

GT-shadows for the gentle version $\widehat{\mathbf{GT}}_{gen}$ of the Grothendieck-Teichmueller group

Vasily A. Dolgushev and Jacob J. Guynee

Abstract

Let B_3 be the Artin braid group on 3 strands and PB_3 be the corresponding pure braid group. In this paper, we construct the groupoid \mathbf{GTSh} of GT-shadows for a (possibly more tractable) version $\widehat{\mathbf{GT}}_0$ of the Grothendieck-Teichmueller group $\widehat{\mathbf{GT}}$ introduced in paper [12] by D. Harbater and L. Schneps. We call this group the gentle version of $\widehat{\mathbf{GT}}$ and denote it by $\widehat{\mathbf{GT}}_{gen}$. The objects of \mathbf{GTSh} are finite index normal subgroups N of B_3 satisfying the condition $N \leq PB_3$. Morphisms of \mathbf{GTSh} are called GT-shadows and they may be thought of as approximations to elements of $\widehat{\mathbf{GT}}_{gen}$. We show how GT-shadows can be obtained from elements of $\widehat{\mathbf{GT}}_{gen}$ and prove that $\widehat{\mathbf{GT}}_{gen}$ is isomorphic to the limit of a certain functor defined in terms of the groupoid \mathbf{GTSh} . Using this result, we get a criterion for identifying genuine GT-shadows.

1 Introduction

Let \widehat{F}_2 be the profinite completion of the free group $F_2 := \langle x, y \rangle$ on two generators and $\widehat{\mathbb{Z}}$ be the profinite completion of the ring of integers. The profinite version $\widehat{\mathbf{GT}}$ of the Grothendieck-Teichmueller group [6, Section 4] is one of the most mysterious objects in mathematics. It consists of pairs $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{F}_2$ satisfying the hexagon relations:

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} = \hat{f}^{-1} \sigma_1 \sigma_2 \sigma_1^{-2\hat{m}} c^{\hat{m}}, \quad (1.1)$$

$$\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} = \sigma_2 \sigma_1 \sigma_2^{-2\hat{m}} c^{\hat{m}} \hat{f}, \quad (1.2)$$

the pentagon relation:

$$\hat{f}(x_{23}, x_{34}) \hat{f}(x_{12}x_{13}, x_{24}x_{34}) \hat{f}(x_{12}, x_{23}) = \hat{f}(x_{12}, x_{23}x_{24}) \hat{f}(x_{13}x_{23}, x_{34}) \quad (1.3)$$

and the invertibility condition. In relations (1.1) and (1.2), σ_1 and σ_2 are the standard generators of the Artin braid group B_3 , $c := (\sigma_1 \sigma_2 \sigma_1)^2$ and \widehat{F}_2 is considered as the subgroup of \widehat{B}_3 ; namely, \widehat{F}_2 is identified with the profinite completion of the subgroup $\langle \sigma_1^2, \sigma_2^2 \rangle$ of B_3 .

In relation (1.3), $x_{12} := \sigma_1^2$, $x_{23} := \sigma_2^2$, \dots are the standard generators [17, Section 1.3] of the pure braid group PB_4 on 4 strands and $\hat{f}(x_{23}, x_{34})$, $\hat{f}(x_{12}x_{13}, x_{24}x_{34})$, $\hat{f}(x_{12}, x_{23})$, \dots are the images of \hat{f} with respect to natural (continuous) group homomorphisms $\widehat{F}_2 \rightarrow \widehat{PB}_4$, e.g. $\hat{f}(x_{12}x_{13}, x_{24}x_{34})$ is the image of the continuous group homomorphism $\widehat{F}_2 \rightarrow \widehat{PB}_4$ which sends x (resp. y) to $x_{12}x_{13} \in PB_4 \leq \widehat{PB}_4$ (resp. $x_{24}x_{34} \in PB_4 \leq \widehat{PB}_4$).

The multiplication on $\widehat{\mathbf{GT}}$ can be defined by an explicit formula (see equations (2.1), (2.4) or [28, Section 1.1]) or by identifying elements of $\widehat{\mathbf{GT}}$ with continuous automorphisms of \widehat{F}_2 (see [12, Introduction]).

The group $\widehat{\mathbf{GT}}$ and its variants are a part of an active area of research¹ [19], [20], [21], [24], [25], [26], [28] and this research is often motivated by fruitful links between operads, moduli of curves and the geometric action of the absolute Galois group $G_{\mathbb{Q}}$ of the field of rational numbers [1], [3], [4], [7], [12], [13], [15], [18], [22].

¹The lists of references in this paragraph are far from complete.

In paper [5], the authors constructed an infinite groupoid closely related to the group $\widehat{\text{GT}}$. The objects of this groupoid are finite index normal subgroups N of the Artin braid group B_4 satisfying the condition $N \leq \text{PB}_4$. The morphisms of this groupoid are called **GT-shadows**. In addition to other things, the authors of [5] proved that $\widehat{\text{GT}}$ is isomorphic to the limit of a certain functor defined in terms of the groupoid of GT-shadows (see [5, Section 3]). In this respect, certain GT-shadows are approximations of elements of the group $\widehat{\text{GT}}$.

The purpose of this paper is to develop a version of the groupoid of GT-shadows for the gentle version $\widehat{\text{GT}}_{\text{gen}}$ of the Grothendieck-Teichmüller group $\widehat{\text{GT}}$. Just as $\widehat{\text{GT}}$, the group $\widehat{\text{GT}}_{\text{gen}}$ consists of pairs $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{F}_2$ satisfying hexagon relations (1.1), (1.2), the invertibility condition and the following consequence of pentagon relation (1.3):

$$\hat{f} \in [\widehat{F}_2, \widehat{F}_2]^{\text{top. cl.}}.$$

For more details, please see Subsection 2.2. It is possible that the group $\widehat{\text{GT}}_{\text{gen}}$ is more tractable and it is often denoted by $\widehat{\text{GT}}_0$ (see, for example, [12, Section 0.1]). The group $\widehat{\text{GT}}_{\text{gen}}$ obviously contains $\widehat{\text{GT}}$ as a subgroup.

The idea of approximating elements of $\widehat{\text{GT}}$ and $\widehat{\text{GT}}_{\text{gen}}$ was originally suggested in paper [11] by D. Harbater and L. Schneps. We would also like to mention papers [9] and [10] in which P. Guillot developed and studied similar constructions for the group $\widehat{\text{GT}}_{\text{gen}}$. Since P. Guillot used a very different definition of $\widehat{\text{GT}}_{\text{gen}}$, it is not easy to compare the groupoid GTSh developed in this paper to the constructions presented in [9], [10].

The groupoid GTSh in a nutshell. Our starting point is the poset $\text{NFI}_{\text{PB}_3}(B_3)$ of finite index normal subgroups N of B_3 such that $N \leq \text{PB}_3$, i.e.

$$\text{NFI}_{\text{PB}_3}(B_3) := \{N \trianglelefteq B_3 \mid |B_3 : N| < \infty, N \leq \text{PB}_3\}. \quad (1.4)$$

Since PB_3 (and B_3) is residually finite, the poset $\text{NFI}_{\text{PB}_3}(B_3)$ is infinite.

For $N \in \text{NFI}_{\text{PB}_3}(B_3)$, we denote by N_{ord} the least common multiple of the orders of elements $x_{12}N$, $x_{23}N$ and cN in PB_3/N . Moreover, we set $N_{F_2} := F_2 \cap N$, where F_2 is identified with the subgroup $\langle x_{12}, x_{23} \rangle$ of PB_3 .

For $N \in \text{NFI}_{\text{PB}_3}(B_3)$, we consider pairs $(m, f) \in \mathbb{Z} \times F_2$ that satisfy the hexagon relations modulo N :

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f N = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m N, \quad (1.5)$$

$$f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} N = \sigma_2 \sigma_1 x_{23}^{-m} c^m f N. \quad (1.6)$$

Due to Proposition 3.2, for every such pair (m, f) , the formulas

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1} N, \quad T_{m,f}(\sigma_2) := f^{-1} \sigma_2^{2m+1} f N \quad (1.7)$$

define a group homomorphism $T_{m,f} : B_3 \rightarrow B_3/N$.

A **GT-shadow** with the target N is a pair

$$(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2}) \in \mathbb{Z}/N_{\text{ord}}\mathbb{Z} \times [F_2/N_{F_2}, F_2/N_{F_2}]$$

that satisfies the following conditions:

- relations (1.5), (1.6) hold,
- $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, and
- the group homomorphism $T_{m,f} : B_3 \rightarrow B_3/N$ is onto.

We denote by $\text{GT}(N)$ the set of GT-shadows with the target N and by $[m, f]$ the GT-shadow represented by a pair $(m, f) \in \mathbb{Z} \times F_2$. The set $\text{GT}(N)$ is finite since it is a subset of a finite set.

Using equation (3.19), it is not hard to see that, for every $[m, f] \in \text{GT}(\mathbf{N})$, $\ker(T_{m,f})$ belongs to the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$. Moreover, since $T_{m,f}$ is onto, it induces an isomorphism of the quotient groups:

$$T_{m,f}^{\text{isom}} : \mathbf{B}_3/\mathbf{K} \xrightarrow{\simeq} \mathbf{B}_3/\mathbf{N},$$

where $\mathbf{K} := \ker(T_{m,f})$.

The set $\text{Ob}(\text{GTSh})$ of objects of the groupoid GTSh is the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$. Moreover, for $\mathbf{K}, \mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, the set $\text{GTSh}(\mathbf{K}, \mathbf{N})$ of morphisms from \mathbf{K} to \mathbf{N} is the subset of GT -shadows $[m, f] \in \text{GT}(\mathbf{N})$ for which $\mathbf{K} = \ker(T_{m,f})$.

The composition of morphisms $[m_1, f_1] \in \text{GTSh}(\mathbf{N}^{(2)}, \mathbf{N}^{(1)})$, $[m_2, f_2] \in \text{GT}(\mathbf{N}^{(3)}, \mathbf{N}^{(2)})$ is defined by the formula

$$[m_1, f_1] \circ [m_2, f_2] := [2m_1m_2 + m_1 + m_2, f_1E_{m_1,f_1}(f_2)],$$

where E_{m_1,f_1} is the endomorphism of \mathbf{F}_2 defined by the equations $E_{m_1,f_1}(x) := x^{2m_1+1}$, $E_{m_1,f_1}(y) := f_1^{-1}y^{2m_1+1}f_1$ (for more details, see Theorem 3.10).

It is important that $\text{Ob}(\text{GTSh})$ is a poset. In Subsection 3.1, we show that, if $\mathbf{N} \leq \mathbf{H}$, $\mathbf{N}, \mathbf{H} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, then we have a natural **reduction map**:

$$\mathcal{R}_{\mathbf{N},\mathbf{H}} : \text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H}). \quad (1.8)$$

In Section 5, this map plays an important role in connecting the groupoid GTSh to the group $\widehat{\text{GT}}_{\text{gen}}$.

Although the groupoid GTSh is infinite, the connected component $\text{GTSh}_{\text{conn}}(\mathbf{N})$ of an object $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ is always a finite groupoid. An object \mathbf{N} of the groupoid GTSh is called **isolated** if its connected component $\text{GTSh}_{\text{conn}}(\mathbf{N})$ has exactly one object. In this case, $\text{GTSh}(\mathbf{N}, \mathbf{N}) = \text{GT}(\mathbf{N})$ is a (finite) group and the groupoid $\text{GTSh}_{\text{conn}}(\mathbf{N})$ may be identified with this group. In Subsection 3.2, we show that the subposet $\text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3) \subset \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ of isolated objects of GTSh is cofinal, i.e., for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, there exists $\tilde{\mathbf{N}} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$ such that $\tilde{\mathbf{N}} \leq \mathbf{N}$. More precisely, due to Proposition 3.14, for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, the intersection of all objects of the connected component $\text{GTSh}_{\text{conn}}(\mathbf{N})$ is an isolated object \mathbf{N}° of GTSh such that $\mathbf{N}^\circ \leq \mathbf{N}$.

The group $\widehat{\text{GT}}_{\text{gen}}$ versus the groupoid GTSh . In Section 4, we define a natural action of the group $\widehat{\text{GT}}_{\text{gen}}$ on the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$. This allows us to introduce the transformation groupoid $\widehat{\text{GT}}_{\text{NFI}}^{\text{gen}}$ and a functor

$$\mathcal{PR} : \widehat{\text{GT}}_{\text{NFI}}^{\text{gen}} \rightarrow \text{GTSh}.$$

More precisely, to every element $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$ and every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, we assign a GT -shadow $[m, f]_{\mathbf{N}}$ with the target \mathbf{N} , and the formula

$$\mathbf{N}^{(\hat{m}, \hat{f})} := \ker(T_{m,f})$$

defines a right action of $\widehat{\text{GT}}_{\text{gen}}$ on the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$. We can think of the GT -shadow $[m, f]_{\mathbf{N}}$ as an approximation of the element (\hat{m}, \hat{f}) . For this reason, the functor \mathcal{PR} is called the **approximation functor**. GT -shadows obtained in this way from elements of $\widehat{\text{GT}}_{\text{gen}}$ are called **genuine** and all the remaining GT -shadows (if any) are called **fake**.

In Section 5, we show how the topological group $\widehat{\text{GT}}_{\text{gen}}$ can be reconstructed from the groupoid GTSh . We observe that, for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$, $\text{GT}(\mathbf{N})$ is a finite group, and the reduction map (1.8) allows us to upgrade the assignment

$$\mathbf{N} \mapsto \text{GT}(\mathbf{N}), \quad \mathbf{N} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$$

to a functor from the poset $\text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$ to the category of finite groups. We call it the **Main Line functor** and denote it by \mathcal{ML} .

Using the approximation functor $\mathcal{PR} : \widehat{\text{GT}}_{\text{NFI}}^{\text{gen}} \rightarrow \text{GTSh}$, it is easy to construct a natural group homomorphism

$$\Psi : \widehat{\text{GT}}_{\text{gen}} \rightarrow \lim(\mathcal{ML}). \quad (1.9)$$

The main result of this paper is Theorem 5.2 which states that Ψ is an isomorphism of groups and a homeomorphism of topological spaces. ($\widehat{\text{GT}}_{gen}$ is considered with the subset topology coming from the topological space $\widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$.)

Thanks to Theorem 5.2, we have the following criterion for identifying genuine GT-shadows: a GT-shadow $[m, f] \in \text{GT}(\mathbf{H})$ is genuine if and only if $[m, f]$ belongs to the image of the reduction map $\mathcal{R}_{\mathbf{N}, \mathbf{H}} : \text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H})$ for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ such that $\mathbf{N} \leq \mathbf{H}$ (see Corollary 5.4). Equivalently, a GT-shadow $[m, f] \in \text{GT}(\mathbf{H})$ is fake if and only if there exists $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ such that $\mathbf{N} \leq \mathbf{H}$ and $[m, f]$ does not belong to the image of the reduction map $\mathcal{R}_{\mathbf{N}, \mathbf{H}} : \text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H})$.

In recent paper [2], the authors considered a subposet Dih of $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ related to the family of dihedral groups and they called Dih the **dihedral poset**. In [2], it was proved that every element of the dihedral poset is an isolated object of the groupoid GTSh and gave an explicit description of the (finite) group $\text{GT}(\mathbf{K})$ for every $\mathbf{K} \in \text{Dih}$. In [2], the authors also proved that, for every pair $\mathbf{N}, \mathbf{H} \in \text{Dih}$ with $\mathbf{N} \leq \mathbf{H}$, the natural map $\text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H})$ is onto. This result implies that one cannot find an example of a fake² GT-shadow using only the dihedral poset Dih .

Organization of the paper. In Section 2, we introduce the group $\widehat{\text{GT}}_{gen}$. We also recall that $\widehat{\text{GT}}_{gen}$ comes with natural injective homomorphisms to the group of continuous automorphisms of $\widehat{\mathbb{F}}_2$ and to the group of continuous automorphisms of $\widehat{\mathbf{B}}_3$.

Section 3 is the core of this paper. In this section, we introduce the groupoid GTSh of GT-shadows (for $\widehat{\text{GT}}_{gen}$), define the reduction map (see (1.8) or (3.60)), discuss connected components of GTSh and introduce isolated objects of GTSh .

In Section 4, we introduce the action of the group $\widehat{\text{GT}}$ on the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ and define the approximation functor \mathcal{PR} from the transformation groupoid $\widehat{\text{GT}}_{\text{NFI}}^{gen}$ to GTSh . The GT-shadows that belong to the image of \mathcal{PR} are called genuine.

In Section 5, we introduce the Main Line functor \mathcal{ML} and prove that $\lim(\mathcal{ML})$ is isomorphic to the group $\widehat{\text{GT}}_{gen}$ (see Theorem 5.2). In this section, we also prove a criterion for identifying genuine GT-shadows (see Corollary 5.4) and show that the group $\widehat{\text{GT}}_{gen}$ is isomorphic to the group $\widehat{\text{GT}}_0$ introduced in [12, Section 0.1] (see Proposition 5.5).

Appendix A is devoted to selected statements about profinite groups.

1.1 Notational conventions

For a set X with an equivalence relation and $a \in X$ we will denote by $[a]$ the equivalence class which contains the element a .

The notation \mathbf{B}_n (resp. PB_n) is reserved for the Artin braid group on n strands (resp. the pure braid group on n strands). S_n denotes the symmetric group on n letters. We denote by σ_1 and σ_2 the standard generators of \mathbf{B}_3 . Furthermore, we set

$$x_{12} := \sigma_1^2, \quad x_{23} := \sigma_2^2, \quad \Delta := \sigma_1 \sigma_2 \sigma_1, \quad c := \Delta^2.$$

We recall [17, Section 1.3] that the element c belongs to the center $\mathcal{Z}(\text{PB}_3)$ of PB_3 (and the center $\mathcal{Z}(\mathbf{B}_3)$ of \mathbf{B}_3). Moreover, $\mathcal{Z}(\mathbf{B}_3) = \mathcal{Z}(\text{PB}_3) = \langle c \rangle \cong \mathbb{Z}$.

We observe that

$$\sigma_1 \Delta = \Delta \sigma_2, \quad \sigma_2 \Delta = \Delta \sigma_1, \quad \sigma_1^{-1} \Delta = \Delta \sigma_2^{-1}, \quad \sigma_2^{-1} \Delta = \Delta \sigma_1^{-1}. \quad (1.10)$$

Using identities (1.10) and $c = \Delta^2$, it is easy to see that the adjoint action of \mathbf{B}_3 on PB_3 is given on generators by the formulas:

$$\sigma_1 x_{12} \sigma_1^{-1} = \sigma_1^{-1} x_{12} \sigma_1 = x_{12}, \quad \sigma_1 x_{23} \sigma_1^{-1} = x_{23}^{-1} x_{12}^{-1} c, \quad \sigma_1^{-1} x_{23} \sigma_1 = x_{12}^{-1} x_{23}^{-1} c, \quad (1.11)$$

$$\sigma_2 x_{12} \sigma_2^{-1} = x_{12}^{-1} x_{23}^{-1} c, \quad \sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{12}^{-1} c, \quad \sigma_2 x_{23} \sigma_2^{-1} = \sigma_2^{-1} x_{23} \sigma_2 = x_{23}. \quad (1.12)$$

²At the time of writing, the authors of this paper do not know a single example of a fake GT-shadow.

Moreover,

$$\Delta x_{12} \Delta^{-1} = x_{23}, \quad \Delta x_{23} \Delta^{-1} = x_{12}. \quad (1.13)$$

It is known [17, Section 1.3] that $\langle x_{12}, x_{23} \rangle$ is isomorphic to the free group F_2 on two generators and we tacitly identify F_2 with the subgroup $\langle x_{12}, x_{23} \rangle$ of PB_3 . Furthermore, PB_3 is isomorphic to $F_2 \times \langle c \rangle$ [17, Section 1.3]. We often by x, y, z the elements x_{12} , x_{23} and $(x_{12}x_{23})^{-1}$, respectively, i.e.

$$x := x_{12}, \quad y := x_{23}, \quad z := y^{-1}x^{-1}.$$

We denote by θ and τ the automorphisms of $F_2 := \langle x, y \rangle$ defined by the formulas

$$\theta(x) := y, \quad \theta(y) := x, \quad (1.14)$$

$$\tau(x) := y, \quad \tau(y) := y^{-1}x^{-1}. \quad (1.15)$$

By abuse of notation, we will use the same letters θ and τ for the corresponding continuous automorphisms of \widehat{F}_2 , respectively. (See Corollary A.2 in Appendix A).

For a group G , the notation $[G, G]$ is reserved for the commutator subgroup of G . For a subgroup $H \leq G$, the notation $|G : H|$ is reserved for the index of H in G . For a normal subgroup $H \trianglelefteq G$ of finite index, we denote by $\text{NFI}_H(G)$ the poset of finite index normal subgroups N in G such that $N \leq H$. Moreover, $\text{NFI}(G) := \text{NFI}_G(G)$, i.e. $\text{NFI}(G)$ is the poset of normal finite index subgroups of a group G . For a subgroup $H \leq G$, $\text{Core}_G(H)$ denotes the normal core of H in G , i.e.

$$\text{Core}_G(H) := \bigcap_{g \in G} gHg^{-1}.$$

For $N \in \text{NFI}(G)$, \mathcal{P}_N denotes the standard (onto) homomorphism

$$\mathcal{P}_N : G \rightarrow G/N. \quad (1.16)$$

Moreover, for $K \in \text{NFI}(G)$ such that $K \leq N$, the notation $\mathcal{P}_{K,N}$ is reserved for the standard (onto) homomorphism

$$\mathcal{P}_{K,N} : G/K \rightarrow G/N. \quad (1.17)$$

Every finite group/set is tacitly considered with the discrete topology.

For a group G , \widehat{G} denotes the profinite completion of G . If G is residually finite, then we tacitly identify G with its image in \widehat{G} . For $N \in \text{NFI}(G)$, $\widehat{\mathcal{P}}_N$ denotes the standard continuous group homomorphism

$$\widehat{\mathcal{P}}_N : \widehat{G} \rightarrow G/N. \quad (1.18)$$

Let G be a residually finite group. Since every group homomorphism $\varphi : G \rightarrow \widehat{H}$ extends uniquely to a continuous group homomorphism from \widehat{G} to \widehat{H} (see Corollary A.2 in Appendix A), we often use the same symbol for this continuous group homomorphism $\widehat{G} \rightarrow \widehat{H}$.

For a prime p , \mathbb{Z}_p denotes the ring of p -adic integers.

For a category \mathcal{C} , the notation $\text{Ob}(\mathcal{C})$ is reserved for the set of objects of \mathcal{C} . For $a, b \in \text{Ob}(\mathcal{C})$, $\mathcal{C}(a, b)$ denotes the set of morphisms in \mathcal{C} from a to b . Every poset J is tacitly considered as the category with J being the set of its objects; if $j_1 \leq j_2$, then we have exactly one morphism $j_1 \rightarrow j_2$; otherwise, there are no morphisms from j_1 to j_2 . A subposet $\tilde{J} \subset J$ is called **cofinal** if $\forall j \in J \exists \tilde{j} \in \tilde{J}$ such that $\tilde{j} \leq j$.

Notational quirks. Paper [5] develops the groupoid of GT-shadows for the original (profinite) version $\widehat{\text{GT}}$ of the Grothendieck-Teichmueller group [6, Section 4]. In consideration of paper [5], we should have denoted the groupoid of GT-shadows for $\widehat{\text{GT}}_{\text{gen}}$ by GTSh_{gen} . However, we decided to omit the subscript “gen” to simplify the notation. This should not lead to a confusion because the main focus of this paper is the group $\widehat{\text{GT}}_{\text{gen}}$ and the corresponding groupoid of GT-shadows. We should also mention that, paper [5] considers GT-shadows $[m, f]$ that may not satisfy the condition

$$f\mathbf{N}_{F_2} \in [F_2/\mathbf{N}_{F_2}, F_2/\mathbf{N}_{F_2}], \quad (1.19)$$

and GT-shadows $[m, f]$ satisfying (1.19) are called charming. (In fact, in paper [5], the authors consider the groupoid GTSh of GT-shadows for $\widehat{\text{GT}}$ and the subgroupoid $\text{GTSh}^\heartsuit \subset \text{GTSh}$ of charming GT-shadows (for $\widehat{\text{GT}}$.) In this paper, we impose condition (1.19) at an earlier stage. Hence we have only one groupoid of GT-shadows for $\widehat{\text{GT}}_{\text{gen}}$.

Acknowledgement. We are thankful to Noémie Combe, Jaclyn Lang, Winnie Li, Adrian Ocneanu, Jessica Radford and Jingfeng Xia for useful discussions. We are thankful to Benjamin Enriquez for suggesting us the idea to use the transformation groupoid of the action of $\widehat{\text{GT}}_{\text{gen}}$ on $\text{NFl}_{\text{PB}_3}(\text{B}_3)$. We are thankful to Florian Pop for showing us paper [14]. We are thankful to Ivan Bortnovskiy, Borys Holikov, Vadym Pashkovskiy and Ihor Pylaiev for their amazing patience with details of this paper. V.A.D. acknowledges a partial support from the project “Arithmetic and Homotopic Galois Theory”. This project is, in turn, supported by the CNRS France-Japan AHGT International Research Network between the RIMS Kyoto University, the LPP of Lille University, and the DMA of ENS PSL. V.A.D. also acknowledges Temple University for 2021 Summer Research Award. J.J.G. acknowledges the Temple University College of Science and Technology Science Scholars Program for their funding in Summer 2020.

2 The gentle version $\widehat{\text{GT}}_{\text{gen}}$ of the Grothendieck-Teichmüller group

2.1 The monoid $(\widehat{\mathbb{Z}} \times \widehat{\text{F}}_2, \bullet)$

To introduce $\widehat{\text{GT}}_{\text{gen}}$, we denote by $E_{\hat{m}, \hat{f}}$ the following group homomorphism from F_2 to $\widehat{\text{F}}_2$

$$E_{\hat{m}, \hat{f}}(x) := x^{2\hat{m}+1}, \quad E_{\hat{m}, \hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f}, \quad (2.1)$$

where $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\text{F}}_2$.

Due to Corollary A.2 from Appendix A, $E_{\hat{m}, \hat{f}}$ extends uniquely to a continuous endomorphism of $\widehat{\text{F}}_2$:

$$E_{\hat{m}, \hat{f}} : \widehat{\text{F}}_2 \rightarrow \widehat{\text{F}}_2. \quad (2.2)$$

By abuse of notation, we use the same symbol $E_{\hat{m}, \hat{f}}$ for the extension of the homomorphism defined in (2.1).

Let $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathbb{Z}} \times \widehat{\text{F}}_2$ and

$$\hat{m} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2, \quad \hat{f} := \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2).$$

A direct computation shows that

$$E_{\hat{m}_1, \hat{f}_1} \circ E_{\hat{m}_2, \hat{f}_2}(x) = E_{\hat{m}, \hat{f}}(x), \quad E_{\hat{m}_1, \hat{f}_1} \circ E_{\hat{m}_2, \hat{f}_2}(y) = E_{\hat{m}, \hat{f}}(y).$$

Hence, applying Corollary A.2, we conclude that

$$E_{\hat{m}_1, \hat{f}_1} \circ E_{\hat{m}_2, \hat{f}_2} = E_{\hat{m}, \hat{f}}. \quad (2.3)$$

This motivates us to define the following binary operation \bullet on $\widehat{\mathbb{Z}} \times \widehat{\text{F}}_2$

$$(\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2) := (2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2, \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)). \quad (2.4)$$

Let us prove that

Proposition 2.1 *The set $\widehat{\mathbb{Z}} \times \widehat{\text{F}}_2$ is a monoid with respect to the binary operation \bullet (see (2.4)) and the pair $(0, 1_{\widehat{\text{F}}_2})$ is the identity element of this monoid. Moreover, the assignment*

$$(\hat{m}, \hat{f}) \mapsto E_{\hat{m}, \hat{f}}$$

defines a homomorphism of monoids $\widehat{\mathbb{Z}} \times \widehat{\text{F}}_2 \rightarrow \text{End}(\widehat{\text{F}}_2)$, where $\text{End}(\widehat{\text{F}}_2)$ is the monoid of continuous endomorphisms of $\widehat{\text{F}}_2$.

Proof. It is easy to see that $(0, 1_{\widehat{\mathbb{F}}_2})$ is the identity element of the magma $(\widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2, \bullet)$. So let us prove the associativity of \bullet .

For $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2), (\hat{m}_3, \hat{f}_3) \in \widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$, we have

$$((\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)) \bullet (\hat{m}_3, \hat{f}_3) = (2\hat{q}\hat{m}_3 + \hat{q} + \hat{m}_3, \hat{g}E_{\hat{q}, \hat{g}}(\hat{f}_3)) \quad (2.5)$$

and

$$(\hat{m}_1, \hat{f}_1) \bullet ((\hat{m}_2, \hat{f}_2) \bullet (\hat{m}_3, \hat{f}_3)) = (2\hat{m}_1\hat{k} + \hat{m}_1 + \hat{k}, \hat{f}_1E_{\hat{m}_1, \hat{f}_1}(\hat{h})), \quad (2.6)$$

where $(\hat{q}, \hat{g}) := (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$ and $(\hat{k}, \hat{h}) := (\hat{m}_2, \hat{f}_2) \bullet (\hat{m}_3, \hat{f}_3)$.

Using $\hat{q} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2$ and $\hat{k} := 2\hat{m}_2\hat{m}_3 + \hat{m}_2 + \hat{m}_3$, it is easy to see that

$$2\hat{q}\hat{m}_3 + \hat{q} + \hat{m}_3 = 2\hat{m}_1\hat{k} + \hat{m}_1 + \hat{k}.$$

Using (2.3) and the fact that $E_{\hat{m}_1, \hat{f}_1}$ is an endomorphism of $\widehat{\mathbb{F}}_2$, we can rewrite $\hat{g}E_{\hat{q}, \hat{g}}(\hat{f}_3)$ as follows

$$\hat{g}E_{\hat{q}, \hat{g}}(\hat{f}_3) = \hat{f}_1E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)E_{\hat{m}_1, \hat{f}_1} \circ E_{\hat{m}_2, \hat{f}_2}(\hat{f}_3) = \hat{f}_1E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2E_{\hat{m}_2, \hat{f}_2}(\hat{f}_3)).$$

Thus $\hat{g}E_{\hat{q}, \hat{g}}(\hat{f}_3) = \hat{f}_1E_{\hat{m}_1, \hat{f}_1}(\hat{h})$ and the associativity of \bullet is proved.

Since $E_{0, 1_{\widehat{\mathbb{F}}_2}} = \text{id}_{\widehat{\mathbb{F}}_2}$, the last statement of the proposition follows from (2.3). \square

Remark 2.2 It is easy to see that, if $(\hat{m}, \hat{f}) = (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$, then

$$2\hat{m} + 1 = (2\hat{m}_1 + 1)(2\hat{m}_2 + 1). \quad (2.7)$$

2.2 The monoid $\widehat{\mathbf{GT}}_{gen, mon}$ and the group $\widehat{\mathbf{GT}}_{gen}$

Let us denote by $\widehat{\mathbf{GT}}_{gen, mon}$ the subset of $\widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$ that consists of pairs

$$(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times [\widehat{\mathbb{F}}_2, \widehat{\mathbb{F}}_2]^{top. cl.} \quad (2.8)$$

satisfying the hexagon relations

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} = \hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}}, \quad (2.9)$$

$$\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} = \sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \hat{f}. \quad (2.10)$$

Let us prove that³,

Proposition 2.3 *For every $(\hat{m}, \hat{f}) \in \widehat{\mathbf{GT}}_{gen, mon}$, the formulas*

$$T_{\hat{m}, \hat{f}}(\sigma_1) := \sigma_1^{2\hat{m}+1}, \quad T_{\hat{m}, \hat{f}}(\sigma_2) := \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \quad (2.11)$$

define a group homomorphism $T_{\hat{m}, \hat{f}} : \mathbf{B}_3 \rightarrow \widehat{\mathbf{B}}_3$ such that

$$T_{\hat{m}, \hat{f}}(c) = c^{2\hat{m}+1}. \quad (2.12)$$

The homomorphism $T_{\hat{m}, \hat{f}}$ extends uniquely to a continuous endomorphism of $\widehat{\mathbf{B}}_3$ and

$$T_{\hat{m}, \hat{f}}|_{\widehat{\mathbb{F}}_2} = E_{\hat{m}, \hat{f}}. \quad (2.13)$$

³See [28, Lemma 1].

Proof. We need to verify that

$$T_{\hat{m},\hat{f}}(\sigma_1)T_{\hat{m},\hat{f}}(\sigma_2)T_{\hat{m},\hat{f}}(\sigma_1) \stackrel{?}{=} T_{\hat{m},\hat{f}}(\sigma_2)T_{\hat{m},\hat{f}}(\sigma_1)T_{\hat{m},\hat{f}}(\sigma_2) \quad (2.14)$$

or equivalently

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} \stackrel{?}{=} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f}. \quad (2.15)$$

Applying (2.9) to the left hand side of (2.15), we get

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} = \hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}} \sigma_1^{2\hat{m}+1} = \hat{f}^{-1} \Delta c^{\hat{m}}. \quad (2.16)$$

To take care of the right hand side of (2.15) we notice that, for every $\hat{t} \in \widehat{\mathbb{Z}}$,

$$\Delta \sigma_1^{\hat{t}} = \sigma_2^{\hat{t}} \Delta. \quad (2.17)$$

Indeed, for every $\mathbf{N} \in \text{NFI}(\mathbf{B}_3)$, there exists $t_{\mathbf{N}} \in \mathbb{Z}$ such that $\widehat{\mathcal{P}}_{\mathbf{N}}(\Delta \sigma_1^{\hat{t}}) = \Delta \sigma_1^{t_{\mathbf{N}}} \mathbf{N}$ and $\widehat{\mathcal{P}}_{\mathbf{N}}(\sigma_2^{\hat{t}} \Delta) = \sigma_2^{t_{\mathbf{N}}} \Delta \mathbf{N}$. Since $\Delta \sigma_1^k = \sigma_2^k \Delta$ for every integer k , relation (2.17) holds.

Applying (2.9) to the right hand side of (2.15) and using (2.17), we get

$$\begin{aligned} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} &= \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}} = \\ &= \hat{f}^{-1} \sigma_2^{2\hat{m}} \Delta x_{12}^{-\hat{m}} c^{\hat{m}} = \hat{f}^{-1} \sigma_2^{2\hat{m}} x_{23}^{-\hat{m}} \Delta c^{\hat{m}} = \hat{f}^{-1} \Delta c^{\hat{m}}. \end{aligned}$$

Combining this result with (2.16), we see that relation (2.15) indeed holds.

Due to (2.16), we have

$$T_{\hat{m},\hat{f}}(\Delta) = \hat{f}^{-1} \Delta c^{\hat{m}}. \quad (2.18)$$

Applying (2.10) to $T_{\hat{m},\hat{f}}(\Delta) = \sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1}$ and using (2.17), we see that

$$T_{\hat{m},\hat{f}}(\Delta) = \sigma_1^{2\hat{m}+1} \sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \hat{f} = \sigma_1^{2\hat{m}} \Delta x_{23}^{-\hat{m}} c^{\hat{m}} \hat{f} = \Delta c^{\hat{m}} \hat{f}. \quad (2.19)$$

Combining (2.18) with (2.19), we get

$$T_{\hat{m},\hat{f}}(c) = T_{\hat{m},\hat{f}}(\Delta) T_{\hat{m},\hat{f}}(\Delta) = \Delta c^{\hat{m}} \hat{f} \hat{f}^{-1} \Delta c^{\hat{m}} = c^{2\hat{m}+1}.$$

Thus (2.12) is proved.

The third statement of the proposition follows from Corollary A.2 and the proof of the last statement is straightforward. \square

Proposition 2.4 *The subset $\widehat{\mathbf{GT}}_{\text{gen},\text{mon}}$ of $\widehat{\mathbb{Z}} \times \widehat{\mathbf{F}}_2$ is a submonoid of $(\widehat{\mathbb{Z}} \times \widehat{\mathbf{F}}_2, \bullet)$. The assignment*

$$(\hat{m}, \hat{f}) \mapsto E_{\hat{m},\hat{f}} \quad (2.20)$$

defines an injective homomorphism of monoids from $\widehat{\mathbf{GT}}_{\text{gen},\text{mon}}$ to the monoid of continuous endomorphisms of $\widehat{\mathbf{F}}_2$. Similarly, the assignment

$$(\hat{m}, \hat{f}) \mapsto T_{\hat{m},\hat{f}} \quad (2.21)$$

defines an injective homomorphism of monoids from $\widehat{\mathbf{GT}}_{\text{gen},\text{mon}}$ to the monoid of continuous endomorphisms of $\widehat{\mathbf{B}}_3$.

Proof. Let $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathbf{GT}}_{gen, mon}$ and $(\hat{m}, \hat{f}) := (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$.

Since $E_{\hat{m}_1, \hat{f}_1}$ is a continuous group homomorphism and $\hat{f}_2 \in [\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}$, $E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)$ also belongs to $[\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}$. Hence

$$\hat{f} := \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2) \in [\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}.$$

Let us prove that the pair (\hat{m}, \hat{f}) satisfies hexagon relations (2.9) and (2.10).

Applying $T_{\hat{m}_1, \hat{f}_1}$ to the first hexagon relation for (\hat{m}_2, \hat{f}_2) and using identities (2.12), (2.13) we get

$$\begin{aligned} & \sigma_1^{(2\hat{m}_2+1)(2\hat{m}_1+1)} E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)^{-1} \hat{f}_1^{-1} \sigma_2^{(2\hat{m}_2+1)(2\hat{m}_1+1)} \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2) = \\ & E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)^{-1} \sigma_1^{2\hat{m}_1+1} \hat{f}_1^{-1} \sigma_2^{2\hat{m}_1+1} \hat{f}_1 x_{12}^{-\hat{m}_2(2\hat{m}_1+1)} c^{\hat{m}_2(2\hat{m}_1+1)}. \end{aligned} \quad (2.22)$$

Using (2.7), the first hexagon relation for (\hat{m}_1, \hat{f}_1) and (2.22), we get

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} = \hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}}.$$

Thus the pair (\hat{m}, \hat{f}) satisfies (2.9).

Similarly, applying $T_{\hat{m}_1, \hat{f}_1}$ to the second hexagon relation for (\hat{m}_2, \hat{f}_2) , and using identities (2.12), (2.13), the second hexagon relation for (\hat{m}_1, \hat{f}_1) and (2.7), one can show that the pair (\hat{m}, \hat{f}) also satisfies (2.10).

We proved that the subset $\widehat{\mathbf{GT}}_{gen, mon}$ is closed with respect to the binary operation \bullet .

It is easy to see that the pair $(0, 1_{\widehat{\mathbf{F}}_2})$ satisfies hexagon relations (2.9) and (2.10). Thus the first statement of the proposition is proved.

Due to the second statement of Proposition 2.1, the assignment in (2.20) is a homomorphism of monoids. To prove that this homomorphism is injective⁴, we will use Theorem B from paper [14] by W. Herfort and L. Ribes.

If $E_{\hat{m}_1, \hat{f}_1} = E_{\hat{m}_2, \hat{f}_2}$, then

$$x^{2\hat{m}_1+1} = x^{2\hat{m}_2+1}, \quad \hat{f}_1^{-1} y^{2\hat{m}_1+1} \hat{f}_1 = \hat{f}_2^{-1} y^{2\hat{m}_2+1} \hat{f}_2. \quad (2.23)$$

The first equation in (2.23) implies that $x^{2(\hat{m}_2-\hat{m}_1)} = 1$ and hence $2(\hat{m}_2-\hat{m}_1) = 0$. Since \mathbb{Z}_p is an integral domain for every prime p and $\widehat{\mathbb{Z}} \cong \prod_{p \text{ is prime}} \mathbb{Z}_p$, we conclude that $\hat{m}_1 = \hat{m}_2$.

We set $\hat{m} := \hat{m}_1 = \hat{m}_2$ and $\hat{w} = \hat{f}_1 \hat{f}_2^{-1}$. The second equation in (2.23) implies that \hat{w} belongs to the centralizer of $y^{2\hat{m}+1}$.

We consider the subgroup $\{y^{\hat{n}} : \hat{n} \in \widehat{\mathbb{Z}}\} \leq \widehat{\mathbf{F}}_2$ and notice that, for every $\hat{m} \in \widehat{\mathbb{Z}}$, $y^{2\hat{m}+1}$ is a non-trivial element of $\{y^{\hat{n}} : \hat{n} \in \widehat{\mathbb{Z}}\}$. Indeed, the component of $2\hat{m}+1$ in \mathbb{Z}_2 is a unit in \mathbb{Z}_2 . Therefore, $2\hat{m}+1$ cannot be zero in $\widehat{\mathbb{Z}}$ and hence $y^{2\hat{m}+1} \neq 1$.

Applying [14, Theorem B] to $\hat{w} \in C_{\widehat{\mathbf{F}}_2}(y^{2\hat{m}+1})$, we conclude that $\hat{w} \in \{y^{\hat{n}} : \hat{n} \in \widehat{\mathbb{Z}}\}$.

Since $\hat{f}_1, \hat{f}_2 \in [\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}$ and the intersection $\{y^{\hat{n}} : \hat{n} \in \widehat{\mathbb{Z}}\} \cap [\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}$ is trivial⁵, we conclude that $\hat{w} = 1$ and hence $\hat{f}_2 = \hat{f}_1$.

We proved that the homomorphism of monoids $\widehat{\mathbf{GT}}_{gen, mon} \rightarrow \text{End}(\widehat{\mathbf{F}}_2)$ is injective.

To prove that the assignment in (2.21) is a homomorphism of monoids, we need to show that,

$$T_{0, 1_{\widehat{\mathbf{F}}_2}} = \text{id}_{\widehat{\mathbf{B}}_3} \quad (2.24)$$

and, for all $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathbf{GT}}_{gen, mon}$, we have

$$T_{\hat{m}_1, \hat{f}_1} \circ T_{\hat{m}_2, \hat{f}_2} = T_{\hat{m}, \hat{f}}, \quad (2.25)$$

⁴A similar statement was mentioned in [12, Section 0.1] without a proof.

⁵To prove that the subgroup $\{y^{\hat{n}} : \hat{n} \in \widehat{\mathbb{Z}}\} \cap [\widehat{\mathbf{F}}_2, \widehat{\mathbf{F}}_2]^{top.cl.}$ is trivial, consider homomorphisms ψ from \mathbf{F}_2 to finite groups such that $\psi(x) = 1$.

where $(\hat{m}, \hat{f}) = (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$.

Applying $T_{\hat{m}_1, \hat{f}_1} \circ T_{\hat{m}_2, \hat{f}_2}$ and $T_{\hat{m}, \hat{f}}$ to the generators σ_1, σ_2 of B_3 , we see that

$$T_{\hat{m}_1, \hat{f}_1} \circ T_{\hat{m}_2, \hat{f}_2} \big|_{B_3} = T_{\hat{m}, \hat{f}} \big|_{B_3}.$$

Since the maps $T_{\hat{m}_1, \hat{f}_1} \circ T_{\hat{m}_2, \hat{f}_2}$ and $T_{\hat{m}, \hat{f}}$ are continuous, they agree on a dense subset B_3 of \widehat{B}_3 and \widehat{B}_3 is Hausdorff, equation (2.25) holds.

The same argument works for (2.24).

The injectivity of the homomorphism $\widehat{GT}_{gen, mon} \rightarrow \text{End}(\widehat{B}_3)$ follows from the injectivity of the homomorphism $\widehat{GT}_{gen, mon} \rightarrow \text{End}(\widehat{F}_2)$ and identity (2.13). \square

Definition 2.5 \widehat{GT}_{gen} is the group of invertible elements of the monoid $\widehat{GT}_{gen, mon}$.

Remark 2.6 As far as we know, the group \widehat{GT}_{gen} was introduced in [12] and, in [12], it is denoted by \widehat{GT}_0 . More precisely, \widehat{GT}_0 consists of elements $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times [\widehat{F}_2, \widehat{F}_2]^{top. cl.}$ satisfying

$$\hat{f}\theta(\hat{f}) = 1_{\widehat{F}_2}, \quad (2.26)$$

$$\tau^2(y^{\hat{m}}\hat{f})\tau(y^{\hat{m}}\hat{f})y^{\hat{m}}\hat{f} = 1_{\widehat{F}_2}, \quad (2.27)$$

and the appropriate invertibility condition. Please see Section 5.1, in which we prove that \widehat{GT}_{gen} indeed coincides with \widehat{GT}_0 introduced in [12, Introduction].

Remark 2.7 Since $\widehat{\mathbb{Z}} \times \widehat{F}_2$ is naturally a topological space and \widehat{GT}_{gen} is a subset of $\widehat{\mathbb{Z}} \times \widehat{F}_2$, the set \widehat{GT}_{gen} is equipped with the subset topology. It is not obvious that \widehat{GT}_{gen} is a topological group with respect to this topology. This statement follows easily from Theorem 5.2 proved in Section 5.

Remark 2.8 Using [8, Theorem 6.2.4] (see also [5, Appendix A.3]), one can show that \widehat{GT}_{gen} is a subgroup of the group $\widehat{GT}_{\leq 3}$ of continuous automorphisms of the truncation $\widehat{PaB}^{\leq 3}$ of the operad \widehat{PaB} .

Remark 2.9 It is easy to see that, for every $(\hat{m}, \hat{f}) \in \widehat{GT}_{gen}$, the endomorphism $E_{\hat{m}, \hat{f}}$ (resp. $T_{\hat{m}, \hat{f}}$) of \widehat{F}_2 (resp. \widehat{B}_3) is invertible. Moreover, due to Proposition 2.4, the assignments

$$(\hat{m}, \hat{f}) \rightarrow E_{\hat{m}, \hat{f}}, \quad (\hat{m}, \hat{f}) \rightarrow T_{\hat{m}, \hat{f}}$$

are injective group homomorphisms from \widehat{GT}_{gen} to the group of continuous automorphisms of \widehat{F}_2 and \widehat{B}_3 , respectively. Due to Remark 2.2, the formula

$$\chi_{vir}(\hat{m}, \hat{f}) := 2\hat{m} + 1 \quad (2.28)$$

defines a group homomorphism $\chi_{vir} : \widehat{GT}_{gen} \rightarrow \widehat{\mathbb{Z}}^\times$, where $\widehat{\mathbb{Z}}^\times$ is the group of units of the ring $\widehat{\mathbb{Z}}$. We call χ_{vir} the **virtual cyclotomic character**. Using the Ihara embedding $\text{Ih} : G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ (see [16, Section 1]) and the surjectivity of the cyclotomic character $\chi : G_{\mathbb{Q}} \rightarrow \widehat{\mathbb{Z}}^\times$, one can show that the group homomorphism $\chi_{vir} : \widehat{GT}_{gen} \rightarrow \widehat{\mathbb{Z}}^\times$ is surjective.

Remark 2.10 Let G be a profinite group with a dense finitely generated subgroup (e.g. $G = \widehat{F}_2$). Due to [23, Theorem 1.1], every endomorphism of G is continuous. Moreover, due to [23, Theorem 1.3], $[G, G]$ is a closed subgroup of G . In particular, $[\widehat{F}_2, \widehat{F}_2]^{top. cl.} = [\widehat{F}_2, \widehat{F}_2]$. However, in this paper, we do not use Theorems 1.1 and 1.3 from [23].

3 The groupoid GTSh

For every $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, we set

$$N_{\text{ord}} := \text{lcm}(\text{ord}(x_{12}N), \text{ord}(x_{23}N), \text{ord}(cN)) \quad (3.1)$$

and

$$N_{F_2} := N \cap F_2. \quad (3.2)$$

It is clear that $N_{F_2} \in \text{NFI}(F_2)$.

We say that a pair $(m, f) \in \mathbb{Z} \times F_2$ satisfies the hexagon relations modulo N if

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f N = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m N, \quad (3.3)$$

$$f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} N = \sigma_2 \sigma_1 x_{23}^{-m} c^m f N. \quad (3.4)$$

Since N_{ord} is the least common multiple of the orders of the elements $x_{12}N$, $x_{23}N$, cN and $N_{F_2} \leq N$, we see that, if a pair $(m, f) \in \mathbb{Z} \times F_2$ satisfies (3.3) and (3.4), then so does the pair $(m + tN_{\text{ord}}, fh)$ for any $t \in \mathbb{Z}$ and any $h \in N_{F_2}$.

Definition 3.1 A GT-pair with the target N is a pair

$$(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2}) \in \mathbb{Z}/N_{\text{ord}}\mathbb{Z} \times F_2/N_{F_2} \quad (3.5)$$

satisfying relations (3.3) and (3.4). A GT-pair (3.5) is called *charming* if

- $2m + 1$ represents a unit in the ring $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ and
- $fN_{F_2} \in [F_2/N_{F_2}, F_2/N_{F_2}]$, or equivalently the coset fN_{F_2} can be represented by an element in the commutator subgroup $[F_2, F_2]$ of F_2 .

We denote by $\text{GT}_{pr}(N)$ (resp. $\text{GT}_{pr}^\heartsuit(N)$) the set of GT-pairs (resp. the set of charming GT-pairs) with the target N . From now on, we denote by $[m, f]$ the GT-pair represented by $(m, f) \in \mathbb{Z} \times F_2$.

The importance of the hexagon relations is emphasized by the following proposition:

Proposition 3.2 For every $[m, f] \in \text{GT}_{pr}(N)$, the formulas

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1} N, \quad T_{m,f}(\sigma_2) := f^{-1} \sigma_2^{2m+1} f N$$

define a group homomorphism $T_{m,f} : B_3 \rightarrow B_3/N$.

Proof. Since $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$, it suffices to verify that

$$T_{m,f}(\sigma_1) T_{m,f}(\sigma_2) T_{m,f}(\sigma_1) \stackrel{?}{=} T_{m,f}(\sigma_2) T_{m,f}(\sigma_1) T_{m,f}(\sigma_2). \quad (3.6)$$

Using (3.3), we rewrite the left hand side of (3.6) as

$$(\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f) \sigma_1^{2m+1} N = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m \sigma_1^{2m+1} N = f^{-1} \Delta c^m N, \quad (3.7)$$

where $\Delta := \sigma_1 \sigma_2 \sigma_1$.

Using (3.3) once again, we rewrite the right hand side of (3.6) as

$$\begin{aligned} f^{-1} \sigma_2^{2m+1} f (\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f) N &= f^{-1} \sigma_2^{2m+1} f (f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m) N = \\ f^{-1} \sigma_2^{2m} \sigma_2 \sigma_1 \sigma_2 x_{12}^{-m} c^m N &= f^{-1} \sigma_2^{2m} \Delta x_{12}^{-m} c^m N = f^{-1} \Delta c^m N. \end{aligned}$$

In the last step, we used the identity $\sigma_2 \Delta = \Delta \sigma_1$.

Relation (3.6) is proved. \square

If we apply both hexagon relations to the left hand side of (3.6), then we get a useful relation on the coset $f\mathbf{N}$. Indeed, due to the calculation in (3.7), we have

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} \mathbf{N} = f^{-1} \Delta c^m \mathbf{N}. \quad (3.8)$$

On the other hand, applying (3.4) and the identity $\sigma_1 \Delta = \Delta \sigma_2$, we get

$$\sigma_1^{2m+1} (f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1}) \mathbf{N} = \sigma_1^{2m+1} \sigma_2 \sigma_1 c^m x_{23}^{-m} f \mathbf{N} = \sigma_1^{2m} \Delta c^m x_{23}^{-m} f \mathbf{N} = \Delta f c^m \mathbf{N}.$$

Comparing this result with (3.8), we conclude that $\Delta f \mathbf{N} = f^{-1} \Delta \mathbf{N}$. Thus, using (1.13), we see that we proved the following statement:

Proposition 3.3 *Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. If a pair $(m, f) \in \mathbb{Z} \times \text{F}_2$ satisfies hexagon relations (3.3) and (3.4) (modulo \mathbf{N}) then*

$$f\theta(f) \in \mathbf{N}, \quad (3.9)$$

where θ is the automorphism of F_2 defined in (1.14). \square

Relation (3.9) can also be written in the form $f(x, y)f(y, x) \in \mathbf{N}$.

Let $(m, f) \in \mathbb{Z} \times [\text{F}_2, \text{F}_2]$ and $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. It turns out that, hexagon relations (3.3), (3.4) for (m, f) (modulo \mathbf{N}) are equivalent to somewhat simpler relations. The following proposition establishes this equivalence.

Proposition 3.4 *Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$ and θ and τ be the automorphisms of F_2 defined in (1.14) and (1.15), respectively. A pair $(m, f) \in \mathbb{Z} \times [\text{F}_2, \text{F}_2]$ satisfies hexagon relations (3.3), (3.4) (modulo \mathbf{N}) if and only if*

$$f\theta(f) \in \mathbf{N}_{\text{F}_2} \quad (3.10)$$

and

$$\tau^2(y^m f)\tau(y^m f)y^m f \in \mathbf{N}_{\text{F}_2}. \quad (3.11)$$

Proof. For our purposes, it is convenient to rewrite (3.10) and (3.11) in the form

$$f(x, y)f(y, x) \in \mathbf{N}_{\text{F}_2} \quad (3.12)$$

and

$$x^m f(z, x)z^m f(y, z)y^m f \in \mathbf{N}_{\text{F}_2}, \quad (3.13)$$

where $z := y^{-1}x^{-1}$.

Using identities (1.11), (1.12) and the property $f \in [\text{F}_2, \text{F}_2]$, one can prove that (3.3) is equivalent to

$$x^m f(z, x)z^m f^{-1}(z, y)y^m f \in \mathbf{N}_{\text{F}_2} \quad (3.14)$$

and (3.4) is equivalent⁶ to

$$x^m f^{-1}(x, z)z^m f(y, z)y^m f^{-1}(y, x) \in \mathbf{N}_{\text{F}_2}, \quad (3.15)$$

where $x := x_{12}$, $y := x_{23}$, $z := x_{23}^{-1}x_{12}^{-1}$.

Moreover, conjugating (3.12) with $\sigma_1 \sigma_2$ and with $(\sigma_1 \sigma_2)^2$, and using the property $f \in [\text{F}_2, \text{F}_2]$ once again, we see that

$$f(z, y)\mathbf{N}_{\text{F}_2} = f^{-1}(y, z)\mathbf{N}_{\text{F}_2} \quad (3.16)$$

and

$$f(x, z)\mathbf{N}_{\text{F}_2} = f^{-1}(z, x)\mathbf{N}_{\text{F}_2}. \quad (3.17)$$

Let us assume that equations (3.3) and (3.4) are satisfied. Due to Proposition 3.3, relation (3.12) is satisfied. Hence relation (3.16) also holds.

Combining (3.14) with (3.16), we conclude that (3.13) is satisfied.

Let us now assume that (3.12) and (3.13) are satisfied. Relation (3.12) implies (3.16) and (3.17).

⁶For this equivalence, we also need (1.13).

Combining (3.12) with (3.13), (3.16) and (3.17), we conclude that (3.14) and (3.15) are satisfied. Since (3.14) and (3.15) are equivalent to (3.3) and (3.4), the desired statement is proved. \square
We call (3.10), (3.11) the **simplified hexagon relations**. (See also [29, Proposition 2.6].)

Let us denote by ρ the standard homomorphism $B_3 \rightarrow S_3$: $\rho(\sigma_1) := (1, 2)$, $\rho(\sigma_2) := (2, 3)$. Since $\mathbf{N} \leq \text{PB}_3$, the formula $\rho_{\mathbf{N}}(w\mathbf{N}) := \rho(w)$ defines the group homomorphism

$$\rho_{\mathbf{N}} : B_3/\mathbf{N} \rightarrow S_3. \quad (3.18)$$

It is easy to see that, for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(B_3)$ and $[m, f] \in \text{GT}_{pr}(\mathbf{N})$,

$$\rho_{\mathbf{N}} \circ T_{m,f} = \rho. \quad (3.19)$$

Hence $T_{m,f}(\text{PB}_3) \subset \text{PB}_3/\mathbf{N}$. We set

$$T_{m,f}^{\text{PB}_3} := T_{m,f}|_{\text{PB}_3} : \text{PB}_3 \rightarrow \text{PB}_3/\mathbf{N}$$

and notice that $\ker(T_{m,f}) = \ker(T_{m,f}^{\text{PB}_3}) \in \text{NFI}_{\text{PB}_3}(B_3)$.

Due to the following proposition, the homomorphism $T_{m,f}^{\text{PB}_3}$ comes from an endomorphism of PB_3 for every $[m, f] \in \text{GT}_{pr}(\mathbf{N})$.

Proposition 3.5 *Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(B_3)$ and $[m, f] \in \text{GT}_{pr}(\mathbf{N})$. Then*

$$T_{m,f}^{\text{PB}_3}(x_{12}) = x_{12}^{2m+1} \mathbf{N}, \quad T_{m,f}^{\text{PB}_3}(x_{23}) = f^{-1} x_{23}^{2m+1} f \mathbf{N}, \quad T_{m,f}^{\text{PB}_3}(c) = c^{2m+1} \mathbf{N}. \quad (3.20)$$

Proof. The first two equations in (3.20) are straightforward consequences of the definitions of $x_{12} := \sigma_1^2$ and $x_{23} := \sigma_2^2$.

To prove the third equation, we will use the calculation in (3.7) and relation (3.9).

Indeed, due to the calculation in (3.7),

$$T_{m,f}(\Delta) = f^{-1} \Delta c^m \mathbf{N}$$

Hence

$$T_{m,f}^{\text{PB}_3}(c) = T_{m,f}(\Delta^2) = f^{-1} \Delta c^m f^{-1} \Delta c^m \mathbf{N} = \Delta f c^m f^{-1} \Delta c^m \mathbf{N} = \Delta^2 c^{2m} \mathbf{N} = c^{2m+1} \mathbf{N}.$$

Proposition (3.5) is proved. \square

Note that, for every $[m, f] \in \text{GT}_{pr}(\mathbf{N})$, the restriction of $T_{m,f}^{\text{PB}_3}$ to $F_2 \leq \text{PB}_3$ gives us a homomorphism

$$T_{m,f}^{F_2} := T_{m,f}^{\text{PB}_3}|_{F_2} : F_2 \rightarrow F_2/\mathbf{N}_{F_2}. \quad (3.21)$$

Let us prove that

Proposition 3.6 *If a pair $(m, f) \in \mathbb{Z} \times F_2$ satisfies hexagon relations (3.3) and (3.4) and $2m+1$ represents a unit in the ring $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, then the following conditions are equivalent:*

- 1) *The homomorphism $T_{m,f} : B_3 \rightarrow B_3/\mathbf{N}$ is onto.*
- 2) *The homomorphism $T_{m,f}^{\text{PB}_3} : \text{PB}_3 \rightarrow \text{PB}_3/\mathbf{N}$ is onto.*
- 3) *The homomorphism $T_{m,f}^{F_2} : F_2 \rightarrow F_2/\mathbf{N}_{F_2}$ is onto.*

Proof. We will start with the implication 1) \Rightarrow 2).

Let $w \in \text{PB}_3$. Since $T_{m,f}$ is onto, there exists $v \in B_3$ such that $T_{m,f}(v) = w\mathbf{N}$. Due to (3.19), $v \in \ker(\rho) = \text{PB}_3$. Thus $T_{m,f}^{\text{PB}_3}$ is indeed surjective.

Now we will take care of the implication $2) \Rightarrow 3)$. We will do so by showing that $x_{12}\mathbf{N}_{F_2}$ and $x_{23}\mathbf{N}_{F_2}$ belong to the image of $T_{m,f}^{F_2}$. First, we have

$$T_{m,f}^{F_2}(x_{12}) = x_{12}^{2m+1}\mathbf{N}_{F_2}. \quad (3.22)$$

Since $2m+1$ is coprime with the order of $x_{12}\mathbf{N}_{F_2}$, $x_{12}^{2m+1}\mathbf{N}_{F_2} \in T_{m,f}^{F_2}(F_2)$ implies that

$$x_{12}\mathbf{N}_{F_2} \in T_{m,f}^{F_2}(F_2). \quad (3.23)$$

Similarly, since $2m+1$ is coprime with $\text{ord}(x_{23}\mathbf{N}_{F_2}) = \text{ord}(f^{-1}x_{23}f\mathbf{N}_{F_2})$ and

$$T_{m,f}^{F_2}(x_{23}) = f^{-1}x_{23}^{2m+1}f\mathbf{N}_{F_2} = (f^{-1}x_{23}f\mathbf{N}_{F_2})^{2m+1},$$

we conclude that

$$f^{-1}x_{23}f\mathbf{N}_{F_2} = T_{m,f}^{F_2}(x_{23}^k) \quad (3.24)$$

for some integer k .

Since $T_{m,f}^{\text{PB}_3}$ is onto, there exists $w \in \text{PB}_3$ such that $T_{m,f}^{\text{PB}_3}(w) = f\mathbf{N}$. Moreover $\text{PB}_3 = F_2 \times \langle c \rangle$, so $w = \tilde{w}c^j$ for some $\tilde{w} \in F_2$ and some integer j . Thus we get

$$T_{m,f}^{\text{PB}_3}(\tilde{w}) = c^{-j(2m+1)}f\mathbf{N}. \quad (3.25)$$

Since $c \in \mathcal{Z}(\text{PB}_3)$, equations (3.24) and (3.25) imply that

$$T_{m,f}^{\text{PB}_3}(\tilde{w}x_{23}^k\tilde{w}^{-1}) = c^{-j(2m+1)}f(f^{-1}x_{23}f)f^{-1}c^{j(2m+1)}\mathbf{N} = x_{23}\mathbf{N}.$$

Note that $T_{m,f}^{F_2} : F_2 \rightarrow F_2/\mathbf{N}_{F_2}$ is the restriction of $T_{m,f}^{\text{PB}_3}$ to $F_2 \leq \text{PB}_3$. Therefore

$$x_{23}\mathbf{N}_{F_2} \in T_{m,f}^{F_2}(F_2). \quad (3.26)$$

Combining (3.23) and (3.26), we see that $F_2 \xrightarrow{T_{m,f}^{F_2}} F_2/\mathbf{N}_{F_2}$ is indeed surjective, i.e. the implication $2) \Rightarrow 3)$ is proved.

Let us now prove the implication $3) \Rightarrow 1)$.

Using $\gcd(2m+1, \text{ord}(x_{12}N)) = \gcd(2m+1, \text{ord}(x_{23}N)) = 1$ and $2 \nmid (2m+1)$, it is easy to show that

$$\gcd(2m+1, \text{ord}(\sigma_1 N)) = \gcd(2m+1, \text{ord}(\sigma_2 N)) = 1. \quad (3.27)$$

Combining (3.27) with

$$T_{m,f}(\sigma_1) = \sigma_1^{2m+1}\mathbf{N}, \quad T_{m,f}(\sigma_2) = f^{-1}\sigma_2^{2m+1}f\mathbf{N} = (f^{-1}\sigma_2f\mathbf{N})^{2m+1},$$

we conclude that

$$\sigma_1\mathbf{N} \in T_{m,f}(\text{B}_3) \quad (3.28)$$

and

$$f^{-1}\sigma_2f\mathbf{N} \in T_{m,f}(\text{B}_3). \quad (3.29)$$

Surjectivity of $T_{m,f}^{F_2}$ implies that $f\mathbf{N}_{F_2} = T_{m,f}^{F_2}(w)$ for some $w \in F_2$. Hence

$$T_{m,f}(w) = f\mathbf{N}. \quad (3.30)$$

Using (3.29) and (3.30), it is easy to see that

$$\sigma_2\mathbf{N} \in T_{m,f}(\text{B}_3). \quad (3.31)$$

Combining (3.28) and (3.31), we conclude that $\text{B}_3 \xrightarrow{T_{m,f}} \text{B}_3/\mathbf{N}$ is indeed surjective, i.e. the implication $3) \Rightarrow 1)$ is also proved.

Proposition 3.6 is proved. \square

Definition 3.7 Let $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. A charming GT-pair $[m, f] \in \text{GT}_{pr}(N)$ is called a **GT-shadow with the target N** if the pair (m, f) satisfies one of the three equivalent conditions of Proposition 3.6. We denote by $\text{GT}(N)$ the set of GT-shadows with the target N .

Using (3.19), it is easy to show that, for every $[m, f] \in \text{GT}(N)$, the kernel K of the homomorphism $T_{m,f} : \text{B}_3 \rightarrow \text{B}_3/N$ belongs to $\text{NFI}_{\text{PB}_3}(\text{B}_3)$, and

$$K = \ker \left(\text{PB}_3 \xrightarrow{T_{m,f}^{\text{PB}_3}} \text{PB}_3/N \right). \quad (3.32)$$

Moreover, the surjectivity of $T_{m,f}$ implies that it factors as follows

$$T_{m,f} = T_{m,f}^{\text{isom}} \circ \mathcal{P}_K, \quad (3.33)$$

where \mathcal{P}_K is the standard onto homomorphism $\text{B}_3 \rightarrow \text{B}_3/K$ and $T_{m,f}^{\text{isom}}$ is the isomorphism $\text{B}_3/K \xrightarrow{\simeq} \text{B}_3/N$ defined by the formula $T_{m,f}^{\text{isom}}(wK) := T_{m,f}(w)$.

Using (3.32), it is easy to prove that, for every $[m, f] \in \text{GT}(N)$,

$$\ker \left(\text{F}_2 \xrightarrow{T_{m,f}^{\text{F}_2}} \text{F}_2/\text{N}_{\text{F}_2} \right) = K_{\text{F}_2}, \quad (3.34)$$

where $K := \ker(T_{m,f})$.

Using (3.32) and (3.34), we get the similar factorizations for the homomorphisms $T_{m,f}^{\text{PB}_3} : \text{PB}_3 \rightarrow \text{PB}_3/N$ and for $T_{m,f}^{\text{F}_2} : \text{F}_2 \rightarrow \text{F}_2/\text{N}_{\text{F}_2}$, i.e.

$$T_{m,f}^{\text{PB}_3} = T_{m,f}^{\text{PB}_3, \text{isom}} \circ \mathcal{P}_K, \quad (3.35)$$

and

$$T_{m,f}^{\text{F}_2} = T_{m,f}^{\text{F}_2, \text{isom}} \circ \mathcal{P}_{K_{\text{F}_2}}, \quad (3.36)$$

where $T_{m,f}^{\text{PB}_3, \text{isom}}$ (resp. $T_{m,f}^{\text{F}_2, \text{isom}}$) is an isomorphism $\text{PB}_3/K \xrightarrow{\simeq} \text{PB}_3/N$ (resp. $\text{F}_2/K_{\text{F}_2} \xrightarrow{\simeq} \text{F}_2/\text{N}_{\text{F}_2}$). For example, the isomorphism $T_{m,f}^{\text{F}_2, \text{isom}} : \text{F}_2/K_{\text{F}_2} \xrightarrow{\simeq} \text{F}_2/\text{N}_{\text{F}_2}$ is defined by the formula:

$$T_{m,f}^{\text{F}_2, \text{isom}}(wK_{\text{F}_2}) := T_{m,f}^{\text{F}_2}(w). \quad (3.37)$$

Thus we proved the first three statements of the following proposition:

Proposition 3.8 Let $K, N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. If there exists $[m, f] \in \text{GT}(N)$ such that $K = \ker(T_{m,f})$, then

- 1) the finite groups B_3/K and B_3/N are isomorphic,
- 2) the finite groups PB_3/K and PB_3/N are isomorphic,
- 3) the finite groups $\text{F}_2/K_{\text{F}_2}$ and $\text{F}_2/\text{N}_{\text{F}_2}$ are isomorphic and, finally,
- 4) $K_{\text{ord}} = N_{\text{ord}}$.

Proof. It remains to prove that $K_{\text{ord}} = N_{\text{ord}}$.

Since $2m+1$ is coprime with the orders of $x_{12}N$, $x_{23}N$, and cN , we have

$$\text{ord}(x_{12}^{2m+1}N) = \text{ord}(x_{12}N), \quad \text{ord}(x_{23}^{2m+1}N) = \text{ord}(x_{23}N), \quad \text{ord}(c^{2m+1}N) = \text{ord}(cN). \quad (3.38)$$

Note that $\text{ord}(x_{23}^{2m+1}N) = \text{ord}(f^{-1}x_{23}^{2m+1}fN)$. Combining this observation with the second equation in (3.38), we conclude that

$$\text{ord}(f^{-1}x_{23}^{2m+1}fN) = \text{ord}(x_{23}N). \quad (3.39)$$

Since

$$T_{m,f}^{\text{PB}_3, \text{isom}}(x_{12}K) = x_{12}^{2m+1}N, \quad T_{m,f}^{\text{PB}_3, \text{isom}}(cK) = c^{2m+1}N,$$

$$T_{m,f}^{\text{PB}_3, \text{isom}}(x_{23}\mathbf{K}) = f^{-1}x_{23}^{2m+1}f\mathbf{N},$$

and $T_{m,f}^{\text{PB}_3, \text{isom}}$ is an isomorphism, equations (3.38) and (3.39) imply that

$$\text{ord}(x_{12}\mathbf{K}) = \text{ord}(x_{12}\mathbf{N}), \quad \text{ord}(x_{23}\mathbf{K}) = \text{ord}(x_{23}\mathbf{N}), \quad \text{ord}(c\mathbf{K}) = \text{ord}(c\mathbf{N}).$$

Thus, $K_{\text{ord}} = N_{\text{ord}}$. □

Our next goal is to show that GT-shadows form a groupoid GTSh with $\text{Ob}(\text{GTSh}) := \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ and

$$\text{GTSh}(\mathbf{K}, \mathbf{N}) := \{[m, f] \in \text{GT}(\mathbf{N}) \mid \ker(T_{m,f}) = \mathbf{K}\}, \quad \mathbf{K}, \mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3). \quad (3.40)$$

To define the composition of morphisms, we need an auxiliary construction.

For every pair $(m, f) \in \mathbb{Z} \times \mathbf{F}_2$, the formulas

$$E_{m,f}(x) := x^{2m+1}, \quad E_{m,f}(y) := f^{-1}y^{2m+1}f \quad (3.41)$$

define an endomorphism $E_{m,f}$ of \mathbf{F}_2 .

A direct computation shows that

$$E_{m_1, f_1} \circ E_{m_2, f_2} = E_{m, f}, \quad (3.42)$$

where

$$m := 2m_1m_2 + m_1 + m_2, \quad f := f_1E_{m_1, f_1}(f_2).$$

It is not hard to see⁷ that the set $\mathbb{Z} \times \mathbf{F}_2$ is a monoid with respect to the binary operation

$$(m_1, f_1) \bullet (m_2, f_2) := (2m_1m_2 + m_1 + m_2, f_1E_{m_1, f_1}(f_2)) \quad (3.43)$$

and the identity element $(0, 1_{\mathbf{F}_2})$. Moreover, the assignment $(m, f) \mapsto E_{m,f}$ defines a homomorphism of monoids $(\mathbb{Z} \times \mathbf{F}_2, \bullet) \rightarrow \text{End}(\mathbf{F}_2)$.

Note that, if $(m, f) \in \mathbb{Z} \times \mathbf{F}_2$ represents a GT-pair with the target $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, then

$$T_{m,f}^{\mathbf{F}_2}(w) = E_{m,f}(w)\mathbf{N}_{\mathbf{F}_2}, \quad \forall w \in \mathbf{F}_2, \quad (3.44)$$

where $T_{m,f}^{\mathbf{F}_2}$ is defined in (3.21).

Let us prove the following auxiliary statement:

Proposition 3.9 *Let $\mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, $[m_1, f_1] \in \text{GTSh}(\mathbf{N}^{(2)}, \mathbf{N}^{(1)})$, $[m_2, f_2] \in \text{GT}(\mathbf{N}^{(3)}, \mathbf{N}^{(2)})$ and $N_{\text{ord}} := N_{\text{ord}}^{(1)} = N_{\text{ord}}^{(2)} = N_{\text{ord}}^{(3)}$. If*

$$m := 2m_1m_2 + m_1 + m_2, \quad f := f_1E_{m_1, f_1}(f_2), \quad (3.45)$$

then

$$(m + N_{\text{ord}}\mathbb{Z}, f\mathbf{N}_{\mathbf{F}_2}^{(1)}) \in \text{GTSh}(\mathbf{N}^{(3)}, \mathbf{N}^{(1)}). \quad (3.46)$$

The pair $[m, f] := (m + N_{\text{ord}}\mathbb{Z}, f\mathbf{N}_{\mathbf{F}_2}^{(1)})$ depends only on the cosets $f_1\mathbf{N}^{(1)}$, $f_2\mathbf{N}^{(2)}$ and residue classes $m_1 + N_{\text{ord}}\mathbb{Z}$, $m_2 + N_{\text{ord}}\mathbb{Z}$. Moreover, the diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{T_{m,f}} & & \\
 & \mathbf{B}_3 & & \mathbf{B}_3 & \\
 & \downarrow \mathcal{P}_{\mathbf{N}^{(3)}} & \searrow T_{m_2, f_2} & \downarrow \mathcal{P}_{\mathbf{N}^{(2)}} & \searrow T_{m_1, f_1} \\
 \mathbf{B}_3/\mathbf{N}^{(3)} & \xrightarrow{T_{m_2, f_2}^{\text{isom}}} & \mathbf{B}_3/\mathbf{N}^{(2)} & \xrightarrow{T_{m_1, f_1}^{\text{isom}}} & \mathbf{B}_3/\mathbf{N}^{(1)} \\
 & & \xleftarrow{T_{m, f}^{\text{isom}}} & &
 \end{array} \quad (3.47)$$

⁷A detailed proof is given in [29, Proposition 2.11].

commutes. In particular,

$$T_{m_1, f_1}^{\text{isom}} \circ T_{m_2, f_2}^{\text{isom}} = T_{m, f}^{\text{isom}}. \quad (3.48)$$

Proof. The first equation in (3.45) implies that

$$2m + 1 = (2m_1 + 1)(2m_2 + 1). \quad (3.49)$$

Our first goal is to show that the pair (m, f) satisfies hexagon relations (3.3), (3.4) (modulo $\mathbf{N}^{(1)}$). The first hexagon relation for (m_2, f_2) (modulo $\mathbf{N}^{(2)}$) reads

$$\sigma_1^{2m_2+1} f_2^{-1} \sigma_2^{2m_2+1} f_2 \mathbf{N}^{(2)} = f_2^{-1} \sigma_1 \sigma_2 x_{12}^{-m_2} c^{m_2} \mathbf{N}^{(2)}. \quad (3.50)$$

Applying $T_{m_1, f_1}^{\text{isom}}$ to the left hand side of (3.50) and using (3.44), (3.49), we get

$$\begin{aligned} \sigma_1^{(2m_1+1)(2m_2+1)} E_{m_1, f_1}(f_2)^{-1} f_1^{-1} \sigma_2^{(2m_1+1)(2m_2+1)} f_1 E_{m_1, f_1}(f_2) \mathbf{N}^{(1)} = \\ \sigma_1^{(2m_1+1)(2m_2+1)} f^{-1} \sigma_2^{(2m_1+1)(2m_2+1)} f \mathbf{N}^{(1)} = \sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \mathbf{N}^{(1)}. \end{aligned} \quad (3.51)$$

Applying $T_{m_1, f_1}^{\text{isom}}$ to the right hand side of (3.50), using (3.20), (3.44), and hexagon relation (3.3) for (m_1, f_1) , we get

$$\begin{aligned} E_{m_1, f_1}(f_2)^{-1} (\sigma_1^{2m_1+1} f_1^{-1} \sigma_2^{2m_1+1} f_1) x_{12}^{-m_2(2m_1+1)} c^{m_2(2m_1+1)} \mathbf{N}^{(1)} = \\ E_{m_1, f_1}(f_2)^{-1} f_1^{-1} \sigma_1 \sigma_2 x_{12}^{-m_1} c^{m_1} x_{12}^{-m_2(2m_1+1)} c^{m_2(2m_1+1)} \mathbf{N}^{(1)} = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m \mathbf{N}^{(1)}. \end{aligned}$$

Combining this result with the final expression in (3.51), we see that the pair (m, f) satisfies (3.3) modulo $\mathbf{N}^{(1)}$.

Applying $T_{m_1, f_1}^{\text{isom}}$ to both sides of the second hexagon relation for (m_2, f_2) and performing similar calculations, we see that the pair (m, f) satisfies (3.4) modulo $\mathbf{N}^{(1)}$.

Since $2m + 1 = (2m_1 + 1)(2m_2 + 1)$ and $2\overline{m}_1 + 1, 2\overline{m}_2 + 1 \in (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$, we conclude that $2m + 1$ represents a unit in the ring $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$.

We may assume, without loss of generality, that $f_1, f_2 \in [\mathbf{F}_2, \mathbf{F}_2]$. Hence $f := f_1 E_{m_1, f_1}(f_2)$ also belongs to the commutator subgroup $[\mathbf{F}_2, \mathbf{F}_2]$.

We proved that (m, f) represents a charming GT-pair with the target $\mathbf{N}^{(1)}$.

Recall that, since the pair (m, f) satisfies hexagon relations (3.3) and (3.4) (modulo $\mathbf{N}^{(1)}$), the formulas

$$T_{m, f}(\sigma_1) := \sigma_1^{2m+1} \mathbf{N}^{(1)}, \quad T_{m, f}(\sigma_2) := f^{-1} \sigma^{2m+1} f \mathbf{N}^{(1)},$$

define a group homomorphism $T_{m, f} : \mathbf{B}_3 \rightarrow \mathbf{B}_3/\mathbf{N}^{(1)}$.

To show that the pair (m, f) represents a GT-shadow with the target $\mathbf{N}^{(1)}$, we need to prove that the group homomorphism $T_{m, f} : \mathbf{B}_3 \rightarrow \mathbf{B}_3/\mathbf{N}^{(1)}$ is onto.

Applying $T_{m_1, f_1}^{\text{isom}} \circ T_{m_2, f_2}$ to the generators σ_1 and σ_2 and using (3.49), we see that

$$T_{m_1, f_1}^{\text{isom}} \circ T_{m_2, f_2}(\sigma_1) = T_{m, f}(\sigma_1), \quad T_{m_1, f_1}^{\text{isom}} \circ T_{m_2, f_2}(\sigma_2) = T_{m, f}(\sigma_2).$$

Therefore,

$$T_{m_1, f_1}^{\text{isom}} \circ T_{m_2, f_2} = T_{m, f}. \quad (3.52)$$

Hence $T_{m, f}$ is onto. Thus the pair (m, f) indeed represents a GT-shadow with the target $\mathbf{N}^{(1)}$.

Combining identity (3.52) with $\mathbf{N}^{(3)} = \ker(T_{m_2, f_2})$, we conclude that $\ker(T_{m, f}) = \mathbf{N}^{(3)}$. Hence, $T_{m, f}$ factors as

$$T_{m, f} = T_{m, f}^{\text{isom}} \circ \mathcal{P}_{\mathbf{N}^{(3)}},$$

where $T_{m, f}^{\text{isom}}$ is the isomorphism $\mathbf{B}_3/\mathbf{N}^{(3)} \xrightarrow{\sim} \mathbf{B}_3/\mathbf{N}^{(1)}$ defined by the formula $T_{m, f}^{\text{isom}}(w\mathbf{N}^{(3)}) := T_{m, f}(w)$.

We proved the first statement of the proposition (see (3.46)).

It is clear that $m + N_{\text{ord}}\mathbb{Z}$ depends only the residue classes of m_1 and m_2 in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$.

Let $h_1 \in \mathbf{N}_{\mathbf{F}_2}^{(1)}$ and $h_2 \in \mathbf{N}_{\mathbf{F}_2}^{(2)}$. It is clear that $T_{m_1+tN_{\text{ord}}, f_1 h_1}^{\mathbf{F}_2} = T_{m_1, f_1}^{\mathbf{F}_2}$ for every $t \in \mathbb{Z}$. Due to (3.44) and $\ker(T_{m_1, f_1}^{\mathbf{F}_2}) = \mathbf{N}_{\mathbf{F}_2}^{(2)}$, we have $E_{m_1, f_1}(h_2) \in \mathbf{N}_{\mathbf{F}_2}^{(1)}$. Hence

$$f_1 h_1 E_{m_1, f_1}(f_2 h_2) \mathbf{N}_{\mathbf{F}_2}^{(1)} = f_1 E_{m_1, f_1}(f_2) \mathbf{N}_{\mathbf{F}_2}^{(1)} = f \mathbf{N}_{\mathbf{F}_2}^{(1)}.$$

We proved that the GT-shadow $[m, f] \in \text{GT}(\mathbf{N}^{(1)})$ depends only on the cosets $f_1 \mathbf{N}^{(1)}$, $f_2 \mathbf{N}^{(2)}$ and residue classes $m_1 + N_{\text{ord}}\mathbb{Z}$, $m_2 + N_{\text{ord}}\mathbb{Z}$.

It should now be clear that diagram (3.47) commutes. Indeed, the inner “straight” triangles commute by definition of $T_{m_1, f_1}^{\text{isom}}$ and $T_{m_2, f_2}^{\text{isom}}$ (see equation (3.33)).

The triangle with the vertices \mathbf{B}_3 , $\mathbf{B}_3/\mathbf{N}^{(2)}$, $\mathbf{B}_3/\mathbf{N}^{(1)}$ and the “curved” arrow $T_{m, f}$ commutes due to identity (3.52).

The definition of $T_{m, f}^{\text{isom}}$ gives us the commutativity of the outer “curved” triangle (i.e. the triangle with the vertices \mathbf{B}_3 , $\mathbf{B}_3/\mathbf{N}^{(3)}$ and $\mathbf{B}_3/\mathbf{N}^{(1)}$). Combining the commutativity of the outer “curved” triangle with identity (3.52), we conclude that the lower “curved” triangle also commutes.

Proposition 3.9 is proved. \square

We are now ready to prove that GTSh is indeed a groupoid.

Theorem 3.10 *Let $\mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, $[m_1, f_1] \in \text{GTSh}(\mathbf{N}^{(2)}, \mathbf{N}^{(1)})$, $[m_2, f_2] \in \text{GT}(\mathbf{N}^{(3)}, \mathbf{N}^{(2)})$ and $N_{\text{ord}} := N_{\text{ord}}^{(1)} = N_{\text{ord}}^{(2)} = N_{\text{ord}}^{(3)}$. The formula*

$$[m_1, f_1] \circ [m_2, f_2] := [2m_1 m_2 + m_1 + m_2, f_1 E_{m_1, f_1}(f_2)] \quad (3.53)$$

defines a composition of morphisms in GTSh. For every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$, the pair $(0, 1_{\mathbf{F}_2})$ represents the identity morphism in $\text{GTSh}(\mathbf{N}, \mathbf{N})$. Finally, for every $[m, f] \in \text{GTSh}(\mathbf{K}, \mathbf{N})$, the formulas

$$\tilde{m} + N_{\text{ord}}\mathbb{Z} := -(2\tilde{m} + 1)^{-1}\tilde{m}, \quad \tilde{f}\mathbf{K}_{\mathbf{F}_2} := (T_{m, f}^{\mathbf{F}_2, \text{isom}})^{-1}(f^{-1}\mathbf{N}_{\mathbf{F}_2}) \quad (3.54)$$

define the inverse $[\tilde{m}, \tilde{f}] \in \text{GTSh}(\mathbf{N}, \mathbf{K})$ of the morphism $[m, f]$.

Proof. Due to Proposition 3.9, formula (3.53) indeed defines a map

$$\text{GTSh}(\mathbf{N}^{(2)}, \mathbf{N}^{(1)}) \times \text{GT}(\mathbf{N}^{(3)}, \mathbf{N}^{(2)}) \rightarrow \text{GT}(\mathbf{N}^{(3)}, \mathbf{N}^{(1)}).$$

Since the binary operation \bullet on $\mathbb{Z} \times \mathbf{F}_2$ defined in (3.43) is associative, the composition of morphisms in GTSh is also associative.

It is easy to see that the pair $(0, 1_{\mathbf{F}_2})$ represents a GT-shadow in $\text{GTSh}(\mathbf{N}, \mathbf{N})$ for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$. Moreover, since $(0, 1_{\mathbf{F}_2})$ is the identity element of the monoid $(\mathbb{Z} \times \mathbf{F}_2, \bullet)$, $[0, 1_{\mathbf{F}_2}]$ is indeed the identity morphism in $\text{GTSh}(\mathbf{N}, \mathbf{N})$ for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$.

To take care of the inverse, we start with $[m, f] \in \text{GTSh}(\mathbf{K}, \mathbf{N})$ and assume that the pair $(\tilde{m} + K_{\text{ord}}\mathbb{Z}, \tilde{f}\mathbf{K}_{\mathbf{F}_2}) \in \mathbb{Z}/K_{\text{ord}}\mathbb{Z} \times \mathbf{F}_2/\mathbf{K}_{\mathbf{F}_2}$ is given by the formulas⁸ (3.54). We denote by \tilde{m} (resp. \tilde{f}) any representative of the coset $-(2\tilde{m} + 1)^{-1}\tilde{m}$ (resp. the coset $(T_{m, f}^{\mathbf{F}_2, \text{isom}})^{-1}(f^{-1}\mathbf{N}_{\mathbf{F}_2})$) in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ (resp. in $\mathbf{F}_2/\mathbf{K}_{\mathbf{F}_2}$).

The equations in (3.54) are equivalent to

$$2m\tilde{m} + \tilde{m} + m \equiv 0 \pmod{N_{\text{ord}}}, \quad T_{m, f}^{\mathbf{F}_2, \text{isom}}(\tilde{f}\mathbf{K}_{\mathbf{F}_2}) := f^{-1}\mathbf{N}_{\mathbf{F}_2}. \quad (3.55)$$

The first equation in (3.55) implies that

$$(2m + 1)(2\tilde{m} + 1) \equiv 1 \pmod{2N_{\text{ord}}}. \quad (3.56)$$

Hence $2\tilde{m} + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$.

Since

$$\sigma_1^{2N_{\text{ord}}}, \sigma_2^{2N_{\text{ord}}} \in \mathbf{N},$$

⁸Since $\text{GTSh}(\mathbf{K}, \mathbf{N})$ is non-empty, $K_{\text{ord}} = N_{\text{ord}}$.

identity (3.56) implies that

$$\sigma_1^{(2m+1)(2\tilde{m}+1)} \mathbf{N} = \sigma_1 \mathbf{N}, \quad \sigma_2^{(2m+1)(2\tilde{m}+1)} \mathbf{N} = \sigma_2 \mathbf{N}. \quad (3.57)$$

Since $f^{-1} \mathbf{N}_{\mathbf{F}_2}$ belongs to $[\mathbf{F}_2/\mathbf{N}_{\mathbf{F}_2}, \mathbf{F}_2/\mathbf{N}_{\mathbf{F}_2}]$, so does $\tilde{f} \mathbf{K}_{\mathbf{F}_2}$.

Let us prove that the pair (\tilde{m}, \tilde{f}) satisfies (3.3) and (3.4) (modulo \mathbf{K}).

Applying $T_{m,f}^{\text{isom}}$ to $\tilde{f}^{-1} \sigma_2^{2\tilde{m}+1} \tilde{f} \sigma_1^{2\tilde{m}+1} \mathbf{K}$ and using the second equation in (3.55) and identities (3.57), we get

$$T_{m,f}^{\text{isom}}(\tilde{f}^{-1} \sigma_2^{2\tilde{m}+1} \tilde{f} \sigma_1^{2\tilde{m}+1} \mathbf{K}) = f f^{-1} \sigma_2^{(2m+1)(2\tilde{m}+1)} f f^{-1} \sigma_1^{(2m+1)(2\tilde{m}+1)} \mathbf{N} = \sigma_2 \sigma_1 \mathbf{N}.$$

Furthermore, applying $T_{m,f}^{\text{isom}}$ to $\sigma_2 \sigma_1 c^{\tilde{m}} x_{23}^{-\tilde{m}} \tilde{f} \mathbf{K}$ and using hexagon relation (3.4) for (m, f) and the first equation in (3.55), we get

$$\begin{aligned} T_{m,f}^{\text{isom}}(\sigma_2 \sigma_1 c^{\tilde{m}} x_{23}^{-\tilde{m}} \tilde{f} \mathbf{K}) &= (f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1}) \mathbf{N} (c^{(2m+1)\tilde{m}} f^{-1} x_{23}^{-(2m+1)\tilde{m}} f) \mathbf{N} = \\ &= \sigma_2 \sigma_1 c^m x_{23}^{-m} f c^{(2m+1)\tilde{m}} f^{-1} x_{23}^{-(2m+1)\tilde{m}} f f^{-1} \mathbf{N} = \sigma_2 \sigma_1 c^{2m\tilde{m}+\tilde{m}+m} x_{23}^{-(2m\tilde{m}+\tilde{m}+m)} \mathbf{N} = \sigma_2 \sigma_1 \mathbf{N}. \end{aligned}$$

Since

$$T_{m,f}^{\text{isom}}(\tilde{f}^{-1} \sigma_2^{2\tilde{m}+1} \tilde{f} \sigma_1^{2\tilde{m}+1} \mathbf{K}) = T_{m,f}^{\text{isom}}(\sigma_2 \sigma_1 c^{\tilde{m}} x_{23}^{-\tilde{m}} \tilde{f} \mathbf{K})$$

and $T_{m,f}^{\text{isom}}$ is an isomorphism, we conclude that the pair (\tilde{m}, \tilde{f}) satisfies hexagon relation (3.4).

Applying $T_{m,f}^{\text{isom}}$ to both sides of

$$\sigma_1^{2\tilde{m}+1} \tilde{f}^{-1} \sigma_2^{2\tilde{m}+1} \tilde{f} \mathbf{K} \stackrel{?}{=} \tilde{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\tilde{m}} c^{\tilde{m}} \mathbf{K}$$

and performing similar calculations, we see that the pair (\tilde{m}, \tilde{f}) also satisfies hexagon relation (3.3).

Using the equations in (3.55) we see that the composition

$$T_{m,f}^{\text{isom}} \circ T_{\tilde{m},\tilde{f}} : \mathbf{B}_3 \rightarrow \mathbf{B}_3/\mathbf{N}$$

coincides with the standard projection $\mathcal{P}_{\mathbf{N}} : \mathbf{B}_3 \rightarrow \mathbf{B}_3/\mathbf{N}$. Hence the group homomorphism $T_{\tilde{m},\tilde{f}} : \mathbf{B}_3 \rightarrow \mathbf{B}_3/\mathbf{K}$ is onto and

$$\ker(T_{\tilde{m},\tilde{f}}) = \mathbf{N}.$$

Thus we proved that

$$(\tilde{m} + N_{\text{ord}} \mathbb{Z}, \tilde{f} \mathbf{K}_{\mathbf{F}_2}) \in \text{GTSh}(\mathbf{N}, \mathbf{K}).$$

The equations in (3.55) imply that

$$[m, f] \circ [\tilde{m}, \tilde{f}] = [0, 1_{\mathbf{F}_2}].$$

Since

$$[\tilde{m}, \tilde{f}] \circ [m, f] = (2\tilde{m}m + m + \tilde{m} + N_{\text{ord}} \mathbb{Z}, \tilde{f} \mathbf{K}_{\mathbf{F}_2} T_{\tilde{m},\tilde{f}}^{\mathbf{F}_2}(f)) = (N_{\text{ord}} \mathbb{Z}, \tilde{f} \mathbf{K}_{\mathbf{F}_2} T_{\tilde{m},\tilde{f}}^{\mathbf{F}_2}(f))$$

it remains to prove that

$$\tilde{f} \mathbf{K}_{\mathbf{F}_2} T_{\tilde{m},\tilde{f}}^{\mathbf{F}_2}(f) \stackrel{?}{=} 1_{\mathbf{F}_2/\mathbf{K}_{\mathbf{F}_2}}. \quad (3.58)$$

Applying $T_{m,f}^{\mathbf{F}_2, \text{isom}}$ to the left hand side of (3.58) and using $T_{m,f}^{\text{isom}} \circ T_{\tilde{m},\tilde{f}} = \mathcal{P}_{\mathbf{N}}$, we get

$$T_{m,f}^{\mathbf{F}_2, \text{isom}}(\tilde{f} \mathbf{K}_{\mathbf{F}_2} T_{\tilde{m},\tilde{f}}^{\mathbf{F}_2}(f)) = f^{-1} \mathbf{N}_{\mathbf{F}_2} f \mathbf{N}_{\mathbf{F}_2} = 1_{\mathbf{F}_2/\mathbf{N}_{\mathbf{F}_2}}.$$

Thus, since $T_{m,f}^{\mathbf{F}_2, \text{isom}}$ is an isomorphism from $\mathbf{F}_2/\mathbf{N}_{\mathbf{F}_2}$ to $\mathbf{F}_2/\mathbf{K}_{\mathbf{F}_2}$, we conclude that identity (3.58) holds. Theorem 3.10 is proved. \square

Remark 3.11 Proposition 3.8 implies that, if $\text{GTSh}(\mathbf{K}, \mathbf{N})$ is non-empty, then

$$|\text{PB}_3 : \mathbf{K}| = |\text{PB}_3 : \mathbf{N}|, \quad |\mathbf{F}_2 : \mathbf{K}_{\mathbf{F}_2}| = |\mathbf{F}_2 : \mathbf{N}_{\mathbf{F}_2}|, \quad K_{\text{ord}} = N_{\text{ord}}.$$

3.1 The reduction map

Let $N, H \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$ and $N \leq H$. In the following proposition, we consider this situation and get a natural map $\mathcal{R}_{N,H} : \text{GT}(N) \rightarrow \text{GT}(H)$.

Proposition 3.12 *Let $N, H \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, $N \leq H$ and $(m, f) \in \mathbb{Z} \times F_2$ represent a GT-pair with the target N . Then $H_{\text{ord}} | N_{\text{ord}}$, $N_{F_2} \leq H_{F_2}$ and*

- a) *the same pair (m, f) also represents an element in $\text{GT}_{pr}(H)$; moreover the resulting GT-pair $[m, f] \in \text{GT}_{pr}(H)$ depends only on $(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2})$;*
- b) *if the GT-pair $[m, f] \in \text{GT}_{pr}(N)$ is charming then so is the corresponding GT-pair in $\text{GT}_{pr}(H)$;*
- c) *if the pair (m, f) represents a GT-shadow with the target N , then (m, f) also represents a GT-shadow with the target H .*

Let us denote by $T_{m,f,H}$ the group homomorphism $\text{B}_3 \rightarrow \text{B}_3/H$ corresponding to $[m, f] \in \text{GT}_{pr}(H)$. In the set-up of statement a), the following diagram

$$\begin{array}{ccc}
 \text{B}_3 & \xrightarrow{T_{m,f}} & \text{B}_3/N \\
 & \searrow T_{m,f,H} & \swarrow \mathcal{P}_{N,H} \\
 & \text{B}_3/H &
 \end{array} \tag{3.59}$$

commutes.

Proof. Since

$$\mathcal{P}_{N,H}(x_{12}N) = x_{12}H, \quad \mathcal{P}_{N,H}(x_{23}N) = x_{23}H, \quad \mathcal{P}_{N,H}(cN) = cH,$$

$\text{ord}(x_{12}H) | \text{ord}(x_{12}N)$, $\text{ord}(x_{23}H) | \text{ord}(x_{23}N)$ and $\text{ord}(cH) | \text{ord}(cN)$. Hence H_{ord} divides N_{ord} . The inclusion $N_{F_2} \leq H_{F_2}$ is obvious.

a) Applying the homomorphism $\mathcal{P}_{N,H} : \text{B}_3/N \rightarrow \text{B}_3/H$ to (3.3) and (3.4), we see that the pair (m, f) satisfies the hexagon relations modulo H if it satisfies the hexagon relations modulo N . Thus (m, f) represents an element in $\text{GT}_{pr}(H)$.

It is obvious that the resulting GT-pair $[m, f] \in \text{GT}_{pr}(H)$ depends only on the residue class of m modulo N_{ord} and the coset fN_{F_2} .

As above, we denote by $T_{m,f,H}$ the group homomorphism $\text{B}_3 \rightarrow \text{B}_3/H$ corresponding to $[m, f] \in \text{GT}_{pr}(H)$. Applying $T_{m,f,H}$ and $\mathcal{P}_{N,H} \circ T_{m,f}$ to the generators σ_1, σ_2 , we see that the diagram in (3.59) indeed commutes.

b) Since $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, $2m + 1$ also represents a unit in $\mathbb{Z}/H_{\text{ord}}\mathbb{Z}$. Since fN_{F_2} belongs to the commutator subgroup $[F_2/N_{F_2}, F_2/N_{F_2}]$, we have

$$fH_{F_2} \in [F_2/H_{F_2}, F_2/H_{F_2}].$$

Thus (m, f) represents a charming GT-pair with the target H .

c) This statement follows easily from the commutativity of the diagram in (3.59) and the surjectivity of the homomorphism $\mathcal{P}_{N,H}$. \square

Due to Proposition 3.12, the formula

$$\mathcal{R}_{N,H}([m, f]) := (m + H_{\text{ord}}\mathbb{Z}, fH_{F_2}) \tag{3.60}$$

defines a map $\mathcal{R}_{N,H} : \text{GT}(N) \rightarrow \text{GT}(H)$. We call $\mathcal{R}_{N,H}$ the **reduction map**.

Just as in [5, Definition 3.12], we say that a GT-shadow $[m, f] \in \text{GT}(H)$ **survives into N** if $[m, f]$ belongs to the image of $\mathcal{R}_{N,H}$.

3.2 Connected components of the groupoid GTSh and its isolated objects

The groupoid GTSh is highly disconnected. Indeed, if $|\text{PB}_3 : \mathbf{N}| \neq |\text{PB}_3 : \mathbf{K}|$, then $\text{GTSh}(\mathbf{K}, \mathbf{N})$ is empty (see Remark 3.11). For $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, we denote by $\text{GTSh}_{\text{conn}}(\mathbf{N})$ the connected component of \mathbf{N} in the groupoid GTSh. Since, for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, $\text{GT}(\mathbf{N})$ is finite, so is the groupoid $\text{GTSh}_{\text{conn}}(\mathbf{N})$.

Definition 3.13 Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. A GT-shadow $[m, f] \in \text{GT}(\mathbf{N})$ is called **settled** if $\ker(T_{m,f}) = \mathbf{N}$, i.e. $[m, f] \in \text{GTSh}(\mathbf{N}, \mathbf{N})$. An object \mathbf{N} of the groupoid GTSh is called **isolated** if every GT-shadow in $\text{GT}(\mathbf{N})$ is settled.

It is clear that $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$ is isolated if and only if the connected component of \mathbf{N} in the groupoid GTSh has exactly one object. Of course, in this case, $\text{GT}(\mathbf{N}) = \text{GTSh}(\mathbf{N}, \mathbf{N})$. In particular, $\text{GT}(\mathbf{N})$ is a group.

Proposition 3.14 For every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, the subgroup

$$\mathbf{N}^\diamond := \bigcap_{\mathbf{K} \in \text{Ob}(\text{GTSh}_{\text{conn}}(\mathbf{N}))} \mathbf{K} \quad (3.61)$$

is an isolated object of the groupoid GTSh.

Proof. Since the groupoid $\text{GTSh}_{\text{conn}}(\mathbf{N})$ has finitely many objects and $\text{NFI}_{\text{PB}_3}(\text{B}_3)$ is closed under finite intersections, \mathbf{N}^\diamond belongs to $\text{NFI}_{\text{PB}_3}(\text{B}_3)$.

To prove that \mathbf{N}^\diamond is isolated, we consider $[m, f] \in \text{GT}(\mathbf{N}^\diamond)$ and $\mathbf{K} \in \text{Ob}(\text{GTSh}_{\text{conn}}(\mathbf{N}))$.

Since $\mathbf{N}^\diamond \leq \mathbf{K}$, Proposition 3.12 implies that the pair (m, f) also represents a GT-shadow with the target \mathbf{K} . Just as in Proposition 3.12, we denote by $T_{m,f,\mathbf{K}}$ the group homomorphism $\text{B}_3 \rightarrow \text{B}_3/\mathbf{K}$ corresponding to the GT-shadow $[m, f] \in \text{GT}(\mathbf{K})$. Let us also recall that

$$T_{m,f,\mathbf{K}} = \mathcal{P}_{\mathbf{N}^\diamond, \mathbf{K}} \circ T_{m,f}. \quad (3.62)$$

Let $w \in \mathbf{N}^\diamond$. Since $w \in \mathbf{H}$ for every $\mathbf{H} \in \text{Ob}(\text{GTSh}_{\text{conn}}(\mathbf{N}))$, we have

$$w \in \ker(T_{m,f,\mathbf{K}})$$

Let $w^\diamond \in \text{B}_3$ be a representative of the coset $T_{m,f}(w) \in \text{B}_3/\mathbf{N}^\diamond$. Using (3.62) we conclude that $w^\diamond \in \mathbf{K}$ for every $\mathbf{K} \in \text{Ob}(\text{GTSh}_{\text{conn}}(\mathbf{N}))$. Therefore $w^\diamond \in \mathbf{N}^\diamond$ and hence $w \in \ker(\text{B}_3 \xrightarrow{T_{m,f}} \text{B}_3/\mathbf{N}^\diamond)$.

We proved that $\mathbf{N}^\diamond \leq \tilde{\mathbf{K}}$, where $\tilde{\mathbf{K}} := \ker(\text{B}_3 \xrightarrow{T_{m,f}} \text{B}_3/\mathbf{N}^\diamond)$. Since $|\text{B}_3 : \tilde{\mathbf{K}}| = |\text{B}_3 : \mathbf{N}^\diamond|$ (see Proposition 3.8) and \mathbf{N}^\diamond has finite index in B_3 , we conclude that $\ker(\text{B}_3 \xrightarrow{T_{m,f}} \text{B}_3/\mathbf{N}^\diamond) = \mathbf{N}^\diamond$. \square

Proposition 3.14 implies that the subposet $\text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$ of isolated elements in $\text{NFI}_{\text{PB}_3}(\text{B}_3)$ is cofinal, i.e. for every $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, there exists $\tilde{\mathbf{N}} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$ such that $\tilde{\mathbf{N}} \leq \mathbf{N}$.

The proof of the following proposition is straightforward and we leave it to the reader:

Proposition 3.15 For all $\mathbf{N}, \mathbf{K} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$, $\mathbf{N} \cap \mathbf{K} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$. \square

Remark 3.16 Let $\mathbf{N}, \mathbf{H} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$ and $\mathbf{N} \leq \mathbf{H}$. Recall that, in this case, $\text{GT}(\mathbf{N}) = \text{GTSh}(\mathbf{N}, \mathbf{N})$ and $\text{GT}(\mathbf{H}) = \text{GTSh}(\mathbf{H}, \mathbf{H})$, i.e. $\text{GT}(\mathbf{N})$ and $\text{GT}(\mathbf{H})$ are (finite) groups. It is easy to see that the reduction map $\mathcal{R}_{\mathbf{N}, \mathbf{H}} : \text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H})$ (see (3.60)) is a group homomorphism. Indeed, both $[m, f] \in \text{GT}(\mathbf{N})$ and $\mathcal{R}_{\mathbf{N}, \mathbf{H}}([m, f]) \in \text{GT}(\mathbf{H})$ are represented by the same pair $(m, f) \in \mathbb{Z} \times \text{F}_2$ and the composition of GT-shadows is defined in terms of their representatives (see equation (3.53) in Theorem 3.10). If $\mathbf{N}, \mathbf{H} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$ and $\mathbf{N} \leq \mathbf{H}$, we call $\mathcal{R}_{\mathbf{N}, \mathbf{H}} : \text{GT}(\mathbf{N}) \rightarrow \text{GT}(\mathbf{H})$ the **reduction homomorphism**.

4 The transformation groupoid $\widehat{\text{GT}}_{\text{NFI}}^{\text{gen}}$ and genuine GT-shadows

Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$ and $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$. Recall that $\hat{\mathcal{P}}_{\mathbf{N}}$ denotes the standard (continuous) group homomorphism from $\hat{\text{B}}_3$ to B_3/\mathbf{N} and $T_{\hat{m}, \hat{f}}$ denotes the continuous automorphism of $\hat{\text{B}}_3$ defined in (2.11). Let us consider the composition

$$\hat{\mathcal{P}}_{\mathbf{N}} \circ T_{\hat{m}, \hat{f}}|_{\text{B}_3} : \text{B}_3 \rightarrow \text{B}_3/\mathbf{N}. \quad (4.1)$$

Using the fact that B_3 is dense in $\hat{\text{B}}_3$, one can easily prove that the homomorphism (4.1) is surjective. In the following proposition, we use (4.1) to define a right action of $\widehat{\text{GT}}_{\text{gen}}$ on $\text{NFI}_{\text{PB}_3}(\text{B}_3)$:

Proposition 4.1 *Let $\mathbf{N} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. For every $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$, the pair*

$$(\hat{\mathcal{P}}_{\text{N}_{\text{ord}}}(\hat{m}), \hat{\mathcal{P}}_{\text{N}_{\text{F}_2}}(\hat{f}))$$

is a GT-shadow with the target \mathbf{N} . Furthermore, the assignment

$$\mathbf{N}^{(\hat{m}, \hat{f})} := \ker(\hat{\mathcal{P}}_{\mathbf{N}} \circ T_{\hat{m}, \hat{f}}|_{\text{B}_3}) \quad (4.2)$$

defines a right action of $\widehat{\text{GT}}_{\text{gen}}$ on $\text{NFI}_{\text{PB}_3}(\text{B}_3)$.

Proof. Let $m \in \mathbb{Z}$ (resp. $f \in \text{F}_2$) be any representative of the residue class $\hat{\mathcal{P}}_{\text{N}_{\text{ord}}}(\hat{m}) \in \mathbb{Z}/\text{N}_{\text{ord}}\mathbb{Z}$ (resp. of the coset $\hat{\mathcal{P}}_{\text{N}_{\text{F}_2}}(\hat{f}) \in \text{F}_2/\text{N}_{\text{F}_2}$).

Since the pair (\hat{m}, \hat{f}) satisfies (2.9) and (2.10), the pair (m, f) satisfies hexagon relations (3.3) and (3.4) modulo \mathbf{N} .

Since $2\hat{m} + 1$ is a unit in $\hat{\mathbb{Z}}$, the integer $2m + 1$ represents a unit in $\mathbb{Z}/\text{N}_{\text{ord}}\mathbb{Z}$.

The property $\hat{f} \in [\hat{\text{F}}_2, \hat{\text{F}}_2]^{\text{top. cl.}}$ implies that

$$f\text{N}_{\text{F}_2} \in [\text{F}_2/\text{N}_{\text{F}_2}, \text{F}_2/\text{N}_{\text{F}_2}].$$

Finally, it is easy to see that the homomorphism $T_{m, f} : \text{B}_3 \rightarrow \text{B}_3/\mathbf{N}$ coincides with $\hat{\mathcal{P}}_{\mathbf{N}} \circ T_{\hat{m}, \hat{f}}$:

$$T_{m, f} = \hat{\mathcal{P}}_{\mathbf{N}} \circ T_{\hat{m}, \hat{f}}|_{\text{B}_3}. \quad (4.3)$$

In particular, $T_{m, f}$ is surjective.

We proved that the pair $(m + \text{N}_{\text{ord}}\mathbb{Z}, f\text{N}_{\text{F}_2})$ is a GT-shadow with the target \mathbf{N} and

$$\mathbf{N}^{(\hat{m}, \hat{f})} = \ker(T_{m, f}).$$

Hence $\mathbf{N}^{(\hat{m}, \hat{f})} \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$.

We say that the GT-shadow $[m, f] \in \text{GT}(\mathbf{N})$ **comes from** the element $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$.

Let us consider the following diagram:

$$\begin{array}{ccccc} & & \hat{\text{B}}_3 & \xrightarrow{T_{\hat{m}, \hat{f}}} & \hat{\text{B}}_3 \\ & \nearrow & \downarrow \hat{\mathcal{P}}_{\mathbf{K}} & & \downarrow \hat{\mathcal{P}}_{\mathbf{N}} \\ \text{B}_3 & \xrightarrow{\mathcal{P}_{\mathbf{K}}} & \text{B}_3/\mathbf{K} & \xrightarrow{T_{m, f}^{\text{isom}}} & \text{B}_3/\mathbf{N} \\ & \searrow & \uparrow & & \\ & & \text{B}_3 & \xrightarrow{T_{m, f}} & \text{B}_3/\mathbf{N} \end{array} \quad (4.4)$$

where $\mathbf{K} := \ker(T_{m, f})$ and the slanted straight arrow is the standard inclusion map $j : \text{B}_3 \rightarrow \hat{\text{B}}_3$.

We claim that the diagram in (4.4) commutes. Indeed, the outer “curved” rectangle commutes due to (4.3). The lower “curved” triangle commutes due to the identity $T_{m, f} = T_{m, f}^{\text{isom}} \circ \mathcal{P}_{\mathbf{K}}$. The left triangle

commutes by definition of \widehat{B}_3 . Finally, the continuous maps $\widehat{\mathcal{P}}_N \circ T_{\widehat{m}, \widehat{f}}$ and $T_{m, f}^{\text{isom}} \circ \widehat{\mathcal{P}}_K$ agree on the dense subset $B_3 \subset \widehat{B}_3$ and B_3/N is Hausdorff. Thus the inner square in (4.4) also commutes.

It is clear that

$$\widehat{\mathcal{P}}_N \circ T_{0, 1_{\widehat{F}_2}}|_{B_3} = \mathcal{P}_N.$$

Hence $N^{(0, 1_{\widehat{F}_2})} = N$.

It remains to prove that, for all $(\widehat{m}_1, \widehat{f}_1), (\widehat{m}_2, \widehat{f}_2) \in \widehat{GT}_{gen}$,

$$(N^{(\widehat{m}_1, \widehat{f}_1)})^{(\widehat{m}_2, \widehat{f}_2)} = N^{(\widehat{m}, \widehat{f})}, \quad (4.5)$$

where $(\widehat{m}, \widehat{f}) := (\widehat{m}_1, \widehat{f}_1) \bullet (\widehat{m}_2, \widehat{f}_2)$.

For this purpose, we will use the inner square of the diagram in (4.4). We set $K := N^{(\widehat{m}_1, \widehat{f}_1)}$ and $H := K^{(\widehat{m}_2, \widehat{f}_2)}$. Then, putting together the “squares” corresponding to $(\widehat{m}_1, \widehat{f}_1)$ and $(\widehat{m}_2, \widehat{f}_2)$, adding the obvious “triangle with the vertex” B_3 , the “curved arrow” $\widehat{\mathcal{P}}_N \circ T_{\widehat{m}, \widehat{f}}|_{B_3}$, and using (2.25), we get the following commutative diagram:

$$\begin{array}{ccccccc} & & \widehat{B}_3 & \xrightarrow{T_{\widehat{m}_2, \widehat{f}_2}} & \widehat{B}_3 & \xrightarrow{T_{\widehat{m}_1, \widehat{f}_1}} & \widehat{B}_3 \\ & \nearrow & \downarrow \widehat{\mathcal{P}}_H & & \downarrow \widehat{\mathcal{P}}_K & & \downarrow \widehat{\mathcal{P}}_N \\ B_3 & \xrightarrow{\mathcal{P}_H} & B_3/H & \xrightarrow{T_{m_2, f_2}^{\text{isom}}} & B_3/K & \xrightarrow{T_{m_1, f_1}^{\text{isom}}} & B_3/N \\ & \searrow & & & \nearrow & & \\ & & & & \widehat{\mathcal{P}}_N \circ T_{\widehat{m}, \widehat{f}}|_{B_3} & & \end{array} \quad (4.6)$$

where $[m_1, f_1] \in GT(N)$ and $[m_2, f_2] \in GT(K)$ are the GT -shadows coming from $(\widehat{m}_1, \widehat{f}_1)$ and $(\widehat{m}_2, \widehat{f}_2)$, respectively.

The commutativity of the lower “curved rectangle” in (4.6) implies that $H = N^{(\widehat{m}, \widehat{f})}$. Thus identity (4.5) holds. \square

For $N \in \text{NFI}_{PB_3}(B_3)$ and $(\widehat{m}, \widehat{f}) \in \widehat{GT}_{gen}$, we denote by $\mathcal{PR}_N(\widehat{m}, \widehat{f})$ the GT -shadow with the target N that comes from $(\widehat{m}, \widehat{f})$, i.e.

$$\mathcal{PR}_N(\widehat{m}, \widehat{f}) := (\widehat{\mathcal{P}}_{N_{\text{ord}}}(\widehat{m}), \widehat{\mathcal{P}}_{N_{F_2}}(\widehat{f})).$$

In view of Corollary 5.4 which is proved later, $\mathcal{PR}_N(\widehat{m}, \widehat{f})$ is called the approximation of the element $(\widehat{m}, \widehat{f}) \in \widehat{GT}_{gen}$.

We denote by $\widehat{GT}_{\text{NFI}}^{gen}$ the transformation groupoid of the action of \widehat{GT}_{gen} on $\text{NFI}_{PB_3}(B_3)$, i.e. $\text{Ob}(\widehat{GT}_{\text{NFI}}^{gen}) = \text{NFI}_{PB_3}(B_3)$ and

$$\widehat{GT}_{\text{NFI}}^{gen}(K, N) := \{(\widehat{m}, \widehat{f}) \in \widehat{GT}_{gen} \mid N^{(\widehat{m}, \widehat{f})} = K\}.$$

Definition 4.2 Let $N \in \text{NFI}_{PB_3}(B_3)$ and $[m, f] \in GT(N)$. We say that the GT -shadow $[m, f]$ is **genuine** if there exists $(\widehat{m}, \widehat{f}) \in \widehat{GT}_{gen}$ such that $[m, f]$ comes from $(\widehat{m}, \widehat{f})$, i.e.

$$m + N_{\text{ord}}\mathbb{Z} = \widehat{\mathcal{P}}_{N_{\text{ord}}}(\widehat{m}), \quad fN_{F_2} = \widehat{\mathcal{P}}_{N_{F_2}}(\widehat{f}).$$

Otherwise, the GT -shadow is called **fake**.

Let $N \in \text{NFI}_{PB_3}(B_3)$. Due to Proposition A.3, the subgroup $\widehat{\mathcal{P}}_N^{-1}(1_{B_3/N}) \leq \widehat{B}_3$ (resp. $\widehat{\mathcal{P}}_{N_{F_2}}^{-1}(1_{F_2/N_{F_2}}) \leq \widehat{F}_2$) coincides with the profinite completion of N (resp. with the profinite completion of N_{F_2}). By abuse of notation, we identify $\widehat{\mathcal{P}}_N^{-1}(1_{B_3/N})$ (resp. $\widehat{\mathcal{P}}_{N_{F_2}}^{-1}(1_{F_2/N_{F_2}})$) with \widehat{N} (resp. with \widehat{N}_{F_2}). We will need the following statement:

Proposition 4.3 *Let $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$ and $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$. If K is the source of the GT-shadow $\mathcal{PR}_N(\hat{m}, \hat{f})$, then*

$$T_{\hat{m}, \hat{f}}(\hat{K}) = \hat{N} \quad (4.7)$$

and

$$E_{\hat{m}, \hat{f}}(\hat{K}_{F_2}) = \hat{N}_{F_2}. \quad (4.8)$$

Proof. Let $(m, f) \in \mathbb{Z} \times F_2$ be a pair that represents the GT-shadow $\mathcal{PR}_N(\hat{m}, \hat{f})$ and $\hat{w} \in \hat{K} = \ker(\hat{B}_3 \xrightarrow{\hat{P}_K} B_3/K)$. Since the diagram in (4.4) commutes,

$$\hat{\mathcal{P}}_N \circ T_{\hat{m}, \hat{f}}(\hat{w}) = 1_{B_3/N}.$$

Hence $T_{\hat{m}, \hat{f}}(\hat{K}) \subset \hat{N} = \ker(\hat{B}_3 \xrightarrow{\hat{P}_N} B_3/N)$.

Since $|\hat{B}_3 : \hat{N}| = |B_3 : N| = |B_3 : K| = |\hat{B}_3 : \hat{K}| = |\hat{B}_3 : T_{\hat{m}, \hat{f}}(\hat{K})|$, the inclusion $T_{\hat{m}, \hat{f}}(\hat{K}) \subset \hat{N}$ implies that $T_{\hat{m}, \hat{f}}(\hat{K}) = \hat{N}$.

Identity (4.8) can be proved in a similar way using the commutative diagram

$$\begin{array}{ccccc} & & \hat{F}_2 & \xrightarrow{T_{\hat{m}, \hat{f}}} & \hat{F}_2 \\ & \nearrow j & \downarrow \hat{P}_{K_{F_2}} & & \downarrow \hat{P}_{N_{F_2}} \\ F_2 & \xrightarrow{\mathcal{P}_{K_{F_2}}} & F_2/K_{F_2} & \xrightarrow{T_{m, f}^{F_2, \text{isom}}} & F_2/N_{F_2} \\ & \searrow & \uparrow T_{m, f}^{F_2} & & \end{array} \quad (4.9)$$

where $T_{m, f}^{F_2, \text{isom}}$ is the isomorphism $F_2/K_{F_2} \xrightarrow{\sim} F_2/N_{F_2}$ defined in (3.37). \square

The following theorem gives us a link between $\widehat{\text{GT}}_{\text{gen}}$ and the groupoid GTSh:

Theorem 4.4 *Let $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. The assignments*

$$\mathcal{PR}(N) := N, \quad \mathcal{PR}_N(\hat{m}, \hat{f}) = (\hat{P}_{N_{\text{ord}}}(\hat{m}), \hat{P}_{N_{F_2}}(\hat{f})) \quad (4.10)$$

define a functor from the transformation groupoid $\widehat{\text{GT}}_{\text{NFI}}^{\text{gen}}$ to GTSh.

Proof. Let $(m, f) \in \mathbb{Z} \times [F_2, F_2]$ be a pair that represents $(\hat{P}_{N_{\text{ord}}}(\hat{m}), \hat{P}_{N_{F_2}}(\hat{f}))$.

Due to the first statement of Proposition 4.1, $[m, f]$ is a GT-shadow with the target N . Moreover, since

$$\ker(T_{m, f}) = N^{(\hat{m}, \hat{f})},$$

$[m, f]$ is indeed a morphism from $N^{(\hat{m}, \hat{f})}$ to N in GTSh.

It is clear that $\mathcal{PR}_N(0, 1_{F_2}) = [0, 1_{F_2}]$ for every $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, i.e. the functor \mathcal{PR} sends the identity morphisms of $\widehat{\text{GT}}_{\text{NFI}}^{\text{gen}}$ to the identity morphisms of GTSh.

It remains to prove that, for all $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\text{GT}}_{\text{gen}}$ and $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$,

$$\mathcal{PR}_N(\hat{m}_1, \hat{f}_1) \bullet \mathcal{PR}_K(\hat{m}_2, \hat{f}_2) = \mathcal{PR}_N(\hat{m}, \hat{f}), \quad (4.11)$$

where $(\hat{m}, \hat{f}) = (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$, (\hat{m}_1, \hat{f}_1) is viewed as a morphism from $K := N^{(\hat{m}_1, \hat{f}_1)}$ to N and (\hat{m}_2, \hat{f}_2) is viewed as a morphism from $K^{(\hat{m}_2, \hat{f}_2)}$ to K .

Let (m_1, f_1) and (m_2, f_2) be pairs that represent the GT-shadows $\mathcal{PR}_N(\hat{m}_1, \hat{f}_1)$ and $\mathcal{PR}_K(\hat{m}_2, \hat{f}_2)$, respectively. Since the source of $[m_1, f_1]$, K , coincides with the target of $[m_2, f_2]$, the GT-shadows $[m_1, f_1]$ and $[m_2, f_2]$ can be composed in this order $[m_1, f_1] \bullet [m_2, f_2]$ and $[m_1, f_1] \bullet [m_2, f_2]$ is an element of GTSh(H, N), where $H := K^{(\hat{m}_2, \hat{f}_2)}$. Recall that $N_{\text{ord}} = K_{\text{ord}} = H_{\text{ord}}$.

We need to prove that

$$m + N_{\text{ord}}\mathbb{Z} = \widehat{\mathcal{P}}_{N_{\text{ord}}}(\hat{m}) \quad (4.12)$$

and

$$fN_{F_2} = \widehat{\mathcal{P}}_{N_{F_2}}(\hat{f}), \quad (4.13)$$

where

$$m := 2m_1m_2 + m_1 + m_2, \quad f := f_1E_{m_1,f_1}(f_2).$$

While (4.12) is obvious, identity (4.13) requires some work.

First, we observe that the diagram

$$\begin{array}{ccc} \widehat{F}_2 & \xrightarrow{E_{\hat{m}_1,\hat{f}_1}} & \widehat{F}_2 \\ j \uparrow & & \downarrow \widehat{\mathcal{P}}_{N_{F_2}} \\ F_2 & \xrightarrow{w \mapsto E_{m_1,f_1}(w)N_{F_2}} & F_2/N_{F_2} \end{array} \quad (4.14)$$

commutes.

Second, since $\widehat{\mathcal{P}}_{K_{F_2}}(\hat{f}_2) = \widehat{\mathcal{P}}_{K_{F_2}}(f_2)$, we have

$$\hat{f}_2 = f_2\hat{b}, \quad (4.15)$$

where⁹ $\hat{b} \in \widehat{K}_{F_2}$. Combining this observation with equation (4.8) in Proposition 4.3 and commutativity of diagram (4.14), we deduce that

$$\widehat{\mathcal{P}}_{N_{F_2}}(E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)) = \widehat{\mathcal{P}}_{N_{F_2}}(E_{\hat{m}_1,\hat{f}_1}(f_2)) = E_{m_1,f_1}(f_2)N_{F_2}. \quad (4.16)$$

Therefore

$$\widehat{\mathcal{P}}_{N_{F_2}}(\hat{f}_1E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)) = \widehat{\mathcal{P}}_{N_{F_2}}(\hat{f}_1)\widehat{\mathcal{P}}_{N_{F_2}}(E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)) = f_1E_{m_1,f_1}(f_2)N_{F_2}.$$

Thus identity (4.13) holds and equation (4.11) follows. \square

In view of Corollary 5.4 which is proved in the next section, we call $\mathcal{P}\mathcal{R}$ the **approximation functor**.

5 The version of the Main Line functor for \widehat{GT}_{gen}

Recall that, for every isolated object N of the groupoid $GTSh$, $GT(N) = GTSh(N, N)$. In particular, $GT(N)$ is a (finite) group.

Let us show that the assignment

$$\mathcal{ML}(N) := GT(N) \quad (5.1)$$

can be upgraded to a functor \mathcal{ML} from the poset $NFI_{PB_3}^{isolated}(B_3)$ to the category of finite groups.

For $N, H \in NFI_{PB_3}^{isolated}(B_3)$, $N \leq H$, we set

$$\mathcal{ML}(N \rightarrow H) := \mathcal{R}_{N,H}. \quad (5.2)$$

Recall that, due to Remark 3.16, the map $\mathcal{R}_{N,H} : GT(N) \rightarrow GT(H)$ is a group homomorphism.

It is obvious that, if $N^{(3)} \leq N^{(2)} \leq N^{(1)}$, then

$$\mathcal{R}_{N^{(2)},N^{(1)}} \circ \mathcal{R}_{N^{(3)},N^{(2)}} = \mathcal{R}_{N^{(3)},N^{(1)}}. \quad (5.3)$$

Thus formulas (5.1), (5.2) define a functor \mathcal{ML} from the poset $NFI_{PB_3}^{isolated}(B_3)$ to the category of finite groups. We call \mathcal{ML} the **Main Line functor**.

Our next goal is to show that the group \widehat{GT}_{gen} is isomorphic to $\lim(\mathcal{ML})$. For this purpose, we need to prove the following auxiliary statement:

⁹Just as in Proposition 4.3, we identify \widehat{K}_{F_2} with $\widehat{\mathcal{P}}_{K_{F_2}}^{-1}(1_{F_2/K_{F_2}})$.

Proposition 5.1 *For every positive integer K , there exists $N \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$ such that $K|N_{\text{ord}}$. Furthermore, for every $H \in \text{NFI}(F_2)$, there exists $N \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$, such that $N_{F_2} \leq H$. Finally, for every pair $(K, H) \in \mathbb{Z}_{\geq 1} \times \text{NFI}(F_2)$, there exists $N \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$ such that $K|N_{\text{ord}}$ and $N_{F_2} \leq H$.*

Proof. The proof of the first statement of the proposition is straightforward, so we leave it to the reader.

Since H is a finite index normal subgroup of F_2 , there exists a group homomorphism ψ from F_2 to a finite group G such that

$$\ker(\psi) = H.$$

Clearly, the formulas

$$\tilde{\psi}(x_{12}) := \psi(x), \quad \tilde{\psi}(x_{23}) := \psi(y), \quad \tilde{\psi}(c) := 1_G \quad (5.4)$$

define a group homomorphism $\tilde{\psi} : \text{PB}_3 \rightarrow G$.

In general, the subgroup $\ker(\tilde{\psi})$ is not normal in B_3 . So we denote by \tilde{N} the normal core of $\ker(\tilde{\psi})$ in B_3 . It is clear that $\tilde{N} \in \text{NFI}_{\text{PB}_3}(B_3)$ and $\tilde{N}_{F_2} \leq \ker(\psi)$.

Let

$$N := \bigcap_{K \in \text{Ob}(\text{GTSh}_{\text{conn}}(\tilde{N}))} K.$$

Due to Proposition 3.14, N is an isolated object of GTSh . Moreover, since $N \leq \tilde{N}$, we have $N_{F_2} \leq \ker(\psi)$.

The second statement of the proposition is proved.

For $(K, H) \in \mathbb{Z}_{\geq 1} \times \text{NFI}(F_2)$, there exist $N^{(1)}, N^{(2)} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$ such that $K|N_{\text{ord}}^{(1)}$ and $N_{F_2}^{(2)} \leq H$. Due to Proposition 3.15,

$$N := N^{(1)} \cap N^{(2)}$$

is an isolated object of GTSh . Using the inclusions $N \subset N^{(1)}$ and $N \subset N^{(2)}$, it is not hard to show that $K|N_{\text{ord}}$ and $N_{F_2} \leq H$, respectively.

The proposition is proved. \square

We are now ready to construct an isomorphism of groups $\widehat{\text{GT}}_{\text{gen}} \xrightarrow{\cong} \lim(\mathcal{ML})$.

Theorem 5.2 *Let $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$ and $N \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$. The formula*

$$\Psi(\hat{m}, \hat{f})(N) := \mathcal{PR}_N(\hat{m}, \hat{f}) \quad (5.5)$$

defines an isomorphism of groups $\Psi : \widehat{\text{GT}}_{\text{gen}} \xrightarrow{\cong} \lim(\mathcal{ML})$. Moreover, Ψ is a homeomorphism (of topological spaces).

Proof. Since N is an isolated object of the groupoid GTSh , $N^{(\hat{m}, \hat{f})} = N$ for every $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}}$. Furthermore, Theorem 4.4 implies that the assignment

$$(\hat{m}, \hat{f}) \mapsto \mathcal{PR}_N(\hat{m}, \hat{f})$$

is a group homomorphism from $\widehat{\text{GT}}_{\text{gen}}$ to the finite group $\text{GT}(N) = \text{GTSh}(N, N)$.

It is clear that, for every $N, H \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(B_3)$, $N \leq H$, we have

$$\mathcal{R}_{N, H} \circ \mathcal{PR}_N(\hat{m}, \hat{f}) = \mathcal{PR}_H(\hat{m}, \hat{f}).$$

Thus the formula in (5.5) indeed defines a group homomorphism $\Psi : \widehat{\text{GT}}_{\text{gen}} \rightarrow \lim(\mathcal{ML})$.

To prove the theorem, we will construct a map $\Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\text{GT}}_{\text{gen}}$ and show that

- Θ is the inverse of Ψ and
- Θ is a homeomorphism of topological spaces.

Let $\hat{T} \in \lim(\mathcal{ML})$, $K \in \mathbb{Z}_{\geq 1}$ and $H \in \mathbf{NFI}(\mathbb{F}_2)$.

Due to Proposition 5.1, there exists $N \in \mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$ such that $K|N_{\text{ord}}$ and $N_{\mathbb{F}_2} \leq H$. Let $(m, f) \in \mathbb{Z} \times \mathbb{F}_2$ be a pair that represents the GT-shadow $\hat{T}(N)$. We set

$$\hat{m}(K) := m + K\mathbb{Z}, \quad \hat{f}(H) := fH. \quad (5.6)$$

Since \hat{T} belongs to $\lim(\mathcal{ML})$, the residue class $\hat{m}(K)$ and the coset $\hat{f}(H)$ do not depend on the choice of $N \in \mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$, and the formulas in (5.6) define $\hat{m} \in \widehat{\mathbb{Z}}$ and $\hat{f} \in \widehat{\mathbb{F}_2}$.

The element \hat{f} belongs to the topological closure of the commutator subgroup $[\widehat{\mathbb{F}_2}, \widehat{\mathbb{F}_2}]$ in $\widehat{\mathbb{F}_2}$ due to these properties:

- for every $N \in \mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$, $\hat{f}(N_{\mathbb{F}_2}) \in [\mathbb{F}_2/N_{\mathbb{F}_2}, \mathbb{F}_2/N_{\mathbb{F}_2}]$,

- the open subsets

$$\widehat{\mathcal{P}}_{N_{\mathbb{F}_2}}^{-1}(1_{\mathbb{F}_2/N_{\mathbb{F}_2}}) \subset \widehat{\mathbb{F}_2}, \quad N \in \mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$$

form a basis of neighborhoods of $1_{\widehat{\mathbb{F}_2}}$ in $\widehat{\mathbb{F}_2}$.

Let us prove that the resulting pair $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathbb{F}_2}$ satisfies hexagon relations (2.9) and (2.10). For this purpose, we consider $L \in \mathbf{NFI}(\mathbb{B}_3)$ and observe that $L \cap \mathbf{PB}_3 \in \mathbf{NFI}_{\mathbf{PB}_3}(\mathbb{B}_3)$. In general, $L \cap \mathbf{PB}_3$ is not an isolated object of the groupoid GTSh. However, due to Proposition 3.14, the subgroup $N := (L \cap \mathbf{PB}_3)^\diamond$ does belong to $\mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$. Moreover, since $N \leq L \cap \mathbf{PB}_3$, N is a subgroup of L .

As above, let $(m, f) \in \mathbb{Z} \times \mathbb{F}_2$ be a pair that represents the GT-shadow $\hat{T}(N)$. For such a pair (m, f) , we have

$$\hat{m}(N_{\text{ord}}) = m + N_{\text{ord}}\mathbb{Z}, \quad \hat{f}(N_{\mathbb{F}_2}) = fN_{\mathbb{F}_2}.$$

Evaluating the left hand side (resp. the right hand side) of the first hexagon relation (2.9) at N , we get the left hand side (resp. the right hand side) of the first hexagon relation (3.3) for (m, f) . Thus

$$(\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f})(N) = (\hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}})(N). \quad (5.7)$$

Similarly, evaluating the left hand side (resp. the right hand side) of the second hexagon relation (2.10) at N , we get the left hand side (resp. the right hand side) of the second hexagon relation (3.4) for (m, f) . Thus

$$(\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1})(N) = (\sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \hat{f})(N). \quad (5.8)$$

Since $N \leq L$, identities (5.7) and (5.8) imply that

$$(\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f})(L) = (\hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}})(L).$$

$$(\hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \sigma_1^{2\hat{m}+1})(L) = (\sigma_2 \sigma_1 x_{23}^{-\hat{m}} c^{\hat{m}} \hat{f})(L).$$

We proved that the pair (\hat{m}, \hat{f}) belongs to $\widehat{\mathbb{Z}} \times [\widehat{\mathbb{F}_2}, \widehat{\mathbb{F}_2}]^{\text{top. cl.}}$ and satisfies hexagon relations (2.9) and (2.10).

Thus the assignment $\hat{T} \mapsto (\hat{m}, \hat{f})$ defines a map

$$\Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\mathbf{GT}}_{\text{gen, mon}}, \quad (5.9)$$

where $\widehat{\mathbf{GT}}_{\text{gen, mon}}$ is the monoid defined in Section 2.2 (see Proposition 2.4).

Let us prove that Θ is a homomorphism of monoids. For this purpose, we consider $N \in \mathbf{NFI}_{\mathbf{PB}_3}^{\text{isolated}}(\mathbb{B}_3)$, $\hat{T}_1, \hat{T}_2 \in \lim(\mathcal{ML})$ and set

$$(\hat{m}_1, \hat{f}_1) := \Theta(\hat{T}_1), \quad (\hat{m}_2, \hat{f}_2) := \Theta(\hat{T}_2), \quad (5.10)$$

$$\hat{m} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2, \quad \hat{f} := \hat{f}_1 E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2). \quad (5.11)$$

Let $(m_1, f_1) \in \mathbb{Z} \times F_2$ (resp. $(m_2, f_2) \in \mathbb{Z} \times F_2$) be a pair that represents the GT-shadow $\hat{T}_1(\mathbf{N}) \in \text{GT}(\mathbf{N})$ (resp. the GT-shadow $\hat{T}_2(\mathbf{N}) \in \text{GT}(\mathbf{N})$) and

$$m := 2m_1m_2 + m_1 + m_2, \quad f := f_1E_{m_1, f_1}(f_2), \quad (5.12)$$

i.e. the pair (m, f) represents the GT-shadow $\hat{T}_1 \bullet \hat{T}_2(\mathbf{N})$.

To prove the compatibility of Θ with the multiplications in $\lim(\mathcal{ML})$ and $\widehat{\text{GT}}_{\text{gen}, \text{mon}}$, we need to show that

$$\hat{m}(N_{\text{ord}}) = m + N_{\text{ord}}\mathbb{Z}. \quad (5.13)$$

and

$$\hat{f}(\mathbf{N}_{F_2}) = f\mathbf{N}_{F_2}. \quad (5.14)$$

Equation (5.13) is clearly satisfied.

As for (5.14), since $\hat{\mathcal{P}}_{\mathbf{N}_{F_2}} : \hat{F}_2 \rightarrow F_2/\mathbf{N}_{F_2}$ is a group homomorphism and $\hat{\mathcal{P}}_{\mathbf{N}_{F_2}}(\hat{f}_1) = f_1\mathbf{N}_{F_2}$, we need to show that

$$\hat{\mathcal{P}}_{\mathbf{N}_{F_2}}(E_{\hat{m}_1, \hat{f}_1}(\hat{f}_2)) = E_{m_1, f_1}(f_2)\mathbf{N}_{F_2}. \quad (5.15)$$

This identity was already established in a more general case in the proof of Theorem 4.4 (see (4.16)).

It is easy to see that Θ sends the identity element of the group $\lim(\mathcal{ML})$ to the identity element of the monoid $\widehat{\text{GT}}_{\text{gen}, \text{mon}}$.

Since $\Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\text{GT}}_{\text{gen}, \text{mon}}$ is a homomorphism of monoids and $\lim(\mathcal{ML})$ is a group, $\Theta(\lim(\mathcal{ML}))$ is a subset of invertible elements of the monoid $\widehat{\text{GT}}_{\text{gen}, \text{mon}}$. Thus Θ is a group homomorphism from $\lim(\mathcal{ML})$ to $\widehat{\text{GT}}_{\text{gen}}$.

It is clear that

$$\Theta \circ \Psi = \text{id}_{\widehat{\text{GT}}_{\text{gen}}} \quad \text{and} \quad \Psi \circ \Theta = \text{id}_{\lim(\mathcal{ML})},$$

i.e. Θ is indeed the inverse of Ψ .

To prove the continuity of Θ , we consider it as the map from $\lim(\mathcal{ML})$ to the topological space $\widehat{\mathbb{Z}} \times \widehat{F}_2$ and denote by $P_{\widehat{\mathbb{Z}}}$ (resp. $P_{\widehat{F}_2}$) the projection $\widehat{\mathbb{Z}} \times \widehat{F}_2 \rightarrow \widehat{\mathbb{Z}}$ (resp. $\widehat{\mathbb{Z}} \times \widehat{F}_2 \rightarrow \widehat{F}_2$). We need to show that the maps $P_{\widehat{\mathbb{Z}}} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\mathbb{Z}}$ and $P_{\widehat{F}_2} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{F}_2$ are continuous.

For a positive integer K , we choose $\mathbf{N} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$ such that $K|N_{\text{ord}}$. Since the map

$$\hat{\mathcal{P}}_K \circ P_{\widehat{\mathbb{Z}}} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \mathbb{Z}/K\mathbb{Z}$$

factors through the continuous map $\lim(\mathcal{ML}) \rightarrow \mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, the composition $\hat{\mathcal{P}}_K \circ P_{\widehat{\mathbb{Z}}} \circ \Theta$ is continuous. Hence the composition $P_{\widehat{\mathbb{Z}}} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\mathbb{Z}}$ is continuous.

Similarly, for $\mathbf{H} \in \text{NFI}(F_2)$, we choose $\mathbf{N} \in \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\mathbf{B}_3)$ such that $\mathbf{N}_{F_2} \leq \mathbf{H}$. Since the map

$$\hat{\mathcal{P}}_{\mathbf{H}} \circ P_{\widehat{F}_2} \circ \Theta : \lim(\mathcal{ML}) \rightarrow F_2/\mathbf{H}$$

factors through the continuous map $\lim(\mathcal{ML}) \rightarrow F_2/\mathbf{N}_{F_2}$, the composition $\hat{\mathcal{P}}_{\mathbf{H}} \circ P_{\widehat{F}_2} \circ \Theta$ is continuous. Hence the composition $P_{\widehat{F}_2} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{F}_2$ is continuous.

Since both maps $P_{\widehat{\mathbb{Z}}} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\mathbb{Z}}$ and $P_{\widehat{F}_2} \circ \Theta : \lim(\mathcal{ML}) \rightarrow \widehat{F}_2$ are continuous, so is the map $\Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\mathbb{Z}} \times \widehat{F}_2$.

Now it is easy to see that $\Theta : \lim(\mathcal{ML}) \rightarrow \widehat{\text{GT}}_{\text{gen}}$ is a homeomorphism. Indeed, Θ is a continuous bijection from the compact topological space $\lim(\mathcal{ML})$ to a Hausdorff space $\widehat{\text{GT}}_{\text{gen}}$. Thus Θ (as well as Ψ) is homeomorphism.

Theorem 5.2 is proved. \square

Remark 5.3 As we mentioned in Remark 2.7, it is not obvious that $\widehat{\text{GT}}_{\text{gen}}$ is a topological group with respect to the subset topology coming from $\widehat{\mathbb{Z}} \times \widehat{F}_2$. However, since $\lim(\mathcal{ML})$ is obviously a topological group, Theorem 5.2 implies that $\widehat{\text{GT}}_{\text{gen}}$ is indeed a topological group with respect to the subset topology coming from $\widehat{\mathbb{Z}} \times \widehat{F}_2$.

Corollary 5.4 *Let $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$. A GT-shadow $[m, f] \in \text{GT}(N)$ is genuine if and only if $[m, f]$ belongs to the image of the map*

$$\mathcal{R}_{K,N} : \text{GT}(K) \rightarrow \text{GT}(N)$$

for every $K \in \text{NFI}_N(\text{B}_3)$.

Proof. If $[m, f] \in \text{GT}(N)$ is genuine, then $[m, f]$ obviously belongs to the image of the map $\mathcal{R}_{K,N} : \text{GT}(K) \rightarrow \text{GT}(N)$ for every $K \in \text{NFI}_N(\text{B}_3)$.

Thus it remains to prove the “if” implication.

For $K \in \text{NFI}_N(\text{B}_3)$, we set

$$\mathcal{F}(K) := \mathcal{R}_{K,N}^{-1}([m, f]) \subset \text{GT}(K).$$

Due to the given condition on $[m, f]$, the set $\mathcal{F}(K)$ is non-empty for every $K \in \text{NFI}_N(\text{B}_3)$.

Property (5.3) implies that the assignment $K \mapsto \mathcal{F}(K)$ upgrades to a functor from the poset $\text{NFI}_N(\text{B}_3)$ to the category of finite sets (since $\text{GT}(K)$ is finite, so is $\mathcal{R}_{K,N}^{-1}([m, f])$). Indeed, if $H, K \in \text{NFI}_N(\text{B}_3)$ and $H \leq K$, then $\mathcal{R}_{H,K}(\mathcal{F}(H)) \subset \mathcal{F}(K)$. So we set

$$\mathcal{F}(H \rightarrow K) := \mathcal{R}_{H,K}|_{\mathcal{F}(H)} : \mathcal{F}(H) \rightarrow \mathcal{F}(K).$$

Since $\mathcal{F}(K)$ is a finite non-empty set for every $K \in \text{NFI}_N(\text{B}_3)$, [27, Proposition 1.1.4] implies that $\lim(\mathcal{F})$ is non-empty.

Taking an arbitrary element in $\lim(\mathcal{F})$ and evaluating it at elements of the poset $\text{NFI}_N(\text{B}_3) \cap \text{NFI}_{\text{PB}_3}^{\text{isolated}}(\text{B}_3)$, we get an element $(\hat{m}, \hat{f}) \in \widehat{\text{GT}}_{\text{gen}} \cong \lim(\mathcal{ML})$ such that

$$\mathcal{PR}_N(\hat{m}, \hat{f}) = [m, f].$$

Thus the GT-shadow $[m, f]$ is indeed genuine. □

5.1 Simplified hexagon relations in the profinite setting

In this section, we prove that

Proposition 5.5 *The group $\widehat{\text{GT}}_{\text{gen}}$ (see Definition 2.5) is isomorphic to the group $\widehat{\text{GT}}_0$ introduced in [12, Section 0.1].*

Proof. According to [12, Section 0.1], $\widehat{\text{GT}}_0$ consists of elements $(\hat{\lambda}, \hat{f}) \in \widehat{\mathbb{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]^{\text{top.cl.}}$ for which the pair $(\hat{m}, \hat{f}) := ((\hat{\lambda} - 1)/2, \hat{f}) \in \widehat{\mathbb{Z}} \times [\widehat{F}_2, \widehat{F}_2]^{\text{top.cl.}}$ satisfies relations (2.26), (2.27) and the endomorphism $E_{\hat{m}, \hat{f}}$ of \widehat{F}_2 is invertible. In fact, the authors of [12] identify elements $(\hat{\lambda}, \hat{f})$ of $\widehat{\text{GT}}_0$ with the corresponding automorphisms $E_{\hat{m}, \hat{f}}$ of \widehat{F}_2 and this is how they get the group structure on $\widehat{\text{GT}}_0$.

Let us start with an element $(\hat{\lambda}, \hat{f}) \in \widehat{\text{GT}}_0$ and consider the corresponding pair

$$(\hat{m}, \hat{f}) := ((\hat{\lambda} - 1)/2, \hat{f}) \in \widehat{\mathbb{Z}} \times [\widehat{F}_2, \widehat{F}_2]^{\text{top.cl.}}.$$

Relations (2.26), (2.27) imply that, for every $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, the pair

$$(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2}) := (\widehat{\mathcal{P}}_{N_{\text{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{N_{F_2}}(\hat{f}))$$

satisfies relations (3.10) and (3.11). In addition, we have $fN_{F_2} \in [F_2/N_{F_2}, F_2/N_{F_2}]$.

Thus Proposition 3.4 implies that, for every $N \in \text{NFI}_{\text{PB}_3}(\text{B}_3)$, the pair $(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2}) := (\widehat{\mathcal{P}}_{N_{\text{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{N_{F_2}}(\hat{f}))$ satisfies relations (3.3), (3.4). Since $\text{NFI}_{\text{PB}_3}(\text{B}_3)$ is a cofinal subposet of $\text{NFI}(\text{B}_3)$, we conclude that the pair (\hat{m}, \hat{f}) satisfies hexagon relations (2.9) and (2.10).

Thus (\hat{m}, \hat{f}) belongs to the submonoid $\widehat{\text{GT}}_{\text{gen}, \text{mon}}$ and we need to show that the element (\hat{m}, \hat{f}) is invertible.

For this purpose we set

$$\hat{k} := -(2\hat{m} + 1)^{-1}\hat{m}, \quad \hat{g} := E_{\hat{m}, \hat{f}}^{-1}(\hat{f}^{-1}). \quad (5.16)$$

A direct computation shows that

$$(\hat{m}, \hat{f}) \bullet (\hat{k}, \hat{g}) = (0, 1_{\hat{F}_2}).$$

Therefore $E_{\hat{m}, \hat{f}} \circ E_{\hat{k}, \hat{g}} = \text{id}_{\hat{F}_2}$ and hence

$$E_{\hat{k}, \hat{g}} = E_{\hat{m}, \hat{f}}^{-1} \quad (5.17)$$

Using (5.16) and (5.17), we get

$$2\hat{k}\hat{m} + \hat{k} + \hat{m} = 0,$$

$$\hat{g} E_{\hat{k}, \hat{g}}(\hat{f}) = \hat{g} E_{\hat{m}, \hat{f}}^{-1}(\hat{f}) = E_{\hat{m}, \hat{f}}^{-1}(\hat{f}^{-1}) E_{\hat{m}, \hat{f}}^{-1}(\hat{f}) = E_{\hat{m}, \hat{f}}^{-1}(1_{\hat{F}_2}) = 1_{\hat{F}_2}.$$

Thus the identity $(\hat{k}, \hat{g}) \bullet (\hat{m}, \hat{f}) = (0, 1_{\hat{F}_2})$ is also satisfied and the element (\hat{m}, \hat{f}) of the monoid $(\widehat{\mathbb{Z}} \times \widehat{F}_2, \bullet)$ is indeed invertible.

Since $\hat{f} \in [\widehat{F}_2, \widehat{F}_2]^{top.cl.}$, the second equation in (5.16) and the continuity of the automorphism $E_{\hat{m}, \hat{f}}^{-1}$ imply that $\hat{g} \in [\widehat{F}_2, \widehat{F}_2]^{top.cl.}$. Thus it remains to prove that the pair (\hat{k}, \hat{g}) satisfies hexagon relations (2.9), (2.10).

Let us rewrite the right hand side of (2.10) for (\hat{k}, \hat{g}) as follows:

$$\sigma_2 \sigma_1 x_{23}^{-\hat{k}} c^{\hat{k}} \hat{g} = \Delta \sigma_2^{-(2\hat{k}+1)} c^{\hat{k}} \hat{g}.$$

Applying $T_{\hat{m}, \hat{f}}$ to the right hand side of (2.10) for (\hat{k}, \hat{g}) and using (2.12), (2.13), (2.19), we get

$$\begin{aligned} T_{\hat{m}, \hat{f}}(\sigma_2 \sigma_1 x_{23}^{-\hat{k}} c^{\hat{k}} \hat{g}) &= T_{\hat{m}, \hat{f}}(\Delta \sigma_2^{-(2\hat{k}+1)} c^{\hat{k}} \hat{g}) = \\ \Delta c^{\hat{m}} \hat{f} \hat{f}^{-1} \sigma_2^{-(2\hat{m}+1)(2\hat{k}+1)} \hat{f} c^{(2\hat{m}+1)\hat{k}} \hat{f}^{-1} &= \Delta \sigma_2^{-1} = \sigma_2 \sigma_1. \end{aligned}$$

Thus

$$T_{\hat{m}, \hat{f}}(\sigma_2 \sigma_1 x_{23}^{-\hat{k}} c^{\hat{k}} \hat{g}) = \sigma_2 \sigma_1. \quad (5.18)$$

Applying $T_{\hat{m}, \hat{f}}$ to the left hand side of (2.10) for (\hat{k}, \hat{g}) , we get

$$T_{\hat{m}, \hat{f}}(\hat{g}^{-1} \sigma_2^{2\hat{k}+1} \hat{g} \sigma_1^{2\hat{k}+1}) = E_{\hat{m}, \hat{f}}(\hat{g})^{-1} \hat{f}^{-1} \sigma_2^{(2\hat{m}+1)(2\hat{k}+1)} \hat{f} E_{\hat{m}, \hat{f}}(\hat{g}) \sigma_1^{(2\hat{m}+1)(2\hat{k}+1)} = \sigma_2 \sigma_1. \quad (5.19)$$

Since $T_{\hat{m}, \hat{f}}$ is an automorphism of \widehat{B}_3 , identities (5.18) and (5.19) imply that

$$\hat{g}^{-1} \sigma_2^{2\hat{k}+1} \hat{g} \sigma_1^{2\hat{k}+1} = \sigma_2 \sigma_1 x_{23}^{-\hat{k}} c^{\hat{k}} \hat{g}.$$

Thus the pair (\hat{k}, \hat{g}) satisfies (2.10).

Using the similar argument, one can show that the pair (\hat{k}, \hat{g}) also satisfies (2.9).

We proved that the pair (\hat{m}, \hat{f}) belongs to the group $\widehat{\mathbf{GT}}_{gen}$.

Let $(\hat{m}, \hat{f}) \in \widehat{\mathbf{GT}}_{gen}$, i.e. (\hat{m}, \hat{f}) is an invertible element of the monoid $\widehat{\mathbf{GT}}_{gen, mon}$. Let us prove that the pair

$$(\hat{\lambda}, \hat{f}), \quad \hat{\lambda} := 2\hat{m} + 1$$

belongs to the group $\widehat{\mathbf{GT}}_0$.

Relations (2.9) and (2.10) imply that, for every $\mathbf{N} \in \mathbf{NFI}_{PB_3}(B_3)$, the pair

$$(m + N_{\text{ord}}\mathbb{Z}, f\mathbf{N}_{F_2}) := (\widehat{\mathcal{P}}_{N_{\text{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{\mathbf{N}_{F_2}}(\hat{f})) \quad (5.20)$$

satisfies hexagon relations (3.3) and (3.4) modulo \mathbf{N} . In addition, $f\mathbf{N}_{F_2} \in [F_2/\mathbf{N}_{F_2}, F_2/\mathbf{N}_{F_2}]$.

Thus Proposition 3.4 implies that, for every $\mathbf{N} \in \mathbf{NFI}_{PB_3}(B_3)$, the pair in (5.20) satisfies relations (3.10), (3.11).

Due to Proposition 5.1, for every $\mathbf{H} \in \mathbf{NFI}(F_2)$, there exists $\mathbf{N} \in \mathbf{NFI}_{PB_3}(B_3)$ such that $\mathbf{N}_{F_2} \leq \mathbf{H}$. Thus the above observation about (3.10) and (3.11) implies that the pair (\hat{m}, \hat{f}) satisfies relations (2.26) and (2.27).

Since $\hat{f} \in [\hat{F}_2, \hat{F}_2]^{top.cl.}$ and $\hat{\lambda} = 2\hat{m} + 1$ is a unit in the ring $\widehat{\mathbb{Z}}$ (see Remarks 2.2, 2.9), it remains to show that the endomorphism $E_{\hat{m}, \hat{f}}$ is invertible. This is an obvious consequence of the second statement of Proposition 2.4. Indeed, if $\varphi : M \rightarrow \tilde{M}$ is a homomorphism of monoids, the restriction of φ to the group M^\times of invertible elements of M gives us a group homomorphism $M^\times \rightarrow \tilde{M}^\times$.

We established a bijection between the set $\widehat{\mathbf{GT}}_0$ (defined in [12, Section 0.1]) and the set $\widehat{\mathbf{GT}}_{gen}$. It remains to prove that this bijection is compatible with the group structures on $\widehat{\mathbf{GT}}_0$ and $\widehat{\mathbf{GT}}_{gen}$. Since the group structure on $\widehat{\mathbf{GT}}_0$ is obtained by identifying elements $(\hat{\lambda}, \hat{f})$ of $\widehat{\mathbf{GT}}_0$ with the corresponding automorphisms $E_{\hat{m}, \hat{f}}$ of \hat{F}_2 , the desired property follows from the second statement of Proposition 2.4.

Proposition 5.5 is proved. \square

Remark 5.6 Relations (2.26) and (2.27) may be interpreted as cocycle conditions and this interpretation was explored successfully in [20].

A Selected statements related to profinite groups

In this appendix, we prove several statements related to profinite groups. These statements are often used in articles about the profinite version of the Grothendieck-Teichmueller group. However, it is hard to find proofs of these statements in the literature.

Let J be a directed poset and \mathcal{F} be a functor from J to the category of finite groups. For $k_1, k_2 \in J$, $k_1 \leq k_2$ we set $\theta_{k_1, k_2} := \mathcal{F}(k_1 \rightarrow k_2)$.

It is convenient to identify elements of the product

$$\prod_{k \in J} \mathcal{F}(k) \tag{A.1}$$

with functions

$$f : J \rightarrow \bigsqcup_{k \in J} \mathcal{F}(k) \tag{A.2}$$

such that $f(k) \in \mathcal{F}(k)$, $\forall k \in J$.

Then $\lim(\mathcal{F})$ consists of functions (A.2) such that

- $f(k) \in \mathcal{F}(k)$, $\forall k \in J$ and
- $\theta_{k_1, k_2}(f(k_1)) = f(k_2)$, $\forall k_1, k_2 \in J$, $k_1 \leq k_2$.

For $k \in J$, η_k denotes the standard projection from $\lim(\mathcal{F})$ to $\mathcal{F}(k)$, i.e.

$$\eta_k(f) := f(k).$$

We consider the product space (A.1) with the standard product topology and we equip $\lim(\mathcal{F})$ with the corresponding subset topology. Let us also recall [27, Proposition 1.1.3] that, as the topological space, $\lim(\mathcal{F})$ is compact and Hausdorff. It is known [27, Section 1.1] that every profinite group is $\lim(\mathcal{F})$ for a functor \mathcal{F} from a directed poset to the category of finite groups.

For every group G , the poset $\mathbf{NFI}(G)$ is clearly directed and the assignments

$$N \mapsto G/N, \quad \theta_{K, N} := \mathcal{P}_{K, N} : G/K \rightarrow G/N, \quad K, N \in \mathbf{NFI}(G), \quad K \leq N$$

define a functor \mathcal{F}_G from $\mathbf{NFI}(G)$ to the category of finite groups. The profinite completion \widehat{G} of G is the limit $\lim(\mathcal{F}_G)$ of this functor.

As we mentioned above, it is convenient to identify elements \hat{g} of \widehat{G} with functions

$$\hat{g} : \mathbf{NFI}(G) \rightarrow \bigsqcup_{N \in \mathbf{NFI}(G)} G/N$$

such that

- $\hat{g}(\mathbf{N}) \in G/\mathbf{N}, \forall \mathbf{N} \in \text{NFI}(G)$ and
- $\mathcal{P}_{\mathbf{K}, \mathbf{N}}(\hat{g}(\mathbf{K})) = \hat{g}(\mathbf{N}), \forall \mathbf{K}, \mathbf{N} \in \text{NFI}(G), \mathbf{K} \leq \mathbf{N}$.

In this set-up, $\eta_{\mathbf{N}} := \hat{\mathcal{P}}_{\mathbf{N}}$.

We denote by j the standard group homomorphism $G \rightarrow \hat{G}$ defined by the formula

$$j(g)(\mathbf{N}) := g\mathbf{N}, \quad \mathbf{N} \in \text{NFI}(G).$$

Recall [27, Lemma 1.1.7] that, for every group G , the subgroup $j(G)$ is dense in \hat{G} . Moreover, the homomorphism $j : G \rightarrow \hat{G}$ is injective if and only if the group G is residually finite.

Lemma A.1 *Let G be a group and j be the standard homomorphism $G \rightarrow \hat{G}$. For every group homomorphism φ from G to a profinite group H , there exists a unique continuous group homomorphism*

$$\hat{\varphi} : \hat{G} \rightarrow H$$

such that $\hat{\varphi} \circ j = \varphi$.

Proof. Since H is a profinite group, there exists a directed poset J and a functor \mathcal{F} from J to the category of finite groups such that $H = \lim(\mathcal{F})$. For $k \in J$, we denote by η_k the standard continuous group homomorphism from H to $\mathcal{F}(k)$.

For every $k \in J$, $\eta_k \circ \varphi$ is a homomorphism from G to the finite group $\mathcal{F}(k)$. Hence $\ker(\eta_k \circ \varphi)$ is a finite index normal subgroup of G . We denote this subgroup by \mathbf{N}_k ,

$$\mathbf{N}_k := \ker(G \xrightarrow{\eta_k \circ \varphi} \mathcal{F}(k)).$$

It is easy to see that the formula

$$\varphi_k(g\mathbf{N}_k) := \eta_k \circ \varphi(g) \tag{A.3}$$

defines a group homomorphism from the finite group G/\mathbf{N}_k to the finite group $\mathcal{F}(k)$.

Let us also observe that, if $k_1, k_2 \in J$ and $k_1 \leq k_2$ then $\mathbf{N}_{k_1} \leq \mathbf{N}_{k_2}$ and the diagram

$$\begin{array}{ccc} G/\mathbf{N}_{k_1} & \xrightarrow{\varphi_{k_1}} & \mathcal{F}(k_1) \\ \mathcal{P}_{\mathbf{N}_{k_1}, \mathbf{N}_{k_2}} \downarrow & & \downarrow \theta_{k_1, k_2} \\ G/\mathbf{N}_{k_2} & \xrightarrow{\varphi_{k_2}} & \mathcal{F}(k_2) \end{array} \tag{A.4}$$

commutes. Here $\theta_{k_1, k_2} := \mathcal{F}(k_1 \rightarrow k_2)$.

We claim that the formula

$$(\hat{\varphi}(\hat{g}))(k) := \varphi_k(\hat{g}(\mathbf{N}_k)), \quad k \in J \tag{A.5}$$

defines a continuous group homomorphism $\hat{\varphi}$ from \hat{G} to H .

Indeed, it is obvious that, for every $k \in J$ and every $\hat{g} \in \hat{G}$, $(\hat{\varphi}(\hat{g}))(k) \in \mathcal{F}(k)$. Thus $\hat{\varphi}(\hat{g})$ belongs to the product

$$\prod_{k \in J} \mathcal{F}(k).$$

The commutativity of the diagram in (A.4) implies that $\hat{\varphi}(\hat{g})$ satisfies the condition

$$\theta_{k_1, k_2}((\hat{\varphi}(\hat{g}))(k_1)) = (\hat{\varphi}(\hat{g}))(k_2)$$

whenever $k_1 \leq k_2$. Thus $\hat{\varphi}(\hat{g})$ belongs to $H \subset \prod_{k \in J} \mathcal{F}(k)$.

It is easy to see that $\hat{\varphi}$ is indeed a group homomorphism $\hat{G} \rightarrow H$.

Equation (A.5) implies that

$$\eta_k \circ \hat{\varphi} = \varphi_k \circ \hat{\mathcal{P}}_{\mathbf{N}_k}.$$

Hence the composition $\eta_k \circ \hat{\varphi}$ is continuous for every $k \in J$.

Thus we proved that equation (A.5) indeed defines a continuous group homomorphism from \widehat{G} to H . Using (A.3), we see that, for every $k \in J$ and $g \in G$, we have

$$(\hat{\varphi} \circ j(g))(k) = \varphi_k(g\mathbf{N}_k) = \eta_k(\varphi(g)).$$

Thus $\hat{\varphi} \circ j = \varphi$.

Let $\psi : \widehat{G} \rightarrow H$ be a continuous group homomorphism such that $\psi \circ j = \varphi$. Since $\hat{\varphi} \circ j = \varphi$, we have

$$\psi|_{j(G)} = \hat{\varphi}|_{j(G)}. \quad (\text{A.6})$$

Since $j(G)$ is dense in \widehat{G} and H is Hausdorff, identity (A.6) implies that $\psi = \hat{\varphi}$. Thus the uniqueness of $\hat{\varphi}$ is established and the lemma is proved. \square

Corollary A.2 *Let G, H be groups and j be the standard homomorphism $G \rightarrow \widehat{G}$. For every group homomorphism $\varphi : G \rightarrow \widehat{H}$, there exists a unique continuous group homomorphism*

$$\hat{\varphi} : \widehat{G} \rightarrow \widehat{H}$$

such that $\hat{\varphi} \circ j = \varphi$. If γ is an automorphism of G then $\widehat{j \circ \gamma}$ is a continuous automorphism of \widehat{G} .

Proof. The first statement of the corollary follows Lemma A.1.

Let $\gamma \in \text{Aut}(G)$ and $\kappa := \gamma^{-1}$. By abuse of notation, we denote by $\hat{\gamma}$ (resp. $\hat{\kappa}$) the continuous group homomorphism $\widehat{G} \rightarrow \widehat{G}$ corresponding to $j \circ \gamma$ (resp. to $j \circ \kappa$).

For $\hat{\gamma}$ and $\hat{\kappa}$, we have

$$\hat{\gamma} \circ j = j \circ \gamma, \quad \hat{\kappa} \circ j = j \circ \kappa.$$

Using these identities, we get

$$\hat{\gamma} \circ \hat{\kappa} \circ j = \hat{\gamma} \circ j \circ \kappa = j \circ \gamma \circ \kappa = j$$

and

$$\hat{\kappa} \circ \hat{\gamma} \circ j = \hat{\kappa} \circ j \circ \gamma = j \circ \kappa \circ \gamma = j.$$

Since $\hat{\gamma} \circ \hat{\kappa}|_{j(G)} = \text{id}|_{j(G)}$, $\hat{\kappa} \circ \hat{\gamma}|_{j(G)} = \text{id}|_{j(G)}$, $j(G)$ is dense in \widehat{G} and \widehat{G} is Hausdorff, we conclude that

$$\hat{\gamma} \circ \hat{\kappa} = \text{id}_{\widehat{G}}, \quad \hat{\kappa} \circ \hat{\gamma} = \text{id}_{\widehat{G}}.$$

Thus $\hat{\gamma}$ is invertible and $\hat{\kappa} = \hat{\gamma}^{-1}$. \square

Let us prove that

Proposition A.3 *For every $\mathbf{N} \in \text{NFI}(G)$, the kernel of the homomorphism $\widehat{\mathcal{P}}_{\mathbf{N}} : \widehat{G} \rightarrow G/\mathbf{N}$ is isomorphic to the profinite completion $\widehat{\mathbf{N}}$ of \mathbf{N} .*

Proof. For every $L \in \text{NFI}(\mathbf{N})$, the normal core $\text{Core}_G(L)$ of L in G is an element of $\text{NFI}_{\mathbf{N}}(G)$. Therefore the subposet $\text{NFI}_{\mathbf{N}}(G)$ of $\text{NFI}(\mathbf{N})$ is cofinal and hence the limit of the functor

$$\mathbf{H} \mapsto \mathbf{N}/\mathbf{H} \quad (\text{A.7})$$

from $\text{NFI}_{\mathbf{N}}(G)$ to the category of finite groups is isomorphic to $\widehat{\mathbf{N}}$ (see [27, Lemma 1.1.9]).

Let

$$\mathbf{K} := \ker(\widehat{G} \xrightarrow{\widehat{\mathcal{P}}_{\mathbf{N}}} G/\mathbf{N}).$$

For every $\mathbf{H} \in \text{NFI}_{\mathbf{N}}(G)$, the restriction of the continuous homomorphism $\widehat{\mathcal{P}}_{\mathbf{H}} : \widehat{G} \rightarrow G/\mathbf{H}$ gives us a continuous homomorphism

$$\widehat{\mathcal{P}}_{\mathbf{H}}|_{\mathbf{K}} : \mathbf{K} \rightarrow \mathbf{N}/\mathbf{H}.$$

Moreover, for all $H_1, H_2 \in \text{NFI}_N(G)$ with $H_1 \leq H_2$, the diagram

$$\begin{array}{ccc} & K & \\ \widehat{\mathcal{P}}_{H_1} \swarrow & & \searrow \widehat{\mathcal{P}}_{H_2} \\ N/H_1 & \xrightarrow{\mathcal{P}_{H_1, H_2}} & N/H_2 \end{array}$$

commutes.

Hence we get a continuous group homomorphism $\gamma : K \rightarrow \widehat{N}$, where \widehat{N} is identified with the limit of functor (A.7). It is not hard to see that γ is a bijection. Since K is compact (K is a closed subset of the compact space \widehat{G}) and γ is a continuous bijection from a compact space K to the Hausdorff space \widehat{N} , γ is a homeomorphism. Since γ is also an isomorphism of groups, we proved that the topological groups K and \widehat{N} are isomorphic. \square

References

- [1] P. Boavida de Brito, G. Horel and M. Robertson, Operads of genus zero curves and the Grothendieck–Teichmueller group, *Geom. Topol.* **23**, 1 (2019) 299–346.
- [2] I. Bortnovskiy and V. Pashkovskiy, Exploration of the Grothendieck–Teichmueller (GT) shadows for the dihedral poset, *The Final Research Paper. MIT Program “Yulia’s Dream”, 2022–23* https://math.temple.edu/~vald/BortnovskiyPashkovskiy_YD_FinalResearchPaper.pdf
- [3] N.C. Combe and A. Kalugin, Hidden symmetries of the Grothendieck–Teichmueller group, <https://arxiv.org/abs/2209.00966>
- [4] N.C. Combe and Y. I. Manin, Genus zero modular operad and absolute Galois group, *North-West. Eur. J. Math.* **8**, i (2022) 25–60.
- [5] V. A. Dolgushev, K.Q. Le and A. Lorenz, What are GT-shadows? accepted to *Algebr. Geom. Topol.*, <https://arxiv.org/abs/2008.00066>
- [6] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *Algebra i Analiz* **2**, 4 (1990) 149–181.
- [7] J. Ellenberg, Galois invariants of dessins d’enfants, *Arithmetic fundamental groups and noncommutative algebra* (Berkeley, CA, 1999), 27–42, *Proc. Sympos. Pure Math.*, **70**, Amer. Math. Soc., Providence, RI, 2002.
- [8] B. Fresse, Homotopy of operads and Grothendieck–Teichmueller groups. Part 1, The algebraic theory and its topological background, *Mathematical Surveys and Monographs*, **217**. AMS, Providence, RI, 2017. xlv+532 pp.
- [9] P. Guillot, The Grothendieck–Teichmueller group of a finite group and G -dessins d’enfants. Symmetries in graphs, maps, and polytopes, 159–191, *Springer Proc. Math. Stat.*, **159**, Springer, 2016; <https://arxiv.org/abs/1407.3112>
- [10] P. Guillot, The Grothendieck–Teichmueller group of $PSL(2, q)$, *J. Group Theory* **21**, 2 (2018) 241–251; <https://arxiv.org/abs/1604.04415>
- [11] D. Harbater and L. Schneps, Approximating Galois orbits of dessins, *Geometric Galois actions*, 1, 205–230, *London Math. Soc. Lecture Note Ser.*, **242**, Cambridge Univ. Press, Cambridge, 1997.
- [12] D. Harbater and L. Schneps, Fundamental groups of moduli and the Grothendieck–Teichmueller group, *Trans. Amer. Math. Soc.* **352**, 7 (2000) 3117–3148.
- [13] A. Hatcher, P. Lochak and L. Schneps, On the Teichmueller tower of mapping class groups, *J. Reine Angew. Math.* **521** (2000) 1–24.

- [14] W. Herfort and L. Ribes, Torsion elements and centralizers in free products of profinite groups, *J. Reine Angew. Math.* **358** (1985) 155–161.
- [15] Y. Hoshi, A. Minamide and S. Mochizuki, Group-theoreticity of numerical invariants and distinguished subgroups of configuration space groups, *Kodai Math. J.* **45**, 3 (2022) 295–348.
- [16] Y. Ihara, On the embedding of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\text{GT}}$, *with an appendix by M. Emsalem and P. Lochak*, London Math. Soc. Lecture Note Ser., **200**, The Grothendieck theory of dessins d'enfants (Luminy, 1993), 289–321, Cambridge Univ. Press, Cambridge, 1994.
- [17] C. Kassel and V. Turaev, Braid groups, *with the graphical assistance of Olivier Dodane*, Graduate Texts in Mathematics, **247**. Springer, New York, 2008. xii+340 pp.
- [18] P. Lochak, H. Nakamura and L. Schneps, On a new version of the Grothendieck-Teichmueller group, *C. R. Acad. Sci. Paris Sér. I Math.* **325**, 1 (1997) 11–16.
- [19] P. Lochak and L. Schneps, Open problems in Grothendieck-Teichmueller theory, *Problems on mapping class groups and related topics*, 165–186, Proc. Sympos. Pure Math., **74**, Amer. Math. Soc., Providence, RI, 2006.
- [20] P. Lochak and L. Schneps, A cohomological interpretation of the Grothendieck-Teichmueller group, *with an appendix by C. Scheiderer*. *Invent. Math.* **127**, 3 (1997) 571–600.
- [21] A. Minamide and H. Nakamura, The automorphism groups of the profinite braid groups, *Amer. J. Math.* **144**, 5 (2022) 1159–1176; <https://arxiv.org/abs/1904.06749>
- [22] H. Nakamura and L. Schneps, On a subgroup of the Grothendieck-Teichmueller group acting on the tower of profinite Teichmueller modular groups, *Invent. Math.* **141**, 3 (2000) 503–560.
- [23] N. Nikolov and D. Segal, Finite index subgroups in profinite groups, *C. R. Math. Acad. Sci. Paris* **337**, 5 (2003), no.5, 303–308.
- [24] F. Pop, Finite tripod variants of I/OM: on Ihara's question/Oda-Matsumoto conjecture, *Invent. Math.* **216**, 3 (2019) 745–797.
- [25] F. Pop, Little survey on I/OM and its variants and their relation to (variants of) $\widehat{\text{GT}}$ – old & new, *Topology Appl.* **313** (2022) Paper No. 107993.
- [26] F. Pop and A. Topaz, A linear variant of GT, <https://arxiv.org/abs/2104.10504>
- [27] L. Ribes and P. Zalesskii, Profinite groups. Second edition. *A Series of Modern Surveys in Mathematics* **40**. Springer-Verlag, Berlin, 2010. xvi+464 pp.
- [28] L. Schneps, The Grothendieck-Teichmueller group $\widehat{\text{GT}}$: a survey, *Geometric Galois actions*, 1, 183–203, London Math. Soc. Lecture Note Ser., **242**, Cambridge Univ. Press, Cambridge, 1997.
- [29] J. Xia, GT-Shadows related to finite quotients of the full modular group, Master Thesis, 2021, Temple University, <https://scholarshare.temple.edu/handle/20.500.12613/6910>

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY,
 WACHMAN HALL RM. 638
 1805 N. BROAD ST.,
 PHILADELPHIA PA, 19122 USA
E-mail address: vald@temple.edu

SCHOOL OF MATHEMATICS,
 GEORGIA INSTITUTE OF TECHNOLOGY
 686 CHERRY STREET
 ATLANTA, GA 30332-0160 USA
E-mail address: jguynnee@gatech.edu