta} \delta_{\dot{a}}^{\dot{\beta}} (\sigma_1)_i^j, \quad (\tilde{\Gamma}_{\dot{a}})_{\mathcal{A}}^{\mathcal{B}} = \delta_a^{\beta} (\gamma_{\dot{a}})_{\dot{\alpha}}^{\dot{\beta}} (i\sigma_2)_i^j, \quad \mathcal{A} = (\dot{\alpha} \alpha i), \quad \mathcal{B} = (\dot{\beta} \beta j). \quad (\text{A.2})$$

Here one starts from the $D = 5$ Euclidean gamma matrices γ_a and $\gamma_{\dot{a}}$, which are identical:

$$\begin{aligned} \gamma_a \gamma_b + \gamma_b \gamma_a &= 2\delta_{ab} \mathbb{1}_4, & \gamma_{\dot{a}} \gamma_{\dot{b}} + \gamma_{\dot{b}} \gamma_{\dot{a}} &= 2\delta_{\dot{a}\dot{b}} \mathbb{1}_4, \\ \tilde{\Gamma}_{\underline{A}} &= \{\tilde{\Gamma}_a, \tilde{\Gamma}_{\dot{a}}\}, & \tilde{\Gamma}_{\underline{A}} \tilde{\Gamma}_{\underline{B}} + \tilde{\Gamma}_{\underline{B}} \tilde{\Gamma}_{\underline{A}} &= 2(\eta_d)_{\underline{AB}}, \\ \tilde{\Gamma}_a \tilde{\Gamma}_b + \tilde{\Gamma}_b \tilde{\Gamma}_a &= 2\delta_{ab} \mathbb{1}_{32}, & \tilde{\Gamma}_{\dot{a}} \tilde{\Gamma}_{\dot{b}} + \tilde{\Gamma}_{\dot{b}} \tilde{\Gamma}_{\dot{a}} &= -2\delta_{\dot{a}\dot{b}} \mathbb{1}_{32}, \quad \tilde{\Gamma}_a \tilde{\Gamma}_{\dot{b}} + \tilde{\Gamma}_{\dot{b}} \tilde{\Gamma}_a &= 0. \end{aligned} \quad (\text{A.3})$$

Similarly, the vector indices are raised as

$$\gamma^a = \delta^{ab} \gamma_b, \quad \gamma^{\dot{a}} = \delta^{\dot{a}\dot{b}} \gamma_{\dot{b}}, \quad \tilde{\Gamma}^{\underline{A}} = (\eta_d)^{\underline{AB}} \tilde{\Gamma}_{\underline{B}} = \{\tilde{\Gamma}^a, \tilde{\Gamma}^{\dot{a}}\} = \{\delta^{ab} \tilde{\Gamma}_b, -\delta^{\dot{a}\dot{b}} \tilde{\Gamma}_{\dot{b}}\}. \quad (\text{A.4})$$

Lowering spinor indices is done in $5D$ with $\Omega_{\alpha\beta}$ and in $10D$ with $\mathbb{C}_{\mathcal{AB}}$

$$\begin{aligned} (\gamma_a)_{\alpha\beta} &= (\gamma_a)_{\alpha}^{\gamma} \Omega_{\gamma\beta} = -(\gamma_a)_{\beta\alpha}, & \Omega_{\beta\alpha} &= -\Omega_{\alpha\beta}, \\ (\tilde{\Gamma}_{\underline{A}})_{\mathcal{A}\mathcal{B}} &= (\tilde{\Gamma}_{\underline{A}})_{\mathcal{A}}^{\mathcal{C}} \mathbb{C}_{\mathcal{C}\mathcal{B}} = (\tilde{\Gamma}_{\underline{A}})_{\mathcal{B}\mathcal{A}}, & \mathbb{C}_{\mathcal{AB}} &= \Omega_{\dot{\alpha}\dot{\beta}} \times \Omega_{\alpha\beta} \times (i\sigma_2)_{ij} = -\mathbb{C}_{\mathcal{BA}}. \end{aligned} \quad (\text{A.5})$$

Lowering only the A index goes thus with

$$C_{AB} = C_{(\alpha\dot{\alpha})(\beta\dot{\beta})} = \Omega_{\dot{\alpha}\dot{\beta}} \times \Omega_{\alpha\beta} = C_{BA}, \quad C^{AB} C_{CB} = \delta_C^A, \quad (\text{A.6})$$

and thus the value of C_{AB} is the same as the value of C^{AB} . The relation can also be written as

$$\begin{aligned} \mathbb{C}_{\mathcal{AB}} &= C_{AB} \varepsilon_{ij} = \begin{pmatrix} 0 & C_{AB} \\ -C_{AB} & 0 \end{pmatrix}_{ij} & \text{for } \mathcal{A} &= (Ai), \mathcal{B} = (Bj), \\ \mathbb{C}^{\mathcal{AB}} &= C^{AB} \varepsilon^{ij} = \begin{pmatrix} 0 & C^{AB} \\ -C^{AB} & 0 \end{pmatrix}_{ij}, & \mathbb{C}^{\mathcal{AC}} \mathbb{C}_{\mathcal{BC}} &= \delta_{\mathcal{C}}^{\mathcal{A}}. \end{aligned} \quad (\text{A.7})$$

We can define

$$\tilde{\Gamma}_* = \tilde{\Gamma}_1 \tilde{\Gamma}_2 \tilde{\Gamma}_3 \tilde{\Gamma}_4 \tilde{\Gamma}_5 \tilde{\Gamma}_1 \tilde{\Gamma}_2 \tilde{\Gamma}_3 \tilde{\Gamma}_4 \tilde{\Gamma}_5 = \mathbb{1}_{16} \times \sigma_3, \\ P_L = \frac{1}{2}(\mathbb{1}_{32} + \tilde{\Gamma}_*) = \begin{pmatrix} \mathbb{1}_{16} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \frac{1}{2}(\mathbb{1}_{32} - \tilde{\Gamma}_*) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{16} \end{pmatrix}. \quad (\text{A.8})$$

Making the i index explicit, we can also define 16×16 matrices $\Gamma_{\underline{A}}$ and $\Gamma'_{\underline{A}}$ as chirally projected matrices from $\tilde{\Gamma}_{\underline{A}}$:

$$\tilde{\Gamma}_{\underline{A}} = \begin{pmatrix} 0 & \Gamma_{\underline{A}} \\ \Gamma'_{\underline{A}} & 0 \end{pmatrix}, \quad \Gamma_{\underline{A}} = \{\Gamma_a, \Gamma_{\dot{a}}\}, \quad \Gamma'_{\underline{A}} = \{\Gamma'_a, \Gamma'_{\dot{a}}\}, \\ \tilde{\Gamma}^{\underline{A}} = \begin{pmatrix} 0 & \Gamma^{\underline{A}} \\ \Gamma'^{\underline{A}} & 0 \end{pmatrix}, \quad \Gamma^{\underline{A}} = (\eta_d)^{AB} \Gamma_{\underline{B}} = \{\Gamma^a, \Gamma^{\dot{a}}\}, \quad \Gamma'^{\underline{A}} = (\eta_d)^{AB} \Gamma'^{\underline{B}} = \{\Gamma'^a, \Gamma'^{\dot{a}}\}, \\ \Gamma_a = \Gamma'_a = \mathbb{1}_4 \times \gamma_a, \quad \Gamma_{\dot{a}} = -\Gamma'_{\dot{a}} = \gamma_{\dot{a}} \times \mathbb{1}_4, \\ \Gamma^a = \Gamma'^a = \mathbb{1}_4 \times \gamma^a, \quad \Gamma^{\dot{a}} = -\Gamma'^{\dot{a}} = -\gamma^{\dot{a}} \times \mathbb{1}_4. \quad (\text{A.9})$$

Note that $\Gamma'^{\underline{A}} = \Gamma_{\underline{A}}$. Their basic relations follow from (A.3):

$$\Gamma_{\underline{A}} \Gamma'^{\underline{B}} + \Gamma_{\underline{B}} \Gamma'^{\underline{A}} = 2(\eta_d)_{AB}. \quad (\text{A.10})$$

Furthermore, the trace rules for the 16×16 matrices is

$$\text{Tr}(\Gamma_{\underline{A}} \Gamma'^{\underline{B}}) = \text{Tr}(\Gamma'_{\underline{A}} \Gamma^{\underline{B}}) = 16\delta_{\underline{A}}^{\underline{B}}. \quad (\text{A.11})$$

The $\Gamma_{\underline{A}}$ (and their primed versions) have naturally indices A, B : $(\Gamma_{\underline{A}})_A{}^B$, and if the latter is lowered they are symmetric:

$$(\tilde{\Gamma}_{\underline{A}})_{(Ai)(Bj)} = \begin{pmatrix} -(\Gamma_{\underline{A}})_{AB} & 0 \\ 0 & (\Gamma'_{\underline{A}})_{AB} \end{pmatrix}_{ij}, \quad (\Gamma_{\underline{A}})_{AB} = (\Gamma_{\underline{A}})_A{}^C C_{CB} = (\Gamma_{\underline{A}})_{BA}. \quad (\text{A.12})$$

The representation matrices of $\text{SO}(5, 5)$ are then

$$\tilde{\Gamma}_{AB} = \tilde{\Gamma}_{[\underline{A}} \tilde{\Gamma}_{\underline{B}]} = \begin{pmatrix} \Gamma_{AB} & 0 \\ 0 & \Gamma'_{AB} \end{pmatrix}, \\ \Gamma_{AB} = \frac{1}{2}(\Gamma_{\underline{A}} \Gamma'^{\underline{B}} - \Gamma_{\underline{B}} \Gamma'^{\underline{A}}), \quad \Gamma'_{AB} = \frac{1}{2}(\Gamma'_{\underline{A}} \Gamma_{\underline{B}} - \Gamma'_{\underline{B}} \Gamma_{\underline{A}}), \\ \Gamma_{ab} = \Gamma'_{ab} = \mathbb{1}_4 \times \gamma_{ab}, \quad \Gamma_{\dot{a}\dot{b}} = \Gamma'_{\dot{a}\dot{b}} = -\gamma_{\dot{a}\dot{b}} \times \mathbb{1}_4, \\ \Gamma_{a\dot{a}} = -\Gamma_{\dot{a}a} = -\gamma_{\dot{a}} \times \gamma_a, \quad \Gamma'_{a\dot{a}} = -\Gamma'_{\dot{a}a} = \gamma_{\dot{a}} \times \gamma_a. \quad (\text{A.13})$$

B Minimal unitary representations of $E_{7(7)}$ and $E_{5(5)}$

In the context of gauge-fixing local H -symmetry in supergravities it is useful to consider also *minimal unitary representations of the group G* as presented in [29, 41] with regard to quantum mechanical system which admit a semisimple Lie algebra. Minimal realization is

the case of the least number of degrees of freedom for which a quantum mechanical system admits a given Lie algebra. For semisimple algebra \mathfrak{g} there is a canonical decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \quad (\text{B.1})$$

where \mathfrak{h} is a Cartan subalgebra and \mathfrak{n}^\pm are the nilpotent subalgebras spanned respectively by the positive and negative root eigenvectors. The procedure of constructing explicitly the minimal unitary representations of a Lie group G was provided in details in [29], where the parabolic subgroups P of G are involved.

The algebraic structure of quantum mechanics and the concept of a spectrum generating algebras emphasised in [29, 41] are based on canonical commutation relations $[p_i, x^j] = -i\delta_i^j$ and p_i is realized as $-i\frac{\partial}{\partial x^i}$. For example, the Dynkin diagram for E_7 in figure 2 corresponds to Cartan generators and simple roots presented in [29] in the form $H_{\beta_0} = -y\partial + x_0\partial_0, \dots$ and $E_{\alpha_1} = x_2\partial_3 + x_4\partial_5 + x_6\partial_8 + x_9\partial_{12}, \dots$, etc. acting on $(y, x_0, x_1, \dots, x_{15})$. An important role in this construction is played by a parabolic subgroups of G where the right node in Dynkin diagrams in figures 1, 2 is removed.

The common feature of all unitary gauges in supergravities, symmetric, and Iwasawa-type is that the number of physical scalars is the same, e.g. 70 in 4D maximal supergravity and 25 in 6D maximal supergravity. However, the difference between these gauges is that only Iwasawa gauges, where coset generators are in a subalgebra of G , qualify for minimal unitary representations of the group G as defined in [29, 41].

The explicit unitary representations for 6D maximal supergravity were given in [42] and for 4D maximal supergravity in [43]. They were obtained by solving differential equations defining representations of $E_{7(7)}$ or $E_{5(5)}$ in a parabolic gauge associated with the so-called decompactification limit.

Therefore these minimal unitary representations for G -duality groups of 4D and 6D maximal supergravities are represented by a field content of 5D and 7D supergravities, respectively, as we have explained in sections 5 and 7. These minimal unitary representations also represent Iwasawa-type class of gauges of a local H -symmetry.

4D maximal supergravity. Minimal unitary representation is presented in the form of the coset representative in a parabolic gauge of 4D maximal supergravity in [43]. It is based on a decomposition of the Lie algebra

$$\mathfrak{e}_{7(7)} \cong \overline{\mathbf{27}}^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{e}_{6(6)})^0 \oplus \mathbf{27}^{(2)}. \quad (\text{B.2})$$

This is the same as the one in 5D4D supergravity in [13] where it was given in the form

$$\mathfrak{e}_{7(7)} = \mathfrak{e}_{6(6)} + \mathfrak{so}(1, 1) + \mathfrak{p}, \quad \mathfrak{p} = \mathbf{27}_{(-2)} + \mathbf{27}'_{(+2)} \quad (\text{B.3})$$

so that under the subgroup $E_{6(6)} \times \text{SO}(1, 1)$ there is a decomposition

$$\mathbf{56} \rightarrow \mathbf{1}_{-3} + \mathbf{27}'_{-1} + \mathbf{1}_{+3} + \mathbf{27}'_{+1}. \quad (\text{B.4})$$

The 56×56 -bein defining the minimal unitary representation of $E_{7(7)}$ was given in [43] in the form

$$\mathcal{V}_{56} = \begin{pmatrix} e^{3\phi} & 0 & 0 & 0 \\ 0 & e^\phi V_{ij}^I & 0 & 0 \\ 0 & 0 & e^{-\phi}(V^{-1})^{ij} & 0 \\ 0 & 0 & 0 & e^{-3\phi} \end{pmatrix} \begin{pmatrix} 1 & a^J & \frac{1}{2}t_{JKL}a^Ka^L & \frac{1}{3}t_{KLP}a^Ka^La^P \\ 0 & \delta_I^J & t_{IJK}a^K & \frac{1}{2}t_{IKL}a^Ka^L \\ 0 & 0 & \delta_J^I & a^I \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{B.5})$$

where V_{ij}^I is a coset representative of $E_{6(6)}/\text{USp}(8)$, and t_{IJK} is the invariant symmetric tensor of $E_{6(6)}$. Here \mathcal{V}_{56} depends on $70 = 1 + 42 + 27$ scalars:

$$\phi, \quad V_{ij}^I, \quad a^I. \quad (\text{B.6})$$

This is in agreement with the construction in [13]. It corresponds to a partial Iwasawa gauge we discussed above. One could have used the same expression for \mathcal{V}_{56} where V_{ij}^I is a representative of $E_{6(6)}/\text{USp}(8)$ in a solvable parametrization, which would give an Iwasawa, triangular gauge for the 56-bein. This is what we did in section 4.

6D maximal supergravity. Here the minimal unitary representation of $E_{5(5)}$ in [42] is based on a decomposition

$$\mathfrak{e}_{5(5)} \cong \overline{\mathbf{10}}^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{sl}_5)^0 \oplus \mathbf{10}^{(2)}. \quad (\text{B.7})$$

This again can be related to 6D supergravity II in [15] which we described in section 7. The corresponding 10×10 and 16×16 matrices are given in [42]

$$\mathcal{V}_{10} = \begin{pmatrix} e^{2\phi}(v^{-1})_J^a & e^{2\phi}(v^{-1})_K^a a^{KJ} \\ 0 & e^{2\phi} v_a^J \end{pmatrix} \quad (\text{B.8})$$

and

$$\mathcal{V}_{16} = \begin{pmatrix} e^{5\phi} & \frac{1}{\sqrt{2}}e^{5\phi}a^{KL} & \frac{1}{8}e^{5\phi}\epsilon_{KPRQS}a^{PQ}a^{RS} \\ 0 & e^\phi v_{[a}^K v_{a]}^L & e^\phi v_{[a}^R v_{a]}^S \frac{1}{2\sqrt{2}}\epsilon_{RSKPQ}a^{PQ} \\ 0 & 0 & e^{-3\phi}(v^{-1})_K^a \end{pmatrix}. \quad (\text{B.9})$$

Matrices \mathcal{V}_{10} and \mathcal{V}_{16} have 35 independent entries

$$\phi, \quad v_a^J, \quad a^{KJ}. \quad (\text{B.10})$$

Here the 24 scalars v_a^J are in $\text{SL}(5, \mathbb{R})$ and there is still one local $\text{SO}(5)$ symmetry in the theory which has to be gauge-fixed. Therefore to bring these expressions, depending on $1 + 24 + 10 = 35$ scalars to the ones which depend only on 25 physical scalars we have to gauge-fix the local $\text{SO}(5)$ by taking a coset representative for $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5)}$. This means replacing the 24 scalars in v_a^J by the coset representative of $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5)}$ which depends on 14 scalars. This can be done either in a solvable parametrization or in a symmetric gauge. In this way one can fully specify the 10-bein and the 16-bein to depend on 25 scalars either in Iwasawa, triangular gauge or in a partial Iwasawa gauge as we did above in section 6.

C Action of maximal 6D supergravity and amplitudes

The action in [3] is the same as the $g \rightarrow 0$ in [4], in slightly different notations. It has $\text{SO}(5) \times \text{SO}(5)$ local symmetry and an on shell $E_{5(5)}$ global symmetry. We have shown bosonic actions in section 7. The fermionic kinetic terms in section (2.6) of [4] are

$$\frac{1}{e} \mathcal{L}_{6D}^{\text{ferm}} = -\frac{1}{2} \bar{\psi}_{+\mu} \gamma^{\mu\nu\rho} D_\nu \psi_{+\rho} - \frac{1}{2} \bar{\psi}_{-\mu} \gamma^{\mu\nu\rho} D_\nu \psi_{-\rho} - \frac{1}{2} \bar{\chi}^a \gamma^\mu D_\mu \chi^a - \frac{1}{2} \bar{\chi}^{\dot{a}} \gamma^\mu D_\mu \chi^{\dot{a}} \quad (\text{C.1})$$

and we skipped terms with fermion interaction with bosons. The spinor fields are $\psi_{+\mu\alpha}$, $\psi_{-\mu\dot{\alpha}}$ and $\chi_{+a\dot{\alpha}}$, $\chi_{-a\alpha}$ where $a, \dot{a} = 1, \dots, 5$ and $\alpha, \dot{\alpha}$ label spinors of $\text{SO}(5) \times \text{SO}(5)$. Here \pm refers to spacetime chirality of the spinors which are 6D symplectic Majorana-Weyl.

When this action is gauge-fixed in a symmetric gauge it is a model for which anomaly computation in [25] is relevant. Note that the action is manifestly invariant under reflection: flipping chirality and $\text{SO}(5)_1$ to $\text{SO}(5)_2$. This symmetry plays an important role in cancellation of 6D supergravity $\text{SO}(5) \times \text{SO}(5) \sim \text{USp}(4) \times \text{USp}(4)$ anomalies in [25].

The maximal 7D6D ungauged supergravity action in eq. (C.2) in [15] has $\text{SO}(5)$ local symmetry. We have shown bosonic terms in the action in section 7. The fermionic kinetic terms are

$$\frac{1}{e} \mathcal{L}_{7D6D}^{\text{ferm}} = -\bar{\psi}_\mu \tau^{\mu\nu\rho} \nabla_\nu \psi_\rho - \bar{\chi}^\mu \nabla_\mu \chi - \bar{\lambda}^i \tau^\mu \nabla_\mu \lambda_i + \dots \quad (\text{C.2})$$

where the \dots involve terms with fermion interaction with bosons. The fermions are $\text{USp}(4)$ 6D Majorana symplectic spinors, they are Lorentz and $\text{SO}(5)$ -covariant. This action has no manifest symmetry under reflection when flipping chirality and $\text{USp}(4)_1$ to $\text{USp}(4)_2$.

In symmetric gauge in [3, 4] there are 25 scalars $\phi^{a\dot{a}}$ with $a, \dot{a} = 1, \dots, 5$ viewed with spinorial indices in $\text{USp}(4) \times \text{USp}(4)$ are

$$W_{\alpha\beta}^{\dot{\alpha}\dot{\beta}}(x) \equiv (\gamma_a)_{\alpha\beta} (\gamma_{\dot{a}})^{\dot{\alpha}\dot{\beta}} \phi^{a\dot{a}}(x), \quad \alpha, \dot{\alpha} = 1, 2, 3, 4. \quad (\text{C.3})$$

At the linear level these scalars are the first components in the linearized BPS superfields $W_{\alpha\beta}^{\dot{\alpha}\dot{\beta}}(x, \theta)$ [44, 45]. For example $W_{12}^{1\dot{2}}(x, \theta)$ is a 1/2 BPS superfield, it depends on half of fermionic directions in superspace

$$D_{\alpha 1} W_{12}^{1\dot{2}} = D_{\alpha 2} W_{12}^{1\dot{2}} = D^{\dot{\alpha} 1} W_{12}^{1\dot{2}} = D^{\dot{\alpha} 2} W_{12}^{1\dot{2}} = 0. \quad (\text{C.4})$$

Using superamplitudes, the structure of the maximal supergravity 4-point tree amplitude was given in [46]. The corresponding on shell superfield depends on 8 Grassmann coordinates and it is directly related to $W_{12}^{1\dot{2}}(x, \theta)$. The four-point superamplitude is given in [46] in the form

$$M_4^{\mathcal{N}=(2,2) \text{ tree}} = \frac{1}{\kappa^2} \delta^6 \left(\sum_{i=1}^4 p_i^{AB} \right) \frac{\delta^8 \left(\sum_{i=1}^4 q_i^{A,I} \right) \delta^8 \left(\sum_{i=1}^4 \tilde{q}_{i,\hat{A}}^{\hat{I}} \right)}{s_{12} s_{23} s_{13}}. \quad (\text{C.5})$$

We have presented in [44] a local linearized superinvariant defining 3-loop 4-point UV divergence found in [47] as follows

$$M_4^{\mathcal{N}=(2,2) \text{ L=3}} = \frac{1}{\epsilon} \frac{5\zeta_3}{(4\pi)^9} \left(\frac{\kappa}{2} \right)^4 \delta^6 \left(\sum_{i=1}^4 p_i^{AB} \right) \delta^8 \left(\sum_{i=1}^4 q_i^{A,I} \right) \delta^8 \left(\sum_{i=1}^4 \tilde{q}_{i,\hat{A}}^{\hat{I}} \right) s_{12} s_{23} s_{34}. \quad (\text{C.6})$$

It is proportional to $M_4^{\mathcal{N}=(2,2) \text{ tree}}$ and numerical factors in eq. (C.6) are taken from [47] where this 3-loop UV divergence was computed.

The presence of a UV divergence in amplitude computations in [47] and the absence of 1-loop anomalies of a global $\text{USp}(4) \times \text{USp}(4)$ symmetry discovered in [25] can be considered as a problem. In the earlier case of a 4D $\mathcal{N} = 4$ supergravity anomalies in [25] and 1-loop $\text{U}(1)$ superamplitude anomalies in [48] were considered as precursors of UV divergences. It was indeed a UV divergence at 4-loop discovered later in [49].

Meanwhile in 6D there are no anomalies in [25] and there was no claim of 1-loop superamplitude anomalies: all based on 6D supergravity [3, 4] with global $\text{USp}(4) \times \text{USp}(4)$ symmetry in a symmetric gauge.

Now we have learned that the local $\text{USp}(4) \times \text{USp}(4)$ symmetry can be gauge-fixed in different gauges. Specifically, 6D supergravity II does not have a global $\text{USp}(4) \times \text{USp}(4)$ symmetry, same as an Iwasawa gauge of the standard 6D supergravity [3, 4]. Therefore it may be not totally surprising that there is a UV divergence. The theory might have local $\text{USp}(4) \times \text{USp}(4)$ anomaly. This would be in agreement with $E_{5(5)}$ anomaly associated with 3-loop UV divergence as discussed in [44, 45, 50].

D Triangular 11D Iwasawa gauge

In terms of figure 1 this is the parabolic group P_{α_2} when the α_2 node is removed.

In 6D supergravity one can use a triangle gauge following up the studies of dualities in [19]. Here we will show how to impose a triangle gauge in the model in [3] to gauge-fix its local $\text{SO}(5) \times \text{SO}(5)$. This gauge is a part of the family of $\frac{\text{SO}(n,n)}{\text{SO}(n) \times \text{SO}(n)}$ coset spaces where $n = 11 - D$, i.e. the gauge related to 11D supergravity compactified on a torus T^{11-D} with $D \geq 6$.

It was found in [19] that the bosonic part of D6 supergravity in a triangle gauge can be associated with the dimensional reduction from 11D supergravity on a T^5 . The relevant scalars are the one defining the torus.

In [19] the matrix $\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is left invariant under infinitesimal $\text{SO}(5,5)$ transformations

$$L = \begin{pmatrix} u & v \\ \tilde{v} & -u^T \end{pmatrix} \quad (\text{D.1})$$

where u is an arbitrary real matrix, $v = -v^T$, $\tilde{v} = -\tilde{v}^T$.

We take the U matrix in [3] and switch to the basis with $\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ instead of the diagonal in eq. (6.1) so that $\mathcal{V} \in \text{SO}(5,5)$ is related to U

$$U^T = C^T \mathcal{V} C, \quad C^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (\text{D.2})$$

\mathcal{V} satisfies an $\text{SO}(5,5)$ condition as given in [19] in appendix C

$$\mathcal{V} \eta \mathcal{V}^T = \eta. \quad (\text{D.3})$$

The relation between \mathcal{V} and U is now established. The triangular gauge, fixing local $\text{SO}(5) \times \text{SO}(5)$ is

$$\mathcal{V}_{\text{gf}} = \begin{pmatrix} S & R \\ 0 & (S^{-1})^T \end{pmatrix}, \quad RS^T + SR^T = 0. \quad (\text{D.4})$$

Thus, the gauge-fixing condition of U in [3] can be given in the form

$$(U_{\text{gf}}^T)^{\text{triangular}} = C^T \begin{pmatrix} S & R \\ 0 & (S^{-1})^T \end{pmatrix} C. \quad (\text{D.5})$$

In this gauge there is a 10×10 matrix

$$\mathcal{M} = \mathcal{V}^T \mathcal{V} = \begin{pmatrix} G_{AB} & -G_{AC}X^{CB} \\ X^{AC}G_{CB} & -X^{AC}G_{CD}X^{DB} \end{pmatrix} \quad (\text{D.6})$$

with

$$(S^T S)_{AB} = G_{AB}, \quad (S^{-1} R)^{AB} = -X^{AB}, \quad X = -X^T. \quad (\text{D.7})$$

The $\frac{O(5,5)}{O(5) \times O(5)}$ coset Lagrangian is

$$\frac{1}{8}e\text{Tr}(\partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M}) = -\frac{1}{4}G_{AC}G_{BD}(\partial_\mu G^{AB} \partial^\mu G^{CD} + \partial_\mu X^{AB} \partial^\mu G^{CD}). \quad (\text{D.8})$$

Thus we have shown here how exactly to fix the triangle $\text{SO}(5) \times \text{SO}(5)$ gauge in [3] to get the 25 moduli in the form

$$G_{(AB)}, \quad X_{[AB]}, \quad A, B = 1, 2, 3, 4, 5 \quad (\text{D.9})$$

which are the torus T^5 moduli.

Note that here the 15 scalars in G_{AB} are on equal footing representing T^5 geometry. But below, in the Iwasawa gauge P_{α_5} related to $D+1$ dimensions, we will see that one of 15 is a size of one extra dimension and 14 are coset representatives of $\frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5, \mathbb{R})}$ coset space inherited from $7D$.

In [19] the triangular gauge for $6D$ maximal supergravity was only given in terms of the 10×10 vielbein, the underlying 16×16 vielbein still has to fit the one in [19]. In fact, in [51] there is a proposal how to gauge-fix V in 16×16 representation.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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