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Numerical Approximation of Riccati-Based Hyperbolic-Like Feedback Controls

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Abstract

This paper provides a (rigorous) theoretical framework for the numerical approximation of Riccati-based feedback control problems of hyperbolic-like dynamics over a finite-time horizon, with emphasis on genuine unbounded control action. Both continuous and approximation theories are illustrated by specific canonical hyperbolic-like equations with boundary control, where the abstract assumptions are actually sharp regularity properties of the hyperbolic dynamics under discussion. Assumptions are divided in two groups. A first group of dynamical assumptions (actually dynamic properties) imply some preliminary critical properties of the control problem, including the definition of the would-be Riccati operator, in terms of the original data. However, in order to guarantee that such an operator is moreover the unique solution (within a specific class) of the corresponding Differential/Integral Riccati Equation, additional smoothing assumptions on the operators defining the performance index are required. The ultimate goal is to show that the the discrete finite dimensional Riccati based feedback operator, when inserted into the original PDE dynamics, provides near optimal performance.

Keywords Riccati equations · Optimal control · Finite dimensional approximation

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1 Introduction: Continuous and Discrete Optimal Control Problems

1.1 Orientation

The continuous theory of the Quadratic Optimal Control Problem over a finite-time horizon and corresponding Differential/Integral Riccati Equations for (linear) abstract differential equations is one of the fundamental disciplines within Optimization Theory. It was introduced within the area of control as a far reaching extension of the very successful finite dimensional control/system theory of the 1960s–80s, in order to model Partial Differential equations (PDEs) defined on a multi-dimensional bounded domain and subject to boundary control action and/or boundary sensing action. At the same time, it pushed the theory of abstract equations far beyond its traditional confines within the mathematical tradition, since in the resulting abstract framework, the boundary term lives in a highly weak topology (a suitable dual space, much larger than the function space of the uncontrolled system).

This application-driven development of mixed problems for PDEs raises a variety of novel and technical issues in the area of both analysis and numerics. Of particular interests are models driven by hyperbolic dynamics. The main reason is that such dynamics has no regularizing effects [in contrast to parabolic systems]. The singularities propagate (hyperbolic case) rather then being smoothed out with the start of the dynamics (parabolic case) [7, 29]. Handling of the problem requires particular methodology which accounts for the specifics of the dynamics [such as “hidden regularity of the traces”, etc]

In the present paper we provide a [rigorous] mathematical framework for the numerical approximations of the continuous theory with unbounded control actions,that has been available in book form [7, 30]. Thus, we first review the main features of such continuous theory, and next discuss the difficulties that arise in constructing a viable approximation theory and accordingly provide complete results thereof.

As noted, the abstract models are motivated by, and ultimately directed to, hyperbolic or hyperbolic-like Partial Differential Equations (PDEs), defined on a multidimensional bounded domain Ω of dimension $d \geq 2$ (the challenging case), subject to boundary (point) control and/or boundary (point) observation. Further extensions of the basic approach in [30] to different dynamics and control systems-such as coupled systems exhibiting both parabolic and hyperbolic behavior with unbounded control inputs has generated significant resonance in the literature [1–3, 8, 9, 18, 36, 37]. Here the goal was to show that there is a parabolic smoothing effect which is propagated onto the hyperbolic part yielding smoother controlled solutions. Feedback Riccati theory for *parabolic* problems alone where the dynamics has strong regularizing effect has been investigated earlier in [7, 29, 39] together with numerical approaches [19, 22–24]. However, purely hyperbolic dynamics with no smoothing effect whatsoever—just sheer propagation—meets with several challenges at both continuous [12, 30] and approximation level as first recognized in [13, 15] and also [16]. While a good theory for the continuous processes is a necessary initial condition, this is far away from being sufficient when it comes to approximations. The main difficulty is that certain properties of the underlying operators need to be replicated on the numerical scheme retaining uniformity in the parameter of discretization. While this process is well understood by now in parabolic dynamics, it is far from clear or even true in the case of hyperbolicity. Properties such as stabilizability, controllability, admissibility [critical for Riccati Theory] while may be known and valid in the continuous case, fail or are not known in the discrete case. This has been noted a long time ago in the case of stabilizability [13] and controllability [15]. The validity of the requisite estimates [uniformly in the parameter of discretization] often depends on the choice of approximation method [splines, nodal functions, finite differences] [15]. The convergence of the so called "gain operators"—the main actors for feedback control—is challenging particularly for finite horizon problems where the gains are time dependent. Correct definition of the domains of these operators lies at the heart of the matter.

As mentioned, the focus of the present paper is to provide a *rigorous* theoretical framework for the numerical approximation of the aforementioned continuous theory. Ultimately, it is the numerical component that provides useful information in engineering design and science problems. This has been the case for the numerical theory of finite dimensional control theory over many years. And this has justified the marked expansion of numerical activities over the past decades in the mathematical community within the PDE framework. The main idea is as follows: we start with the original system [on line] and a given observation. Based on this information, we construct a finite dimensional algorithm for computing the approximated *feedback* control which is finite dimensional. This algorithm is based on the numerical approximation of the Riccati operator—the so called gain operator—which is then inserted into the real system. The hope is that the resulting construct would yield "near" optimal performance with an error going to zero as the rate of approximation scheme tends to zero. See more relevant references such as [4–6, 10, 14, 17, 22, 23, 26, 27, 31–33, 42–46]. Various aspects of discrete approximations of continuous dynamical control systems modeled by differential evolution inclusions, possibly to include time delays, have been investigated in [35, Chapter 6 & 7].

1.2 Dynamical Model

In this paper we return to the abstract differential equation

$$\dot{y} = Ay + Bu \quad \text{on, say, } [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y, \tag{1.2.1}$$

A^* being the Y-adjoint of A, Y a Hilbert space, subject to the assumptions:

(H.1): $A : Y \supset \mathcal{D}(A) \to Y$ is the generator of a s.c. semigroup e^{At} on Y, $t \geq 0$, satisfying (without loss of generality) $A^{-1} \in \mathcal{L}(Y)$;

(H.2): B is a (linear) continuous operator $U \to [\mathcal{D}(A^*)]'$, U another Hilbert space; equivalently,

$$A^{-1}B \in \mathcal{L}(U; Y), \tag{1.2.2}$$

so that $B^* \in \mathcal{L}(\mathcal{D}(A^*); U)$ is defined by $(B^*x, u)_U = (x, Bu)_Y$, $u \in U$, $x \in \mathcal{D}(A^*)$;

(H.3): The (closable) operator $B^*e^{A^*t}$ can be extended as a map

$$B^*e^{A^*t}: \text{ continuous } Y \to L_2(0,T;U), \text{ i.e.,}$$
$$\int_0^T \|B^*e^{A^*t}x\|_U^2 dt \le c_T \|x\|_Y^2, \quad x \in Y \tag{1.2.3}$$

for all $0 < T < \infty$. Thus, by duality on (1.2.3), we obtain that

$$(L_{0,T}u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau : L_2(0,T;U) \to C([0,T];Y). \tag{1.2.4}$$

continuously. The solution to problem (1.2.1) is then given by

$$y(t) = y(t;y_0) = e^{At}y_0 + (L_{0,T}u)(t) :$$
$$\text{continuous } \{y_0, u\} \in Y \times L_2(0,T;U) \to C([0,T];Y). \tag{1.2.5}$$

1.3 Optimal Control Problem: Interval [0, *T*]

We associate with the dynamics (1.2.1), or (1.2.5), the quadratic cost functional (where we now specify endpoints of the interval, as well as the final state penalization G)

$$J_{0,T;G}(u,y) = \int_0^T [\|Ry(t)\|_Z^2 + \|u(t)\|_U^2]dt + \|Gy(T)\|_{Z_f}^2, \tag{1.3.1}$$

where $Z,\ Z_f$ are two other Hilbert spaces, and

(H.4):

$$R \in \mathcal{L}(Y;Z); \quad G \in \mathcal{L}(Y;Z_f). \tag{1.3.2}$$

(H.5):

$$R^*Re^{At}B: \text{ continuous } U \to L_1(0,T;Y) \tag{1.3.3a}$$
$$\int_0^T \|R^*Re^{At}Bu\|_Y dt \le c_T\|u\|_U, \qquad u \in U. \tag{1.3.3b}$$

(H.6):

$$G^*Ge^{At}B \in L(U;Y) \text{ and } \sup_{0\le t\le T} \|G^*Ge^{At}B\|_{\mathcal{L}(U;Y)} < \infty. \tag{1.3.4}$$

The corresponding optimal control problem $O.C.P._{0,T;G}$ is:

$$\textbf{Minimize } J_{0,T;G}(u,y) \text{ over all } u \in L_2(0,T;U)$$
$$\text{where } y \text{ is the solution of (1.2.1) due to } u. \tag{1.3.5}$$

Assumptions (H.1)–(H.6) are in force throughout the present work and they can be found in [30, pp. 766–767]. Assumption (H.5) and (H.6) will be needed for the representation of the Riccati operator [value function] as a solution of the Differential Riccati Equation. By using Bellman's optimality principle, one may also consider the optimal control problem starting at time $s > 0$.
Interval $[s, T]$. If the initial time for the dynamics (1.2.1) is $t = s \geq 0$, with corresponding initial condition $y(s) = y_0$, the resulting solution is now denoted by $y(t) \equiv y(t; s; y_0)$, so that $y(t; 0; y_0)$ may be simply denoted by $y(t; y_0)$. The corresponding optimal control problem $O.C.P._{s,T;G}$ over the interval $[s, T]$, $T < \infty$, is then:

Minimize the functional cost over all $u \in L_2(s, T; U)$:

$$J_{s,T;G}(u, y) = \int_s^T \left[\|Ry(t, s)\|_Z^2 + \|u(t, s)\|_U^2 \right] dt + \|Gy(T, s)\|_{Z_f}^2. \tag{1.3.6}$$

Remark 1.1 By duality on (H.5) and (H.6), we obtain

(H.5*): The map $v \to \int_0^T B^* e^{A^* t} R^* Rv(t)dt$ can be extended as a map continuous $L_\infty(0, T; Y) \to U$,

$$\left\| \int_0^T B^* e^{A^* t} R^* Rv(t)dt \right\|_U \leq c_T \|v\|_{L_\infty(0,T;Y)}. \tag{1.3.7}$$

(H.6*): For each $0 \leq t \leq T$,

$$B^* e^{A^* t} G^* G \in \mathcal{L}(Y; U) \text{ and } \sup_{0 \leq t \leq T} \left\| B^* e^{A^* t} G^* G \right\|_{\mathcal{L}(Y;U)} < \infty, \tag{1.3.8}$$

i.e., $B^* e^{A^* t} G^* G \in \mathcal{B}([0, T]; \mathcal{L}(Y; U))$ in the notation of [30, Remark 8.1.1].

The corresponding approximating assumptions of the discrete problem are (H.7) = (2.2.1) and (H.8)=(2.2.2) in Sect. 2.1.
Summary of known results. Results (1) through (4) below are in [30, Chapter 9, Theorem 9.2.1, p. 773], under the sole assumptions (H.1), (H.2), (H.3), (H.4); while results (5) and (6) below asserting that the operator $P_T(\cdot)$ satisfies the D.R.E. (1.3.21) and, moreover, is the unique such solution to satisfy properties (1.3.10b), (1.3.19), (1.3.20) are in [30, Theorem 9.2.2, p. 775] under the additional assumptions (H.5), (H.6).

(1) There exists a unique optimal pair $\{u_T^0(\cdot; s; y_0), y^0(\cdot; s; y_0)\}$ of the optimal control problem $O.C.P._{s,T;G}$ in (1.3.6):

$$u_T^0(\cdot; s; y_0) \in L_2(s, T; U); \quad y_T^0(\cdot; s; y_0) \in C([s, T]; Y). \tag{1.3.9}$$

(2) There exists an operator $P_T(t) \in \mathcal{L}(Y),\ 0 \leq s \leq t$, given explicitly by

$$P_T(t)x = \int_t^T e^{A^*(\tau - t)} R^* Ry^0(\tau; t; x)d\tau + e^{A^*(T-t)} G^* Gy^0(T; t; x) \tag{1.3.10a}$$

$$: \text{ continuous } Y \to C([0,T];Y), \quad x \in Y,\ 0 \le s \le t \le T \tag{1.3.10b}$$

such that

(i)

$$u_T^0(t;s;y_0) = -B^* P_T(t) y_T^0(t,s;y_0) \in L_2(s,T;U); \tag{1.3.11}$$

(ii)

$$P_T(t) \equiv P_T^*(t) \ge 0, \quad 0 \le s \le t \le T; \tag{1.3.12}$$

(iii)

$$\begin{aligned}(P_T(t)x,x)_Y &= \int_t^T \left[\|Ry_T^0(\tau;t;x)\|_Z^2 + \|u^0(\tau;t;x)\|_U^2\right] d\tau + \|Gy_T^0(T;t;x)\|_{Z_f}^2 \\ &= J_{t,T;G}^0(x) = J_{t,T;G}(u_T^0(\cdot;t;x), y_T^0(\cdot;t;x)).\end{aligned} \tag{1.3.13}$$

(3) Setting

$$\Phi_T(t,s)x \equiv y_T^0(t;s;x) \in C([s,T];Y), \quad x \in Y \tag{1.3.14}$$

(strong continuity in the first variable) $\Phi_T(t,s) \in \mathcal{L}(Y)$ is an evolution operator satisfying

$$\Phi_T(t,t) = I \text{ on } Y; \quad \Phi_T(t,s) = \Phi_T(t,\tau)\Phi_T(\tau,s),\ s \le \tau \le t \le T. \tag{1.3.15}$$

Moreover (see [30, Chapter 9, Eq. (9.2.24), p. 776]), the following transition property holds true

$$\Phi_T(t+\sigma,t)x = \Phi_{T-t}(\sigma,0)x \in C([0,T];Y). \tag{1.3.16}$$

We also recall that [30, Chapter 9, Eq. (9.2.13), p. 774]

$$\text{the map } s \to \Phi_T(\cdot,s)x = y_T^0(\cdot,s;x) \text{ is continuous on } Y,\ x \in Y \tag{1.3.17}$$

(strong continuity in the second variable).

(4) Finally, the operator $P_T(\cdot)$ in (1.3.10) satisfies the uniform bound:

$$\sup_{0 \le t \le T} \|P_T(t)\|_{\mathcal{L}(Y)} \le M < \infty. \tag{1.3.18}$$

by the Principle of Uniform Boundedness.

(5) Moreover (under the additional assumption (H.5), (H.6)), the above operator $P_T(t)$ satisfies the following additional regularity properties

$$B^* P_T(t) \in \mathcal{L}(Y; U), \quad 0 \le t \le T \tag{1.3.19a}$$

$$\text{or } B^* P_T(t) \in \mathcal{B}([0, T]; \mathcal{L}(Y; U)) \tag{1.3.19b}$$

$$B^* P_T(\cdot) e^{A(\cdot - s)} B : \text{ continuous } U \to L^2(s, T; U) \tag{1.3.20}$$

for any $s, 0 \le s < T$, with norm that may be taken independent of s.

(6) As a consequence of properties (5), the operator $P_T(t)$ is a solution of the Differential Riccati Equation with $(\cdot, \cdot) = Y$-inner product:for all $x, y \in \mathcal{D}(A),\ 0 \le t \le T$

$$\begin{aligned}\left(\frac{d}{dt} P_T(t)x, y\right) &= -(R^* R x, y) - (P_T(t)x, Ay) - (P_T(t)Ax, y), \\ &+ \left(B^* P_T(t)x, B^* P_T(t)y\right)_U, \quad \forall x, y \in \mathcal{D}(A),\ 0 \le t \le T\end{aligned} \tag{1.3.21}$$

with the terminal data $P_T(T) = G^* G$, and hence of the corresponding Integral Riccati Equation

$$\begin{aligned}(P_T(t)x, y) = &\int_t^T \left(R e^{A(\tau - t)} x, R e^{A(\tau - t)} y\right)_Z d\tau + \left(G^* G e^{A(T-t)} x, e^{A(T-t)} y\right) \\ &- \int_t^T \left(B^* P_T(\tau) e^{A(\tau - t)} x,\ B^* P_T(\tau) e^{A(\tau - t)} y\right)_U d\tau, \quad \forall x, y \in Y.\end{aligned} \tag{1.3.22}$$

In fact, such $P_T(t)$ is the unique solution of the I.R.E. (1.3.22) and hence of the D.R.E. (1.3.21) to satisfy properties (1.3.10b), (1.3.19), (1.3.20).

Our goal is to introduce a *finite dimensional approximation* of the Riccati Equation which would produce a finite dimensional solution $P_h(t) \in R^n$, and a *finite dimensional control*

$$u_h(t) = -B_h^* P_h(t) y_h(t) \in L_2(0, T; U) \tag{1.3.23}$$

which, when inserted into the original dynamics

$$y_{h,t}(t) = A y_h(t) + B u_h(t) = A y_h(t) - B B_h^* P_h(t) y_h(t) \tag{1.3.24}$$

would provide an *almost optimal performance*. In order to accomplish this, we need to first approximate the control problem.

Note that the algorithm allows to control the original plant (A, B), in a feedback form with the gain operator $B_h^* P_h$, where P_h is computed from a nonlinear ODE [the finite dimensional Riccati equation] and B_h is a suitable approximation of the control operator B. Note, we do not approximate the original plant (A, B)-and this is the whole point. The diagram below provides an illustration of the process.

Remark 1.2 This remark is in response to questions raised by a referee.

1. The path or general strategy in the present hyperbolic work and in the parabolic work [29, Chapters 4 & 5] is similar: we seek a finite dimensional approximation of the Riccati operator which would produce a corresponding finite dimensional feedback control, such that—once inserted into the original dynamics—produces an almost optimal performance. However, very different techniques are needed in order to implement the common strategy in the two cases. In a few words:
   (i) Regarding the control-free dynamics: analyticity of the free dynamic semigroup in the parabolic case versus the group or only semigroup in the hyperbolic case;
   (ii) Regarding the controlled dynamics: the drastic difference of the optimal regularity in the two cases with boundary control. More technically, in the parabolic case, see Assumption (A.1) = (4.1.2.2) of uniform analyticity in the t-domain or the corresponding equivalent version in the frequency domain (4.1.2.3) at [29, pp. 434–435]. And compare this against the discrete version of assumption (B.1) = (1.5.8) (sharp regularity sometimes called admissibility) of the present paper dealing with the hyperbolic case.
   To further elaborate: the two main properties needed for feedback synthesis of the optimal control with unbounded actuation are: (i) properties of the operator $B^*e^{A^*t}$ and (ii) properties of the operator Re^{At}. In both cases, the regularity of these operators depends on the uncontrolled dynamics e^{At}. In the parabolic case, said dynamics is described by an analytic semigroup. Hence the dynamics itself regularizes the control/observation actions. In the hyperbolic case, this is no longer true. Thus, one seeks "special properties" [e.g. cancellation of singularities] which still will produce the assumptions postulated where unbounded control or observation action is "mitigated" by the behavior of the dynamics—without expecting regularization of the entire state. This has to do with what is often called *hidden trace regularity* which can be discovered by PDE methods [25] capable of capturing "better" regularity at the boundary than can be inferred from the optimal interior regularity.

   At the level of approximations, the issue becomes whether these "better" properties can be reconstructed uniformly with respect to the parameters of approximations. In the parabolic case, the analyticity of the free dynamics is a strong enough property which can be reconstructed on a majority of common approximation schemes [splines, spectral methods, etc]. This, in turn, leads to the mentioned desirable properties to be satisfied uniformly by the approximation schemes. In the hyperbolic case, instead, special care is necessary in order to have the reconstruction of special "boundary behavior" represented by the "uniformly approximated hidden trace regularity". The examples provided in this paper deals with this hyperbolic situation.

2. We note that the present paper can be applied also to transport equations such as, in the simplest case, a one dimensional first order hyperbolic system with Dirichlet control. Here, the admissibility condition follows from energy method applied to the controlled dynamics, see [30, Section 10.6]. And this property can be replicated on typical approximations such as finite elements, spectral, or finite differences.
3. It is likely that the cases of using local spatial average or nodal measurements for the system output may be included. The issue needs further attention.

1.4 Examples of Controlled Dynamics

In order to illustrate the type of PDE-controlled dynamics covered by the above abstract setting (H.1), (H.2), (H.3), (H.4), we provide two canonical examples of hyperbolic dynamics::one involving wave and another one a plate. Let Ω be a multi-dimensional bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega \equiv \Gamma \equiv \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, $\Gamma_0 \neq \emptyset$.
Example #1: *boundary damped wave equation in the unknown $w(x,t)$ with Neumann boundary control*:

$$\begin{cases} w_{tt} = \Delta w & \text{in } \Omega \times (0,T) & (1.4.1a) \\ w\big|_{\Gamma_0} = 0 & \text{in } \Gamma_0 \times (0,T) & (1.4.1b) \\ \dfrac{\partial w}{\partial \nu} + a w_t = u & \text{on } \Gamma_1 \times (0,T),\ a > 0 & (1.4.1c) \\ w(0) = w_0 \text{ in } H^1_{\Gamma_0}(\Omega), \quad w_t(0) = w_1 & \text{in } L_2(\Omega). & (1.4.1d) \end{cases}$$

The state space, control space, and state variable are

$$Y \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega); \quad U \equiv L_2(\Gamma_1); \quad y \equiv (w, w_t), \tag{1.4.2}$$

where

$$f \in H^1_{\Gamma_0}(\Omega) = \left\{ f \in H^1(\Omega),\ f\big|_{\Gamma_0} = 0 \right\}.$$

With the dynamics given by (1.4.1), we associate the functional cost given by

$$J(u,y) \equiv \int_0^T \left[\|w\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Gamma_1)} \right] dt + \|w(T)\|^2_{L_2(\Omega)}. \tag{1.4.3}$$

Appendix A will justify that: (i) the PDE-dynamics (1.4.1) (where the assumption $a > 0$ in (1.4.1c) is essential) can be re-written in the abstract form (1.2.1) while satisfying the required dynamical assumptions (H.1), (H.2), (H.3); (ii) the cost function (1.4.3) can be rewritten as in (1.3.1) with observation operators R and G satisfying the regularity assumption (H.4) on the spaces $Z \equiv Z_f \equiv Y \equiv H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. Assumptions (H.5), (H.6) involve a combination of properties of the observations operators R and G in the performance index and of the dynamics. Next, Appendix B will provide a detailed account of the corresponding numerical approximation.

Example #2 *plate equation with boundary controls via moments*

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } \Omega \times (0,T) & (1.4.4a) \\ w = 0 \; and \; \Delta w + a\dfrac{\partial}{\partial \nu} w_t = u \quad (a > 0) & \text{on } \Gamma \times (0,T) & (1.4.4b) \\ w(0) = w_0 \;\; \text{in } H^2(\Omega) \cap H^1_0(\Omega), \quad w_t(0) = w_1 & \text{in } L_2(\Omega), & (1.4.4c) \end{cases}$$

The dynamics is considered on the space $Y \equiv H^2(\Omega) \cap H^1_0(\Omega) \times L_2(\Omega)$[28] with the functional cost given by

$$J(u, y) \equiv \int_0^T \left[\|\nabla w\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Gamma)} \right] dt + \|\nabla w(T)\|^2_{L_2(\Omega)}. \tag{1.4.5}$$

A continuous/numerical treatment is given in Appendix C.
What is the critical difficulty. Since we are dealing with PDEs with boundary control, the operator B has its range in a large dual space, see (H.2), and thus can be more readily identified via its adjoint B^*. While it is natural to expect that the term $B^* y^0(t, y_0)$ will be well defined (the latter follows from the optimality principle, see (1.3.8)), it is not so natural to expect that $B^* P_T(t)$ will be well defined—even as a densely defined operator. Though $P_h(t)$ is sought to be finite dimensional, it is not clear whether its range is included in the domain of B^*, see the Orientation below. Not to mention issues of convergence and suboptimality. This will become more appreciated during the development of the approximation theory. Clearly, the continuous theory will have a fundamental bearing on the approximating analysis.

1.5 Approximation of Continuous Dynamics and Related Properties

1.5.1 Orientation

Our goal here is to introduce a discrete finite dimensional approximation of the Riccati operator, so that the control based on such approximated $P_h(t)$ through the formula

$$u_h(t) = -B_h^* P_h(t) y_h(t) \tag{1.5.1}$$

would yield an almost optimal performance when applied to the original dynamics $\{A, B\}$

$$y_{h,t} = A y_h + B u_h = A y_h - B B_h^* P_h(t) y_h, \tag{1.5.2}$$

where P_h is a solution of a nonlinear ODE (finite dimensional Riccati Equation). See Fig. 1.

1.5.2 Approximation Assumptions: Approximating Subspaces

We introduce a family of finite dimensional approximating subspaces $V_h \subset \mathcal{D}(B^*)$, where h is a parameter of discretization, $0 < h \leq h_0$ which tends to zero $h \downarrow 0$. Let

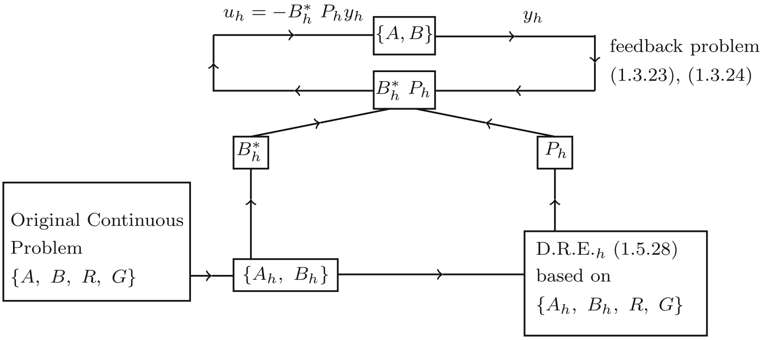

Fig. 1 Illustration of the feedback form control algorithm

Π_h be the Y-orthogonal projection of Y onto V_h so that $(\Pi_h x, \phi_h)_Y = (x, \phi_h)_Y$, $x \in Y$, $\phi_h \in V_h$ with the usual approximating property

$$\|\Pi_h x - x\|_Y \to 0, \text{ as } h \downarrow 0, \text{ for all } x \in Y \tag{1.5.3}$$

to be used freely below.
Approximation of A. Let $A_h : V_h \to V_h$ be an approximation of A which satisfies the following requirements (A.1) through (A.3):

(A.1): there exist constants $C \geq 1$, $\omega_1 > 0$ (independent of h) such that the approximating semigroups are uniformly bounded:

$$\left\| e^{A_h t} \right\|_{\mathcal{L}(Y)} \leq C e^{\omega_1 t}, \quad t \geq 0. \tag{1.5.4}$$

(A.2):

$$\left\| [A_h^{-1} \Pi_h - A^{-1}] x \right\|_Y \to 0 \text{ as } h \downarrow 0, \quad x \in Y. \tag{1.5.5}$$

(Without loss of generality, we may assume that $A_h^{-1} \in \mathcal{L}(V_h)$, $\forall\ h$; indeed, as noted in [30, Chapter 5], condition (1.5.5) may be equivalently formulated at any point of the common resolvent set of A and A_h.)

(A.3): the adjoint operator A^* satisfies the counterpart of assumption (A.2) for A, i.e.,

$$\left\| [A_h^{*-1} \Pi_h - A^{*-1}] y \right\|_Y \to 0 \text{ as } h \downarrow 0, \quad y \in Y. \tag{1.5.6}$$

(A.4):

$$\left\| [A_h^* \Pi_h A^{*-1} - \Pi_h] y \right\|_Y \to 0 \text{ as } h \downarrow 0, \quad y \in Y. \tag{1.5.7}$$

Approximation of B. We shall assume that the operators $B : U \to [\mathcal{D}(A^*)]'$ and $B_h : U \to V_h$ satisfy the following approximating properties:

(B.1): Given $0 < T < \infty$, there exists a constant $c_T > 0$ such that the following discrete version of (1.2.3) holds true, uniformly in h:

$$\int_0^T \left\| B_h^* e^{A_h^* t} \Pi_h x \right\|_U^2 dt \le c_T \|x\|_Y^2, \quad x \in Y; \tag{1.5.8}$$

(B.2):

$$\left\| A^{-1}[B_h - B]u \right\|_Y \to 0 \text{ as } h \downarrow 0, \quad u \in U; \tag{1.5.9}$$

(B.3):

$$\left\| [A_h^{-1} - \Pi_h A^{-1}] B_h u \right\|_Y \to 0 \text{ as } h \downarrow 0, \quad u \in U; \tag{1.5.10}$$

(B.4):

$$\left\| [B_h^* \Pi_h - B^*] A^{*-1} x \right\|_U \to 0 \text{ as } h \downarrow 0, \quad x \in Y; \tag{1.5.11}$$

(B.5):

$$\left\| B_h^* [A_h^{*-1} \Pi_h - \Pi_h A^{*-1}] x \right\|_U \to 0 \text{ as } h \downarrow 0, \quad x \in Y. \tag{1.5.12}$$

Remark 1.3 The usual duality argument as in [30, Chapter 7, Theorem 7.2.1, p. 648], as applied to assumption (B.1) = (1.5.8), yields the following statement:

(B.1*): For any $0 < T < \infty$, there exists a constant $c_T > 0$ such that

$$\left\| \int_0^{\cdot} e^{A_h(\cdot - \tau)} B_h u(\tau) d\tau \right\|_{C([0,T];Y)} \le c_T \|u\|_{L_2(0,T;U)}, \quad \forall\ h. \tag{1.5.13}$$

1.5.3 Consequences of Approximating Assumptions

In this section we collect, for easy reference in the sequel, some direct consequences of the approximating assumptions of the preceding subsection. We begin with a well-known, elementary consequence of the Principle of Uniform Boundedness, which we state as a lemma for convenience (see, e.g., [20]).

Lemma 1.1 *Let X_i be Banach spaces, $i = 1, 2, 3$. Let $S_n \in \mathcal{L}(X_1; X_2)$, $Q_n \in \mathcal{L}(X_2; X_3)$ converge strongly to $S \in \mathcal{L}(X_1; X_2)$ and $Q \in \mathcal{L}(X_2; X_3)$, respectively. Then, $Q_n S_n$ converge strongly to QS.*

As a first application of Lemma 1.1, we have

Proposition 1.1 *Assume hypotheses (1.5.3) and (B.2) = (1.5.9). Then (B.3) $\Longleftrightarrow$ (B.3′), where statement (B.3′) is defined by*

(B.3′):

$$\|[A_h^{-1} - A^{-1}]B_h u\|_Y \to 0 \text{ as } h \downarrow 0, \ u \in U. \tag{1.5.14}$$

Proof Let $u \in U$. Consider the identity

$$[A_h^{-1} - A^{-1}]B_h u = [A_h^{-1} - \Pi_h A^{-1}]B_h u + [\Pi_h - I]A^{-1}B_h u. \tag{1.5.15}$$

By virtue of Lemma 1.1, assumptions (1.5.3) and (B.2) imply that $[\Pi_h - I]A^{-1}B_h \to 0$ strongly; in which case, identity (1.5.15) readily yields the equivalence (B.3) $\Longleftrightarrow$ (B.3′). ◻

Proposition 1.2 *Assume hypotheses (1.5.3), (B.2) = (1.5.9), and (B.3) = (1.5.10). Then*

$$\|[A_h^{-1}B_h - A^{-1}B]u\|_Y \to 0 \text{ as } h \downarrow 0, \quad u \in U. \tag{1.5.16}$$

Proof For $u \in U$ we estimate

$$\begin{aligned} &\|[A_h^{-1}B_h - A^{-1}B]u\|_Y \\ &\quad \le \|[A_h^{-1}B_h - A^{-1}B_h]u\|_Y + \|[A^{-1}B_h - A^{-1}B]u\|_Y \to 0, \end{aligned} \tag{1.5.17}$$

where convergence on the right-hand side of (1.5.17) stems from (B.3′) = (1.5.14) via Proposition 1.1, and from (B.2) = (1.5.9). ◻

Proposition 1.3 *Assume (1.5.3), (B.2) = (1.5.9) and (B.3) = (1.5.10). Then*

$$\|[A_h^{-1}B_h - \Pi_h A^{-1}B]u\|_Y \to 0 \text{ as } h \downarrow 0, \quad u \in U. \tag{1.5.18}$$

Proof For $u \in U$, we estimate

$$\begin{aligned} &\|[A_h^{-1}B_h - \Pi_h A^{-1}B]u\|_Y \\ &\quad \le \|[A_h^{-1} - \Pi_h A^{-1}]B_h u\|_Y + \|\Pi_h A^{-1}[B_h - B]u\|_Y \to 0, \end{aligned} \tag{1.5.19}$$

where convergence on the right-hand side of (1.5.19) stems from (B.3) and (B.2) via $\|\Pi_h\| \le$ const. by (1.5.3), via the Principle of Uniform Boundedness. ◻

Proposition 1.4 *Assume (B.4) = (1.5.11) and (B.5) = (1.5.12). Then*

$$\|[B_h^* A_h^{*-1}\Pi_h - B^* A^{*-1}]x\|_U \to 0 \text{ as } h \downarrow 0, \quad x \in Y. \tag{1.5.20}$$

Proof For $x \in Y$, we estimate

$$\|[B_h^* A_h^{*-1}\Pi_h - B^* A^{*-1}]x\|_U \le \|B_h^*[A_h^{*-1}\Pi_h - \Pi_h A^{*-1}]x\|_U + \|[B_h^*\Pi_h - B^*]A^{*-1}x\|_U \to 0, \tag{1.5.21}$$

where convergence on the right-hand side of (1.5.21) follows from (B.4) and (B.5). □

Proposition 1.5 *Assume hypotheses (1.5.3) and (A.4) = (1.5.7). Then*

$$\|A_h^* \Pi_h A^{*-1}\|_{\mathcal{L}(Y)} \leq \; const., \quad \forall \; h. \tag{1.5.22}$$

Proof Let $y \in Y$. Then, as $h \downarrow 0$,

$$\|A_h^* \Pi_h A^{*-1} y\|_Y - \|\Pi_h y\|_Y \leq \|[A_h^* \Pi_h A^{*-1} - \Pi_h]y\| \to 0 \tag{1.5.23}$$

by invoking (A.4) = (1.5.7) on the right-hand side of (1.5.23), which is therefore uniformly bounded by a constant, say c_y, depending on y but not on h. Thus, from (1.5.23) we obtain

$$\|A_h^* \Pi_h A^{*-1} y\|_Y \leq \|\Pi_h y\|_Y + c_y \leq \; \text{const.}_y \tag{1.5.24}$$

via (1.5.3), and the Principle of Uniform Boundedness applied to (1.5.24) yields (1.5.22). □

Henceforth, the basic assumption (1.5.3) on Π_h will be tacitly assumed.

1.5.4 Approximation of Continuous Dynamics and Control Problem

Discrete Optimal Control problem OCP_h: Given the approximating dynamics $y_h(t) \subset V_h$ satisfying the semi-discrete model

$$\dot{y}_h(t) = A_h y_h(t) + B_h u(t), \quad y_h(s) = \Pi_h y_0, \tag{1.5.25}$$

minimize over all $u \in L_2(s, T; U)$ the cost functional

$$J_{s,T;G}(u, y_h) = \int_s^T [\|R y_h(t, s; \Pi_h y_0)\|_Z^2 + \|u(t)\|_U^2] dt + \|G y_h(T, s; \Pi_h y_0)\|_{Z_f}^2. \tag{1.5.26}$$

It is a standard result (on V_h)—which is, in fact, contained in [30, Chapter 9, Theorem 9.2.1 and Theorem 9.2.2] when specialized to the finite dimensional V_h—that problem (1.5.26) for the dynamics (1.5.25) admits a unique optimal pair $\{u_{h,T}^0(\cdot, s; \Pi_h y_0), y_{h,T}^0(\cdot, s; \Pi_h y_0)\}$ satisfying

$$u_{h,T}^0(t, s; \Pi_h y_0) = -B_h^* P_{h,T}(t) y_{h,T}^0(t, s; \Pi_h y_0) \in L_2(s, \infty; U), \tag{1.5.27}$$

where $0 \leq P_{h,T}(t) = P_{h,T}^*(t) \in \mathcal{L}(Y)$ is the unique, non-negative, self-adjoint solution of the following discrete *Differential Riccati Equation (D.R.E.)* on V_h

$$\begin{aligned}(\dot{P}_{h,T}(t) x_h, z_h)_Y = &- (R x_h, R z_h)_Z - (P_{h,T}(t) x_h, A_h z_h)_Y - (P_{h,T}(t) A_h x_h, z_h)_Y \\ &+ (B_h^* P_{h,T}(t) x_h, B_h^* P_{h,T}(t) z_h)_U, \; \forall \; x_h, z_h \in V_h\end{aligned} \tag{1.5.28}$$

$$(P_{h,T}(T)x_h, z_h)_Y = (G^*Gx_h, z_h)_Y. \tag{1.5.29}$$

Moreover, the optimal cost for problem (1.5.26) is given by

$$\begin{aligned}\min_{u\in L_2(0,T;U)} J_{s,T;G}(u, y_h) &= J_T(u^0_{h,T}(\cdot; s; \varPi_h y_0), y^0_{h,T}(\cdot; s; \varPi_h y_0)) \\ &= (P_{h,T}(s)\varPi_h y_0, \varPi_h y_0)_Y = (\varPi_h P_{h,T}(s)\varPi_h y_0, y_0)_Y. \end{aligned}\tag{1.5.30}$$

Finally, $P_{h,T}(t)\varPi_h$ is explicitly given by the following formula

$$\begin{aligned}&P_{h,T}(t)\varPi_h x \\ &= \int_t^T e^{A_h^*(\tau-t)}\varPi_h R^*Ry^0_{h,T}(\tau, t; \varPi_h x)d\tau + e^{A_h^*(T-t)}\varPi_h G^*Gy^0_{h,T}(T, t; \varPi_h x).\end{aligned}\tag{1.5.31}$$

discrete version of formula (1.3.10) (see also [30, Chapter 9, Eq. (9.2.14)]). The operator (recall (1.3.14)):

$$\varPhi_{h,T}(\tau, t)x_h \equiv y^0_{h,T}(\tau, t; x_h), \quad x_h \in V_h \tag{1.5.32}$$

is an evolution operator on V_h satisfying properties such as (1.3.15), (1.3.16) in the continuous case.
The main Result is near optimality of the construction: $J_T(u_h, y_h) - J_T(u^0, y^0) \to 0$, as $h \to 0$ with $u_h(t) = -B_h^* P_h(t)y_h(t)$ and $\dot{y_h} = Ay_h + Bu_h$.

2 Main Results

2.1 Convergence of Optimal Solutions: Theorem 2.1

Our first approximating result is the following:

Theorem 2.1 *Assume hypotheses (H.1) through (H.4) on the continuous dynamics (1.2.1), as well as the approximating hypotheses (A.1) through (A.4) on the operator A, and (B.1) through (B.5) on the operator B. Let*

$$\{u^0_T(\cdot, s; y_0),\ y^0_T(\cdot, s; y_0)\} \text{ and } \{u^0_{h,T}(\cdot, s; \varPi_h y_0),\ y^0_{h,T}(\cdot, s; \varPi_h y_0)\}$$

be the optimal solution pairs of the optimal problem (1.3.6) for the continuous dynamics (1.2.1), and, respectively, of the optimal problem (1.5.26) for the corresponding approximating dynamics (1.5.25). Then, for all $y_0 \in Y$ and $h \downarrow 0$ the folowing convergence is uniform in s.:

$$(i) \qquad \|u^0_{h,T}(\cdot, s; \varPi_h y_0) - u^0_T(\cdot; s; y_0)\|_{L_2(s,T;U)} \to 0 \tag{2.1.1}$$

(ii)
$$\begin{aligned}&\|\Phi_{h,T}(\cdot,s)\Pi_h y_0 - \Phi_T(\cdot,s)y_0\|_{C([s,T];Y)}\\ &\quad = \|y^0_{h,T}(\cdot,s;\Pi_h y_0) - y^0_T(\cdot,s;y_0)\|_{C([s,T];Y)} \to 0\end{aligned} \tag{2.1.2}$$

(iii) $$J_{s,T;G}(u^0_{h,T}(\cdot,s;\Pi_h y_0), y^0_{h,T}(\cdot,s;\Pi_h y_0)) \to J_{s,T;G}(u^0_T(\cdot,s;y_0), y^0_T(\cdot,s;y_0)) \tag{2.1.3}$$

Finally, let $P_T(t)$ be the (Riccati) operator defined by (1.3.10) (and satisfying (1.3.12)–(1.3.13), and let $P_{h,T}(t)$ be the unique, non-negative, self-adjoint operator satisfying (1.3.21) and (1.3.22), given explicitly by (1.5.31). Then

(iv) $$\|P_{h,T}(\cdot)\Pi_h y_0 - P_T(\cdot)y_0\|_{C([s,T];Y)} \to 0 \text{ as } h \downarrow 0, \quad y_0 \in Y. \tag{2.1.4}$$

2.2 Convergence of Gain Operators: Theorem 2.2

Regarding the discrete problem, we next state two additional approximation assumptions, invoking also the regularity properties of the observation operators R and G. They are the approximating counterparts of the continuous assumptions (H.5) = (1.3.3) and (H.6) = (1.3.4) in their dual version (H.5*) = (1.3.7) and (H.6*) = (1.3.8). The corresponding assumptions, in the regular case, for the approximating problems are (when written in the dual form corresponding to (H.5*) and (H.6*)):

(H.7):
$$\begin{aligned}&\left\| \int_0^T \left[B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t} \right] R^* R v(t)\,dt \right\|_U\\ &\to 0 \quad \text{as } h \downarrow 0, \forall\ v \in L_\infty(0,T;Y).\end{aligned} \tag{2.2.1}$$

(H.8):
$$\begin{aligned}&\sup_{0\le t\le T} \left\| \left[B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t} \right] G^* G x \right\|_U\\ &\to 0 \quad \text{as } h \downarrow 0, \quad \forall\ x \in Y.\end{aligned} \tag{2.2.2}$$

Implications of (H.5), (H.6), (H.7), and (H.8). In view of the continuous hypotheses (H.5*) = (1.3.7) and (H.6*) = (1.3.8), we readily see that the discrete hypotheses (H.7) = (2.2.1) and (H.8) = (2.2.2) imply, respectively, the stability estimates

$$\left\| \int_0^T B_h^* e^{A_h^* t} \Pi_h R^* R v(t)\,dt \right\|_U \le c_v, \text{ uniformly in } h, v \in L_\infty(0,T;Y); \tag{2.2.3}$$
$$\left\| B_h^* e^{A_h^* t} \Pi_h G^* G x \right\|_{L_\infty(0,T;U)} \le c_v, \text{ uniformly in } h, \quad x \in Y. \tag{2.2.4}$$

These, then, in turn, imply by the Principle of Uniform Boundedness

$$\left\| \int_0^T B_h^* e^{A_h^* t} \Pi_h R^* R v(t)\,dt \right\|_U \le c_T \|v\|_{L_\infty(0,T;Y)}; \tag{2.2.5}$$
$$\sup_{0\le t\le T} \left\| B_h^* e^{A_h^* t} \Pi_h G^* G x \right\|_U \le c_T \|x\|_Y. \tag{2.2.6}$$

Under these additional assumptions of smoothness, we may now claim convergence of the gain operators.

Theorem 2.2 *As in Theorem 2.1, assume hypotheses (H.1) through (H.3) on the continuous dynamics (1.2.1), as well as the approximating assumptions (A.1) through (A.4) on the operator A, and (B.1) through (B.5) on the operator B. In addition, assume the smoothing assumptions (H.5) and (H.6) for R and G, and the discrete versions (H.7) and (H.8). Then, the operator* $P_T(t)$ *[defined by (1.3.10), which is now the (unique) solution of the integral/differential Riccati equation, as noted below (1.5.9)], and the operators* $P_{h,T}(t)$ *[defined by (1.5.31) and the (unique) solution of the D.R.E. (1.3.21) and I.R.E. (1.3.22)], satisfy the following convergence of gains operators*

$$\sup_{0\le t\le T} \|B_h^* P_{h,T}(t)\Pi_h - B^* P_T(t)]x\|_U \to 0 \text{ as } h \downarrow 0. \tag{2.2.7}$$

and the original system when driven by the feedback control $B_h^* P_h(t)$ *– see (1.3.24) – converges, as* $h \to 0$*, to the optimal solution: (i)* $u_h \to u^0$ *in* $L_2(0T; U)$*; (ii)* $y_h \to y^0$ *in* $C([0, T], Y)$*; (iii)* $J(u_h, y_h) \to J(u^0, y^0)$*.*

The reminder of the paper is devoted to the proofs of Theorem 2.1 and Theorem 2.2.

3 Background Material

3.1 Continuous Problem: Representation Formulae for Optimal Pair

Theorem 2.1 and 2.2 will generally follow, at the discrete level, the strategy of the continuous theory presented in [30, Chapter 9]. Thus, for convenience, as well as to emphasize the parallelism between continuous and discrete proofs, we shall collect relevant quantities and relations from [30, Chapter 9] to be used in the sequel. With reference to the optimal problem (1.3.6) on $[s, T]$, the following operators play a role

$$\{L_{s,T}u\}(t) = \begin{cases} \int_s^t e^{A(t-\tau)}Bu(\tau)d\tau, & s \le t \\ 0, & 0 \le t < s \end{cases} \tag{3.1.1a}$$

$$: \text{ continuous } L_2(s, T; U) \to C([s, T]; Y) \text{ with norm uniform,} \tag{3.1.1b}$$

$$\text{with respect to } s, \text{ i.e. } \quad \|L_{s,T}u\|_{C([s,T];Y)} \le c_T \|u\|_{L_2(s,T;U)} \tag{3.1.1c}$$

[30, Chapter 9, Eq. (9.1.17)] where the regularity in (3.1.1b) is a consequence of assumption (H.3) = (1.2.3). Its adjoint $L^*_{s,T}$ in the $L_2(s, T; \cdot)$-sense is

$$(L^*_{s,T}f)(t) = \begin{cases} \int_t^T B^* e^{A^*(\tau-t)} f(\tau)d\tau, & s \le t \le T \\ 0, & 0 \le t < s \end{cases} \tag{3.1.2a}$$

$$: \text{ continuous } L_1(s, T; Y) \to L_2(s, T; U) \text{ with norm uniform w.r.t. } s., \tag{3.1.2b}$$

[30, Chapter 9, Eq. (9.1.18)]. By specializing (3.1.1a) at $t = T$, we get

$$L_{(sT)}u \equiv (L_{s,T}u)(T) = \int_s^T e^{A(T-\tau)} Bu(\tau)d\tau \tag{3.1.3a}$$

$$\text{continuous } L_2(s, T; U) \to Y \text{ with norm uniform with respect to } s., \tag{3.1.3b}$$

[30, Chapter 9, Eq. (9.1.20)]. with its adjoint

$$\{L^*_{(sT)}y\}(t) = \begin{cases} B^* e^{A^*(T-t)}y, & s \le t \le T,\ y \in Y \\ 0, & 0 \le t < s \end{cases} \tag{3.1.4a}$$

$$: \text{continuous } Y \to L_2(s, T; U). \tag{3.1.4b}$$

Hence, putting together (3.1.1) through (3.1.4) we get that

$$\Lambda_{s,T} = I_s + L^*_{s,T} R^* R L_{s,T} + L^*_{(sT)} G^* G L_{(sT)} \tag{3.1.5a}$$

$$: \text{ isomorphism on } L_2(s, T; U) \text{ with norms uniform w.r.t. } s., \tag{3.1.5b}$$

$$\|\Lambda_{s,T}\|_{\mathcal{L}(L_2(s,T;U))} \le c_T; \qquad \|\Lambda^{-1}_{s,T}\|_{\mathcal{L}(L_2(s,T;U))} \le 1, \tag{3.1.5c}$$

where I_s is the identity operator on $L^2(s, T; U)$. Next, with reference to the optimal control problem (1.3.6) over $[s, T]$, we have that the optimal pair in (1.3.9) is expressed by the following representation formulae [30, Chapter 9, Eq. (9.2.2)],

$$-u^0_T(\cdot, s; y_0) = \Lambda^{-1}_{s,T}\left[L^*_{(sT)} G^* G e^{A(T-s)} y_0 + L^*_{s,T} R^* R e^{A(\cdot-s)} y_0\right] \in L_2(s, T; U) \tag{3.1.6}$$

$$y^0_T(t, s; y_0) \equiv \Phi_T(t, s) y_0 = e^{A(t-s)} y_0 + \{L_{s,T} u^0_T(\cdot, s; y_0)\}(t) \tag{3.1.7}$$

$$= e^{A(t-s)} y_0 - L_{s,T} \Lambda^{-1}_{s,T}\left[L^*_{(sT)} G^* G e^{A(T-s)} y_0 + L^*_{s,T} R^* R e^{A(\cdot-s)} y_0\right] \in C([s, T]; Y). \tag{3.1.8}$$

3.2 Discrete Problem

In order to describe the solution to the discrete problem (1.5.26), (1.5.25), we similarly introduce the operators

$$\{L_{h;s,T}u\}(t) = \begin{cases} \int_s^t e^{A_h(t-\tau)} B_h u(\tau)d\tau, & s \le t \\ 0, & 0 \le t < s \end{cases} \tag{3.2.1a}$$

$$: \text{continuous } L_2(s, T; U) \to C([s, T]; V_h). \tag{3.2.1b}$$

$$L_{h;(sT)}u \equiv (L_{h;s,T}u)(T) = \int_s^T e^{A_h(T-\tau)} B_h u(\tau)d\tau \; : \tag{3.2.2a}$$

$$: \text{ continuous } L_2(s, T; U) \to V_h, \tag{3.2.2b}$$

and their L_2-duals

$$(L^*_{h;s,T}\Pi_h f)(t) = \begin{cases} \int_t^T B_h^* e^{A_h^*(\tau-t)} \Pi_h f(\tau)d\tau, & s \le t \le T \\ 0, & 0 \le t < s \end{cases} \tag{3.2.3a}$$

$$: \text{continuous } L_1(s, T; Y) \to L_2(s, T; U). \tag{3.2.3b}$$

$$\{L^*_{h;(sT)}\Pi_h y\}(t) \equiv \begin{cases} B_h^* e^{A_h^*(T-t)} \Pi_h y, & s \le t \le T, \; y \in Y \\ 0, & 0 \le t < s \end{cases} \tag{3.2.4a}$$

$$: \text{continuous } Y \to L_2(s, T; U). \tag{3.2.4b}$$

Thus, from (3.2.1)–(3.2.4), we obtain the discrete versions of (3.1.5)

$$\Lambda_{h;s,T} = I_s + L^*_{h;s,T}\Pi_h R^* R \Pi_h L_{h,s;T} + L^*_{h;(sT)}\Pi_h G^* G \Pi_h L_{h;(sT)} \tag{3.2.5a}$$

$$: \text{ self-adjoint isomorphism on } L_2(s, T; U) \text{ with norms uniform w.r.t. } s; \tag{3.2.5b}$$

$$\|\Lambda_{h,s;T}\|_{\mathcal{L}(L_2(s,T;U))} \le c_T; \quad \|\Lambda^{-1}_{h,s;T}\|_{\mathcal{L}(L_2(s,T;U))} \le 1. \tag{3.2.5c}$$

$$\begin{aligned} -u^0_{h,T}(\cdot, s; \Pi_h y_0) &= \Lambda^{-1}_{h;s,T}\Big[L^*_{h;(sT)}\Pi_h G^* G e^{A_h(T-s)}\Pi_h y_0 \\ &\quad + L^*_{h;s,T}\Pi_h R^* R e^{A_h(\cdot-s)}\Pi_h y_0\Big] \in L_2(s, T; U). \end{aligned} \tag{3.2.6}$$

$$y^0_{h,T}(t, s; \Pi_h y_0) = \Phi_{h,T}(t, s)\Pi_h y_0 = e^{A_h(t-s)}\Pi_h y_0 + \{L_{h;s,T}u^0_{h,T}(\cdot, s; \Pi_h y_0)\}(t) \tag{3.2.7}$$

$$\begin{aligned} \text{(by (3.2.6))} \quad &= e^{A_h(t-s)}\Pi_h y_0 - L_{h;s,T}\Lambda^{-1}_{h;sT}\Big[L^*_{h;(sT)}G^* G e^{A_h(T-s)}\Pi_h y_0 \\ &\quad + L^*_{h;s,T}R^* R e^{A_h(\cdot-s)}\Pi_h y_0\Big]. \end{aligned} \tag{3.2.8}$$

4 Convergence Properties of Control Operators

4.1 Convergence Properties of the Operators $L_{h;s,T}$ and $L^*_{h;s,T}\Pi_h$

Proposition 4.1 *Assume the standing hypotheses (H.1), (H.2), (H.3) on the dynamics as well as the approximating hypothesis (A.1) = (1.5.4). Then, for any* $0 < T < \infty$, *the following convergence properties hold true as* $h \downarrow 0$*:*

(i) First, if (A.2) = (1.5.5) is assumed, then

$$\|e^{A_h t}\Pi_h x - e^{At}x\|_{C([0,T];Y)} \to 0, \quad x \in Y. \tag{4.1.1}$$

(ii) If, instead, (A.3) = (1.5.6) is assumed, then

$$\left\| e^{A_h^* t} \Pi_h x - e^{A^* t} x \right\|_{C([0,T];Y)} \to 0, \quad x \in Y. \tag{4.1.2}$$

Proof (i) Assumptions (A.1) and (A.2) allow us to invoke the Trotter-Kato Theorem [P.1, p.85] on $e^{A_h t}\Pi_h$ and conclude with the convergence in (4.1.1). (ii) The adjoint version of (A.1), along with (A.3) = (1.5.6), prove similarly conclusion (4.1.2) for $e^{A_h^* t}\Pi_h$. □

The next lemma provides a stability result for $L_{h,s}$.

Lemma 4.1 *Assume the approximating hypothesis (B.1) = (1.5.8). Then, there is a constant c_T, depending on T but not on h or s, such that the following bounds, uniform in h and s, hold true for the operators $L_{h;s,T}$ and $L_{h;(sT)}$ defined in (3.2.1) and (3.2.2), respectively, as well as $\Lambda_{h;s,T}$ defined in (3.2.5):*

(i)
$$\|L_{h;s,T} u\|_{C([0,T];Y)} \le c_T \|u\|_{L_2(0,T;U)}; \tag{4.1.3a}$$

hence,

$$\|L^*_{h;s,T} v_h\|_{L_2(0,T;U)} \le c_T \|v_h\|_{L_1(0,T;V_h)}; \tag{4.1.3b}$$

(ii)
$$\|L_{h;(sT)} u\|_Y \le c_T \|u\|_{L_2(0,T;U)}; \tag{4.1.4a}$$

equivalently,

$$\|L^*_{h;(sT)} y_h\|_{L_2(0,T;U)} \le c_T \|y_h\|_Y, \quad y_h \in V_h; \tag{4.1.4b}$$

(iii)
$$\|\Lambda_{h;s,T} u\|_{L_2(s,T;U)} \le c_T \|u\|_{L_2(s,T;U)}. \tag{4.1.5}$$

Proof We first show (4.1.3) with $C([0,T];Y)$ replaced by $L_\infty(0,T;Y)$. To this end, let $f \in L_1(0,T;Y)$ and $u \in L_2(0,T;U)$. Then, we compute from (3.2.1),

$$\left| \int_0^T ((L_{h;s,T} u)(t), f(t))_Y dt \right| = \left| \int_0^T \left(\int_s^t e^{A_h(t-\tau)} B_h u(\tau) d\tau, f(t) \right)_Y dt \right|$$
$$= \left| \int_0^T \int_s^t (u(\tau), B_h^* e^{A_h^*(t-\tau)} f(t))_U d\tau\, dt \right| \le \int_0^T \int_s^t \|u(\tau)\|_U \|B_h^* e^{A_h^*(t-\tau)} f(t)\|_U d\tau\, dt$$
$$\le \int_0^T \left\{ \int_s^t \|u(\tau)\|_U^2 d\tau \right\}^{\frac{1}{2}} \left\{ \int_s^t \|B_h^* e^{A_h^*(t-\tau)} f(t)\|_U^2 d\tau \right\}^{\frac{1}{2}} dt \tag{4.1.6}$$
$$\le c_T \|u\|_{L_2(0,T;U)} \int_0^T \|f(t)\| dt = c_T \|u\|_{L_2(0,T;U)} \|f\|_{L_1(0,T;Y)}, \tag{4.1.7}$$

by (1.5.8)), where in going from (4.1.6) to (4.1.7) we have used assumption (B.1) = (1.5.8). Then (4.1.7) shows (4.1.3) with $C([0,T];Y)$ replaced by $L_\infty(0,T;Y)$. But, for $u \in L_2(0,T;U)$, we already know from (3.2.1b) that $L_{h;s,T}u \in C([0,T];Y)$. Thus, the desired estimate (4.1.3) is proved. Since $L_{h;(sT)}u = (L_{h;s,T}u)(T)$, see (3.2.2), (4.1.4) is a specialization of (4.1.3). Finally, (4.1.5) is an immediate corollary of (4.1.3), (4.1.4) via (3.2.5). □

The main results of this subsection are the following two theorems.

Theorem 4.1 *Assume the standing hypotheses (H.1), (H.2), (H.3) on the dynamics. Furthermore, assume the approximating hypotheses (A.1) = (1.5.4), (A.2) = (1.5.5), as well as (B.1) = (1.5.8), (B.2) = (1.5.9), (B.3) = (1.5.10). Then, as* $h \downarrow 0$, *the following convergence results hold true, uniformly in s:*

(i)
$$\|L_{h;s,T}u - L_{s,T}u\|_{C([0,T];Y)} \to 0, \quad u \in L_2(0,T;U); \tag{4.1.8}$$

(ii)
$$\|L_{h;(sT)}u - L_{(sT)}u\|_Y \to 0, \quad u \in L_2(0,T;U); \tag{4.1.9}$$

Proof (i) We already know from (4.1.3) of Lemma 3.2, by (B.1), that the uniform bound

$$\|L_{h;s,T}\|_{\mathcal{L}(L_2(0,T;U);C([0,T];Y))} \le c_T \tag{4.1.10}$$

holds true. Thus, in view of (4.1.10), in order to prove (4.1.8), it suffices to show that the uniform convergence (4.1.8) takes place for all u in the subspace

$$\mathcal{D} = C^1([0,T];U) \tag{4.1.11}$$

dense in $L_2(0,T;U)$. Accordingly, we show that

$$\|L_{h;s,T}u - L_{s,T}u\|_{C([0,T];Y)} \to 0 \text{ as } h \downarrow 0 \text{ uniformly in } s, \quad \forall\, u \in \mathcal{D}. \tag{4.1.12}$$

After this (i) is proved. To establish 4.1.12, we take $u \in \mathcal{D}$, and integrate 3.2.1a by parts, thus obtaining

$$\begin{aligned}(L_{h;s,T}u)(t) &= \Big[-A_h^{-1}e^{A_h(t-\tau)}B_hu(\tau)\Big]_{\tau=s}^{\tau=t} + \int_s^t e^{A_h(t-\tau)}A_h^{-1}B_h\dot{u}(\tau)d\tau \\ &= A_h^{-1}e^{A_h(t-s)}B_hu(s) - A_h^{-1}B_hu(t) \\ &\quad + \int_s^t e^{A_h(t-\tau)}A_h^{-1}B_h\dot{u}(\tau)d\tau, \quad u \in \mathcal{D}. \end{aligned} \tag{4.1.13}$$

[As noted below 1.5.5, we may assume without loss of generality that $A_h^{-1} \in \mathcal{L}(V_h)$, $\forall\ h$.] Similarly, for $u \in \mathcal{D}$, we integrate 3.1.1a by parts, thus obtaining

$$\begin{aligned}(L_{s,T}u)(t) &= A^{-1}e^{A(t-s)}Bu(s) - A^{-1}Bu(t) \\ &\quad + \int_s^t e^{A(t-\tau)}A^{-1}B\dot{u}(\tau)d\tau, \quad u \in \mathcal{D},\ s \le t \le T \end{aligned} \tag{4.1.14}$$

(where we recall $A^{-1} \in \mathcal{L}(Y)$ from (H.1)). Thus, for $u \in \mathcal{D}$, we obtain from (4.1.13) and (4.1.14) after adding and subtracting $e^{A_h(t-s)}\Pi_h A^{-1}Bu(s)$:

$$(L_{h;s,T}u)(t) - (L_{s,T}u)(t) = I_{1,hs}(t) + I_{2,hs}(t) + I_{3,h}(t) + I_{4,hs}(t); \tag{4.1.15}$$

$$I_{1,hs}(t) = e^{A_h(t-s)}[A_h^{-1}B_h - \Pi_h A^{-1}B]u(s) \tag{4.1.16}$$

$$I_{2,hs}(t) = [e^{A_h(t-s)}\Pi_h - e^{A(t-s)}]A^{-1}Bu(s) \tag{4.1.17}$$

$$I_{3,h}(t) = [A^{-1}B - A_h^{-1}B_h]u(t) \tag{4.1.18}$$

$$I_{4,hs}(t) = \int_s^t [e^{A_h(t-\tau)}A_h^{-1}B_h - e^{A(t-\tau)}A^{-1}B]\dot{u}(\tau)d\tau. \tag{4.1.19}$$

- $I_{1,hs}(t)$. As to $I_{1,hs}(t)$, recalling hypothesis (A.1) = (1.5.4), we obtain from (4.1.16):

$$\begin{aligned}\|I_{1,hs}(t)\|_Y &= \|e^{A_h(t-s)}[A_h^{-1}B_h - \Pi_h A^{-1}B]u(s)\|_Y \\ &\leq Ce^{\omega_1 T}\|[A_h^{-1}B_h - \Pi_h A^{-1}B]u(s)\|_Y, \quad s \leq t \leq T.\end{aligned} \tag{4.1.20}$$

  For fixed s, the right-hand side of (4.1.20) goes to zero as $h \downarrow 0$ by simply invoking (1.5.18) of Proposition 1.3(which requires (B.2) and (B.3)). Moreover, for $u \in \mathcal{D}$, $\{Q_h u(s)\}$ is an equicontinuous family on the compact interval $0 \leq s \leq T$, where $Q_h \equiv A_h^{-1}B_h - \Pi_h A^{-1}B$: given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\|Q_h u(s') - Q_h u(s'')\|_Y \leq C\|u(s') - u(s'')\|_U < \epsilon$$

  for all $|s - s''| < \delta_\epsilon$, since $u(\cdot) \in \mathcal{D}$ is uniformly continuous on $0 \leq s \leq T$, where $\|Q_h\|_{\mathcal{L}(Y)} \leq C$, by the Principle of Uniform Boundedness. Then, the equicontinuous family $Q_h u(s)$, which converges at each s to zero, converges uniformly to zero on the compact interval $0 \leq s \leq T$ as $h \downarrow 0$ [38, Lemma 32, p. 176]. Returning to (4.1.20) all this means that

$$\|I_{1,hs}(\cdot)\|_{C([s,T];Y)} \leq Ce^{\omega_1 T}\|Q_h u(\cdot)\|_{C([0,T];Y)} \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s. \tag{4.1.21}$$

- $I_{3,h}(t)$. Exactly the same argument above following (4.1.20), this time with $Q_h = A^{-1}B - A_h^{-1}B_h$ converging strongly to zero by (1.5.16) of Proposition 1.2, shows that, as $h \downarrow 0$,

$$\|I_{3,h}(\cdot)\|_{C([0,T];Y)} = \|[A^{-1}B - A_h^{-1}B_h]u(\cdot)\|_{C([0,T];Y)} \to 0. \tag{4.1.22}$$

- $I_{2,hs}(t)$. From (4.1.17) we obtain for s fixed,

$$\begin{aligned}\|I_{2,hs}(\cdot)\|_{C([s,T];Y)} &= \|[e^{A_h(\cdot-s)}\Pi_h - e^{A(\cdot-s)}]A^{-1}Bu(s)\|_{C([s,T];Y)} \\ &\equiv M_h(s) \to 0 \text{ as } h \downarrow 0\end{aligned} \tag{4.1.23}$$

by recalling (4.1.1) and (H.2) = (1.2.2). To show that the convergence in (4.1.23) is uniform in $s \in [0, T]$; i.e., to show that

$$\sup_{0 \leq s \leq T} M_h(s) = \sup_{0 \leq s \leq T} \|I_{2,hs}(\cdot)\|_{C([s,T];Y)} \to 0 \text{ as } h \downarrow 0, \tag{4.1.24}$$

we set for convenience

$$Q_h(t-s) \equiv [e^{A_h(t-s)}\Pi_h - e^{A(t-s)}]A^{-1}B \in \mathcal{L}(U; Y), \tag{4.1.25}$$

so that

$$M_h(s) = \|Q_h(\cdot - s)u(s)\|_{C([s,T];Y)} \to 0 \text{ as } h \downarrow 0. \tag{4.1.26}$$

We next compute with s_1 a point of $[0, T]$:

$$\begin{aligned} M_h(s) &= \|Q_h(\cdot - s)[(u(s) - u(s_1)) + u(s_1)]\|_{C([s,T];Y)} \\ &\leq \|Q_h(\cdot - s)[u(s) - u(s_1)]\|_{C([s,T];Y)} \\ &\quad + \|Q_h(\cdot - s)u(s_1)\|_{C([s,T];Y)} & (4.1.27) \\ &\leq C_T\|u(s) - u(s_1)\|_U + \|Q_h(\cdot)u(s_1)\|_{C([0,T];Y)}, & (4.1.28) \end{aligned}$$

where in going from (4.1.27) to (4.1.28) we have recalled the uniform bound (A.1) = (1.5.4) in (4.1.25). Now, as to the first term in (4.1.28) we have that $u(\cdot) \in \mathcal{D}$ is uniformly continuous on $0 \leq s \leq T$, while the second term in (4.1.28) converges to zero as $h \downarrow 0$ by (4.1.1) via (4.1.25). Thus, given $\epsilon > 0$ and s_1, there exist constants $\delta_\epsilon > 0$ (independent of s_1), and $\bar{h}_{\epsilon,s_1} > 0$ such that:

$$\text{if } |s - s_1| < \delta_\epsilon \implies \|u(s) - u(s_1)\|_U < \frac{\epsilon}{C_T}; \tag{4.1.29}$$

$$\text{if } h < \bar{h}_{\epsilon,s_1} \implies \|Q_h(\cdot)u(s_1)\|_{C([0,T];Y)} < \epsilon. \tag{4.1.30}$$

Thus, using (4.1.29) and (4.1.30) in (4.1.28), we obtain

$$\text{if } |s - s_1| < \delta_\epsilon \text{ and } h < \bar{h}_{\epsilon,s_1} \implies M_h(s) < 2\epsilon. \tag{4.1.31}$$

There are finitely many intervals $\mathcal{I}_{s_i,\delta_\epsilon}$ of length δ_ϵ centered at suitable points $s_1, \ldots, s_N$, which cover the interval $[0, T]$: for each of them, the above argument yields the counterpart of (4.1.31):

$$\text{if } s \in \mathcal{I}_{s_i,\delta_\epsilon} \text{ and } h < \bar{h}_{\epsilon,s_i} \implies M_s(s) < 2\epsilon. \tag{4.1.32}$$

Taking $\bar{h}_\epsilon = \min\{\bar{h}_{\epsilon,s_1}, \ldots, \bar{h}_{\epsilon,s_N}\} > 0$, we then obtain from (4.1.32),

$$\text{if } h < \bar{h}_\epsilon \implies \sup_{0 \leq s \leq T} M_h(s) < 2\epsilon, \tag{4.1.33}$$

and (4.1.24) is proved.

- $I_{4,hs}(t)$. Finally, regarding $I_{4,hs}(t)$, we estimate by (4.1.19), after adding and subtracting, with $u \in \mathcal{D}$, hence $\dot{u} \in C([0,T];U)$:

$$\begin{aligned}\|I_{4,hs}(t)\|_Y &= \|\int_s^t [e^{A_h(t-\tau)}A_h^{-1}B_h - e^{A(t-\tau)}A^{-1}B]\dot{u}(\tau)d\tau\|_Y \\ &\le \|\int_s^t e^{A_h(t-\tau)}\Pi_h[A_h^{-1}B_h - A^{-1}B]\dot{u}(\tau)d\tau\|_Y \\ +\|\int_s^t [e^{A_h(t-\tau)}\Pi_h &- e^{A(t-\tau)}]A^{-1}B\dot{u}(\tau)d\tau\|_Y \equiv a_{h,s}(t) + b_{h,s}(t).\end{aligned} \tag{4.1.34}$$

As to the first term $a_{h,s}(t)$ in (4.1.34), we estimate

$$\|a_{h,s}(\cdot)\|_{C([s,T];Y)} \equiv \left\|\int_s^{\cdot} e^{A_h(\cdot-\tau)}\Pi_h[A_h^{-1}B_h - A^{-1}B]\dot{u}(\tau)d\tau\right\|_{C([s,T];Y)} \tag{4.1.35}$$

$$(\text{by } (1.5.4)) \le TCe^{\omega_1 T}\|[A_h^{-1}B_h - A^{-1}B]\dot{u}(\cdot)\|_{C([0,T];Y)} \tag{4.1.36}$$

$$\to 0 \text{ as } h \downarrow 0 \text{ uniformly in } s,$$

where in going from (4.1.35) to (4.1.36) we have invoked (A.1) = (1.5.4), and where in asserting convergence to zero in (4.1.36), we have used the same reasons as in (4.1.22) (or in (4.1.21)) with $u \in C([0,T];U)$ there replaced by $\dot{u} \in C([0,T];Y)$ now: i.e., the family $[A_h^{-1}B_h - A^{-1}B]\dot{u}(t)$ is equicontinuous and converges to zero at each t in the compact interval $[0,T]$, so that uniform convergence is guaranteed by [38, Lemma 32, p. 176]. As to the second term $b_{h,s}(t)$ in (4.1.34), we estimate

$$\|b_{h,s}(\cdot)\|_{C([s,T];Y)} = \left\|\int_s^{\cdot} [e^{A_h(\cdot-\tau)}\Pi_h - e^{A(\cdot-\tau)}]A^{-1}B\dot{u}(\tau)d\tau\right\|_{C([s,T];Y)}$$

$$\le \int_0^T \|[e^{A_h(\cdot-\tau)}\Pi_h - e^{A(\cdot-\tau)}]A^{-1}B\dot{u}(\tau)\|_{C([s,T];Y)}d\tau \tag{4.1.37a}$$

$$\to 0 \text{ as } h \downarrow 0 \text{ uniformly in } s, \tag{4.1.37b}$$

where convergence to zero is achieved by invoking the Lebesgue Dominated Convergence Theorem along with (4.1.1) of Proposition 4.1 (for each τ fixed). Combining (4.1.36) with (4.1.37) in (4.1.34), we finally obtain

$$\|I_{4,hs}(\cdot)\|_{C([s,T];Y)} \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s. \tag{4.1.38}$$

To complete the proof, we return to (4.1.15) and use here (4.1.21) for $I_{1,hs}$, (4.1.22) for $I_{3,h}$, (4.1.24) for $I_{2,hs}$, and (4.1.38) for $I_{4,hs}$ to obtain

$$\|L_{h;s,T}u - L_{s,T}u\|_{C([s,T];Y)} \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s, \quad \forall\, u \in \mathcal{D} = C^1([0,T];U). \tag{4.1.39}$$

Since $\{L_{h;s,T}u\}(t) = \{L_{s,T}\}(t) \equiv 0$ for $0 \le t < s$, see (3.2.1a) and (3.1.1a), respectively, then conclusion (4.1.39) proves (4.1.12), as desired. Thus, part (i) is proved.

(ii) Part (ii), Eq. (4.1.9), is a specialization of part (i), by virtue of the definitions (3.1.3a) and (3.2.2a). Theorem 4.1 is proved.

The adjoint version of Theorem 4.1 is the following

Theorem 4.2 *Assume the standing hypotheses (H.1), (H.2), (H.3) on the dynamics. Furthermore, assume the approximating hypotheses (A.1) = (1.5.4), (A.3) = (1.5.6), and (A.4) = (1.5.7), as well as (B.1) = (1.5.8), (B.4) = (1.5.11), and (B.5) = (1.5.12). Then, as $h \downarrow 0$, the following convergence results hold true, uniformly in s.*

(i)
$$\|L^*_{h;(sT)}\Pi_h x - L^*_{(sT)}x\|_{L_2(0,T;U)} \to 0, \quad x \in Y; \tag{4.1.40}$$

(ii)
$$\|L^*_{h;s,T}\Pi_h f - L^*_{s,T}f\|_{L_2(0,T;U)} \to 0, \quad f \in L_1(0,T;Y). \tag{4.1.41}$$

Proof The proof is conceptually similar (with some technical differences, however) to the one of Theorem 4.1.

(i) From the definitions (3.2.4) and (3.1.4), we compute, for $t \ge s$, and initially for $y \in \mathcal{D}(A^*)$:

$$\begin{aligned}
\{L^*_{h;(sT)}\Pi_h y\}(t) - \{L^*_{(sT)}y\}(t) &= B^*_h e^{A^*_h(T-t)}\Pi_h y - B^* e^{A^*(T-t)}y \\
&= B^*_h A^{*-1}_h e^{A^*_h(T-t)} A^*_h \Pi_h y - B^* A^{*-1} e^{A^*(T-t)} A^* y \\
&= [B^*_h A^{*-1}_h \Pi_h - B^* A^{*-1}] e^{A^*(T-t)} A^* y \\
&\quad + B^*_h A^{*-1}_h \Pi_h [e^{A^*_h(T-t)}\Pi_h - e^{A^*(T-t)}] A^* y \\
&\quad + B^*_h A^{*-1}_h e^{A^*_h(T-t)} [A^*_h \Pi_h - \Pi_h A^*] y, \quad t \ge s, \ y \in \mathcal{D}(A^*),
\end{aligned} \tag{4.1.42}$$

after adding and subtracting two quantities. Now, for each t fixed in $[s, T]$, each of the three terms in (4.1.42) goes to zero as $h \downarrow 0$: the first term by virtue of the convergence (1.5.20) of Proposition 1.4, whereby we use assumptions (B.4) and (B.5); the second term by virtue of the convergence (4.1.2) (whereby we use (A.1) and (A.3)), and of the uniform bound $\|B^*_h A^{*-1}_h \Pi_h\|_{\mathcal{L}(Y;U)} \le$ const., which follows from (1.5.20) via the Principle of Uniform Boundedness (whereby we use (B.4) and (B.5)); the third term by virtue of the assumption (A.4) = (1.5.7), along with the uniform bound (A.1) and the uniform bound of $B^*_h A^{*-1}_h$ on V_h just recalled above. Thus, we have just shown that for each $t \in [s, T]$ fixed, we have

$$\{L^*_{h;(sT)}\Pi_h y\}(t) - \{L^*_{(sT)}y\}(t) \to 0 \text{ as } h \downarrow 0, \ y \in \mathcal{D}(A^*). \tag{4.1.43}$$

Moreover, by the preceding arguments on the three terms of (4.1.42), in particular the uniform bound (A.1), we have

$$\|\{L^*_{h;(sT)}\Pi_h y\}(t) - \{L^*_{(sT)}y\}(t)\|_U \le Ce^{ct}\|y\|_Y, \quad y \in \mathcal{D}(A^*). \tag{4.1.44}$$

Thus, by (4.1.43) and (4.1.44), we can apply the Lebesgue Dominated Convergence Theorem and obtain

$$\|L^*_{h;(sT)}\Pi_h y - L^*_{(sT)}y\|^2_{L_2(0,T;U)} = \int_0^T \|\{L^*_{h;(sT)}\Pi_h y\}(t) - \{L^*_{(sT)}y\}(t)\|^2_U dt \\ \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s, \ y \in \mathcal{D}(A^*), \tag{4.1.45}$$

as each term on the right-hand side of (4.1.42) does not depend on s, while for $0 \le t < s$, the operator on the left-hand side of (4.1.42) is identically zero by (3.2.4a) and (3.1.4a). But $\mathcal{D}(A^*)$ is dense in Y, and by (4.1.4b) of Lemma 3.2 we have

$$\|L^*_{h;(sT)}\Pi_h\|_{\mathcal{L}(Y;L_2(0,T;U))} \le \text{ const.}_T. \tag{4.1.46}$$

Then, (4.1.45) and (4.1.46) yield (4.1.40), as desired.
(ii) To begin with, we already know from (4.1.3b) of Lemma 3.2 via(B.1) that

$$\|L^*_{h;s,T}\Pi_h\|_{\mathcal{L}(L_1(0,T;Y);L_2(0,T;U))} \le \text{ const.}_T. \tag{4.1.47}$$

Thus, in view of (4.1.47), in order to prove the desired convergence (4.1.41), it suffices to show that the uniform convergence (4.1.41) takes place for all f in the subspace

$$\mathcal{S} = L_1(0, T; \mathcal{D}(A^*)) \tag{4.1.48}$$

dense in $L_1(0, T; Y)$. Accordingly, we show that $\forall f \in \mathcal{S}$,

$$\|L^*_{h;s,T}\Pi_h f - L^*_{s,T} f\|_{L_2(0,T;U))} \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s. \tag{4.1.49}$$

After this, (ii) = (4.1.41) is proved. To establish (4.1.49), we take $f \in \mathcal{S}$ and compute from (3.2.3a) and (3.1.1a) for $t \ge s$:

$$(L^*_{h;s,T}\Pi_h f)(t) - \{L^*_{s,T} f\}(t) = \int_s^T [B^*_h e^{A^*_h(\tau-t)}\Pi_h - B^* e^{A^*(\tau-t)}] f(\tau) d\tau \\ = i_{h,1}(t) + i_{h,2}(t) + i_{h,3}(t). \tag{4.1.50}$$

$$i_{h,1}(t) = \int_t^T [B^*_h A^{*-1}_h \Pi_h - B^* A^{*-1}] e^{A^*(\tau-t)} A^* f(\tau) d\tau; \tag{4.1.51}$$

$$i_{h,2}(t) = \int_t^T B^*_h A^{*-1}_h \Pi_h [e^{A^*_h(\tau-t)}\Pi_h - e^{A^*(\tau-t)}] A^* f(\tau) d\tau; \tag{4.1.52}$$

$$i_{h,3}(t) = \int_t^T B^*_h A^{*-1}_h e^{A^*_h(\tau-t)} [A^*_h \Pi_h - \Pi_h A^*] f(\tau) d\tau. \tag{4.1.53}$$

In getting (4.1.50) via (4.1.51)–(4.1.53), we have used the same identity as in (4.1.42) (with T and y there replaced by τ and $f(\tau)$ now). The argument given in part (i), from (4.1.42) to (4.1.45), and based on the Lebesgue Dominated Convergence Theorem, shows likewise that as $h \downarrow 0$ we have

$$\begin{aligned}&\|i_{h,1}(t)\|_U \to 0; \quad \|i_{h,2}(t)\|_U \to 0; \quad \|i_{h,3}(t)\|_U \to 0,\\ &t \geq s, \ \forall \ f \in L_1(0,T;\mathcal{D}(A^*));\end{aligned} \tag{4.1.54}$$

i.e., by (4.1.50) that as $h \downarrow 0$,

$$\|\{L^*_{h;s,T}\Pi_h f\}(t) - \{L^*_{s,T} f\}(t)\|_U \to 0, \ t \geq s, \ \forall f \in L_1(0,T;\mathcal{D}(A^*)). \tag{4.1.55}$$

But, for $0 \leq t < s$, the operator in (4.1.55) is identically zero by (3.2.3a) and (3.1.2a). Thus, (4.1.55) yields the desired convergence (4.1.41) uniformly in s. Theorem 4.2 is proved. □

4.2 Strong Convergence of $\Lambda^{-1}_{h;s,T}$ to $\Lambda^{-1}_{s,T}$

As a corollary of both Theorem 4.1 and 4.2, we obtain

Theorem 4.3 *We assume the hypotheses of Theorem 4.1 and of Theorem 4.2; i.e., we assume the standing assumptions (H.1), (H.2), (H.3) on the dynamics, as well as the approximating assumptions (A.1) = (1.5.4) through (A.4) = (1.5.7), and (B.1) = (1.5.8) through (B.5) = (1.5.12). Then, with reference to the operators* $\Lambda_{h;s,T}$ *in (3.2.5) and* $\Lambda_{s,T}$ *in (3.1.5), the following convergence results hold true as* $h \downarrow 0$*, uniformly in* s*:*

(i)
$$\|\Lambda_{h;s,T}u - \Lambda_{s,T}u\|_{L_2(s,T;U)} \to 0, \quad u \in L_2(0,T;U); \tag{4.2.1}$$

(ii)
$$\|\Lambda^{-1}_{h;s,T}u - \Lambda^{-1}_{s,T}u\|_{L_2(s,T;U)} \to 0, \quad u \in L_2(0,T;U); \tag{4.2.2}$$

Proof (i) By (3.2.5) and (3.1.5), we rewrite the term in (4.2.1) explicitly as

$$\begin{aligned}&\|\Lambda_{h;s,T}u - \Lambda_{s,T}u\|_{L_2(s,T;U)}\\ &\quad = \|[I_s + L^*_{h,s;T}\Pi_h R^* R \Pi_h L_{h;s,T} + L^*_{h;(sT)}\Pi_h G^* G \Pi_h L_{h;(sT)}]u\\ &\qquad -[I_s + L^*_{s,T}R^*RL_{s,T} + L^*_{(sT)}G^*GL_{(sT)}]u\|_{L_2(s,T;U)}\\ &\quad = \|[L^*_{h,s;T}\Pi_h R^* R\Pi_h L_{h;s,T} - L^*_{s,T}R^*RL_{s,T}]u\\ &\qquad +[L^*_{h;(sT)}\Pi_h G^*G\Pi_h L_{h;(sT)} - L^*_{(sT)}G^*GL_{(sT)}]u\|_{L_2(s,T;U)}\end{aligned} \tag{4.2.3}$$

$$\begin{aligned}&\quad \leq \|[L^*_{h,s;T}\Pi_h R^* R\Pi_h L_{h;s,T} - L^*_{s,T}R^*RL_{s,T}]u\|_{L_2(s,T;U)}\\ &\qquad +\|[L^*_{h;(sT)}\Pi_h G^*G\Pi_h L_{h;(sT)} - L^*_{(sT)}G^*GL_{(sT)}]u\|_{L_2(s,T;U)} \to 0,\end{aligned} \tag{4.2.4}$$

uniformly in s, as desired, as it follows by applying the convergence results (4.1.8), (4.1.41), for the first term in (4.2.4), and (4.1.9), (4.1.40) for the second term, along with Lemma 1.1 about the strong convergence of the product of strongly convergent sequences of operators.

(ii) The usual identity (second resolvent equation)

$$[\Lambda_{h;s,T}^{-1} - \Lambda_{s,T}^{-1}]u = \Lambda_{h;s,T}^{-1}[\Lambda_{s,T} - \Lambda_{h;s,T}]\Lambda_{s,T}^{-1}u \tag{4.2.5}$$

allows us to fall into part (i), by use of (3.2.5b). Indeed, from (4.2.5) we readily compute for $u \in L_2(0, T; U)$:

$$\begin{aligned} \|[\Lambda_{h;s,T}^{-1} - \Lambda_{s,T}^{-1}]u\|_{L_2(s,T;U)} &= \|\Lambda_{h;s,T}^{-1}[\Lambda_{s,T} - \Lambda_{h;s,T}]\Lambda_{s,T}^{-1}u\|_{L_2(s,T;U)} \\ \text{(by (3.2.5b))} \quad &\leq \|[\Lambda_{s,T} - \Lambda_{h;s,T}]\Lambda_{s,T}^{-1}u\|_{L_2(s,T;U)} \to 0, \end{aligned} \tag{4.2.6}$$

where uniform in s convergence as $h \to 0$ takes place after invoking (4.2.1), since $\Lambda_{sT}^{-1}u \in L_2(s, T; U)$ by (3.1.5b). □

5 Completion of the Proofs of Theorem 2.1 and Theorem 2.2

5.1 Proof of Theorem 2.1

We can now establish the main Theorem 2.1 as a corollary of Theorem 4.3.

Proof of (2.1.1) We rewrite here (3.2.6) and (3.1.6) for convenience, where $y_0 \in Y$,

$$\begin{aligned} -u_{h,T}^0(\cdot, s; \Pi_h y_0) &= \Lambda_{h;s,T}^{-1}[L_{h;(sT)}^* \Pi_h G^* G e^{A_h(T-s)} \Pi_h y_0 + L_{h;s,T}^* \Pi_h R^* R e^{A_h(\cdot-s)} \Pi_h y_0] \\ -u_T^0(\cdot, s; y_0) &= \Lambda_{s,T}^{-1}[L_{(sT)}^* G^* G e^{A(T-s)} y_0 + L_{s,T}^* R^* R e^{A(\cdot-s)} y_0]. \end{aligned} \tag{5.1.1}$$

We now recall the (strong) convergence results (uniformly in s) (4.1.1) for $e^{A_h \cdot}\Pi_h$; (4.1.8) and (4.1.9) for $L_{h;s,T}$ and $L_{h;(sT)}$; (4.1.41) and (4.1.40) for $L_{h;s,T}^*\Pi_h$ and $L_{h;(sT)}^*\Pi_h$; finally (4.2.2) for $\Lambda_{h;s,T}^{-1}$, along with Lemma 1.1 to conclude from (5.1.1) that, as desired

$$\|u_{h,T}^0(\cdot, s; \Pi_h y_0) - u_T^0(\cdot, s; y_0)\|_{L_2(s,T;U)} \to 0 \text{ as } h \downarrow 0, \text{ uniformly in } s. \tag{5.1.2}$$

Proof of (2.1.2) We rewrite here (3.2.7) for convenience

$$y_{h,T}^0(\cdot, s; \Pi_h y_0) = \Phi_{h,T}(t, s)\Pi_h y_0 = e^{A_h(t-s)}\Pi_h y_0 + L_{h;s,T} u_{h,T}^0(\cdot, s; \Pi_h y_0) \tag{5.1.3}$$

$$y_T^0(\cdot, s; y_0) = \Phi_T(t, s) y_0 = e^{A(t-s)} y_0 + L_{s,T} u_T^0(\cdot, s; y_0). \tag{5.1.4}$$

We then recall the convergence results (4.1.1) on $e^{A_h \cdot}\Pi_h$; (4.1.8) on $L_{h;s,T}$, and (5.1.2) on $u^0_{h,T}$ along with Lemma 1.1, to conclude from (5.1.3) and (5.1.4) that, as desired

$$\|y^0_{h,T}(\cdot, s; \Pi_h y_0) - y^0_T(\cdot, s; y_0)\|_{C([s,T];Y)} \to 0, \text{ as } h \downarrow 0, \text{ uniformly in } s. \tag{5.1.5}$$

Proof of (2.1.3) By recalling from 1.5.26 and (1.3.1) that

$$\begin{aligned} &J_{s,T;G}(u^0_{h,T}(\cdot, s; \Pi_h y_0), y^0_{h,T}(\cdot, s; \Pi_h y_0)) \equiv J^0_{h;s,T;G}(\Pi_h y_0) \\ &= \int_s^T [\|R y^0_{h,T}(t, s; \Pi_h y_0)\|^2_Z + \|u^0_{h,T}(t, s; \Pi_h y_0)\|^2_U] dt + \|G y^0_{h,T}(T, s; \Pi_h y_0\|^2_{Z_f}; \\ &J_{s,T;G}(u^0_T(\cdot, s; y_0), y^0_T(\cdot, s; y_0)) \equiv J^0_{s,T;G}(y_0) \\ &= \int_s^T [\|R y^0_T(t, s; y_0)\|^2_Z + \|u^0_T(t, s; y_0)\|^2_U] dt + \|G y^0_T(T, s; y_0)\|^2_{Z_f}, \end{aligned} \tag{5.1.6}$$

we obtain *a fortiori* from (5.1.2) = (2.1.1) and (5.1.5) = (2.1.2) (with $s = 0$) that

$$(P_{h,T}(s)\Pi_h y_0, y_0) = J^0_{h;s;T;G}(\Pi_h y_0) \to J^0_{s,T;G}(y_0) = (P_T(s) y_0, y_0) \tag{5.1.7}$$

for $y_0 \in Y$ as desired, and (2.1.3) is proved, recalling (1.5.30) and (1.3.13).

Proof of (2.1.4) We first notice that, since $\Pi_h P_{h,T}(s)\Pi_h$ and $P_T(s)$ are self-adjoint operators in $\mathcal{L}(Y)$, convergence (5.1.7) is equivalent to weak convergence of $\Pi_h P_{h,T}(s)\Pi_h = P_{h,T}(s)\Pi_h$ to $P_T(s)$. To obtain the strong convergence (2.1.4), we use formulas (1.5.31) and (1.3.10) which we rewrite, with $x \in Y$,

$$P_{h,T}(t)\Pi_h x = \int_t^T e^{A^*_h(\tau - t)} \Pi_h R^* R y^0_{h,T}(\tau; t; \Pi_h x) d\tau + e^{A^*_h(T-t)} \Pi_h G^* G y^0_{h,T}(T, t; \Pi_h x) \tag{5.1.8}$$

$$P_T(t) x = \int_t^T e^{A^*(\tau - t)} R^* R y^0_T(\tau; t; x) d\tau + e^{A^*(T-t)} G^* G y^0_T(T, t; x). \tag{5.1.9}$$

We then invoke the convergence results (uniform in s, which in (5.1.8), (5.1.9) is replaced by t) (4.1.2) on $e^{A^*_h \cdot}\Pi_h$; (5.1.5) on $y^0_{h,T}$ to conclude via Lemma 1.1 that $P_{h,T}(t)\Pi_h x$ converges to $P_T(t)x$, uniformly in $t \in [0, T]$, for $x \in Y$; i.e., (2.1.4).

Theorem 2.1 is fully proved.

5.2 Proof of Theorem 2.2

The proof is similar to that of (2.1.4) given above, and uses the additional assumptions (H.5*) = (1.3.7) and (H.6*) = (1.3.8) for the continuous dynamics, and (H.7) = (2.2.1) and (H.8) = (2.2.2) for the corresponding approximations. In fact, from (5.1.8) and

(5.1.9), we obtain

$$
\begin{aligned}
B_h^* P_{h,T}(t)\Pi_h x &= \int_t^T B_h^* e^{A_h^*(\tau-t)}\Pi_h R^* R y_{h,T}^0(\tau,t;\Pi_h x)d\tau \\
&\quad + B_h^* e^{A_h^*(T-t)}\Pi_h G^* G y_{h,T}^0(T,t;\Pi_h x) && (5.2.1)\\
B^* P_T(t)x &= \int_t^T B^* e^{A^*(\tau-t)} R^* R y_T^0(\tau,t;x)d\tau \\
&\quad + B^* e^{A^*(T-t)} G^* G y_T^0(T,t;x). && (5.2.2)
\end{aligned}
$$

For each t fixed in $[0, T]$ and $x \in Y$, formula (5.2.2) is well defined by (H.5*) = (1.3.7), (H.6*) = (1.3.8), and (1.2.5), and yields $B^* P_T(t)x \in \mathcal{L}(Y; U)$, which, moreover, acts continuously $Y \to L_\infty(0, T; U)$; see (1.3.19). Similarly, formula (5.2.1) yields $B_h^* P_{h,T}(t)\Pi_h \in \mathcal{L}(Y; U)$ at each t, by use of consequences (2.2.5) and (2.2.6) of assumptions (H.7) and (H.8), respectively. Indeed, even more, using (2.2.5), (2.2.6) in combination with the convergence (2.1.2) uniform in s, we obtain from (5.2.1)

$$
\begin{aligned}
\|B_h^* P_{h,T}(t)\Pi_h x\|_U &\le C_T \|y_{h,T}^0(\cdot,t;\Pi_h x)\|_{C([t,T];Y)} && (5.2.3)\\
\text{(by (2.1.2))} \qquad &\le \text{const.}_T \|\Pi_h x\|_Y \le \text{const.}_T \|x\|_Y && (5.2.4)
\end{aligned}
$$

so that $B_h^* P_{h,T}(t)\Pi_h :$ continuous $Y \to L_\infty(0, T; U)$, uniformly in h. From (5.2.1) and (5.2.2), we obtain, after adding and subtracting,

$$B_h^* P_{h,T}(t)\Pi_h x - B^* P_T(t)x = 1_h(t) + 2_h(t) + 3_h(t) + 4_h(t). \tag{5.2.5}$$

$$1_h(t) = \int_t^T B_h^* e^{A_h^*(\tau-t)}\Pi_h R^* R[y_{h,T}^0(\tau,t;\Pi_h x) - y_T^0(\tau,t;x)]d\tau \tag{5.2.6}$$

$$2_h(t) = \int_t^T [B_h^* e^{A_h^*(\tau-t)}\Pi_h - B^* e^{A^*(\tau-t)}]R^* R y_T^0(\tau,t;x)d\tau \tag{5.2.7}$$

$$3_h(t) = B_h^* e^{A_h^*(T-t)}\Pi_h G^* G[y_{h,T}^0(T,t;\Pi_h x) - y_T^0(T,t;x)] \tag{5.2.8}$$

$$4_h(t) = [B_h^* e^{A_h^*(T-t)}\Pi_h - B^* e^{A^*(T-t)}]G^* G y_T^0(T,t;x). \tag{5.2.9}$$

Starting from (5.2.6) and (5.2.8), and using consequences (2.2.5) and (2.2.6), as well as the convergence (2.1.2) uniform in the initial time (as in (5.2.3)), we obtain

$$
\begin{aligned}
&\sup_{0\le t\le T} \{\|1_h(t)\|_U + \|3_h(t)\|_U\} \qquad \text{(by (2.1.2))}\\
&\le C_T \sup_{0\le t\le T} \|y_{h,T}^0(\cdot,t;\Pi_x x) - y_T^0(\cdot,t;x)\|_{C([t;T];Y)} \to 0 \text{ as } h \downarrow 0.
\end{aligned}
\tag{5.2.10}
$$

Moreover, for each $t \in [0, T]$ fixed, assumptions (H.7) = (2.2.1) and (H.8) = (2.2.2), along with (1.2.5) yield

$$2_h(t) \to 0 \quad \text{and} \quad 4_h(t) \to 0 \quad \text{as } h \downarrow 0. \tag{5.2.11}$$

We proceed now as in the proof from (4.1.23) to (4.1.33). Moreover, from (5.2.9), if $t_1 \in [0, T]$,

$$\begin{aligned}
\|4_h(t)\|_U \leq\ & \|[B_h^* e^{A_h^*(T-t)} \Pi_h - B^* e^{A^*(T-t)}] G^* G [y_T^0(T, t; x) - y_T^0(T, t_1; x)]\|_U \\
& + \|[B_h^* e^{A_h^*(T-t)} \Pi_h - B^* e^{A^*(T-t)}] G^* G y_T^0(T, t_1; x)\|_U \qquad (5.2.12) \\
\leq\ & C_T \|y_T^0(T, t; x) - y_T^0(T, t_1; x)\|_Y \\
& + \|[B_h^* e^{A_h^*(T-\cdot)} \Pi_h - B^* e^{A^*(T-\cdot)}] G^* G y_T^0(T, t_1; x)\|_{L_\infty(0,T;U)}, \qquad (5.2.13)
\end{aligned}$$

where in going from (5.2.12) to (5.2.13) we have used the uniform bound (2.2.6) and (1.3.8). But, as to the first term in (5.2.13), we recall that the map $t \to y_T^0(T, t; x)$ is continuous (in Y), hence uniformly continuous for $0 \leq t \leq T$; while the second term in (5.2.13) converges to zero as $h \downarrow 0$ by (H.8) = (2.2.2). Thus, given $\epsilon > 0$ and t_1, there exists constants $\delta_\epsilon > 0$ (independent of t_1), and $\bar{h}_{\epsilon,t_1} > 0$ such that

$$\text{if } |t - t_1| < \delta_\epsilon \Longrightarrow \|y_T^0(T, t; x) - y_T^0(T, t_1; x)\|_Y < \frac{\epsilon}{C_T}; \qquad (5.2.14)$$

$$\begin{aligned}
&\text{if } h < \bar{h}_{\epsilon,t_1} \Longrightarrow \\
&\|[B_h^* e^{A_h^*(T-\cdot)} \Pi_h - B^* e^{A^*(T-\cdot)}] G^* G y_T^0(T, t_1; x)\|_{L_\infty(0,T;U)} < \epsilon \qquad (5.2.15)
\end{aligned}$$

so that, using (5.2.14) and (5.2.15) in (5.2.13), yields that

$$\text{if } |t - t_1| < \delta_\epsilon \text{ and } h < \bar{h}_{\epsilon,t_1} \Longrightarrow \|4_h(t)\|_U < 2\epsilon. \qquad (5.2.16)$$

Using finitely many intervals of length δ_ϵ and centered in t_i, $i = 1, \ldots, N$, which cover $[0, T]$, we obtain as below (4.1.31) that there exists $\bar{h}_\epsilon > 0$ such that

$$\text{if } h < \bar{h}_\epsilon \Longrightarrow \sup_{0 \leq t \leq T} \|4_h(t)\|_U < 2\epsilon. \qquad (5.2.17)$$

5.3 Proof of Feedback Finite Dimensional Suboptimal Convergence Parts (i)–(iii) of Theorem 2.2

With the notation as above we compare the optimal response u^0, y^0

$$u^0(t) = -B^* P(t) y^0(t) \qquad (5.3.1)$$

$$\begin{aligned}
y^0(t) &= e^{At} y_0 + \int_0^t e^{A(t-s)} B u^0(s)\, ds \qquad (5.3.2) \\
&= e^{At} y_0 - \int_0^t e^{A(t-s)} B B^* P(s) y^0(s)\, ds = e^{At} y_0 + (L u^0)(t) \qquad (5.3.3)
\end{aligned}$$

in (1.3.1) (in simplified notation). with suboptimal discrete version given by

$$\begin{aligned} u_h(t) &= -B_h^* P_h(t) y_h(t); \\ y_h(t) &= e^{At} y_0 - \int_0^t e^{A(t-s)} B B_h^* P_h(s) y_h(s) ds = e^{At} y_0 + (L u_h)(t). \end{aligned} \tag{5.3.4}$$

We study the convergence of $y_h(t)$ to $y^0(t)$. In other words we look at

$$(y_h - y^0)(t) = -L\left[(B_h^* P_h - B^* P) y_h\right] - L B^* P(s)[y_h - y^0] \tag{5.3.5}$$

obtained after adding and subtracting $B^* P(s) y_h$. Setting $z \equiv y_h - y^0$, we have

$$z = -L\left[B_h^* P_h - B^* P\right] y_h - L B^* P z. \tag{5.3.6}$$

By assumption (H.3)=(1.2.4), L is bounded from $L_2(0, T; U) \to C([0, T]; Y)$ while $B^* P$ is continuous $Y \to L_\infty(0, T; U)$ by (1.3.19). Thus, a standard fixed point [contraction] applies, yielding

$$\|z(t)\|_Y \le C_T \|B_h^* P_h - B^* P\|_{\mathcal{L}(Y \to C(U))} \|y_h\|_{C(Y)}. \tag{5.3.7}$$

Thus, in view of the convergence in (2.2.7), $z(t) \to 0$ as long as $\|y_h\|_{C(Y)}$ is uniformly bounded in $h > 0$. The latter follows from an aplication of the fixed point to the equation

$$y_h(t) = e^{At} y_0 + L(B_h^* P_h y_h)(t) \tag{5.3.8}$$

with the information that $\|B_h^* P_h\|_{\mathcal{L}(Y \to U)}$ is bounded uniformly in time and $h > 0$. Thus, we have

$$\sup_{h>0,\ t \in [0.T]} \|y_h(t)\|_Y \le C_T \|y_0\|_Y. \tag{5.3.9}$$

The convergence of suboptimal controls follows now in a straightforward way.

$$u^0 - u_h = -B^* P y^0 + B_h^* P_h y_h = -B^* P[y^0 - y_h] + [B_h^* P_h - B^* P] y_h. \tag{5.3.10}$$

Finally, the convergence of the functional cost follows from the arguments above.

6 Conclusions

The continuous Optimal Control Theory with a Quadratic Cost Functional and corresponding Algebraic Riccati Equation (infinite time-horizon) or Differential/Integral Riccati Equation (finite time-horizon) for hyperbolic-like dynamics defined on a multidimensional bounded domain and subject to boundary control action and/or boundary sensing action has been available for many years now ([30] and the authors' quoted references). The purpose of the present paper was to provide a corresponding (rigorous) numerical approximation theory in the finite time-horizon case. The ultimate goal is to introduce a finite dimensional approximation of the Riccati Equation, which would produce a finite dimensional solution $P_h(t) \in \mathbb{R}^n$, and a finite dimensional control $u_h(t)$ given in feedback form by (1.3.23), such that, when inserted into the original dynamics as given by (1.3.24) provides an almost optimal performance. This means the convergence Theorem 2.1 for several basic quantities and the convergence Theorem 2.2 for the gain operators. To this end, the assumptions are divided into two groups. A first group includes assumptions (H.1) through (H.4) for the continuous problem and the approximating hypotheses (A.1) through (A.4) for the operator A and (B.1) through (B.5) for the operator B. This first group then yields the convergence Theorem 2.1 for several basic quantities of the control problem, including the definition of the would-be Riccati operator, in terms of the original data. A second group of assumptions (H.5) and (H.6) for the smoothing operators R and G and the discrete versions (H.7) and (H.8) is then needed in order to guarantee three conclusions: (i) that such operator is moreover the unique solution (within a specified class) of the corresponding Differential/Integral Riccati Equation; (ii) the convergence of the gains operators as in Theorem 2.2; (iii) the convergence of suboptimal solutions obtained by applying finite dimensional feedback gain operator to the original plant [system].

The theory is illustrated by two genuine multidimensional PDE-dynamics: the boundary damped wave equation of Example #1 (treated in Appendix B); and the plate equation with moments boundary controls treated in Appendix C; where all the required assumptions are dynamical properties.

Appendix A: Abstract Model for Problem (1.4.1) with Neumann Control

Verification of dynamical assumptions (H.1), (H.2), (H.3). Here we give a short account producing (i) the abstract model of the damped wave equation PDE-dynamics (1.4.1) with Neumann control. (ii) Moreover, we verify the required dynamical assumptions (H.1), (H.2), (H.3). For more extensive details we refer to [41]. Without loss of generality, we take the damped constant $a = 1$ in (1.4.1b).

1. Let $A_N : L^2(\Omega) \supset \mathcal{D}(A_N) \to L^2(\Omega)$ be the Neumann Laplacian operator defined by

$$A_N = -\Delta f, \qquad \mathcal{D}(A_N) \equiv \left\{ f \in H^2(\Omega) : f\big|_{\Gamma_0} = 0, \ \frac{\partial f}{\partial \nu}\bigg|_{\Gamma_1} = 0 \right\}. \tag{A.1}$$

   Then, A_N is positive self-adjoint and has compact resolvent $R(\cdot, A_N)$ on $L^2(\Omega)$ and $A_N^{-1} \in \mathcal{L}(L^2(\Omega))$, as $\left\{ \Delta h = 0 \text{ in } \Omega, \ h\big|_{\Gamma_0} = \frac{\partial h}{\partial \nu}\big|_{\Gamma_1} = 0 \right\}$ implies $h \equiv 0$ by Green's theorem applied to $(\Delta h, h) = 0$.

2. Next, define the operator $\widetilde{N}$ by [30]

$$h = \widetilde{N}g \iff \left\{ \Delta h = 0 \text{ in } \Omega,\ h\big|_{\Gamma_0} = 0,\ \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_1} = g \right\}. \tag{A.2}$$

Elliptic theory with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ gives [34]and [29] p.195

$$\widetilde{N} : \text{continuous } L^2(\Gamma_1) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\rho}(\Omega) \equiv \mathcal{D}(A_N^{\frac{3}{4}-\rho}),\ \rho > 0, \tag{A.3}$$

or

$$A_N^{\frac{3}{4}-\rho}\widetilde{N} \in \mathcal{L}(L^2(\Gamma_1); L^2(\Omega)) \Longleftarrow \widetilde{N}^* A_N^{\frac{3}{4}-\rho} \in \mathcal{L}(L^2(\Omega); L^2(\Gamma_1)) \tag{A.4}$$

where $\widetilde{N}^*$ is the adjoint of $\widetilde{N}$: $(Nv, u)_{L^2(\Omega)} = (v, \widetilde{N}^* u)_{L^2(\Gamma_1)}$. We have [29] p/196

$$\widetilde{N}^* A_N y = \begin{cases} y\big|_{\Gamma_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0 \end{cases}, \qquad y \in \mathcal{D}(A_N). \tag{A.5}$$

and (A.5) can be extended to all $y \in H^{3/2+\epsilon}(\Omega)$ with $\frac{\partial y}{\partial \nu} = 0$ on Γ_1 and $y = 0$ on Γ_0.

3. Consider at first the following undamped case

$$\begin{cases} \phi_{tt} = \Delta\phi & \text{in } \Omega \times (0, T] & \text{(A.6a)} \\ \phi\big|_{\Gamma_0} = 0 & \text{in } \Gamma_0 \times (0, T] & \text{(A.6b)} \\ \dfrac{\partial \phi}{\partial \nu}\bigg|_{\Gamma_1} = g & \text{in } \Gamma_1 \times (0, T]\,. & \text{(A.6c)} \end{cases}$$

Using the definition (A.2) of $\widetilde{N}$, we can rewrite Eq. (A.6a) as

$$\phi_{tt} = \Delta(\phi - \widetilde{N}g) \quad \text{or} \quad \phi_{tt} = -A_N(\phi - \widetilde{N}g) \quad \text{in } L^2(\Omega) \tag{A.7}$$

as $[\phi - \widetilde{N}g]_{\Gamma_0} = 0 - 0 = 0$, and $[\phi - \widetilde{N}g]_{\Gamma_1} = g - g = 0$. Let now $A_{N,\text{ext}}$ denote the isomorphism extension $A_{N,\text{ext}} : L^2(\Omega) \to \left[\mathcal{D}(A_N^*)\right]' \equiv [\mathcal{D}(A_N)]'$ of the original operator A_N in (A.1). Then (A.6) can be rewritten as the following abstract second order equation

$$\phi_{tt} = -A_N\phi + A_N\widetilde{N}g \quad \text{in } [\mathcal{D}(A_N)]'\,, \tag{A.8}$$

where in (A.8) and below we use the symbol A_N to denote also the extension $A_{N,\text{ext}}$, for simplicity of notation. Its first order system is given by

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \phi_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_N & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \phi_t \end{bmatrix} + \begin{bmatrix} 0 \\ A_N\widetilde{N}g \end{bmatrix}. \tag{A.9}$$

4. Return now to the original damped problem (1.4.1a)–(1.4.1c) with $u \equiv 0, a = 1$ (w.l.o.g.) and $g = -w_t = -N^* A_N w_t$ on $\Gamma_1 \times (0, T]$ by (A.5). Thus, via (A.8), the second order abstract model for (1.4.1a)–(1.4.1c) is

$$w_{tt} = -A_N w - A_N \widetilde{N} N^* A_N w_t \quad \text{ in } [\mathcal{D}(A_N)]' . \tag{A.10}$$

Its corresponding first order version is

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_N & -A_N \widetilde{N}\widetilde{N}^* \end{bmatrix}\begin{bmatrix} w \\ w_t \end{bmatrix} = A\begin{bmatrix} w \\ w_t \end{bmatrix}, \tag{A.11a}$$

$$A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_N & -A_N \widetilde{N}\widetilde{N}^* A_N \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{(formally)},$$

$$Az = \begin{bmatrix} 0 & I \\ -A_N & 0 \end{bmatrix}\begin{bmatrix} z_1 + \widetilde{N}\widetilde{N}^* A_N z_2 \\ z_2 \end{bmatrix},$$

$$\begin{aligned} \mathcal{D}(A) = \Big\{ [z_1, z_2] \in Y;\ z_2 \in \mathcal{D}(A_N^{\frac{1}{2}}) \equiv H^1_{\Gamma_0}(\Omega) \\ \equiv \Big\{ z_2 \in H^1(\Omega), z_2 = 0 \text{ on } \Gamma_0 \Big\}; \\ z_1 + \widetilde{N}\widetilde{N}^* A_N z_2 \in \mathcal{D}(A_N) \Big\}, \quad Y \equiv \mathcal{D}(A_N^{\frac{1}{2}}) \times L^2(\Omega). \end{aligned} \tag{A.11b}$$

By Poincare inequality, we have

$$\|h\|^2_{\mathcal{D}(A_N^{1/2})} = \|A_N^{\frac{1}{2}} h\|^2_{L^2(\Omega)} \equiv \int_\Omega |\nabla h|^2 \, d\Omega = \|\nabla h\|^2_{L^2(\Omega)}. \tag{A.11c}$$

5. Finally, by (A.11) and (A.9), the abstract version of the original PDE Neumann problem (1.4.1a)–(1.4.1d) is given by

$$\frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = A\begin{bmatrix} w \\ w_t \end{bmatrix} + Bu, \quad Bu = \begin{bmatrix} 0 \\ A_N \widetilde{N} u \end{bmatrix} \in \begin{bmatrix} \otimes \\ \left[\mathcal{D}(A_N^{\frac{1}{4}+\rho})\right]' \end{bmatrix}. \tag{A.12}$$

One readily solves $A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \in Y$ and obtains

$$A^{-1} = \begin{bmatrix} -\widetilde{N}\widetilde{N}^* A_N & -A_N^{-1} \\ I & 0 \end{bmatrix} \in \mathcal{L}(Y). \tag{A.13}$$

Thus, (A.12), (A.13) imply for $u \in U = L^2(\Gamma)$

$$A^{-1} B u = \begin{bmatrix} -\widetilde{N}\widetilde{N}^* A_N & -A_N^{-1} \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 \\ A_N \widetilde{N} u \end{bmatrix} = \begin{bmatrix} -\widetilde{N} u \\ 0 \end{bmatrix} \in Y, \tag{A.14}$$

i.e. $\widetilde{N} u \in \mathcal{D}(A_N^{\frac{1}{2}})$ for $u \in L^2(\Gamma_1)$.

Next, let $u \in L^2(\Gamma_1)$, $f = [f_1, f_2]$, with $f_2 \in \mathcal{D}(A)$. Then, via (A.12), from

$$\begin{aligned}(Bu, f)_Y &= \left(A_N \widetilde{N} u, f_2\right)_{L^2(\Omega)} = \left(u, \widetilde{N}^* A_N f_2\right)_{L^2(\Gamma_1)} \\ &= \left(u, B^* f\right)_{L^2(\Gamma_1)},\end{aligned} \tag{A.15}$$

we obtain

$$B^* \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \widetilde{N}^* A_N f_2 = \begin{cases} 0 & \text{on } \Gamma_0 \\ f_2\big|_{\Gamma_1} & \text{on } \Gamma_1 \end{cases}, \qquad f \in \mathcal{D}(A_N). \tag{A.16}$$

6. Verification of the abstract dynamical assumptions (H.1), (H.2), (H.3) for the abstract model (A.12) of the damped PDE (1.4.1a)–(1.4.1d) with Neumann boundary control, with

$$Y \equiv \mathcal{D}(A_N^{\frac{1}{2}}) \equiv H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \qquad U = L^2(\Gamma_1). \tag{A.17}$$

(H.1): it is proved, e.g., in [41] that the operator A in (A.11a)–(A.11b) generates a s.c. contraction semigroup e^{At} on Y, which moreover is uniformly stable.
(H.2): This assumption is verified in (A.14).
(H.3): The standard energy method of multiplying Eq. (1.4.1a) (we are taking $a = 1$ w.l.o.g.) by w_t, and integrating in time and space after setting $\frac{1}{2}\frac{\partial}{\partial t}|w_t|^2 = w_{tt}w_t$ and $\frac{1}{2}\frac{\partial}{\partial t}(\nabla w \cdot \nabla w) = \nabla w \cdot \nabla w_t$ yields the usual identity

$$\begin{aligned}\|y(T)\|_Y^2 &= \|\nabla w(T)\|^2_{L^2(\Omega)} + \|w_t(T)\|^2_{L^2(\Omega)} + 2\int_0^T \int_{\Gamma_1} w_t^2 d\Gamma_1 dt \\ &= \|y(0)\|_Y^2 = \|\nabla w(0)\|^2_{L^2(\Omega)} + \|w_t(0)\|^2_{L^2(\Omega)}.\end{aligned} \tag{A.18}$$

Hence (we are using $a > 0$), we obtain recalling B^* in (A.16):

$$\begin{aligned}2\int_0^T \int_{\Gamma_1} w_t^2 d\Gamma_1 dt &= 2\int_0^T \left\| B^* \begin{bmatrix} w \\ w_t \end{bmatrix} \right\|^2_{U=L^2(\Gamma_1)} dt \\ &\le 2\int_0^T \left\| B^* e^{At} y_0 \right\|^2_{L^2(\Gamma_1)} dt \le \|y(0)\|_Y^2.\end{aligned} \tag{A.19}$$

Similarly for e^{A^*t}:

$$2\int_0^T \left\| B^* e^{A^*t} y_0 \right\|^2_{U=L^2(\Gamma_1)} dt \le \|y(0)\|_Y^2, \tag{A.20}$$

where in fact

$$A^* = \begin{bmatrix} 0 & -I \\ -A_N & -A_N \widetilde{N}\widetilde{N}^* A_N \end{bmatrix}. \tag{A.21}$$

(A.20) verified assumption on (H.3) (due to $a > 0$).

Verification of dynamical assumptions (H.4), (H.5), (H.6) in the performance index (1.4.3). We recall (1.4.3)

$$J(u, y) \equiv \int_0^T \left[\|w\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Gamma_1)} \right] + \|w(T)\|^2_{L_2(\Omega)} \tag{A.22}$$

(H.4): Verification of assumption (H.4) for R defined as

$$Ry = R \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_N^{-\frac{1}{2}} y_1 \\ 0 \end{bmatrix} \in Z \equiv Y, \quad R = \begin{bmatrix} A_N^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{A.23}$$

In fact, we write via (A.23)

$$\begin{aligned} \left\| R \begin{bmatrix} w \\ w_t \end{bmatrix} \right\|_{Z \equiv Y} &= \left\| \begin{bmatrix} A_N^{-\frac{1}{2}} w \\ 0 \end{bmatrix} \right\|_Y \\ &= \|A_N^{-\frac{1}{2}} w\|_{\mathcal{D}(A_N^{\frac{1}{2}})} = \|A_N^{\frac{1}{2}} A_N^{-\frac{1}{2}} w\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega)} \end{aligned} \tag{A.24}$$

as required by (A.22) via the abstract form (1.3.1). Then (A.23) implies

$$R^* = R \in \mathcal{L}(Z; Y) \equiv \mathcal{L}(Y), \qquad R^* \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_N^{-\frac{1}{2}} y_1 \\ 0 \end{bmatrix} \in Y, \tag{A.25a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in Y, \qquad R^* = \begin{bmatrix} A_N^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \tag{A.25b}$$

and

$$R^* R \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_N^{-1} y_1 \\ 0 \end{bmatrix}, \qquad R^* R = \begin{bmatrix} A_N^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{A.26}$$

Similarly from (A.22) recalling the abstract form (1.3.1),

$$G = R, \qquad G^* = R^*. \tag{A.27}$$

Thus assumption (H.4) has been verified.

(H.5): Verification of assumption (H.5) = (1.3.3a). Let $u \in U \equiv L^2(\Gamma_1)$. We shall show that

$$R^* R e^{At} B u \in C([0, T]; Y) \tag{A.28}$$

and hence (H.5) = (1.3.3a) is a-fortiori verified. In fact, we first rewrite

$$R^* R e^{At} B u \equiv R^* R A e^{At} A^{-1} B u, \tag{A.29}$$

where by (A.14)

$$A^{-1}Bu = \begin{bmatrix} -\widetilde{N}u \\ 0 \end{bmatrix} \in Y. \tag{A.30}$$

Moreover, by (A.26) and (A.11) for $[y_1, y_2]^T \in Y \equiv \mathcal{D}(A_N^{\frac{1}{2}}) \times L^2(\Omega)$, we compute

$$\begin{aligned} R^*RA \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} A_N^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -A_N & -A_N\widetilde{N}\widetilde{N}^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & A_N^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} A_N^{-1}y_2 \\ 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{D}(A_N) \\ 0 \end{bmatrix} \subset Y. \end{aligned} \tag{A.31}$$

Combing (A.31) with (A.30) in (A.29) shows (A.28).

(H.6): Verification of assumption (H.6) = (1.3.4) is the same via (A.27).

Appendix B: A Numerical Look at Example #1: Neumann Boundary Control

Let $\mathcal{V}_h \subset H^1_{\Gamma_0}(\Omega)$ be the standard finite dimensional approximation space defined either through splines defined on a quasiuniform mesh of Ω [11, 40] or through nodal functions (eigenfunctions of Laplacian with appropriate boundary conditions as in (1.4.1b)–(1.4.1c) with $u \equiv 0$). Then, the dynamics of the wave equation is defined through a standard variational form. We recall that (,) denotes the $L_2(\Omega)$-inner product, while $\langle\ ,\ \rangle$ denotes the $L_2(\Gamma)$-inner product. With π_h the orthogonal $L^2(\Omega)$-projection, one has

$$A_h = \begin{bmatrix} 0 & \pi_h \\ -\mathcal{A}_h & -a\mathcal{B}_h\mathcal{B}_h^* \end{bmatrix} = \text{approximation of } A, \tag{B.1}$$

where $\mathcal{A}_h$ = approximation of A_N, is given by

$$(\mathcal{A}_h v_h, z_h) = (\nabla v_h, \nabla z_h)\,, \quad v_h, z_h \in \mathcal{V}_h. \tag{B.2}$$

(As a motivation, recall [21] that $(A_N u, v) = (\nabla u, \nabla v)\,, \forall\, u, v \in H^1_{\Gamma_0}(\Omega)$ implies $A_N u = 0$, or $\left\{\Delta u = 0 \text{ in } \Omega, u\big|_{\Gamma_0 = 0}, \frac{\partial u}{\partial \nu}\big|_{\Gamma_1} = 0\right\}$) and where

$$B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix}, \quad B_h^* \begin{bmatrix} x_h^1 \\ x_h^2 \end{bmatrix} = \begin{cases} 0 & \text{on } \Gamma_0 \\ x_h^2\big|_{\Gamma_1} & \text{on } \Gamma_1 \end{cases}, \quad \begin{bmatrix} x_h^1 \\ x_h^2 \end{bmatrix} \in V_h \equiv \mathcal{V}_h \times \mathcal{V}_h, \tag{B.3}$$

counterpart of the continuous relation (A.16). It is reasonably straightforward to verify that the approximating assumptions in the group A and also the approximating assumptions (B.2)–(B.5) are satisfied. However, (B.1) [discrete abstract trace condition] requires an explanation. The discrete counterpart of (A.18) with $y_h = (w_h, w_{h,t})$ is

$$\begin{aligned} &\|w_{h,t}(T)\|^2_{L^2(\Omega)} + \|\nabla w_h(T)\|^2_{L^2(\Omega)} + 2a \int_0^T \|w_{h,t}\|^2_{L^2(\Gamma_1)} \\ &= \|w_{h,t}(0)\|^2_{L_2(\Omega)} + \|\nabla w_h(0)\|^2_{L_2(\Omega)}. \end{aligned} \tag{B.4}$$

This is the same as

$$\|y_h(T)\|^2_Y + 2a \int_0^T \|B_h^* y_h\|^2_U dt = \|y_h(0)\|^2_Y. \tag{B.5}$$

In particular, since $a > 0$,

$$\int_0^T \|B_h^* y_h(t)\|^2 dt \leq C \|y_h(0)\|^2_Y, \tag{B.6}$$

which is a desired result testing the abstract trace condition of the discrete control operator.

The above result is valid for any approximating scheme that is consistent [splines or modes]. In the case of the second example—Dirichlet case—it is known that the abstract trace condition may not hold for splines. It holds, however for nodal [eigenfunction] approximations.

Next, define as in (A.23)

$$R_h = \Pi_h R = \begin{bmatrix} \pi_h \mathcal{A}_h^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{so } R^* R = \begin{bmatrix} -\mathcal{A}_h^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{B.7}$$

counterpart of the continuous relation (A.26). Thus, we compute (as in the continuous case in (A.29)

$$R_h^* R_h e^{A_h t} B_h u = (R_h^* R_h A_h)(e^{A_h t})(A_h^{-1} B_h u) \in C([0, T]; Y), u \in U, \tag{B.8}$$

counterpart of continuous version (A.28). We shall show below that (B.8) is well defined. With $[x_1, x_2], [y_1, y_2] \in V_h \equiv \mathcal{V}_h \times \mathcal{V}_h$, we compute via (B.1)

$$A_h^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_h \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \pi_h y_2 = y_2 \\ -\mathcal{A}_h y_1 - \mathcal{B}_h \mathcal{B}_h^* y_2 \end{bmatrix} \tag{B.9a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_h^{-1} \mathcal{B}_h \mathcal{B}_h^* & -\mathcal{A}_h^{-1} \\ \pi_h & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_h^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{B.9b}$$

counterpart of (A.13). Thus, by A_h^{-1} in (B.9b) and $B_h u$ in (B.3), we obtain

$$A_h^{-1} B_h u = \begin{bmatrix} -\mathcal{A}_h^{-1} \mathcal{B}_h \mathcal{B}_h^* & -\mathcal{A}_h^{-1} \\ \pi_h & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_h^{-1} \mathcal{B}_h u \\ 0 \end{bmatrix}. \tag{B.10}$$

From (B.2), with $v_h = z_h \in \mathcal{V}_h$, we have

$$\left\| \mathcal{A}_h^{\frac{1}{2}} v_h \right\| = \|\nabla v_h\| = \|v_h\|_{H^1_{\Gamma_0}(\Omega)}. \tag{B.11}$$

By (B.11),

$$\left\| \mathcal{A}_h^{-1} B_h u \right\|_{H^1_{\Gamma_0}(\Omega)} = \left\| \mathcal{A}_h^{\frac{1}{2}} \mathcal{A}_h^{-1} \mathcal{B}_h u \right\|_{L^2(\Omega)} = \left\| \mathcal{A}_h^{-\frac{1}{2}} \mathcal{B}_h u \right\|_{L^2(\Omega)}. \tag{B.12}$$

Hence, by (B.10) and (B.12),

$$A_h^{-1} B_h u \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \qquad u \in L^2(\Gamma_1) \tag{B.13}$$

as desired in (B.8). More specifically, by (B.3), for $u \in L^2(\Gamma_1)$, $\varphi_h \in \mathcal{V}_h$,

$$\begin{aligned} \left| \left(\mathcal{A}_h^{-\frac{1}{2}} \mathcal{B}_h u,\ \varphi_h \right) \right| &= \left| \left\langle u,\ \mathcal{B}_h^* \mathcal{A}_h^{-\frac{1}{2}} \varphi_h \right\rangle \right| = \left| \left\langle u,\ \mathcal{A}_h^{-\frac{1}{2}} \varphi_h \big|_{\Gamma_1} \right\rangle_{L^2(\Gamma_1)} \right| \\ &\le c \|u\|_{L^2(\Gamma_1)} \left\| \mathcal{A}_h^{-\frac{1}{2}} \varphi_h \right\|_{H^1_{\Gamma_0}(\Omega)} && \text{(B.14)} \\ &= c \|u\|_{L^2(\Gamma_1)} \left\| \mathcal{A}_h^{\frac{1}{2}} \mathcal{A}_h^{-\frac{1}{2}} \varphi_h \right\|_{L^2(\Omega)} && \text{(B.15)} \\ &= c \|u\|_{L^2(\Gamma_1)} \|\varphi_h\|_{L^2(\Omega)} && \text{(B.16)} \end{aligned}$$

as desired.

Finally, from (B.7) on $R_h^* R_h$ and (B.1) on A_h, we compute

$$\begin{aligned} R_h^* R_h A_h \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -\mathcal{A}_h^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_h x_2 = x_2 \\ -\mathcal{A}_h x_1 - \mathcal{B}_h \mathcal{B}_h^* x_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_h^{-1} x_2 \\ 0 \end{bmatrix} \\ &\in \begin{bmatrix} \mathcal{D}(\mathcal{A}_h) \\ \otimes \end{bmatrix} \subset V_h \end{aligned} \tag{B.17}$$

as desired (counterpart of the continuous relation (A.31)).

In conclusion, relation (B.8) is well-defined.

Appendix C: Plate Equation with Boundary Control via Moments

To the plate equation dynamics (1.4.4) with control via bending moment [28], we associate the following performance index

$$J_{0,T;G}\left(u, y = \begin{bmatrix} w \\ w_t \end{bmatrix}\right) = \int_0^T \left[\left\| R \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y^2 + \|u(t)\|_U^2 \right] dt$$

$$+\left\| G\begin{bmatrix} w(T) \\ w_t(T) \end{bmatrix} \right\|_Y^2 \tag{C.1}$$

$$Y \equiv H^2(\varOmega) \cap H_0^1(\varOmega) \times L_2(\varOmega), \quad U = L^2(\varGamma), \quad \text{thus } Z \equiv Y \text{ and } Z_f \equiv Y \tag{C.2}$$

according to the abstract model (1.3.1). In particular let's consider the performance index given by (1.4.5). The present example is essentially described and analyzed in [30, Chapter 9 and 10], see details below.

1. Let $A_D : L^2(\varOmega) \supset \mathcal{D}(A_D) \to L^2(\varOmega)$ be the Dirichlet Laplacian operator defined by

$$A_D h = -\Delta h, \qquad \mathcal{D}(A_D) = H^2(\varOmega) \times H_0^1(\varOmega). \tag{C.3}$$

   Then, A_D is positive self-adjoint and has compact resolvent and $A_D^{-1} \in \mathcal{L}(L^2(\varOmega))$.
2. Define the Dirichlet map D [30, p. 948]:

$$h = Dg \Longleftrightarrow \left\{ \Delta h = 0 \text{ in } \varOmega, h\big|_{\varOmega} = g \right\}. \tag{C.4}$$

   Elliptic theory gives [30, p. 948]

$$\begin{cases} D: \text{ continuous } L^2(\varGamma) \to H^{\frac{1}{2}}(\varOmega) \subset H^{\frac{1}{2}-2\rho}(\varOmega) = \mathcal{D}\left(A_D^{\frac{1}{4}-\rho}\right), \ \rho > 0 \\ A_D^{\frac{1}{4}-\rho} D \in \mathcal{L}(L^2(\varGamma); L^2(\varOmega)) \Longleftrightarrow D^* A_D^{\frac{1}{4}-\rho} \in \mathcal{L}(L^2(\varOmega); L^2(\varGamma)), \end{cases} \tag{C.5}$$

   where D^* is the adjoint of D: $(Dv, u)_{L^2(\varOmega)} = (v, D^* u)_{L^2(\varGamma)}$.
   We have [29, p. 181]

$$D^* A_D h = -\left.\frac{\partial h}{\partial \nu}\right|_{\varGamma}, \qquad h \in \mathcal{D}(A_D). \tag{C.6}$$

3. The second order abstract model of the PDE-problem (1.4.4) is [28], [30, p. 1019]

$$w_{tt} + A_D^2 w + a A_D D D^* A_D w_t = A_D D u, \tag{C.7}$$

   where A_D denotes the isomorphic extension $A_{D,\text{ext}} : L^2(\varOmega) \to \left[\mathcal{D}(A_D^*)\right]' = [\mathcal{D}(A_D)]'$ of the original operator in (C.3). The corresponding first order equation is

$$\frac{d}{dt} y = \frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} = A\begin{bmatrix} w \\ w_t \end{bmatrix} + Bu \tag{C.8}$$

$$A = \begin{bmatrix} 0 & I \\ -A_D^2 & -a A_D D D^* A_D \end{bmatrix}, \qquad Bu = \begin{bmatrix} 0 \\ A_D D u \end{bmatrix} \in \begin{bmatrix} \otimes \\ \left[\mathcal{D}(A_D^{\frac{3}{4}+\rho})\right]' \end{bmatrix}. \tag{C.9}$$

Verification of the dynamical assumptions (H.1), (H.2), (H.3) with $a \geq 0$

(H.1): The operator A generates a s.c. contraction semi group e^{At} in Y.
(H.2): We have

$$A^{-1}Bu = \begin{bmatrix} -aA_D^{-1}DD^*A_D & -A_D^{-2} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ A_D Du \end{bmatrix} = \begin{bmatrix} -A_D^{-1}Du \\ 0 \end{bmatrix}$$
$$\in \begin{bmatrix} \mathcal{D}(A_D^{\frac{5}{4}-\rho}) \\ \otimes \end{bmatrix} \subset Y, \tag{C.10}$$

thus a-fortiori $A^{-1}B$: continuous $U \to Y$.

(H.3): It is well-known by now that assumption (H.3) = (1.2.3) is satisfied in the case of the plate equation with bending moment control (abstract trace regularity). This is, in fact a purely PDE-property [28] which was established by PDE-methods, see [25].

Verification of the observation assumptions (H.4), (H.5), (H.6) in the performance index (1.4.3). It readily follows that with the choice of observation as above, the operator $R(w, w_t) = [A_D^{-1/2}w, 0]$ so that

$$R^*R = \begin{bmatrix} A_D^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{C.11}$$

then the L_1-condition (H.5) = (1.3.3) is satisfied. Similarly, choosing G so that

$$G^*G = \begin{bmatrix} A_D^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{C.12}$$

assumption (H.6) = (1.3.4) is then fulfilled. A-fortiori (H.4) = (1.3.2) is satisfied.

The verification of approximating assumptions follow now the same line of arguments as before-in Appendix B. It suffices to notice that the discrete generator A_h takes the form with $(u_h, v_h) \in V_h = \mathcal{V}_h \times \mathcal{V}_h \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$A_h(u_h, v_h) \equiv \begin{bmatrix} 0 & \Pi_h \\ -\mathcal{A}_h u_h & -a\mathcal{B}_h\mathcal{B}_h^* \end{bmatrix} \tag{C.13}$$

where

$$(\mathcal{A}_h u_h, v_h) = (\Delta u_u, \Delta v_h)$$

and

$$\mathcal{B}_h^*(u_h) = \left.\frac{\partial u_h}{\partial \nu}\right|_\Gamma .$$

Remark C.1 For the construction of approximating space $\mathcal{V}_h$, see [47–49].

Remark C.2 It should be noted that the presence of $a > 0$ is critical for the verification of the approximating assumptions [approximate admissibility]. The role of this boundary term is to "regularize" the dynamics from the boundary-see [30]. On the other hand, this term also provides a stabilizing effect-so it can be i=used in the study of Algebraic equations. From the numerical point of view, it can also be viewed as a regularization parameter tending to zero. This approach has been explored within the context of finite dimensional compensator designs [18].

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Declarations

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