

A PROOF OF GROMOV'S CUBE INEQUALITY ON SCALAR CURVATURE

JINMIN WANG*, ZHIZHANG XIE[†] & GUOLIANG YU[‡]

Abstract

Gromov proved a cube inequality on the bound of distances between opposite faces of a cube equipped with a positive scalar curvature metric in dimension ≤ 8 using a minimal surface method. He conjectured that the cube inequality also holds in dimension ≥ 9 . In this paper, we prove Gromov's cube inequality in all dimensions with the optimal constant via a Dirac operator method.

1. Introduction

In [12], Gromov proved a cube inequality on scalar curvature in dimension ≤ 8 using a minimal surface method [14, 15, 16]. The cube inequality plays an important role in the recent proof of nonexistence of positive scalar curvature metrics on 5-dimensional aspherical manifolds in [11] (cf. another proof of the same result in [4]). By applying a quantitative relative index theorem, the second author proved Gromov's cube inequality in all dimensions but with a suboptimal constant [21]. In this paper, we prove Gromov's cube inequality with the optimal constant in all dimensions.

Theorem 1.1. *Let g be a Riemannian metric on the cube $I^n = [0, 1]^n$. Suppose $\ell_i = \text{dist}(\partial_{i-}, \partial_{i+})$ is the g -distance between the i -th pair of opposite faces ∂_{i-} and ∂_{i+} of the cube. If the scalar curvature of g satisfies $\text{Sc}_g \geq k > 0$, then*

$$(1.1) \quad \sum_{i=1}^n \frac{1}{\ell_i^2} \geq \frac{kn}{4\pi^2(n-1)}.$$

Consequently, we have

$$(1.2) \quad \min_{1 \leq i \leq n} \text{dist}(\partial_{i-}, \partial_{i+}) \leq 2\pi \sqrt{\frac{n-1}{k}}.$$

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We remark that the constant $4\pi^2$ in the inequality (1.1) is optimal. This can be seen by considering warped product metrics on large flat $(n-1)$ -dimensional cubes over an interval (see Remark 2.1 for more details).

The proof of Theorem 1.1 is based on an explicit index computation of the Dirac operator with an appropriately chosen potential on $\mathbb{R}^n$. Our proof also works for a more general version of Gromov's cube inequality in the spin case. More precisely, the following Gromov's $\square^{n-m}$ inequality holds.

Theorem 1.2. *Let X be an n -dimensional compact connected Riemannian spin manifold with corners. Suppose $f: X \rightarrow [-1, 1]^m$ is a continuous map that preserves corner structures.¹ Let $\partial_{i\pm}, i = 1, \dots, m$, be the pullbacks of the pairs of the opposite faces of the cube $[-1, 1]^m$. Suppose $Y_{\cap}$ is an $(n-m)$ -dimensional closed submanifold (without boundary) in X that satisfies the following conditions:*

- (1) $\pi_1(Y_{\cap}) \rightarrow \pi_1(X)$ is injective;
- (2) $Y_{\cap}$ is the transversal intersection² of m orientable hypersurfaces $\{Y_i\}_{i=1}^m$ of X , where each Y_i separates ∂_{i-} from ∂_{i+} ;
- (3) the higher index $\text{Ind}_{\Gamma}(D_{Y_{\cap}})$ does not vanish in $KO_{n-m}(C_{\max}^*(\Gamma; \mathbb{R}))$, where $\Gamma = \pi_1(Y_{\cap})$ and $C_{\max}^*(\Gamma; \mathbb{R})$ is the maximal group C^* -algebra of Γ with real coefficients.

If $\text{Sc}(X) \geq k > 0$, then the distances $\ell_i = \text{dist}(\partial_{i-}, \partial_{i+})$ satisfy the following inequality:

$$(1.3) \quad \sum_{i=1}^m \frac{1}{\ell_i^2} \geq \frac{kn}{4\pi^2(n-1)}.$$

Consequently, we have

$$(1.4) \quad \min_{1 \leq i \leq m} \text{dist}(\partial_{i-}, \partial_{i+}) \leq 2\pi \sqrt{\frac{m(n-1)}{kn}}.$$

We point out that Cecchini [3] and Zeidler [23, 24] proved a special case of Theorem 1.2 when $m = 1$, i.e., Gromov's bandwidth conjecture.

We remark that, by extending the index theory for manifolds with corners (and more generally for manifolds with polyhedral boundary) developed by the authors in [19], one can show that the inequalities (1.1) and (1.3) are in fact strict inequalities for manifolds with corners whose dihedral angles are $< \pi$. The proofs of the strict inequalities will be given elsewhere in order for the present paper to be self-contained.

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¹This means f maps codimension k faces to codimension k faces.

²In particular, this implies that the normal bundle of $Y_{\cap}$ is trivial.

2. Proof of Gromov's cube inequality

In this section, we prove Theorem 1.1. Our proof is inspired in part by Zeidler's proof for Gromov's bandwidth conjecture [24, Theorem 1.4]. Throughout this section, we assume the constant k in Theorem 1.1 to be $n(n-1)$, while the general case follows by rescaling the metric.

Proof of Theorem 1.1. Assume to the contrary that

$$(2.1) \quad \sum_{i=1}^n \frac{1}{\ell_i^2} < \frac{n^2}{4\pi^2}.$$

Then there exists $\varepsilon > 0$ such that

$$(2.2) \quad \sum_{i=1}^n \frac{\pi^2(1+\varepsilon)^2}{(\ell_i - 2\varepsilon)^2} < \frac{n^2}{4}.$$

Now we shall extend the metric on the cube I^n to a complete metric on $\mathbb{R}^n$. We first give a construction of the metric for $n = 2$, and then briefly describe the general case.

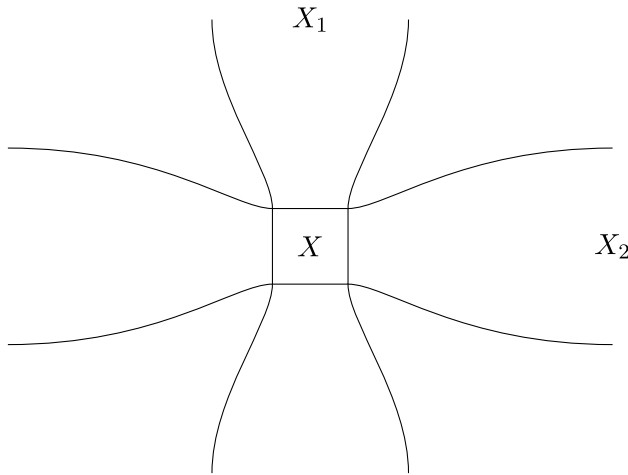


Figure 1. Extend the metric from $[0, 1]^2$ to $X_1 \cup X_2$ and rescale the metric.

Denote $X = [0, 1]^2$, $X_2 = \mathbb{R} \times [0, 1]$ and $X_1 = [0, 1] \times \mathbb{R}$. Note that $X_1 \cap X_2 = X$, and $X_1 \cup X_2$ has a shape of cross (see Figure 1). By the Whitney's extension theorem [20] and a smooth partition of unity, there is a smooth metric g_1 on $X_1 \cup X_2$ such that $g_1 = g$ on X and $g_1 = dx_1^2 + dx_2^2$ outside a compact set of $X_1 \cup X_2$.

Suppose that Ψ is a smooth positive function on $X_1 \cup X_2$ such that $\Psi(x_1, x_2)$ is equal to 1 if $(x_1, x_2) \in X$, and $\Psi(x_1, x_2)$ is a constant when $x_1^2 + x_2^2$ is sufficiently large. Set $g_2 = \Psi \cdot g_1$. By choosing Ψ to be large

enough outside X , we obtain a smooth metric g_2 on $X_1 \cup X_2$ with the following properties (see Figure 1).

- (1) The g_2 -distance $\text{dist}_{g_2}(\partial_-X_1, \partial_+X_1)$ between the two boundary components $\partial_-X_1 = \{0\} \times \mathbb{R}$ and $\partial_+X_1 = \{1\} \times \mathbb{R}$ is $\geq \ell_1 - \varepsilon/3$, where ε is given in line (2.2).
- (2) The g_2 -distance $\text{dist}_{g_2}(\partial_-X_2, \partial_+X_2)$ between the two boundary components $\partial_-X_2 = \mathbb{R} \times \{0\}$ and $\partial_+X_2 = \mathbb{R} \times \{1\}$ is $\geq \ell_2 - \varepsilon/3$.
- (3) $g_2 = R^2(dx_1^2 + dx_2^2)$ on $X_1 \cup X_2$ outside a compact set for some $R > 0$.

Finally, using Whitney’s extension theorem, we extend the metric g_2 on $X_1 \cup X_2$ to obtain a complete metric $\bar{g}$ on $\mathbb{R}^2$ such that $\bar{g} = R^2(dx_1^2 + dx_2^2)$ on $\mathbb{R}^2$ outside a compact subset. Since $\bar{g}$ is flat outside a compact set, there exists $\sigma > 0$ such that $\text{Sc}_{\bar{g}} \geq -\sigma$ on $\mathbb{R}^2$.

For $x = (x_1, x_2) \in \mathbb{R}^2$, we define

$\varphi_1(x) = \text{sgn}(x_1) \cdot \text{dist}_{\bar{g}}(x, \{0\} \times \mathbb{R})$ and $\varphi_2(x) = \text{sgn}(x_2) \cdot \text{dist}_{\bar{g}}(x, \mathbb{R} \times \{0\})$,

where sgn is the sign function. Set $\psi_i = \varphi_i - \ell_i/2$. For each $i = 1, 2$, let z_i be a smooth approximation of ψ_i such that $|z_i - \psi_i| \leq \varepsilon/3$ and $\|\nabla z_i\| \leq 1 + \varepsilon$ (cf. [5, Proposition 2.1]). Here ∇ is the gradient with respect to the metric $\bar{g}$.

To summarize, we have constructed a complete metric $\bar{g}$ and smooth functions z_1 and z_2 on $\mathbb{R}^2$ with the following properties.

- The metric $\bar{g}$ restricts to the metric g on $[0, 1]^2$, and coincides with the Euclidean metric rescaled by R on $\mathbb{R}^2$ outside a compact set. In particular, there exists a positive number $\sigma > 0$ such that $\text{Sc}_{\bar{g}} \geq -\sigma$ on $\mathbb{R}^2$.
- For each $x \in \mathbb{R}^2$, if both $|z_1(x)| \leq \ell_1/2 - \varepsilon$ and $|z_2(x)| \leq \ell_2/2 - \varepsilon$, then x lies in the cube $[0, 1]^2$.
- For each $C > 0$, the subset $\{x \in \mathbb{R}^2 : z_1(x)^2 + z_2(x)^2 \leq C\}$ is compact.
- $\|\nabla z_i\| \leq 1 + \varepsilon$, where ∇ is the gradient with respect to the metric $\bar{g}$.

When $n > 2$, we extend the metric g on $I^n = [0, 1]^n$ to a complete metric $\bar{g}$ on $\mathbb{R}^n$ by a similar induction argument. let us briefly describe the construction below.

For each $k = 0, 1, \dots, n$, let Λ_k be the family of all subsets of $\{1, 2, \dots, n\}$ with k -elements. For $\lambda \in \Lambda_k$, set

$$X_\lambda = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \text{if } j \notin \lambda, \text{ then } 0 \leq x_j \leq 1\}.$$

In particular, $X_\emptyset = [0, 1]^n$ for $\lambda = \emptyset \in \Lambda_0$. For each $\lambda \in \Lambda_k$, the boundary of X_λ has $(n - k)$ pairs of opposite faces, denoted by $\partial_{j\pm}X_\lambda$ for each $j \notin \lambda$.

Now we shall extend the metric g on $X_\emptyset = [0, 1]^n$ smoothly to a complete metric $\bar{g}$ on $\mathbb{R}^n$ by induction on k so that for each given $k \in \{0, 1, \dots, n - 1\}$, we have

(2.3) $\text{dist}_{\bar{g}}(\partial_{j-}X_\lambda, \partial_{j+}X_\lambda) \geq \ell_j - k\varepsilon/(n+1)$ for $\forall \lambda \in \Lambda_k$ and $\forall j \notin \lambda$,

and the metric $\bar{g}$ coincides with a multiple of the Euclidean metric on $\mathbb{R}^n$ outside a compact set. For $k = 0$, this is trivial. Now assume that the metric has been constructed on X_λ satisfying the condition (2.3), for each $\lambda \in \Lambda_k$. For each $\mu \in \Lambda_{k+1}$, it suffices to extend the metric on the union $\cup_{\lambda \subset \mu} X_\lambda$ to X_μ , where λ runs through all k -element subsets of μ . Note that the condition (2.3) is already satisfied on $\cup_{\lambda \subset \mu} X_\lambda$. Therefore such an extension of metric always exists on X_μ by the Whitney's extension theorem, a smooth partition of unity, and rescaling. By induction, we obtain a metric $\bar{g}$ on $\mathbb{R}^n$ with the required properties.

For each $i = 1, 2, \dots, n$, set

$$E_i = \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_i = 0\}.$$

For each $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we define

$$(2.4) \quad \varphi_i(x) := \text{sgn}(x_i) \cdot \text{dist}_{\bar{g}}(x, E_i),$$

where sgn is the sign function. Set $\psi_i = \varphi_i - \ell_i/2$. For each $i = 1, 2, \dots, n$, let z_i be a smooth approximation of ψ_i such that $|z_i - \psi_i| \leq \varepsilon/(n+1)$ and $\|\nabla z_i\| \leq 1 + \varepsilon$. Here ∇ is the gradient with respect to the metric $\bar{g}$. To summarize, we have constructed a smooth metric $\bar{g}$ and smooth functions $\{z_i\}_{i=1,2,\dots,n}$ on $\mathbb{R}^n$ with the following properties.

- The metric $\bar{g}$ restricts to the metric g on $[0, 1]^n$, and coincides with a multiple of the Euclidean metric on $\mathbb{R}^n$ outside a compact set. In particular, there exists a positive number $\sigma > 0$ such that $\text{Sc}_{\bar{g}} \geq -\sigma$ on $\mathbb{R}^n$.
- For each $x \in \mathbb{R}^n$, if $|z_i(x)| \leq \ell_i/2 - \varepsilon$ for $1 \leq i \leq n$, then x lies in the cube $[0, 1]^n$.
- For each $C > 0$, the subset $\{x \in \mathbb{R}^n : z_1(x)^2 + \dots + z_n(x)^2 \leq C\}$ is compact.
- $\|\nabla z_i\| \leq 1 + \varepsilon$, where ∇ is the gradient with respect to the metric $\bar{g}$.

With the constant $\varepsilon > 0$ and $\sigma > 0$ given above, there exist positive numbers $\delta > 0$ and $\{\bar{\ell}_i\}_{1 \leq i \leq n}$ such that for all $1 \leq i \leq n$, we have

$$(2.5) \quad 0 < \bar{\ell}_i < \ell_i - 2\varepsilon < \ell_i,$$

$$(2.6) \quad \frac{n^2}{4} - \sum_{i=1}^n \frac{\pi^2(1+\varepsilon)^2}{(\ell_i - 2\varepsilon)^2} > \delta,$$

and for $1 \leq j \leq n$,

$$(2.7) \quad \frac{-n\sigma}{4(n-1)} + \frac{\pi^2(1+\varepsilon)^2}{(\ell_j - 2\varepsilon)^2} \tan^2 \frac{\pi \bar{\ell}_j}{2(\ell_j - 2\varepsilon)} - (1+\varepsilon) - \sum_{i \neq j} \frac{\pi^2(1+\varepsilon)^2}{(\ell_i - 2\varepsilon)^2} > \delta.$$

Set

$$r_i = \frac{\pi(1+\varepsilon)}{\ell_i - 2\varepsilon}.$$

Fix a positive number ℓ such that $\ell > \ell_i/2 + 1$ for all $1 \leq i \leq n$. We choose smooth functions $\{\xi_i\}_{1 \leq i \leq n}$ on $\mathbb{E}^n$, the flat Euclidean space, such that

- (1) $0 \leq (1 + \varepsilon)\xi_i(x, y) \leq y^2 + r_i^2$ for $\forall (x, y) \in \mathbb{E}$,
- (2) $0 \leq \xi_i(x, y) \leq \varepsilon$ for $|x| \geq \ell_i/2$, where ε is the constant given in line (2.6),
- (3) and

$$\xi_i(x, y) = \begin{cases} \frac{y^2 + r_i^2}{1 + \varepsilon} & \text{if } |x| \leq \ell_i/2, \\ 0 & \text{if } \ell_i/2 - \varepsilon \leq |x| \leq \ell, \\ 1 & \text{if } |x| \geq \ell + 1. \end{cases}$$

Let f_i be the unique solution to the following differential equation with initial condition

(2.8)
$$\begin{cases} f'(x) = \xi_i(x, f(x)), \\ f(0) = 0. \end{cases}$$

The solution always exists in a small neighborhood of $x = 0$. By the comparison theorem, we have

(2.9)
$$f_i(x) \leq r_i \tan\left(\frac{r_i x}{1 + \varepsilon}\right),$$

since the function $h_i(x) = r_i \tan(\frac{r_i x}{1 + \varepsilon})$ is the unique solution to the differential equation

$$\begin{cases} f'(x) = \frac{f(x)^2 + r_i^2}{1 + \varepsilon}, \\ f(0) = 0. \end{cases}$$

Therefore the solution f_i exists at least for $|x| < \ell_i/2$. On the other hand, by assumption, $\xi_i(x, y) = 0$ for all $\ell_i/2 - \varepsilon \leq |x| \leq \ell$ and $0 \leq \xi_i(x, y) \leq 1$ for $|x| \geq \ell_i/2$. It follows that the solution f_i exists on the whole real line (cf. [17, §9, Theorem XIII]). Roughly speaking, $f_i(x)$ is equal to the function $h_i(x) = r_i \tan(\frac{r_i x}{1 + \varepsilon})$ when $|x|$ is small, and is equal to the linear function $x \pm c$ for some constant c when $|x|$ is large.

Let $\mathcal{C}\ell_{n,0}$ be the Clifford algebra generated by $\{e_i\}_{1 \leq i \leq n}$ subject to the relation $e_i^2 = -1$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$. Similarly, we define $\mathcal{C}\ell_{0,n}$ to be the Clifford algebra generated by $\{\hat{e}_i\}_{1 \leq i \leq n}$ subject to the relation $\hat{e}_i^2 = 1$ and $\hat{e}_i \hat{e}_j + \hat{e}_j \hat{e}_i = 0$ for $i \neq j$. Let $T\mathbb{R}^n \oplus \mathbb{E}^n$ be the direct sum of the tangent bundle of $(\mathbb{R}^n, \bar{g})$ and the trivial bundle $\mathbb{E}^n$, and

$$E := S(T\mathbb{R}^n \oplus \mathbb{E}^n)$$

the spinor bundle of $T\mathbb{R}^n \oplus \mathbb{E}^n$. By construction, E is equipped with the Clifford action of $\mathcal{C}\ell_{n,0}(T\mathbb{R}^n) \hat{\otimes} \mathcal{C}\ell_{0,n}(\mathbb{E}^n)$.

Let D be the twisted Dirac operator on $L^2(\mathbb{R}^n, E)$, where $L^2(\mathbb{R}^n, E)$ is the space of L^2 sections of the bundle E . In terms of a local orthonormal frame $\{w_i\}_{1 \leq i \leq n}$ of the tangent bundle $T\mathbb{R}^n$, the operator D can be locally expressed by

$$D = \sum_{i=1}^n w_i \nabla_{w_i},$$

where w_i denotes the Clifford action of $C\ell_{n,0}(T\mathbb{R}^n)$ on E .

Consider the following Callias-type operator

$$B = D \hat{\otimes} 1 + \sum_{i=1}^n 1 \hat{\otimes} f_i(z_i) \hat{e}_i$$

where $\{\hat{e}_i\}$ is the flat orthonormal basis of $\mathbb{E}^n$. A direct computation shows that

$$(2.10) \quad B^2 = D^2 \hat{\otimes} 1 + \sum_{i=1}^n f_i(z_i)^2 + \sum_{i=1}^n [D, f_i(z_i)] \hat{\otimes} \hat{e}_i.$$

As $[D, f_i(z_i)] \hat{\otimes} \hat{e}_i$ is a bounded operator on $L^2(\mathbb{R}^n, E)$ for all $1 \leq i \leq n$ and $\sum_{i=1}^n f_i(z_i)^2$ is a proper³ function on $\mathbb{R}^n$, it follows that B is essentially self-adjoint and Fredholm [1].

Alternatively, the essentially self-adjointness and Fredholmness of B can be seen as follows. Clearly, there is a smooth path of Riemannian metrics g_t on $\mathbb{R}^n$ connecting the metric $\bar{g}$ to a multiple of the standard Euclidean metric g_o on $\mathbb{R}^n$. For simplicity, we may assume that the rescaling factor is equal to 1. Suppose that there exists a compact set K such that each metric g_t coincides with g_o on $\mathbb{R}^n$ outside K . Furthermore, there is a homotopy⁴ of functions $\{\theta_{t,i} : 0 \leq t \leq 1\}$ between f_i and the standard coordinate function x_i of $(\mathbb{R}^n, g_o)$ such that $\theta_{i,t} = f_i$ outside the compact set K for all i and $t \in [0, 1]$. In particular, outside a compact set K , the operator B coincides with the standard Bott-Dirac operator

$$(2.11) \quad B_o = \sum_{i=1}^n \frac{\partial}{\partial x_i} e_i \hat{\otimes} 1 + \sum_{i=1}^n 1 \hat{\otimes} x_i \hat{e}_i$$

acting on $L^2(\mathbb{R}^n, S(\mathbb{E}^n \oplus \mathbb{E}^n))$. Since B_o is essentially self-adjoint and Fredholm, it follows that B is also essentially self-adjoint and Fredholm. Moreover, it follows from the homotopy invariance of the Fredholm index that $\text{Ind}(B) = \text{Ind}(B_o)$.

We have

$$B_o^2 = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n x_i^2 + \sum_{i=1}^n e_i \hat{\otimes} \hat{e}_i.$$

Observe that $\{e_i \hat{\otimes} \hat{e}_i\}_{1 \leq i \leq n}$ is a commuting family of symmetries, i.e., $(e_i \hat{\otimes} \hat{e}_i)^2 = 1$ and

³A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be proper if the preimage $f^{-1}(K)$ of each compact subset K is compact.

⁴For example, consider the homotopy obtained by the linear combinations between f_i and x_i .

$$(e_i \widehat{\otimes} \widehat{e}_i)(e_j \widehat{\otimes} \widehat{e}_j) = (e_j \widehat{\otimes} \widehat{e}_j)(e_i \widehat{\otimes} \widehat{e}_i)$$

for $i \neq j$. Let Λ be the set of all maps from $\{1, 2, \dots, n\}$ to $\{1, -1\}$. The family of symmetries $\{e_i \widehat{\otimes} \widehat{e}_i\}_{1 \leq i \leq n}$ decomposes $S(\mathbb{E}^n \oplus \mathbb{E}^n)$ into 2^n subspaces V_λ indexed by $\lambda \in \Lambda$, where

$$(e_i \widehat{\otimes} \widehat{e}_i)f = \lambda(i)f \text{ for } \forall f \in V_\lambda.$$

As the rank of $S(\mathbb{E}^n \oplus \mathbb{E}^n)$ is equal to 2^n , each V_λ has rank 1.

The operator B_o^2 also decomposes correspondingly into an orthogonal direct sum of operators

$$-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \lambda(i)$$

each of which acts on $L^2(\mathbb{R}^n, V_\lambda)$. It is known that the harmonic oscillator

$$-\frac{d^2}{dx^2} + x^2$$

has a complete system of eigenvectors in $L^2(\mathbb{R})$ with the set of eigenvalues $\mathcal{O} = \{1, 3, 5, \dots\}$. Therefore the operator

$$-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n x_i^2$$

has a complete system of eigenvectors in $L^2(\mathbb{R}^n)$ with the set of eigenvalues

$$\mathcal{O}_n := \{k_1 + k_2 + \dots + k_n : k_i \in \mathcal{O}\}.$$

It follows that B^2 restricted on $L^2(\mathbb{R}^n, V_\lambda)$ is one-to-one and onto unless λ is the constant map with $\lambda(i) = -1$ for all $i \in \{1, 2, \dots, n\}$. In particular, when $\lambda \equiv -1$, the kernel of B^2 is $\{\exp(-(x_1^2 + \dots + x_n^2)/2) \cdot v : v \in V_\lambda\}$, which is of dimension 1. As such V_λ is located in the even part of $S(\mathbb{E}^n \oplus \mathbb{E}^n)$ with respect to its even-odd grading, the (graded) Fredholm index of B is equal to 1.

On the other hand, we shall show that B is an invertible operator, hence has $\text{Ind}(B) = 0$, which will lead to a contradiction. Indeed, by the Lichnerowicz formula, we have

$$D^2 = \nabla^* \nabla + \frac{\text{Sc}_{\bar{g}}}{4}.$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned} n \langle \nabla v, \nabla v \rangle &= n \left(\sum_{i=1}^n \langle \nabla_{e_i} v, \nabla_{e_i} v \rangle \right) = n \left(\sum_{i=1}^n \|e_i \nabla_{e_i} v\|^2 \right) \\ &\geq \left\| \sum_{i=1}^n e_i \nabla_{e_i} v \right\|^2 = \langle Dv, Dv \rangle \end{aligned}$$

for all $v \in C_c^\infty(\mathbb{R}^n, E)$. Therefore, we have

$$(2.12) \quad D^2 \geq \frac{\text{Sc}_{\bar{g}}}{4} \cdot \frac{n}{n-1} \text{ on } C_c^\infty(\mathbb{R}^n, E).$$

By line (2.10), we have

$$\begin{aligned} B^2 &\geq D^2 + \sum_{i=1}^n f_i(z_i)^2 - \sum_{i=1}^n \|[D, f_i(z_i)]\| \\ &\geq \frac{\text{Sc}_{\bar{g}}}{4} \cdot \frac{n}{n-1} + \sum_{i=1}^n f_i(z_i)^2 - \sum_{i=1}^n f'_i(z_i) \|\nabla z_i\| \\ &\geq \frac{\text{Sc}_{\bar{g}}}{4} \cdot \frac{n}{n-1} + \sum_{i=1}^n f_i(z_i)^2 - \sum_{i=1}^n f'_i(z_i)(1+\varepsilon). \end{aligned}$$

Case (1): inside the cube. By assumption, we have $\text{Sc}_{\bar{g}} \geq n(n-1)$ inside the cube $I^n = [0, 1]^n$. Hence

$$\begin{aligned} B^2 &\geq \frac{n^2}{4} + \sum_{i=1}^n f_i(z_i)^2 - \sum_{i=1}^n f'_i(z_i)(1+\varepsilon) \\ &= \frac{n^2}{4} + \sum_{i=1}^n f_i(z_i)^2 - \sum_{i=1}^n \xi_i(z_i, f_i(z_i))(1+\varepsilon) \\ &\geq \frac{n^2}{4} - \sum_{i=1}^n r_i^2 = \frac{n^2}{4} - \sum_{i=1}^n \frac{\pi^2(1+\varepsilon)^2}{(\ell_i - 2\varepsilon)^2} > \delta \end{aligned}$$

where the positivity is deduced from line (2.6).

Case (2): outside the cube. In this case, we see that at least one of the $|z_i|$'s is $\geq \ell_i/2 - \varepsilon$. Without loss of generality, we assume that $|z_1| \geq \ell_1/2 - \varepsilon > \bar{\ell}_1/2$. Note that $f_1(x) = r_1 \tan(\frac{r_1 x}{1+\varepsilon})$ for $|x| \leq \bar{\ell}_1/2$. Since f_1 is an increasing function, we have

$$|f_1(x)| \geq r_1 \tan \frac{r_1 \bar{\ell}_1}{2(1+\varepsilon)} = \frac{\pi(1+\varepsilon)}{\ell_1 - 2\varepsilon} \tan \frac{\pi \bar{\ell}_1}{2(\ell_1 - 2\varepsilon)} \text{ for all } |x| \geq \ell_1/2 - \varepsilon.$$

Furthermore, since $|z_1| \geq \ell_1/2 - \varepsilon$, we have $|f'_1(z_1)| \leq 1$. Therefore, we see that

$$\begin{aligned} B^2 &\geq \frac{-n\sigma}{4(n-1)} + f_1(z_1)^2 - (1+\varepsilon) - \sum_{i=2}^n r_i^2 \\ &\geq \frac{-n\sigma}{4(n-1)} + \frac{\pi^2(1+\varepsilon)^2}{(\ell_1 - 2\varepsilon)^2} \tan^2 \frac{\pi \bar{\ell}_1}{2(\ell_1 - 2\varepsilon)} - (1+\varepsilon) - \sum_{i=2}^n r_i^2 > \delta, \end{aligned}$$

where the positivity is deduced from line (2.7).

To summarize, we have proved that $B^2 > \delta$ for some positive constant $\delta > 0$. It follows that B is invertible, hence $\text{Ind}(B) = 0$, which contradicts the fact that $\text{Ind}(B) = 1$. This finishes the proof. q.e.d.

Remark 2.1. The constant $4\pi^2$ in line (1.1) of Theorem 1.1 is optimal. In fact, for any $\varepsilon > 0$, there is a Riemannian metric g on $[0, 1]^n$ with scalar curvature equal to $n(n - 1)$, and

(2.13)
$$\sum_{i=1}^n \frac{1}{\ell_i^2} \leq \frac{n^2}{4\pi^2} + \varepsilon.$$

For any $l < \pi/n$, consider the warped metric

$$g = dx_1^2 + \varphi(x_1)^2 \cdot \sum_{i=2}^n R^2 dx_i^2$$

on $[-l, l] \times [0, 1]^{n-1}$, where R is a positive constant and

$$\varphi(x_1) = \left(\cos \frac{nx_1}{2}\right)^{2/n}.$$

As computed in [7, page 653], the scalar curvature of g is given by

$$Sc_g = -2(n - 1)\frac{\varphi''}{\varphi} - (n - 1)(n - 2)\frac{\varphi'^2}{\varphi^2} = n(n - 1).$$

Note that there is $C_l > 0$ (depending on l) such that

$$\sum_{i=1}^n \frac{1}{\ell_i^2} \leq \frac{1}{(2l)^2} + \frac{C_l}{R^2}.$$

The optimality (2.13) follows by first choosing l to be close to π/n and then choosing R to be large enough.

3. Proof of Gromov’s $\square^{n-m}$ inequality

In this section, we prove Theorem 1.2. We assume that the constant k in Theorem 1.2 to be $n(n - 1)$, while the general case follows by rescaling the metric.

Proof of Theorem 1.2. Let us assume to the contrary that

$$\sum_{i=1}^m \frac{1}{\ell_i^2} < \frac{n^2}{4\pi^2}.$$

We first show that the general case where $\iota: \pi_1(Y_{\hbar}) \rightarrow \pi_1(X)$ is injective can be reduced to the case where $\iota: \pi_1(Y_{\hbar}) \rightarrow \pi_1(X)$ is split injective.⁵ This reduction step was shown in [21]. We repeat the proof here for the convenience of the reader. Let X_u be the universal cover of X . Since by assumption $\iota: \pi_1(Y_{\hbar}) \rightarrow \pi_1(X)$ is injective, we can view $\Gamma = \pi_1(Y_{\hbar})$ as a subgroup of $\pi_1(X)$. Let $X_{\Gamma} = X_u/\Gamma$ be the covering space of X corresponding to the subgroup $\Gamma \subset \pi_1(X)$. Then the inverse image of $Y_{\hbar}$ under the projection $p: X_{\Gamma} \rightarrow X$ is a disjoint union of

⁵We say $\iota: \pi_1(Y_{\hbar}) \rightarrow \pi_1(X)$ is split injective if there exists a group homomorphism $\varpi: \pi_1(X) \rightarrow \pi_1(Y_{\hbar})$ such that $\varpi \circ \iota = \mathbf{1}$, where $\mathbf{1}$ is the identity morphism of $\pi_1(Y_{\hbar})$.

covering spaces of $Y_{\cap}$, at least one of which is a diffeomorphic copy of $Y_{\cap}$. Fix such a copy of $Y_{\cap}$ in X_{Γ} and denote it by $\widehat{Y}_{\cap}$. Roughly speaking, the space X_{Γ} equipped with the lifted Riemannian metric from X could serve as a replacement of the original space X , except that X_{Γ} is not compact in general. To remedy this, we shall choose a “fundamental domain” around $\widehat{Y}_{\cap}$ in X_{Γ} as follows.

By assumption, $Y_{\cap} \subset X$ is the transversal intersection of m orientable hypersurfaces $Y_i \subset X$. Let r_i be the distance function⁶ from ∂_{i-} , that is $r_i(x) = \text{dist}(x, \partial_{i-})$. Without loss of generality, we can assume $Y_i = r_i^{-1}(a_i)$ for some regular value $a_i \in [0, \ell_i]$. Let $Y_i^{\Gamma} = p^{-1}(Y_i)$ be the inverse image of Y_i in X_{Γ} . Denote by $\bar{r}_i$ the lift of r_i from X to X_{Γ} . Let $\nabla \bar{r}_i$ be the gradient vector field associated to $\bar{r}_i$. A point $x \in X_{\Gamma}$ said to be *permissible* if there exist a number $s \geq 0$ and a piecewise smooth curve $c: [0, s] \rightarrow X_{\Gamma}$ satisfying the following conditions:

- (i) $c(0) \in \widehat{Y}_{\cap}$ and $c(s) = x$;
- (ii) there is a subdivision of $[0, s]$ into finitely many subintervals $\{[t_k, t_{k+1}]\}$ such that, on each subinterval $[t_k, t_{k+1}]$, the curve c is either an integral curve or a reversed integral curve⁷ of the gradient vector field $\nabla \bar{r}_{j_k}$ for some $1 \leq j_k \leq m$, where we require j_k 's to be all distinct from each other;
- (iii) furthermore, when c is an integral curve of the gradient vector field $\nabla \bar{r}_{j_k}$ on the subinterval $[t_k, t_{k+1}]$, we require the length of $c|_{[t_k, t_{k+1}]}$ to be less than or equal to $(\ell_{j_k} - a_{j_k} - \frac{\varepsilon}{4})$; and when c is a reversed integral curve of the gradient vector field $\nabla \bar{r}_{j_k}$ on the subinterval $[t_k, t_{k+1}]$, we require the length of $c|_{[t_k, t_{k+1}]}$ to be less than or equal to $(a_{j_k} - \frac{\varepsilon}{4})$.

Let T be the set of all permissible points. Now T may not be a manifold with corners. To fix this, we choose an open cover $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in \Lambda}$ of T by geodesically convex metric balls of sufficiently small radius $\delta > 0$. Now take the union of members of $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in \Lambda}$ that do not intersect the boundary ∂T of T , and denote by Z the closure of the resulting subset. Then Z is a manifold with corners which, together with the subspace $\widehat{Y}_{\cap} \subset Z$, satisfies all the conditions of the theorem, provided that ε and δ are chosen to be sufficiently small. In particular, the intersection $Y_i^{\Gamma} \cap Z$ of each hypersurface Y_i^{Γ} with Z gives a hypersurface of Z . The transversal intersection of the resulting hypersurfaces is precisely $\widehat{Y}_{\cap} \subset Z$. Furthermore, note that the

⁶To be precise, let r_i be a smooth approximation of the distance function from ∂_{i-} .

⁷By definition, an integral curve of a vector field is a curve whose tangent vector coincides with the given vector field at every point of the curve. A reversed integral curve is an integral curve with the reversed parametrization, that is, the tangent vector field of a reversed integral curve coincides with the negative of the given vector field at every point of the curve.

isomorphism $\Gamma = \pi_1(Y_\hbar^\Gamma) \rightarrow \pi_1(X^\Gamma) = \Gamma$ factors as the composition $\pi_1(Y_\hbar^\Gamma) \rightarrow \pi_1(Z) \rightarrow \pi_1(X^\Gamma)$, where the morphisms $\pi_1(Y_\hbar^\Gamma) \rightarrow \pi_1(Z)$ and $\pi_1(Z) \rightarrow \pi_1(X^\Gamma)$ are induced by the obvious inclusions of spaces. It follows that $\pi_1(Y_\hbar^\Gamma) \rightarrow \pi_1(Z)$ is a split injection. Therefore, without loss of generality, it suffices to prove the theorem under the additional assumption that $\iota: \pi_1(Y_\hbar) \rightarrow \pi_1(X)$ is a split injection.

From now on, let us assume $\iota: \pi_1(Y_\hbar) \rightarrow \pi_1(X)$ is a split injection with a splitting morphism $\bar{\omega}: \pi_1(X) \rightarrow \pi_1(Y_\hbar) = \Gamma$. Let $\tilde{X}$ be the Galois Γ -covering space determined by $\bar{\omega}: \pi_1(X) \rightarrow \Gamma$. In particular, the restriction of the covering map $\tilde{X} \rightarrow X$ on $Y_\hbar$ gives the universal covering space of $Y_\hbar$.

For any sufficiently small $\varepsilon > 0$ and for each $1 \leq i \leq m$, there exists a real-valued smooth function φ_i on X such that (cf. [5, Proposition 2.1])

- 1) $\|\nabla \varphi_i\| \leq 1 + \varepsilon$,
- 2) and $\varphi_i = 0$ on ∂_{i-} and $\varphi_i \geq \ell_i$ on ∂_{i+} .

Set $\psi_i = \varphi_i - \ell_i/2$. We may assume that $Y_\hbar = \cap_{i=1}^m \psi_i^{-1}(0)$. Since $Y_\hbar$ has a trivial normal bundle, there is $c > 0$ such that $\cap_{i=1}^m \psi_i^{-1}([-c, c])$ is diffeomorphic to $Y_\hbar \times I^m = Y_\hbar \times [0, 1]^m$. This shows that ∂X is cobordant to $Y_\hbar \times \partial I^m$ via a manifold W . Set

$$Z = X \cup_{\partial X} W^{\text{op}} \cup_{Y_\hbar \times \partial I^m} (Y_\hbar \times \partial I^m \times \mathbb{R}_{\geq 0}),$$

where W^{op} is the manifold W with reversed orientation and $\mathbb{R}_{\geq 0} = [0, \infty)$.

Fix a metric $g_{Y_\hbar}$ on $Y_\hbar$. We shall construct a metric $\bar{g}$ on Z that extends g on X and is equal to the product metric $g_{Y_\hbar} + dx_1^2 + \cdots + dx_m^2$ on $Y_\hbar \times \mathbb{R}^m$ outside a compact subset of Z .

Since the construction is similar to the one that appeared in the proof of Theorem 1.1, we shall be brief. For simplicity, let us describe the construction of the metric $\bar{g}$ on Z for the case where $m = 2$. The general case of $m > 2$ is completely similar by iterating the same construction below. By Sard's theorem, without loss of generality, we may assume that

$$E_+ := \{x \in X : \varphi_1(x) = \varphi_2(x)\} \text{ and } E_- := \{x \in X : \varphi_1(x) = -\varphi_2(x)\}$$

are both submanifolds in X . For each $i = 1, 2$, the subspaces E_+ , E_- , $Y_\hbar \times \partial_{i\pm} I^2$ and $\partial_{i\pm} X$ together bound a submanifold with corners in W , denoted by $W_{i\pm}$ (see Figure 2). Let $W_{i\pm}^{\text{op}}$ be the corresponding copy of $W_{i\pm}$ in W^{op} . Set W_2 to be the following union of subspaces in Z :

$$W_2 = (Y_\hbar \times \mathbb{R}_{\leq 0} \times [0, 1]) \cup W_{1-}^{\text{op}} \cup X \cup W_{1+}^{\text{op}} \cup (Y_\hbar \times \mathbb{R}_{\geq 0} \times [0, 1])$$

Then W_2 is a manifold with boundary, whose boundary consists of two components $\partial_- W_2$ and $\partial_+ W_2$. One can extend the metric g on X to a metric g_2 on W_2 such that the g_2 -distance $\text{dist}_{g_2}(\partial_- W_2, \partial_+ W_2)$ of $\partial_- W_2$ and $\partial_+ W_2$ is $\geq \ell_2 - \varepsilon/3$. Furthermore, we require that $\bar{g}$ coincides with

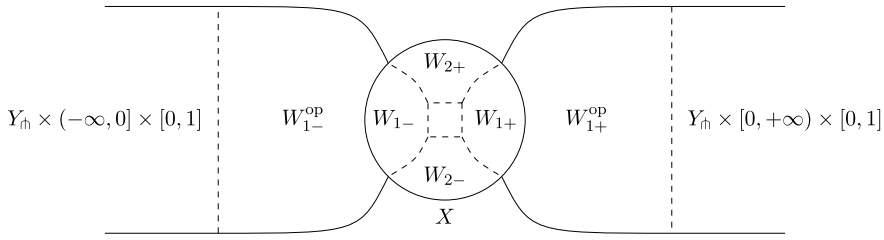


Figure 2. Extend the metric from X to W_2 .

the product metric $g_{Y_{\hat{\eta}}} + dx_1^2 + dx_2^2$ on $Y_{\hat{\eta}} \times \mathbb{R} \times [0, \ell_2]$ outside a compact set of W_2 .

We construct W_1 and a metric g_1 on W_1 with similar properties. By viewing $W_1 \cup_X W_2$ as a submanifold of Z , we finally extend the metric on $W_1 \cup_X W_2$ to obtain a metric $\bar{g}$ on Z .

For $x \in Z$, let $\bar{\varphi}_i(x)$ be the signed distance from x to $\partial_- W_i$. Let z_i be an appropriate smooth approximation of $\bar{\varphi}_i - \ell_i/2$. To summarize, we have constructed a complete metric $\bar{g}$ and real-valued smooth functions z_1 and z_2 on Z with the following properties.

- The metric $\bar{g}$ on Z restricts to the metric g on X , and coincides with the product metric $R^2(g_{Y_{\hat{\eta}}} + dx_1^2 + dx_2^2)$ outside a compact subset of Z for some $R > 0$. In particular, there exists a positive number $\sigma > 0$ such that $\text{Sc}_{\bar{g}} \geq -\sigma$ on Z .
- For each $x \in Z$, if both $|z_1(x)| \leq \ell_1/2 - \varepsilon$ and $|z_2(x)| \leq \ell_2/2 - \varepsilon$ for some sufficiently small $\varepsilon > 0$, then x lies in X .
- For each $C > 0$, the subset $\{x \in \mathbb{R}^2 : z_1(x)^2 + z_2(x)^2 \leq C\}$ is compact.
- $\|\nabla z_i\| \leq 1 + \varepsilon$ for both $i = 1, 2$. Here ∇ is the gradient with respect to the metric $\bar{g}$ and $\|\nabla z_i\|$ is measured with respect to the metric $\bar{g}$.

For the general case where $m \geq 2$, the same induction argument above produces a complete metric $\bar{g}$ on Z and real-valued smooth functions $\{z_i\}_{1 \leq i \leq n}$ on Z with similar properties. By the construction of Z , it is not difficult to see that the Γ -covering space $\tilde{X} \rightarrow X$ naturally extends to a Γ -covering space $\tilde{Z}$ of Z . The metric $\bar{g}$ and the functions z_i on Z lift to $\tilde{Z}$, which will be denoted by $\tilde{g}$ and $\tilde{z}_i$ respectively.

Now let $T\tilde{Z}$ be the tangent bundle of $\tilde{Z}$, and $\mathbb{E}^m$ the trivial bundle over $\tilde{Z}$. Let

$$E = S(T\tilde{Z} \oplus \mathbb{E}^m)$$

be the spinor bundle of $T\tilde{Z} \oplus \mathbb{E}^m$. Consider the following Callias-type operator

$$B = D \hat{\otimes} 1 + \sum_{i=1}^m 1 \hat{\otimes} f_i(\tilde{z}_i) \hat{e}_i$$

acting on $L^2(\widetilde{Z}, E)$, where $\hat{e}_i$ denotes the Clifford action of $\text{Cl}_{0,m}(\mathbb{E}^m)$. Note that B is a $\mathbb{Z}/2\mathbb{Z}$ -graded Γ -equivariant essentially self-adjoint operator. Since we have assumed to the contrary that

(3.1)
$$\sum_{i=1}^m \frac{1}{\ell_i^2} < \frac{n^2}{4\pi^2},$$

it follows from the same estimates as in the proof of Theorem 1.1 that B is invertible. In particular, the higher index $\text{Ind}_\Gamma(B)$ of B vanishes in $KO_{n-m}(C^*_{\max}(\Gamma; \mathbb{R}))$.

On the other hand, let $S_{\widetilde{Y}_{\mathfrak{h}}}$ be the Clifford bundle over $\widetilde{Y}_{\mathfrak{h}}$ with respect to the given metric $\widetilde{g}_{Y_{\mathfrak{h}}}$, and $D_{\widetilde{Y}_{\mathfrak{h}}}$ the associated Clifford-linear Dirac operator. We consider the following generalized Bott-Dirac operator

$$B_o = D_{\widetilde{Y}_{\mathfrak{h}}} \widehat{\otimes} 1 \widehat{\otimes} 1 + \sum_{i=1}^m 1 \widehat{\otimes} \frac{\partial}{\partial x_i} e_i \widehat{\otimes} 1 + \sum_{i=1}^m 1 \widehat{\otimes} 1 \widehat{\otimes} x_i \hat{e}_i$$

acting on $L^2(\widetilde{Y}_{\mathfrak{h}} \times \mathbb{E}^m, S_{\widetilde{Y}_{\mathfrak{h}}} \widehat{\otimes} S(\mathbb{E}^m \oplus \mathbb{E}^m))$. By the product formula of the higher index, we see that

$$\text{Ind}_\Gamma(B_o) = \text{Ind}_\Gamma(D_{\widetilde{Y}_{\mathfrak{h}}}) \in KO_{n-m}(C^*_{\max}(\Gamma, \mathbb{R})),$$

where $\text{Ind}_\Gamma(D_{\widetilde{Y}_{\mathfrak{h}}})$ is assumed to be non-zero. Furthermore, by applying the relative higher index theorem [2, 22], we have

$$\text{Ind}_\Gamma(B_o) - \text{Ind}_\Gamma(B) = \text{Ind}_\Gamma(D_{\mathscr{D}\widetilde{X}})$$

in $KO_{n-m}(C^*_{\max}(\Gamma; \mathbb{R}))$, where $D_{\mathscr{D}\widetilde{X}}$ is the associated Clifford-linear Dirac operator on the double $\mathscr{D}\widetilde{X}$ of $\widetilde{X}$. Since $\text{Ind}_\Gamma(D_{\mathscr{D}\widetilde{X}}) = 0$ (cf. [22, Theorem 5.1]), it follows that

$$\text{Ind}_\Gamma(B_o) = \text{Ind}_\Gamma(B) + \text{Ind}_\Gamma(D_{\mathscr{D}\widetilde{X}}) = 0.$$

This leads to a contradiction, hence finishes the proof. q.e.d.

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INSTITUTE OF MATHEMATICS
CHINESE ACADEMY OF SCIENCES
BEIJING, 100190
CHINA

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, 77843
UNITED STATES

E-mail address: jinmin@amss.ac.cn

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, 77843
UNITED STATES

E-mail address: xie@tamu.edu

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, 77843
UNITED STATES

E-mail address: guoliangyu@tamu.edu