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Boundary Stabilization for a Heat-Kelvin-Voigt Unstable Interaction Model, with Control and Partial Observation Localized at the Interface Only

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Abstract

A prototype model for a Fluid–Structure interaction is considered. We aim to stabilize [enhance stability of] the model by having access only to a portion of the state. Toward this goal we shall construct a compensator-based Luenberger design, with the following two goals: (1) reconstruct the original system asymptotically by tracking partial information about the full state, (2) stabilize the original unstable system by feeding an admissible control based on a system which is obtained from the compensator. The ultimate result is boundary control/stabilization of partially observed and originally unstable fluid–structure interaction with restricted information on the current state and without any knowledge of the initial condition. This prevents applicability of known methods in either open-loop or closed loop stabilization/control.

Keywords Luenberger compensator · Stability · Control · Partial observation · Fluid–structure interaction · Interface · Boundary control

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1 Introduction, Qualitative Description of the Results

1.1 Motivation

A 2-or 3-dimensional Fluid (Heat)-Structure Interaction model is considered, which couples a fluid (heat) dynamics with an elastic body (structure), the coupling taking place at the interface between the two media, where the respective dynamics evolve. The original problem is unstable on its natural functional setting. The goal is then to design a feedback control system endowed with the following features. The boundary control operator is constrained to be active only on an arbitrarily small portion of the interface, while the feedback operator cannot have access to the entire state, but only to an accessible part of the state through a partial observation operator (partial observer) acting itself only on another arbitrarily small portion of the interface. Thus, control operator and observation operator are uncollocated, as they act on disjoint arbitrarily small portions of the interface. It is the dynamics (called "compensator") determined by the only accessible partial observation of the state, that feeds the control function through the feedback operator. Justification of the name "compensator" is that this known dynamics asymptotically approximates ("recovers") the unknown inaccessible full state; that is, it compensates for the lack of knowledge of the whole state at a given time, or even of the initial condition, in line with the Luenberger compensator theory (originally introduced in the finite dimensional setting).

1.2 Premise

To provide a first qualitative description of the problem studied in this paper, we first review the Luenberger compensator theory in its original finite dimensional setting [29]. So all operators A, B, C, K below in this section are finite dimensional, with the operator A being unstable. The theory rests on two assumptions: (i) stabilizability and (ii) detectability, possibly with the same exponential decay.

(A.1) (Stabilizabilty) Given the pair $\{A, B\}$, with A unstable, there exists a (feedback) operator F, such that $(A + BF)$ is exponentially stable.

(A.2) (Detectability) Given the pair $\{A, C\}$, with A unstable, there exists an operator K such that $(A - KC)$ is exponentially stable.

Thus, under (A.1) and (A.2), we conclude that there exist constants $M \geq 1$ and $k > 0$, such that

$$\|e^{(A+BF)t}\| + \|e^{(A-KC)t}\| \leq Me^{-kt}, \quad t \geq 0. \tag{1.1}$$

Standard representation of Luenberger's compensator theory is as follows (continuous theory): It is based on the following coupled system:

$$\dot{y} = Ay + Bg, \quad g = Fz = \text{control}, \quad y(0) = y_0, \tag{1.2a}$$
$$\dot{z} = (A + BF - KC)z + K(Cy), \quad z(0) = z_0. \tag{1.2b}$$

where y_0 is the initial state of the original plant [may be unknown] and z_0 is arbitrary. It can be taken $z_0 = 0$.
The basic idea is that the full state y is inaccessible, unknown, beyond any measurement, as is often the case in applications. What we have instead at our disposal is the *partial observation* (Cy), where C is the known observation operator. Examples abound: (i) the actual state within a furnace or (ii) the true distribution of 'noise' within an acoustic chamber are not exactly accessible, and only some information from the boundary may be available in each case. Thus, the (compensator) z-equation (1.2b) is fed, or determined, by only the available partial observation (Cy). Subtracting (1.2b) from (1.2a) with $Bg = BFz$, we obtain after a cancellation of the term BFz:

$$\frac{d}{dt}\left[y(t) - z(t)\right] = (A - KC)\left[y(t) - z(t)\right], \tag{1.3a}$$

$$\left[y(t) - z(t)\right] = e^{(A-KC)t}[y_0 - z_0], \quad t \geq 0. \tag{1.3b}$$

One next invokes the detectability assumption (A.2) for the pair $\{A, C\}$: there exist $M \geq 1$ and k such that $\|e^{(A-KC)}\| \leq Me^{-kt}$, $k > 0, t \geq 0$. Thus, from (1.3b), we finally obtain

$$\|y(t) - z(t)\| = \|e^{(A-KC)t}[y_0 - z_0]\| \leq Me^{-kt}\|y_0 - z_0\|, \quad t \geq 0, \tag{1.4}$$

and the *dynamic compensator* $z(t)$, which is fed only by the known *partial observation* (Cy) of the inaccessible state y, and with possibly $z_0 = 0$, asymptotically approximates such state $y(t)$, at an exponential rate. The ultimate goal is to show that such control $g \equiv Fz$ provides the sought-after uniform stability of the original plant described by (1.2). This is the key of Luenberger's theory [29] in the lumped case where the state of the system is a finite dimensional vector. Non-trivial extensions were subsequently introduced and studied in the case of distributed parameter systems modeled by Partial Differential Equations [10, 11] and with boundary control/boundary observations [7, 15–18, 22, 23], [24, p 495]. In the present paper we shall develop this theory within the context of Fluid-Structure interaction with an interface, where both control and observation are restricted to the first component state on just arbitrarily small, disjoint portion of the interface. This setup presents not only physical interest driven by applications, but also leads to considerable challenges at the mathematical level when dealing with two different environments separated by an interface.

1.3 The Coupled PDE-System Corresponding to (1.2 a-b)

Let $\Omega_f \subset \mathbb{R}^n$, $n = 2, 3$ denote the bounded domain on which the heat component of the coupled PDE system evolves. The boundary of Ω_f consists of two disjoint parts Γ_f and Γ_s. The domain Ω_s [elastic body] immersed within Ω_f is the domain on which the structural component evolves in time. see Fig. 1. On this domain we consider a heat-structure interaction in the variables $y = [w, w_t, u]$ and $z = [z_1, z_2 = z_{1t}, z_3]$, where $u \in \mathbb{R}^n$ denotes the velocity of the fluid, w on $\mathbb{R}^n$ displacement of the body and w_t its velocity. The common interface between the two environments is Γ_s. On the interface Γ_s one imposes matching of the velocities $u = w_t$.

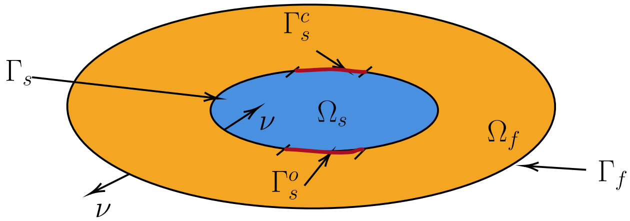

Fig. 1 The physical interaction model

Let Γ_s^c and Γ_s^o (c =control; o =observation) be two arbitrarily small, disjoint, connected, subsets of positive measure, of the interface Γ_s, see Fig. 1. The function χ_D will denote the characteristic function of the domain D. The vector ν denotes the outward unit normal to the domain Ω_f, hence toward Ω_s on Γ_s. We introduce at the outset the PDE-coupled system corresponding to the motivating finite dimensional system (1.2a, b):

$$u_t - \Delta u \equiv 0 \qquad \text{in } (0, T] \times \Omega_f, \tag{1.5a}$$

$$w_{tt} - \Delta w - \Delta w_t \equiv 0 \qquad \text{in } (0, T] \times \Omega_s, \tag{1.5b}$$

$$u|_{\Gamma_f} = 0 \qquad \text{in } (0, T] \times \Gamma_f \tag{1.5c}$$

$$u|_{\Gamma_s} = w_t|_{\Gamma_s} \qquad \text{in } (0, T] \times \Gamma_s, \tag{1.5d}$$

$$\frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} + \chi_{\Gamma_s^c} g \qquad \text{in } (0, T] \times \Gamma_s. \tag{1.5e}$$

The control function g acts on a small portion of Γ_s denoted by Γ_s^c. This will be later modeled as a control action given by a respective operator Bg which enters the system via Neumann type boundary conditions [balance of the stresses]. The system is observed via a displacement w localised on another [disjoint] portion of Γ_s denoted by Γ_s^o. The above will lead to compensator construction in the variable z

$$z_{3t} - \Delta z_3 \equiv 0 \qquad \text{in } (0, T] \times \Omega_f, \tag{1.6a}$$

$$z_{1tt} - \Delta z_1 - \Delta z_{1t} \equiv 0 \qquad \text{in } (0, T] \times \Omega_s, \tag{1.6b}$$

$$z_3|_{\Gamma_f} = 0 \qquad \text{in } (0, T] \times \Gamma_f, \tag{1.6c}$$

$$z_3|_{\Gamma_s} = z_{1t}|_{\Gamma_s} \qquad \text{in } (0, T] \times \Gamma_s, \tag{1.6d}$$

$$\frac{\partial(z_1 + z_{1t})}{\partial \nu} = \frac{\partial z_3}{\partial \nu} + \chi_{\Gamma_s^c} z_1 - \chi_{\Gamma_s^o} z_1 + \chi_{\Gamma_s^o} w \qquad \text{in } (0, T] \times \Gamma_s, \tag{1.6e}$$

with I.C. $y_0 = \{w_0, w_1, u_0\} \in \mathbf{H}$, $z_0 = \{z_{10}, z_{20}, z_{30}\} \in \mathbf{H}$. We stress that while y_0 is an initial condition corresponding to the original system, z_0 can be taken arbitrary-including $z_0 = 0$. Coupling to both systems is exercised at a portion of the interface Γ_s: $\chi_{\Gamma_s^c} z_1$ in (1.5e) and $\chi_{\Gamma_s^o} w$ in (1.6e). It will be documented below-see (2.3), (2.4),(4.1), (4.2)-that the abstract model for the coupled PDE-system (1.5 a–e), (1.6 a–e) is given by,

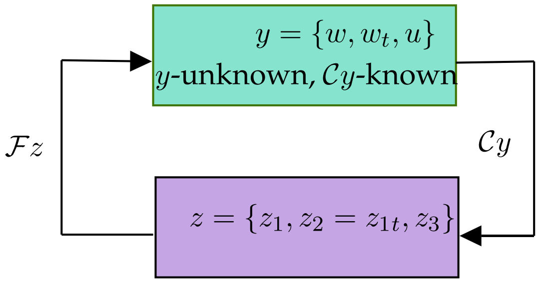

Fig. 2 Illustration of problem (1.7 a-b)

$$\dot{y} = \mathcal{A}y + \mathcal{B}\mathcal{F}z, \tag{1.7a}$$
$$\dot{z} = (\mathcal{A} + \mathcal{B}\mathcal{F} - \mathcal{K}\mathcal{C})\, z + \mathcal{K}(\mathcal{C}y), \tag{1.7b}$$

for operators $\mathcal{A}, \mathcal{B}, \mathcal{F}, \mathcal{K}, \mathcal{C}$ to be identified below ($[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ given by the physical problem, while $[\mathcal{F}, \mathcal{K}]$ to be designed accordingly); or

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \mathscr{A}\begin{bmatrix} y \\ z \end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix} \mathcal{A} & \mathcal{B}\mathcal{F} \\ \mathcal{K}\mathcal{C} & \mathcal{A} + \mathcal{B}\mathcal{F} - \mathcal{K}\mathcal{C} \end{bmatrix}; \tag{1.8a}$$
$$\mathcal{H} = \mathbf{H} \times \mathbf{H} \supset \mathcal{D}(\mathscr{A}) \to \mathcal{H}, \tag{1.8b}$$

$\mathcal{D}(\mathscr{A}) = \{(\eta, \zeta) \in \mathcal{H}, \mathcal{A}\eta + \mathcal{B}\mathcal{F}\zeta \in \mathbf{H};\ \mathcal{K}\mathcal{C}\eta + (\mathcal{A} + \mathcal{B}\mathcal{F} - \mathcal{K}\mathcal{C})\zeta \in \mathbf{H}\}$. With the suitable feedback operator $\mathcal{F}$, the control g given by $g = \mathcal{F}z$ will be shown to provide a stabilizing effect on the original system described in the variable y in the natural finite energy space for the model:

$$\mathbf{H} = H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f), \tag{1.9}$$

say for $y \equiv \{w, w_t, u\}$ noted in (1.5). (For simplicity of notation, component spaces are not bold faced. So we write, say $L^2(\Omega_s)$ rather than $\mathbf{L}^2(\Omega_s)$ or $\left[L^2(\Omega_s)\right]^n$). See the symbolic Fig. 2.

In the analysis, it will be critical to replace the space $\mathbf{H}$ in (1.9) with the norm equivalent space:

$$\mathbf{H}_e = H_e^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f), \tag{1.10}$$

where the inner product, and square of the norm of $v_1 \in H_e^1(\Omega_s)$, are defined by,

$$(v_1, \tilde{v_1})_{H_e^1(\Omega_s)} = (\nabla v_1, \nabla \tilde{v_1})_{\Omega_s} + (v_1, \tilde{v_1})_{\widetilde{\Gamma_s}}\,, \tag{1.11a}$$
$$\|v_1\|^2_{H_e^1(\Omega_s)} \equiv \|\nabla v_1\|^2_{L^2(\Omega_s)} + \|v_1|_{\widetilde{\Gamma_s}}\|^2_{L^2(\widetilde{\Gamma_s})}, \tag{1.11b}$$

so that $\|v_1\|_{H_e^1(\Omega_s)} = 0 \implies v_1 \equiv 0$ in Ω_s. In (1.11a), $\widetilde{\Gamma_s}$ is a fixed portion of Γ_s. Norm-equivalence between $\mathbf{H}$ and $\mathbf{H}_e$ is justified in (4.16) below. The key reason for introducing $\mathbf{H}_e$ is explained by Lemma 4.1 (ii) followed by Proposition 4.2, (4.20) below.

2 Main Results

2.1 Formulation and Comments

The main result of the present paper is the following one. (We recall that the operator $\mathcal{A}$ is unstable on $\mathbf{H}$: $\lambda = 0$ is an eigenvalue of $\mathcal{A}$ on $\mathbf{H}$ with eigenvector $e = [1, 0, 0]$, Remark 3.2.)

Theorem 2.1 *The coupled PDE-system (1.5 a–e), (1.6 a–e) defines the operator $\mathscr{A}$ as in (1.8 a-b), which is the generator of a s.c semigroup $e^{\mathscr{A}t}$ on the finite energy space $\mathcal{H} = \mathbf{H} \times \mathbf{H}$, $\mathbf{H}$ in (1.9):*

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = e^{\mathscr{A}t} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in C([0, T]; \mathcal{H}). \tag{2.1}$$

Moreover, $e^{\mathscr{A}t}$ is analytic and uniformly stable on $\mathcal{H}$: there exist constants $M \geq 1$, $k > 0$, such that,

$$\|e^{\mathscr{A}t}\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-kt}, \quad thus \ \| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \|_{\mathcal{H}} \leq Me^{-kt} \| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \|_{\mathcal{H}}. \tag{2.2}$$

Remark 2.1

- We note that the compensator system can be solved with an arbitrary initial condition, in particular $z_0 = 0$. The only information needed is the value of the displacement w of the body on a portion of the boundary Γ_s^o which is assumed to be the observed quantity. The control acting on the original plant $g = \mathcal{F}z = z_1$ feeds the information from the compensator into the system. It acts also on a an arbitrary small portion of the interface Γ_s^c which may be far away from the observed part Γ_s^o. The final result provides exponential stabilization of the original plant on a finite energy space.
- In [39] a similar model has been considered under the assumption of suitable stability. The goal of that work was to track asymptotically the solution via restricted observation fed to a compensator. However, this work did not involve control part or stabilization, where the latter is the main issue of the present contribution.

Remark 2.2

- As already mentioned, a compensator design has been considered for a variety of PDE models also with boundary observation or control. The interest of the present model is that this is an interface model with interaction of two different environments. As we shall see, this provides for several challenges at the level of mathematical [PDE] analysis, where unboundedness of traces occurring at the interface is the main game changer. A more comprehensive model will be by replacing "heat equation" by the dynamic Stokes equation -which also accounts for the presence of pressure in the system [13, 34]. This can be done by using a well known by now representation of the pressure in terms of a nonlocal operator collecting the information from the interface [1–3, 37]. Similarly, more general forms of elliptic operators, where Δ is replaced by strain–stress tensors could be easily considered without changing the mathematical technicalities of the present analysis.

- Other heat-plate interaction models are given in [38, 40].
- The problem of stabilization subject to partial information on the state has acquired a lot of recent attention in the literature and it is often referred to as "data assimilations" or "determining modes" techniques [4, 6, 12, 30]. Here the principle is somewhat different where asymptotic tracking of the trajectory is based on collecting information on "finitely" many modes determining the system. The above method reconstructs asymptotically trajectories and remarkably may even provide information on additional regularity of solutions [4]. However, there is no control aspect in the problem.
- It would be interesting to consider effects of nonlinearity in the model. In fact, this has been done within the context of data assimilation [4, 6]. Within the present framework, this could be done as in [5] where the control mechanism $\mathcal{F}z$ is fed to the original plant and provides a stabilizing effect also in the presence of nonlinear effects for suitably small data.
- The stabilizing feedback operator $\mathcal{F}$ on the compensated system can be also selected by using a Riccati operator $\mathcal{F} = -\mathcal{B}^*\mathcal{P}$ where $P \in L(\mathbf{H})$ satisfies the Algebraic Riccati Equation of the form $(\mathcal{A}^* Px, y)_{\mathbf{H}} + (P\mathcal{A}x, y)_{\mathbf{H}} + (x, y)_{\mathbf{H}} = (\mathcal{B}^* Px, \mathcal{B}^* Py)_{L_2(\Gamma_s^c)}$ associated with the cost functional $J(g, y(g)) = \int_0^\infty [||y||^2 + ||g||^2]dt$ and the trajectory $y(g)$ given by (1.5). The analyticity of the "y " dynamics, along with the stabilizability condition allow to apply Riccati Theory with unbounded controls, in order to assert [24] that $P \in L(\mathbf{H})$ exists, it is unique, $\mathcal{B}^* P \in L(\mathbf{H}; L_2(\Gamma_s))$ and $\mathcal{F}(y) = -\mathcal{B}^* P(y)$ is uniformly stabilizing the original dynamics. Appealing now to the result of our main Theorem, we can claim that the partially observed [compensated] feedback $g = -B^* Pz$ provides "almost" optimal performance asymptotically. The latter follows from Bellman's optimality principle. This construct provides a minimal cost control feedback, at the price of using the entire state z given by the compensator [16].
- Another aspect of the problem for consideration is the numerical approximation of the compensator system in line with past work in [16, 18]. For discretization of a fluid structure interactions see [20, 21].

Critical ingredients in the proof of Theorem 2.1 are the following results, which are also of independent interest on their own.

Theorem 2.2 *The (feedback) operator $\mathcal{A}_F = \mathcal{A} + \mathcal{B}\mathcal{F}$ (to be defined in (4.3),(4.5)) is maximally dissipative on the space $\mathbf{H}_e$ (e =equivalent) norm-equivalent to the space $\mathbf{H}$, and hence generates a s.c. contraction semigroup $e^{\mathcal{A}_F t}$ on $\mathbf{H}_e$, which moreover is analytic and uniformly stable on $\mathbf{H}_e$, and hence analytic and uniformly stable on $\mathbf{H}$ as well. The PDE-version of the abstract (feedback) system $\dot{y} = \mathcal{A}_F y,\ y = [w, w_t, u]$ is given by (4.4 a–e):*

$$u_t - \Delta u \equiv 0 \qquad in\ (0, T] \times \Omega_f, \tag{2.3a}$$
$$w_{tt} - \Delta w - \Delta w_t \equiv 0 \qquad in\ (0, T] \times \Omega_s, \tag{2.3b}$$
$$u|_{\Gamma_f} = 0 \qquad in\ (0, T] \times \Gamma_f, \tag{2.3c}$$
$$u|_{\Gamma_s} = w_t|_{\Gamma_s} \qquad in\ (0, T] \times \Gamma_s, \tag{2.3d}$$

$$\frac{\partial(w+w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} + \chi_{\Gamma_s^c} w \qquad \text{in } (0,T]\times\Gamma_s. \tag{2.3e}$$

Theorem 2.3 *Similarly, the operator $(\mathcal{A}-\mathcal{KC})$ (to be defined below in (4.6), (4.8)) is maximally dissipative on the space $\mathbf{H}_e$ norm-equivalent to the space $\mathbf{H}$, and hence generates a s.c. contraction semigroup on $\mathbf{H}_e$, which moreover is analytic and uniformly stable on $\mathbf{H}_e$, and hence analytic and uniformly stable on $\mathbf{H}$ as well. The PDE-version of the abstract system, $\dot{y} = (\mathcal{A}-\mathcal{KC})y,\ y = [w, w_t, u]$ is given by (4.7 a-e): eqivalently: (2.3a), (2.3b), (2.3c), (2.3d) with (2.3e) replaced by*

$$\frac{\partial(w+w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} - \chi_{\Gamma_s^o} w \text{ in } (0,T]\times\Gamma_s. \tag{2.4}$$

Given the pair $\{\mathcal{A},\mathcal{B}\}$ from the physical problem, Theorem 2.2 designs a suitable feedback stabilizing operator $\mathcal{F}$ to provide justification of the stabilizability assumption (A.1). Similarly, given the pair $\{\mathcal{A},\mathcal{C}\}$ from the physical problem, Theorem 2.3 designs a suitable detectable operator $\mathcal{K}$ to provide justification of the detectability assumption (A.2).

2.2 Strategy for the Proofs

Orientation and qualitative description of the present paper We next provide, at first qualitatively, a description of the results of the present paper, following Luenberger's script.

Uncontrolled problem The operator $\mathcal{A}$ in Section (3.1a–e). The uncontrolled problem is the Heat-Kelvin-Voight interaction model (3.1a–e), in the variable $\{w, w_t\}$ for the structure, and u for the heat component. It is modeled by the abstract operator $\mathcal{A}$ in (3.5), (3.6) on the natural function space $\mathbf{H} = H^1(\Omega_s)\times L^2(\Omega_s)\times L^2(\Omega_f)$, for $\{w, w_t, u\}$, as noted in (1.9) or (3.2). In this setting, the operator $\mathcal{A}$ is *unstable*: $\lambda = 0$ is an simple eigenvalue with corresponding eigenvector $e = [1, 0, 0]$. There is therefore the need to stabilize it.

Controlled problem Section 3.2. The pair $\{\mathcal{A},\mathcal{B}\}$ with controlled operator $\mathcal{B}$ applied at the boundary (interface). In order to stabilize by feedback control the original uncontrolled system (3.1a-e)[i.e. the operator $\mathcal{A}$ in (3.5), (3.6)], we need first to apply to it a control operator. This is the operator $\mathcal{B}$ in (3.8), (3.9), whose boundary action on the original system is given by Eq. (3.9), as acting on the portion Γ_s^c of the interface Γ_s. This controlled equation on Γ_s in (3.7e) replaces the uncontrolled version (3.1e) which is part of the operator $\mathcal{A}$. With the pair $\{\mathcal{A},\mathcal{B}\}$ in place, one next addresses the stabilizability condition to satisfy assumption (A.1), and hence counteract the instability of $\mathcal{A}$. Section 4 is dedicated to this issue. It defines a feedback operator $\mathcal{F}$ in (4.1), which is active only on the trace of the first coordinate on an arbitrary small portion Γ_s^c (of positive measure) of the full interface Γ_s. The corresponding PDE-version is given by problem (2.3 a-e) or (4.4a–e). Its abstract formulation, i.e. the abstract feedback operator $\mathcal{A}_F = \mathcal{A}+\mathcal{BF}$ is given in (4.3). The key issue is to establish the validity of assumption (A.1): that is, that the feedback problem $\mathcal{A}_F$ given by (4.4a–e) in PDE-form is well-posed and uniformly stable. This is a delicate issue that will be described below in Sect. 4.1, after treating the selection of the partial observation operator $\mathcal{C}$.

The partial observation operator $\mathcal{C}$ *and the detectability condition of the pair* $\{\mathcal{A}, \mathcal{C}\}$ *Section* 4. In line with the core of the Luenberger theory, the full state y is not available, only the observation $(\mathcal{C}y)$ is accessible through a known partial observation operator $\mathcal{C}$. This is defined in (3.23) as an operator that, of the whole state $[w, w_t, u]$ in $\mathbf{H}$, picks up only the trace of the first component on an arbitrary small portion Γ_s^o (of positive measure) of the interface Γ_s. It is desirable to allow $\Gamma_s^c \cap \Gamma_s^o = \emptyset =$ empty set (uncollocated control and observation actions).

The detectable operator $\mathcal{K}$ is selected in (4.1) [compare with $\mathcal{B}$ in (3.9)]. The result of the design for $\mathcal{BF}$ and $-\mathcal{KC}$ is that showing uniform stabilization of $\mathcal{A} + \mathcal{BF}$ (problem (4.4 a-e)) or of $\mathcal{A} - \mathcal{KC}$ (problem (4.7 a-e)) amounts essentially to the same task, save for Γ_s^c replacing Γ_s^o.

Well-posedness in the sense of analytic semigroup well-posedness, and uniform stability of $\mathcal{A}+\mathcal{BF}$ or of $\mathcal{A}-\mathcal{KC} = \mathcal{A}+\mathcal{BC}$: *Section* 4.1–4.2 The common problem of well-posedness (analyticity) and uniform stability of $\mathcal{A}+\mathcal{BF}$ and $\mathcal{A}-\mathcal{KC} = \mathcal{A}+\mathcal{BC}$ is taken up in the Sect. 4.1 and 4.2, respectively. Here a generic portion $\widetilde{\Gamma}_s$ of the interface Γ_s is selected, to stand for either Γ_s^c or Γ_s^o in the two cases. The PDE-problem with $\widetilde{\Gamma}_s$ given by (4.9a-e) and its evolution abstract operator is denoted by $\mathbb{A}$ in (4.11),(4.12), to serve for $\mathcal{A}_F = \mathcal{A} + \mathcal{BF}$ ($\widetilde{\Gamma}_s = \Gamma_s^c$) or for $\mathcal{A} - \mathcal{KC}$, ($\widetilde{\Gamma}_s = \Gamma_s^o$).

The first consequence is positive: now $0 \in \rho(\mathbb{A})$, the resolvent set of $\mathbb{A}$, say on the original space $\mathbf{H}$, thus overcoming the original problem $0 \in \sigma_p(\mathcal{A})$ lamented for the uncontrolled problem. However, at the outset, one faces a negative feature: it turns out that the key operator $\mathbb{A}$ (image of $\mathcal{A} + \mathcal{BF}$ and $\mathcal{A} - \mathcal{KC}$) is *not* dissipative on the original state space $\mathbf{H}$. To overcome this difficulty, we introduce a new state space $\mathbf{H}_e$ (e=equivalent) which is norm-equivalent to $\mathbf{H}$. More precisely, the first component space $H^1(\Omega_s)$ with squared norm $\|\nabla v_1\|^2_{L^2(\Omega_s)} + \|v_1\|^2_{L^2(\Omega_s)}$, is replaced by the equivalent space $H^1_e(\Omega_s)$ with squared norm $\|\nabla v_1\|^2_{L^2(\Omega_s)} + \|v_1|_{\widetilde{\Gamma}_s}\|^2_{L^2(\widetilde{\Gamma}_s)}$ as noted in (1.11 a-b). It then turns out that in the new equivalent space $\mathbf{H}_e$, $\mathbb{A}$ is dissipative, hence maximal dissipative (as $0 \in \rho(\mathbb{A})$) and thus generates a s.c. contraction semigroup $e^{\mathbb{A}t}$ on $\mathbf{H}_e$ (hence a s.c. semigroup on $\mathbf{H}$ as well). This is Proposition 4.3. Similar results holds for the adjoint $\mathbb{A}^*$ of $\mathbb{A}$ on $\mathbf{H}_e$ defined in Sect. 4.1.2. With this preliminary contraction semigroup result at hand, Sect. 4.2, then uses delicate energy estimates to show the bound (4.43) in the resolvant $R(i\omega, \mathbb{A})$ in the imaginary axis $i\omega$, $\omega \in \mathbb{R}$ for $|\omega| \geq \omega_0 > 0$ and appeal to the abstract result in [24, p 334] to conclude that $e^{\mathbb{A}t}$ is analytic on $\mathbf{H}_e$, hence on $\mathbf{H}$ as well.

Finally, in Sect. 4.3, the resolvent bound (4.43) combined with $0 \in \rho(\mathbb{A})$ allows one to invoke Pruss's result [33] and conclude that $e^{\mathbb{A}t}$ is moreover, uniformly stable on $\mathbf{H}_e$, hence on $\mathbf{H}$ as well. As $\mathbb{A}$ stands for both $(\mathcal{A}+\mathcal{BF})$ and $(\mathcal{A}-\mathcal{KC})$, we then obtain the desired results that both $e^{(\mathcal{A}+\mathcal{BF})t}$ and $e^{(\mathcal{A}-\mathcal{KC})t}$ are analytic and uniformly stable on $\mathbf{H}_e$, hence on $\mathbf{H}$. But as described, by the very core of the Luenberger's theory, the difference between the real (inaccessible) state y and its compensator z, which is fed only by the known observation $(\mathcal{C}y)$ is given by: $y(t) - z(t) = e^{(\mathcal{A}-\mathcal{KC})t}[y_0 - z_0]$ as in the finite dimensional case (1.4) and hence,

$$\|y(t) - z(t)\|_{(\cdot)} \le Me^{-kt}\|y_0 - z_0\|_{(\cdot)}, \tag{2.5}$$

$(\cdot)$ either $\mathbf{H}_e$ or $\mathbf{H}$, so that the known compensator $z(t)$ approximates asymptotically with exponential rate the original unknown state $y(t)$.

The last part of the program is to show that the control operator $\mathcal{F}z$ does the "job" of uniform stabilization on the original plant. This is done in section 6-Proof of Theorem 3.1. The guiding idea is to use perturbation theory for analytic semigroups where one can show that the perturbation $\mathcal{KC}$ -though unbounded can be handled via appropriate duality method. Techniques from [14, 24] with critical us of the transformation in [32], [24, p 497] are used.

3 The Original (free) HSI Model: Unstable on the Natural State Space $\mathbf{H} = [H^1(\Omega_s)]^n \times [L^2(\Omega_s)]^n \times [L^2(\Omega_f)]^n$ for $\{w, w_t, u\}$

3.1 PDE and Related Semigroup Version

Throughout, $\Omega_f \subseteq \mathbb{R}^n$, $n = 2$ or 3, will denote the bounded domain on which the heat component of the coupled PDE system evolves. Its boundary will be denoted here as $\partial\Omega_f = \Gamma_s \cup \Gamma_f$, $\Gamma_s \cap \Gamma_f = \emptyset$, with each boundary piece being sufficiently smooth. Moreover, the geometry Ω_s, immersed within Ω_f, will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct fluid and elastic dynamics occurs across boundary interface $\Gamma_s = \partial\Omega_s$; see Figure 1. In addition, the unit normal vector $\nu(x)$ will be directed away from Ω_f; thus on Γ_s, *toward* Ω_s. (This specification of the direction of ν will influence the computations to be done below.)

On this geometry in Fig. 1, we thus consider the following fluid–structure PDE model in solution variables $u = [u_1(t,x), u_2(t,x), \dots, u_n(t,x)]$ (the heat component here replacing the usual velocity field), and $w = [w_1(t,x), w_2(t,x), \dots, w_n(t,x)]$ (the structural displacement field):

$$u_t - \Delta u \equiv 0 \quad \text{in } (0,T] \times \Omega_f, \tag{3.1a}$$
$$w_{tt} - \Delta w - \Delta w_t \equiv 0 \quad \text{in } (0,T] \times \Omega_s, \tag{3.1b}$$
$$u|_{\Gamma_f} = 0 \quad \text{in } (0,T] \times \Gamma_f, \tag{3.1c}$$
$$u|_{\Gamma_s} = w_t|_{\Gamma_s} \quad \text{in } (0,T] \times \Gamma_s, \tag{3.1d}$$
$$\frac{\partial(w + w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} \quad \text{in } (0,T] \times \Gamma_s, \tag{3.1e}$$
$$\text{(IC)} \ [w(0,\cdot), w_t(0,\cdot), u(0,\cdot)] = [w_0, w_1, u_0] \in \mathbf{H}, \tag{3.1f}$$

$$\mathbf{H} = H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f), \tag{3.2}$$

for the variable $\{w, w_t, u\}$. $\mathbf{H}$ is a Hilbert space with the following norm inducing inner product, where $(f, g)_\Omega \equiv \int_\Omega f\bar{g}\, d\Omega$:

$$\left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \right)_{\mathbf{H}} = (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}. \quad (3.3)$$

Abstract model for the free dynamics (3.1 *a–e*) *The operator* $\mathcal{A}$. We rewrite problem (3.1 a–e) as a first-order equation:

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \quad (3.4)$$

where we introduce the operator $\mathcal{A} : \mathbf{H} \supset \mathcal{D}(\mathcal{A}) \to \mathbf{H}$:

$$\mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix}, \quad (3.5)$$

for $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A})$, described as follows [28]:

(i) $v_1, v_2 \in H^1(\Omega_s)$, so that $v_2|_{\Gamma_s} = h|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$,

$$\Delta(v_1 + v_2) \in L^2(\Omega_s). \quad (3.6a)$$

(ii) $h \in H^1(\Omega_f)$, $\Delta h \in L^2(\Omega_f)$, $h|_{\Gamma_f} = 0$, $h|_{\Gamma_s} = v_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$,

$$\frac{\partial(v_1 + v_2)}{\partial \nu} = \frac{\partial h}{\partial \nu} \in H^{-1/2}(\Gamma_s). \quad (3.6b)$$

(3.6b) is justified in [26, Section 1].

Remark 3.1 As noted in this same reference, the above description of $\mathcal{D}(\mathcal{A})$ shows that the point $\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A})$ enjoys a smoothing of regularity by one Sobolev unit—from $L^2(\cdot)$ to $H^1(\cdot)$ — but only of the coordinates v_2 and h, with respect to the original finite energy state space $\mathbf{H}$ in (3.2). In contrast, the first coordinate v_1 experiences no smoothing: it is in $H^1(\Omega_s)$, the first coordinate component of the space $\mathbf{H}$. This amounts to the fact that $\mathcal{A}$ has *non-compact resolvent* $R(\lambda, \mathcal{A})$ on $\mathbf{H}$. Consistently, it was shown in [26, Proposition 2.4] that the point $\lambda = -1$ belongs to the continuous spectrum of $\mathcal{A} : -1 \in \sigma_c(\mathcal{A})$.

Remark 3.2 *Orientation* Reference [26] provides a rather detailed and comprehensive account of the properties of system (3.1 a–e)-equivalently of the operator $\mathcal{A}$ in (3.5), (3.6) not in the energy space $\mathbf{H}$ in (3.2) with full $H^1(\Omega_s)$ first component, but on the space $\mathbf{H}_0$ where the first component is given by $H^1(\Omega_s)/\mathbb{R}$, i.e. with squared norm $\|\nabla v_1\|^2_{L^2(\Omega_s)}$ rather than $\left[\|\nabla v_1\|^2_{L^2(\Omega_s)} + \|v_1\|^2_{L^2(\Omega_s)}\right]$. In particular, it was shown in [26] that, in the space $\mathbf{H}_0$, the operator $\mathcal{A}$ generates a s.c. contraction semigroup $e^{\mathcal{A}t}$, which moreover is *analytic* and *uniformly stable* here. The choice of $\mathbf{H}_0$ over $\mathbf{H}$ was induced by the fact that, as readily pointed out in [39, Section 0], $\lambda = 0$ is a simple eigenvalue of $\mathcal{A}$ with corresponding eigenvector $e = [1, 0, 0]$ on $\mathbf{H}$. Thus, moving from $\mathbf{H}$ to $\mathbf{H}_0$, allowed one to factor out the 1-dimensional eigenspace generated by $e = [1, 0, 0]$. Plainly analyticity of $e^{\mathcal{A}t}$ is preserved on $\mathbf{H} = \mathbf{H}_0 +$ span $\{e\}$; however, uniform stablity is lost. Paper [39] on the Luenberger compensator exploited uniform stability and thus studied problem (3.1 a-e) on $\mathbf{H}_0$.

3.2 The Localized Control Operator $\mathcal{B}$; the Localized Observation Operator $\mathcal{C}$

The key factor of the present problem is that control action and state observation are localized on arbitrary, small, disjoint, portions of the interface Γ_s, say Γ_s^c and Γ_s^o, respectively of positive measures. See Fig 1.

The localized control portion Γ_s^c We consider the following non-homogeneous variation of the unstable free problem (3.1 a–e), with control exercised only on the portion Γ_s^c of the interface Γ_s.

$$u_t - \Delta u \equiv 0 \quad \text{in } (0, T] \times \Omega_f, \tag{3.7a}$$
$$w_{tt} - \Delta w - \Delta w_t \equiv 0 \quad \text{in } (0, T] \times \Omega_s, \tag{3.7b}$$
$$u|_{\Gamma_f} = 0 \quad \text{in } (0, T] \times \Gamma_f \tag{3.7c}$$
$$u|_{\Gamma_s} = w_t|_{\Gamma_s} \quad \text{in } (0, T] \times \Gamma_s, \tag{3.7d}$$
$$\frac{\partial(w + w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} + \chi_{\Gamma_s^c} g \quad \text{in } (0, T] \times \Gamma_s. \tag{3.7e}$$

$\chi_{\Gamma_s^c}$ = characteristic function of Γ_s^c; $\chi_{\Gamma_s^c} \equiv 1$ on Γ_s^c, $\chi_{\Gamma_s^c} \equiv 0$ on $\Gamma_s \backslash \Gamma_s^c$. Optimal well-posed results of the map g (on the entireΓ_s) $\to [w, w_t, u]$ are given in [35].

Abstract model of the controlled problem (3.7a–e). One may show as in [39] that the abstract model is:

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathcal{A}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} + \mathcal{B}g, \tag{3.8}$$

$$\mathcal{B}g = \begin{bmatrix} 0 \\ A_{N,tr} N \chi_{\Gamma_s^c} g \\ 0 \end{bmatrix} : \text{ continuous } L^2(\Gamma_s^c) \to \begin{bmatrix} \otimes \\ \left[\mathcal{D}(A_{N,tr})^{1/4+\varepsilon}\right]' \\ \otimes \end{bmatrix}, \tag{3.9}$$

where $-A_{N,tr}$ is the negative self-adjoint Neumann Laplacian translated by 1:

$$-A_{N,tr}\varphi = (\Delta - 1)\varphi, \quad \mathcal{D}(A_{N,tr}) = \left\{\varphi \in H^2(\Omega_s);\ \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_s} = 0\right\}, \tag{3.10}$$

and where N is the corresponding Neumann map:

$$\psi = N\mu \iff \begin{cases} (\Delta - 1)\psi = 0 & \text{in } \Omega_s, \\ \frac{\partial\psi}{\partial\nu}\Big|_{\Gamma_s} = \mu. \end{cases} \tag{3.11}$$

In fact, we return to (3.1b) and rewrite it as:

$$w_{tt} = \Delta(w + w_t) = (\Delta - 1)\left[(w + w_t) - N\left(\frac{\partial u}{\partial \nu} + g\right)\right] + (w + w_t), \quad (3.12)$$

where the term $\left[(w + w_t) - N\left(\frac{\partial u}{\partial \nu} + g\right)\right]$ satisfies the zero Neumann B.C. of $-A_{N,tr}$ in (3.9), by (3.11) so we can rewrite (3.12) as:

$$w_{tt} = -A_{N,tr}\left[(w + w_t) - N\left(\frac{\partial u}{\partial \nu} + g\right)\right] + (w + w_t) \in L^2(\Omega_s), \quad (3.13)$$

or

$$w_{tt} = -\widetilde{A}_N(w + w_t) + \widetilde{A}_{N,tr}N\left(\frac{\partial u}{\partial \nu} + g\right) \in \left[\mathcal{D}(A_{N,tr})\right]', \quad (3.14)$$

with $\widetilde{A}_{N,tr}$ the isomorphic extension $L^2(\Omega_s) \to \left[\mathcal{D}(A_{N,tr})\right]' =$ dual of $\mathcal{D}(A_{N,tr})$ w.r.t. $L^2(\Omega_s)$ and $-\widetilde{A}_N$ the isomorphic extension $L^2(\Omega_s) \to [\mathcal{D}(A_N)]'$ of the operator $-A_N = \Delta$, $\mathcal{D}(A_N) = \mathcal{D}(A_{N,tr})$. Henceforth, we shall drop the $\widetilde{}$ for the extension $\widetilde{A}_{N,tr}$. As in [39] the adjoint operator $\mathcal{B}^*$ is given by,

$$\mathcal{B}^*\begin{bmatrix} x_1, \\ x_2 \\ x_3 \end{bmatrix} = -\chi_{\Gamma_s^c} x_2|_{\Gamma_s} = -x_2|_{\Gamma_s^c}, \quad x_2 \in H^{1/2+\varepsilon}(\Omega_s). \quad (3.15)$$

In fact, as in [39], by (3.9),

$$\left(\mathcal{B}g, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{\mathbf{H}} = \left(\begin{bmatrix} 0 \\ A_{N,tr}N\chi_{\Gamma_s^c}g \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{\mathbf{H}} \quad (3.16)$$

$$= \left(A_{N,tr}N\chi_{\Gamma_s^c}g, x_2\right)_{L^2(\Omega_s)} = \left(g, N^*A_{N,tr}x_2\right)_{L^2(\Gamma_s^c)} \quad (3.17)$$

$$= \left(g, -x_2|_{\Gamma_s^c}\right)_{L^2(\Gamma_s^c)} = \left(\chi_{\Gamma_s^c}g, \mathcal{B}^*\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)_{L^2(\Gamma_s)}, \quad (3.18)$$

since, in the usual way, we have,

$$N^*A_{N,tr}x_2 = -x_2|_{\Gamma_s}, \text{ for } x_2 \in H^{1/2+\varepsilon}(\Omega_s). \quad (3.19)$$

In fact, take initially $\varphi \in \mathcal{D}(A_{N,tr})$ as in (3.10) and refer to (3.11) with $\mu \in L^2(\Gamma_s)$ to obtain via Green's second theorem, since ν is inward to Ω_s;

$$-\left(N^*A_{N,tr}\varphi, \mu\right)_{L^2(\Gamma_s)} = \left(-A_{N,tr}\varphi, N\mu\right)_{L^2(\Omega_s)} = ((\Delta - I)\varphi, N\mu)_{L^2(\Omega_s)} \quad (3.20)$$

$$= \left(\varphi, \cancel{(\Delta - I)\psi}\right)_{L^2(\Omega_s)} - \int_{\Gamma_s} \cancel{\frac{\partial \varphi}{\partial \nu}} \psi \, d\Gamma_s + \int_{\Gamma_s} \varphi \frac{\partial \psi}{\partial \nu} \, d\Gamma_s \tag{3.21}$$

$$= (\varphi, \mu)_{L^2(\Gamma_s)} \,. \tag{3.22}$$

Thus (3.19) follows from (3.21) initially for $\varphi \in \mathcal{D}(A_{N,tr})$ and later extended to $\varphi \in H^{1/2+2\varepsilon}(\Omega_s)$ [24].

The localized portion Γ_s^o of the observer Let now Γ_s^o be a (say, connected) arbitrary portion of the boundary Γ_s, also of positive measure, and generally $\Gamma_s^o \cap \Gamma_s^c = \emptyset =$ empty set . Let $\chi_{\Gamma_s^o}$ be the characteristic function: Γ_s^o. We define the operator $\mathcal{C}$ by:

$$\mathcal{C} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \chi_{\Gamma_s^o} v_1, \quad \mathcal{C} : \mathbf{H} \to L^2(\Gamma_s^o), \tag{3.23}$$

thus, with $v_1 \in H^1(\Omega_s)$. Thus, the two critical operators to be employed on the free system (3.1 a–e) or the operator $\mathcal{A}$ in (3.5), (3.6)-the control operator $\mathcal{B}$ and the observation operator $\mathcal{C}$-are acting on two arbitrary small disjoint portion Γ_s^c and Γ_s^o of the interface Γ_s. Next, we need to design a feedback stabilizing operator $\mathcal{F}$ and a detectablity operator $\mathcal{K}$.

4 Uniform Stabilization of $\{\mathcal{A}, \mathcal{B}\}$ by a Feedback Stabilizing Operator $\mathcal{F}$ and Detectability of $\{\mathcal{A}, \mathcal{C}\}$ by an Operator $\mathcal{K}$

We design the feedback stabilizing operator $\mathcal{F}$ and the detectability operator $\mathcal{K}$ by setting

$$\mathcal{F} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1|_{\Gamma_s^c}, \quad \mathcal{F} : \mathbf{H} \to L^2(\Gamma_s^c), \quad \mathcal{K}\mu = \begin{bmatrix} 0 \\ -A_{N,tr} N \mu \\ 0 \end{bmatrix} : \\ L^2(\Gamma_s) \to \left[\mathcal{D}(A_{N,tr})^{1/4+\epsilon}\right]', \tag{4.1}$$

(compare $\mathcal{K}$ with $\mathcal{B}$ in (3.9)), so that by (3.9) and (4.1), we obtain:

$$\mathcal{B}\mathcal{F} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ A_{N,tr} N \chi_{\Gamma_s^c} v_1 \\ 0 \end{bmatrix}, \quad -\mathcal{K}\mathcal{C} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ A_{N,tr} N \chi_{\Gamma_s^o} v_1 \\ 0 \end{bmatrix}. \tag{4.2}$$

Since $v_1 \in H^1(\Omega_s)$, we have $v_1|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$, hence $N v_1 \in H^2(\Omega_s)$. Our task is to show that, with this selection of $\mathcal{F}$ and $\mathcal{C}$, the stabilizability condition for $\{\mathcal{A}, \mathcal{B}\}$ and the detectability condition for $\{\mathcal{A}, \mathcal{C}\}$ as defined by (A.1) and (A.2) hold true. To this end, we need to introduce the PDE-problems corresponding to the operator models $(\mathcal{A} + \mathcal{B}\mathcal{F})$ and $(\mathcal{A} - \mathcal{K}\mathcal{C})$.

Thus, the PDE-version of the abstract (feedback) problem:

$$\dot{y} = \mathcal{A}y + \mathcal{B}\mathcal{F}y = \mathcal{A}_F y, \quad y = \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \; g = \mathcal{F}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \tag{4.3}$$

is given by,

$$\begin{aligned}
u_t - \Delta u &\equiv 0 && \text{in } (0,T] \times \Omega_f, && (4.4a)\\
w_{tt} - \Delta w - \Delta w_t &\equiv 0 && \text{in } (0,T] \times \Omega_s, && (4.4b)\\
u|_{\Gamma_f} &= 0 && \text{in } (0,T] \times \Gamma_f, && (4.4c)\\
u|_{\Gamma_s} &= w_t|_{\Gamma_s} && \text{in } (0,T] \times \Gamma_s, && (4.4d)\\
\frac{\partial(w+w_t)}{\partial\nu} &= \frac{\partial u}{\partial \nu} + \chi_{\Gamma_s^c} w && \text{in } (0,T] \times \Gamma_s, && (4.4e)
\end{aligned}$$

with I.C. $w(0,\cdot) = w_0, w_t(0,\cdot) = w_1, u(0,\cdot) = u_0$. A mathematical analysis of problem (4.4 a–e) is deferred to Sect. 4.1. We have: $(\mathcal{A}+\mathcal{B}\mathcal{F}) : \mathbf{H} \supset \mathcal{D}(\mathcal{A}+\mathcal{B}\mathcal{F}) \to \mathbf{H}$.

$$(\mathcal{A}+\mathcal{B}\mathcal{F})\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1+v_2) \\ \Delta h \end{bmatrix} \in \mathbf{H} \tag{4.5a}$$

$$v_1, v_2 \in H^1(\Omega_s),\; \Delta(v_1+v_2) \in L^2(\Omega_s),\; \Delta h \in L^2(\Omega_f); \tag{4.5b}$$

$$h|_{\Gamma_f} = 0, u|_{\Gamma_s} = v_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s),\; \left.\frac{\partial(v_1+v_2)}{\partial\nu}\right|_{\Gamma_s} = \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_s} + \chi_{\Gamma_s^c} v_1 \in H^{-1/2}(\Gamma_s). \tag{4.5c}$$

The PDE-version of the abstract observed system:

$$\dot{y} = \mathcal{A}y - \mathcal{K}\mathcal{C}y,\; y = \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \tag{4.6}$$

is given by,

$$\begin{aligned}
u_t - \Delta u &\equiv 0 && \text{in } (0,T] \times \Omega_f, && (4.7a)\\
w_{tt} - \Delta w - \Delta w_t &\equiv 0 && \text{in } (0,T] \times \Omega_s, && (4.7b)\\
u|_{\Gamma_f} &= 0 && \text{in } (0,T] \times \Gamma_f, && (4.7c)\\
u|_{\Gamma_s} &= w_t|_{\Gamma_s} && \text{in } (0,T] \times \Gamma_s, && (4.7d)\\
\frac{\partial(w+w_t)}{\partial\nu} &= \frac{\partial u}{\partial \nu} - \chi_{\Gamma_s^o} w && \text{in } (0,T] \times \Gamma_s, && (4.7e)
\end{aligned}$$

with I.C. $[w_0, w_1, u_0]$. We have: $(\mathcal{A} - \mathcal{K}\mathcal{C}) : \mathbf{H} \supset \mathcal{D}(\mathcal{A} - \mathcal{K}\mathcal{C}) \to \mathbf{H}$.

$$(\mathcal{A} - \mathcal{K}\mathcal{C}) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix} \in \mathbf{H}; \tag{4.8a}$$

$$v_1, v_2 \in H^1(\Omega_s),\ \Delta(v_1 + v_2) \in L^2(\Omega_s),\ \Delta h \in L^2(\Omega_f); \tag{4.8b}$$

$$h|_{\Gamma_f} = 0, u|_{\Gamma_s} = v_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s),\ \frac{\partial(v_1 + v_2)}{\partial\nu}\Big|_{\Gamma_s} = \frac{\partial h}{\partial\nu}\Big|_{\Gamma_s} - \chi_{\Gamma_s^o} v_1 \in H^{-1/2}(\Gamma_s). \tag{4.8c}$$

Thus, the feedback control operator $\mathcal{F}$ acts on Γ_s^c while the partial observation operator $\mathcal{C}$ picks up the information w on Γ_s^o.

4.1 Analysis of the Critical PDE-System: Analytic Semigroup, Well-Posedness and Uniform Stability

In this section we denote by $\widetilde{\Gamma}_s$ an arbitrary portion of the boundary Γ_s; thus $\widetilde{\Gamma}_s = \Gamma_s^c$ for the controlled portion of Γ_s, $\widetilde{\Gamma}_s = \Gamma_s^o$ for the observed portion of Γ_s. We consider the following PDE problem:

$$u_t - \Delta u \equiv 0 \quad \text{in } (0, T] \times \Omega_f, \tag{4.9a}$$
$$w_{tt} - \Delta w - \Delta w_t \equiv 0 \quad \text{in } (0, T] \times \Omega_s, \tag{4.9b}$$
$$u|_{\Gamma_f} = 0 \quad \text{in } (0, T] \times \Gamma_f, \tag{4.9c}$$
$$u|_{\Gamma_s} = w_t|_{\Gamma_s} \quad \text{in } (0, T] \times \Gamma_s, \tag{4.9d}$$
$$\frac{\partial(w + w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} + \chi_{\widetilde{\Gamma}_s} w \quad \text{in } (0, T] \times \Gamma_s. \tag{4.9e}$$

Its abstract version is given by,

$$\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \mathbb{A} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \tag{4.10}$$

with action of $\mathbb{A}$ given by,

$$\mathbb{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix}, \tag{4.11}$$

where the domain of the operator $\mathbb{A} : \mathbf{H} \supset \mathcal{D}(\mathbb{A}) \to \mathbf{H}$ is as follows:
$\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A})$

(i) $v_1, v_2 \in H^1(\Omega_s)$, so that $v_2|_{\Gamma_s} = h|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$,

$$\Delta(v_1 + v_2) \in L^2(\Omega_s), \tag{4.12a}$$

(ii) $h \in H^1(\Omega_f)$, $\Delta h \in L^2(\Omega_f)$, $h|_{\Gamma_f} = 0$, $h|_{\Gamma_s} = v_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$,

$$\frac{\partial(v_1 + v_2)}{\partial\nu} = \frac{\partial h}{\partial\nu} + \chi_{\widetilde{\Gamma}_s} v_1 \in H^{-1/2}(\Gamma_s). \tag{4.12b}$$

[One cannot invoke the abstract results [8, 9], because of the coupled boundary conditions.]

4.1.1 Well-Posedness of (4.9 a–e) or (4.10)

Lemma 4.1 *With reference to the operator $\mathbb{A}$ in (4.10), (4.11), we have:*

(i)

$$0 \in \rho(\mathbb{A}). \tag{4.13}$$

(ii) $\mathbb{A}$ is not dissipative on $\mathbf{H}$ in (3.2).

Verification of (ii) will appear from the proof of Proposition 4.2. The proof of (i) is given in Appendix 7.1, which provides a explicit expression for $\mathbb{A}^{-1} : \mathbf{H} \to \mathcal{D}(\mathbb{A})$ in Eqt. (7.11). To remedy the shortcoming in (ii), we replace the space: $\mathbf{H} = H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f)$ in (3.2), with the space

$$\mathbf{H}_e \equiv H_e^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f), \tag{4.14}$$

where the inner product on $H_e^1(\Omega_s)$ (e = equivalent) is defined by,

$$(v_1, \tilde{v_1})_{H_e^1(\Omega_s)} = (\nabla v_1, \nabla \tilde{v_1})_{\Omega_s} + (v_1, \tilde{v_1})_{\widetilde{\Gamma}_s}, \tag{4.15a}$$

$$\|v_1\|^2_{H_e^1(\Omega_s)} \equiv \|\nabla v_1\|^2_{L^2(\Omega_s)} + \|v_1|_{\widetilde{\Gamma}_s}\|^2_{L^2(\widetilde{\Gamma}_s)}, \tag{4.15b}$$

so that $\|v_1\|_{H_e^1(\Omega_s)} = 0 \implies v_1 \equiv 0$ in Ω_s. In (4.15a), the portion $\widetilde{\Gamma}_s$ of Γ_s is the same as in (4.9e).

Claim: The $H_e^1(\Omega_s)-$ norm of v_1 in (4.15b) is equivalent to the $H^1(\Omega_s)$-norm of v_1. This follows from [27, p 260]:

Let $\psi \in H^1(\Omega_s)$. Let $\widetilde{\Gamma}$ be an arbitrary portion of $\partial\Omega = \Gamma$ of positive measure. Then, there exist constants $0 < k_1 < k_2$, such that,

$$k_1 \int_\Omega \left[|\psi|^2 + |\nabla\psi|^2\right] d\Omega \le \int_\Omega |\nabla\psi|^2 d\Omega + \int_{\widetilde{\Gamma}} |\psi|^2 d\widetilde{\Gamma} \le k_2 \int_\Omega \left[|\psi|^2 + |\nabla\psi|^2\right] d\Omega. \tag{4.16}$$

Henceforth, we shall study problem (4.9) (the operator $\mathbb{A}$ in (4.11), (4.12) on the function space $\mathbf{H}_e$ in (4.14) which is norm equivalent to the function space $\mathbf{H}$ in (3.2). The reason is justified by the following result.

Proposition 4.2 *(i) Let* $[v_1, v_2, h], \left[\tilde{v_1}, \tilde{v_2}, \tilde{h}\right] \in \mathcal{D}(\mathbb{A})$. *Then, with reference to the space* $\mathbf{H}_e$ *in (4.14) we have, in the* L^2*−norms:*

$$\left(\mathbb{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = (\nabla v_2, \nabla \tilde{v_1})_{\Omega_s} - (\nabla v_1, \nabla \tilde{v_2})_{\Omega_s} + (v_2, \tilde{v_1})_{\tilde{\Gamma_s}} - (v_1, \tilde{v_2})_{\tilde{\Gamma_s}} - (\nabla v_2, \nabla \tilde{v_2})_{\Omega_s} - (\nabla h, \nabla \tilde{h})_{\Omega_f}. \tag{4.17}$$

(ii) Thus, if now $[v_1, v_2, h] = \left[\tilde{v_1}, \tilde{v_2}, \tilde{h}\right] \in \mathcal{D}(\mathbb{A})$, *then (4.17) specializes to:*

$$\left(\mathbb{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = (\nabla v_2, \nabla v_1)_{\Omega_s} - (\nabla v_1, \nabla v_2)_{\Omega_s} + (v_2, v_1)_{\tilde{\Gamma_s}} - (v_1, v_2)_{\tilde{\Gamma_s}} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f} \tag{4.18}$$

$$= 2Im(\nabla v_2, \nabla v_1)_{\Omega_s} + 2Im(v_2, v_1)_{\tilde{\Gamma_s}} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}, \tag{4.19}$$

thus, $\mathbb{A}$ *is dissipative on* $\mathbf{H}_e$

$$Re\left(\mathbb{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = -\|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}, \quad [v_1, v_2, h] \in \mathcal{D}(\mathbb{A}). \tag{4.20}$$

Since $0 \in \rho(\mathbb{A})$ by Lemma 4.1 (also on $\mathbf{H}_e$), then dissipativity of $\mathbb{A}$ in $\mathbf{H}_e$ in (4.20) implies its maximal dissipativity. The Lumer-Phillips Theorem [31] then yields:

Theorem 4.3 *The operator* $\mathbb{A}$ *in (4.11), (4.12) is maximal dissipative on* $\mathbf{H}_e$*; hence, it generates a s.c. contraction semigroup* $e^{\mathbb{A}t}$ *on* $\mathbf{H}_e$*. Since* $\mathbf{H}$ *and* $\mathbf{H}_e$ *are norm-equivalent, then* $\mathbb{A}$ *generates a s.c semigroup* $e^{\mathbb{A}t}$ *on* $\mathbf{H}$ *as well.*

Proof of Proposition 4.2 As in Part 1, take $[v_1, v_2, h], \left[\tilde{v_1}, \tilde{v_2}, \tilde{h}\right] \in \mathcal{D}(\mathbb{A})$ and compute via (4.11) and the topology in (4.15)

$$\left(\mathbb{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = \left(\begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix}, \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = (\nabla v_2, \nabla \tilde{v_1})_{\Omega_s} + (v_2, \tilde{v_1})_{\tilde{\Gamma_s}} + (\Delta(v_1 + v_2), \tilde{v_2})_{\Omega_s} + \left(\Delta h, \tilde{h}\right)_{\Omega_f}. \tag{4.21}$$

Next we apply Green's first Theorem to each of the last two terms in (4.21), recall that the unit normal ν is inward to Ω_s, as well as the B.C. (4.12a),(4.12b) in $\mathcal{D}(\mathbb{A})$. We obtain:

$$(\Delta(v_1+v_2), \tilde{v_2})_{\Omega_s} + \left(\Delta h, \tilde{h}\right)_{\Omega_f} = -\int_{\Gamma_s} \frac{\partial(v_1+v_2)}{\partial\nu} \bar{\tilde{v}}_2 \, d\Gamma_s - (\nabla(v_1+v_2), \nabla\tilde{v_2})_{\Omega_s}$$
$$+ \int_{\Gamma_s} \frac{\partial h}{\partial\nu} \bar{\tilde{h}} \, d\Gamma_s - \left(\nabla h, \nabla\tilde{h}\right)_{\Omega_f}$$
$$= -\int_{\Gamma_s} \frac{\partial h}{\partial\nu} \bar{\tilde{h}} \, d\Gamma_s - \int_{\widetilde{\Gamma}_s} v_1 \bar{\tilde{v}}_2 \, d\widetilde{\Gamma}_s + \int_{\Gamma_s} \frac{\partial h}{\partial\nu} \bar{\tilde{h}} \, d\Gamma_s - \left(\nabla h, \nabla\tilde{h}\right)_{\Omega_f}. \tag{4.22}$$

Substituting (4.22) in the RHS of (4.21) yields (4.17), from which (4.19)-(4.20) readily follow. □

Time-domain version of dissipativity We next provide (as in [26]) a time-domain version of dissipativity. To this end, with reference to problem (4.9 a–e), define:

$$E_u(t) = \int_{\Omega_f} u^2(t,\cdot)\, d\Omega_f, \tag{4.23}$$

$$E_w(t) = \int_{\Omega_s} w^2(t,\cdot)\, d\Omega_s + \left[\int_{\Omega_s} |\nabla w(t,\cdot)|^2\, d\Omega_s + \int_{\widetilde{\Gamma}_s} w^2(t,\cdot)\, d\widetilde{\Gamma}_s\right]. \tag{4.24}$$

Proposition 4.4 *With reference to problem (4.9 a–e), the following energy equality holds true, in the notation of (4.23), (4.24),*

$$E_u(t) + E_w(t) + 2\int_0^t \int_{\Omega_f} |\nabla u|^2\, d\Omega_f d\tau$$
$$+2\int_0^t \int_{\Omega_s} |\nabla w_t|^2\, d\Omega_s d\tau = E_u(0) + E_w(0). \tag{4.25}$$

Hence, the following dissipativity inequality holds

$$2\int_0^\infty \int_{\Omega_f} |\nabla u|^2\, d\Omega_f dt + 2\int_0^\infty \int_{\Omega_s} |\nabla w_t|^2\, d\Omega_s dt \le E_u(0) + E_w(0) \tag{4.26a}$$

as well as contraction of the semigroup $e^{\mathbb{A}t}$ on $\mathbf{H}_e$,

$$E_u(t) + E_w(t) \le E_u(0) + E_w(0) \quad or \quad \|e^{\mathbb{A}t}\|_{\mathcal{L}(\mathbf{H}_e)} \le 1,\ t \ge 0. \tag{4.26b}$$

Proof Multiply Eq (4.9a) by u, use $u_t u = \frac{1}{2}\frac{\partial(u^2)}{\partial t}$ and integrate in time and space, use Green's first lemma and $u|_{\Gamma_f} = 0$, to obtain:

$$E_u(t) + 2\int_0^t \int_{\Omega_f} |\nabla u|^2\, d\Omega_f d\tau = E_u(0) + 2\int_0^t \int_{\Gamma_s} \frac{\partial u}{\partial\nu} u\, d\Gamma_s. \tag{4.27}$$

Next, multiply Eq. (4.9b) by w_t, use $w_{tt}w_t = \frac{1}{2}\frac{\partial(w_t^2)}{\partial t}$, and Greens First Lemma with unit normal ν inward to Ω_s, use the B.C $\frac{\partial(w+w_t)}{\partial\nu} = \frac{\partial u}{\partial\nu} + \chi_{\widetilde{\Gamma}_s} w$ on Γ_s and obtain:

$$\frac{1}{2}\int_{\Omega_s}\left[w_t^2(t,\cdot) - w_t^2(0,\cdot)\right] d\Omega_s + \int_0^t\int_{\Gamma_s}\frac{\partial u}{\partial\nu} w_t\, d\Gamma_s d\tau + \int_{\widetilde{\Gamma}_s}\int_0^t \frac{1}{2}\frac{\partial(w_t^2)}{\partial\tau}\, d\tau d\widetilde{\Gamma}_s$$
$$+\int_{\Omega_s}\int_0^t\frac{1}{2}\frac{\partial(|\nabla w|^2)}{\partial\tau}\, d\tau d\Omega_s + \int_0^t\int_{\Omega_s}|\nabla w_t|^2\, d\Omega_s d\tau = 0. \tag{4.28}$$

Recalling $u = w_t$ on Γ_s, we see that (4.28) yields via (4.24),

$$E_w(t) + 2\int_0^t\int_{\Omega_s}|\nabla w_t|^2\, d\Omega_s d\tau = E_w(0) - 2\int_0^t\int_{\Gamma_s}\frac{\partial u}{\partial\nu} u\, d\Gamma_s d\tau. \tag{4.29}$$

Finally sum up (4.27) and (4.29) and obtain (4.25) after a cancellation of the boundary terms, $\int_0^t\int_{\Gamma_s}\frac{\partial u}{\partial\nu} u\, d\Gamma_s d\tau$. □

4.1.2 Spectral Properties on $i\mathbb{R}$ of $\mathbb{A}$ and its Adjoint $\mathbb{A}^*$

Complementing $0 \in \rho(\mathbb{A})$ in (4.13), we now obtain:

Lemma 4.5 *With reference to the operator $\mathbb{A}$ in (4.11), (4.12), we have*

$$i\mathbb{R} \notin \sigma_p(\mathbb{A}) = \textit{ point spectrum of } \mathbb{A}. \tag{4.30}$$

Proof Consider the eigenproblem of $\mathbb{A}$ with potential eigenvalue ir, $r \neq 0 \in \mathbb{R}$, for $[v_1, v_2, h] \in \mathcal{D}(\mathbb{A})$, with norm=1:

$$\mathbb{A}\begin{bmatrix}v_1\\v_2\\h\end{bmatrix} = ir\begin{bmatrix}v_1\\v_2\\h\end{bmatrix} \Longrightarrow \left(\mathbb{A}\begin{bmatrix}v_1\\v_2\\h\end{bmatrix}, \begin{bmatrix}v_1\\v_2\\h\end{bmatrix}\right)_{\mathbf{H}_e} = ir\left(\begin{bmatrix}v_1\\v_2\\h\end{bmatrix}, \begin{bmatrix}v_1\\v_2\\h\end{bmatrix}\right)_{\mathbf{H}_e} = ir. \tag{4.31}$$

Thus recalling the dissipativity condition (4.20), we obtain from (4.31):

$$Re\left(\mathbb{A}\begin{bmatrix}v_1\\v_2\\h\end{bmatrix}, \begin{bmatrix}v_1\\v_2\\h\end{bmatrix}\right)_{\mathbf{H}_e} = -\|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f} = 0. \tag{4.32}$$

Hence $\nabla h \equiv 0$ or $h \equiv$constant in Ω_f, in fact $h \equiv 0$ in Ω_f, since $h|_{\Gamma_f} = 0$. Moreover $\nabla v_2 \equiv 0$ or $v_2 \equiv$ constant in Ω_s, in fact $v_2 \equiv 0$ in Ω_s, since $v_2|_{\Gamma_s} = h|_{\Gamma_s} = 0$. Finally, the eigen expression in (4.31) yields via (4.11) $v_2 = irv_1$, hence $v_1 \equiv 0$ in Ω_s for $r \neq 0$. For $r = 0$, we have the stronger statement (4.13). □

Adjoint$\mathbb{A}^*$*of* $\mathbb{A}$ *in* $\mathbf{H}_e$4 Though not strictly necessary for the analysis of the present paper, we wish to introduce the adjoint $\mathbb{A}^*$ of $\mathbb{A}$ in $\mathbf{H}_e$.

Proposition 4.6 *The adjoint $\mathbb{A}^*$ of $\mathbb{A}$ in $\mathbf{H}_e$ is given by*

$$\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ -\Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) \\ \Delta h \end{bmatrix}, \tag{4.33}$$

with $[v_1, v_2, h] \in \mathcal{D}(\mathbb{A})$, meaning:

(i) $v_1, v_2 \in H^1(\Omega_s)$, *so that* $v_2|_{\Gamma_s} = h|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$

$$\Delta(v_2 - v_1) \in L^2(\Omega_s) \tag{4.34a}$$

(ii) $h \in H^1(\Omega_f)$, $\Delta h \in L^2(\Omega_f)$, $h|_{\Gamma_f} = 0$, $h|_{\Gamma_s} = v_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$

$$\frac{\partial(v_2 - v_1)}{\partial\nu} = \frac{\partial h}{\partial\nu} - \chi_{\widetilde{\Gamma}_s} v_1 \in H^{-1/2}(\Gamma_s). \tag{4.34b}$$

The proof is given in Appendix 7.2. The counterpart of Proposition 4.2, whose proof is also given in Appendix B is

Lemma 4.7 *(i) Let $[v_1, v_2, h]$, $\left[\tilde{v_1}, \tilde{v_2}, \tilde{h}\right] \in \mathcal{D}(\mathbb{A}^*)$. Then with reference to the space $\mathbf{H}_e$ in (4.14) we have, in the L^2-norms:*

$$\left(\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v_1} \\ \tilde{v_2} \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = -(\nabla v_2, \nabla \tilde{v_1})_{\Omega_s} + (\nabla v_1, \nabla \tilde{v_2})_{\Omega_s} - (v_2, \tilde{v_1})_{\widetilde{\Gamma}_s} + (v_1, \tilde{v_2})_{\widetilde{\Gamma}_s} - (\nabla v_2, \nabla \tilde{v_2})_{\Omega_s} - (\nabla h, \nabla \tilde{h})_{\Omega_f}. \tag{4.35}$$

(ii) Thus, if now $[v_1, v_2, h] = \left[\tilde{v_1}, \tilde{v_2}, \tilde{h}\right] \in \mathcal{D}(\mathbb{A}^*)$, then (4.35) specializes to

$$\left(\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = -(\nabla v_2, \nabla v_1)_{\Omega_s} + (\nabla v_1, \nabla v_2)_{\Omega_s} - (v_2, v_1)_{\widetilde{\Gamma}_s} + (v_1, v_2)_{\widetilde{\Gamma}_s} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f} \tag{4.36}$$

$$= 2Im(\nabla v_1, \nabla v_2)_{\Omega_s} + 2Im(v_1, v_2)_{\widetilde{\Gamma}_s} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f} \tag{4.37}$$

thus, $\mathbb{A}^*$ is dissipative on $\mathbf{H}_e$:

$$Re\left(\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = -\|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}, \quad [v_1, v_2, h] \in \mathcal{D}(\mathbb{A}^*). \tag{4.38}$$

The counterpart of Lemma 4.5 is now

Lemma 4.8 *With reference to the operator $\mathbb{A}^*$ in (4.33), (4.34), we have*

$$i\mathbb{R} \notin \sigma_p(\mathbb{A}^*) = \text{ point spectrum of } \mathbb{A}^*, \text{ hence } i\mathbb{R} \notin \sigma_r(\mathbb{A}), \\ \text{the residual spectrum of } \mathbb{A}. \tag{4.39}$$

Proof Consider the eigenproblem of $\mathbb{A}^*$ with potential eigenvalue ir, $0 \neq r \in \mathbb{R}$, for $[v_1, v_2, h] \in \mathcal{D}(\mathbb{A}^*)$, with norm=1:

$$\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = ir \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \Longrightarrow \left(\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_e} = ir \left(\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_e} = ir. \tag{4.40}$$

Thus recalling the dissipativity condition (4.38), we obtain from (4.40):

$$Re \left(\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathbf{H}_e} = -\|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f} = 0. \tag{4.41}$$

Hence $\nabla h \equiv 0$ or $h \equiv$constant in Ω_f, in fact $h \equiv 0$ in Ω_f, since $h|_{\Gamma_f} = 0$ by (4.34b). Moreover $\nabla v_2 \equiv 0$ or $v_2 \equiv$ constant in Ω_s, in fact $v_2 \equiv 0$ in Ω_s, since $v_2|_{\Gamma_s} = h|_{\Gamma_s} = 0$ by (4.34b). Finally, the eigen expression in (4.40) yields via (4.33), $-v_2 = irv_1$, hence $v_1 \equiv 0$ in Ω_s for $r \neq 0$. For $r = 0$, $0 \notin \sigma_p(\mathbb{A}^*)$. In fact, let

$$\mathbb{A}^* \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) \\ \Delta h \end{bmatrix} = 0, \tag{4.42}$$

hence $v_2 \equiv 0$ in Ω_s. Moreover $\Delta h \equiv 0$ in Ω_f which along woth $h|_{\Gamma_f} \equiv 0$ and $v_2|_{\Gamma_s} = h|_{\Gamma_s} = 0$ via (4.34b) yields $h \equiv 0$ in Ω_f. Finally

$$\begin{aligned} &\Delta v_1 \equiv 0 \quad \text{in } \Omega_s \\ &-\frac{\partial v_1}{\partial \nu} = \frac{\partial h}{\partial \nu} - \chi_{\tilde{\Gamma}_s} v_1 = 0 \quad \text{or} \quad \frac{\partial v_1}{\partial \nu} - \chi_{\tilde{\Gamma}_s} v_1 = 0 \end{aligned}$$

implies $v_1 \equiv 0$. So $0 \notin \sigma_p(\mathbb{A}^*)$. □

4.2 Analyticity of $e^{\mathbb{A}t}$ (and $e^{\mathbb{A}^*t}$) on $\mathbf{H}_e$ (hence on H)

The following is a main result of the present paper.

Theorem 4.9 *The s.c. contraction semigroup $e^{\mathbb{A}t}$ on the space $\mathbf{H}_e$ is moreover analytic. Then $e^{\mathbb{A}t}$ is also analytic on $\mathbf{H}$.*

Proof We have already established that the operator $\mathbb{A}$ in (4.11), (4.12), possesses the following two features: (i) it is the generator of a s.c. (C_0-) semigroup $e^{\mathbb{A}t}$ of contractions on the finite energy space $\mathbf{H}_e$in (4.14) (Theorem 4.3); (ii) $0 \in \rho(\mathbb{A})$, the resolvent set of $\mathbb{A}$, and hence there is a small open disk $\mathcal{S}_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that is all contained in $\rho(\mathbb{A}) : \mathcal{S}_{r_0} \subset \rho(\mathbb{A})$ (Lemma 4.1 (i)). Accordingly, to conclude that $e^{\mathbb{A}t}$ is, moreover, analytic on $\mathbf{H}_e$ (and hence on $\mathbf{H}$), all we need to show [24, Thm. 3E.3, p. 334] is that ($\mathbb{A}$ has no spectrum on the imaginary axis, and):

$$\|R(i\omega, \mathbb{A})\|_{\mathcal{L}(\mathbf{H}_e)} \leq \frac{C}{|\omega|}, \quad \forall\, |\omega| \geq \text{ some } \omega_0 > 0. \tag{4.43}$$

Equivalently, in this section we shall show that

$$\|\mathbb{A}R(i\omega, \mathbb{A})\|_{\mathcal{L}(\mathbf{H}_e)} \leq C, \quad \forall\, |\omega| \geq \text{ some } \omega_0 > 0. \tag{4.44}$$

since $\mathbb{A}R(i\omega, \mathbb{A}) = -I + i\omega R(i\omega, \mathbb{A})$.
Step 1 Given $\{v_1^*, v_2^*, h^*\} \in \mathbf{H}_e$, and $\omega \in \mathbb{R}\backslash\{0\}$. Initially, we seek to solve the equation

$$((i\omega)I - \mathbb{A})\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \left\{(i\omega)I - \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\right\}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \tag{4.45}$$

in terms of $\{v_1, v_2, h\} \in \mathcal{D}(\mathbb{A})$ uniquely, and establish, in fact, the analyticity estimate (4.43), equivalently estimate (4.44). For $i\omega \in \rho(\mathbb{A})$, we have

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(i\omega, \mathbb{A})\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}; \quad \mathbb{A}R(i\omega, \mathbb{A})\begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix}. \tag{4.46}$$

We then see that the analyticity condition (4.44) is equivalent to showing the following estimate (recall (4.15b)) (all norms are L_2-norms on the respective domains): there exists a constant $C > 0$ such that

$$\begin{aligned} &\|\nabla v_2\|^2_{\Omega_s} + \|v_2|_{\widetilde{\Gamma}_s}\|^2_{\widetilde{\Gamma}_s} + \|\Delta(v_1 + v_2)\|^2_{\Omega_s} + \|\Delta h\|^2_{\Omega_f} \\ &\quad \leq C\left\{\|\nabla v_1^*\|^2_{\Omega_s} + \|v_1^*|_{\widetilde{\Gamma}_s}\|^2_{\widetilde{\Gamma}_s} + \|v_2^*\|^2_{\Omega_s} + \|h^*\|^2_{\Omega_f}\right\} \end{aligned} \tag{4.47}$$

$\forall\, |\omega| \geq$ some $\omega_0 > 0$. Explicitly (4.45) is re-written as

$$i\omega v_1 - v_2 = v_1^* \in H^1_e(\Omega_s); \tag{4.48a}$$

$$i\omega v_2 - \Delta(v_1 + v_2) = v_2^* \in L^2(\Omega_s) \tag{4.48b}$$

$$i\omega h - \Delta h = h^* \in L^2(\Omega_f). \tag{4.48c}$$

Step 2 We take the $L^2(\Omega_f)$-inner product of Eqn. (4.48c) against Δh, use Green's First Theorem to evaluate $\int_{\Omega_f} h \, \Delta \, \overline{h} \, d\Omega_f$, recall the B.C. $h|_{\Gamma_f} = 0$ in $\mathcal{D}(\mathbb{A})$ and obtain

$$i\omega \int_{\Gamma_s} h \, \frac{\partial \overline{h}}{\partial \nu} \, d\Gamma_s - i\omega \|\nabla h\|^2_{\Omega_f} - \|\Delta h\|^2_{\Omega_f} = (h^*, \Delta h)_{\Omega_f}. \tag{4.49}$$

Similarly, we take the $L^2(\Omega_s)$-inner product of (4.48b) against $\Delta(v_1 + v_2)$, use Green's First Theorem to evaluate $\int_{\Omega_s} v_2 \Delta(\overline{v}_1 + \overline{v}_2) d\Omega_s$, recalling that the normal vector ν is inward w.r.t. Ω_s, and obtain

$$-i\omega \int_{\Gamma_s} v_2 \, \frac{\partial(\overline{v}_1 + \overline{v}_2)}{\partial \nu} \, d\Gamma_s - i\omega \\ (\nabla v_2, \nabla(v_1 + v_2))_{\Omega_s} - \|\Delta(v_1 + v_2)\|^2_{\Omega_s} = (v_2^*, \Delta(v_1 + v_2))_{\Omega_s}. \tag{4.50}$$

Invoking now the B.C., $h|_{\Gamma_s} = v_2|_{\Gamma_s}$ and $\frac{\partial(v_1+v_2)}{\partial \nu}\big|_{\Gamma_s} = \frac{\partial h}{\partial \nu}\big|_{\Gamma_s} + \chi_{\widetilde{\Gamma}_s} v_1$ in $\mathcal{D}(\mathbb{A})$ (see (4.12b)), we rewrite (4.50) as

$$-i\omega \int_{\Gamma_s} h \, \frac{\partial \overline{h}}{\partial \nu} \, d\Gamma_s - i\omega \int_{\widetilde{\Gamma}_s} v_2 \tilde{v_1} \, d\widetilde{\Gamma}_s - i\omega \|\nabla v_2\|^2_{\Omega_s} - i\omega(\nabla v_2, \nabla v_1)_{\Omega_s} \\ -\|\Delta(v_1 + v_2)\|^2_{\Omega_s} = (v_2^*, \Delta(v_1 + v_2))_{\Omega_s}. \tag{4.51}$$

Summing up (4.50) and (4.51) yields, after a cancellation of the boundary terms,

$$-i\omega \left[\|\nabla v_2\|^2_{\Omega_s} + \|\nabla h\|^2_{\Omega_s} \right] = \|\Delta(v_1 + v_2)\|^2_{\Omega_s} + \|\Delta h\|^2_{\Omega_f} \\ + i\omega \left[(\nabla v_2, \nabla v_1)_{\Omega_s} + (v_2, v_1)_{\widetilde{\Gamma}_s} \right] \\ + (v_2^*, \Delta(v_1 + v_2))_{\Omega_s} + (h^*, \Delta h)_{\Omega_f}. \tag{4.52}$$

or recalling (4.12b),

$$\|\Delta(v_1 + v_2)\|^2_{\Omega_s} + \|\Delta h\|^2_{\Omega_f} + i\omega \left[(v_2, v_1)_{H^1_e(\Omega_s)} + \|\nabla v_2\|^2_{\Omega_s} + \|\nabla h\|^2_{\Omega_f} \right] \\ = -(v_2^*, \Delta(v_1 + v_2))_{\Omega_s} - (h^*, \Delta h)_{\Omega_f} \tag{4.53}$$

Next, by (4.48a), $v_1 = -\frac{i\omega}{\omega^2} \left[v_2 + v_1^* \right]$, hence,

$$i\omega(v_2, v_1)_{H^1_e(\Omega_s)} = i\omega \left(v_2, -\frac{i\omega}{\omega^2} \left[v_2 + v_1^* \right] \right)_{H^1_e(\Omega_s)} \tag{4.54}$$

$$= -\|v_2\|^2_{H^1_e(\Omega_s)} - (v_2, v_1^*)_{H^1_e(\Omega_s)}. \tag{4.55}$$

Substituting (4.55) into the LHS of (4.53) yields:

$$\|\Delta(v_1+v_2)\|_{\Omega_s}^2 + \|\Delta h\|_{\Omega_f}^2 + i\omega\left[\|\nabla v_2\|_{\Omega_s}^2 + \|\nabla h\|_{\Omega_f}^2\right]$$
$$= \|v_2\|_{H_e^1(\Omega_s)}^2 + (v_2, v_1^*)_{H_e^1(\Omega_s)} - (v_2^*, \Delta(v_1+v_2))_{\Omega_s} - (h^*, \Delta h)_{\Omega_f}. \quad (4.56)$$

Step 3 We take the real part of identity (4.56), thus obtaining a new identity.

$$\|\Delta(v_1+v_2)\|_{\Omega_s}^2 + \|\Delta h\|_{\Omega_f}^2$$
$$= \|v_2\|_{H_e^1(\Omega_s)}^2 + \mathrm{Re}(v_2, v_1^*)_{H_e^1(\Omega_s)} - \mathrm{Re}(v_2^*, \Delta(v_1+v_2)) - \mathrm{Re}(h^*, \Delta h), \quad (4.57)$$

from which we obtain:

$$(1-\epsilon)\left[\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2\right] \le (1+\epsilon)\|v_2\|_{H_e^1(\Omega_s)}^2$$
$$+ C_\epsilon\{\|v_1^*\|_{H_e^1(\Omega_s)}^2 + \|v_2^*\|^2 + \|h^*\|^2\}. \quad (4.58)$$

Step 4 We take the imaginary part of identity (4.56), thus obtaining the new identity,

$$\omega\left[\|\nabla v_2\|^2 + \|\nabla h\|^2\right] = \mathrm{Im}(v_2, v_1^*)_{H_e^1(\Omega_s)} - \mathrm{Im}(v_2^*, \Delta(v_1+v_2)) - \mathrm{Im}(h^*, \Delta h) \quad (4.59)$$

or for $\omega \neq 0$,

$$\left[\|\nabla v_2\|^2 + \|\nabla h\|^2\right] = \frac{1}{\omega}\left\{\mathrm{Im}(v_2, v_1^*)_{H_e^1(\Omega_s)} - \mathrm{Im}(v_2^*, \Delta(v_1+v_2)) - \mathrm{Im}(h^*, \Delta h)\right\}. \quad (4.60)$$

Step 5 Next, by Poincare inequality, since $h|_{\Gamma_f} = 0$ and $v_2|_{\Gamma_s} = h|_{\Gamma_s}$, we have,

$$\|v_2|_{\Gamma_s}\|_{L^2(\Gamma_s)}^2 = \|h|_{\Gamma_s}\|_{L^2(\Gamma_s)}^2 \le c\left[\|\nabla h\|^2 + \|h\|^2\right] \le \tilde{c}\|\nabla h\|^2. \quad (4.61)$$

Using (4.61) on the LHS of identity (4.60) yields:

$$\|\nabla v_2\|^2 + \frac{1}{\tilde{c}}\|v_2|_{\tilde{\Gamma}_s}\|_{\tilde{\Gamma}_s}^2 \le \|\nabla v_2\|^2 + \frac{1}{\tilde{c}}\|v_2|_{\Gamma_s}\|_{L^2(\Gamma_s)}^2 \quad (4.62)$$
$$\le \|\nabla v_2\|^2 + \|\nabla h\|^2 \quad (4.63)$$
$$\le \frac{1}{|\omega|}\left\{\varepsilon\left[\|v_2\|_{H_e^1(\Omega_s)}^2 + \|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2\right]\right.$$
$$\left.+\frac{1}{\varepsilon}\left[\|v_1^*\|_{H_e^1(\Omega_s)}^2 + \|v_2^*\|^2 + \|h^*\|^2\right]\right\}. \quad (4.64)$$

Thus, taking $c_1 = min\left\{1, \frac{1}{\bar{c}}\right\}$ and $|\omega| \geq$ some $\omega_0 > 0$, we obtain from (4.64),

$$\|v_2\|^2_{H^1_e(\Omega_s)} = \|\nabla v_2\|^2 + \|v_2|_{\widetilde{\Gamma}_s}\|^2_{L^2(\widetilde{\Gamma}_s)} \leq \frac{\varepsilon}{c_1\omega_0}\|v_2\|^2_{H^1_e(\Omega_s)} + \frac{\varepsilon}{c_1\omega_0}\left[\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2\right] + \frac{1}{\varepsilon c_1\omega_0}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right]. \tag{4.65}$$

Finally, by taking $\varepsilon < \frac{c_1\omega_0}{2}$, we get

$$\frac{1}{2}\|v_2\|^2_{H^1_e(\Omega_s)} \leq \left[1 - \frac{\varepsilon}{c_1\omega_0}\right]\|v_2\|^2_{H^1_e(\Omega_s)} \leq \frac{\varepsilon}{c_1\omega}\left[\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2\right] + \frac{1}{\varepsilon c_1\omega_0}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right]. \tag{4.66}$$

Step 6 We substitute $\|v_2\|^2_{H^1_e(\Omega_s)}$ from (4.66) into the RHS of inequality (4.58) with $\frac{1}{2} < (1-\varepsilon)$, thus obtaining,

$$\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2 \leq 2(1+\varepsilon)\|v_2\|^2_{H^1_e(\Omega_s)} + \widetilde{c_\varepsilon}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right]$$

$$\leq \frac{2(1+\varepsilon)2\varepsilon}{c_1\omega_0}\left[\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2\right] \tag{4.67}$$

$$+ \widehat{c_\varepsilon}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right], \tag{4.68}$$

from which we obtain the estimate:

$$\|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2 \leq const_{\varepsilon,\omega_0}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right], \quad \forall\ |\omega| \geq \text{ some } \omega_0 > 0. \tag{4.69}$$

Step 7 Substituting estimate (4.69) on the RHS of estimate (4.66), we obtain likewise,

$$\|v_2\|^2_{H^1_e(\Omega_s)} \leq Const_{\varepsilon,\omega_0}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right], \quad \forall\ |\omega| \geq \text{ some } \omega_0 > 0 \tag{4.70}$$

Step 8 Summing up (4.69) and (4.70) yields the sought after estimate (4.47):

$$\|v_2\|^2_{H^1_e(\Omega_s)} + \|\Delta(v_1+v_2)\|^2 + \|\Delta h\|^2 \leq \mathcal{C}_{\varepsilon,\omega_0}\left[\|v_1^*\|^2_{H^1_e(\Omega_s)} + \|v_2^*\|^2 + \|h^*\|^2\right], \quad \forall\ |\omega| \geq \text{ some } \omega_0 > 0, \tag{4.71}$$

and Theorem 4.9 is proved. $e^{\mathbb{A}t}$ is analytic on $\mathbf{H}_e$, hence on $\mathbf{H}$. □

4.3 Uniform Stability of $e^{\mathbb{A}t}$ on $\mathbf{H}_e$, Hence on $\mathbf{H}$

Theorem 4.10 *The s.c contraction, analytic semigroup $e^{\mathbb{A}t}$ on $\mathbf{H}_e$ is moreover uniformly stable here: there exist constants $M \geq 1, \delta > 0$ such that*

$$\|e^{\mathbb{A}t}\|_{\mathcal{L}(\mathbf{H}_e)} \leq Me^{-\delta t}, \quad t \geq 0, \tag{4.72}$$

hence, as $\mathbf{H}_e$ and $\mathbf{H}$ are norm-equivalent,

$$\|e^{\mathbb{A}t}\|_{\mathcal{L}(\mathbf{H})} \leq M_1 e^{-\delta t}, \quad t \geq 0. \tag{4.73}$$

Proof The resolvent bound (4.43) combined with $0 \in \rho(\mathbb{A})$ (or $\mathbb{A}^{-1} \in \mathcal{L}(\mathbf{H}_e)$), hence $\mathcal{S}_{r_0} \subset \rho(\mathbb{A})$ by (4.13), allows one to conclude that the resolvent operator is uniformly bounded on the imaginary axis $i\mathbb{R}$:

$$\|R(i\omega, \mathbb{A})\|_{\mathcal{L}(\mathbf{H})} \leq \text{const}, \quad \forall \omega \in \mathbb{R}. \tag{4.74}$$

Hence, [33] the s.c. analytic semigroup $e^{\mathbb{A}t}$ is, moreover. (uniformly) exponentially stable and (4.72) is proved. □

5 Consequences of Sect. 4.3: The pair $\{\mathcal{A}, \mathcal{B}\}$ is *stabilizable* by feedback $\mathcal{F}$ in (4.1); the pair $\{\mathcal{A}, \mathcal{C}\}$ is *detectable* by $\mathcal{K}$ in (4.1). Assumptions (A.1) and (A.2) are verified, and so is analyticity of $(\mathcal{A} + \mathcal{B}\mathcal{F})$ and $(\mathcal{A} - \mathcal{K}\mathcal{C})$

The results of semigroup well-posedness, in particular analyticity, and uniform stability, refer to the PDE-problem (4.9 a–e) for an arbitrary choice of a portion of the boundary $\widetilde{\Gamma}_s$ of Γ_s of positive measure.

Next, if we specialize to $\widetilde{\Gamma}_s = \Gamma_s^c$, we obtain the PDE-problem (4.4 a–e), whose abstract version is given by $\dot{y} = (\mathcal{A} + \mathcal{B}\mathcal{F})y$ in (4.3), (4.5). This means that the open loop pair $\{\mathcal{A}, \mathcal{B}\}$,whose PDE-version given by (3.7 a–e) is stabilizable by the feedback operator $\mathcal{F}$ in (4.1); in other words, on the space $\mathbf{H}_e$, the s.c. contraction analytic semigroup $e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}$ is moreover, uniformly stable. This verifies assumption (A.1), with the feedback operator $\mathcal{F}$.

Similarly, if we specialize to $\widetilde{\Gamma}_s = \Gamma_s^o$, we obtain the PDE-problem (4.7 a–e), whose abstract version is given by $\dot{y} = (\mathcal{A} - \mathcal{K}\mathcal{C})y$ in (4.6), with $\mathcal{K}$ defined by (4.1). The partial observation $\mathcal{C}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \chi_{\Gamma_s^o} w$ in (3.23) picks up only the first coordinate at the portion Γ_s^o of the interface Γ_s. This means that the pair $\{\mathcal{A}, \mathcal{C}\}$ is detectable, i.e. there exists an operator $\mathcal{K}$ in (4.1) such that the corresponding s.c. contraction analytic semigroup $e^{(\mathcal{A}-\mathcal{K}\mathcal{C})t}$ is moreover uniformly stable on $\mathbf{H}_e$. This verifies assumption (A.2). Moreover, the analyticity Theorem 4.9 for $e^{\mathbb{A}t}$ applies also to $e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}$ and $e^{(\mathcal{A}-\mathcal{K}\mathcal{C})t}$ for $\widetilde{\Gamma}_s$ specialized to $\widetilde{\Gamma}_s = \Gamma_s^c$ or $\widetilde{\Gamma}_s = \Gamma_s^o$ respectively. Thus Theorem 2.2 is established.

We next recall that:

(1) The PDE-problem corresponding to the abstract equation $\dot{y} = \mathcal{A}y + \mathcal{B}g,\ g = \mathcal{F}z$, $y = [w, w_t, u]\,,\ z = [z_1, z_2 = z_{1t}, z_3]$ is given by (4.4 a–e);
(2) The PDE-problem corresponding to $\dot{z} = (\mathcal{A} + \mathcal{B}\mathcal{F} - \mathcal{K}\mathcal{C})z + \mathcal{K}(\mathcal{C}y)$ is given by (1.6 a–e).

5.1 The PDE-Problem Corresponding to $\dot{d} = (\mathcal{A} - \mathcal{K}\mathcal{C})d,\ d = y - z$

In the Luenberger's theory, the ultimate goal is to establish uniform stability of the problem $\dot{d} = (\mathcal{A} - \mathcal{K}\mathcal{C})d$, where,

$$\begin{aligned} d(t) = y(t) - z(t) &= [w(t) - z_1(t), w_t(t) - z_2(t), u(t) - z_3(t)] \\ &= [d_1(t), d_2(t), d_3(t)]\,. \end{aligned} \tag{5.1}$$

Recalling (4.6), (4.7 a–e) and relabeling the notation $[d_1(t), d_2(t), d_3(t)] = [\widehat{w}(t), \widehat{w}_t(t), \widehat{u}(t)]$, we see that the PDE-problem corresponding to $\dot{d} = (\mathcal{A} - \mathcal{K}\mathcal{C})d$ is:

$$\widehat{u}_t - \Delta\widehat{u} \equiv 0 \qquad \text{in } (0, T] \times \Omega_f, \tag{5.2a}$$

$$\widehat{w}_{tt} - \Delta\widehat{w} - \Delta\widehat{w}_t \equiv 0 \qquad \text{in } (0, T] \times \Omega_s, \tag{5.2b}$$

$$\widehat{u}|_{\Gamma_f} = 0 \qquad \text{in } (0, T] \times \Gamma_f, \tag{5.2c}$$

$$\widehat{u}|_{\Gamma_s} = \widehat{w}_t|_{\Gamma_s} \qquad \text{in } (0, T] \times \Gamma_s, \tag{5.2d}$$

$$\frac{\partial(\widehat{w} + \widehat{w}_t)}{\partial\nu} = \frac{\partial\widehat{u}}{\partial\nu} - \chi_{\Gamma_s^o}\widehat{w} \qquad \text{in } (0, T] \times \Gamma_s. \tag{5.2e}$$

This is precisely problem (4.7 a–e) that was shown in Sect. 4.3/5 to be uniformly stable, so that the Luenberger theory ultimate goal:

$$\|y(t) - z(t)\| \leq Me^{-\delta t}\|y_0 - z_0\|, \tag{5.3}$$

is achieved on either $\mathbf{H}_e$ or $\mathbf{H}$. It remains to establish Theorem 2.1.

6 Proof of Theorem 2.1

Theorem 2.2 was established in Sects. 4.1, 4.2, 4.3 as noted in Sect. 5: thus the operator $(\mathcal{A} + \mathcal{B}\mathcal{F})$ defined in (4.5) and the operator $(\mathcal{A} - \mathcal{K}\mathcal{C})$ defined in (4.8) are generators of s.c. semigroups on $\mathbf{H}$ and $\mathbf{H}_e$, which moreover are analytic and uniformly stable in these spaces (and moreover contraction on $\mathbf{H}_e$). These results will be now critically used to prove Theorem 2.1. The next point is to use the transformation introduced in [32], see also [24, p 497]

$$\mathcal{J} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \quad \text{with inverse} \quad \mathcal{J}^{-1} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}, \tag{6.1}$$

to transform $\mathscr{A}$ into its similarity form:

$$\widehat{\mathscr{A}} = \mathcal{J}^{-1}\mathscr{A}\mathcal{J} = \begin{bmatrix} \mathcal{A}-\mathcal{KC} & 0 \\ \mathcal{KC} & \mathcal{A}+\mathcal{BF} \end{bmatrix}. \tag{6.2}$$

This leads us to the study of the transformed system. For $\lambda \in \mathbb{C}$ to be further identified below, we compute the resolvent $R\left(\lambda, \widehat{\mathscr{A}}\right)$ of the operator $\widehat{\mathscr{A}}$ in (6.2). For $\{\widetilde{y}, \widetilde{z}\} \in \mathbf{H} \times \mathbf{H} = \mathcal{H}$, we seek to solve:

$$\left(\lambda I_{\mathcal{H}} - \widehat{\mathscr{A}}\right)\begin{bmatrix} y \\ z \end{bmatrix} = \lambda I_{\mathbf{H}\times\mathbf{H}} - \begin{bmatrix} \mathcal{A}-\mathcal{KC} & 0 \\ \mathcal{KC} & \mathcal{A}+\mathcal{BF} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \widetilde{y} \\ \widetilde{z} \end{bmatrix}, \tag{6.3}$$

for $[y, z] \in \mathcal{D}\left(\widehat{\mathscr{A}}\right)$. Thus,

$$[\lambda I_{\mathbf{H}} - (\mathcal{A}-\mathcal{KC})]\, y = \widetilde{y} \quad \text{or} \quad y = R\,(\lambda, \mathcal{A}-\mathcal{KC})\,\widetilde{y}, \tag{6.4}$$

and

$$[\lambda I_{\mathbf{H}} - (\mathcal{A}+\mathcal{BF})]\, z = \mathcal{KC}y + \widetilde{z}, \tag{6.5}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = R\left(\lambda, \widehat{\mathscr{A}}\right)\begin{bmatrix} \widetilde{y} \\ \widetilde{z} \end{bmatrix} = \begin{bmatrix} R\,(\lambda, \mathcal{A}-\mathcal{KC}) & 0 \\ R\,(\lambda, \mathcal{A}+\mathcal{BF})\,\mathcal{KC}R\,(\lambda, \mathcal{A}-\mathcal{KC}) & R\,(\lambda, \mathcal{A}+\mathcal{BF}) \end{bmatrix} \begin{bmatrix} \widetilde{y} \\ \widetilde{z} \end{bmatrix}. \tag{6.6}$$

The resolvents $R\,(\lambda, \mathcal{A}-\mathcal{KC})$ and $R\,(\lambda, \mathcal{A}+\mathcal{BF})$ are well-defined for all $\lambda \in \mathbb{C}$ outside the usual triangular sector with vertex on the negative real axis, which contains the spectrum of the analytic, uniformly stable generators $(\mathcal{A}-\mathcal{KC})$ and $(\mathcal{A}+\mathcal{BF})$. We shall next establish that the term $R\,(\lambda, \mathcal{A}+\mathcal{BF})\,\mathcal{KC}$ is likewise well defined in $\mathbf{H}$. To this end we shall establish the following.

Lemma 6.1 *With reference to the operator $(\mathcal{KC})$ in (4.2), its adjoint $(\mathcal{KC})^*$ is given by,*

$$(\mathcal{KC})^*\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} NN^*A_{N,tr}f_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -N\left(f_2|_{\Gamma_s}\right) \\ 0 \\ 0 \end{bmatrix}, \tag{6.7}$$

for $[f_1, f_2, f_3] \in \mathcal{D}\left((\mathcal{KC})^*\right)$, and thus requires $f_2 \in H^{1/2+\epsilon}(\Omega_s)$.

Proof For $[v_1, v_2, v_3] \in \mathcal{D}(\mathcal{KC})$ and $[f_1, f_2, f_3] \in \mathcal{D}\left((\mathcal{KC})^*\right)$, we compute, recalling (4.2)

$$\begin{aligned}\left(\mathcal{KC}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right)_{\mathbf{H}} &= \left(\begin{bmatrix} 0 \\ -A_{N,tr}N\left(\chi_{\Gamma_s^o}v_1|_{\Gamma_s}\right) \\ 0 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right)_{\mathbf{H}} \\ &= \left(-A_{N,tr}N\left(\chi_{\Gamma_s^o}v_1|_{\Gamma_s}\right), f_2\right)_{L^2(\Omega_s)}\end{aligned} \tag{6.8}$$

$$= \left(-\left(\chi_{\Gamma_s^o} v_1|_{\Gamma_s}\right), N^* A_{N,tr} f_2\right)_{L^2(\Gamma_s)} \tag{6.9}$$

$$(by\ (3.19)) = \left(N^* A_{N,tr} v_1|_{\Gamma_s}, N^* A_{N,tr} f_2\right)_{L^2(\Gamma_s)} \tag{6.10}$$

$$= \left(A_{N,tr}^{1/2} v_1, A_{N,tr}^{1/2} N N^* A_{N,tr} f_2\right)_{L^2(\Omega_s)} \tag{6.11}$$

$$= \left(v_1, N N^* A_{N,tr} f_2\right)_{H^1(\Omega_s)} \tag{6.12}$$

$$(by\ (3.19)) = \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} N N^* A_{N,tr} f_2 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathbf{H}} = \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} -N\left(f_2|_{\Gamma_s}\right) \\ 0 \\ 0 \end{bmatrix}\right)_{\mathbf{H}} \tag{6.13}$$

$$= \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} 0 & N N^* A_{N,tr} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right)_{\mathbf{H}} \tag{6.14}$$

$$= \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, (\mathcal{K}\mathcal{C})^* \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right)_{\mathbf{H}}, \tag{6.15}$$

and (6.15) proves (6.7). □

In order to show that the terms $R(\lambda, \mathcal{A} + \mathcal{B}\mathcal{F})(\mathcal{K}\mathcal{C})$ is well defined on $\mathbf{H}$ in the second row of (6.6), we re-write it as:

$$R(\lambda, \mathcal{A} + \mathcal{B}\mathcal{F})\,\mathcal{K}\mathcal{C} = [\lambda - (\mathcal{A} + \mathcal{B}\mathcal{F})]^{1/2}\, R(\lambda, \mathcal{A}+\mathcal{B}\mathcal{F})\, [\lambda - (\mathcal{A}+\mathcal{B}\mathcal{F})]^{-1/2} (\mathcal{K}\mathcal{C}) \tag{6.16}$$

whereby all we need to show is that:

Lemma 6.2

$$(\mathcal{K}\mathcal{C})^* \left[\lambda - (\mathcal{A} + \mathcal{B}\mathcal{F})^*\right]^{-1/2} \in \mathcal{L}(\mathbf{H}). \tag{6.17}$$

Proof Let $y \in \mathbf{H}$. We want to show that:

$$(\mathcal{K}\mathcal{C})^* f \in \mathbf{H}, \text{ where } \quad f = [\lambda - (\mathcal{A} + \mathcal{B}\mathcal{F})]^{-1/2}\, y, \tag{6.18}$$

so that $f \in \mathcal{D}\left([\lambda - (\mathcal{A} + \mathcal{B}\mathcal{F})]^{1/2}\right) = \mathcal{D}\left([-(\mathcal{A} + \mathcal{B}\mathcal{F})]^{1/2}\right)$. In other words, we want to show that:

$$(\mathcal{K}\mathcal{C})^* \left[-(\mathcal{A} + \mathcal{B}\mathcal{F})^*\right]^{-1/2} \in \mathcal{L}(\mathbf{H}). \tag{6.19}$$

i.e. that $(\mathcal{K}\mathcal{C})^*$ is well-defined in $\mathcal{D}\left(\left[-(\mathcal{A} + \mathcal{B}\mathcal{F})^*\right]^{1/2}\right)$. To see this, we recall [25, 26, 36] that,

$$[f_1, f_2, f_3] \in \mathcal{D}\left(\left[-(\mathcal{A} + \mathcal{B}\mathcal{F})^*\right]^{1/2}\right), \tag{6.20a}$$

means that f_1, f_2, f_3 have the same regularity of the operator $(\mathcal{A}+\mathcal{B}\mathcal{F})^*$ or of $\mathcal{A}$, i.e.

$$f_1 \in H^1(\Omega_s), \quad f_2 \in H^1(\Omega_s), \quad f_3 \in H^1(\Omega_f), \tag{6.20b}$$

while only the lower order B.C. of $\mathcal{D}((\mathcal{A}+\mathcal{B}\mathcal{F})^*)$, i.e. of $\mathcal{D}(\mathcal{A})$ apply; i.e.

$$f_1|_{\Gamma_f} = 0, \quad f_1|_{\Gamma_s} = f_2|_{\Gamma_s}. \tag{6.20c}$$

But the definition (6.7) of $(\mathcal{K}\mathcal{C})^*$ shows that $(\mathcal{K}\mathcal{C})^*$ is well-defined a-fortiori for $f_2 \in H^1(\Omega_s)$ as is the case by (6.20b). Thus, Lemma 6.2 is established.
In conclusion The resolvent expression $R(\lambda, \widehat{\mathscr{A}})$ in (6.6) is well defined for all $\lambda \in \mathbb{C}$ for which $R(\lambda, \mathcal{A}+\mathcal{B}\mathcal{F})$ and $R(\lambda, \mathcal{A}-\mathcal{K}\mathcal{C})$ are well defined, with $(\mathcal{A}+\mathcal{B}\mathcal{F})$ and $(\mathcal{A}-\mathcal{K}\mathcal{C})$ uniformly stable, analytic generators. □

Proof *Analyticity* A few proofs may be given. Perhaps simplest is to consider the perturbation of an analytic generator view point [19, 31]: We want show that

$$\begin{aligned} P = \begin{bmatrix} 0 & 0 \\ \mathcal{K}\mathcal{C} & 0 \end{bmatrix} &\text{ is relatively bounded with respect to } \widehat{\mathscr{A}}^{1/2} \\ = &\begin{bmatrix} [-(\mathcal{A}-\mathcal{K}\mathcal{C})]^{1/2} & 0 \\ 0 & [-(\mathcal{A}+\mathcal{B}\mathcal{F})]^{1/2} \end{bmatrix}, \end{aligned} \tag{6.21}$$

or passing to the adjoints, that

$$\begin{aligned} P^* = \begin{bmatrix} 0 & (\mathcal{K}\mathcal{C})^* \\ 0 & 0 \end{bmatrix} &\text{ is relatively bounded with respect to } \widehat{\mathscr{A}}^{*1/2} \\ = &\begin{bmatrix} \left[-(\mathcal{A}-\mathcal{K}\mathcal{C})^*\right]^{1/2} & 0 \\ 0 & \left[-(\mathcal{A}+\mathcal{B}\mathcal{F})^*\right]^{1/2} \end{bmatrix} \end{aligned} \tag{6.22}$$

more precisely that,

$$\|(\mathcal{K}\mathcal{C})^* y\|_{\mathbf{H}} \leq C \| \left[-(\mathcal{A}+\mathcal{B}\mathcal{F})^*\right]^{1/2} y \|_{\mathbf{H}}, \quad y \in \mathcal{D}((\mathcal{K}\mathcal{C})^*), \tag{6.23}$$

or, explicitly, setting $z = (\mathcal{A}+\mathcal{B}\mathcal{F})^{*1/2} y$:

$$\|(\mathcal{K}\mathcal{C})^* \left[-(\mathcal{A}+\mathcal{B}\mathcal{F})^*\right]^{-1/2} z\|_{\mathbf{H}} \leq C \|z\|_{\mathbf{H}}, \quad z \in \mathbf{H}. \tag{6.24}$$

But this is precisely what (6.19) shows. Thus the classical perturbation of an analytic generator applies [19, 31].

Uniform Boundedness We have concluded below (6.20c) that the resolvent expression $R\left(\lambda, \widehat{\mathscr{A}}\right)$ in (6.6) is well-defined for all $\lambda \in \mathbb{C}$, save in the triangular sector containing the spectrum of $(\mathcal{A}-\mathcal{K}\mathcal{C})$ and $(\mathcal{A}+\mathcal{B}\mathcal{F})$. Thus,

$$\operatorname{Re} \sigma\left(\widehat{\mathscr{A}}\right) \leq -\delta, \quad \text{for some } \delta > 0. \tag{6.25}$$

Hence uniform stability of $e^{\widehat{\mathscr{A}}t}$ follows via spectral theorem. □

7 Appendix

7.1 Lemma 4.1: Explicit Expression of $\mathbb{A}^{-1}$

We reference to the operator $\mathbb{A}$ in (4.10),(4.11), we have established that:

$$\mathbb{A}^{-1} \in \mathcal{L}(\mathbf{H}) \quad \text{or} \quad 0 \in \rho(\mathbb{A}), \tag{7.1}$$

with explicit expression given in (7.11) below. Let $\left[\widetilde{v}_1, \widetilde{v}_2, \widetilde{h}\right] \in \mathbf{H}$. We seek to solve:

$$\mathbb{A}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix} = \begin{bmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \\ \widetilde{h} \end{bmatrix}, \tag{7.2}$$

uniquely for $[v_1, v_2, h] \in \mathcal{D}(\mathbb{A})$ in (4.12). From (7.2)-(i), we first obtain, $v_2 = \widetilde{v}_1 \in H^1(\Omega_s)$, thus $v_2|_{\Gamma_s} = \widetilde{v}_1|_{\Gamma_s} \in H^{1/2}(\Omega_s)$. From (7.2)-(iii), we next obtain:

$$\Delta h = \widetilde{h} \in L^2(\Omega_f), \tag{7.3a}$$

$$h|_{\Gamma_f} = 0,\ h|_{\Gamma_s} = v_2|_{\Gamma_s} = \widetilde{v}_1|_{\Gamma_s} \in H^{1/2}(\Gamma_s). \tag{7.3b}$$

Hence, the unique solution is:

$$h = A_{D,f}^{-1}\widetilde{h} + \widetilde{D}\left(\widetilde{v}_1|_{\Gamma_s}\right) \in \mathcal{D}(A_{D,f}) + H^1(\Omega_f), \tag{7.4}$$

where $A_{D,f}\varphi = \Delta\varphi$ in Ω_f, $\mathcal{D}(A_{D,f}) = H^2(\Omega_s) \cap H_0^1(\Omega_f)$, and $\widetilde{D}$ is the Dirichlet map in Ω_f, acting from Γ_s:

$$\widetilde{D}g = \varphi \iff \left\{\Delta\varphi = 0 \text{ in } \Omega_f, \varphi|_{\Gamma_f} = 0, \varphi|_{\Gamma_s} = g\right\}. \tag{7.5}$$

Finally, from (7.2)-(iii), recalling $v_2 = \widetilde{v}_1$ and the B.C., we have,

$$\Delta(v_1 + v_2) = \widetilde{v}_2 \text{ in } \Omega_s, \quad \Delta v_1 = \widetilde{v}_2 - \Delta\widetilde{v}_1 \in H^{-1}(\Omega_s), \tag{7.6a}$$

$$\frac{\partial(v_1 + v_2)}{\partial\nu} = \frac{\partial h}{\partial\nu} + \chi_{\widetilde{\Gamma}_s} v_1 \text{ on } \Gamma_s, \quad \text{or} \quad \left[\frac{\partial v_1}{\partial\nu} - \chi_{\widetilde{\Gamma}_s} v_1\right]_{\Gamma_s} = \left[\frac{\partial h}{\partial\nu} - \frac{\partial\widetilde{v}_1}{\partial\nu}\right]_{\Gamma_s} \in H^{-1/2}(\Gamma_s), \tag{7.6b}$$

a Robin problem, whose unique solution is:

$$v_1 = A_{R,s}^{-1}\left[\widetilde{v}_2 - \Delta\widetilde{v}_1\right] + R_s\left[\frac{\partial h}{\partial\nu} - \frac{\partial\widetilde{v}_1}{\partial\nu}\right]_{\Gamma_s}, \tag{7.7}$$

In (7.7), $A_{R,s}$ is the Robin Laplacian on Ω_s, and R_s is the Robin map:

$$A_{R,s}\varphi = \Delta\varphi, \quad \mathcal{D}(A_{R,s}) = \left\{\varphi \in H^2(\Omega_s) : \left[\frac{\partial\varphi}{\partial\nu} - \chi_{\widetilde{\Gamma}_s}\varphi\right]_{\Gamma_s} = 0\right\} \tag{7.8}$$

$$R_s\mu = f \iff \left\{\Delta f = 0 \quad \text{in } \Omega_s, \ \frac{\partial f}{\partial\nu} - \chi_{\widetilde{\Gamma}_s} f = \mu\right\}. \tag{7.9}$$

Recalling (7.4) in (7.7), we rewrite it explicitly in terms of $\left[\tilde{v}_1, \tilde{v}_2, \tilde{h}\right]$ as

$$v_1 = A_{R,s}^{-1}\left[\tilde{v_2} - \Delta\tilde{v_1}\right] + R_s\left\{\frac{\partial}{\partial\nu}\left[A_{D,f}^{-1}\tilde{h} + \tilde{D}(\tilde{v}_1|_{\Gamma_s})\right] - \frac{\partial\tilde{v}_1}{\partial\nu}\bigg|_{\Gamma_s}\right\}, \tag{7.10}$$

Ultimately we obtain:

$$\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -A_{R,s}^{-1}\Delta + R_s\frac{\partial\widetilde{D}}{\partial\nu}(\cdot|_{\Gamma_s}) - \frac{\partial}{\partial\nu} & A_{R,s}^{-1} & R_s\frac{\partial A_{D,f}^{-1}}{\partial\nu} \\ I & 0 & 0 \\ \widetilde{D}(\cdot|_{\Gamma_s}) & 0 & A_{D,f}^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \\ \widetilde{h} \end{bmatrix}. \tag{7.11}$$

7.2 Adjoint $\mathbb{A}^*$ of $\mathbb{A}$ on $\mathbf{H}_e$. Proof of Proposition 4.6

Let $\{w_1, w_2, u\} \in \mathcal{D}(\mathbb{A})$, hence subject to the conditions (4.12 a, b), and let $\{v_1, v_2, h\} \in \mathbf{H}_e$, subject to the conditions in (4.34 a, b). We compute:

$$\begin{aligned}\left(\mathbb{A}\begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} &= \left(\begin{bmatrix} w_2 \\ \Delta(w_1 + w_2) \\ \Delta u \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} \\ &= (\nabla w_2, \nabla v_1)_{\Omega_s} + (w_2, \widetilde{v}_1)_{\widetilde{\Gamma}_s} + (\Delta(w_1 + w_2), v_2)_{\Omega_s} \\ &\quad + (\Delta u, h)_{\Omega_f}.\end{aligned} \tag{7.12}$$

By Green's Second Theorem on Ω_f, recalling $u|_{\Gamma_f} = 0$, $h|_{\Gamma_f} = 0$, we obtain:

$$(\Delta u, h) = (u, \Delta h) + \int_{\Gamma_s} \frac{\partial u}{\partial\nu}\,\overline{h}\,d\Gamma_s - \int_{\Gamma_s} u\,\frac{\partial\overline{h}}{\partial\nu}\,d\Gamma_s. \tag{7.13}$$

Similarly, by Green's Second Theorem on Ω_s, recalling that the unit normal vector ν is inward w.r.t. Ω_s,

$$\begin{aligned}(\Delta(w_1 + w_2), v_2)_{\Omega_s} = (w_1 + w_2, \Delta v_2)_{\Omega_s} - \int_{\Gamma_s} \frac{\partial(w_1 + w_2)}{\partial\nu}\,\overline{v}_2\,d\Gamma_s \\ + \int_{\Gamma_s} (w_1 + w_2)\frac{\partial\overline{v}_2}{\partial\nu}\,d\Gamma_s\end{aligned}$$

$$= (w_1 + w_2, \Delta v_2)_{\Omega_s} - \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \overline{v}_2 \, d\Gamma_s \\ - \int_{\widetilde{\Gamma}_s} w_1 \overline{v}_2 \, d\widetilde{\Gamma}_s + \int_{\Gamma_s} (w_1 + w_2) \frac{\partial \overline{v}_2}{\partial \nu} \, d\Gamma_s, \tag{7.14}$$

recalling, $\frac{\partial(w_1+w_2)}{\partial \nu} = \frac{\partial u}{\partial \nu} + \chi_{\widetilde{\Gamma}_s} w_1$ on Γ_s. Summing up (7.13) and (7.14) and recalling that $v_2|_{\Gamma_s} = h|_{\Gamma_s}$, $w_2|_{\Gamma_s} = u|_{\Gamma_s}$, we obtain

$$\text{RHS of (7.12)} = (\nabla w_2, \nabla v_1)_{\Omega_s} \\ + (w_2, v_1)_{\widetilde{\Gamma}_s} + (w_1 + w_2, \Delta v_2)_{\Omega_s} - \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \overline{h} \, d\Gamma_s - (w_1, v_2)_{\widetilde{\Gamma}_s} \\ + \int_{\Gamma_s} (w_1 + w_2) \frac{\partial \overline{v_2}}{\partial \nu} \, d\Gamma_s + (u, \Delta h) + \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \overline{h} \, d\Gamma_s \\ - \int_{\Gamma_s} w_2 \frac{\partial \overline{h}}{\partial \nu} \, d\Gamma_s \tag{7.15}$$

Next, recalling again that ν is inward to Ω_s, we obtain:

$$(w_1, \Delta v_2)_{\Omega_s} = \int_{\Omega_s} \Delta \overline{v}_2 w_1 \, d\Omega_s = - \int_{\Gamma_s} \frac{\partial \overline{v}_2}{\partial \nu} w_1 \, d\Gamma_s - (\nabla w_1, \nabla w_2)_{\Omega_s} \tag{7.16}$$

$$(\nabla w_2, \nabla v_1)_{\Omega_s} = - (w_2, \Delta v_1)_{\Omega_s} - \int_{\Gamma_s} \frac{\partial \overline{v}_1}{\partial \nu} w_2 \, d\Gamma_s. \tag{7.17}$$

Substituting (7.16) and (7.17) on the RHS of (7.15) yields:

$$\text{RHS of (7.12)} = - (w_2, \Delta v_1)_{\Omega_s} - \int_{\Gamma_s} \frac{\partial \overline{v}_1}{\partial \nu} w_2 \, d\Gamma_s + (w_2, v_1)_{\widetilde{\Gamma}_s} \\ - \int_{\Gamma_s} \frac{\partial \overline{v}_2}{\partial \nu} w_1 \, d\Gamma_s - (\nabla w_1, \nabla w_2)_{\Omega_s} + (w_2, \Delta v_2)_{\Omega_s} - (w_1, v_2)_{\widetilde{\Gamma}_s} \\ + \int_{\Gamma_s} w_1 \frac{\partial \overline{v}_2}{\partial \nu} \, d\Gamma_s \\ + \int_{\Gamma_s} w_2 \frac{\partial \overline{v}_2}{\partial \nu} \, d\Gamma_s + (u, \Delta u)_{\Omega_f} - \int_{\Gamma_s} w_2 \frac{\partial \overline{h}}{\partial \nu} \, d\Gamma_s \\ = - (\nabla w_1, \nabla w_2)_{\Omega_s} - (w_1, v_2)_{\widetilde{\Gamma}_s} + (w_2, \Delta(v_2 - v_1))_{\Omega_s} \\ + (u, \Delta u)_{\Omega_f} + \int_{\Gamma_s} w_2 \left[\frac{\partial(\overline{v}_2 - \overline{v}_1)}{\partial \nu} - \frac{\partial \overline{h}}{\partial \nu} + \chi_{\widetilde{\Gamma}_s} \overline{v}_1 \right] d\Gamma_s \\ = - (w_1, v_2)_{H_e^1(\Omega_s)} + (w_2, \Delta(v_2 - v_1))_{\Omega_s} + (u, \Delta u)_{\Omega_f}, \tag{7.18}$$

since by (4.34b),

$$\frac{\partial(\overline{v}_2 - \overline{v}_1)}{\partial \nu} = - \frac{\partial \overline{h}}{\partial \nu} - \chi_{\widetilde{\Gamma}_s} \overline{v}_1, \quad \text{on } \Gamma_s. \tag{7.19}$$

Finally (7.18) can be rewritten as:

$$\left(\mathbb{A}\begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = \left(\begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} 0 & -I & 0 \\ -\Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} \tag{7.20}$$

or

$$\mathbb{A}^*\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ -\Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) \\ \Delta h \end{bmatrix}, \tag{7.21}$$

for $[v_1, v_2, h]$ satisfying conditions (4.34) characterizing $\mathcal{D}(\mathbb{A}^*)$.

7.3 Dissipativity of $\mathbb{A}^*$ on the Space $\mathbf{H}_e$

Proposition 7.1 *Let* $[v_1, v_2, h]\,, \left[\widetilde{v}_1, \widetilde{v}_2, \widetilde{h}\right] \in \mathcal{D}(\mathbb{A}^*)$ *defined in (4.34). Then,*

1.

$$\left(\mathbb{A}^*\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \widetilde{v}_1 \\ \widetilde{v}_2 \\ \widetilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = -(\nabla v_2, \nabla\widetilde{v}_1)_{\Omega_s} + (\nabla v_1, \nabla\widetilde{v}_2)_{\Omega_s} - (v_2, \widetilde{v}_1)_{\widetilde{\Gamma}_s} + (v_1, \widetilde{v}_2)_{\widetilde{\Gamma}_s} - (\nabla v_2, \nabla\widetilde{v}_2)_{\Omega_s} - \left(\nabla h, \nabla\widetilde{h}\right)_{\Omega_f}. \tag{7.22}$$

2. Let now $[v_1, v_2, h] = \left[\widetilde{v}_1, \widetilde{v}_2, \widetilde{h}\right] \in \mathcal{D}(\mathbb{A}^*)$,

$$\left(\mathbb{A}^*\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = 2\,Im\,(\nabla v_1, \nabla v_2)_{\Omega_s} + 2\,Im\,(v_1, v_2)_{\widetilde{\Gamma}_s} - \|\nabla v_2\|^2 - \|\nabla h\|^2. \tag{7.23}$$

3. $\mathbb{A}^*$ *is dissipative on* $\mathbf{H}_e$*:*

$$Re\left(\mathbb{A}^*\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}\right)_{\mathbf{H}_e} = -\|\nabla v_2\|^2 - \|\nabla h\|^2, \quad [v_1, v_2, h] \in \mathcal{D}(\mathbb{A}^*). \tag{7.24}$$

Proof Recalling $\mathbb{A}^*$ from (4.33) and the topology of $\mathbf{H}_e$ from (4.14), we compute:

$$\left(\mathbb{A}^*\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e} = \left(\begin{bmatrix} -v_2 \\ \Delta(v_2 - v_1) \\ \Delta h \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix}\right)_{\mathbf{H}_e}$$
$$= -(\nabla v_2, \nabla\tilde{v}_1)_{\Omega_s} - (v_2, \tilde{v}_1)_{\widetilde{\Gamma}_s} + (\Delta(v_2 - v_1), \tilde{v}_2)_{\Omega_s}$$

$$
\begin{aligned}
&+\left(\Delta h, \tilde{h}\right)_{\Omega_f} \\
&= -(\nabla v_2, \nabla \tilde{v_1})_{\Omega_s} - (v_2, \tilde{v_1})_{\widetilde{\Gamma}_s} - \int_{\Gamma_s} \frac{\partial (v_2 - v_1)}{\partial \nu} \overline{\tilde{v_2}}\, d\Gamma_s \\
&- (\nabla (v_2 - v_1), \nabla \tilde{v_2})_{\Omega_s} + \int_{\Gamma_s} \frac{\partial h}{\partial \nu} \overline{\tilde{h}}\, d\Gamma_s - \left(\nabla h, \nabla \tilde{h}\right)_{\Omega_f},
\end{aligned}
\tag{7.25}
$$

where we have used the unit ν inward to Ω_s and $h|_{\Gamma_f} = 0$. Next, recalling $\frac{\partial (v_2 - v_1)}{\partial \nu} = \frac{\partial h}{\partial \nu} - \chi_{\widetilde{\Gamma}_s} v_1$ on Γ_s as well as $h|_{\Gamma_s} = v_2|_{\Gamma_s}$, we obtain:

$$
\begin{aligned}
\text{RHS of (7.25)} = &-(\nabla v_2, \nabla \tilde{v_1})_{\Omega_s} - (v_2, \tilde{v_1})_{\widetilde{\Gamma}_s} - \cancel{\int_{\Gamma_s} \frac{\partial h}{\partial \nu} \overline{\tilde{h}}\, d\Gamma_s} + \int_{\widetilde{\Gamma}_s} v_1 \overline{\tilde{h}}\, d\widetilde{\Gamma}_s \\
&- (\nabla v_2, \nabla \tilde{v_2})_{\Omega_s} + (\nabla v_1, \nabla \tilde{v_2})_{\Omega_s} + \cancel{\int_{\Gamma_s} \frac{\partial h}{\partial \nu} \overline{\tilde{h}}\, d\Gamma_s} - \left(\nabla h, \nabla \tilde{h}\right)_{\Omega_f}
\end{aligned}
\tag{7.26}
$$

and (7.26) proves (7.22).

Parts (2) and (3) follow readily. □

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