Properties and Comparisons of Various Graphs and Their Codes

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***Abstract*—It is established in literature that finding stabilizer quantum error correcting codes (QECCS) is the same as finding self-dual additive codes over the finite field $\mathbb{F}_4$ under the Hermitian trace inner product. Additionally, every self-dual additive code can be represented by a graph adjacency matrix. Many self-dual additive codes are constructed from circulant graphs. We introduce new graph code constructions: the Toeplitz Graph, the Multidimensional Toeplitz Graph (MDT), and the Generalized Toeplitz (GT) graph constructions. We consider some of the properties of the Toeplitz and Multidimensional Toeplitz Graphs and compare the Circulant and GT code constructions.**

***Index Terms*—Circulant Graphs, Toeplitz Graphs, Circulant Codes, Quantum Codes, Generalized Toeplitz Graphs**

I. Introduction

In a groundbreaking paper, Calderbank et al. established that 0-dimensional quantum stabilizer (qubit) codes can be represented by special types of additive subgroups of $GF(4)^n$ called self-dual additive codes [1]. Self-dual additive codes can be classified by their length, whether they contain a codeword of odd length (Type I) or not (Type II), and their minimum distance (a quantity proportional to the number of errors the code can correct). Self-dual additive codes can be generated by graph[2], though many such current codes in literature are constructed from circulant graphs and their variations.

In this paper, we define a new type of graph, denoted the multidimensional Toeplitz (MDT) graph, that is a generalization of the classical Toeplitz graph. We demonstrate different properties of this graph and its relation to circulant and Toeplitz graphs. We then use a subcase of the generalized Toeplitz (GT) graph, first defined by Sheng Bau, to generate 0-dimensional qubit codes. We show that generalized Toeplitz graphs are capable of creating new 0-dimensional qubit codes that are of optimal length and are distinct from those created by Grassl [3]. Additionally, we create codes using the GT graph that are of a type that cannot be created using circulant graphs [4].

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II. Background

A. Graph Theory

A *graph* $\Gamma = (V, E)$ is a ordered pair consisting of a *vertex set* V and an *edge set* E. Common notation to denote the vertex and edge sets, respectively, are $V(\Gamma)$ and $E(\Gamma)$. Given that Γ is undirected, for vertices $v, w \in V$, v is a *neighbor* of w if $\{v, w\} \in E$; furthermore, v and w are then said to be *adjacent*. The number of neighbors some $v \in V(\Gamma)$ has is called its *degree*, denoted $deg(v)$. The graph Γ is *regular* if for all $v, w \in V(\Gamma)$, $deg(v) = deg(w)$. This constant is known as the graph's *valency*. The graph Γ is the *complete graph on n vertices*, denoted K_n, if for all distinct $v, w \in V(\Gamma)$, $\{v, w\} \in E(\Gamma)$, where $|V(\Gamma)| = n$.

The *complement* of Γ, $\overline{\Gamma}$, is defined such that

$$V(\overline{\Gamma}) = V(\Gamma)$$

$$E(\overline{\Gamma}) \cup E(\Gamma) = E(K_{|V(\Gamma)|}).$$

In other words, we consider simple graphs, so taking the complement of a graph doesn't create self-loops or multiple edges.

Assuming Γ is *simple* (that is, there is at most a single edge between any distinct vertices and vertices are never self-adjacent), the *adjacency matrix* of Γ, denoted in the paper by $Adj(\Gamma)$, is an $|V(\Gamma)| \times |V(\Gamma)|$ matrix with entries:

$$Adj(G)_{ij} = \begin{cases} 0 & : \{v_j, v_i\} \notin E(\Gamma) \\ 1 & : \{v_j, v_i\} \in E(\Gamma) \end{cases}$$

Since we work only with simple, undirected graphs, any and all of our adjacency matrices are symmetric and have zeroes along the entire diagonal, indicating that there are no self-loops and that vertex i is linked to vertex j if and only if j is linked to i. Two graphs Γ_G and Γ_H are *isomorphic* if and only if there exists a bijection $\varphi : V(\Gamma_G) \mapsto V(\Gamma_H)$ that preserves neighbors; in this case, for any $v_i, v_j \in V(G)$, $(v_i, v_j) \in E(\Gamma_G)$ if and only if $(\varphi(v_i), \varphi(v_j)) \in E(\Gamma_H)$.

B. Coding Theory

Let $\mathbb{F}_q$ be the finite field of order q. A *code* $\mathcal{C}$ of length n over $\mathbb{F}_q$ is an *additive code* if $\mathcal{C}$ is a finite additive subgroup of $\mathbb{F}_q^n$.

We specifically consider additive codes over $\mathbb{F}_4$. Let $\mathbb{F}_4 := \{0, 1, \omega, \overline{\omega}\}$ such that $\overline{\omega} = \omega^2 = \omega + 1$. Each additive code has a *generator matrix* M whose rows additively span the entire code. The **conjugates** of elements of F_4 are given by: $\overline{0} = 0$, $\overline{1} = 1$, $\overline{\overline{\omega}} = \omega$. The *trace map* $Tr : \mathbb{F}_4 \longmapsto \mathbb{F}_2$ is defined as:

$$Tr(x) = x + x^2.$$

From the trace map, the *Hermitian trace inner product*, denoted by $*$, between $\mathbf{x}, \mathbf{y} \in \mathbb{F}_4^n$ is defined as:

$$\mathbf{x} * \mathbf{y} := \sum_{i=1}^{n} Tr(x_i \overline{y_i}) = \sum_{i=1}^{n} (x_i^2 y_i + x_i y_i^2) \mod 2$$

Where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_4^n$ For some code C, we define its *dual-code* C^* under the Hermitian trace inner product as:

$$C^* := \{\mathbf{x} \in \mathbb{F}_4^n : \forall \mathbf{c} \in C, \mathbf{c} * \mathbf{x} = 0\}$$

C is called *self-orthogonal* if $C \subseteq C^*$ and *self-dual* if $C = C^*$.

On $\mathbb{F}_4^n$, we impose a metric known as the *Hamming distance* between $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}_4^n$, denoted $d(\mathbf{x}, \mathbf{y})$, which is defined as:

$$d(\mathbf{x}, \mathbf{y}) = |\{i : x_i \neq y_i\}|,$$

That is, it's the count of the components where $\mathbf{x}$ and $\mathbf{y}$ differ. The *weight* of some code word $\mathbf{x}$, denoted $w(\mathbf{x})$, is then defined as:

$$w(\mathbf{x}) = d(\mathbf{x}, \mathbf{0}),$$

or the number of nonzero elements in a code word. To each code $\mathcal{C}$, the *minimum distance*, denoted $d_{min}(C)$, the minimum weight of any given code word in the code. Since the Hamming distance is invariant under translation of vectors, the minimum distance between any two vectors is exactly equal to the minimum weight of any single vector. Thus, since Hamming distance measures "distinctness" of code words, a main goal in coding theory is the production of codes with maximum minimum distances. The weights of the code words defines the type of code: a *Type II* code is one where all code words in the code have even weight, while a *Type I* code is a code where at least one code word is of odd weight. Type II codes are the only codes that are capable of also being linear codes. Type I and Type II arise naturally in the study of codes due to the following well known upper bounds on their minimum distances:

$$d_1 \leq \begin{cases} 2\left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{if } n \equiv 0 \mod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 3, & \text{if } n \equiv 5 \mod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor, & \text{otherwise} \end{cases}$$

$$d_2 \leq 2\left\lfloor \frac{n}{6} \right\rfloor + 2$$

Additionally, Danielsen and Parker showed that any self-dual additive code can be generated by $A + \omega I$, where A is the adjacency matrix of a graph, and I is the identity matrix.

III. Graphs

Self-dual additive codes are frequently generated by circulant graphs [3], [4]. Seneviratne et al. considered a multidimensional generalization of the circulant graph and their codes [5]. In this section, we outline their generalization of circulant graphs. Additionally, we introduce the Toeplitz graph, our analogous multidimensional generalization of the Toeplitz graph and some of their properties. Finally, we introduce the Generalized Toeplitz graph, from which we generate self-dual additive codes.

A. Circulant Graphs

A common class of graphs used to generate self-dual additive codes is that of circulant graphs, so although not explicitly used, a few definitions have been included for the reader.

One definition of a circulant graph is a graph whose adjacency matrix is *circulant*; That is, it's a matrix where each row is the preceding row shifted over to the right by 1 element:

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_n & c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ c_1 & \cdots & c_{n-1} & c_n & c_0 \end{bmatrix}$$

Alternatively, we may equivalently define a circulant graph as follows. Let $\mathbb{Z}_n$ denote the ring of integers modulo n. Let $S \subset \mathbb{Z}_n$ such that $0 \notin S$ and $S = -S$. Then the *circulant graph* Γ has $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(\Gamma) = \{v_i v_j : (v_i - v_j) \mod n \in S\}$.

Since circulant graphs are Cayley graphs on cyclic groups and all Cayley graphs are generalized Toeplitz graphs [6], all circulant graphs are generalized Toeplitz graphs.

B. MDC Graphs

For the sake of compactness, we employ the following notation. Suppose that $\mathbf{N} = (n_1, \ldots, n_k)$. Then we'll write $Z_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ instead as $\mathbb{Z}_{\mathbf{N}}$. Furthermore, let $\mathbf{t} \in \mathbb{Z}_{\mathbf{N}}$. We'll interpret $\mathbf{t} \mod \mathbf{N}$ as $(t_1 \mod n_1, \ldots, t_k \mod n_k)$.

Definition III.1. Following Leighton [7], let $\mathbf{N} = (n_1, \ldots, n_k)$ and let $S \subset \mathbb{Z}_{\mathbf{N}}$ such that $S = -S$ and $\mathbf{0} \notin S$. A *multidimensional circulant graph* Γ has

$$V(\Gamma) := \mathbb{Z}_{\mathbf{N}}$$
$$E(\Gamma) := \{\mathbf{vw} : (\mathbf{w} - \mathbf{v}) \mod \mathbf{N} \in S\}$$

Seneviratne et al. established some properties of MDC graphs, including properties relating to complements and when an MDC graph is isomorphic to a circulant graph [5].

C. Toeplitz Graphs

Toeplitz matrices are a generalization of circulant matrices where all diagonals parallel to the main diagonal of the matrix-are composed of the same element:

$$\begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} & c_n \\ c_{n+1} & c_0 & c_1 & \dots & c_{n-1} \\ c_{n+2} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{2n-1} & \dots & c_{n+2} & c_{n+1} & c_0 \end{bmatrix}$$

A matrix can be Toeplitz without being circulant, as shown here:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 \\ 2 & 5 & 0 & 1 \end{bmatrix}$$

Just like the circulant graph, a Toeplitz graph is a graph whose adjacency matrix is Toeplitz, and it can be directly defined as follows:

Definition III.2. Let A be a subset of $\mathbb{Z}_n$ and let $T_n(A)$ be a Toeplitz graph. Then

$$V(T_n(A)) = Z_{\mathbf{N}}$$
$$E(T_n(A)) = \{\{v_i, v_j\} : |v_i - v_j| \mod n \in A\}$$

where $A = \{a_1, \dots, a_r\} \subseteq \mathbb{Z}_n \setminus \{0\}$ and $\mathbf{N} = (n_1, \dots, n_k)$.

D. MDT Graphs

We define our Multidimensional Toeplitz Graph (MDT) and explore a few of its properties.

Definition III.3. A **Multidimensional Toeplitz Graph** Γ over vertices $\mathbf{N} = (n_1, \dots, n_k)$ with defining set A, denoted by $MDT_{\mathbf{N}}(A)$, is defined as

$$V(\Gamma) = \mathbb{Z}_{\mathbf{N}}$$
$$E(\Gamma) = \{\mathbf{xy} : (|x_1 - y_1| \mod n_1, \dots, |x_k - y_k| \mod n_k) \in A\}$$

where $A \subseteq \mathbb{Z}_{\mathbf{N}} \setminus \{\mathbf{0}\}$.

If we take $|(v_1, \dots, v_n)|$ to be $(|v_1|, \dots, |v_n|)$, then we can recast the definition in more compact notation as:

$$V(\Gamma) = \{\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}_{\mathbf{N}}\}$$
$$E(\Gamma) = \{\mathbf{xy} \mid |\mathbf{x} - \mathbf{y}| \mod \mathbf{N} \in A\}$$

Definition III.4. A $k \times k$ **Toeplitz nested block matrix** is a Toeplitz matrix of the form

$$\begin{bmatrix} B_1 & B_2 & \dots & B_{k-1} & B_k \\ B_{k+1} & B_1 & B_2 & \dots & B_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{2k-2} & \dots & \dots & B_1 & B_2 \\ B_{2k-1} & B_{2k-2} & \dots & \dots & B_1 \end{bmatrix}$$

where each block $B_1, B_2, \dots, B_{k_0}, \dots, B_{2k_0}$ can be recursively partitioned into blocks $B_1^i, B_2^i, \dots, B_{l_1}^i$ where $1 \le i \le r$ for some r and each matrix is Toeplitz with respect to the block matrices formed from partitioning.

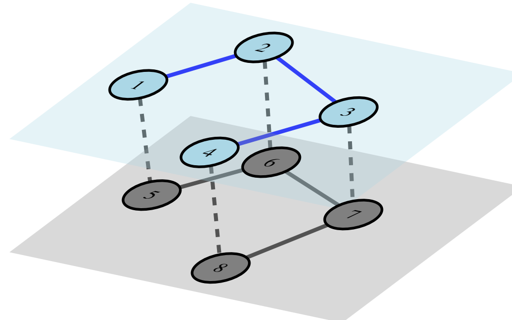

Figure 1. The MDT graph $T([2,4], \{(1,0),(0,1)\})$

To make accessing various blocks of a Toeplitz nested block matrix easier, we use the following notational conventions.

Suppose $\Gamma = MDT_{\mathbf{N}}(A)$ and let $\mathbf{v}_1 = (a_1, \dots, a_k), \mathbf{v}_2 = (b_1, \dots, b_k) \in V(\Gamma)$. Then, the entry of the adjacency matrix M of Γ corresponding to the edge $\mathbf{v_1 v_2}$ will be denoted by $M_{\mathbf{v_2}}^{\mathbf{v_1}} \equiv M_{(b_1,\dots,b_k)}^{(a_1,\dots,a_k)}$.Furthermore, $M_{(x_1,\dots,x_q,\cdots)}^{(y_1,\dots,y_q,\dots)}$ will refer to the entire nested matrix with specified indices in the qth nesting level within the Toeplitz nested matrix M.

Proposition 1. Any MDT has a Toeplitz nested block matrix as its adjacency matrix.

Proof. Let $\Gamma = MDT_{\mathbf{N}}(A)$ with nested adjacency matrix M and $A \subseteq \mathbb{Z}_{\mathbf{N}} \setminus \{\mathbf{0}\}$, $\mathbf{N} = (n_1, \dots, n_k)$ where $n_i \in \mathbb{Z}_i$. Take two arbitrary vertices $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^k), \mathbf{y} = (\mathbf{y}^1, \dots, \mathbf{y}^k) \in \mathbb{Z}_{\mathbf{N}}$ where $\mathbf{xy} \in E(\Gamma)$. Then, by definition of MDT, it follows that $|\mathbf{x} - \mathbf{y}| \mod \mathbf{N} \in A$. We take $\hat{e}_i$ to be the unit vector with all zeroes except for a 1 as its i-th entry. By assumption that $|\mathbf{x} - \mathbf{y}| \mod \mathbf{N} \in A$, it follows then that for any $q \in \{1, \dots, k\}$, $|(\mathbf{x} \pm \hat{e}_q) - (\mathbf{y} \pm \hat{e}_q)| \mod \mathbf{N} \in A$. Thus, for any nesting level q, we may make the following claim:

$$M_{\mathbf{x}}^{\mathbf{y}} = M_{(\mathbf{y}^1,\dots,\mathbf{y}^q,\dots,\mathbf{y}^k)}^{(\mathbf{x}^1,\dots,\mathbf{x}^q,\dots,\mathbf{x}^k)} = M_{(\mathbf{y}^1,\dots,\mathbf{y}^q,\dots,\mathbf{y}^k)\pm\hat{e}_q}^{(\mathbf{x}^1,\dots,\mathbf{x}^q,\dots,\mathbf{x}^k)\pm\hat{e}_q}$$

Thus, along any diagonal parallel to the main diagonal at any nesting level q, the entries will always match, making it a nested block Toeplitz matrix. $\square$

Proposition 2. The graph complement of a MDT is itself an MDT.

Proof. Let $G = MDT_{\mathbf{N}}(A)$ be a multidimensional Toeplitz graph with vertices $V(G) \setminus \{\mathbf{0}\}$ and edges $E(G)$. Define $\overline{G}$ to be the complement to G. Let $\overline{A}$ be the set such that $A \cup \overline{A} = V(G) \setminus \{\mathbf{0}\}$ and $A \cap \overline{A}$ is the empty set. Let $\mathbf{x}, \mathbf{y} \in V(G)$ with $\mathbf{x} = (x_1, x_2, \dots, x_N)$ and $\mathbf{y} = (y_1, y_2, \dots, y_N)$, $\mathbf{x} \neq \mathbf{y}$. An edge is present between two vertices of G if and only if $|\mathbf{x} - \mathbf{y}| \in A$; thus, if $|\mathbf{x} - \mathbf{y}| \notin A$, $(\mathbf{x}, \mathbf{y}) \in E(G)$. Therefore, $G = MDT_{\mathbf{N}}(\overline{A})$. $\square$

Example 3. The MDT graph $T([2,4], \{(1,0),(0,1)\})$ and its complement $T([2,4], \{(0,2),(0,3,(1,1),(1,2),(1,3)\})$.

Proposition 4. Each possible edge is unique to a single defining element of $\mathbb{Z}_{\mathbf{N}}$

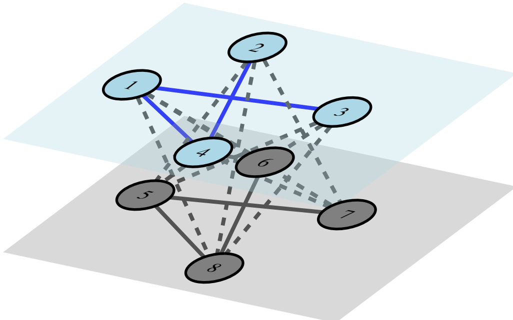

Figure 2. Compliment of $T([2,4],\{(1,0),(0,1)\})$

Proof. Consider the edge $\mathbf{xy} \in \mathbb{Z}_{\mathbf{N}}$ for $\mathbf{N} = (n_1, \ldots, n_k)$. $\mathbf{xy} \in E(\Gamma)$ for $\Gamma = MDT_{\mathbf{N}}(A)$ iff $|\mathbf{x}-\mathbf{y}| \in A$. Since $|\mathbf{x}-\mathbf{y}| = |\mathbf{y}-\mathbf{x}|$, we may define a new vector $\mathbf{Q}$ as follows:

$$|\mathbf{y}-\mathbf{x}| = \mathbf{Q} = \sum_{i\in\{1,\ldots,k\}} \hat{e}_i \begin{cases} \mathbf{x}^i - \mathbf{y}^i & : \mathbf{x}^i \geq \mathbf{y}^i \\ \mathbf{y}^i - \mathbf{x}^i & : \mathbf{x}^i \leq \mathbf{y}^i \end{cases}$$

Thus, we've found a unique $\mathbf{Q} \in \mathbb{Z}_{\mathbf{N}}$ such that $\mathbf{xy}, \mathbf{yx} \in E(\Gamma) \Longleftrightarrow \mathbf{Q} \in A$. □

Proposition 5. For every MDT T, there exists an MDC C such that T is a spanning subgraph of C.

Proof. Define $T = MDT_{\mathbf{N}}(A)$ and $C = MDT_{\mathbf{N}}(S)$ such that $S = V(T)\setminus\{\mathbf{0}\}$. Since $S = \mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_k}\setminus\{(0,\ldots,0)\}$ and for all $0 < i \leq k$, $\mathbb{Z}_{n_i}$ is a group under modular addition, it follows that S contains all of its inverses. Thus, $S = -S$, and C is a multidimensional circulant graph. Since S is the set of all vertices except for the zero vector, it follows that $A \subseteq S$. By Proposition 7, every element of N will create unique edges between vertices, so all edges added by A will also be added by elements in S; thus, $E(T) \subseteq E(C)$ and T is a spanning subgraph of C. Therefore, for any MDT T, there exists an MDC for which T is a spanning subgraph. □

E. Generalized Toeplitz (GT) Graphs

In this section we outline the definition of generalized toeplitz graphs [6] and a few of their properties.

Definition III.5. Let S be the semigroup $(X,\cdot)$, A be the defining set, and V be the vertex set such that $A \subset V \subseteq S$ where V is finite. Then, the **Generalized Toeplitz Graph**, $\Gamma = T_V(A)$, is defined as:

$$V(\Gamma) = V$$

$$E(\Gamma) = \{(v, av) : a \in A, av \in V\}$$

In this work, we will only consider generalized Toeplitz graphs where $1 \notin A$ and $A = A^{-1}$. These two restrictions ensure that $T_V(A)$ is both simple and undirected.

F. Future Directions

Following Seneviratne et al.'s generalization of circulant graphs [5], we outline our generalizations of the Toeplitz graph and some of their properties.

Future directions of this research could involve examining other properties of the MDT graph. Specifically, an area that could be of interest would be connectivity and regularity of the MDT graph. We tentatively found properties of a regular MDT graph, but it would be useful to create a schematic to determine when an MDT graph is regular, which we were not able to do. Additionally, Heuberger assigned conditions under which a Toelitz graph is connected [8] ; it would be interesting to examine how these conditions connect to the MDT graph, and if an MDT graph is connected when Heuberger's conditions hold true for components of the vertices.

IV. GT and Circulant Code Comparison

In this section we list optimal $[[n,0,d]]_2$ codes generated by the GT construction and compare them to existing codes [3]. The following codes were generated from a randomized search using the MAGMAprogramming language.

In Table I, we display codes of a Type unattainable by Saito [4], which were constructed using Circulant graphs. We list their minimum distance, $d_min(C\Gamma_n)$, their GroupID in the SmallGroupDatabase provided by MAGMA, denoted $G_{n,i}$, where n is the order of G, and i is the i-th group of order n in MAGMA's SmallGroupDatabase.

In Table II, we display codes of lengths 50-80 generated by the GT construction, and list some of their properties. In some cases, the GT construction produces codes absent from Grassl's table[3]. In other cases, Grassl's table produces codes GT construction does not produce. Note that the codes, whose automorphism groups are of different orders, are not isomorphic. Table II displays the orders of the automorphism of codes produced from the GT construction. Therefore, the GT codes are distinct from the circulant codes listed in Grassl's table [3].

V. Conclusion

We presented two constructions inspired by Seneviratne et al.'s MDC construction [5] as well as GT construction to generate new self-dual additive codes. Additionally, we presented future directions to explore the properties of such graphs. It would be interesting to continue filling out the table: by constructing longer codes or by examining codes generated from modifications of the GT construction (such as puncturing).

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Table II
PROPERTIES OF THE GRAPHS Γ_n GENERATED BY THE GENERAL TOEPLITZ DIRECT OR BORDERED CONSTRUCTION

Γ_n	$d_{\min}(C(\Gamma_n))$	Construction	Group ID	A	$k(\Gamma_n)$	$d(\Gamma_n)$	$g(\Gamma_n)$	$\omega(\Gamma_n)$	$\|Aut(\Gamma_n)\|$
Γ_{50}	14	Direct	$G_{50,3}$	A_{50}	25	2	3	6	50
Γ_{51}	14	Bordered	$G_{50,3}$	A_{51}	-	2	3	6	50
Γ_{52}	14	Direct	$G_{52,3}$	A_{52}	28	2	3	5	52
Γ_{53}	15	-	-	-	-	-	-	-	-
Γ_{54}	16	-	-	-	-	-	-	-	-
Γ_{55}	14	Direct	$G_{55,1}$	A_{54}	44	2	3	12	55
Γ_{56}	15	Direct	$G_{56,11}$	A_{56}	26	2	3	5	56
Γ_{57}	15	Direct	$G_{57,1}$	A_{57}	28	2	3	6	57
Γ_{58}	16	Direct	$G_{58,1}$	A_{58}	39	2	3	11	58
Γ_{59}	15	Bordered	$G_{58,1}$	A_{59}	-	2	3	13	58
Γ_{60}	16	Direct	$G_{60,1}$	A_{60}	35	2	3	8	60
Γ_{61}	17	-	-	-	-	-	-	-	-
Γ_{62}	18	-	-	-	-	-	-	-	-
Γ_{63}	16	Direct	$G_{63,1}$	A_{63}	48	2	3	12	63
Γ_{64}	16	Direct	$G_{64,71}$	A_{64}	43	2	3	10	64
Γ_{65}	16	Bordered	$G_{64,1}$	A_{65}	-	2	3	14	128
Γ_{66}	16	Direct	$G_{66,1}$	A_{66}	32	2	3	7	66
Γ_{67}	17	Bordered	$G_{66,1}$	A_{67}	-	2	3	10	66
Γ_{68}	18	Direct	$G_{68,3}$	A_{68}	47	2	3	8	68
Γ_{69}	16	Bordered	$G_{68,2}$	A_{69}	-	2	3	15	136
Γ_{70}	18	Direct	$G_{70,1}$	A_{70}	37	2	3	6	70
Γ_{71}	18	-	-	-	-	-	-	-	-
Γ_{72}	18	Direct	$G_{72,16}$	A_{72}	43	2	3	9	72
Γ_{73}	18	-	-	-	-	-	-	-	-
Γ_{74}	18	Direct	$G_{74,1}$	A_{74}	18	2	3	12	74
Γ_{75}	18	Bordered	$G_{74,1}$	A_{74}	-	2	3	17	74
Γ_{76}	18	Direct	$G_{76,3}$	A_{76}	17	3	3	4	76
Γ_{77}	19	-	-	-	-	-	-	-	-
Γ_{78}	19	Direct	$G_{78,3}$	A_{78}	18	3	3	4	78
Γ_{79}	19	-	-	-	-	-	-	-	-

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