Relaxation of Wegner's Planar Graph Conjecture for maximum degree 4

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Abstract

The famous Wegner's Planar Graph Conjecture asserts sharp upper bounds on the chromatic number of the square G^2 of a planar graph G, depending on the maximum degree $\Delta(G)$ of G. The only case that the conjecture is known to be true is when $\Delta(G) \leq 3$. Even the case when $\Delta(G) = 4$ is still open; the conjecture states that the chromatic number of G^2 is at most 9.

We take a completely different approach from previous partial results, and show that a relaxation of properly coloring the square of a planar graph G with $\Delta(G) \leq 4$ can be achieved with 9 colors. Instead of requiring every color in the neighborhood of a vertex to be unique, which is equivalent to a proper coloring of G^2, we seek a proper coloring of G such that at most one color is allowed to be repeated in the neighborhood of each vertex of degree 4, but not at vertices of other degrees.

1 Introduction

Given a simple graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively, of G. For each $v \in V(G)$, the *neighborhood* of v, denoted $N_G(v)$, is the set of vertices adjacent to v, and the *degree* of v, denoted $d_G(v)$, is the number of neighbors of v. A *proper coloring* φ of a graph G assigns colors to vertices of G so that $\varphi(x) \neq \varphi(y)$ for every edge xy of G.

Given a graph G, the *square* of G, denoted G^2, is the graph obtained from G by adding edges between every pair of vertices at distance 2. The famous and very popular Wegner's Planar Graph Conjecture [22], first raised in 1977, asserts sharp upper bounds on the chromatic number of the square G^2 of a planar graph G, depending on the maximum degree $\Delta(G)$ of G. We state the conjecture below, and refer the reader to [22] for illustrations of the sharpness examples.

Wegner's Planar Graph Conjecture ([22])**.** *If G is a planar graph, then*

$$\chi(G^2) \leq \begin{cases} 7 & \text{if } \Delta(G) = 3, \\ \Delta(G) + 5 & \text{if } \Delta(G) \in \{4, 5, 6, 7\}, \\ \frac{3}{2}\Delta(G) + 1 & \text{if } \Delta(G) \geq 8. \end{cases}$$

For sufficiently large maximum degree, Havet, van den Heuvel, McDiarmid, and Reed [12] proved that the above conjecture is true asymptotically. For exact results, Molloy and Salavatipour [19] proved the current best bound.

Theorem 1 ([19])**.** *If G is a planar graph, then* $\chi(G^2) \leq \left\lceil \frac{5}{3}\Delta(G) \right\rceil + 78$.

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The only case that the conjecture is resolved is when $\Delta(G) = 3$, which was proven to be true by Thomassen [20], and independently by Hartke, Jahanbekam, Thomas [11]; the former proof uses a meticulous induction argument, and the latter uses a simple discharging argument with a computer assisted proof of its reducible configurations.

In general, Wegner's Planar Graph Conjecture is still wide open, and we refer the reader to the following references for various partial results, oftentimes with restrictions on the maximum degree [1, 2, 14, 15, 18, 21, 23, 24]. In particular, when the maximum degree is exactly 4, after a series of improvements in [7, 8, 25] by various authors, Bousquet, Deschamps, de Meyer, and Pierron [3] recently established an upper bound of 12. Note that the conjectured upper bound is 9.

The following relaxation of proper coloring the square of a graph was formally defined in [6]: a *proper h-conflict-free* (h-PCF for short) *k-coloring* of a graph G is a proper k-coloring of G such that the neighborhood of every vertex v has at least $\min\{h, d_G(v)\}$ unique colors. (This concept is a generalization of proper conflict-free coloring, defined recently by Fabrici, Lužar, Rindošová, and Soták [10], and has received considerable attention, we refer the interested reader to [4, 5, 13, 16].) Note that proper coloring the square of a graph requires every color in the neighborhood of each vertex to be unique, whereas in an h-PCF coloring the restriction on the colors of the neighborhood of a vertex is for vertices with degree at least $h + 2$.

In this paper, we take a completely different approach from previous authors for attacking Wegner's Planar Graph Conjecture for planar graphs G with maximum degree at most 4. Instead of allowing more colors than the conjectured bound, we show that replacing the condition that the square of G must be properly colored by the condition that the coloring of G is a 2-PCF coloring, we can obtain the bound predicted in Wegner's Planar Graph Conjecture. When restricted to graphs G with maximum degree at most 4, a 2-PCF 9-coloring is equivalent to a proper coloring of G such that at most one color is allowed to be repeated in the neighborhood of each vertex of degree 4, but not at vertices of other degrees. Note that requiring $\min\{3, d_G(v)\}$ unique colors in the neighborhood of every vertex v is equivalent to a proper coloring of the square of the graph when it has maximum degree 4. We now state our main result:

Theorem 2. *Every planar graph with maximum degree at most 4 has a 2-PCF 9-coloring. In other words, every planar graph G with maximum degree at most 4 has a proper 9-coloring such that the neighborhood of each vertex v has at least $\min\{2, d_G(v)\}$ unique colors.*

We end this section with a subsection on definitions and notation used for the proof of Theorem 2, which is in section 2. We end the paper with some future directions in section 3.

1.1 Definitions and notation

A *k-vertex*, *k^--vertex*, *k^+-vertex* is a vertex of degree k, at most k, at least k, respectively.

Given a vertex v of a graph G with a 2-PCF coloring φ, the *unique colors of v* are the unique colors appearing in the neighborhood of v; in particular, let $\varphi_1(v)$ and $\varphi_2(v)$ denote two (different) unique colors of v, if they exist. We say *v has k unique colors* if there are k unique colors in the neighborhood of v. For $X \subseteq V(G)$, we abuse the notation and define $\varphi(X) = \{\varphi(v) : v \in X\}$.

For $S \subseteq V(G)$ where each vertex in S has at most two neighbors not in S, define $G * S$ to be the graph obtained from G by removing S (as well as edges with an endvertex in S) and adding an edge uv for $u, v \in V(G) \setminus S$ if u and v have a common neighbor in S and uv is not an edge already; $G * S$ is called the *S-reduced graph*. Note that $G * S$ is planar whenever G is planar, and the maximum degree of $G * S$ is at most the maximum degree of G.

For a 2-PCF coloring φ of $G * S$, let $v \in S$ and $u \in N_G(v) \setminus S$. If all vertices in $N_G(u) \setminus S$ receive distinct colors (in particular if u is a 3-vertex in G), then let $B_S(u) = \{\varphi(u), \varphi_1(u), \varphi_2(u)\}$. If either $\varphi_1(u)$ or $\varphi_2(u)$ is not defined, then do not include it in $B_S(u)$. (Recall that $\varphi_1(v)$ and $\varphi_2(v)$ denote two (different) unique colors of v, if they exist.) If there is a repeated color among vertices in $N_G(u) \setminus S$, then let $B_S(u) = \{\varphi(u)\} \cup \varphi(N_{G-S}(u))$. Notice that for $u \in V(G * S)$ with a neighbor in S

$$|B_S(u)| \leq 3 \tag{1}$$

if G has maximum degree at most 4. Let $C_{G*S}(v) = \bigcup_{u\in N_G(v)\setminus S} B_S(u)$. By (1), $|C_{G*S}(v)| \le 3|N_G(v) \setminus S|$ when G has maximum degree at most 4. Moreover, for a partial coloring φ of G, if we extend φ to G by assigning a color not in $C_{G*S}(v)$ to v, then two unique colors are guaranteed for vertices in $N_G(v) \setminus S$ under φ and φ is still a (partial) proper coloring of G.

2 Proof of Theorem 2

Let G be a counterexample to Theorem 2 with the minimum number of vertices. If G has maximum degree at most 3, then $\chi(G^2) \le 7$, by Thomassen [20] and independently by Hartke, Jahanbekam, and Thomas [11], so G has a 2-PCF 7-coloring. Thus, we may assume G has maximum degree exactly 4. We first prove a sequence of claims regarding the structure of G.

Claim 1. *G does not have a 2^--vertex.*

Proof. Let v be a vertex of minimum degree in G. Suppose v is a 2^--vertex. For $S = \{v\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Extend φ to all of G by coloring v with a color not in $C_H(v)$. Now φ is a 2-PCF 9-coloring of G, which is a contradiction. □

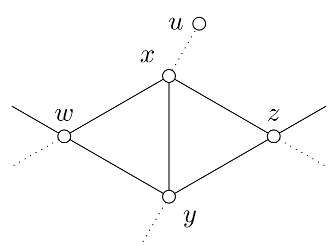

Figure 1: Two 3-cycles sharing an edge

Claim 2. *G does not have two 3-cycles sharing an edge.*

Proof. Suppose G contains two 3-cycles xyz and xyw that share an edge xy. See Figure 1.

Let H be the graph obtained from G by removing x and adding the edge wz. Since H is a planar graph with maximum degree at most 4, by the minimality of G, H has a 2-PCF 9-coloring φ. Because φ gives w, y, z distinct colors, regardless of the color on u (if it exists), x is already guaranteed (at least) two unique colors at this point. Let $B_1 = \{\varphi_1(v), \varphi_2(v) : v \in N_G(x) \setminus \{y, z, w\}\}$ and $B_2 = \{\varphi(N_G(v)) : |\varphi(N_G(v))| = 2 \text{ and } v \in \{w, z\}\}$. Note that B_1 or B_2 might be empty. Moreover, $|\varphi(N_G(v))| \ge 2$ for $v \in \{w, z\}$ since otherwise φ is not a 2-PCF coloring of H. When coloring x, avoiding colors in B_1 and B_2 will guarantee two unique colors for u (if it exists) and w, z, respectively. Let $B = \varphi(N_G(x)) \cup B_1 \cup B_2$. Since $|B| \le 8$, we can extend φ to x by using a color not in B. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. □

Claim 3. *G does not have a 3-cycle with a 3-vertex.*

Proof. Suppose G contains a 3-cycle $T : xyz$ where x is a 3-vertex. Let x_1, y_1, z_1 be neighbors of x, y, z, respectively, not on T. Let y_2 (resp. z_2) be the neighbor of y (resp. z) that is neither on T nor y_1 (resp. z_1) if y (resp. z) is a 4-vertex. See Figure 2(a). By Claim 2, all vertices in the figure are distinct.

Suppose x_1 is a 3-vertex. For $S = \{x, x_1\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Now extend φ to all of G as follows: color x_1 with a color not in $C_H(x_1) \cup \{\varphi(y), \varphi(z)\}$ to guarantee two (actually three) unique colors for x, and color x with a color not in $\{\varphi(x_1), \varphi_1(x_1), \varphi_2(x_1), \varphi(y), \varphi(z), \varphi(y_1), \varphi(z_1)\}$ to guarantee two (actually three) unique colors for x_1. Thus φ is a 2-PCF 9-coloring of G, which is a contradiction.

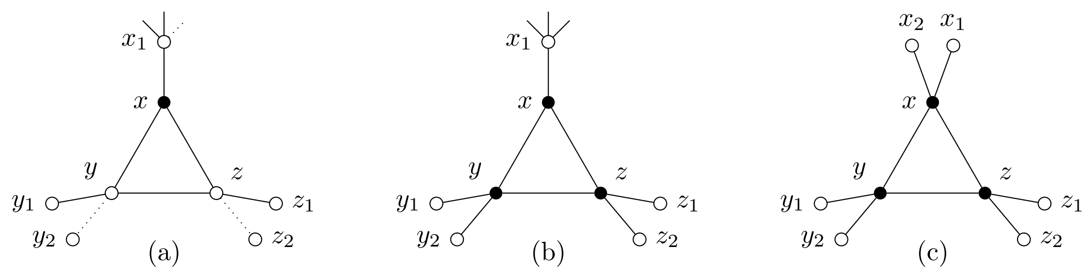

Figure 2: (a), (b): A 3-cycle with a 3-vertex (c): 3-cycle with no 3-vertex

Now we know x_1 is a 4-vertex. For $S' = \{x, y, z\}$, let H' be the S'-reduced graph. By the minimality of G, H' has a 2-PCF 9-coloring φ'. Since x_1 has three unique colors at this point, at least two unique colors for x_1 are guaranteed regardless of the color assigned to x.

Suppose y is a 3-vertex. Then color z with a color not in $C_{H'}(z) \cup \{\varphi'(x_1), \varphi'(y_1)\}$, and color y with a color not in $C_{H'}(y) \cup \varphi'(N_G(z) \setminus S') \cup \{\varphi'(x_1), \varphi'(z)\}$. At this point x has two (actually three) unique colors. Color x with a color not in $\{\varphi'(x_1), \varphi'(y), \varphi'(z), \varphi'(y_1)\} \cup \varphi'(N_G(z) \setminus S')$ to guarantee two unique colors for each of y and z. Thus φ' is a 2-PCF 9-coloring of G, which is a contradiction.

By symmetry, we may assume both y and z are 4-vertices. See Figure 2(b). Now, color y with a color not in $C_{H'}(y) \cup \{\varphi'(x_1)\}$, and color z with a color not in $C_{H'}(z) \cup \{\varphi'(y), \varphi'(x_1)\}$. This guarantees two (actually three) unique colors for x. Color x with a color not in $\{\varphi'(x_1), \varphi'(y), \varphi'(y_1), \varphi'(y_2), \varphi'(z), \varphi'(z_1), \varphi'(z_2)\}$ to guarantee an additional unique color for each of y and z. Note that each of y and z already had a unique color in $N_G(y) \setminus \{x\}$ and $N_G(z) \setminus \{x\}$, respectively, since H' is an S'-reduced graph. Then φ' is a 2-PCF 9-coloring of G, which is a contradiction. □

Claim 4. *G does not have a 3-cycle.*

Proof. Suppose G contains a 3-cycle $T : xyz$. By Claim 3, all vertices on T are 4-vertices. Let x_1, x_2, and y_1, y_2, and z_1, z_2 be the neighbors of x and y and z, respectively, not on T. See Figure 2(c). By Claim 2, all vertices in the figure are distinct. For $S = \{x, y, z\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Let $C' = C_H(x) \cup \{\varphi(y_1), \varphi(y_2), \varphi(z_1), \varphi(z_2)\}$.

Suppose $|C'| \le 8$. First color x with a color not in C' to guarantee three unique colors for each of y and z, so at least two unique colors are guaranteed for y and z regardless of the colors assigned to y and z.

If $|C_H(y) \cup \{\varphi(x), \varphi(x_1), \varphi(x_2)\}| \le 8$, then color y with a color not in $C_H(y) \cup \{\varphi(x), \varphi(x_1), \varphi(x_2)\}$, guaranteeing three unique colors for x, so at least two unique colors are guaranteed for x regardless of the color assigned to z. Now color z with a color not in $C_H(z) \cup \{\varphi(x), \varphi(y)\}$. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction.

Thus, by symmetry, we may assume $|C_H(y) \cup \{\varphi(x), \varphi(x_1), \varphi(x_2)\}| = |C_H(z) \cup \{\varphi(x), \varphi(x_1), \varphi(x_2)\}| = 9$. Without loss of generality, assume $\varphi(x_i) = i$ for $i \in \{1, 2\}$, $\varphi(x) = 3$, and $C_H(y) = C_H(z) = \{4, 5, 6, 7, 8, 9\}$. Delete the color on x and color y with 3 and z with 1 to guarantee two unique colors for x, y, z. Now color x with a color not in $C_H(x) \cup \{\varphi(y), \varphi(z)\}$ to obtain a 2-PCF 9-coloring of G, which is a contradiction.

Now we know, $|C'| = 9$, so either $\varphi(y_1)$ or $\varphi(y_2)$ appears only once on C'. Without loss of generality, assume $\varphi(y_1)$ does not appear in C' except on y_1. Color x with $\varphi(y_1)$, guaranteeing the three unique colors for z. Color z with a color not in $C_H(z) \cup \{\varphi(y_1), \varphi(y_2)\}$, guaranteeing two unique colors for y.

If $\varphi(z) \notin \{\varphi(x_1), \varphi(x_2)\}$, then x has three unique colors, so coloring y with a color not in $C_H(y) \cup \{\varphi(z)\}$ guarantees at least two unique colors for x. If $\varphi(z) \in \{\varphi(x_1), \varphi(x_2)\}$, then color y with a color not in $C_H(y) \cup \{\varphi(x_1), \varphi(x_2)\}$, guaranteeing an additional unique color for x. Note that $N_G(x) \setminus \{y\}$ already has a unique color since H is an S-reduced graph. In all cases, φ is a 2-PCF 9-coloring of G, which is a contradiction. □

Claim 5. *G does not have a path on three 3-vertices where the middle vertex is adjacent to a 4-vertex.*

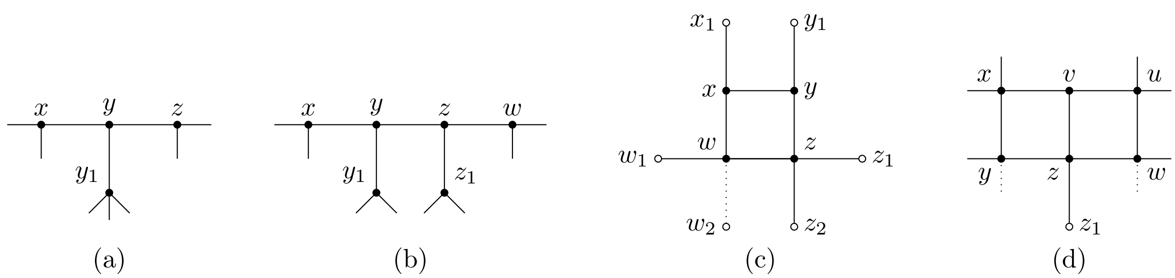

Figure 3: Figures for Claims 5, 6, 7, and 8

Proof. Suppose G has a path xyz on three 3-vertices where the neighbor y_1 of y other than x and z is a 4-vertex. See Figure 3(a). For $S = \{x, y, z\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Color x with a color not in $C_H(x) \cup \{\varphi(y_1)\}$, and color z with a color not in $C_H(z) \cup \{\varphi(x), \varphi(y_1)\}$ to guarantee two (actually three) unique colors for y. Since y_1 is a 3-vertex in H, $\varphi(N_G(y_1))$ consists of three distinct colors and at least two unique colors for y_1 are guaranteed regardless of the color assigned to y. Color y with a color not in $\varphi((N_G(x) \cup N_G(z)) \setminus S) \cup \{\varphi(x), \varphi(y_1), \varphi(z)\}$ to obtain a 2-PCF 9-coloring φ of G, which is a contradiction. □

Claim 6. *G does not have a path on four 3-vertices.*

Proof. Suppose G has a path $xyzw$ on four 3-vertices, and let y_1 (resp. z_1) be the neighbor of y (resp. z) that is not on the path. See Figure 3(b). By Claim 4, all vertices in the figure are distinct. Here, we may assume xw is not an edge; the case where xw is an edge is analogous. For $S = \{x, y, z, w\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Note that $|C_H(x) \cup \{\varphi(y_1)\}| \leq 7$. If $|C_H(x) \cup \{\varphi(y_1)\}| = 7$, then color y with a color in $(C_H(x) \cup \{\varphi(y_1)\}) \setminus (C_H(y) \cup \varphi((N_G(x) \setminus S) \cup \{z_1\}))$, and if $|C_H(x) \cup \{\varphi(y_1)\}| \leq 6$, then color y with a color not in $C_H(y) \cup \varphi((N_G(x) \setminus S) \cup \{z_1\})$. This guarantees two (actually three) unique colors for x, and in both cases, $|C_H(x) \cup \{\varphi(y), \varphi(y_1)\}| \leq 7$. Color w with a color not in $C_H(w) \cup \{\varphi(y), \varphi(z_1)\}$ to guarantee two (actually three) unique colors for z, and color z with a color not in $C_H(z) \cup \varphi(N_G(w) \setminus S) \cup \{\varphi(y), \varphi(y_1), \varphi(w)\}$ to guarantee two (actually three) unique colors for w. Finally, color x with a color not in $C_H(x) \cup \{\varphi(y), \varphi(y_1), \varphi(z)\}$ to guarantee two (actually three) unique colors for y, and now φ is a 2-PCF 9-coloring of G, which is a contradiction. □

Claim 7. *G does not have a 4-cycle $xyzw$ where x and y are 3-vertices.*

Proof. Suppose G has a 4-cycle $F : xyzw$ where x and y are 3-vertices. Let x_1, y_1, z_1, w_1 be a neighbor of x, y, z, w, respectively, that is not on F. By Claim 6, we may assume z is a 4-vertex, so let z_2 be the neighbor of z that is neither on F nor z_1, and if w is a 4-vertex, then let w_2 be the neighbor of w that is neither on F nor w_1. See Figure 3(c). Note that x_1 and $z_i \in \{z_1, z_2\}$ may coincide, and y_1 and $w_i \in \{w_1, w_2\}$ may coincide, but all other vertices in the figure are distinct by Claim 4. For $S = \{x, y, z, w\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Color z with a color not in $C_H(z) \cup \{\varphi(y_1), \varphi(w_1)\}$, and color w with a color not in $C_H(w) \cup \{\varphi(z), \varphi(x_1)\}$. Color y with a color not in $C_H(y) \cup \{\varphi(x_1), \varphi(w), \varphi(z), \varphi(z_1), \varphi(z_2)\}$ to guarantee two unique colors for each of x and z. Finally, color x with a color not in $C_H(x) \cup \{\varphi(y), \varphi(y_1), \varphi(z), \varphi(w), \varphi(w_1)\}$ to guarantee two unique colors for each of y and w. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. □

Claim 8. *G does not have a 3-vertex incident with two 4-faces.*

Proof. Suppose G has a 3-vertex v incident with two 4-cycles $xyzv$ and $uwzv$. By Claim 7, x, z, u are 4-vertices. Let z_1 be the neighbor of z that is not y, v, w. See Figure 3(d). Let H be the graph obtained from G by removing v and adding the edge xu. Note that xu did not exist beforehand by Claim 4. Note that H is still planar and the maximum degree did not increase. By the minimality of G, H has a 2-PCF 9-coloring

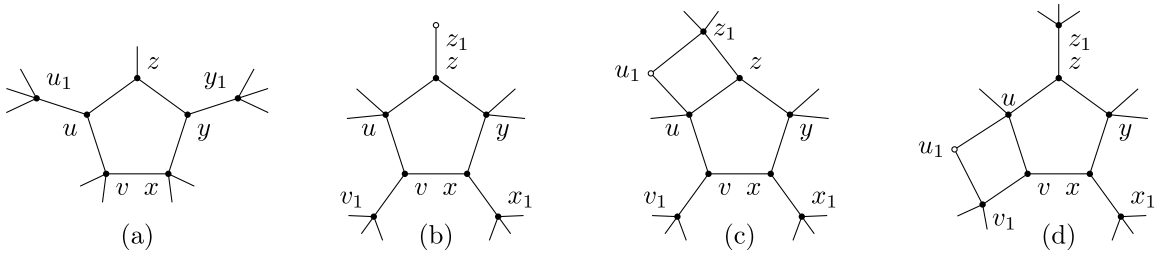

Figure 4: Figures for Claims 9, 10, and 11

φ, so each of $\varphi(N_G(x))$ and $\varphi(N_G(u))$ must consist of at least two distinct colors. Let α, β be two distinct colors in $\varphi(N_G(x))$, and let γ, δ be two distinct colors in $\varphi(N_G(u))$. Note that there are three unique colors for z, so regardless of the color assigned to v, at least two unique colors are guaranteed for z.

If $\varphi(N_G(v))$ consists of three distinct colors, then color v with a color not in $\{\varphi(x), \alpha, \beta, \varphi(z), \varphi(u), \gamma, \delta\}$ to guarantee two unique colors for each of x and u. Now φ is a 2-PCF 9-coloring of G, which is a contradiction. Thus, $\varphi(N_G(v))$ consists of two distinct colors.

Without loss of generality, assume $\varphi(x) = \varphi(z)$. There must be two unique colors for y, so y must be a 4-vertex and the two neighbors of y other than x and z received different colors that is also different from $\varphi(x)$. Thus, by reassigning a color to z, two unique colors are guaranteed for y. Note that since $\varphi(N_G(w))$ contains at least three colors, there is a color $a \in \varphi(N_G(w)) \setminus \{\varphi(u), \varphi(z)\}$. Let b and c be two distinct colors in $\varphi(N_G(z_1) \setminus \{z\})$. Recolor z with a color not in $\{\varphi(x), \varphi(u), \varphi(y), \varphi(z_1), b, c, \varphi(w), a\}$ to guarantee two unique colors for each of v, z_1 and w, and color v with a color not in $\{\varphi(x), \alpha, \beta, \varphi(z), \varphi(u), \gamma, \delta\}$ to guarantee two unique colors for each of x and u. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. ☐

Claim 9. *G does not have a 5-cycle with three consecutive 3-vertices.*

Proof. Suppose G has a 5-cycle $F : xyzuv$ with three consecutive 3-vertices y, z, and u. Let y_1 and u_1 be the neighbor of y and u, respectively, that is not on F. By Claim 6, x, v, y_1, u_1 are 4-vertices. See Figure 4(a). By Claims 4 and 7, all vertices in the figure are distinct. For $S = \{x, y, z, u, v\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Color v with a color not in $C_H(v) \cup \{\varphi(u_1)\}$, and color x with a color not in $C_H(x) \cup \{\varphi(v), \varphi(y_1)\}$. Color z with a color not in $C_H(z) \cup \{\varphi(u_1), \varphi(v), \varphi(y_1), \varphi(x)\}$ to guarantee two (actually three) unique colors for each of y and u. Color y with a color not in $C_H(y) \cup \varphi((N_G(x) \cup N_G(z)) \setminus S) \cup \{\varphi(z), \varphi(x)\}$ to guarantee two unique colors for x. Note that u_1 already has three unique colors, so regardless of the color assigned to u, at least two unique colors are guaranteed for u_1. Color u with a color not in $\varphi((N_G(v) \cup N_G(z)) \setminus S) \cup \{\varphi(y), \varphi(z), \varphi(v), \varphi(u_1)\}$ to guarantee two unique colors for each of z and v. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. ☐

Claim 10. *If G has a 5-cycle F incident with three 3-vertices, then every 3-vertex on F has a 4-neighbor that is not on F.*

Proof. Let $F : xyzuv$ be a 5-cycle of G incident with three 3-vertices. By Claim 9, we may assume x, z, v are 3-vertices and y, u are 4-vertices. Let x_1, z_1, and v_1 be the neighbor of x, z, and v, respectively, that is not on F. See Figure 4(b). By Claim 5, x_1 and v_1 are 4-vertices. Note that z_1 may coincide with $w \in \{x_1, v_1\}$ if z_1 is a 4-vertex, but all other vertices in the figure are distinct by Claim 4.

Suppose z_1 is a 3-vertex. For $S = \{x, y, z, u, v, z_1\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Color y with a color not in $C_H(y) \cup \{\varphi(x_1)\}$, and color u with a color not in $C_H(u) \cup \{\varphi(y), \varphi(v_1)\}$. Color z_1 with a color not in $C_H(z_1) \cup \{\varphi(y), \varphi(u)\}$ to guarantee two (actually three) unique colors for z. Color z with a color not in $\varphi(N_G(z_1) \setminus S) \cup \{\varphi(y), \varphi(u), \varphi(z_1)\}$ to guarantee two (actually three) unique colors for z_1. Note that there are three unique colors for each of x_1 and v_1, so regardless of the color assigned to x and v, at least two unique colors are guaranteed for x_1 and v_1. Color x with a color not in $\varphi(N_G(y) \setminus S) \cup \{\varphi(y), \varphi(u), \varphi(v_1), \varphi(x_1)\}$ to guarantee two unique colors for each of y and v. Color v

with a color not in $\varphi(N_G(u) \setminus S) \cup \{\varphi(u), \varphi(v_1), \varphi(x), \varphi(x_1), \varphi(y)\}$ to guarantee two unique colors for each of x and u. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. □

Let us define a 3-vertex on a 4-cycle to be **bad**, and a 3-vertex on no 4-cycles to be **good**.

Claim 11. *If G has a 5-cycle F incident with three 3-vertices, then every 3-vertex on F is a good 3-vertex.*

Proof. Let $F : xyzuv$ be a 5-cycle with three 3-vertices. By Claim 9, we may assume x, z, v are 3-vertices and y, u are 4-vertices. Let x_1, z_1, and v_1 be the neighbor of x, z, and v, respectively, that is not on F. By Claim 10, x_1, z_1, and v_1 are 4-vertices.

Suppose z is a bad 3-vertex. Without loss of generality, assume uzz_1u_1 is a 4-cycle where u_1 is a neighbor of u not on F. See Figure 4(c). Note that z_1 and $w \in \{v_1, x_1\}$ may coincide, but all other vertices in the figure are distinct by Claims 4 and 7. For $S = \{x, y, z, u, v, z_1, u_1\}$, let H be the S-reduced graph. By the minimality of G, H has a 2-PCF 9-coloring φ. Color z_1 with a color not in $C_H(z_1) \cup \varphi(N_G(u_1) \setminus S)$, color y with a color not in $C_H(y) \cup \{\varphi(x_1), \varphi(z_1)\}$, and color u with a color not in $C_H(u) \cup \varphi(N_G(u_1) \setminus S) \cup \{\varphi(v_1), \varphi(y), \varphi(z_1)\}$ to guarantee two unique colors for each of z and u_1. Color u_1 with a color not in $C_H(u_1) \cup \{\varphi(z_1), \varphi(u)\}$, and color z with a color not in $\varphi(N_G(z_1) \setminus S) \cup \{\varphi(z_1), \varphi(u), \varphi(y), \varphi(u_1)\}$. At this point, two unique colors for z_1 are guaranteed if $z_1 \notin \{x_1, v_1\}$; if $z_1 \in \{x_1, v_1\}$, then two unique colors for z_1 will be guaranteed when coloring x and v. Color x with a color not in $\varphi(N_G(y) \setminus S) \cup \{\varphi(y), \varphi(x_1), \varphi(u_1), \varphi(z), \varphi(v_1), \varphi(u)\}$ to guarantee two unique colors for each of y and v, and also z_1 if $x_1 = z_1$. Note that if $x_1 \neq z_1$, then there are three unique colors for x_1, so regardless of the color assigned to x, at least two unique colors are guaranteed for x_1. Finally, color v with a color not in $\{\varphi(x), \varphi(x_1), \varphi(y), \varphi(u), \varphi(u_1), \varphi(z), \varphi(v_1)\}$ to guarantee two unique colors for each of x and u, and also z_1 if $v_1 = z_1$. Note that if $v_1 \neq z_1$, then there are three unique colors for v_1, so regardless of the color assigned to v, at least two unique colors are guaranteed for v_1. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction.

Suppose v or x is a bad 3-vertex. Without loss of generality, assume uvv_1u_1 is a 4-cycle where u_1 is a neighbor of u not on F. See Figure 4(d). Note that z_1 and $w \in \{v_1, x_1\}$ may coincide, but all other vertices in the figure are distinct by Claims 4 and 7. For $S' = \{x, y, z, u, v, u_1, v_1\}$, let H' be the S'-reduced graph. By the minimality of G, H' has a 2-PCF 9-coloring φ'. Color y with a color not in $C_{H'}(y) \cup \{\varphi'(z_1), \varphi'(x_1)\}$, color v_1 with a color not in $C_{H'}(v_1) \cup \varphi'(N_G(u_1) \setminus S')$, and color u with a color not in $C_{H'}(u) \cup \varphi'(N_G(u_1) \setminus S') \cup \{\varphi'(y), \varphi'(z_1), \varphi'(v_1)\}$ to guarantee two unique colors for each of z and u_1. Color u_1 with a color not in $C_{H'}(u_1) \cup \{\varphi'(u), \varphi'(v_1)\}$, and color v with a color not in $\varphi'(N_G(v_1) \setminus S') \cup \{\varphi'(v_1), \varphi'(u_1), \varphi'(u), \varphi'(x_1), \varphi'(y)\}$ to guarantee two unique colors for x. At this point two unique colors for v_1 are guaranteed if $v_1 \neq z_1$; if $v_1 = z_1$, then two unique colors for v_1 will be guaranteed when coloring z. Color x with a color not in $C_{H'}(x) \cup \{\varphi'(v), \varphi'(v_1), \varphi'(u), \varphi'(y)\}$ to guarantee two (actually three) unique colors for v. Color z with a color not in $\varphi'(N_G(y) \setminus S') \cup \{\varphi'(y), \varphi'(z_1), \varphi'(u), \varphi'(u_1), \varphi'(v)\}$ to guarantee two unique colors for each of u and y, and also v_1 if $z_1 = v_1$. Note that if $z_1 \neq v_1$, then there are three unique colors for z_1, so regardless of the color assigned to z, at least two unique colors are guaranteed for z_1. Now, φ' is a 2-PCF 9-coloring of G, which is a contradiction. □

Using the above claims, we now explicitly state and prove the essential reducible configurations. For a face f of a plane graph, let $d(f)$ denote the length of a boundary walk of f. We now fix an embedding of G.

Lemma 3. *In G, the following holds:*

(1) every vertex has degree at least 3,

(2) every cycle has length at least 4,

(3) every 3-vertex is incident with at most one 4-face,

(4) if a 5-face is incident with exactly three 3-vertices, then they are all good 3-vertices.

(5) every 5^+-face f is incident with at most $\left\lfloor \frac{3d(f)}{4} \right\rfloor$ 3-vertices.

(6) G has no cut-vertex.

Proof. By Claim 1, every vertex has degree at least 3 so (1) holds. By Claim 4, every cycle has length at least 4 so (2) holds. By Claim 8, every 3-vertex is incident with at most one 4-face so (3) holds. By Claim 11, if a 5-face is incident with exactly three 3-vertices, then they are all good 3-vertices, hence (4) holds. By Claim 6, every 5^+-face f does not have four consecutive 3-vertices, so f is incident with at most $\left\lfloor \frac{3d(f)}{4} \right\rfloor$ 3-vertices, hence (5) holds.

Suppose G has a cut-vertex v. Let H be a component of $G-v$, $H_1 = G[V(H) \cup \{v\}]$, and $H_2 = G[V(G) \setminus V(H)]$. By the minimality of G, for each $i \in \{1, 2\}$, H_i has a 2-PCF 9-coloring ψ^i. Since v is a 4^--vertex, we can permute the colors of the vertices of H_1 under ψ^1 so that $\{\psi^1(u) \mid u \in N_G(v)\} \cap \{\psi^2(u) \mid u \in N_G(v)\} = \emptyset$ and $\psi^1(v) = \psi^2(v)$. Define a function φ on $V(G)$ such that for each vertex v, if $v \in V(H_i)$ then $\varphi(v) = \psi^i(v)$. Now, φ is a 2-PCF 9-coloring of G, which is a contradiction. □

We use the well-known discharging method to finish off the proof. See [9] for a nice expository survey of the method. Let $F(G)$ denote the set of faces of G. For each $z \in V(G) \cup F(G)$, let the *initial charge* $\mu(z)$ of z be $d(z)-4$. By Euler's formula the sum of all initial charge is negative: $\sum_{v \in V(G)}(d(v)-4) + \sum_{f \in F(G)}(d(f)-4) = 2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)| = -8$.

Here are the discharging rules:

[R1] Every 5-face sends charge 1/3 to each incident good 3-vertex.

[R2] Every 5-face sends charge 1/2 to each incident bad 3-vertex.

[R3] Every 6^+-face sends charge 1/2 to each incident 3-vertex.

We recount the charge after applying the discharging rule. We will obtain that the final charge is non-negative for each vertex and face, to conclude that the sum of the final charge is non-negative. This is a contradiction since the initial charge sum is negative and the discharging rule preserved the total charge sum. We conclude that a counterexample could not have existed in the first place.

Only 3-vertices have negative initial charge since G has no 2^--vertices by Lemma 3 (1). Note that G has no 3-faces by Lemma 3 (2). Also, each face is incident with a vertex at most once by Lemma 3 (6).

Each good 3-vertex v is incident with three 5^+-faces, each of which sends charge at least $\frac{1}{3}$ to v by **[R1]** and **[R3]**, so the final charge of v is at least $-1 + \frac{1}{3} \cdot 3 = 0$. Each bad 3-vertex v is incident with at least two 5^+-faces by Lemma 3 (3), so v receives charge $\frac{1}{2}$ at least twice by **[R2]** and **[R3]**, so the final charge of v is at least $-1 + \frac{1}{2} \cdot 2 = 0$. Each 4-vertex and 4-face is not involved in the discharging process, so the final charge is the initial charge, which is 0. If f is a 5-face incident with a bad 3-vertex, then f is incident with at most one other 3-vertex by Lemma 3 (4) and (5), so the final charge of f is at least $1 - \frac{1}{2} \cdot 2 = 0$ by **[R1]** and **[R2]**. If f is a 5-face not incident with a bad 3-vertex, then f is incident with at most three good 3-vertices by Lemma 3 (5), so the final charge of f is at least $1 - \frac{1}{3} \cdot 3 = 0$ by **[R1]**. Each 6^+-face f has at most $\left\lfloor \frac{3d(f)}{4} \right\rfloor$ incident 3-vertices by Lemma 3 (5). Thus, the final charge of f is at least $d(f) - 4 - \left\lfloor \frac{3d(f)}{4} \right\rfloor \frac{1}{2}$ by **[R3]**, which is non-negative since $d(f) \geq 6$.

3 Further discussion

As mentioned in the introduction, Wegner's Planar Graph Conjecture is true for planar graphs with maximum degree at most 3. Recall that for a graph G (not necessarily planar) with maximum degree 3, properly coloring G^2 is equivalent to a 2-PCF coloring of G. One could also ask what the 1-PCF chromatic number is for planar graphs with maximum degree 3, yet this is already known to be at most 4 by a result of Liu and Yu [17]. Their result actually applies to all graphs (not necessarily planar) of maximum degree 3; see also the discussion in the last section of [4]. Caro, Petruševski, and Škrekovski [4] conjectured that every graph G that is not the 5-cycle is 1-PCF $(\Delta(G)+1)$-colorable; this conjecture is known to be true for only $\Delta(G) \leq 3$.

For planar graphs with maximum degree 4, Wegner's Planar Graph Conjecture is unresolved, so we proved a result in the flavor of 2-PCF colorings. One could also ask what the maximum 1-PCF chromatic number is for a planar graph with maximum degree 4. By the conjecture mentioned in the previous paragraph, one guess is that the bound is at most 5.

We also remark that in [10], Fabrici et al. constructed a planar graph that is not 1-PCF 5-colorable, conjectured that all planar graphs are 1-PCF 6-colorable, and proved that all planar graphs are 1-PCF 8-colorable.

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