ON TWO-GENERATOR SUBGROUPS OF MAPPING TORUS GROUPS

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WITH AN APPENDIX BY PETER SHALEN

ABSTRACT. We prove that if $G_\varphi = \langle F, t | txt^{-1} = \varphi(x), x \in F\rangle$ is the mapping torus group of an injective endomorphism $\varphi : F \to F$ of a free group F (of possibly infinite rank), then every two-generator subgroup H of G_φ is either free or a (finitary) sub-mapping torus. As an application we show that if $\varphi \in \mathrm{Out}(F_r)$ is a fully irreducible atoroidal automorphism then every two-generator subgroup of G_φ is either free or has finite index in G_φ.

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1. Introduction

Characterizing subgroups with a given number of generators in a group from some natural class is usually too difficult. However, two-generator subgroups often have more restricted algebraic structure, particularly in various geometric contexts. A well-known result of Delzant [Del91, Théorème 2.1] shows that in a torsion-free word-hyperbolic group G there are only finitely many conjugacy classes of non-elementary freely indecomposable two-generator subgroups. In the setting of right-angled Artin groups, a classical result of Baudisch states that any two-generator subgroup is either free-abelian or free [Bau81, Corollary 1.3]. Antolín and Minasyan generalize this to a structure theorem for two-generator subgroups of graph products of torsion-free groups in a product and subgroup closed family [AM15, Corollary 1.5]. An even older result of Jaco and Shalen from 1979 [JS79, Theorem VI.4.1] proves that if M is an orientable[1] atoroidal Haken 3-manifold and H is a two-generator subgroup of $G = \pi_1(M)$ then either H is free (of rank ≤ 2) or H is free abelian (again of rank ≤ 2) or H has finite index in G. (A result similar to Jaco and Shalen's was proved independently and contemporaneously by Thomas Tucker in [Tuc77].) This theorem applies, for example, to the case where M is a complete finite-volume hyperbolic Haken 3-manifold without boundary. Note that in this case if M is closed then the Jaco–Shalen result implies that every two-generator subgroup of $G = \pi_1(M)$ is either free or has finite index in G.

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[1]All 3-manifolds considered by Jaco and Shalen are assumed to be orientable [JS79, convention p. 2]. See the appendix to this paper for the extension to the non-orientable setting.

In this paper we obtain an analog of the Jaco–Shalen result for two-generator subgroups of mapping tori of free group endomorphisms.

Let F be a free group (of possibly infinite rank), let $\varphi : F \to F$ be an injective endomorphism of F. We call the group

$$G_\varphi = \langle F, t|\, txt^{-1} = \varphi(x), x \in F\rangle$$

the *mapping torus group of* φ. Here φ may be an automorphism of F in which case G_φ is a free-by-cyclic group $G_\varphi = F \rtimes_\varphi \mathbb{Z}$. A finitely generated subgroup $H \le G_\varphi$ is called a *finitary sub-mapping torus of* G_φ or just *sub-mapping torus of* G_φ if there exist $u \in F$, $m \ge 1$ and a nontrivial finitely generated subgroup $V \le F$ such that with $s = ut^m$, the group $H = \langle s, V\rangle$ and $sVs^{-1} \le V$. For brevity we will typically use the term "sub-mapping torus of G_φ", but it is important to remember that in this definition V is required to be finitely generated.

As we see in Lemma 5.2 below, if H is a sub-mapping torus of G_φ as above, then H has the presentation

$$H = \langle s, V|sys^{-1} = \psi(y), y \in V\rangle$$

where $\psi : V \to V$ is an injective endomorphism of the free group V given by $\psi(y) = u\varphi^m(y)u^{-1}$ for all $y \in V$. Thus $H = G_\psi$ is the mapping torus group of the injective endomorphism $\psi : V \to V$ of a finite rank free group V. This fact explains the term "sub-mapping torus."

Theorem A. Let F be a free group, let $\varphi : F \to F$ be an injective endomorphism of F and let G_φ be the mapping torus group of φ.

Then for every two-generator subgroup $H \le G_\varphi$ either H is free or H is conjugate to a finitary sub-mapping torus subgroup of G_φ.

Remark 1.1. If $\varphi : F \to F$ is an injective endomorphism of a free group F such that G_φ is finitely generated, and we include the additional assumption that H is *normal* in G_φ, then (even without requiring the finite rank of H be two), it is well-known that we can get a stronger conclusion. In fact, G_φ has cohomological dimension at most two [Bie81, Proposition 6.2], and coherence of G_φ [FH99] entails that H is finitely presented. Therefore by a theorem of Bieri H is either free or finite index [Bie76, Theorem B].

Note that standard examples of $\mathbb{Z}\times\mathbb{Z}$ and, more generally, Baumslag-Solitar subgroups of G_φ are covered by Theorem A and appear as sub-mapping tori. These subgroups arise from the situations where there exist some $1 \ne x \in F$, $p \ne 0$, $u \in F$, $m \ge 1$ such that $sxs^{-1} = x^p$, where $s = ut^m$. There are also more complicated two-generated sub-mapping torus subgroups that may arise in G_φ, as demonstrated by the following example.

Example 1.2. Let $F = F(a, b, c, d, e)$ and consider an automorphism $\varphi \in \mathrm{Aut}(F)$ given by

$$\varphi(a) = b,\ \varphi(b) = c,\ \varphi(c) = ab^2c^2,\ \varphi(d) = dea,\ \varphi(e) = ede.$$

Put $G_\varphi = \langle F, t|txt^{-1} = \varphi(x), x \in F\rangle$. Note that $K = \langle a, b, c\rangle$ is a proper free factor of F with $\varphi(K) = K$. Consider the subgroup $H = \langle t, a\rangle \le G_\varphi$. Then $H = G_\psi$ is the mapping torus of the automorphism ψ of K where $\psi = \varphi|_K$, and H is a sub-mapping torus of G_φ.

The proof of Theorem A relies on the proof of coherence of mapping tori of free group endomorphisms given by Feighn and Handel [FH99]. In their proof, Feighn and Handel introduced the notion of an *invariant graph pair* (see Section 3) for representing finitely generated subgroups of such mapping tori. They defined the notion of *relative complexity* for invariant graph pairs and proved that there exist *tight* (or *folded*) invariant graph pairs of minimal complexity and that one can read-off the presentation of a subgroup from having such a pair. In general, finding a tight invariant subgroup pair of minimal complexity is a non-constructive step in Feighn and Handel's proof of coherence. However, for two-generator subgroups we are able to overcome this difficulty and analyze precisely what happens algebraically because the two-generator assumption implies that the minimal relative complexity for subgroups under consideration is either 0 or 1.

If $\varphi \in \mathrm{Aut}(F)$, then the isomorphism class of G_φ depends only on the outer automorphism class of φ in $\mathrm{Out}(F)$. For that reason, if $\varphi \in \mathrm{Out}(F)$, by a slight abuse of notation we will denote by G_φ the group G_ψ, where $\psi \in \mathrm{Aut}(F)$ is any representative of the outer automorphism class φ.

In the case where $\varphi \in \mathrm{Aut}(F)$ then every subgroup of G_φ conjugate to a sub-mapping torus subgroup of G_φ is itself a sub-mapping torus subgroup of G_φ. Therefore in the case where φ is an automorphism of F, the "conjugate to" phrase in the conclusion of Theorem A can be dropped. Further, the structure and geometry of a sub-mapping torus can be determined.

Corollary B. Suppose G_φ is a free-by-cyclic group $F_r \rtimes_\varphi \mathbb{Z}$. Then a two-generator subgroup of G_φ is either free of rank 2, or an undistorted (f.g. free)-by-cyclic sub-mapping torus subgroup of G_φ.

Recall that for an integer $r \geq 2$, an element $\varphi \in \mathrm{Aut}(F_r)$ (or the corresponding outer automorphism class in $\mathrm{Out}(F_r)$) is called *fully irreducible* if there do not exist a power $p \geq 1$ and a proper free factor $1 \neq U \lneq F_r$ of F_r such that $\varphi^p([U]) = [U]$, where $[U]$ is the conjugacy class of U in F_r. Also an element $\varphi \in \mathrm{Aut}(F_r)$ (or the corresponding outer automorphism class in $\mathrm{Out}(F_r)$) is called *atoroidal* if there do not exist a power $p \geq 1$ and an element $1 \neq u \in F_r$ such that $\varphi^p([u]) = [u]$, where again $[u]$ is the conjugacy class of u in F_r. The notion of being fully irreducible provides the main $\mathrm{Out}(F_r)$ analog of being a pseudo-Anosov mapping class group element [CH12]. By a result of Brinkmann [Bri00, Theorem 1.2], for $\varphi \in \mathrm{Out}(F_r)$ the group G_φ is word-hyperbolic if and only if φ is atoroidal. It is known that for most reasonable random walks on $\mathrm{Out}(F_r)$, where $r \geq 2$, a "random" element of $\mathrm{Out}(F_r)$ is fully irreducible, and for $r \geq 3$ it is also atoroidal by [Riv10, Theorem 4.1] or [Sis18, Theorem 1.4, Proposition 3.9].

As an application of Theorem A we obtain:

Theorem C. Let F_r be a free group of finite rank $r \geq 2$, let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible outer automorphism and let G_φ be the mapping torus group of φ.

Then for every two-generator subgroup $H \leq G_\varphi$ either H is free, free abelian, a Klein bottle group, or H has finite index in G_φ.

Here by the Klein bottle group we mean the Baumslag-Solitar group $BS(1,-1) = \langle a, b | bab^{-1} = a^{-1}\rangle$.

Remark 1.3. In Theorem C (and similar statements below), $H \leq G_\varphi$ can be finite index only if $b_1(G_\varphi) \leq 2$, as by the existence of transfer, the first Betti number is non-decreasing over finite index subgroups (see Lemma 6.4 below).

Remark 1.4. An obvious way to get 2-generated free subgroups in a free-by-cyclic group G_φ such as the one in Theorem C is as subgroups of fibers of algebraic fibrations of G_φ. In fact, one can see that not all 2-generated free subgroups of G_φ arise this way: For instance, one can construct a group G_φ as in Theorem C (for $r \geq 3$, so that G_φ is word-hyperbolic) such that $b_1(G_\varphi) = 1$ in which case the subgroup $H = \langle x^n, t^n\rangle$ is free, for sufficiently large n, but is not contained in the unique fiber of G_φ. On the other hand, it is not clear whether one could find a finite-index subgroup of G_φ admitting a fibration whose fiber contains H.

If $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible but not atoroidal, then G_φ is a 3-manifold group. In this case the conclusion of Theorem C might be derived using the methods of Jaco and Shalen mentioned above. As we explain below, the non-orientable case presents some extra complication to the 3-manifold approach, requiring additional work.

In fact, using properties of the pseudo-Anosov homeomorphism we generalize one case of the Jaco–Shalen result by removing the orientability assumption:

Corollary D. Let G be the fundamental group of a fibered cusped hyperbolic 3-manifold. Then every two-generator subgroup $H \leq G$ is either free, free abelian, a Klein bottle group, or H has finite index in G.

One might expect an easier alternate argument for the proof of Theorem C in the case where $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible but not atoroidal, using 3-manifold methods. Since $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible but not atoroidal, a result of Bestvina and Handel [BH95] implies that φ is induced by a pseudo-Anosov homeomorphism f of a compact connected surface Σ with a single boundary component. Then the mapping torus M of f is a (truncated) finite volume complete hyperbolic 3-manifold with a single cusp which has $G_\varphi = \pi_1(M)$. Note that M is orientable if and only if Σ is orientable and f is orientation preserving. With this set-up one might hope to use the result of Jaco and Shalen [JS79, Theorem VI.4.1] that motivates this paper to obtain the same conclusion about two-generator subgroups of G_φ: that every such subgroup is either free or free abelian or the Klein bottle group or has finite index in G_φ. However, it is not straightforward to apply that result to the non-orientable case, and some further techniques from Jaco and Shalen's work are needed to handle non-orientable fibers. (A more recent paper of Boileau and Weidmann [BW05] about 3-manifolds with two-generated fundamental group also only considers the orientable case.) The same issue arises in trying to apply the results of Jaco and Shalen [JS79] to obtain a 3-manifold proof of Corollary D.

Nevertheless, in the appendix to the present paper Shalen remedies this situation and generalizes the relevant results to include $\mathbb{P}^2$-irreducible 3-manifolds that may be non-orientable (and may fail to be Haken manifolds, i.e. to be "sufficiently large"). In particular, Corollary A.2 in the appendix implies the conclusion of Corollary D (and thus also of Theorem C in the fully irreducible but non-atoroidal case).

Corollary A.2 (P. Shalen)**.** Let N be a compact, $\mathbb{P}^2$-irreducible 3-manifold such that every rank-2 free abelian subgroup of $\pi_1(N)$ is peripheral. Let H be a subgroup of $\pi_1(N)$ that has rank at most 2. Then H is either a free group, a free abelian group, a Klein bottle group, or a finite-index subgroup of $\pi_1(N)$.

This result is in turn deduced from a more general result showing that for any fixed integer $k \geq 2$, under some additional algebraic assumptions about the fundamental group of a 3-manifold N, every k-generated subgroup of $G = \pi_1(M)$ is isomorphic to a free product of some free group, some number of copies of $\mathbb{Z}^2$ and some number of copies of the Klein bottle group.

Theorem A.1 (P. Shalen)**.** Let N be a compact, $\mathbb{P}^2$-irreducible 3-manifold, and let $k \geq 2$ be an integer. Suppose that every rank-2 free abelian subgroup of $\pi_1(N)$ is peripheral. Suppose also that either

(a) N is orientable and $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a closed, orientable surface S such that $1 < \operatorname{genus} S < k$, or

(b) $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a closed surface S such that $\chi(S)$ is even and $2 - 2k < \chi(S) < 0$.

Let H be a subgroup of $\pi_1(N)$ that has rank at most k. Then either H has finite index in $\pi_1(N)$, or H is a free product of finitely many subgroups, each of which is a free abelian group of rank at most 2 or a Klein bottle group.

It would be interesting to understand if there exist any reasonable analogues of Theorem A.1 in the context of free-by-cyclic groups or, more generally, mapping tori of free group endomorphisms.

Returning to outer automorphisms, when $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible and atoroidal (which necessarily implies that $r \geq 3$), G_φ is a word-hyperbolic group [Bri00, Theorem 1.2] and hence G_φ contains no $\mathbb{Z} \times \mathbb{Z}$ subgroups (and therefore G_φ cannot contain a Klein bottle subgroup either). Therefore in this case Theorem C implies a sharper corollary.

Corollary E. Let F_r be a free group of finite rank $r \geq 3$, let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible atoroidal outer automorphism.

Then every two-generator subgroup $H \leq G_\varphi$ of the mapping torus group is either free or has finite index in G_φ.

It turns out that in the above applications of Theorem A one can remove the "or has finite index in G" alternative in the conclusion of these results by replacing G by a suitable subgroup G_0 of finite index at the outset:

Corollary F. For each of the following groups G there exists a finite index subgroup G_0 such that every two-generator subgroup of G_0 is either free, free abelian, or the Klein bottle group.

(1) If $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible, $r \geq 2$ and $G = G_\varphi$. If φ is additionally atoroidal, every two-generator subgroup of G_0 is free.
(2) If G is the fundamental group of a fibered cusped hyperbolic 3–manifold.

Another application of Theorem A (where cyclic extensions of infinite rank free groups play a role) concerns one-relator groups with torsion. Pride [Pri77] proved that if $G = \langle x_1, \ldots, x_r | w^n = 1\rangle$, where w is a nontrivial freely and cyclically reduced word and where $n \geq 2$, then every two-generator subgroup of G is either a free product of two cyclic groups or itself is a one-relator group with torsion. Since G as above is known to be virtually torsion-free, it follows that G has a subgroup of finite index G_0 such that every two-generator subgroup of G_0 is free. We recover a weaker version of this result, under the extra assumption that n is big enough.

Theorem G. Let $G = \langle x_1, \ldots, x_r | w^n = 1\rangle$ where $r \geq 2$, $w \in F(x_1, \ldots, x_r)$ is a nontrivial freely and cyclically reduced word and n is an integer such that $n \geq |w| \geq 2$. Then G contains a subgroup of finite index G_0 such that every two-generator subgroup of G_0 is free.

Note that the subgroup $G_0 \leq G$ is free-by-cyclic as well. We also obtain a similar in spirit application for RFRS groups.

Corollary H. Suppose G is a finitely generated RFRS group with $b_2^{(2)}(G) = 0$ and $\mathrm{cd}_{\mathbb{Q}}(G) = 2$. Then there is a finite index subgroup G_0 of G such that every two generator subgroup H of G_0 is free or the mapping torus of an endomorphism of a finitely generated free group.

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2. Labelled graphs and subgroups of free groups

Let F be a free group with a free basis A. We will utilize the standard machinery of Stallings subgroup graphs and Stallings folds for working with subgroups of $F = F(A)$. We direct the reader to standard references [Sta83, KM02] and only briefly recall some relevant notions here. An *A-graph* is a graph (i.e. a 1-complex) X where every oriented edge e is given a label $\mu(e) \in A^{\pm 1}$ so that for every e we have $\mu(e^{-1}) = (\mu(e))^{-1}$. The *$A$-rose* R_A is an A-graph R_A with a single vertex v_0 and one loop edge at v_0 labelled by $a \in A$ for each $a \in A$. For an A-graph X, the edge-labeling μ for X defines a map $f_X : X \to R_A$ that send all vertices of X to v_0 and maps each open 1-cell of X labelled by $a \in A$ homeomorphically, respecting orientation, to the open 1-cell labelled by a in R_A. An A-graph X is *tight* or *folded* if the map $f_X : X \to R_A$ is an immersion; that is, if there do not exist two distinct oriented edges e_1, e_2 in X with the same initial vertex $o(e_1) = o(e_2)$ and with $\mu(e_1) = \mu(e_2)$.

We naturally identify $F = F(A) = \pi_1(R_A, v_0)$. If an A-graph X has a base-vertex $*$, we denote by $X^\#$ the subgroup

$$X^\# = (f_X)_\#(\pi_1(X, *)) \le F = F(A).$$

Thus $X^\#$ consists of all the elements of $F = F(A)$ represented by labels of closed loops in X at $*$. Note that if X is folded then $(f_X)_\# : \pi_1(X, *) \to X^\#$ is an isomorphism.

Suppose X is an A-graph that is not tight and that e_1, e_2 are two distinct oriented edges in X with $o(e_1) = o(e_2)$ and $\mu(e_1) = \mu(e_2)$. A *Stallings fold* on the pair e_1, e_2 is a map $q : X \to X'$ that identifies e_1, e_2 into a single edge with label $\mu(e_1) = \mu(e_2)$, and identifies the terminal vertices $t(e_1)$, $t(e_2)$ if they were distinct in X.

3. Standard subgroups of mapping torus groups and invariant graph pairs

Convention 3.1. For the remainder of this paper, unless specified otherwise we assume that F is a free group, $\varphi : F \to F$ is an injective endomorphism of F and

$$G_\varphi = \langle F, t | \, txt^{-1} = \varphi(x), x \in F\rangle$$

is the mapping torus group of φ. We also fix a free basis A of F.

Definition 3.2. We say that a finitely generated subgroup $H \le G_\varphi$ is *standard* if $H = \langle t, V\rangle$ where $V \le F$ is a nontrivial finitely generated subgroup of F.

Definition 3.3 (Invariant graph pairs)**.** Let $H \le G_\varphi$ be a standard finitely generated subgroup. Let Z be a finite connected A-graph with a base-vertex $*$ and let $X \subseteq Z$ be a connected subgraph containing $*$. We say that the pair (Z, X) is an *invariant graph pair for H* if the following conditions hold:

(1) $H = \langle t, X^\#\rangle$;
(2) $Z^\# = \langle X^\#, \varphi(X^\#)\rangle$;
(3) $Z^\# \le H \cap F$;

4. Relative complexity

For an invariant graph pair (Z, X), its *relative rank* $\mathrm{rr}(Z, X)$ is defined as

$$\mathrm{rr}(Z, X) = \mathrm{rank}(\pi_1(Z)) - \mathrm{rank}(\pi_1(X)) = \chi(X) - \chi(Z).$$

Let (Z, X) be an invariant graph pair for H. Let $T \subseteq Z$ be a maximal subtree in Z such that $T' = T \cap X$ is a maximal subtree in X. For each (oriented) edge e of $Z - T$ labelled by a letter from the free basis A of F we get a loop $\gamma_e = [*, o(e)]_T e[t(e), *]_T$. The labels of these loops give us generating sets $\{c_1, \dots, c_k\}$ of $X^\#$ and $\{c_1, \dots, c_k, d_1, \dots, d_q\}$ of $Z^\#$. We call them the *associated* generating sets of $X^\#$ and $Z^\#$ for (Z, X). Note that $q = \mathrm{rr}(Z, X)$.

Definition 4.1 (Minimal relative complexity)**.** Let $H \le G_\varphi$ be a standard finitely generated subgroup. Denote by $\mathrm{rr}_0(H)$ the minimum relative rank among all invariant graph pairs for H.

The following useful observation is an immediate consequence of the definitions:

Lemma 4.2. *Let $H \le G_\varphi$ be a standard finitely generated subgroup. Suppose that $\mathrm{rr}_0(H) = 0$. Then H is a standard sub-mapping torus of G_φ.*

Proof. Let (Z, X) be an invariant graph pair for H with $\mathrm{rr}(Z, X) = 0$. Hence the inclusion of X into Z is a homotopy equivalence and we may make Z smaller if needed and assume that $X = Z$. Hence $Z^\# = \langle X^\#, \varphi(X^\#)\rangle = X^\#$, and therefore $\varphi(X^\#) \le X^\#$. Thus $H = \langle t, X^\#\rangle$ is a standard sub-mapping torus of G_φ, as claimed. □

Feighn and Handel define a *tightening* process for reducing the relative rank of invariant graph pairs [FH99]. Moves in this process are either ordinary Stallings folds or “adding-of-loop” moves, where the latter can only occur after the so-called “exceptional” folds. The relative rank is monotone non-increasing throughout the tightening process. The details of the tightening process are not important for our purposes, but we record here several key results about it.

Proposition 4.3. *Let $H \leq G_\varphi$ be a standard finitely generated subgroup.*

(1) There exists a tight invariant graph pair (Z, X) for H with $\mathrm{rr}(Z, X) = \mathrm{rr}_0(H)$.
(2) Suppose that (Z, X) is a tight invariant graph pair for H with $\mathrm{rr}(Z, X) = \mathrm{rr}_0(H)$. Let $\{c_1, \ldots, c_k\}$ and $\{c_1, \ldots, c_k, d_1, \ldots, d_q\}$ be associated generating sets of $X^\#$ and $Z^\#$ accordingly. Thus for $i = 1, \ldots, k$ there exists a word $w_i = w_i(c_1, \ldots, c_k, d_1, \ldots, d_q)$ such that $tc_it^{-1} = w_i$.
Then the subgroup H has the presentation

$$H = \langle c_1, \ldots, c_k, d_1, \ldots, d_q, t | tc_it = w_i, i = 1, \ldots, k\rangle.$$

Proof. These statements are reformulations or immediate consequences from Feighn and Handel’s paper [FH99], this proof refers to their results.

Part (1) follows from [FH99, Proposition 5.4], which states that relative rank is monotone non-increasing under tightening.

Part (2) is a restatement of the proof of the Main Proposition in [FH99, Section 6]. □

We also need the following classic 1939 result of Magnus [Mag39, Satz 1].[2]

Proposition 4.4. *Let $m \geq n \geq 0$ be integers and let $G = \langle a_1, \ldots, a_m | r_1, \ldots r_n\rangle$ be a group given by a presentation with m generators and n defining relators. Suppose also that G can be generated by $m - n$ elements. Then G is free of rank $m - n$.*

Corollary 4.5. *If H s a standard subgroup of G_φ of the form $H = \langle t, v\rangle$, where $v \in F$, and such that $\mathrm{rr}_0(H) = 1$, then H is free of rank 2.*

Proof. By Proposition 4.3 (1) there is a tight invariant graph pair (Z, X) for H with $\mathrm{rr}(Z, X) = \mathrm{rr}_0(H) = 1$. The associated generating sets of $X^\#$ and $Z^\#$ have the form $\{c_1, \ldots, c_k\}$ and $\{c_1, \ldots, c_k, d_1\}$ where $k = \mathrm{rank}(\pi_1(X))$. Proposition 4.3 (2) then implies that there exists words $w_i = w_i(c_1, \ldots c_k, d_1)$ where $tc_it^{-1} = w_i$ and H has the presentation

$$H = \langle t, c_1, \ldots, c_k, d_1 | tc_it = w_i, i = 1, \ldots, k\rangle.$$

This presentation has $k + 2$ generators and k defining relators. By assumption $H = \langle t, v\rangle$ is two-generated. Therefore, by Proposition 4.4, H is free of rank 2. □

Proposition 4.6. *Let $1 \neq v \in F$ and $H = \langle t, v\rangle \leq G_\varphi$. Then either H is free of rank 2 or H is a standard sub-mapping torus of G_φ.*

Proof. Let X' be a loop at $*$ labelled by the freely reduced form of v and let Z' be obtained from X' by wedging it at $*$ with a loop labelled by the freely reduced form of $\varphi(v)$ in F. Then (Z', X') is an invariant graph pair for H with $\mathrm{rr}(Z', X') = 1$. Therefore $\mathrm{rr}_0(H) \leq 1$.

If $\mathrm{rr}_0(H) = 0$, then by Lemma 4.2 H is a standard sub-mapping torus of G_φ. If $\mathrm{rr}_0(H) = 1$, then because $H = \langle t, v\rangle$ is a two-generated standard subgroup of G_φ by hypothesis, Corollary 4.5 implies H is free of rank 2. □

[2]An English presentation can be found in Magnus, Karrass, and Solitar [MKS04, p. 354, Corollary 5.14.2]. Magnus’ result can also be proved using homological methods [Sta67] or representation variety techniques [Lir97]

5. Subgroups of mapping torus groups

Recall that in Convention 3.1 we have fixed a mapping torus group G_φ for an injective endomorphism $\varphi : F \to F$. First we record the following standard consequences of the definition of G_φ:

Lemma 5.1. *With G_φ and $\varphi : F \to F$ as in Convention 3.1,*

(1) For any $u \in F$ and $s = ut$ the group $G_\varphi = \langle F, s\rangle$ has the presentation

$$G_\varphi = \langle F, s|\, sxs^{-1} = \xi(x), x \in F\rangle$$

where $\xi : F \to F$ is an injective endomorphism given by $\xi(x) = u\varphi(x)u^{-1}$ for all $x \in F$.

(2) For any integer $m \geq 1$ the subgroup $\langle t^m, F\rangle$ has index m in G_φ and with $s = t^m$ this subgroup has the presentation

$$\langle F, s|\, sxs^{-1} = \varphi^m(x), x \in F\rangle \cong G_{\varphi^m}$$

(3) For any $u \in F$, $m \geq 1$ the subgroup $\langle ut^m, F\rangle = \langle t^m, F\rangle$ has index m in G_φ and with $s = ut^m$ this subgroup has the presentation

$$\langle F, s|\, sxs^{-1} = \psi(x), x \in F\rangle.$$

where $\psi : F \to F$ is an injective endomorphism given by $\psi(x) = u\varphi^m(x)u^{-1}$.

Proof. Part (1) follows by performing a Tietze transformation (introducing s, then eliminating t) on the presentation of G_φ from Convention 3.1.

For part (2), note that for $m \geq 1$ we have $\langle t^m, F\rangle = \ker\theta$ where $\theta : G_\varphi \to \mathbb{Z}_m$ is the epimorphism defined as $\theta|_F = 0$ and $\theta(t) = [1]_m \in \mathbb{Z}_m$. Hence $\langle t^m, F\rangle$ is a subgroup of index m in G_φ. Moreover, $F \lhd G_\varphi$ and $t^m xt^{-m} = \varphi^m(x)$ for all $x \in F$, which implies that inside $G_\varphi = F \rtimes_\varphi \langle t\rangle$ we have $\langle t^m, F\rangle = F \rtimes_{\varphi^m} \langle t^m\rangle$, and part (2) follows.

Part (3) now follows from part (2) by a Tietze transformation. □

Lemma 5.2. *Let $H = \langle ut^m, V\rangle \leq G_\varphi$ be a sub-mapping torus subgroup of G_φ, where $u \in F$, $m \geq 1$ and $V \leq F$ is a nontrivial finitely generated subgroup such that $\varphi(V) \leq V$. Then with $s = ut^m$ the subgroup H has the presentation*

$$H = \langle s, V|sys^{-1} = \psi(y), y \in V\rangle$$

where $\psi : V \to V$ is an injective endomorphism defined as $\psi(x) = u\varphi^m(x)u^{-1}$.

Proof. Consider a group K given by the presentation $K = \langle s, V|sys^{-1} = \psi(y), y \in V\rangle$. Then there is a natural homomorphism $\alpha : K \to G_\varphi$, $\alpha(s) = s$, $\alpha(y) = y$ for $y \in V$ with $\alpha(K) = H$. We claim that α is injective.

Indeed, it is not hard to see that every element of K has the form $s^{-p}ys^q$ for some $p, q \geq 0$ and $y \in V$. Hence every nontrivial element of K is conjugate in K to either some $1 \neq y \in V$ or to ys^n with $y \in V$ and $n \neq 0$. If $1 \neq y \in V$ then $\alpha(y) = y \neq 1$ in G_φ. Similarly, if $y \in V$ and $n \neq 0$ then $\alpha(ys^n) = ys^n \neq 1$ in G_φ. Therefore α is injective, as claimed.

Hence $\alpha : K \to H$ is an isomorphism and the statement of the lemma follows. □

Proposition 5.3. *Let $H = \langle s, V\rangle \leq G_\varphi$ be a sub-mapping torus subgroup of G_φ, where $u \in F$, $m \geq 1$, $s = ut^m$, and $V \leq F$ is a nontrivial finitely generated subgroup such that $\varphi(V) \leq V$. Let $\psi : V \to V$ be an injective endomorphism of V defined as in Lemma 5.2, that is $\psi(x) = u\varphi^m(x)u^{-1}$ for $x \in V$.*

Suppose that $\varphi : F \to F$ is an automorphism of F.

Then $sVs^{-1} = V$, ψ is an automorphism of V and $H = V \rtimes_\psi \langle s\rangle$ is a free-by-cyclic group.

Proof. Since H is a sub-mapping torus of G_φ, we have $sVs^{-1} \leq V$. Conjugation by s is an automorphism φ of F. Thus if $sVs^{-1} \leq V$ is a proper subgroup of V then

$$V \lneq s^{-1}Vs \lneq s^{-2}Vs^2 \lneq \cdots \lneq s^{-n}Vs^n \lneq \dots$$

is an infinite strictly ascending chain of subgroups of F of fixed finite rank. This is impossible by the Higman-Takahasi Theorem (e.g. see [KM02, Theorem 14.1]). Thus $sVs^{-1} = V$ and the statement of the proposition follows from Lemma 5.2. ☐

Theorem A. Let F be a free group, let $\varphi : F \to F$ be an injective endomorphism of F and let G_φ be the mapping torus group of φ.

Then for every two-generator subgroup $H \leq G_\varphi$ either H is free or H is conjugate to a sub-mapping torus subgroup of G_φ.

Proof. Let $H = \langle x, y\rangle \leq G_\varphi$ be a two-generator subgroup of G. Recall that every element of G_φ has the form $t^{-p}zt^q$ for some $z \in F$ and $p, q \geq 0$. Let $x = t^{-p_1}z_1t^{q_1}$ and $y = t^{-p_2}z_2t^{q_2}$ as above. By inverting x, y if necessary we may assume that $q_1 \geq p_1 \geq 0$ and $q_2 \geq p_2 \geq 0$. Put $n = \max\{q_1, q_2\}$ and let $x' = t^nxt^{-n}, y' = t^nyt^{-n}$. Then $x' = v_1t^{n_1}, y' = v_2t^{n_2}$ where $v_1, v_2 \in F$ and $n_1, n_2 \geq 0$. More precisely, $n_i = q_i - p_i$ and $v_i = t^{n-p_i}z_it^{-(n-p_i)}$ for $i = 1, 2$. Moreover $H' = \langle x', y'\rangle = t^nHt^{-n}$ is conjugate to H in G_φ. If $n_1 = n_2 = 0$ then $H' = \langle v_1, v_2\rangle \leq F$. Therefore H' is free and hence H is free as well.

Thus we may assume that $n_1 + n_2 > 0$. We may also assume that $n_1 > 0$ and $n_1 \geq n_2$. We can then perform the Euclidean algorithm on the exponents n_1, n_2 of t in the generators x', y' and transform the generating set x', y' of H' to a generating set $H' = \langle ut^m, v\rangle$ where $u, v \in F$ and where $m \geq 1, m = gcd(n_1, n_2)$. If $v = 1$ then H' is infinite cyclic and therefore free, as required. Thus we may assume that $v \neq 1$.

Put $s = ut^m$. By Lemma 5.1 (3), the subgroup $\langle s, F\rangle$ of G_φ has index m in G_φ and has the presentation

$$G_\psi = \langle F, s |\, sxs^{-1} = \psi(x), x \in F\rangle.$$

where $\psi : F \to F$ is given by $\psi(x) = u\varphi^m(x)u^{-1}$ for all $x \in F$. Now $H' = \langle s, v\rangle \leq G_\psi$ is a standard two-generator subgroup of G_ψ with $v \neq 1$. Hence, by Proposition 4.6, either H' is free of rank 2 or H is a standard sub-mapping torus of G_ψ.

If H' is free of rank 2 then $H \cong H'$ is free of rank 2 as well. If H' is a standard sub-mapping torus of G_ψ then H' is a sub-mapping torus of G_φ and H is conjugate to a sub-mapping torus of G_φ, as required. ☐

6. Application to free-by-cyclic groups

When G_φ is the mapping torus group of an automorphism of a rank r free group F_r, more structure is available. First we observe an immediate corollary on the distortion of a two-generator subgroup.

Corollary B. Suppose G_φ is a free-by-cyclic group $F_r \rtimes_\varphi \mathbb{Z}$. Then a two-generator subgroup of G_φ is either free of rank 2, or an undistorted (f.g. free)-by-cyclic sub-mapping torus subgroup of G_φ.

Proof. Suppose $H \leq G_\varphi$ is two-generated and not isomorphic to F_2. Then by Theorem A it is either cyclic or conjugate to a sub-mapping torus subgroup of H. In the second case, the sub-mapping torus subgroup $\langle V, s\rangle$ is itself a (f.g. free)-by-cyclic group: since s induces an automorphism of F_r, in fact we have $sVs^{-1} = V$, see Proposition 5.3 above.

It then follows from work of Mutanguha ([Mut24, Lemma 4.1] in the sub-mapping torus case, and [Mut24, Lemma 4.2] in the cyclic case) that H is undistorted in G_φ. ☐

Next we also impose conditions on the automorphism φ. Recall that an outer automorphism $\varphi \in \mathrm{Out}(F_r)$ is *fully irreducible* if no conjugacy class of a proper free factor is φ-periodic, and *atoroidal* if no conjugacy class of a nontrivial element is φ-periodic. Note that these properties are invariant under taking nonzero powers. The possible sub-mapping torus groups are highly constrained by the structure of fully irreducible outer automorphisms.

Lemma 6.1. *Suppose F_r is a free group of finite rank $r \geq 2$ and let $\varphi \in Out(F_r)$ be a fully irreducible outer automorphism. Let $V \leq F_r$ be a nontrivial finitely generated subgroup such that $\varphi(V) \subseteq V$ and such that $V \neq \langle u \rangle$ where the conjugacy class $[u]$ is φ-periodic. Then V is of finite index in F_r.*

Proof. This lemma is a consequence of Bestvina, Feighn, and Handel's work on lamination theory for $\mathrm{Out}(F_r)$. For brevity we do not include the definition of a lamination or weak convergence. The assumptions on V imply that there exists a nontrivial element $v \in V$ whose conjugacy class is not φ-periodic and such that iterates $\varphi^n(v) \in V$ for all $n \geq 1$. These iterates $\varphi^n(v)$ can be used to construct a leaf of Λ^+_φ, satisfying the definition of V carrying Λ^+_φ [BFH97, Definition 2.2]. If a finitely generated subgroup $V \leq F_r$ carries the stable lamination Λ^+_φ of an irreducible φ, then V is finite index in F_r, as claimed [BFH97, Proposition 2.4]. ☐

The following consequence of Lemma 6.1 is a well-known folklore result that is commonly attributed to Bestvina, Feighn, and Handel [BFH97].

Lemma 6.2. *Let $\varphi \in Aut(F_r)$ be a fully irreducible atoroidal automorphism. Let $U \leq F_r$ be a subgroup of finite index and let $n \geq 1$ be such that $\varphi^n(U) = U$. Put $\psi = \varphi^n|_U \in Aut(U)$. Then ψ is again fully irreducible and atoroidal.*

Proof. Since φ has no nontrivial periodic conjugacy classes in F_r, it follows that ψ has no nontrivial periodic conjugacy classes in U, so that $\psi \in \mathrm{Aut}(U)$ is atoroidal. Suppose that ψ is not fully irreducible. Then there exists a nontrivial proper free factor V of U such that $\psi^m(V) = u^{-1}Vu$ for some $u \in U$ and $m \geq 1$. Thus $u\varphi^{mn}(V)u^{-1} = V$. Note that V has infinite index in U and hence in F_r as well. The automorphism φ^{mn} of F_r is fully irreducible in $\mathrm{Aut}(F_r)$ since φ is fully irreducible. Hence $\varphi' \in \mathrm{Aut}(F_r)$, $\varphi'(w) = u\varphi^{mn}(w)u^{-1}$, where $w \in F_r$, is also fully irreducible in $\mathrm{Aut}(F_r)$ as it equals φ^{mn} in $\mathrm{Out}(F_r)$. For φ' we have $\varphi'(V) = V$, where $V \leq F_r$ is a nontrivial subgroup of infinite index in F_r. Note that φ' is also atoroidal. Then Lemma 6.1 implies that V has finite index in F_r, yielding a contradiction. ☐

Theorem C. Let F_r be a free group of finite rank $r \geq 2$, let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible outer automorphism and let G_φ be the mapping torus group of φ.

Then for every two-generator subgroup $H \leq G_\varphi$ either H is free, free abelian, a Klein bottle group, or H has finite index in G_φ.

Proof. Let $H \leq G_\varphi$ be a two-generator subgroup. By Theorem A, either H is free or H is conjugate to a subgroup $H' \leq G_\varphi$ such that H' is a sub-mapping torus subgroup.

If H is free then the conclusion of Theorem C holds.

Suppose now that H is conjugate to a subgroup $H' \leq G_\varphi$ such that H' is a sub-mapping torus subgroup of G_φ. Thus $H' = \langle s, V\rangle$ where $V \leq F_r$ is a nontrivial finitely generated subgroup, $s = ut^m$ with $u \in F_r$ and $m \geq 1$ and, moreover, $sVs^{-1} \leq V$. By Lemma 5.1 (3), the subgroup $\langle s, F_r\rangle$ has index m in G_φ and, moreover, $\langle s, F_r\rangle = G_\psi$ where $\psi : F_r \to F_r$ is the automorphism given by $\psi(x) = u\varphi^m(x)u^{-1}$ for all $x \in F_r$. Then $\psi = \varphi^m$ in $\mathrm{Out}(F_r)$ and in particular ψ is also fully irreducible. We also have $\psi(V) \leq V$ for a nontrivial finitely generated subgroup $V \leq F_r$. Since ψ is an automorphism of F_r, it follows that $\psi(V) = V$.

If V has rank 1, then $V = \langle c\rangle$ is infinite cyclic and either $\psi(c) = c$ or $\psi(c) = c^{-1}$. If $\psi(c) = c$, then $H \cong H' = \langle s, V\rangle \cong \mathbb{Z} \times \mathbb{Z}$. If $\psi(c) = c^{-1}$ then

$$H \cong H' = \langle s, V\rangle = \langle s, c | scs^{-1} = c^{-1}\rangle,$$

the Klein bottle group.

If V has rank at least 2 then, by Lemma 6.1 V has finite index in F_r. Therefore $H' = \langle s, V\rangle$ has finite index in G_ψ and therefore also in G_φ. Since H is conjugate to H' in G_φ, it follows that H has finite index in G_φ, as required. □

For automorphisms induced by a pseudo-Anosov homeomorphism on a punctured surface, we have an alternative to Lemma 6.1. The following lemma is likely well understood in low-dimensional topology, and the specific proof given below was suggested to us by Chris Leininger.

Lemma 6.3. *Suppose F_r is a free group of finite rank $r \geq 2$ and let $\varphi \in Out(F_r)$ be induced by a pseudo Anosov homeomorphism on some punctured surface Σ with $\pi_1(\Sigma) \cong F_r$. If $V \leq F_r$ is a nontrivial finitely generated subgroup that is not conjugate into a subgroup generated by a boundary curve and $\varphi(V) \subseteq V$ then V is finite index in F_r.*

Proof. Suppose V is infinite index. Then subgroup separability of free groups (or indeed of surface groups) implies that there is a finite index subgroup of F_r with V as a retract. In terms of the surface, this means that there is a finite degree cover Σ_0 of Σ with V the fundamental group of a proper subsurface of Σ_0. Let $\mathcal{C}$ be the isotopy classes of curves on the boundary of this subsurface but not of Σ_0.

Let f be the pseudo-Anosov homeomorphism inducing φ, and recall that every power of f is again pseudo-Anosov, and that if a pseudo-Anosov homeomorphism lifts to a cover, it induces a pseudo-Anosov homeomorphism of the cover. In particular, since Σ_0 is a finite degree cover, some power f^k lifts to a pseudo-Anosov on Σ_0, and will still preserve V. But then f^k permutes the curves in $\mathcal{C}$, and must be reducible, a contradiction. □

Using this lemma, we can reproduce algebraically a special case of the Jaco–Shalen result for 3-manifold groups, needing no assumption on orientability.

Corollary D. Let G be the fundamental group of a fibered cusped hyperbolic 3-manifold M. Then every two-generator subgroup $H \leq G$ is either free, free abelian, a Klein bottle group, or H has finite index in G.

Proof. Since M is cusped and fibered, it is a mapping torus of a punctured surface Σ, and G is a free-by-cyclic group G_φ, where the underlying free group F_r is the fundamental group of Σ. Since M is hyperbolic, the homeomorphism in question is pseudo-Anosov. Applying Theorem A we see that H is either free or conjugate to a sub-mapping torus. As in the proof of Theorem C suppose H is conjugate to a sub-mapping torus $H' = \langle s, V\rangle$, where $V \leq F_r$ is a non-trivial finitely generated subgroup, and $s = ut^m$ with $u \in F_r$, $m \geq 1$ and $sVs^{-1} \leq V$. Considering the index m subgroup G_ψ with $\psi(x) = u\varphi^m(x)u^{-1}$, again we see $\psi(V) = V$.

Now apply Lemma 6.3 to conclude that either V is rank 1, and contained in a boundary subgroup, or of finite index in F_r. In the rank 1 case, as before we have $V = \langle c\rangle$ and $H \cong \mathbb{Z}^2$ if $\psi(c) = c$ or a Klein bottle group if $\psi(c) = c^{-1}$. In each case, H is a cusp subgroup. Otherwise V is of finite index in F_r, and H' is finite index in G_ψ and hence in $G_\varphi = G$. Since H is conjugate to H', H is of finite index in G, as required. The final part of the statement follows as in Theorem C. □

Note that for $F_2 = F(a, b)$ and $u = [a, b]$, for every $\varphi \in \mathrm{Aut}(F_2)$ we have $\varphi^2([u]) = [u]$. Thus F_2 has no atoroidal automorphisms.

Corollary E. Let F_r be a free group of finite rank $r \geq 3$, let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible atoroidal outer automorphism. Then every two-generator subgroup $H \leq G_\varphi$ of the mapping torus group is either free or has finite index in G_φ.

Proof. If φ is atoroidal, then so is ψ in the proof above, therefore the case where V is rank 1 is excluded, and the conclusion follows. Alternately, since φ is fully irreducible and atoroidal, G_φ is hyperbolic [Bri00, Theorem 1.2]. By Theorem C the subgroup H is free, free abelian, a Klein bottle group, or has finite index. Since G_φ is hyperbolic, H cannot be $\mathbb{Z}\times\mathbb{Z}$ or a Klein bottle group. The last part of the statement follows as usual. □

The next corollary shows that, passing to a suitable finite index subgroup of G_φ, one can dispense with the possibility that a two-generator subgroup is finite index. One ingredient of the proof is the following well-known lemma, whose proof we include for completeness.

Lemma 6.4. *Let G be a group and $G_0 \leq G$ a subgroup of finite index. Then the inclusion-induced map in homology $H_1(G_0;\mathbb{Q}) \to H_1(G;\mathbb{Q})$ is surjective; in particular, $b_1(G_0) \geq b_1(G)$.*

Proof. Denote by $\alpha\colon G_0 \to G$ the inclusion map, so that the inclusion-induced map in integral homology is $\alpha_*\colon H_1(G_0;\mathbb{Z}) \to H_1(G;\mathbb{Z})$. As $[G : G_0] < \infty$, there exists a *transfer* map $\alpha^*\colon H_1(G;\mathbb{Z}) \to H_1(G_0;\mathbb{Z})$ with the property that $\alpha_* \circ \alpha^*\colon H_1(G;\mathbb{Z}) \to H_1(G;\mathbb{Z})$ is multiplication by $[G : G_0]$ [Bro94, Section III.9]. Changing the coefficients to the rational numbers, this entails that $\alpha_*\colon H_1(G_0;\mathbb{Q}) \to H_1(G;\mathbb{Q})$ admits a right-inverse, hence it is surjective. □

Corollary F. For each of the following groups G there exists a finite index subgroup G_0 such that every two-generator subgroup of G_0 is either free, free abelian, or the Klein bottle group.

(1) If $\varphi \in \mathrm{Out}(F_r)$ is fully irreducible, $r \geq 2$ and $G = G_\varphi$. Moreover, if φ is additionally atoroidal, every two-generator subgroup of G_0 is free.
(2) If G is the fundamental group of a fibered cusped hyperbolic 3–manifold.

Proof. Note that the moreover of Case (1) directly follows from the first statement, since for an atoroidal fully irreducible φ the group G_φ is word-hyperbolic and hence cannot contain $\mathbb{Z}\times\mathbb{Z}$ or Klein bottle subgroups.

In both of these cases $G = F \rtimes \mathbb{Z}$ where F is a finitely generated free group of rank at least two. Therefore G is *large*, namely it admits a finite index subgroup which surjects to a free noncyclic group [But08, But13, HW15]. It follows that G contains a subgroup G_0 of finite index such that $b_1(G_0) > 2$.

Now let $H \leq G_0$ be a two-generator subgroup. Since $H \leq G$, applying Theorem C to G in Case (1) and Corollary D to G in Case (2) we conclude that H is either free, free abelian, the Klein bottle group or H has finite index in G and hence in G_0 as well.

However, the latter option is actually not possible. Indeed, if H were finite index in G_0 then $2 \geq b_1(H) \geq b_1(G_0) > 2$, with the first inequality dictated by the fact that H is two-generator, and the middle inequality due to the fact that the first Betti number is non-decreasing over finite index subgroups, by Lemma 6.4. That yields a contradiction, and hence H is either free, free abelian, or the Klein bottle group, as required. □

7. Applications to one-relator groups with torsion and RFRS groups

Recall that for an infinite subgroup H of a group G the *height* of H in G [GMRS98] is the supremum of all $n \geq 1$ such that there exist $g_1, \dots, g_n \in G$ with $Hg_1, \dots, Hg_n$ pairwise distinct and such that the intersection

$$\bigcap_{i=1}^{n} g_i^{-1} H g_i$$

is infinite.

Theorem G. Let $G = \langle x_1, \dots, x_r | w^n = 1\rangle$ where $r \geq 2$, $w \in F(x_1, \dots, x_r)$ is a nontrivial freely and cyclically reduced word and n is an integer such that $n \geq |w| \geq 2$. Then G contains a subgroup of finite index G_0 such that every two-generator subgroup of G_0 is free.

Proof. A recent result of Kielak and Linton [KL24] shows that G has a subgroup of finite index G_0 such that $G_0 \cong F \rtimes_\varphi \mathbb{Z}$, where F is a free group (necessarily of infinite rank) and where $\varphi \in \mathrm{Aut}(F)$ is some automorphism of F. Let H be a two-generator subgroup of $G_0 = G_\varphi$. By Theorem A, either H is free or H is conjugate in G_φ to a subgroup $H' \le G_\varphi$ such that H' is a sub-mapping torus of G_φ.

If H is free, the conclusion of Theorem G holds, as required.

Suppose now that H is conjugate in G_φ to a subgroup $H' \le G_\varphi$ such that H' is a sub-mapping torus of G_φ. Then $H' = \langle s, V\rangle$, where $s = ut^m$, $u \in F$, $m \ge 1$ and where $V \le F$ is a nontrivial finitely generated subgroup such that $sVs^{-1} \le V$ and so $V \le s^{-1}Vs$. Hence $V \le s^{-(p-1)}Vs^{p-1} \le s^{-p}Vs^p$ for all $p \ge 1$. For the homomorphism $\theta : G_\varphi \to \mathbb{Z}$, $\theta|_F = 0$, $\theta(t) = 1$ we have $\theta(V) = 0$, $\theta(s) = m$ and $\theta(Vs^p) = pm$ where $p \ge 0$ is arbitrary. Therefore the cosets $\{Vs^p | p = 0, 1, 2, 3, \dots\}$ are pairwise distinct for every $p \ge 0$, while the intersection $\cap_{i=0}^p s^{-i}Vs^i = V$ is infinite. Hence $V \le G$ has infinite height in G. However, G is a one-relator group with torsion and thus is word-hyperbolic. Moreover, the assumption $n \ge |w| \ge 2$ implies that G is locally quasiconvex [HW01, Theorem 1.2]. In particular, $V \le G$ is an infinite quasiconvex subgroup and hence has finite height in G [GMRS98], yielding a contradiction. □

We also obtain a corollary about certain RFRS groups. These are groups where there is a residual chain such that each G_{i+1} is contained in the kernel of the map $G_i \to G_i^{\mathrm{fab}}$, the maximal torsion free quotient of the abelianization of G_i [Ago08, Definition 2.1].

Corollary H. Suppose G is a finitely generated RFRS group with $b_2^{(2)}(G) = 0$ and $\mathrm{cd}_{\mathbb{Q}}(G) = 2$. Then there is a finite index subgroup G_0 of G such that every two generator subgroup H of G_0 is free or the mapping torus of an endomorphism of a finitely generated free group.

Proof. By a theorem of Fisher [Fis24] Theorem A, such a group is virtually free-by-cyclic (in the more general sense, where the free kernel need not be finitely generated) and the group G_0 is precisely this finite index subgroup. This is exactly the situation of our Theorem A, and so any two generator subgroup of G_0 is free or conjugate to a sub-mapping torus of G_0. The description of sub-mapping tori implies that any such subgroup is the mapping torus of an injective endomorphism of a finite rank free group (the map being given by the conjugation action of s on V), and passing to a conjugate does not change this. □

Appendix A. The 3–manifold way *by Peter Shalen*

The immediate goal of this appendix is to prove a generalization of Theorem VI.4.1 of [JS79] (or of the similar result proved by Tucker in [Tuc77]) which does not require the 3-manifold in question to be orientable or "sufficiently large." (A compact 3-manifold is said to be sufficiently large if it contains a properly embedded, 2-sided, incompressible surface. Compact, orientable, irreducible 3-manifolds that are sufficiently large are often called Haken manifolds.) This generalization is stated below as Corollary A.2. I have chosen to deduce it from a still more general result, Theorem A.1 below, which includes [ACCS96, Corollary 7.2] and seems to be of independent interest.

I will work in the PL category. It is understood that a "manifold" may have a boundary. The Euler characteristic of a compact PL space X will be denoted by $\chi(X)$. Base points will generally be suppressed in statements whose truth is independent of the choice of base points.

A 3-manifold N is termed *irreducible* if N is connected and every 2-sphere in N bounds a ball. One says that N is $\mathbb{P}^2$*-irreducible* if N is irreducible and contains no two-sided projective plane. These are the same definitions as the ones given in [Hem04], except that the connectedness condition is not made explicit there.

As on page 172 of [Hem04], a subgroup of $\pi_1(N)$, where N is a connected 3-manifold, will be termed *peripheral* if there is a connected subsurface X of ∂N such that the inclusion homomorphism $\pi_1(X) \to \pi_1(N)$ is injective, and P is conjugate to a subgroup of the image of this inclusion homomorphism.

The *rank* of a group H is the minimum cardinality of a generating set for H.

Here are the main results:

Theorem A.1. Let N be a compact, $\mathbb{P}^2$-irreducible 3-manifold, and let $k \geq 2$ be an integer. Suppose that every rank-2 free abelian subgroup of $\pi_1(N)$ is peripheral. Suppose also that either

(a) N is orientable and $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a closed, orientable surface S such that $1 < \operatorname{genus} S < k$, or

(b) $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a closed surface S such that $\chi(S)$ is even and $2 - 2k < \chi(S) < 0$.

Let H be a subgroup of $\pi_1(N)$ that has rank at most k. Then either H has finite index in $\pi_1(N)$, or H is a free product of finitely many subgroups, each of which is a free abelian group of rank at most 2 or a Klein bottle group.

Theorem A.1 will be seen to have the following corollary:

Corollary A.2. Let N be a compact, $\mathbb{P}^2$-irreducible 3-manifold such that every rank-2 free abelian subgroup of $\pi_1(N)$ is peripheral. Let H be a subgroup of $\pi_1(N)$ that has rank at most 2. Then H is either a free group, a free abelian group, a Klein bottle group, or a finite-index subgroup of $\pi_1(N)$.

We emphasize that if N is assumed to be orientable and sufficiently large, Theorem A.1 and Corollary A.2 become essential equivalent, respectively, to [ACCS96, Corollary 7.2] and to [JS79, Theorem VI.4.1] or the main result of [Tuc77]. Thus the new contribution of this appendix is to remove the hypotheses "sufficiently large" and (more significantly) "orientable" from these results.

As was mentioned in the introduction, Corollary A.2 includes Corollary C of the present paper as a special case, since the compact core of every finite-volume hyperbolic 3-manifold (fibered or not) satisfies the topological hypotheses of the corollary.

Proof that Theorem A.1 implies Corollary A.2. To prove the corollary we apply the theorem with $k = 2$. With this choice of k, Alternative (b) of the hypothesis of the theorem is vacuously true, since there are no even integers strictly between -2 and 0. (If N happens to be orientable, then Alternative (a) is also vacuously true; otherwise it obviously is not.) Hence, by the theorem, either H has finite index in $\pi_1(N)$, or H is isomorphic to a free product $X_1 \star \cdots \star X_q$, where each X_i is either a free abelian group of rank at most 2 and a Klein bottle group. In the latter case, by Grushko's Theorem, we have $\sum_{i=1}^{q} \operatorname{rank} X_i = \operatorname{rank} H \leq 2$. Hence either each X_i is a free abelian group of rank at most 1, in which case H is free; or $m = 1$ and X_1 has rank 2, in which case H is a rank-two free abelian group or a Klein bottle group. ☐

The proof of Theorem A.1 is a refinement of the proof of Theorem VI.4.1 of [JS79]. In order to emphasize this, I will be citing only results that were available when [JS79] was written.

I will say that a group G is *freely indecomposable* if G is neither trivial nor infinite cyclic, and is not isomorphic to the free product of two non-trivial groups. It follows from Grushko's Theorem that every non-trivial finitely generated group H is isomorphic to a free product $X_1 \star \cdots \star X_q$, where $q \geq 1$ is an integer, and each X_i has rank at most k and is either freely indecomposable or infinite cyclic. If H arises as a subgroup of a group G, and if some X_i has finite index in G, then $q = 1$ and H has finite index in G. Hence Theorem A.1 will follow immediately from the following result:

Proposition A.3. *Let N be a 3-manifold for which the hypotheses of Theorem A.1 hold. Let H be a freely indecomposable subgroup of $\pi_1(N)$ that has rank at most k. Then either H has finite index in $\pi_1(N)$, or H is isomorphic either to a free abelian group of rank 2 or a Klein bottle group.*

The rest of this appendix is devoted to the proof of Proposition A.3. It will be understood that we are given a manifold N for which the hypotheses of Theorem A.1 hold, and a freely indecomposable subgroup H of $\pi_1(N)$ that has rank at most k. We shall denote by $\widetilde{N} = \widetilde{N}_H$ the connected covering space of N determined by the subgroup H, and by $p : \widetilde{N} \to N$ the covering projection.

Since N is $\mathbb{P}^2$-irreducible, it follows from the Projective Plane Theorem [Hem04, Theorem 4.12] that $\pi_2(N) = 0$. If $\pi_1(N)$ is finite, then H has finite index in $\pi_1(N)$; hence we may assume that

$\pi_1(N)$ is infinite. This, together with the triviality of $\pi_2(N)$, shows that the universal cover of N is contractible, so that N is aspherical. Hence $\widetilde{N}$ is also aspherical.

According to [Hem04, Theorem 8.6], $\widetilde{N}$ has a *compact core*, i.e. a compact, connected three-dimensional submanifold $K \subset \operatorname{int} \widetilde{N}$ such that the inclusion homomorphism $\pi_1(K) \to \pi_1(\widetilde{N})$ is an isomorphism. For any compact core K of $\widetilde{N}$, we have $\pi_1(K) \cong H$. Among all compact cores of $\widetilde{N}$ we fix one, K_0, having the smallest possible number of boundary components.

Lemma A.4. *The manifold* $\operatorname{int} K_0$ *contains no two-sided projective plane, and every* 2*-sphere in* K_0 *is the boundary of a compact, contractible submanifold of* K_0*. Furthermore, if* $\widetilde{N}$ *is irreducible then* K_0 *is* $\mathbb{P}^2$*-irreducible.*

Proof. We have observed that $\widetilde{N}$ is aspherical; since it is finite-dimensional, it follows that it has a torsion-free fundamental group. Any two-sided projective plane in $\widetilde{N}$ would therefore lift to a two-sided projective plane in the orientation cover $\widetilde{N}'$ of $\widetilde{N}$, contradicting the orientability of $\widetilde{N}'$. In particular, $\operatorname{int} K_0$ contains no two-sided projective plane.

To prove the remaining assertions, it suffices to show that if $S \subset \operatorname{int} K_0$ is a 2-sphere, then S is the boundary of a compact, contractible submanifold B of K_0; and that if $\widetilde{N}$ is irreducible then we can take B to be a 3-ball.

Since $\pi_2(\widetilde{N}) = 0$, it follows from the proof of [Mil62, Theorem 2] that the sphere $S \subset \operatorname{int} K_0 \subset \operatorname{int} N$ is the boundary of a compact submanifold B of $\widetilde{N}$ which is simply connected and therefore contractible. If $\widetilde{N}$ is irreducible we may take B to be a ball. Set $K = K_0 \cup B$. The inclusion homomorphism $\pi_1(K_0) \to \pi_1(K)$ is surjective; since the inclusion homomorphism $\pi_1(K_0) \to \pi_1(\widetilde{N})$ is an isomorphism, the inclusion homomorphism $\pi_1(K) \to \pi_1(\widetilde{N})$ must also be an isomorphism, i.e. K is a compact core of $\widetilde{N}$. Our choice of K_0 then guarantees that ∂K has at least as many components as ∂K_0. But ∂K is the union of all components of ∂K_0 that are not contained in $\operatorname{int} B$. Hence $\operatorname{int} B$ contains no component of ∂K_0, and therefore $B \subset K_0$. □

A connected 3-manifold Q is termed *boundary-irreducible* if for every boundary component T of Q, the inclusion homomorphism $\pi_1(T) \to \pi_1(Q)$ is injective; otherwise, Q is said to be *boundary-reducible.*

Lemma A.5. *The manifold* K_0 *is boundary-irreducible.*

Proof. Assume that K_0 is not boundary-irreducible. Then according to the Loop Theorem [Hem04, Theorem 4.2], there is a properly embedded disk D in K_0 such that $C \doteq \partial D \subset \partial K_0$ does not bound a disk in ∂K_0. Consider the case in which D separates K_0, and let A and B denote the components of $K_0 - D$. Then $H \cong \pi_1(K_0)$ is isomorphic to the free product $\pi_1(A) \star \pi_1(B)$. Since C does not bound a disk in ∂K_0, each of the compact manifolds $\overline{A}$ and $\overline{B}$ has a boundary component which is not a 2-sphere. Hence A and B are not simply connected, and H is a free product of two non-trivial subgroups; this contradicts the free indecomposability of H. Now consider the case in which D does not separate K_0. In this case, $H \cong \pi_1(K_0)$ is isomorphic to the free product $\pi_1(K_0 - D) \star \mathbb{Z}$. Hence H is either an infinite cyclic group or a free product of two non-trivial subgroups. Again we have a contradiction to the free indecomposability of H. □

Lemma A.6. *Every component of* ∂K_0 *is a torus or a Klein bottle.*

Proof. It follows from Lemma A.4 that no component of ∂K_0 is a projective plane. If some component T of ∂K_0 is a sphere, then by Lemma A.4, C is the boundary of a compact, contractible submanifold of K_0, which must be all of K_0; this implies that H is a trivial group, a contradiction to free indecomposability. thus Alternative (i) of the lemma holds. Thus every component of ∂K_0 has non-positive Euler characteristic.

We may assume that K_0 is not closed, as otherwise the conclusion of the lemma is vacuously true. Hence if h_i denotes the dimension of $H_i(K_0;\mathbb{Q})$ for $i = 0, 1, 2$, then $\chi(K_0) = h_0 - h_1 + h_2 = 1-h_1+h_2 \geq 1-h_1$. Since $\pi_1(K_0) \cong H$ has rank at most k, we have $h_1 \leq k$ and hence $\chi(K_0) \geq 1-k$.

We claim that the latter inequality is strict. Suppose to the contrary that $\chi(K_0) = 1 - k$. Since the compact 3-manifold K_0 is not closed, K_0 admits some compact PL space L of dimension at most 2 as a deformation retract. We may give L the structure of a finite CW complex with only one 0-cell. Let m and n denote, respectively, the number of 1-cells and the number of 2-cells of this CW complex. Then $1 - k = \chi(K_0) = \chi(L) = 1 - m + n$, so that $m - n = k$. But $H \cong \pi_1(K_0) \cong \pi_1(L)$ admits a presentation with m generators and n relations; thus the deficiency of this presentation is equal to k (in the sense that the number of generators exceeds the number of relations by k). On the other hand, the rank of H is at most k by hypothesis. We now invoke a theorem due to Magnus [Mag39] which asserts that if for some integer k, a given group has rank at most k and admits a presentation with deficiency k, then the group is free of rank k. Thus in this case H is free of rank k, a contradiction to free indecomposability. This proves that $\chi(K_0) > 1 - k$.

Assume that the conclusion of the lemma is false, and let us denote the components of ∂K_0 with non-zero Euler characteristic as $S_1, \dots, S_r$, where $r \geq 1$. Since every component of ∂K_0 has non-positive Euler characteristic, we have $\chi(S_i) < 0$ for $i = 1, \dots, r$. We may take the S_i to be indexed so that $0 > \chi(S_1) \geq \ldots \geq \chi(S_r)$. We have $\sum_{i=1}^{r} \chi(S_i) = \chi(\partial K_0) = 2\chi(K_0) > 2-2k$. (The equality $\chi(\partial K_0) = 2\chi(K_0)$ holds because K_0 is a compact, odd-dimensional manifold with boundary: see for example [Mas91, p. 379, Exercise 7.3].) In particular we have $0 > r\chi(S_1) > 2 - 2k$. Note also that by Lemma A.5, $\pi_1(S_1)$ is isomorphic to a subgroup of $\pi_1(K_0)$. Since K_0 is a compact core of the covering space $\widetilde{N}$ of N, it follows that $\pi_1(N)$ has a subgroup isomorphic to $\pi_1(S_1)$.

By hypothesis, one of the alternatives (a) or (b) of Theorem A.1 holds. In the case where (a) holds, $\widetilde{N}$ is orientable, and hence the compact core K_0 of the covering space $\widetilde{N}$ is orientable. Since $0 > r\chi(S_1) > 2 - 2k$ and $r \geq 1$, we have $1 < \operatorname{genus} S_1 < k$. As $\pi_1(N)$ has a subgroup isomorphic to $\pi_1(S_1)$, this contradicts (a).

Now suppose that Alternative (b) holds and that $r = 1$. Then we have $\chi(S_1) = 2\chi(K_0)$, and hence $\chi(S_1)$ is even. Since $0 > \chi(S_1) > 2 - 2k$, and $\pi_1(N)$ has a subgroup isomorphic to $\pi_1(S_1)$, this contradicts (b).

There remains the case in which (b) holds and $r \geq 2$. Since $0 > r\chi(S_1) > 2 - 2k$, we have $0 > \chi(S_1) > 1-k$. Since the closed surface S_1 has negative Euler characteristic, it has a two-sheeted covering space $\widetilde{S}_1$, and $\pi_1(N)$ has a subgroup isomorphic to $\pi_1(\widetilde{S}_1)$. But since $\chi(\widetilde{S}_1) = 2\chi(S_1)$, the integer $\chi(\widetilde{S}_1)$ is even and satisfies $0 > \chi(\widetilde{S}_1) > 2 - 2k$; again we have a contradiction to (b). $\square$

Proof of Proposition A.3. Let N be a 3-manifold for which the hypotheses of Theorem A.1 hold. Let H be a freely indecomposable subgroup of $\pi_1(N)$ that has rank at most k. Then $\widetilde{N} = \widetilde{N}_H$ and K_0 may be defined as above, and Lemmas A.4, A.5 and A.6 may be applied.

If K_0 is closed, then since $K_0 \subset \widetilde{N}$ and the 3-manifold $\widetilde{N}$ is connected, we must have $K_0 = \widetilde{N}$; thus $\widetilde{N}$ is closed, and $p : \widetilde{N} \to N$ is a finite-sheeted covering. Hence H has finite index in $\pi_1(N)$. For the rest of the proof I shall assume that K_0 has non-empty boundary.

By Lemmas A.5 and A.6, K_0 is boundary-irreducible, and each component of ∂K_0 is a torus or a Klein bottle.

We define a covering space M of K_0 as follows. If every component of ∂K_0 is a torus (in which case K_0 may or may not be orientable) we take $M = K_0$ to be the trivial covering. If some component of ∂K_0 is a Klein bottle, so that in particular K_0 is non-orientable, we define M to be the orientation covering of K_0. Note that in any event M is connected, that each component of ∂M is a torus, and that the covering M of K_0 has degree at most 2. We let $p' : M \to K_0$ denote the covering projection. Letting $\iota : K_0 \to \widetilde{N}$ denote the inclusion map, we set $\ell = p \circ \iota \circ p' : M \to N$. Since p and p' are covering maps, and K_0 is a compact core of $\widetilde{N}$, the maps p', ι and p all induce injections of fundamental groups, and hence so does ℓ. Now since K_0 is boundary-irreducible, M

is also boundary-irreducible. Hence for every component T of ∂M the inclusion homomorphism $\pi_1(T) \to \pi_1(M)$ is injective, and therefore $(\ell|T)_\sharp : \pi_1(T) \to \pi_1(N)$ is injective. Since T is a torus, it follows that the image of $(\ell|T)_\sharp$, which is a subgroup of $\pi_1(N)$ defined up to conjugacy, is a free abelian group of rank 2. By the hypothesis of the theorem it now follows that the image P of $(\ell|T)_\sharp$ (a subgroup of $\pi_1(N)$ defined up to conjugacy) is peripheral; that is, there is a connected subsurface X of ∂N such that (1) the inclusion homomorphism $\pi_1(X) \to \pi_1(N)$ is injective, and (2) P is conjugate to a subgroup of the image of this inclusion homomorphism. It follows from (1) and (2), together with the asphericity of N, that $\ell|T$ is homotopic to a map whose image is contained in ∂N. As this holds for every component T of ∂M, it now follows from the homotopy extension property for compact PL pairs that ℓ is homotopic to a map $f : M \to N$ which is *boundary-preserving* in the sense that $f(\partial M) \subset \partial N$.

Since $\partial K_0 \neq \emptyset$, we have $\partial M \neq \emptyset$. Since f is boundary-preserving it then follows that $\partial N \neq \emptyset$. It follows from Lemmas 6.7 and 6.6 of [Hem04] that every compact, $\mathbb{P}^2$-irreducible 3-manifold with non-empty boundary is sufficiently large, in the sense defined on page 125 of [Hem04]: that is, that such a manifold contains a properly embedded, 2-sided, incompressible surface. According to [Hem04, Corollary 13.5], every covering space of a sufficiently large, $\mathbb{P}^2$-irreducible, compact 3-manifold is $\mathbb{P}^2$-irreducible. Thus, in the present situation, $\widetilde{N}$ is $\mathbb{P}^2$-irreducible and in particular irreducible. It then follows from Lemma A.4 that K_0 is $\mathbb{P}^2$-irreducible. Since $\partial K_0 \neq \emptyset$, the manifold K_0 is also sufficiently large. Another application of [Hem04, Corollary 13.5] now shows that the covering space M of K_0 is $\mathbb{P}^2$-irreducible; and since $\partial M \neq \emptyset$, the manifold M is sufficiently large.

Since $f : M \to N$ is homotopic to ℓ, it induces an injection of fundamental groups. It follows from [Hem04, Theorem 13.6] that if $f : M \to N$ is a boundary-preserving map between sufficiently large, $\mathbb{P}^2$-irreducible, compact 3-manifolds, and if M is boundary-irreducible and $f_\sharp : \pi_1(M) \to \pi_1(N)$ is injective, then either f is homotopic to a covering map or M is an I-bundle over a closed surface.

In the present context, in the case where f is homotopic to a covering map, then the image of $f_\sharp : \pi_1(M) \to \pi_1(N)$ is a finite-index subgroup H' of $\pi_1(N)$. If we choose basepoints in M, $K_0 \subset \widetilde{N}$ and N which are compatible in the sense that p' and p map the respective basepoints of M and K_0 to those of K_0 and N, we have $H' \leq H \leq \pi_1(N)$. Hence H has finite index in $\pi_1(N)$ in this case.

Now consider the case in which M is an I-bundle over a closed surface. In the subcase where every component of ∂K_0 is a torus, our construction of the covering space M of K_0 gives that $M = K_0$, so that K_0 is an I-bundle over a closed surface; since each component of ∂K_0 is a torus, the base of the I-bundle is either a torus or a Klein bottle, and hence $H \cong \pi_1(K_0)$ is either a rank-2 free abelian group or a Klein bottle group.

There remains the subcase in which some component J of ∂K_0 is a Klein bottle. In this subcase, according to our construction, M is the orientation cover of K_0. Hence $\widetilde{J} \doteq (p')^{-1}(J)$ is the orientation cover of the Klein bottle J. Let us fix compatible basepoints in $\widetilde{J}$ and J, which will also serve as basepoints for M and K_0 respectively. In terms of these basepoints, we have a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(\widetilde{J}) & \xrightarrow{\widetilde{j}_\sharp} & \pi_1(M) \\ \downarrow{\scriptstyle (p'|\widetilde{J})_\sharp} & & \downarrow{\scriptstyle p'_\sharp} \\ \pi_1(J) & \xrightarrow{j_\sharp} & \pi_1(K_0) \end{array}$$

where $\widetilde{j}$ and j denote inclusion maps. The vertical homomorphisms in the diagram are injective because they are induced by covering maps, and $j_\sharp$ is injective because K_0 is irreducible; in view of the commutativity of the diagram, it follows that $\widetilde{j}_\sharp$ is also injective.

Since $\widetilde{J}$ is a boundary component of M, which is an I-bundle over a closed surface, the image of $\widetilde{j}_\sharp$ has index 1 or 2 in $\pi_1(M)$; and since p is a two-sheeted covering map between connected spaces, the image of $p'_\sharp$ has index 2 in $\pi_1(K_0)$. Hence the image of $p'_\sharp \circ \widetilde{j}_\sharp = j_\sharp \circ (p'|\widetilde{J})_\sharp$ has index

2 or 4 in $\pi_1(K_0)$. Since the image of $(p'|\widetilde{J})_\sharp$ has index 2 in $\pi_1(J)$, it follows that the image of $j_\sharp : \pi_1(J) \to \pi_1(K_0)$, which I shall denote by A, has index 1 or 2 in $\pi_1(K_0)$. Since K_0 is boundary-irreducible, the homomorphism $j_\sharp$ is injective and hence $A \cong \pi_1(J)$.

If $|\pi_1(K_0) : A| = 1$ then $H \cong \pi_1(K_0) \cong \pi_1(J)$, i.e. H is a Klein bottle group. Finally, suppose that $|\pi_1(K_0) : A| = 2$. Let K_0^* denote the degree-2 covering space of K_0 determined by the subgroup A of $\pi_1(K_0)$. Since K_0 is $\mathbb{P}^2$-irreducible and sufficiently large, K_0^* is $\mathbb{P}^2$-irreducible by [Hem04, Corollary 13.5]. The boundary of K_0^* contains a surface J^* which is mapped homeomorphically onto J by the covering projection $q : K_0^* \to K_0$; furthermore, the inclusion homomorphism $\pi_1(J^*) \to \pi_1(K_0^*)$ is an isomorphism, which by [Hem04, Theorem 10.2] and the irreducibility of K_0^* implies that K_0^* is a trivial I-bundle over a closed surface. Since K_0 is doubly covered by a trivial I-bundle over a closed surface, it follows from [Hem04, Theorem 10.3] that K_0 is itself an I-bundle over a closed surface. Since the component J of K_0 is a Klein bottle, the base of the I-bundle K_0 is doubly covered by a Klein bottle, and must therefore itself be a Klein bottle; hence $H \cong \pi_1(K_0)$ is again a Klein bottle group. □

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