Wave equation with nonlinear boundary dissipation and boundary/interior sources of critical exponents is asymptotically finite dimensional and smooth

Article *in* Discrete and Continuous Dynamical Systems · January 2025

DOI: 10.3934/dcds.2025023

Xiaoyue Han
Donghua University

Irena Lasiecka
University of Memphis

Discrete and Continuous Dynamical Systems
Vol. 45, No. 9, September 2025, pp. 3375-3410
doi:10.3934/dcds.2025023

WAVE EQUATION WITH NONLINEAR BOUNDARY DISSIPATION AND BOUNDARY/INTERIOR SOURCES OF CRITICAL EXPONENTS IS ASYMPTOTICALLY FINITE DIMENSIONAL AND SMOOTH

Xiaoyue Han[1], Irena Lasiecka[2,3] and Yuming Qin[*1,4]

[1]School of Mathematics and Statistics, Donghua University, Shanghai 201620, China

[2]Univ Memphis, Dept Math Sci, Memphis, TN 38152 USA

[3]Polish Acad Sci, IBS, Warsaw, Poland

[4]Institute for Nonlinear Science, Donghua University, Shanghai 201620, China

(Communicated by Joachim Krieger)

Abstract. The long time behavior of the wave equation subjected to *boundary dissipation* with interior and boundary nonlinear sources of *critical exponents* is studied. Of particular interest is a characterization of the asymptotic behavior of the solutions in terms of attracting sets. As it is known, the corner stones of any theory of attractors are the following two properties of the dynamics: dissipation and compactness of the trajectories. Neither of the two is present in our model. Since the damping is only partial and localized on the boundary, the entire system lacks dissipation.This, then implies challenges to be overcome. The criticality of both *boundary and interior sources* brings major difficulties already for the proof of asymptotic compactness of the trajectories. While the existence of global attractors for a related dynamics has been dealt with recently [4, 14, 30], the issue of *smoothness and finite dimensionality of the attracting sets has been open in the critical cases* [6, 30] . The present paper fills in this gap by showing that the global attractors for the dynamics with "critical exponents" are *finite dimensional and smooth*. The obtained result allows to reduce (asymptotically) the dynamics from a PDE to an ODE. It is this aspect of the problem which presents the biggest challenge due to the criticality of both sources anda reduced dissipation which needs to be propagated through the geometric region. To obtain the above stated result, a new methodology will be developed. It is based on suitable Carleman's estimates together with "sharp" boundary trace results and dissipation integrals method [11].

1. **Introduction.**

We shall study the following wave equation with nonlinear boundary dissipation and boundary and interior sources f_0, f_1 of *critical exponents*.

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + f_0(u) = p(x) & in\ Q_0^T, \\ \frac{\partial u}{\partial \nu} + g(u_t) + f_1(u) = 0 & in\ \Sigma_0^T, \\ u|_{t=0} = u_0(x),\ u_t|_{t=0} = u_1(x) & in\ \Omega, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification.* Primary: 35B33, 35B40; Secondary: 35L70, 37L05, 37L30.

Key words and phrases. Wave equation, boundary dissipation, quasi-stability, finite dimension, regularity.

*Corresponding author: Yuming Qin.

where $\Omega \subset \mathbb{R}^3$ is a smooth, bounded connected domain with boundary Γ. By criticality, we mean critical exponents with respect to Sobolev's embeddings $u \in H^1(\Omega) \to f_0(u) \in L_2(\Omega), f_1(u|_\Gamma) \in L_2(\Gamma)$.

Let $Q_0^T = \Omega \times [0, T]$ and $\Sigma_0^T = \Gamma \times [0, T]$. As said,the nonlinearities $f_0(u)$ and $f_1(u)$ are of critical exponents representing nonlinear sources, while $g(u_t)$ acts as a nonlinear frictional damping and $p(x)$ is an autonomous external force and bounded in $L^2(\Omega)$.

The model equation in (1) is a classical semilinear wave equation subject to forcing and boundary damping. This is a canonical, benchmark model which exhibits the main features of semilinear hyperbolic dynamics for which long time behavior analysis is sought after. This includes existence, regularity and characterization of attracting sets. As is known, dissipation and compactness of the trajectories play a major role in the analysis. While such properties are natural when external forcing enjoys some compactness properties, much less is known in the critical cases when the dynamics itself does not have any smoothing properties (unlike parabolic problems), dissipation is only partial and the effect of forcing leads to a non-compact contribution. In such critical cases, we need to look for some compensation or cancellations of singularities, which is particularly demanding when boundary effects are considered. The need to propagate dissipation from the boundary into the interior is one of the hurdles. As we shall see, methods previously developed within the context of hyperbolic dynamics with boundary dissipation [1, 2, 4, 11, 13, 21, 34] and references therein are no longer adaptable. The aim of this work is to elucidate this type of situation on a simplest possible model which however retains the main critical features. Thus, we deal with "the old" model for which "new" results are obtained which answer several open questions raised in the past. We shall develop a new, and rather general, methodology which could be also applied to other models enjoying similar properties.

In order to proceed, we make the following assumptions and introduce the definitions of several useful concepts.

Assumption I

(i) (The internal source) $f_0 \in C^2(\mathbb{R})$ satisfies

$$|f_0^{''}(s)| \leq C_{f_0}(1 + |s|), \ \lim inf_{|s|\to\infty} f_0(s)/s > 0. \tag{2}$$

(ii) (The boundary source) $f_1 \in C^2(\mathbb{R})$ satisfies

$$|f_1^{''}(s)| \leq C_{f_1}, \ \lim inf_{|s|\to\infty} f_1(s)/s \geq 0. \tag{3}$$

(iii) (the damping) $g(\cdot) \in C^1(\mathbb{R})$, $g(0) = 0$, is a strictly increasing function such that there exist constants $0 < m_1 < m_2 < \infty$

$$0 < m_1 < g'(s) < m_2 < \infty, \ \forall\, s \in \mathbb{R}. \tag{4}$$

Remark 1.1. The assumptions imposed on the sources and the damping could be relaxed with respect to differentiability. In order to streamline the exposition, by focusing on the main challenges, we shall work within the framework assumed above. The linear bound of the damping (4) is typical in problem with boundary dissipation being the sole source of damping. Otherwise, one has counterexamples regarding uniform convergence to equilibria of linear dynamics [33].

The growth conditions imposed on the sources are in line with critical growth in three dimension.

Let $(\cdot\,,\cdot\,)$ or $<\cdot\,,\cdot>$ denote the L^2- inner product and $\|\cdot\|$ represent the $L^2(\Omega)-$ norm, Q_s^t denote $\Omega\times[s,t]$ and Σ_s^t denote $\Gamma\times[s,t]$, for all $s\le t\in\mathbb{R}$. Also, we denote by $H^s(\Omega)$ the L^2 based Sobolev space with norm $\|\cdot\|_{H^s(\Omega)}$. In the following, we consider the Hilbert space

$$\mathcal{H}=H^1(\Omega)\times L^2(\Omega).$$

It will be convenient to introduce the following norm on $H^1(\Omega)$. Let $2\lambda\equiv\lim inf_{|s|\to\infty}\ f_0(s)/s>0$. We write

$$-\Delta u+f_0(u)=[-\Delta+\lambda I]u+f_0(u)-\lambda u=-\Delta_\lambda u+f_0(u)-\lambda u.$$

Note that Δ_λ with the homogenous Neumann's boundary data is a positive operator on $L^2(\Omega)$ which can be denoted by A_N. Thus $||A_N^{1/2}u||^2=||\nabla u||^2+\lambda||u||^2$. The above formula defines the $H^1(\Omega)$ norm by

$$||u||^2_{H^1(\Omega)}=||\nabla u||^2+\lambda||u||^2.$$

Additionally, we define the norm of the phase space $\mathcal{H}$.

$$\|U(\cdot,t)\|^2_{\mathcal{H}}=\|u(t)\|^2_{H^1(\Omega)}+\|u_t(t)\|^2,\quad\forall\,U(\cdot,t)=(u(t),u_t(t))\in\mathcal{H}.$$

The space $\mathcal{H}$ will denote the phase space for problem (1). It is known [30] that under the assumptions imposed on the data, (1) generates a dynamical system which is described by a continuous nonlinear semigroup $S(t)U\in\mathcal{H}$. The corresponding dynamical system $(\mathcal{H},S(t))$ admits a global attractor-see [30] and [4, 14] in the case the boundary forcing is zero,$g\equiv 0$. Our goal is to show that the attractor generated by (1) is *smooth and finite-dimensional.*

About the problem. Model (1) has attracted a lot of attention in recent years. This is a benchmark model for semilinear waves with nonlinear boundary damping defined on a 3 dimensional bounded domain. Due to a restricted support of the dissipation and the hyperbolic nature of the problem with infinitely many unstable modes [in contrast to parabolic models], establishing stability of the solutions along with uniform decay rates to equilibria is a challenging endeavor. This leads to the development of various techniques allowing for geometric propagation of the damping - see [22] and references therein. The optimal decay rates to the equilibria were shown in [22] for *subcritical sources* with unquantified damping at the origin. This led to a description of stability properties in terms of comparison with solutions of an appropriate nonlinear ODE's. The theory developed in [22] later has continued with the analysis of long time behavior of solutions subjected to dissipative internal forces. This includes theories of attractors including some critical cases [5, 11, 13, 30]. Criticality of internal forcing gave rise to new techniques in the area of hyperbolic dynamical systems which are based on compensated compactness [8, 11]. These techniques were further developed in [30], in order to treat fully critical forces-both in the interior and on the boundary -the latter much more demanding due to the limited regularity of the Neumann hyperbolic map in dimension higher than one. In spite of this recent progress, the daunting question whether the sole boundary damping reduces the asymptotic behavior of a wave solutions with critical sources to a *finite dimensional coherent structure* has been awaiting for an answer. And this is the question which is resolved positively in the present paper.

More specifically, the main goal of this paper is to establish estimates of finite dimension and also regularity of global attractors for the system (1) with *both internal and boundary sources* f_i $(i=0,1)$ *of critical exponents.*

Past literature. Let us contextualize the above result within rather large existing literture. In recent years, there have been a lot of studies on the variation of wave equations with Dirichlet homogenous boundary conditions. However, relatively fewer papers dealt with Neumann or Robin boundary conditions of wave equations, see [1, 2, 4, 6, 8, 9, 13, 14, 15, 16, 17, 18, 20, 21, 23, 24, 30, 31]. Moreover, most of them addressed linear dissipation supported in the interior of the domain with nonlinear forces of subcritical growth, see, for instance, [1, 2, 6, 15, 17, 18, 30]. As we all have known, when exploring properties of attractors, criticality of the source in hyperbolic problems brings the major difficulty due to the lack of compactness. Indeed, the existence of attractors with finite dimension for a "critical" wave equation with nonlinear damping supported on the entire domain has been studied [8]. Earlier researches on critical exponents were limited to certain specific situations, such as one-dimensional model [16] and so on. In addition, Lasiecka et al. [4, 13] also studied the wave equation with nonlinear boundary dissipation and nonlinear *interior* force of critical growth, where finite dimension and regularity of global attractors were established. However, the analysis in [4, 13] did not account for boundary forces. Only recent article [30] dealt with boundary sources.The results presented there are limited to existence of global attractors. Properties of smoothness and finite dimension of global attractor are left open there. Our present work has solved this open problem. That is, we have established an existence of global attractor which is smooth and finite-dimensional, thus completing the program of the study of attractors for a doubly critical benchmark model of semilinear waves with boundary dissipation.

What is new and challenging-strategy of the proof. The presence of critical sources *both in the interior and the boundary* provides for major challenge in establishing "asymptotic attractiveness" of the dynamics. While criticality of interior sources has been dealt with via compensated compactness method in the presence of "interior" damping [8, 9, 11], the dissipation acting on the boundary presents a series of new issues due to necessity of propagating the damping. The first work in this direction was [4], which was followed by [13] and most recently [30] where boundary critical sources are treated with a help of "hidden" boundary regularity caused by the dissipation. However, the methods used in these papers are *not adequate to treat the issue of smoothness and dimension* of the attractor. Here are the reasons.

(•) Since the nonlinear damping and nonlinear forces of critical exponents exist simultaneously in the interior and on the boundary, the needed estimates require "sharp" trace regularity of hyperbolic traces on the boundary, which otherwise remain unbounded above energy level. This is accomplished by developing suitable tangential estimates for the boundary traces, see Lemma 4.2 which takes the origin in microlocal analysis of hyperbolic traces [23]. The above estimate is critical in propagating dissipation from the boundary into the interior in the presence of critical forcings.

(•) The main novelty and challenge are to prove that the attractor is "smooth" and asymptotic behavior is finite dimensional. We note that for "subcritical" sources, this was accomplished in [6]. Criticality of the sources either interior or boundary provides the main difficulty when trying to prove "quasi-stability". To cope with these, one needs three main new ingredients: (1) handle propagation of the damping from the boundary into the interior in the presence of critical sources. For this, it is necessary to work with flow multipliers and criticality of the sources. Carleman's type of estimates [19] help in handling the issue at the level of potential energy. It is with the help of large parameter in Carleman's weights [internal sources] and sharp [hidden] tangential boundary estimates for the boundary sources, that eventually lead to recovery of potential energy but modulo kinetic energy contribution of the sources -see Lemma 5.1. This leads to the second issue: (2) how to handle criticality of both interior and boundary sources at the level of kinetic energy? At this point, different strategy must be used for handling internal sources and the boundary ones. For internal sources we use backward trajectory method by estimating solutions near equilibria and then propagating forward the improved regularity [14]. However, this method does not apply to boundary sources-simply because there is a loss of dimensionality when going from the intrerior to the boundary-trajectories. Velocities near stationary point are "small" with respect to $L_2(\Omega)$ but not with respect to $L_2(\Gamma)$-which is needed. The above manifests the fundamental difficulty of the problem. To handle this part, method of "dissipative integrals" [11] is suitably adopted to the problem. Combining both principles leads to the final quasi-stability estimate obtained for the full time scale.

Remark 1.2. It is believed that the method developed for the benchmark problem of hyperbolic dynamics with boundary damping and fully critical interior/boundary sources can be applied to many other systems sharing the same properties.

2. Main results.

2.1. Known results.

We shall start with listing several relatively recent known results on the problem. This will provide a better perspective and context for the main contribution of this work.

Let's begin with a fundamental result related to the well-posedness of the dynamical system corresponding to (1). This result has been shown in [30] with a proof based on an extension of the semigroup-approximation method developed in [22].

Theorem 2.1. *([30]) Consider the dynamics described in (1) under the assumptions $(i)-(iii)$. Then, there exists a unique solution $U(t) \in \mathcal{H}$ with the following properties. For every $T > 0$ $U \in C([0,T];\mathcal{H})$ with the additional boundary regularity*

$$u_t, \partial_\nu u \in L^2_{loc}(0,\infty; L_2(\Gamma)).$$

Moreover, for smooth initial data $U(0) \in H^2(\Omega) \cap H^1(\Omega)$ subject to boundary compatibility data, the corresponding solutions $U(t)$ with $U \in C([0,T]; H^2(\Omega) \times H^1(\Omega))$ are strong.

As a consequence of Theorem 2.1, we obtain the existence of a well-posed dynamical system $(\mathcal{H}, S(t))$ with the additional information on "hidden" regularity on the boundary. These properties will become critical for further development.

It has been recently proved [30] that the dynamical system $(\mathcal{H}, S(t))$ admits a global attractor. The corresponding result pertaining to long time behavior and the existence of a global attractor is formulated below. For reader's convenience, the definition of attractor and other concepts related to long time behavior are given in the Appendix I.

Theorem 2.2. [30]. *Under the Assumpttion I, the dynamical system $(\mathcal{H}, S(t))$ is asymptotically smooth and gradient, thus it possesses a global, compact attractor*

$\mathcal{A} \subset \mathcal{H}$ that coincides with the unstable manifold of stationary points

$$\mathcal{A} = M^u(\mathcal{N}) \quad and \quad \lim_{t\to\infty} dist_{\mathcal{H}}(S(t)U, \mathcal{N}) = 0, \ \ \forall U \in \mathcal{H}.$$

where $\mathcal{N}$ denotes the set of stationary solutions which is also shown to be bounded.

Remark 2.3. It should be noted that the actual assumptions imposed in the reference cited [30] on the sources and the damping are slightly weaker than the ones imposed in Assumption I-particularly with respect to the differentiability. In order to focus the analysis on the main difficulties, we shall not dwell on this.

2.2. **New results.** Once a global attractor for the dynamics iis available, it is only natural to investigate its properties, with an eye on the main question "*whether the orbits on the attractor are described by finitely many degrees of freedom-thus reducing asymptotically the PDE to anODE?*". And the answer to this question is positive, as documented by the theorem below. It should also be noted that arriving at this result met with a number of challenges. The methods most recently developed run into a number of obstructions due to the critical nature of the nonlinear forces-both in the interior and on the boundary- combined with geometrically restricted dissipation and difficulties encountered in propagating it. In fact, the estimates developed earlier in [30] (along with established methods [1, 3, 11, 13, 26]) can not handle this issue due to the obvious loss of fraction of derivative encountered when treating boundary damping and forcing. The hyperbolic Neumann map looses 1/3 derivative with respect to the energy space in dimension higher than one [23, 25]. And this result is sharp [32]. The goal of this paper is to develop a new methodology which allows to deal with criticality not only in the interior but also on the boundary and in the presence of very weak dissipation restricted to the boundary only. The ultimate result is formulated below. With reference to Theorem 2.2

Theorem 2.4. *The following properties of the global attractor $\mathcal{A}$ of the dynamical system $(\mathcal{H}, S(t))$ hold.*
(1) *Said attractor $\mathcal{A}$ has finite fractal dimension.*
(2) *$\mathcal{A}$ is bounded in $H^2(\Omega) \times H^1(\Omega)$.*
(3) *There exists an exponential attractor $\mathcal{A}_e \supset \mathcal{A}$ for the dynamics, which is weakly compact in $\mathcal{H}$. Moreover, for any $\delta \in (0, 1]$, the dynamical system $(\mathcal{H}, S(t))$ possesses generalized fractal exponential attractor $\mathcal{A} \subset \mathcal{A}_{e,\delta} \subset \mathcal{H}$, with finite fractal dimension in the extended space $\tilde{\mathcal{H}}_{-\delta}$, defined as an interpolation of*

$$\tilde{\mathcal{H}}_0 := \mathcal{H} \quad and \quad \tilde{\mathcal{H}}_{-1} := [H^1(\Omega)]' \times L^2(\Omega).$$

Organization of the paper. The rest of the paper is organized as follows. The main results are formulated in Section 2. In Section 3, some background materials and supporting estimates are given. Section 4 and section 5 are devoted to the proof of the main results. The backbone of the proof is the "quasi-stability" estimate which is valid for the double critical cases considered in this paper, see Proposition 4.1 with the proof accomplished via a string of lemmas and propositions. Section 6 contains some additional materials which are helpful to the reader.

Hereafter, it is noteworthy that all letters in the following may be different, which may vary from line to line to each step and for convenience, we have omitted the integral variable of many integral equations or inequalities.

3. **Background material and preliminary estimates.** In this section, we shall give some estimates and facts which play an important role in the proofs of the subsequent theorems and lemmas.

3.1. **Energy estimates.** We define the energy functional by

$$\tilde{E}(t) = E(t) + \int_\Omega \tilde{f}_0(u) + \int_\Gamma \tilde{f}_1(u) - \int_\Omega p(x)u \tag{5}$$

where $E(t) = \frac{1}{2}\|U(x,t)\|^2_{\mathcal{H}}$ and $\tilde{f}_0(s) = \int_0^s [f_0(\tau) - \lambda\tau]d\tau$, $\tilde{f}_1(s) = \int_0^s f_1(\tau)d\tau$, for $s \in \mathbb{R}$.

The result below provides the energy *equality* rather than inequality valid for the forced wave equation with weak forcing and initial data, see [22].

Proposition 3.1. *([22]) Let u be a given function in $C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega))$ such that*

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = f \in L^1(0,T;L^2(\Omega)), \\ u|_{t=0} = u_0 \in H^1(\Omega),\ u_t|_{t=0} = u_1 \in L^2(\Omega), \\ u_t \text{ and } \frac{\partial u}{\partial \nu}|_\Gamma \text{ are in } L^2(0,T;L^2(\Gamma)). \end{cases} \tag{6}$$

Then the following energy identity holds for any $0 \le s \le t$,

$$E_1(t) - \int_s^t \int_\Gamma \frac{\partial u}{\partial \nu}|_\Gamma u_t - \int_s^t \int_\Omega f u_t = E_1(s)$$

where $E_1(t) = \frac{1}{2}\left(\|\nabla u(t)\|^2 + \|u_t\|^2\right)$.

Remark 3.2. The importance of the Proposition 3.1 lies in the fact that typically one obtains the energy *inequality* only. It is the additional information on the behavior of the solutions on the boundary which allows to obtain the *equality*.

We now give the energy identity valid for the nonlinear problem whose proof is based on the above Proposition 3.1, see [22] and also Proposition 2, see [30].

Lemma 3.3. *Let Assumption I holds. If $u(x,t)$ is a solution of (1) satisfying the extra boundary regularity $u_t|_\Gamma$, $\partial_\nu u|_\Gamma \in L^2(0,\infty;L^2(\Gamma))$, then for all $s \le t$*

$$E(t) - \int_s^t \int_\Gamma \frac{\partial u}{\partial \nu}|_\Gamma u_t - \frac{\lambda}{2}\int_\Omega [u^2(t) - u^2(s)]dx + \int_s^t \int_\Omega f_0 u_t - \int_s^t \int_\Omega p(x)u_t = E(s). \tag{7}$$

In addition, if u and v are (weak) solutions of (1), then $z = u - v$ verifies the identity

$$\begin{aligned} E(z(t)) + \int_s^t \int_\Gamma g^{uv} z_t|_\Gamma = & E(z(s)) - \int_s^t \int_\Omega f_0^{uv} z_t - \int_s^t \int_\Gamma f_1^{uv} z_t \\ & + \frac{\lambda}{2}\int_\Omega [z^2(t) - z^2(s)]dx, \quad s \le t \end{aligned} \tag{8}$$

where $f_i^{uv} = f_i(u) - f_i(v)\ (i = 0,1)$, and $g^{uv} = g(u_t) - g(v_t)$.

The proof of Lemma 3.3 follows from the above Proposition 3.1, see [22].

Using (5) and (7), we conclude

Corollary 3.4.

$$\tilde{E}(u(t)) + \int_s^t \int_\Gamma g(u_t)u_t|_\Gamma = \tilde{E}(u(s)), s \le t. \tag{9}$$

Lemma 3.5. *Let Asuumption I holds, then there exist positive constants C_0 and C_1 such that*

$$C_0 E(u(t)) - C_1 \le \tilde{E}(t) \le \tilde{E}(0), \qquad \forall\, t > 0, \quad (u, u_t) \in \mathcal{H}. \tag{10}$$

Proof. Let $(u, u_t) \in \mathcal{H}$. The RHS of (10) follows from (4) and (9). Next, we estimate the LHS of (10). Since $2\lambda \equiv \liminf_{|s|\to\infty} f_0(s)/s$, there exists a positive constant K, such that

$$\frac{f_0(s)}{s} \ge 2\lambda, \quad for\ |s| \ge K.$$

Let's consider first $s > K$ [similar argument applies for $s < -K$].

$$\begin{aligned}
\tilde{f}_0(s) &\ge \int_0^K f_0(\tau) + \int_K^s f_0(\tau) - \frac{\lambda}{2} s^2 \\
&\ge K \min_{0\le\tau\le K} f_0(\tau) + \lambda(s^2 - K^2) - \frac{\lambda}{2} s^2 \\
&\ge -C_{f_0,K,\lambda} + \frac{\lambda}{2} s^2,
\end{aligned} \tag{11}$$

where $C_{f_0,K,\lambda} = -K \min_{0\le\tau\le K} f_0(\tau) + \lambda K^2$, which implies

$$\int_\Omega \tilde{f}_0(u) \ge -C_{f_0,K,\lambda}|\Omega| + \frac{\lambda}{2}\|u\|^2. \tag{12}$$

Similarly, using (3), one obtains

$$\tilde{f}_1(s) = \int_0^s f_1(\tau) \ge -C_{f_1,K,\lambda} \tag{13}$$

Then, using (11)-(13), and Young's inequality we derive

$$\begin{aligned}
\tilde{E}(t) &\ge E(t) - \int_\Omega p(x)u - C_{f_0,K,\lambda}|\Omega| + \frac{\lambda}{2}\|u\|^2 - C_{f_1,K,\lambda}|\Gamma| \\
&\ge E(t) - \frac{||p(x)||^2}{2\lambda} - C_{f_0,K,\lambda}|\Omega| - C_{f_1,K,\lambda}|\Gamma|.
\end{aligned} \tag{14}$$

This leads to the LHS of (10), completing the proof of Lemma 3.5. □

Corollary 3.6. *(Uniform bound) Suppose that Assumption I holds. If $U(x,t) = (u, u_t)$ is a solution of (1) with initial data $(u_0, u_1) \in \mathcal{B}$, where $\mathcal{B} \subset \mathcal{H}$ is a bounded set, then for any $s < t$, there exists a positive constant $C_\mathcal{B}$, such that for all $s < t$*

$$\tilde{E}(t) \le C_\mathcal{B}, \quad E(t) \le C_\mathcal{B} \quad and \quad \int_s^t \int_\Gamma g(u_t)u_t \le C_\mathcal{B}. \tag{15}$$

4. **Quasi-stability on the attractor $\mathcal{A}$.** In order to establish both: finite dimensionality and additional regularity of the attractor, the following form of *quasi-stability* estimate established on the attractor is critical.

Proposition 4.1. *There exist constants $C_1, C_2, \omega > 0$, which may depend on the attractor $\mathcal{A}$, such that for any $y_1, y_2 \in \mathcal{A}$, one has with $(z(t), z_t(t)) = S(t)y_1 - S(t)y_2$, for all $t > 0$*

$$||S(t)y_1 - S(t)y_2||^2_\mathcal{H} \le C_1 e^{-\omega t}||y_1 - y_2||^2_\mathcal{H} + C_2 sup_{s\in[0,t]}||z(s)||^2. \tag{16}$$

Stability estimate formulated above is the key estimate which allows to conclude both: finite dimensionality, smoothness of the attractor and an existence of exponential attractor. The corresponding results are given in Theorem 3.11 [7], Theorem 2.14 [11] and [8]. See also [3] for a comprehensive treatment of quasi-stability theory. The crux of this estimate is that it shows that difference of any two solutions on the attractor can be stabilized exponentially modulo compact perturbations but of *quadratic* structure. The latter is very important and essentially used for the proofs. We recall that the for proving just existence of the attractor, the structure of lower order perturbation is immaterial.-rendering the task of the proof much simpler.

Thus the goal of the paper is precisely to show the validity of this estimate for the model under study. What are the hurdles and obstacles requiring novel methods to enter the game?

1. Reconstruction of potential energy. 2. Reconstruction of kinetic energy due to internal forces. 3. Reconstruction of kinetic energy due to boundary forces.

The main difficulty is caused by the criticality of the exponents in *both* sources. Propagation of dissipation from the boundary to the interior requires the so called "flux" multipliers which are differential operators of the energy level. The finite dimension or smoothness of attractors requires working on the difference of two solutions. Thus, any hamiltonian based cancellations occurring for a single solution is out of question. This, combined with the criticality of the exponents, leaves the source related terms of energy level without any hope for a "smallness" [absorbtion] or compactness. How to deal with this issue? This is the point when Carleman's estimates with large parameter come to picture. The large parameter $\tau > 0$ allows to absorb the effect of internal source. However, boundary source needs an additional "sharp tangential estimate" which allows to handle criticality on the boundary (this is the situation when trace theory is not sufficient and loss of derivative in Neumann hyperbolic map enters the game). The described method allows to obtain good estimate - modulo critical terms resulting from potential energy. And these will be handled by a method originally developed by Zelik [34] and referred to "backward trajectories method". The latter is applicable because the system is gradient. However, this method is not applicable to "absorb" the boundary sources. The reason is simple, small $L_2(\Omega)$ neighborhood of velocities are not "small" in $L_2(\Gamma)$ topology. Handling this hurdle would require a different approach -ultimately based on the theory of dissipative integrals [11]. And, at the end - both strategies need to cooperate for the final estimate in Proposition 4.1.

4.1. **Tangential trace estimates.** In order to handle tangential derivatives on the boundary above the energy level, which will appear due to the application of Carleman's estimates in section 4.2, the following "hidden" trace regularity will be employed.

Lemma 4.2. *Under Assumption I and u, v be any two solutions of (1) satisfying $(u(0), u_t(0))$, $(v(0), v_t(0)) \in \mathcal{B}$, where $\mathcal{B} \subset \mathcal{H}$ is a bounded set, and the boundary regularity $u_t|_\Gamma$, $\partial_\nu u|_\Gamma \in L^2(0,\infty; L^2(\Gamma))$. Then, for any $t > 0$, $\eta > 0$, $z = u - v$ satisfies the following estimate*

$$\int_\eta^{t+\eta}\int_\Gamma |\partial_\nu z|^2 + \int_\eta^{t+\eta}\int_\Gamma |\nabla_\tau z|^2$$
$$\leq C_\eta \int_0^{t+2\eta} \Big(\|z_t\|^2_{L^2(\Gamma)} + \|g^{uv}\|^2_{L^2(\Gamma)} \Big)$$

$$+\varepsilon C_{\mathcal{B}}\int_0^{t+2\eta}E(z(\tau))+C_{t,\eta,\varepsilon}l.o.t._{[0,t+2\eta]}\{z\}, \tag{17}$$

for any $\varepsilon > 0$, where C_η, $C_{\mathcal{B}}$ and $C_{t,\eta,\varepsilon}$ are positive constants, and

$$l.o.t._{[0,t+2\eta]}\{z\} := \sup_{\tau\in[0,t+2\eta]}\left\{\|z(\tau)\|^2_{H^s(\Omega)},\ \ 0\le s<1\right\}$$

is defined as a "lower order term". $\nabla_\tau z$ is the tangential derivative of z.

The lemma above states that tangential derivatives on the boundary are controlled by time boundary derivatives, Neumann traces and the lower order terms. Note that such estimate can not follow from any trace theorem, as there is a gain of 1/2 derivative with respect to the trace theory. The proof relies on microlocal analysis and it exploits, in a crucial manner, hyperbolicity of the waves dynamics.

Proof. We shall begin with quoting a related result from [23] applicable to a single wave solution.

Lemma 4.3. *([23]) Let w be a solution of the problem (6). Then, for any $t > 0$, $\eta \in R$ and $0 < \varsigma < \frac{1}{2}$, there exist positive constants $C_{\varsigma,\eta}$ and $C_{t,\varsigma,\eta}$ such that the following estimate holds*

$$\int_\eta^{t+\eta}\int_\Gamma|\nabla_\tau w|^2 \le C_{\varsigma,\eta}\int_\eta^{t+\eta}(\|w_t\|^2_{L^2(\Gamma)}+\|\partial_\nu w\|^2_{L^2(\Gamma)})$$
$$+ C_{t,\varsigma,\eta}(\|w\|^2_{H^{\frac{1}{2}+\varsigma}(Q_0^{t+2\eta})}+\|f\|^2_{H^{-\frac{1}{2}+\varsigma}(Q_0^{t+2\eta})}) \tag{18}$$

This estimate goes back to Lemma 7.2 in [23] and was also used in [30].

By using Lemma 4.3, we shall establish the relevant tangential estimate valid for the nonlinear problem in hand. Using the boundary condition $\partial_\nu z = -(f_1^{uv}+g^{uv})$ and (18), we derive

$$\int_\eta^{t+\eta}\int_\Gamma|\partial_\nu z|^2+\int_\eta^{t+\eta}\int_\Gamma|\nabla_\tau z|^2$$
$$\le C_\eta\int_\eta^{t+\eta}\left(\|z_t\|^2_{L^2(\Gamma)}+\|g^{uv}\|^2_{L^2(\Gamma)}\right)$$
$$+ C_{t,\eta}\int_0^{t+2\eta}\left(\|z\|^2_{H^{\frac{1}{2}+\varsigma}(\Omega)}+\|f_0^{uv}\|^2_{H^{-\frac{1}{2}+\varsigma}(\Omega)}\right)+C_\eta\int_\eta^{t+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)}. \tag{19}$$

From (3), we can derive

$$\int_\eta^{t+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)} \le C_{f_1}\int_\eta^{t+\eta}\int_\Gamma|z|^2(1+|u|+|v|)^2 \tag{20}$$

where $C_{f_1} = 2[1+\max_{|\tau|\le 1}|f_1'(\tau)|]^2$.

Therefore, using the embedding $H^{\frac{1}{2}}(\Gamma)\hookrightarrow L^4(\Gamma)$ and (20), we have

$$\int_\eta^{t+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)} \le C_{f_1}\int_\eta^{t+\eta}\|z\|^2_{L^4(\Gamma)}\left(|\Gamma|^{\frac{1}{4}}+\|u\|_{L^4(\Gamma)}+\|v\|_{L^4(\Gamma)}\right)^2$$
$$\le C_{f_1}\int_\eta^{t+\eta}\|z\|^2_{L^4(\Gamma)}\left(|\Gamma|^{\frac{1}{4}}+\|u\|_{H^{\frac{1}{2}}(\Gamma)}+\|v\|_{H^{\frac{1}{2}}(\Gamma)}\right)^2. \tag{21}$$

Furthermore, from the trace theorem and (15), we can derive

$$\int_\eta^{t+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)} \le C_{f_1,\mathcal{B}}\int_\eta^{t+\eta}\|z\|^2_{H^{\frac{1}{2}}(\Gamma)}. \tag{22}$$

Next, applying interpolation between $L^2(\Gamma)$ and $H^1(\Gamma)$ to (22), for any $\varepsilon_0 > 0$, we have

$$\|z\|^2_{H^{\frac{1}{2}}(\Gamma)} \le \|z\|_{L^2(\Gamma)}\|z\|_{H^1(\Gamma)} \le \frac{\varepsilon_0/C_\eta}{C_{f_1,\mathcal{B}}}\|\nabla_\tau z\|^2_{L^2(\Gamma)} + C_{\mathcal{B},\varepsilon_0,\alpha}\|z\|^2_{L^2(\Gamma)} \tag{23}$$

where $C_{\mathcal{B},\varepsilon_0,\eta} = \frac{\varepsilon_0/C_\eta}{C_{f_1,\mathcal{B}}} + \frac{C_{f_1,\mathcal{B}}}{4\varepsilon_0/C_\eta}$ and we have used $\|z\|^2_{H^1(\Gamma)} = \|z\|^2_{L^2(\Gamma)} + \|\nabla_\tau z\|^2_{L^2(\Gamma)}$.

Combining (21)-(22) and (23), we conclude

$$\int_\eta^{t+\eta} \|f_1^{uv}\|^2_{L^2(\Gamma)} \le \frac{\varepsilon_0}{C_\eta}\int_\eta^{t+\eta}\int_\Gamma |\nabla_\tau z|^2 + C_{\mathcal{B},\varepsilon_0,t,\eta} l.o.t_{[0,t+2\eta]}\{z\}, \tag{24}$$

for any $\varepsilon_0 > 0$, where $C_{\mathcal{B},\varepsilon_0,t,\eta} = tC_{f_1} + (t+2\eta)C_{\mathcal{B},\varepsilon_0,\eta}$.

Substituting (24) into (19), we obtain

$$\begin{aligned}
&\int_\eta^{t+\eta}\int_\Gamma |\partial_\nu z|^2 + (1-\varepsilon_0)\int_\eta^{t+\eta}\int_\Gamma |\nabla_\tau z|^2 \\
&\le C_\eta \int_\eta^{t+\eta} \|z_t\|^2_{L^2(\Gamma)} + C_{t,\eta,\varepsilon_0} l.o.t_{[0,t+2\eta]}\{z\} \\
&\le C_{t,\eta}\int_0^{t+2v}\left(\|z\|^2_{H^{\frac{1}{2}+\varsigma}} + \|f_0^{uv}\|^2_{H^{-\frac{1}{2}+\varsigma}}\right).
\end{aligned} \tag{25}$$

We next use interpolation for $0 \le \varsigma \le \frac{1}{2}$ to derive

$$\begin{aligned}
\|z\|^2_{H^{\frac{1}{2}+\varsigma}(Q_0^{t+2\eta})} &\le \|z\|_{L^2(Q_0^{t+2\eta})}\|z\|_{H^1_{(Q_0^{t+2\eta})}} \\
&\le \frac{1}{4\varepsilon}\|z\|_{L^2(Q_0^{t+2\eta})} + \varepsilon\|z\|^2_{H^1(Q_0^{t+2\eta})} \\
&\le \frac{(t+2\varsigma)C_{t,\eta}}{4\varepsilon} l.o.t_{[0,t+2\eta]}\{z\} + \frac{\varepsilon}{C_{t,\eta}}\int_0^{t+2\eta} E(z)
\end{aligned} \tag{26}$$

where $\varepsilon > 0$ will be choosen later.

Additionally, using (2), the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, along with a standard duality argument we conclude

$$\|f_0(u)\|_{[H^1(\Omega)]'} \le C_{\mathcal{B},\Omega}\|u\|, \quad for\ all\ u \in \mathcal{B}. \tag{27}$$

Then, from the Lipschitz property of f_0, interpolation and (27), we derive

$$\begin{aligned}
\|f_0^{uv}\|^2_{H^{-\frac{1}{2}+\varsigma}(Q_0^{t+2\eta})} &\le \frac{1}{4\varepsilon}\|f_0^{uv}\|^2_{H^{-1+\varsigma}(Q_0^{t+2\eta})} + \varepsilon\|f_0^{uv}\|^2_{L^2(Q_0^{t+2\eta})} \\
&\le \frac{C_{\mathcal{B},\Omega}}{4\varepsilon}\int_0^{t+2\eta}\|z\|^2_{L^2(\Omega)} + \varepsilon C_{\mathcal{B},\Omega}\int_0^{t+2\eta} E(z) \\
&\le \frac{C_{\mathcal{B},\Omega}(t+2\eta)C_{t,\eta}}{4\varepsilon} l.o.t_{[0,t+2\eta]}\{z\} + \frac{\varepsilon C_{\mathcal{B},\Omega}}{C_{t,\eta}}\int_0^{t+2\eta} E(z)
\end{aligned} \tag{28}$$

where $\varepsilon > 0$ will be determined later on.

Substituting (26) and (28) into (25) and choosing $\varepsilon = \varepsilon_0 = \frac{1}{2}$, we thus get

$$\begin{aligned}
&\int_\eta^{t+\eta}\int_\Gamma |\partial_\nu z|^2 + (1-\varepsilon_0)\int_\eta^{t+\eta}\int_\Gamma |\nabla_\tau z|^2 \\
&\le C_\eta \int_0^{t+2\eta}\left(\|z_t\|^2_{L^2(\Gamma)} + \|g^{uv}\|^2_{L^2(\Gamma)}\right)
\end{aligned}$$

$$\leq \varepsilon(1+C_{\mathcal{B}})\int_0^{t+2\eta} E(z(\tau)) + C_{t,\eta,\varepsilon,\varepsilon_0} l.o.t._{[0,t+2\eta]}\{z\}. \tag{29}$$

which concludes the estimate in (17). □

4.2. **Preliminaries on Carleman's estimates applied to waves and the resulting recovery estimate for the energy function.** We begin with a construction of the appropriate vector field which captures the behavior on the boundary and provides a suitable weight function for Carleman's estimate.

Proposition 4.4. *(Vector field)([14, 25]) There exist a strictly convex scalar function $d(x) \in C^3(\overline{\Omega})$ and a positive constant ρ, such that $\nabla d(x)$ is tangent to Γ and $J_h \geq \rho I$, where $h := \nabla d$ and J_h denotes the Jacobian of h. In particular, considering the damping being effective over the boundary Γ, then the vector field can be considered radial, i.e., $h(x) = x - x_0$.*

We shall also need the following preliminaries (see [14]) which are related to Carleman's estimates.

Choose any $\eta \geq 0$ and $c > 0$ satisfying $0 < c < \min\{1, \frac{\rho}{2}\}$, where ρ is determined in Proposition 4.4 . Let $\bar{T}$ be large enough [related to the speed of propagation] such that

$$c\bar{T} > 2\max_{y\in\overline{\Omega}}\{\sqrt{d(y)}\} \tag{30}$$

where d is defined in Proposition 4.4. The first task is to prove the following recovery estimate which accounts for tangenetial spillowers due to criticality of the sources. The latter is due to flow multipliers applied to the equation.

Proposition 4.5. *With the notation of Proposition 4.4, $(z, z_t) = (u - v, u_t - v_t)$, $y_1 = (u_0, u_1)$, $y_2 = (v_0, v_1)$, $y_i \in \mathcal{B}$, where $\mathcal{B} \subset \mathcal{H}$ is a bounded set, $T > \bar{T}$, one has*

$$\begin{aligned} E_z(T) + C_T\int_0^T E_z(r)dr \leq C_{T,\mathcal{B}}\int_0^T |< g^{u,v}, z_t >| + C_{T,\mathcal{B}}F \\ + C_{T,\mathcal{B}} sup_{s\in[0,T]}||z(s)||^2 \end{aligned} \tag{31}$$

where $F \leq |\int_0^T (f_0^{u,v}, z_t)_\Omega dt| + |\int_0^T < f_1^{u,v}, z_t >_\Gamma dt|$.

In order to prove the proposition we shall need several intermediate estimates.

4.2.1. *Carleman weights.* Set

$$\Phi(x,t) = \Phi(x,t;\eta,\bar{T}) := d(x) - c\left(t - \eta - \frac{\bar{T}}{2}\right)^2, \tag{32}$$

then we can conclude that there exists a constant $\delta > 0$ such that

$$\Phi(x,\eta) = \Phi(x,\eta+\bar{T}) := d(x) - c\frac{\bar{T}}{4} < -\delta. \tag{33}$$

In particular, we can always redefine $d(x)$ (only by adding a constant to it) so that there exists subinterval $[t_0, t_1]$ with $\eta < t_0 < t_1 < \eta + \bar{T}$ such that

$$\Phi(x,t) > 0 \quad for\ \ all\ \ t \in [t_0, t_1]. \tag{34}$$

The function $\Phi(x,t)$ is the important weight function in Carleman's estimates.

Remark 4.6. Indeed, we only need $\Phi(x,t) > -\delta\ \ for\ \ t \in [t_0, t_1]$, considering $d(x)$ to be taken any value and for convenience, we can choose $\Phi(x,t) > 0$. In addition $\nabla\Phi = \nabla d(x) = h$.

The starting and critical part of the proof of Proposition 4.5 is the following **Carleman's estimate** written for the difference z of two solutions corresponding to the original equation. We use the weight function $\Phi(t,x)$ and the vector field $h = \nabla d$.

Lemma 4.7. *Recall that J_h is defined in Proposition 4.4 and $c, \bar{T}, \rho, h, \Phi$ are as defined above (see (30), Proposition 4.4, (32)). Let the Assumptions I be satisfied and $U^1 = (u, u_t)$, $U^2 = (v, v_t)$ with initial data $U_0^1 = (u_0, u_1), U_0^2 = (v_0, v_1) \in \mathcal{B} \subset \mathcal{H}$, respectively, be strong solutions of (1), where $\mathcal{B}$ is a bounded set. Set $z = u - v$, and choose any $\eta \geq 0$, $\tau > 0$, then we have*

$$\begin{aligned}\int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi}(J_h - \rho I_{\mathbb{R}^3})\nabla z \cdot \nabla z + (\frac{\rho}{2} - c)\int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi}\left(|\nabla z|^2 + |z_t|^2\right) \\ = (M\Sigma_\eta^{\bar{T}+\eta})_\tau - (\mathcal{L}_\eta^{\bar{T}+\eta})_\tau - \tau\int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi}\Psi_1^2 - \int_{Q_\eta^{\bar{T}+\eta}} f_0^{uv}\Psi_1 \\ + [Almost\ lower\ order] \end{aligned} \tag{35}$$

where

$$\begin{aligned}(M\Sigma_\eta^{\bar{T}+\eta})_\tau := \int_{\Sigma_\eta^{\bar{T}+\eta}} \frac{\partial z}{\partial \nu}\Psi_1 + \int_{\Sigma_\eta^{\bar{T}+\eta}} \frac{\partial z}{\partial \nu} z\left(\frac{1}{2}\omega - (\frac{\rho}{2} + c)e^{\tau\Phi}\right) \\ + \frac{1}{2}\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}\left(|z_t|^2 - |\nabla z|^2\right)(h\cdot\nu); \end{aligned} \tag{36}$$

$$\Psi_1 := e^{\tau\Phi}(h \cdot \nabla z - \Phi_t z_t); \quad \omega(x,t) := div\{e^{\tau\Phi}h\} - \frac{d}{dt}(\Phi_t e^{\tau\Phi});$$

$$\begin{aligned}[Almost\ lower\ order] := \int_{Q_\eta^{\bar{T}+\eta}} z z_t \frac{d}{dt}\left(\frac{\omega}{2} - (\frac{\rho}{2} + c)e^{\tau\Phi}\right) \\ - \int_{Q_\eta^{\bar{T}+\eta}} z(\nabla z \cdot z + f_0^{uv})\left(\frac{\omega}{2} - (\frac{\rho}{2} + c)e^{\tau\Phi}\right);\end{aligned}$$

and

$$\begin{aligned}(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau := \int_\Omega e^{\tau\Phi} z_t(h\nabla z - \frac{1}{2}\Phi_t z_t)\Big|_\eta^{\bar{T}+\eta} - \frac{1}{2}\int_\Omega e^{\tau\Phi}\Phi_t|\nabla z|^2\Big|_\eta^{\bar{T}+\eta} \\ + \frac{1}{2}\int_\Omega z_t z\omega\Big|_\eta^{\bar{T}+\eta} - (\frac{\rho}{2} + c)\int_\Omega z_t z e^{\tau\Phi}\Big|_\eta^{\bar{T}+\eta}.\end{aligned} \tag{37}$$

Proof. We note that since the estimate in Lemma 4.7 does not specify the boundary conditions, we can borrow much of calculations from [13]. However, since some of the estimates will be used for our specific model, we provide a quick and fairly complete account of the estimates involved, including critical dependence on large parameter.

We begin by applying weighted multipliers to the equation:

$$\begin{cases} z_{tt} - \Delta z = -f_0^{uv} & in\ (0,\infty)\times\Omega, \\ \partial_\nu z = -(f_1^{uv} + g^{uv}) & on\ (0,\infty)\times\Gamma, \\ z|_{t=0} = z_0,\ z_t|_{t=0} = z_1 & in\ \Omega, \end{cases} \tag{38}$$

Taking any $\tau > 0$ and defining "flux" multiplier $\Psi_1 := e^{\tau\Phi}(h \cdot \nabla z - \Phi_t z_t)$, then multiplying $(4.16)_1$ by Ψ_1 and integrating by parts over $[0,T]\times\Omega$, hence, we have the following equalities for every term in $(4.16)_1$. (Note that, here, we take the time interval $[0,T]$ in the calculation, and in the corresponding results, we only need to replace $[0,T]$ with $[\eta,\bar{T}+\eta]$ to obtain the final proof of (35).)

Calculations are as follows:

For $\int_{Q_0^T} z_{tt}\Psi_1$, we get

$$\begin{aligned}\int_{Q_0^T} & z_{tt}e^{\tau\Phi}(h\cdot\nabla z-\Phi_t z_t)\\ &=\frac{1}{2}\int_{Q_0^T} z_t^2\left(div\{e^{\tau\Phi}h\}+\frac{d}{dt}(e^{\tau\Phi}\Phi_t)\right)-\frac{1}{2}\int_{\Sigma_0^T} z_t^2 e^{\tau\Phi}h\cdot\nu\\ &\quad -\tau\int_{Q_0^T} z_t h\Phi_t e^{\tau\Phi}\nabla z+\mathcal{L}_1(z,z_t)\end{aligned}\tag{39}$$

where

$$\mathcal{L}_1(z,z_t):=\int_\Omega e^{\tau\Phi}z_t(h\cdot\nabla z-\frac{1}{2}\Phi_t z_t)|_0^T.\tag{40}$$

Indeed, we have

$$\begin{aligned}\int_{Q_0^T} & z_{tt}e^{\tau\Phi}(h\cdot\nabla z-\Phi_t z_t)\\ &=\int_{Q_0^T} z_{tt}e^{\tau\Phi}h\cdot\nabla z-\frac{1}{2}\int_{Q_0^T} e^{\tau\Phi}\Phi_t\frac{d}{dt}(z_t^2)\\ &=\int_\Omega z_t e^{\tau\Phi}h\cdot\nabla z|_0^T-\int_{Q_0^T} z_t\frac{d}{dt}(e^{\tau\Phi}h\cdot\nabla z)-\frac{1}{2}\int_\Omega e^{\tau\Phi}\Phi_t z_t^2|_0^T\\ &\quad+\frac{1}{2}\int_{Q_0^T} z_t^2\frac{d}{dt}(e^{\tau\Phi}\Phi_t),\end{aligned}\tag{41}$$

and

$$\begin{aligned}\int_{Q_0^T} & z_t\frac{d}{dt}(e^{\tau\Phi}h\cdot\nabla z)\\ &=\int_{Q_0^T} z_t h(\tau\Phi_t e^{\tau\Phi}\cdot\nabla z+e^{\tau\Phi}\nabla z_t)\\ &=\tau\int_{Q_0^T} z_t h\Phi_t e^{\tau\Phi}\cdot\nabla z+\frac{1}{2}\int_{\Sigma_0^T} z_t^2 e^{\tau\Phi}h\cdot\nu-\frac{1}{2}\int_{Q_0^T} z_t^2 div\{e^{\tau\Phi}h\}.\end{aligned}\tag{42}$$

Therefore, combining (41) and (42), we can derive (39).

Next, for $\int_{Q_0^T}(-\Delta z)\Psi_1$, we have

$$\begin{aligned}\int_{Q_0^T} & (-\Delta z)e^{\tau\Phi}(h\cdot\nabla z-\Phi_t z_t)\\ &=-\int_{\Sigma_0^T}\frac{\partial z}{\partial\nu}e^{\tau\Phi}(h\cdot\nabla z-\Phi_t z_t)+\tau\int_{Q_0^T}(h\cdot\nabla z)e^{\tau\Phi}(h\cdot\nabla z-\Phi_t z_t)\\ &\quad-\int_{Q_0^T} e^{\tau\Phi}\Phi_t\nabla z\cdot\nabla z_t+\int_{Q_0^T} e^{\tau\Phi}\nabla z\cdot\nabla(h\cdot\nabla z).\end{aligned}\tag{43}$$

For the last term of RHS in (43), we obtain

$$\int_{Q_0^T} e^{\tau\Phi}\nabla z\cdot\nabla(h\cdot\nabla z)$$

$$= \int_{Q_0^T} e^{\tau\Phi} J_h \nabla z \cdot \nabla z + \frac{1}{2}\int_{Q_0^T} e^{\tau\Phi} h \nabla |\nabla z|^2$$
$$= \int_{Q_0^T} e^{\tau\Phi} J_h \nabla z \cdot \nabla z + \frac{1}{2}\int_{\Sigma_0^T} e^{\tau\Phi} |\nabla z|^2 (h\cdot\nu) - \frac{1}{2}\int_{Q_0^T} div\{e^{\tau\Phi} h\} |\nabla z|^2. \quad (44)$$

In addition, we also have

$$\int_{Q_0^T} e^{\tau\Phi}\Phi_t \nabla z\cdot\nabla z_t = \frac{1}{2}\int_{Q_0^T} e^{\tau\Phi}\Phi_t \frac{d}{dt}|\nabla z|^2$$
$$= \frac{1}{2}\int_{\Omega} e^{\tau\Phi}\Phi_t |\nabla z|^2|_0^T - \frac{1}{2}\int_{Q_0^T} |\nabla z|^2|_0^T \frac{d}{dt}(e^{\tau\Phi}\Phi_t). \quad (45)$$

Then, substituting (44)-(45) into (43), we thus derive

$$\int_{Q_0^T} (-\Delta z) e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t)$$
$$= -\int_{\Sigma_0^T} \frac{\partial z}{\partial\nu} e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t) + \tau\int_{Q_0^T} (\nabla z\cdot h) e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t) - \mathcal{L}_2(z, z_t)$$
$$+ \int_{Q_0^T} e^{\tau\Phi} J_h \nabla z\cdot\nabla z + \frac{1}{2}\int_{\Sigma_0^T} e^{\tau\Phi}|\nabla z|^2 (h\cdot\nu)$$
$$- \frac{1}{2}\int_{Q_0^T} |\nabla z|^2 \left(div\{e^{\tau\Phi} h\} - \frac{d}{dt}(\Phi_t e^{\tau\Phi})\right) \quad (46)$$

where

$$\mathcal{L}_2(z, z_t) := \frac{1}{2}\int_{\Omega} e^{\tau\Phi}\Phi_t |\nabla z|^2|_0^T. \quad (47)$$

Finally, combining (39) and (46), we get

$$\int_{\Sigma_0^T} \frac{\partial z}{\partial\nu} e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t) + \frac{1}{2}\int_{\Sigma_0^T} e^{\tau\Phi}\left(|z_t|^2| - \nabla z|^2\right)(h\cdot\nu)$$
$$= \frac{1}{2}\int_{Q_0^T} \left(|z_t|^2 + |\nabla z|^2\right)\frac{d}{dt}(\Phi_t e^{\tau\Phi}) + \frac{1}{2}\int_{Q_0^T}\left(|z_t|^2 - |\nabla z|^2\right) div\{e^{\tau\Phi} h\}$$
$$+ \int_{Q_0^T} e^{\tau\Phi} J_h\nabla z\cdot\nabla z + \tau\int_{Q_0^T}(\nabla z\cdot h) e^{\tau\Phi}(h\cdot\nabla z - 2\Phi_t z_t)$$
$$+ \int_{Q_0^T} f_0^{uv} e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t) + \mathcal{L}_1 - \mathcal{L}_2. \quad (48)$$

Next, multiplying (38) by the "equipartition" multiplier $\Psi_2 := z\omega(x,t)$ and integrating by parts over $[0,T]\times\Omega$, where $\omega(x,t) := div\{e^{\tau\Phi}h\} - \frac{d}{dt}(\Phi_t e^{\tau\Phi})$, we obtain

$$\int_{Q_0^T} z_{tt} z\omega - \int_{Q_0^T} \Delta z\cdot z\omega + \int_{Q_0^T} f_0^{uv}\cdot z\omega = 0. \quad (49)$$

Calculating some terms of (49), we derive

$$\int_{Q_0^T} z_{tt} z\omega = \int_{\Omega} z_t z\omega|_0^T - \int_{Q_0^T}(|z_t|^2\omega + z_t z\omega_t), \quad (50)$$

$$\int_{Q_0^T} \Delta z\cdot z\omega = \int_{\Sigma_0^T} \frac{\partial z}{\partial\nu} z\omega - \int_{Q_0^T}(|\nabla z|^2\omega + z\cdot\nabla z\cdot\nabla\omega). \quad (51)$$

Substituting (50)-(51) into (49) and multiplying the resultant by 1/2, hence, (49) can be rewritten as

$$\begin{aligned}&\frac{1}{2}\int_{Q_0^T}\left(|z_t|^2-|\nabla z|^2\right)div(e^{\tau\Phi}h)\\&\quad=\frac{1}{2}\int_{Q_0^T}\left(|z_t|^2-|\nabla z|^2\right)\frac{d}{dt}(\Phi_te^{\tau\Phi})-\frac{1}{2}\int_{\Sigma_0^T}\frac{\partial z}{\partial\nu}z\omega\\&\qquad+\frac{1}{2}\int_{Q_0^T}z(\nabla z\cdot\nabla\omega-z_t\omega_t)+\frac{1}{2}\int_{Q_0^T}f_0^{uv}z\omega+\frac{1}{2}\int_\Omega z_tz\omega|_0^T.\end{aligned}\tag{52}$$

Moreover, considering the definitions of Ψ_1 and Φ in (33), we have

$$\begin{aligned}&\int_{Q_0^T}z_t^2\frac{d}{dt}(\Phi_te^{\tau\Phi})+\tau\int_{Q_0^T}(\nabla z\cdot h)e^{\tau\Phi}(h\cdot\nabla z-2\Phi_tz_t)\\&=\int_{Q_0^T}z_t^2\frac{d}{dt}(\Phi_te^{\tau\Phi})+\tau\int_{Q_0^T}(\nabla z\cdot h)(\Psi_1-e^{\tau\Phi}\Phi_tz_t)\\&=\tau\int_{Q_0^T}e^{\tau\Phi}|z_t|^2\Phi_t^2-2c\int_{Q_0^T}e^{\tau\Phi}|z_t|^2+\tau\int_{Q_0^T}e^{-\tau\Phi}(\Psi_1+e^{\tau\Phi}\Phi_tz_t)(\Psi_1-e^{\tau\Phi}\Phi_tz_t)\\&=\tau\int_{Q_0^T}e^{-\tau\Phi}\Psi_1^2-2c\int_{Q_0^T}e^{\tau\Phi}|z_t|^2.\end{aligned}\tag{53}$$

Now, combining (48), (52) and (53), we can notice that our present result is irrelevant to the value of ρ. We thus add the term $\int_{Q_0^T}\rho e^{\tau\Phi}|\nabla z|^2$ to the result and consider $J_h\geq\rho I$ (see Proposition 4.4), to derive

$$\begin{aligned}&\int_{\Sigma_0^T}\frac{\partial z}{\partial\nu}\Psi_1+\frac{1}{2}\int_{\Sigma_0^T}\frac{\partial z}{\partial\nu}z\omega+\frac{1}{2}\int_{\Sigma_0^T}e^{\tau\Phi}\left(|z_t|^2-|\nabla z|^2\right)(h\cdot\nu)\\&\quad=\int_{Q_0^T}e^{\tau\Phi}(J_h-\rho I_{\mathbb{R}^3})\nabla z\cdot\nabla z+\int_{Q_0^T}\rho e^{\tau\Phi}|\nabla z|^2-2c\int_{Q_0^T}e^{\tau\Phi}|z_t|^2\\&\qquad+\frac{1}{2}\int_{Q_0^T}z(\nabla z\cdot\nabla\omega-z_t\omega_t)+\tau\int_{Q_0^T}e^{-\tau\Phi}\Psi_1^2\\&\qquad+\int_{Q_0^T}f_0^{uv}(\frac{z\omega}{2}+\Psi_1)+\mathcal{L}_3(z,z_t)\end{aligned}\tag{54}$$

where

$$\mathcal{L}_3(z,z_t)=\mathcal{L}_1(z,z_t)-\mathcal{L}_2(z,z_t)+\frac{1}{2}\int_\Omega z_tz\omega|_0^T,\tag{55}$$

in which $\mathcal{L}_1(z,z_t)$ and $\mathcal{L}_2(z,z_t)$ are defined in (40) and (47), respectively.

Additionally, we apply the following equipartition relation to reconstruct the quadratic energy $\|\nabla z\|^2+\|z_t\|^2$ (see (54)),

$$\begin{aligned}&\int_{Q_0^T}\rho e^{\tau\Phi}|\nabla z|^2-2c\int_{Q_0^T}e^{\tau\Phi}|z_t|^2\\&\quad=(\frac{\rho}{2}+c)\int_{Q_0^T}e^{\tau\Phi}(|\nabla z|^2-|z_t|^2)+(\frac{\rho}{2}-c)\int_{Q_0^T}e^{\tau\Phi}(|\nabla z|^2+|z_t|^2).\end{aligned}\tag{56}$$

Moreover, replacing ω with $-e^{-\tau\Phi}$ in (49)-(51), we can obtain

$$\int_{Q_0^T}e^{\tau\Phi}(|\nabla z|^2-|z_t|^2)=\int_{\Sigma_0^T}\frac{\partial z}{\partial\nu}ze^{\tau\Phi}-\int_{Q_0^T}z\left(\nabla z\cdot\nabla e^{\tau\Phi}-z_t(\frac{d}{dt}e^{\tau\Phi})\right)$$

$$- \int_{Q_0^T} f_0^{uv} z e^{\tau\Phi} - \int_\Omega z_t e^{\tau\Phi} z|_0^T. \tag{57}$$

Therefore, substituting (56)-(57) into (54), then replacing the time interval $[0,T]$ with $[\eta, \bar{T}+\eta]$, and defining

$$(M\Sigma_\eta^{\bar{T}+\eta})_\tau := \int_{\Sigma_\eta^{\bar{T}+\eta}} \frac{\partial z}{\partial \nu}\Psi_1 + \int_{\Sigma_\eta^{\bar{T}+\eta}} \frac{\partial z}{\partial \nu} z \left(\frac{1}{2}\omega - (\frac{\rho}{2}+c)e^{\tau\Phi}\right) + \frac{1}{2}\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}\left(|z_t|^2 - |\nabla z|^2\right)(h\cdot\nu); \tag{58}$$

$$\Psi_1 := e^{\tau\Phi}(h\cdot\nabla z - \Phi_t z_t); \quad \omega(x,t) := div\{e^{\tau\Phi}h\} - \frac{d}{dt}(\Phi_t e^{\tau\Phi});$$

$$[\textit{Almost lower order}] := \int_{Q_\eta^{\bar{T}+\eta}} z z_t \frac{d}{dt}\left(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi}\right) - \int_{Q_\eta^{\bar{T}+\eta}} z(\nabla z\cdot z + f_0^{uv})\left(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi}\right);$$

and

$$(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau := \int_\Omega e^{\tau\Phi} z_t (h\nabla z - \frac{1}{2}\Phi_t z_t)|_\eta^{\bar{T}+\eta} - \frac{1}{2}\int_\Omega e^{\tau\Phi}\Phi_t|\nabla z|^2|_\eta^{\bar{T}+\eta} + \frac{1}{2}\int_\Omega z_t z\omega|_\eta^{\bar{T}+\eta} - (\frac{\rho}{2}+c)\int_\Omega z_t z e^{\tau\Phi}|_\eta^{\bar{T}+\eta}. \tag{59}$$

we obtain (35) and thus prove Lemma 4.7. □

5. Proof of Proposition 4.5.

In what follows, we shall use Carleman's estimates to absorb -via the large parameter τ - critical contribution of internal source along with tangential estimates to absorb by the damping tangential derivatives on the boundary resulting from applications of flow multipliers within Carleman's estimates. We begin with the potential energy first.

5.1. Potential energy and the sources.

In the next lemma, we shall rewrite Carleman's inequality in terms of the energy function and the sources acting only on velocity (part of kinetic energy). The latter will be later eliminated by a combination of sharp trace tangential estimate and "backwards trajectory" method. The just announced result reads as follows.

Lemma 5.1. *Under the assumptions of Lemma 4.7, for $t_0, t_1 > 0$ (see (34)) and $T := \bar{T}+2\eta$, we have*

$$\begin{aligned} &E(z(T)) + C_{\mathcal{B}}\int_0^T E(z) \\ &\le C_{T,\mathcal{B}}\int_0^T |<g^{uv}, z_t>| + C_{T,\mathcal{B}} l.o.t._{[0,T]}\{z\} \\ &\quad + C_1\left(\int_0^T + \int_{t+\eta}^T\right)\int_\Gamma f_1^{uv} z_t + C_2\left(\int_0^T + \int_{t+\eta}^T\right)\int_Q f_0^{uv} z_t \\ &\quad + C_3\left(\int_{\Sigma_0^T} f_1^{uv} z_t + \int_{Q_0^T} f_0^{uv} z_t\right) - C_4\int_0^T\left(\int_{\Sigma_0^\theta} f_1^{uv} z_t + \int_{Q_0^\theta} f_0^{uv} z_t\right) \end{aligned}$$

$$- C_5 \int_{t_0}^{t_1} \left(\int_{\Sigma_\theta^T} f_1^{uv} z_t + \int_{Q_\theta^T} f_0^{uv} z_t \right) \tag{60}$$

where $C_i > 0 (i = 1, \ldots, 5)$ *depend only on specific parameters and the diameter of* $\mathcal{B}$*.*

Proof. The main task for the proof is to eliminate critical terms entering the potential energy. This will be accomplished by using the Carleman's estimate in the previous lemma along with appropriate scaling of large parameter τ.

Considering $\frac{\rho}{2} - c > 0 (0 < c < \min\{1, \frac{\rho}{2}\})$, $e^{\tau\Phi} \geq 1$ on $[t_0, t_1]$ (see (34)) and J_h (see Proposition 4.4) is strictly positive definite, then for $C_\rho > 0$(small enough), we can get

$$C_\rho \int_{t_0}^{t_1} E(z(t)) + C_\rho \int_\eta^{\bar{T}+\eta} e^{\tau\Phi} E(z(t)) \leq \left(\int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi} (J_h - \rho I_{\mathbb{R}^3}) \nabla z \cdot \nabla z \right) + (\frac{\rho}{2} - c) \int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi} (|\nabla z|^2 + |z_t|^2). \tag{61}$$

Then, in order to further obtain the estimate (60) we want, we shall first perform each term of RHS in (35) (see Lemma 4.7). Estimates as follows ($\mathbf{(1)} - \mathbf{(4)}$):

(1) For $(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau$, from the conditions of Φ in (32) and (33), applying Schwartz's inequality to $(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau$, we can get

$$\begin{aligned}(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau =& \int_\Omega e^{\tau\Phi} z_t (h\nabla z - \frac{1}{2}\Phi_t z_t)|_\eta^{\bar{T}+\eta} - \frac{1}{2}\int_\Omega e^{\tau\Phi} \Phi_t |\nabla z|^2|_\eta^{\bar{T}+\eta} \\ &+ \frac{1}{2}\int_\Omega z_t z \omega|_\eta^{\bar{T}+\eta} - (\frac{\rho}{2} + c)\int_\Omega z_t z e^{\tau\Phi}|_\eta^{\bar{T}+\eta} \\ \leq& C e^{-\delta\tau} \left(E(z(\eta)) + E(z(\eta + \bar{T})) \right).\end{aligned} \tag{62}$$

Indeed, for every terms in $(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau$, we have the following estimates

$$\begin{aligned}&\left| \int_\Omega e^{\tau\Phi} z_t h \nabla z|_\eta^{\bar{T}+\eta} \right| \\ =& \left| \int_\Omega e^{\tau\Phi(\bar{T}+\eta)} z_t(\bar{T}+\eta) h\nabla z(\bar{T}+\eta) - \int_\Omega e^{\tau\Phi(\eta)} z_t(\eta) h \nabla z(\eta) \right| \\ \leq& C e^{-\delta\tau} \left(\|z_t(\bar{T}+\eta)\|^2 + \|\nabla z(\bar{T}+\eta)\|^2 + \|z_t(\eta)\|^2 + \|\nabla z(\eta)\|^2 \right).\end{aligned} \tag{63}$$

Applying the similar estimates as (63) to the orther terms in $(\mathcal{L}_\eta^{\bar{T}+\eta})_\tau$, we can thus conclude (62).

(2) Next, using Young's inequality, we derive

$$\begin{aligned}&- \tau \int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi} \Psi_1^2 - \int_{Q_\eta^{\bar{T}+\eta}} f_0^{uv} \Psi_1 \\ &\leq -\tau \int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi} \Psi_1^2 + \left| \int_{Q_\eta^{\bar{T}+\eta}} f_0^{uv} \Psi_1 \right| \\ &\leq -\tau \int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi} \Psi_1^2 + \varepsilon_0 \int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi} (f_0^{uv})^2 + \varepsilon_0^{-1} \int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi} \Psi_1^2.\end{aligned} \tag{64}$$

Choosing $\varepsilon_0^{-1} < \tau$ in the above estimate (64), (*note the role of large parameter τ which allows to think of ε_0 as a small quantity*), we derive

$$-\tau\int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi}\Psi_1^2 - \int_{Q_\eta^{\bar{T}+\eta}} f_0^{uv}\Psi_1 \le \varepsilon_0\int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi}(f_0^{uv})^2. \tag{65}$$

From (2), we obtain

$$e^{\tau\Phi/2}|f_0^{uv}| \le e^{\tau\Phi/2}\int_0^1 |f_0'(\lambda u+(1-\lambda)v)|d\lambda|u-v| \le C(1+|u|+|v|)e^{\tau\Phi/2}|z|. \tag{66}$$

Hence, by Hölder's inequality and the embedding theorem $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and $H^1(\Omega)\hookrightarrow L^4(\Gamma)$, we can further get

$$\begin{aligned}\|e^{\tau\Phi}(f_0^{uv})^2\| &\le C\|1+|u|+|v|\|_{L^3(\Omega)}^2\|e^{\tau\Phi}z^2\|_{L^3(\Omega)} \le C_{\mathcal{B}}\|e^{\tau\Phi/2}z\|_{1,\Omega}^2\\ &\le C_{\mathcal{B}}\|e^{\tau\Phi/2}\nabla z\|^2 + C_{\mathcal{B},T,\tau}\|z\|^2,\end{aligned} \tag{67}$$

and similarly we can also have

$$\begin{aligned}\|e^{\tau\Phi}(f_1^{uv})^2\|_{L^2(\Gamma)} &\le C\|1+|u|+|v|\|_{L^4(\Gamma)}^2\|e^{\tau\Phi}z^2\|_{L^2(\Gamma)} \le C_{\mathcal{B}}\|e^{\tau\Phi/2}z\|_{1,\Omega}^2\\ &\le C_{\mathcal{B}}\|e^{\tau\Phi/2}\nabla z\|^2 + C_{\mathcal{B},T,\tau}\|z\|^2,\end{aligned} \tag{68}$$

which plays an important role in the below estimates about $(M\Sigma_\eta^{\bar{T}+\eta})_\tau$.

Then, based on the estimetes (66) and (67), (65) can thus be rewritten as

$$\begin{aligned}&-\tau\int_{Q_\eta^{\bar{T}+\eta}} e^{-\tau\Phi}\Psi_1^2 - \int_{Q_\eta^{\bar{T}+\eta}} f_0^{uv}\Psi_1\\ &\le \varepsilon_0\int_{Q_\eta^{\bar{T}+\eta}} e^{\tau\Phi}(f_0^{uv})^2\\ &\le \varepsilon_0 C_{\mathcal{B}}\int_\eta^{\bar{T}+\eta}\|e^{\tau\Phi/2}\nabla z\|^2 + C_{\mathcal{B},T,\tau}\int_\eta^{\bar{T}+\eta}\|z\|^2\\ &\le \tilde{\varepsilon}_0\int_\eta^{\bar{T}+\eta} e^{\tau\Phi}E(z) + C_{\tau,\eta,\bar{T},\varepsilon_0,\mathcal{B}}\, l.o.t._{[0,\bar{T}+2\eta]}\{z\}.\end{aligned} \tag{69}$$

We note that $\varepsilon_0, \tilde{\varepsilon}_0$ can be taught as an arbitrary small quantity due to the choice of large parameter τ. Particularly, we can take $0 < \tilde{\varepsilon}_0 < C_\rho/2$.

(3) Now, we deal with each term of [*Almost lower order*] in (35) (see Lemma 4.7). Firstly, applying integration by parts, Young's inequality and the embedding theorem $H^s(\Omega)\hookrightarrow L^2(\Omega)(0\le s<1)$, we have

$$\begin{aligned}&\int_{Q_\eta^{\bar{T}+\eta}} zz_t\frac{d}{dt}\left(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi}\right)\\ &= \int_\Omega \frac{1}{2}|z|^2(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi})|_\eta^{\bar{T}+\eta} - \int_{Q_\eta^{\bar{T}+\eta}} zz_t(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi})\\ &\le \varepsilon_{0_1}\int_\eta^{\bar{T}+\eta} E(z) + C_{\tau,\eta,\bar{T},\varepsilon_{0_1}}\, l.o.t._{[0,\bar{T}+2\eta]}\{z\}\end{aligned} \tag{70}$$

where ε_{0_1} can be taken small enough.

Then, we emply the local Lipschitz property of f_0 in (2), (67), Young's inequality and the embedding theorem $H^s(\Omega) \hookrightarrow L^2(\Omega)(0 \le s < 1)$ to obtain

$$\begin{aligned}
&\int_{Q_\eta^{\bar{T}+\eta}} z f_0^{uv}(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi}) \\
&= \int_{Q_\eta^{\bar{T}+\eta}} \frac{\omega}{2} z f_0^{uv} - \int_{Q_\eta^{\bar{T}+\eta}} z(\frac{\rho}{2}+c)e^{\tau\Phi} f_0^{uv} \\
&\le \varepsilon_{0_2} \int_\eta^{\bar{T}+\eta} E(z) + C_{\tau,\eta,\bar{T},\varepsilon_{0_1}} l.o.t._{[0,\bar{T}+2\eta]}\{z\} + \left| \int_{Q_\eta^{\bar{T}+\eta}} z(\frac{\rho}{2}+c)e^{\tau\Phi} f_0^{uv} \right| \\
&\le \varepsilon_{0_2} \int_\eta^{\bar{T}+\eta} E(z) + C_{\tau,\eta,\bar{T},\varepsilon_{0_1},\tilde{\varepsilon}_0} l.o.t._{[0,\bar{T}+2\eta]}\{z\} + \tilde{\varepsilon}_0 \int_\eta^{\bar{T}} e^{\tau\Phi} E(z)
\end{aligned} \tag{71}$$

where we can take ε_{0_2} small enough and choose $0 < \tilde{\varepsilon}_0 < C_\rho/2$.

As for the orther terms, using Young's inequality and the embedding theorem, we derive

$$\int_{Q_\eta^{\bar{T}+\eta}} z\nabla z \cdot z(\frac{\omega}{2} - (\frac{\rho}{2}+c)e^{\tau\Phi}) \le \varepsilon_{0_3} \int_\eta^{\bar{T}+\eta} E(z) + C_{\tau,\eta,\bar{T},\varepsilon_{0_3}} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \tag{72}$$

where ε_{0_3} is sufficiently small.

Finally, combining (70)-(72), and taking $\varepsilon_0 = \varepsilon_{0_1} + \varepsilon_{0_2} + \varepsilon_{0_3}$, which can be choosen small enough, we thus get

$$\begin{aligned}
[\textit{Almost lower order}] \le \tilde{\varepsilon}_0 \int_0^{\bar{T}+2\eta} e^{\tau\Phi} E(z) + \varepsilon_0 \int_0^{\bar{T}+2\eta} E(z) \\
+ C_{\tau,T,\varepsilon_0,\tilde{\varepsilon}_0} l.o.t._{[0,\bar{T}+2\eta]}\{z\}
\end{aligned} \tag{73}$$

where $0 < \tilde{\varepsilon}_0 < C_\rho/2$.

(4) We next estimate each term of $(M\Sigma_\eta^{\bar{T}+\eta})_\tau$ in (35), respectively. It follows from the boundary condition, the estimate inequality (68), Young's inequality and sharp trace estimate in Lemma 4.2 that

$$\begin{aligned}
&\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi} \frac{\partial z}{\partial \nu} h \cdot \nabla z \\
&= \int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi} \left((\frac{\partial z}{\partial \nu})^2 (h\cdot\nu) - f_1^{uv}(h\cdot\nabla_\tau z) - g^{uv}(h\cdot\nabla_\tau z) \right) \\
&\le \tilde{C}_{1,\tau,h} \int_{\Sigma_\eta^{\bar{T}+\eta}} (\frac{\partial z}{\partial \nu})^2 + \varepsilon_0 \int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}|f_1^{uv}|^2 + C\varepsilon_0^{-1} \int_{\Sigma_\eta^{\bar{T}+\eta}} |\nabla_\tau z|^2 \\
&\quad + \tilde{C}_{2,\tau,h} \int_{\Sigma_\eta^{\bar{T}+\eta}} |g^{uv}|^2 + \tilde{C}_{3,\tau,h} \int_{\Sigma_\eta^{\bar{T}+\eta}} |\nabla_\tau z|^2 \\
&\le \tilde{C}_{\tau,\eta,h,\varepsilon_0} \int_0^{\bar{T}+2\eta} \left(\|z_t\|^2_{L^2(\Gamma)} + \|g^{uv}\|^2_{L^2(\Gamma)} \right) + \tilde{C}_{t,\eta,\varepsilon_0,\varepsilon} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \\
&\quad + (\varepsilon_0 C_{\mathcal{B}} + \varepsilon_0^{-1}\varepsilon \tilde{C}_{\mathcal{B},\tau,h}) \int_0^{\bar{T}+2\eta} E(z) \\
&\le \tilde{C}_{\tau,\eta,h,\varepsilon_0} \int_0^{\bar{T}+2\eta} \left(\|z_t\|^2_{L^2(\Gamma)} + \|g^{uv}\|^2_{L^2(\Gamma)} \right) + \tilde{C}_{t,\eta,\varepsilon_0,\varepsilon} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \\
&\quad + \varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z)
\end{aligned} \tag{74}$$

where $\varepsilon, \varepsilon_0 > 0$ satisfy $\varepsilon = \varepsilon_0^2$. We apply Lemma 4.2 with $\varepsilon > 0$ and choose $\varepsilon_0 = \sqrt{\varepsilon}$ which can be arbitrarily small due to the parameter τ.

Then, considering the definition of Φ in (32), we have

$$\begin{aligned}\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}\frac{\partial z}{\partial\nu}\Phi_t z_t &= -\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi} f_1^{uv}\Phi_t z_t - \int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi} g^{uv}\Phi_t z_t \\ &\le C_{\mathcal{B}}\int_{\Sigma_\eta^{\bar{T}+\eta}} f_1^{uv} z_t + C_{\tau,\eta,\bar{T}}\int_\eta^{\bar{T}+\eta}\|z_t\|^2_{L^2(\Gamma)} \\ &\le C_{\mathcal{B}}\int_{\Sigma_0^{\bar{T}+2\eta}} f_1^{uv} z_t + C_{\tau,\eta,\bar{T},\alpha}\int_0^{\bar{T}+2\eta}|g^{uv}z_t| \qquad (75)\end{aligned}$$

where we have used the following estimate

$$\int_0^{\bar{T}+2\eta}\|z_t\|^2_{L^2(\Gamma)} \le \alpha^{-1}\int_0^{\bar{T}+2\eta} < |g^{uv}, z_t> |, \qquad (76)$$

under the assumption of g in (4).

Using the boundary condition and Lemma 4.2 again and (76), we derive the “equipartition” of the energy on the boundary,

$$\begin{aligned}&\int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}(|z_t|^2 - |\nabla z|^2)(h\cdot\nu) \\ &= \int_{\Sigma_\eta^{\bar{T}+\eta}} e^{\tau\Phi}\left(|z_t|^2 - (\frac{\partial z}{\partial\nu})^2 - |\nabla_\tau z|^2\right)(h\cdot\nu) \\ &\le C_{\tau,\eta,\bar{T}}\int_0^{\bar{T}+2\eta}|g^{uv}z_t|^2 + \varepsilon_0\tilde{C}_{\mathcal{B}}\int_0^{\bar{T}+2\eta}E(z) \\ &\quad + \tilde{C}_{t,\eta,\varepsilon_0} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \qquad (77)\end{aligned}$$

where $\varepsilon_0 > 0$ depends on $c, \bar{T}, \eta$, where ε_0 can be taken arbitrarily small.

Finally, applying the boundary condition and Young's inequality, we obtain

$$\begin{aligned}&\int_{\Sigma_\eta^{\bar{T}+\eta}}\frac{\partial z}{\partial\nu}z\left(\frac{\omega}{2} - (\rho/2+c)e^{\tau\Phi}\right) \\ &= \frac{1}{2}\int_{\Sigma_\eta^{\bar{T}+\eta}}\frac{\partial z}{\partial\nu}z\left(\frac{\omega}{2} - (\rho/2+c)e^{\tau\Phi}\right) + \frac{1}{2}\int_{\Sigma_\eta^{\bar{T}+\eta}}(-f_1^{uv} - g^{uv})z\left(\frac{\omega}{2} - (\rho/2+c)e^{\tau\Phi}\right) \\ &\le \varepsilon_1\int_\eta^{\bar{T}+\eta}\int_\Gamma|\frac{\partial z}{\partial\nu}|^2 \\ &\quad + \varepsilon_2\int_\eta^{\bar{T}+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)} + C_{\tau,\eta,\varepsilon_1,\varepsilon_2}\int_0^{\bar{T}+2\eta}(\|g^{uv}\|^2_{L^2(\Gamma)} + \|z\|^2_{L^2(\Gamma)}) \qquad (78)\end{aligned}$$

where $\varepsilon_i (i = 1, 2)$ can be small enough.

Next, we manage to deal with each term of RHS in (78) to get our desired estimates. Applying (17) (see, Lemma 4.2) and the similar estimate as (24), we derive

$$\begin{aligned}&\varepsilon_1\int_\eta^{\bar{T}+\eta}\int_\Gamma|\frac{\partial z}{\partial\nu}|^2 + \varepsilon_2\int_\eta^{\bar{T}+\eta}\|f_1^{uv}\|^2_{L^2(\Gamma)} \\ &\quad \le \varepsilon\int_\eta^{\bar{T}+\eta}\int_\Gamma(|\frac{\partial z}{\partial\nu}|^2 + |\nabla_\tau z|^2) + C_{\tau,\varepsilon,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\}\end{aligned}$$

$$\leq \varepsilon_0 C_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) + C_\eta \int_0^{\bar{T}+2\eta} (\|z_t\|^2_{L^2(\Gamma)} + \|g^{uv}\|^2_{L^2(\Gamma)}) \\ + C_{\tau,\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \tag{79}$$

where ε depends on $\varepsilon_i (i = 1, 2)$, and ε_0 is sufficiently small.

Additionally, using assumptions for g in $(iii) - (iv)$, we get

$$\int_0^{\bar{T}+2\eta} \|g^{uv}\|^2_{L^2(\Gamma)} \leq m_2 \int_0^{\bar{T}+2\eta} |g^{uv} z_t| \\ \int_0^{\bar{T}+2\eta} \|z_t\|^2_{L^2(\Gamma)} \leq \alpha^{-1} \int_0^{\bar{T}+2\eta} |g^{uv} z_t|, \tag{80}$$

and by the interpolation, we have

$$C_{\tau,\eta,\varepsilon_1,\varepsilon_2} \|z\|_{L^2(\Gamma)} \leq C_{\tau,\eta,\varepsilon_1,\varepsilon_2} \|z\|_{H^{1/2+\varsigma}(\Omega)} \leq C_{\varepsilon_0} \|z\| + \varepsilon_0 \|\nabla z\|, \tag{81}$$

for some $\varepsilon_0 > 0$, which gives

$$C_{\tau,\eta,\varepsilon_1,\varepsilon_2} \int_0^{\bar{T}+2\eta} \|z\|^2_{L^2(\Gamma)} \leq \varepsilon_0 \int_0^{\bar{T}+2\eta} E(z(\tau)) + C_{\tau,\bar{T},\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\}. \tag{82}$$

Therefore, now we substitute (79)-(82) into (78) to obtain the following our desired estimate

$$\int_{\Sigma_\eta^{\bar{T}+\eta}} \frac{\partial z}{\partial \nu} z \left(\frac{\omega}{2} - (\rho/2 + c)e^{\tau\Phi}\right) \leq \varepsilon_0 C_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) + C_{\tau,\bar{T},\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \\ + C_{\tau,\bar{T}} \int_0^{\bar{T}+2\eta} |g^{uv} z_t| \tag{83}$$

where $\varepsilon_0 > 0$ depends on η, τ, h and can be small enough.

Then, combining (74)-(77) and (83), we can conclude

$$(M\Sigma_\eta^{\bar{T}+\eta})_\tau \leq C_{\tau,\bar{T},\eta} \int_0^{\bar{T}+2\eta} |g^{uv} z_t| + \tilde{C}_{\mathcal{B}} \int_{\Sigma_0^{\bar{T}+2\eta}} f_1^{uv} z_t \\ + \varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) + C_{\tau,\bar{T},\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \tag{84}$$

where $\varepsilon_0 > 0$ depends on $\eta, \tau, h, c, \bar{T}$ and is sufficiently small.

Considering the above **(1)** $-$ **(4)**, we thus complete the estimates of every term of the RHS in (35).

Now combining (35)(see, Lemma 4.7) and (61), and applying the above estimates (62), (69)-(73) and (84) (in **(1)** $-$ **(4)**), to (61), we thus get

$$C_\rho \int_{t_0}^{t_1} E(z) + (C_\rho - \tilde{\varepsilon}_0) \int_\eta^{\bar{T}+\eta} e^{\tau\Phi} E(z) \\ \leq C_{\tau,\bar{T},\eta} \int_0^{\bar{T}+2\eta} |g^{uv} z_t| + \tilde{C}_{\mathcal{B}} \int_{\Sigma_0^{\bar{T}+2\eta}} f_1^{uv} z_t + \varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) \\ + Ce^{-\delta\tau} \left(E(z(\eta)) + E(z(\eta + \bar{T}))\right) + C_{\tau,\bar{T},\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \tag{85}$$

where $\tilde{\varepsilon}_0$ can be choosen as $\tilde{\varepsilon}_0 < C_\rho$, (*see* (69) *and* (73) *and* $\tilde{\varepsilon}_0$ *is only from* (69) *and* (73)).

Next, we try to eliminate the term $C_\rho \int_{t_0}^{t_1} E(z)$ in (85). To this end, we recall (8), from (8), we have for $0 \le s \le \bar{T} + 2\eta$,

$$E(z(\bar{T}+2\eta)) + \int_{\Sigma_s^{\bar{T}+2\eta}} g^{uv} z_t = E(z(s)) - \int_{\Sigma_s^{\bar{T}+2\eta}} f_1^{uv} z_t - \int_{Q_s^{\bar{T}+2\eta}} f_0^{uv} z_t + \frac{\lambda}{2}\int_\Omega [z^2(\bar{T}+2\eta) - z^2(s)], \tag{86}$$

Then, intergrating (86) over $[t_0, t_1]$, multiplying the result by C_ρ and using Assumption 1 imposed on g , we can obtain for $0 \le \theta \le \bar{T} + 2\eta$,

$$C_\rho(t_1-t_0)E(z(\bar{T}+2\eta)) \le C_\rho \int_{t_0}^{t_1} E(z) - C_\rho \int_{t_0}^{t_1}\int_{\Sigma_\theta^{\bar{T}+2\eta}} f_1^{uv} z_t - C_\rho \int_{t_0}^{t_1}\int_{\Sigma_\theta^{\bar{T}+2\eta}} f_0^{uv} z_t + C_{\lambda,\bar{T},\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\}. \tag{87}$$

Finally, combining (85) and (87) and considering $\tilde{\varepsilon}_0 < C_\rho$ in (85) (see (69) and (73)), then we derive

$$\begin{aligned} &C_\rho(t_1-t_0)E(z(\bar{T}+2\eta)) + \varepsilon_0 \tilde{C}_{\mathcal{B}} \int_\eta^{\bar{T}+\eta} E(z) \\ &\le 2\varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) + Ce^{-\delta\tau}\left(E(z(\eta)) + E(z(\eta+\bar{T}))\right) \\ &\quad + C_{\tau,\bar{T},\eta}\int_0^{\bar{T}+2\eta} |g^{uv} z_t| + \tilde{C}_{\mathcal{B}} \int_{\Sigma_0^{\bar{T}+2\eta}} f_1^{uv} z_t - C_\rho \int_{t_0}^{t_1}\int_{\Sigma_\theta^{\bar{T}+2\eta}} f_1^{uv} z_t \\ &\quad - C_\rho \int_{t_0}^{t_1}\int_{\Sigma_\theta^{\bar{T}+2\eta}} f_0^{uv} z_t + C_{\bar{T},\lambda,\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\}. \end{aligned} \tag{88}$$

We now need to deal with the first two terms in RHS of (88), respectively. We first use (8) again and integrate (8) over $[0, \bar{T}+2\eta]$, then multiply the result by $2\varepsilon_0 \tilde{C}_{\mathcal{B}}$ to get the following for $0 \le \theta \le \bar{T} + 2\eta$,

$$2\varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) \le 2\varepsilon_0 \tilde{C}_{\mathcal{B}}(\bar{T}+2\eta)E(z(0)) - 2\varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta}\int_{\Sigma_0^\theta} f_1^{uv} z_t - 2\varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta}\int_{Q_0^\theta} f_0^{uv} z_t + C_{\bar{T},\lambda,\varepsilon_0,\eta} l.o.t._{[0,\bar{T}+2\eta]}\{z\}. \tag{89}$$

In addition, from (8), we can obtain

$$E(z(s)) \le E(z(\bar{T}+2\eta)) + \int_{\Sigma_s^{\bar{T}+2\eta}} g^{uv} z_t|_\Gamma + \int_{\Sigma_s^{\bar{T}+2\eta}} f_1^{uv} z_t|_\Gamma + \int_{Q_s^{\bar{T}+2\eta}} f_0^{uv} z_t + C_{\lambda,\eta,\bar{T}} l.o.t._{[0,\bar{T}+2\eta]}\{z\}, \quad 0 \le s \le \bar{T}+2\eta. \tag{90}$$

Replacing s with η and $t+\eta$ in (90), respectively, we also have

$$E(z(\eta)) \le E(z(\bar{T}+2\eta)) + \int_{\Sigma_0^{\bar{T}+2\eta}} g^{uv} z_t + \int_{\Sigma_\eta^{\bar{T}+2\eta}} f_1^{uv} z_t + \int_{Q_\eta^{\bar{T}+2\eta}} f_0^{uv} z_t + C_{\lambda,\eta,\bar{T}} l.o.t._{[0,\bar{T}+2\eta]}\{z\},$$

$$E(z(t+\eta)) \leq E(z(\bar{T}+2\eta)) + \int_{\Sigma_0^{\bar{T}+2\eta}} g^{uv} z_t + \int_{\Sigma_{t+\eta}^{\bar{T}+2\eta}} f_1^{uv} z_t + \int_{Q_{t+\eta}^{\bar{T}+2\eta}} f_0^{uv} z_t + C_{\lambda,\eta,\bar{T}} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \tag{91}$$

where $0 \leq \eta, \eta + t \leq \bar{T} + 2\eta$.

Therefore, taking $s = 0$ in (90) and multiplying the result by $2\varepsilon_0 \tilde{C}_{\mathcal{B}}(\bar{T}+2\eta)$, then adding the final result to (89), we can get

$$\begin{aligned} &2\varepsilon_0 \tilde{C}_{\mathcal{B}} \int_0^{\bar{T}+2\eta} E(z) \\ \leq &2\varepsilon_0(\bar{T}+2\eta)\tilde{C}_{\mathcal{B}} E(z(\bar{T}+2\eta)) + C_{\mathcal{B},h,\lambda,\eta,\bar{T}} l.o.t._{[0,\bar{T}+2\eta]}\{z\} \\ &+ 2\varepsilon_0(\bar{T}+2\eta)\tilde{C}_{\mathcal{B}} \left(\int_{\Sigma_0^{\bar{T}+2\eta}} g^{uv} z_t + \int_{\Sigma_0^{\bar{T}+2\eta}} f_1^{uv} z_t + \int_{Q_0^{\bar{T}+2\eta}} f_0^{uv} z_t \right) \\ &- 2\varepsilon_0 \tilde{C}_{\mathcal{B}} \left(\int_0^{\bar{T}+2\eta} \int_{\Sigma_0^\theta} f_1^{uv} z_t + \int_0^{\bar{T}+2\eta} \int_{Q_0^\theta} f_0^{uv} z_t \right) \end{aligned} \tag{92}$$

and from (91), we derive

$$\begin{aligned} E(z(\eta)) + E(z(t+\eta)) \leq &2E(z(\bar{T}+2\eta)) + \left(\int_0^{\bar{T}+2\eta} + \int_{t+\eta}^{\bar{T}+2\eta} \right) \int_\Gamma f_1^{uv} z_t \\ &+ 2\int_{\Sigma_0^{\bar{T}+2\eta}} g^{uv} z_t + \left(\int_0^{\bar{T}+2\eta} + \int_{t+\eta}^{\bar{T}+2\eta} \right) \int_\Omega f_0^{uv} z_t \\ &+ C_{\lambda,\eta,\bar{T}} l.o.t._{[0,\bar{T}+2\eta]}\{z\}. \end{aligned} \tag{93}$$

Now, we can substitute (92)-(93) into (88), take $2(\varepsilon_0 \tilde{C}_{\mathcal{B}}(\bar{T}+2\eta) + Ce^{-\delta\tau}) << C_\rho(t_1 - t_0)$, and $T = \bar{T} + 2\eta$, then the final result can imply (60). We thus complete the proof of Lemma 5.1. □

Proof of Proposition 4.5 follows now from Lemma 5.1 after adjusting the constants.

5.2. **Kinetic energy and the sources.**

5.2.1. *Estimates for F terms in Proposition 4.5.* This is the final key part of the proof which together with Proposition 4.5 and Lemma 5.1, will lead to the quasi-stability of the system $(\mathcal{H}, S(t))$ for the problem (1) -as stated in Proposition 4.1. The latter will imply finite dimension and regularity of the global attractor $\mathcal{A}$. How to go about it? From Lemma 5.1, we see that the kinetic contribution of **critical sources** needs to be eliminated. This step requires in depth analysis of "how does the damping propagates in order to absorb the critical sources". It turns out that the situation is *very different for the cases of internal source and boundary source.* Let's explain-first at the quailitative level. Having criticality in the interior and dissipation localised only on the boundary-the issue is of course that of propagation. While propagation via Carleman's estimate was helpful to control criticality of the energy at the potential level [by playing with large parameter τ], this "trick" will have no effect on kinetic part of the energy. The control of $\int_0^T (f_0^{uv}, z_t)$ which is critical and non-structured term -due to the fact that the difference of two solutions is considered- can not be achieved by the usual cancellations via energy methods. So, the idea is that of "backward trajectories" which exploits two facts: (i) we already work on the attractor and (ii) the system is gradient. Thus estimating the energy of difference of two trajectories near equilibrium [where $u_t = 0$ *in* Ω] provides badly needed "epsilons" in the estimates. This will be executed later via the a priori estimate in Lemma 5.2 along with the "backward trajectory" method introduced originally by Zelik in [34] . The idea is that the velocities of the trajectories are "small" near equilibria points (the system is gradient). So for time $\to -\infty$, one obtains small quantities generated by external force f_0. This will produce the needed regularity that will be propagated forward on the attractor. Topological characterization of the generated "smoothness" will follow from the higher order energy estimates. Clearly, this method will fail when dealing with "boundary critical" sources. The reason is that "smallness" of u_t in $L_2(\Omega)$ topology [near the equilibrium] by no means implies smallness of $u_t|_{\Gamma_1}$. Therefore, being near equilibrium can not be taken advantage of when dealing with the boundary source.

The question thus arises how to handle the "boundary criticality"? Indeed, one needs to devise a different mechanism which will take an advantage of the fact that the dissipation and criticality are "collocated". This idea will lead to a different estimate- with a first preliminary step in Lemma 5.3. The results of Lemma 5.2 and Lemma 5.3, when combined yield the estimate in Lemma 5.5 and eventually *"quasi-stability"* estimate on negative scale (103).

Lemma 5.2. *Assume that hypotheses of Theorem 2.4 have been satisfied. Then, for a given two trajectories* $(u(t), u_t(t))$ *and* $(v(t), v_t(t))$ *through the attractor* $\mathcal{A}$, *and any* $\epsilon > 0$, $s < t$, *the difference of trajectories* $z(t) = u(t) - v(t)$, *satisfies*

$$\left|\int_s^t (f_0^{uv}, z_t)\right| \le \epsilon\,(E(z(s)) + E(z(t))) + C_{\mathcal{A},\epsilon} \sup_{\theta\in[s,t]} \|z(\theta)\|^2 + C_{\mathcal{A}} \int_s^t (\|u_t\| + \|v_t\|)\|\nabla z\|^2 \tag{94}$$

where $C_{\mathcal{A},\epsilon} > 0$ *is independent of* u *and* v.

Proof. We begin by applying the Newton-Leibniz formula to get

$$\begin{aligned}\int_s^t (f_0^{uv}, z_t) &= \int_s^t \int_\Omega \int_0^1 f_0'(\varpi z + v) z z_t d\varpi dQ_s^t \\ &= \frac{1}{2}\int_s^t \int_\Omega \int_0^1 f_0'(\varpi z + v)\frac{d}{dt} z^2 d\varpi dQ_s^t \\ &= \frac{1}{2}\int_\Omega \int_0^1 f_0'(\varpi z + v) z^2|_s^t d\varpi d\Omega \\ &\quad - \frac{1}{2}\int_s^t \int_\Omega \int_0^1 f_0''(\varpi z + v)(\varpi z_t + v_t) z^2 d\varpi d\Omega.\end{aligned} \tag{95}$$

Using (2), the embedding theorem and Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega \int_0^1 f_0'(\varpi z + v) z^2 d\varpi d\Omega &\le C\int_\Omega (1 + |u|^2 + |v|^2) z^2 d\Omega \\ &\le C\|1 + |u| + |v|\|^2_{L^6(\Omega)} \|z\|^2_{L^3(\Omega)} \\ &\le C_{\mathcal{A},\epsilon}\|z(t)\|^2 + \epsilon\|\nabla z(t)\|^2\end{aligned} \tag{96}$$

where $\epsilon > 0$ is arbitrary. Applying Hölder's inequality twice to the other integral in (95), we derive

$$\begin{aligned}\int_\Omega \int_0^1 f_0''(\varpi z + v)(\varpi z_t + v_t) z^2 d\varpi d\Omega &\le C_{\mathcal{A}} \left(\int_\Omega (|u_t| + |v_t|)^{6/5} z^{12/5} d\Omega \right)^{5/6} \\ &\le C_{\mathcal{A}} (\|u_t\| + \|v_t\|) \|z\|^2_{L^6(\Omega)} \\ &\le C_{\mathcal{A}} (\|u_t\| + \|v_t\|) \|\nabla z(t)\|^2. \end{aligned} \tag{97}$$

Combining (96)-(97) and (95), we can get (94). □

Lemma 5.3. *Under the assumptions stated above in Lemma 5.2, the following estimate holds with $s < t$,*

$$\begin{aligned}\left| \int_s^t \int_{\Gamma_1} f_1^{u,v} z_t|_{\Gamma_1} dx dr \right| \le & \epsilon \left(E(z(s)) + E(z(t)) \right) + C_t sup_{\theta \in [s,t]} ||z(\theta)||^2 \\ & \epsilon \int_s^t ||z||^2_{H^1(\Omega)} + C_{\epsilon, \mathcal{A}} \int_s^t K(r) ||z||^2_{H^1(\Omega)} dr \end{aligned}$$

where $K(r) =< g(u_t), u_t > + < g(v_t), v_t > \in L_1(R)$, and $\epsilon > 0$ is arbitrary.

Proof. We apply the Newton-Leibniz formula, as above, to get

$$\begin{aligned}\int_s^t < f_1^{uv}, z_t > &= \int_s^t \int_\Gamma \int_0^1 f_1'(\varpi z + v) z z_t d\varpi d\Sigma_s^t \\ &= \frac{1}{2} \int_s^t \int_\Gamma \int_0^1 f_1'(\varpi z + v) \frac{d}{dt} z^2 d\varpi d\Sigma_s^t \\ &= \frac{1}{2} \int_\Gamma \int_0^1 f_1'(\varpi z + v) z^2|_s^t d\varpi d\Gamma \\ &\quad - \frac{1}{2} \int_s^t \int_\Gamma \int_0^1 f''(\varpi z + v)(\varpi z_t + v_t) z^2 d\varpi d\Gamma. \end{aligned} \tag{98}$$

Using (3), the embedding theorem and Hölder's inequality and interpolating the lower order terms, we obtain

$$\begin{aligned}\int_\Gamma \int_0^1 f_1'(\varpi z + v) z^2 d\varpi d\Gamma &\le C \int_\Gamma (1 + |u|^2 + |v|^2) z^2 d\Gamma \\ &\le C \|1 + |u| + |v|\|^2_{L^1(\Gamma)} \|z\|^2_{L^1(\Gamma)} \\ &\le C_{\mathcal{A}} \|z\|^2_{L^2(\Gamma)} \le C_{\mathcal{A}} \|z\|^2_{H^\eta(\Omega)} \\ &\le C_{\mathcal{A},\epsilon} \|z\|^2 + \epsilon \|\nabla z\| \end{aligned} \tag{99}$$

where $\epsilon > 0$ and $0 < \eta < 1$.

From assumption (iii) about g, Hölder's inequality, Young's inequality and the embedding $H^1(\Omega) \hookrightarrow L^4(\Gamma)$, we have for every $\epsilon > 0$.

$$\begin{aligned}\int_\Gamma \int_0^1 f_1''(\varpi z + v)(\varpi z_t + v_t) z^2 d\varpi d\Gamma \quad &\le C_{\mathcal{A}} \left(\int_\Gamma (|u_t| + |v_t|)^2 z^4 d\Gamma \right)^{1/2} \\ &\le C_{\mathcal{A}} (\|u_t\|_{L^2(\Gamma)} + \|v_t\|_{L^2(\Gamma)}) \|z\|^2_{L^4(\Gamma)} \\ &\le C_{\mathcal{A},\epsilon} (\|u_t\|^2_{L^2(\Gamma)} + \|v_t\|^2_{L^2(\Gamma)}) \|z\|^2_{L^4(\Gamma)} + \epsilon \|z\|^2_{L^4(\Gamma)} \\ &\le C_{\mathcal{A},\epsilon} (< g(u_t), u_t > + < g(v_t), v_t >) \|z(t)\|^2_{H^1(\Omega)} + \epsilon \|z(t)\|^2_{H^1(\Omega)} \end{aligned} \tag{100}$$

Substituting (99) and (100) into and (98), we can obtain

$$\left|\int_s^t \int_{\Gamma_1} f_1^{u,v} z_t|_{\Gamma_1} dxdr\right| \leq \epsilon\left(E(z(s)) + E(z(t))\right) + C_t sup_{\theta\in[s,t]}||z(\theta)||^2 + \epsilon\int_s^t \|z\|^2_{H^1(\Omega)} + C_{\epsilon,\mathcal{A}}\int_s^t K(r)\|z\|^2_{H^1(\Omega)}dr. \quad (101)$$

where $K(r) =< g(u_t), u_t > + < g(v_t), v_t >\in L_1(R)$. The proof of Lemma 5.3 is thus completed. □

Remark 5.4. It is important that we have worked on the attractor already. The critical term in Lemma 5.2 is accompanied by $||u_t|| + ||v_t||$ -which is small near the equilibrium. This suggest "backward trajectory method". The critical term in Lemma 5.3 has the term $< g(u_t), u_t > + < g(v_t), v_t >$ which has finite $L_1(0, T)$ norm due to the boundary dissipation. Thus, we have different mechanisms for controlling criticality in internal and boundary sources. This will become more clear later in the development.

In fact, the result of Lemma 5.2 is strengthen when considering time scale near $-\infty$. This means taking $s, t < T^{u,v}$, where $T^{u,v}$ is close to $-\infty$. This result is formulated below in the Lemma 5.5, which is a revisited version of Lemma 5.2 and Lemma 5.3. We also note that all the estimates in the previous lemmas are valid on extended time scale $s < t$.

Lemma 5.5. *Assume that hypotheses of Theorem 2.4 have been satisfied. Then, for a given two trajectories $(u(t), u_t(t))$ and $(v(t), v_t(t))$ through the attractor $\mathcal{A}$, and for any $\epsilon > 0$, there exists a time $T^{u,v,\epsilon}$, such that for $s < t < T^{u,v,\epsilon}$, the difference of trajectories $z(t) = u(t) - v(t)$, satisfies*

$$\left|\int_s^t (f_0^{uv}, z_t)\right| + \left|\int_s^t < f_1^{u,v}, z_t >\right| \leq \epsilon\left(E(z(s)) + E(z(t))\right) + C_{\mathcal{A},\varepsilon}\sup_{\theta\in[s,t]}\|z(\theta)\|^2 + \epsilon\int_s^t \|z\|^2_{H^1(\Omega)} + \epsilon\int_s^t ||\nabla z\|^2 + C_{\epsilon,\mathcal{A}}\int_s^t K(r)||z||^2_{H^1(\Omega)}dr \quad (102)$$

where $i = 0, 1$, $C_{\mathcal{A},\epsilon} > 0$ is independent of u and v, and $T^{u,v,\epsilon} := T(u, v, \epsilon)$.

Proof. Since the system is gradient and asymptotically smooth one obtains

$$\forall Y \in \mathcal{A}, \lim_{\to -\infty} dist(S(t)Y, \mathcal{N}) = 0$$

Therefore, for every $\epsilon > 0$ there exists a time $T^{u,v,\epsilon}$ such that

$$||u_t(t)|| + ||v_t(t)|| \leq \epsilon/C_{\mathcal{A}}, \quad \forall\ t \leq T^{u,v,\epsilon},\ (where\ C_{\mathcal{A}}\ is\ as\ in\ (94)).$$

Inserting the above inequality into the estimate in Lemma 5.2 and Lemma 5.3 and rescaling ϵ gives the result of Lemma 5.5. □

5.2.2. *Handling of $K(r)$ term and Quasistability for negative times.*

Lemma 5.6. *Under the previous assumptions there exists a constant $\omega > 0$, such that for $s < T + s < T^{u,v,\epsilon}$*

$$E(z(s+T)) + C_{\mathcal{A}}\int_s^{s+T} E(z)$$

$$\leq C_1[E(z(s))e^{-\omega T} + C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2]exp[C_{\epsilon,\mathcal{A}} \int_s^{T+s} K(r)dr]. \tag{103}$$

Proof. Using (91), we have

$$\int_{\Sigma_0^T} g^{uv} z_t + E(z(T)) \leq E(z(0)) - \int_{\Sigma_0^T} f_1^{uv} z_t - \int_{\Omega_0^T} f_0^{uv} z_t + \lambda T l.o.t._{[0,T]}\{z\}. \tag{104}$$

Hence, from (60) and (104), we derive

$$\begin{aligned}(C_{T,\mathcal{A}}+1)E(z(T)) &+ C_{\mathcal{A}} \int_0^T E(z)\\ &\leq C_{T,\mathcal{A}} E(z(0)) + C_{T,\mathcal{A}} l.o.t._{[0,T]}\{z\} + C_{T,\mathcal{A}}[source\ terms]\end{aligned} \tag{105}$$

where $i = 0,1$ and [source terms] denote all integrals involving $(f_i^{uv}, z_t)(i = 0,1)$ from the RHS of (60) (see Lemma 5.1).

Furthermore, taking $\sigma = \frac{C_{T,\mathcal{A}}}{C_{T,\mathcal{A}}+1} \leq 1$, then from (105), we can derive

$$\begin{aligned}E(z(s+T)) + C_{\mathcal{A}} \int_s^{s+T} E(z) \leq \sigma E(z(s)) &+ C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2\\ &+ C_{T,\mathcal{A}}[source\ terms].\end{aligned} \tag{106}$$

From Lemma 5.5, applied to time scale $s < t < T^{u,v,\epsilon}$, one obtains with $s + T \leq T^{u,v,\epsilon}$,

$$\begin{aligned}E(z(s+T)) + C_{\mathcal{A}} \int_s^{s+T} E(z) \leq &\sigma E(z(s)) + C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2\\ &+ \epsilon C_{T,\mathcal{A}} \int_s^{T+s} ||z||^2_{H^1(\Omega)} + \epsilon C_{T,\mathcal{A}} \int_s^{T+s} ||\nabla z||^2\\ &+ C_{T,\mathcal{A}}(\epsilon[E(z(s) + E(z(s+T)]))\\ &+ C_{\epsilon,\mathcal{A}} \int_s^{s+T} K(r)||z||^2_{H^1(\Omega)} dr.\end{aligned} \tag{107}$$

The ϵ terms are easily absorbed leaving [after adjusting the constants and taking into consideration $\sigma < 1$], so that we obtain for $s < T + s < T^{u,v,\epsilon}$,

$$\begin{aligned}E(z(s+T)) + C_{\mathcal{A}} \int_s^{s+T} E(z) \leq &\sigma E(z(s)) + C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2\\ &+ C_{\epsilon,\mathcal{A}} \int_s^{s+T} K(r)\|z\|^2_{H^1(\Omega)} dr.\end{aligned} \tag{108}$$

The last term in (108) is still at the critical level. In order to handle this, the fact that the kernel $K(r)$ is in $L_1(R)$ plays a fundamental role. Proceeding like in [8] page 175 and using discrete version of Gronwall's inequality one obtains with some $\omega > 0$ the inequality in (103). This is "quasi-stabilty" estimate established on negative time scale i.e., $s < T + s < T^{u,v,\epsilon}$. □

5.2.3. *Smoothness of the orbits for negative time.*

Lemma 5.7. *Under the assumptions of Lemma 5.5, the following regularity holds on the attractor for negative times.*

$$\|u_{tt}(s)\|^2 + \|u_t(s)\|^2_{H^1(\Omega)} \leq C_{\mathcal{A}}, \qquad \forall\ s \leq T^{u,v,\epsilon} - T. \tag{109}$$

Proof. Note that a given trajectory (u, u_t) through attractor $\mathcal{A}$ can be extended to a full trajectory for all $t \in \mathbb{R}$. Let $v(t) = u(t+h)$, then $z^h(t) = v(t) - u(t)$. Using (103) and virtue of $K \in L_1(R)$, we obtain

$$E(z^h(s+T)) + C_{\mathcal{B}} \int_s^{s+T} E(z) \leq C_1[E(z^h(s))e^{-\omega T} + C_{T,\mathcal{A}} \sup_{\theta \in [0,T]} \|z(s+\theta)\|^2]. \tag{110}$$

Multiplying (110) by h^{-2}, letting $y^h = h^{-1}z^h$, we get for all $s \leq T^{u,v,\epsilon} - T$,

$$E(y^h(s+T)) \leq C_{\mathcal{A}} E(y^h(s))e^{-\omega T} + C_{\epsilon,T,\mathcal{A}} \sup_{\theta \in [0,T]} \|y^h(s+\theta)\|^2. \tag{111}$$

Then, for small h and all $\theta \geq 0$, the term $\|y^h(s+\theta)\|^2$ is uniformly bounded by a constant $C_{\mathcal{A}}$ (depending on the initial data) because $y^h \to u_t$ as $h \to 0^+$ in $L^2(\Omega)$. Hence, we can obtain

$$E(y^h(s+T)) \leq C_{\mathcal{A}} e^{-\omega T} E(y^h(s)) + C_{\mathcal{A}}, \qquad \forall\, s \leq T^{u,v,\epsilon} - T. \tag{112}$$

Consequently

$$E(y^h(s+T)) \leq C_{\mathcal{A}}, \qquad \forall\, s \leq T^{u,v,\epsilon} - T \tag{113}$$

where $C_{\mathcal{A}}$ does not depend on $h \in [0,1]$.

Finally, we take the limit $h \to 0^+$, and can obtain that (u, u_t) belongs to the domain of the differential operator ∂_t and

$$\|u_{tt}(s)\|^2 + \|u_t(s)\|^2_{H^1(\Omega)} \leq C_{\mathcal{A}}, \qquad \forall\, s \leq T^{u,v,\epsilon} - T, \tag{114}$$

which completes the proof of Lemma 5.7. □

5.2.4. *Propagation of the regularity forward.*

Lemma 5.8. *$\mathcal{A} \subset H^2(\Omega) \times H^1(\Omega)$ is bounded.*

Proof. (Forward propagation of the regularity) It follows from the system (1), the bounds of finite energy and (114) that $\|\Delta u(t)\|$ is uniformly bounded for $t \in (-\infty, T^{u,v,\epsilon} - T]$, i.e., the trajectory is strong for $t \in (-\infty, T^{u,v,\epsilon} - T]$. Using forward well-posedness of strong solutions, we infer that $t \mapsto (u(t), u_t(t))$ is a strong solution to the system (1), which implies that all trajectories though the global attractor $\mathcal{A}$ are strong trajectories. Therefore the global attractor $\mathcal{A}$ resides in $D(\mathcal{A}) \subset H^2(\Omega) \times H^1(\Omega)$. □

5.2.5. *Quasi-stability for all times on the attractor.* Our next goal is to obtain "quasi-stability" estimate valid on the attractor for all times. This will be possible due to enhanced regularity of the attractor.

Lemma 5.9. *Quasi-stability estimate in (103) can be extended to all $s < t$.*

Proof. Considering $\mathcal{A}$ is compact, we can thus pick $t \in \mathbb{R}$, then there exist two velocity trajectories $u_t(t), v_t(t)$ through $\mathcal{A}$ belonging to a compact set J in $L^2(\Omega)$, (where J consists of the elements from $D(A^{1/2}) \subset H^1(\Omega)$). Hence, for any $\epsilon_1 > 0$, we can always find a finite set $\{w_i\}_{i=1}^{N(\epsilon)} \subset D(A^{1/2})$, such that indices i_1, i_2 satisfy

$$\|u_t - w_{i_1}\| + \|v_t - w_{i_1}\| \leq \epsilon_1,$$

and

$$\sup_{1 \leq i \leq N(\varepsilon)} \|w_i\|_{H^1(\Omega)} \leq C_{\epsilon_1}.$$

We refine the estimate (97) as

$$\begin{aligned}
&\int_\Omega \int_0^1 f_0''(\varpi z+v)(\varpi z_t+v_t)z^2 d\varpi d\Omega \\
&\quad= \int_\Omega \int_0^1 f_0''(\varpi z+v)z^2\left((\varpi z_t+v_t)-(\varpi w^{i_1,i_2}+w_{i_2})\right)d\varpi d\Omega \\
&\qquad+\int_\Omega \int_0^1 f_1''(\varpi z+v)z^2(\varpi w^{i_1,i_2}+w_{i_2})d\varpi d\Omega
\end{aligned} \tag{115}$$

where $w^{i_1,i_2}=w_{i_1}-w_{i_2}$. Then, we have the following estimates

$$\begin{aligned}
&\int_\Omega \int_0^1 f_0''(\varpi z+v)(\varpi z_t+v_t)z^2 d\varpi d\Omega \\
&\quad\le \int_\Omega C(1+|u|+|v|)z^2(|z_t-w^{i_1,i_2}|+|v_t-w_{i_2}|)d\Omega \\
&\quad\le C\|1+|u|+|v|\|_{L^6(\Omega)}\|z\|^2_{L^6(\Omega)}(\|z_t-w^{i_1,i_2}\|+\|v_t-w_{i_2}\|) \\
&\quad\le \epsilon_1 C_{\mathcal{A}}\|\nabla z\|^2,
\end{aligned} \tag{116}$$

and

$$\begin{aligned}
\left|\int_\Omega \int_0^1 f_0''(\varpi z+v)z^2(\varpi w^{i_1,i_2}+w_{i_2})d\varpi d\Omega\right| &\le C_{\epsilon_1,\mathcal{A}}\|z\|^2_{H^{1-\eta}(\Omega)} \\
&\le \epsilon_2\|\nabla z\|^2+C_{\epsilon_1,\epsilon_2,\mathcal{A}}\|z\|^2
\end{aligned} \tag{117}$$

where $0<\eta<1$ and $\epsilon_2>0$. Substituting (116) and (117) into (115) and choosing $\epsilon_1=\epsilon_2$, we obtain

$$\left|\int_\Omega \int_0^1 f_0''(\varpi z+v)(\varpi z_t+v_t)z^2 d\varpi d\Gamma\right| \le \epsilon_1 C_{\mathcal{A}}\|\nabla z\|^2+C_{\epsilon_1,\mathcal{A}}\|z\|^2. \tag{118}$$

Therefore, applying (118) and (96) to (95), we get for all $s\le t\in\mathbb{R}$,

$$\begin{aligned}
\left|\int_s^t (f_0^{uv},z_t)\right| \le&\epsilon\left(E(z(s))+E(z(t))\right)+C_{\mathcal{A},\epsilon,\epsilon_1}\sup_{\theta\in[s,t]}\|z(\theta)\|^2 \\
&+\epsilon_1 C_{\mathcal{A}}\int_s^t\|\nabla z\|^2.
\end{aligned} \tag{119}$$

Finally, substituting (101) and (119) into (106), we derive for all $s\le s+T\in\mathbb{R}$,

$$\begin{aligned}
E(z(s+T))+C_{\mathcal{A}}\int_s^{s+T}E(z) \le&\epsilon\left(E(z(s))+E(z(s+T))\right)+\sigma E(z(s)) \\
&+(\epsilon+\epsilon_1 C_{\mathcal{A}})\int_s^{s+T}E(z)+C_{T,\mathcal{A}}\sup_{\theta\in[0,T]}\|z(s+\theta)\|^2 \\
&+C_{\epsilon,\mathcal{A}}\int_s^{T+s}K(r)E(z(r))dr.
\end{aligned} \tag{120}$$

Then, for above estimate (120), we notice that for sufficiently small ϵ, the term $\epsilon\left(E(z(s))+E(z(s+T))\right)$ can be absorbed by $E(z(s+T))$ and $\sigma E(z(s))$, and the term $(\epsilon+\epsilon_1 C_{\mathcal{A}})\int_s^{s+T}E(z)$ can be absorbed by $C_{\mathcal{A}}\int_s^{s+T}E(z)$. Hence, we have (possibly for a different $\sigma = \sigma(\epsilon) < 1$)

$$E(z(s+T)) \leq \sigma E(z(s)) + C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2 + C_{\epsilon,\mathcal{A}} \int_s^{T+s} K(r)E(z(r))dr. \tag{121}$$

Proceeding as before and taking advantage of the fact that $K \in L_1(R)$ we apply discrete Gronwall's inequality to obtain

$$E(z(s+T)) \leq [E(z(s))e^{-\omega T} + C_{T,\mathcal{A}} \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2]e^{\int_R K(r)dr}, \tag{122}$$

which by virtue of $K \in L_1(R)$ implies the desired estimate

$$E(z(s+T)) \leq C_1 e^{-\omega T} E(z(s)) + C_2 \sup_{\theta\in[0,T]} \|z(s+\theta)\|^2 \tag{123}$$

with non constraints on time scale. We have thus completed the proof of "quasi-stability estimate" which is stated in Proposition 4.1, below in Lemma 5.10. □

Lemma 5.10. *Let the Assumptions I hold and the initial data be from the attractor $\mathcal{A}$, then there exist constants $C_1, C_2, \omega > 0$, so for every $s < t \in \mathbb{R}$, the following quasi-stabilty inequality holds*

$$E(z(t)) \leq C_1 e^{-\omega t} E(z(s)) + C_2 \sup_{\theta\in[s,t]} \|z(\theta)\|^2. \tag{124}$$

5.3. **Completion of the proof of Theorem 2.4.** The estimate (124) essentially states that trajectories through the attractor converge exponentially to each other up to a compact perturbation. Then from (124), we are in a position to claim finite dimension of the attractor $\mathcal{A}$ (see Lemma 6.10 in the Appendix I), this completes the first statement (1) in Theorem 2.4. From Lemma 5.7 and Lemma 5.8, we can complete the proof of the first two statement (2) in Theorem 2.4. In order to prove the last statement (3) in Theorem 2.4, we shall use just obtained "quasi-stability" estimate (124) along with the arguments in [5, 11].

Proof. Define $\tilde{\mathcal{H}}_{-1} := [H^1(\Omega)]' \times L^2(\Omega)$ and

$$\mathcal{B}_R := \{U \in \mathcal{H}; \tilde{E} \leq R\}$$

where $R > 0$ and $\tilde{E}$ is the strict Lyapunov function.

Note that, for any $R > 0$ and some given $U_0 \in \mathcal{B}_R$, there exists a unique solution $U \in C(0, \infty; \mathcal{H})$, such that $U(t) = S(t)U_0$. In addition, considering $\tilde{E}$ is the strict Lyapunov function, we have $\tilde{E}(U(t)) \leq \tilde{E}(U_0) \leq R$, for every $t \geq 0$, which implies that the set $\mathcal{B}_R$ is a positive invariant set for all $R > 0$. Since the attractor $\mathcal{A}$ is a compact set, there exists $R_0 > 0$ satisfying

$$\mathcal{A} \subset \mathcal{B} = B(0, R_0) \subset \{U \in \mathcal{H}; \tilde{E} \leq R\} = \mathcal{B}_R$$

where R is large enough and $\mathcal{B} = B(0, R_0) \subset \mathcal{H}$ is a bounded absorbing set with radius R_0.

Then, from the boundedness of the set $\mathcal{B}$, we have

$$\lim_{t\to\infty} dist_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{A}) = 0,$$

hence, there exists $T_{\mathcal{B}} > 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_R$, for any $\forall t \geq T_{\mathcal{B}}$. Based on the above analysis, we can thus conclude that the set $\mathcal{B}_R$ is absorbing and positively invariant. Besides, it follows from Corollary 3.6 that $\mathcal{B}_R$ is bounded. Therefore, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on $\mathcal{B}_R$. Then, for any $T > 0$ and $U_0 \in \mathcal{B}_R$, $U(t) = S(t)U_0$, we obtain

$$\|U(t)\|_{\mathcal{H}} = \|S(t)U_0\|_{\mathcal{H}} \le C_{\mathcal{B}}, \quad \forall t \in [0,T],$$

where $C_{\mathcal{B}} > 0$.

From (1) and the Lipschitz conditions of f_i $(i = 0,1)$ in (2) and (3), we get

$$\left\|\frac{d}{dt}S(t)U(0)\right\|_{\tilde{\mathcal{H}}_{-1}} \le C_{\mathcal{B}}, \quad \forall\, t \in [0,T]. \tag{125}$$

Therefore, we have for all $t_1, t_2 \in [0,T]$,

$$\|S(t_1)U(0) - S(t_2)U(0)\|_{\tilde{\mathcal{H}}_{-1}} \le \left|\int_{t_1}^{t_2} \left\|\frac{d}{dt}S(t)U(0)\right\|_{\tilde{\mathcal{H}}_{-1}} dt\right| \le C_{\mathcal{B}}|t_1 - t_2|, \tag{126}$$

which gives us that the map $U \mapsto S(t)U$ is Hölder continuous in $\tilde{\mathcal{H}}_{-1}$ for any $U(0) \in \mathcal{B}_R$ with exponent $\delta = 1$. From Lemma 6.11, it follows that $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor $\mathcal{A}_{1,e}$ with finite fractal dimension.

Next, we shall prove that there exists an exponential attractor in $\tilde{\mathcal{H}}_{-\delta}$ for system $(\mathcal{H}, S(t))$ and any $\delta \in (0,1)$. Since

$$\|U(0)\|_{\tilde{\mathcal{H}}_{-\delta}} \le C\|U(0)\|_{\mathcal{H}}^{1-\delta}\|U(0)\|_{\tilde{\mathcal{H}}_{-1}}^{\delta} \le C_{\mathcal{B}}^{1-\delta}\|U(0)\|_{\tilde{\mathcal{H}}_{-\delta}}^{\delta}, \tag{127}$$

we have, for any $t_1, t_2 \in [0,T]$,

$$\|S(t_1)U(0) - S(t_2)U(0)\|_{\tilde{\mathcal{H}}_{-\delta}} \le C_{\mathcal{B}}^{1-\delta}\|S(t_1)U(0) - S(t_2)U(0)\|_{\tilde{\mathcal{H}}_{-1}}^{\delta}. \tag{128}$$

Using (126), we get

$$\|S(t_1)U(0) - S(t_2)U(0)\|_{\tilde{\mathcal{H}}_{-\delta}} \le C_{\mathcal{B}}|t_1 - t_2|^{\delta}, \quad \forall\, t_1, t_2 \in [0,T]. \tag{129}$$

Hence, the map $U \mapsto S(t)U$ is Hölder continuous in $\tilde{\mathcal{H}}_{-\delta}$ for any $U(0) \in \mathcal{B}_R$ and $\delta \in (0,1)$. It follows from Lemma 6.11 again that $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor $\mathcal{A}_{\delta,e}$ with finite fractal dimension. We thus complete the proof of (3) in Theorem 2.4. □

6. Appendices.

6.1. Appendix I.

Let $S(t)$ be a strongly continuous semigroup and $(\mathcal{H}, S(t))$ be a dynamical system related to $S(t)$.

Definition 6.1. ([29]) A bounded set $B_0 \subseteq E$ is called to be an absorbing set if for any $B \in \mathbb{B}(E)$, there exists a time $t_B = t(B) > 0$, such that $S(t)B \subseteq B_0$ for any $t \ge t_B$, where $\{S(t)\}_{t\ge 0}$ is a semigroup in the complete metric space E, $\mathbb{B}(E)$ is the collection of all bounded sets in E.

Definition 6.2. ([3, 4, 5, 7, 10, 11, 12]) (1) $(\mathcal{H}, S(t))$ is called to be asymptotically smooth if for any bounded positively invariant set $\mathcal{B} \subset \mathcal{H}$, there exists a compact set $K \subset \overline{\mathcal{B}}$ such that

$$\lim_{t\to\infty} dist_{\mathcal{H}}(S(t)\mathcal{B}, K) = 0, \tag{130}$$

where $dist_{\mathcal{H}}$ is the Hausdorff semi-distance in $\mathcal{H}$.

(2) A global attractor for $(\mathcal{H}, S(t))$ is a compact set $\mathcal{A}$ of $\mathcal{H}$ if it satisfies $S(t)\mathcal{A} = \mathcal{A}$ for all $t \ge 0$ and

$$dist_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{A}) = \sup_{x\in\mathcal{B}}\inf_{y\in\mathcal{A}} \|S(t)x - y\|_{\mathcal{H}} \to 0, \quad as \;\; t \to \infty. \tag{131}$$

(3) We say that $(\mathcal{H}, S(t))$ is dissipative if it possesses a bounded absorbing set $\mathcal{B} \subset \mathcal{H}$ such that for any bounded set $\mathcal{B} \subset \mathcal{H}$, there exists a time $t_{\mathcal{B}} \geq 0$ satisfying

$$S(t)\mathcal{B} \subset \mathcal{B}, \quad \forall\, t > t_{\mathcal{B}}. \tag{132}$$

(4) The fractal dimension of a compact set $\mathcal{A} \subset \mathcal{H}$ is given by

$$dim_{\mathcal{H}}\mathcal{A} = \lim_{\varepsilon \to 0} \sup \frac{\ln n(\mathcal{A}, \varepsilon)}{\ln(1/\varepsilon)} \tag{133}$$

where $n(\mathcal{A}, \varepsilon)$ is the minimal number of closed balls in $\mathcal{H}$ of radius ε which covers $\mathcal{A}$.

(5) A full trajectory in $\mathcal{H}$ is a continuous curve $\varpi = \{u(t)|t \in \mathbb{R}\}$ satisfying $S(t)u(\tau) = u(t + \tau)$ for all $t \geq 0$ and $\tau \in \mathbb{R}$.

Definition 6.3. ([3, 4, 5, 7, 10, 11, 12])(**Lyapunov function**) Let $(\mathcal{H}, S(t))$ be a dynamical system with the phase space $\mathcal{H}$ and evolution semigroup $S(t)$.

- The continuous functional $V(Y)$ defined on $\mathcal{H}$ is said to be the Lyapunov function for the dynamical system $(\mathcal{H}, S(t))$ if and only if $t \mapsto V(S(t)Y)$ is a non-increasing function for any $Y \in \mathcal{H}$.
- The Lyapunov function $V(Y)$ is said to be strict if and only if $V(S(t)Y) = V(Y)$ for all $t > 0$ and for some $Y \in \mathcal{H}$ implies that $S(t)Y = Y$ for all $t > 0$, i.e., Y is a stationary point of $(\mathcal{H}, S(t))$.
- The dynamical system $(\mathcal{H}, S(t))$ is said to be gradient if and only if there exists a strict Lyapunov function on $\mathcal{H}$.

Definition 6.4. ([3, 4, 5, 7, 10, 11, 12, 27, 28]) (**Exponential attractor**) A compact set $\mathcal{A}_e \subset \mathcal{H}$ is called a fractal exponential attractor if it has finite fractal dimension, is positively invariant, and for any bounded set $\mathcal{B} \subset \mathcal{H}$, there exist constants $T_{\mathcal{B}}, C_{\mathcal{B}} > 0$, and $\gamma_{\mathcal{B}} > 0$ such that for all $t \geq T_{\mathcal{B}}$,

$$dist_H(S(t)B, \mathcal{A}_e) \leq C_{\mathcal{B}} e^{-\gamma_{\mathcal{B}}(t - T_{\mathcal{B}})}.$$

In some cases, one can prove the existence of an exponential attractor whose dimension is finite in some extended space $\tilde{\mathcal{H}} \supset \mathcal{H}$ only. We frequently call this exponentially attracting set a generalized exponential attractor.

Lemma 6.5. *([3, 4, 5, 7, 10, 11, 12, 27, 28]) Let $\mathcal{B}$ be any bounded positively invariant set for $\mathcal{H}$. Suppose that for any $\varepsilon > 0$ and for any $\mathcal{B}$, there exists $T = T(\varepsilon, \mathcal{B}) > 0$ such that*

$$\|S(T)x - S(T)y\|_{\mathcal{H}} \leq \varepsilon + \phi_T(x, y), \quad \forall x, y \in \mathcal{B}, \tag{134}$$

where $\phi_T : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ verifies for any sequence $\{U_n\}$ in $\mathcal{B}$,

$$\lim_{n \to \infty} \inf \lim_{m \to \infty} \inf \phi_T(U_n, U_m) = 0. \tag{135}$$

Then $S(t)$ is asypmtotically smooth in $\mathcal{H}$.

Lemma 6.6. *([3, 4, 5, 7, 10, 11, 12]) Suppose $(\mathcal{H}, S(t))$ is a gradient asymptotically smooth dynamical system. Assume its Lyapunov function $V(Y)$ is bounded from above on any bounded subset of $\mathcal{B}$ and the set $V_R = \{Y : V(Y) \leq R\}$ is bounded for every $R > 0$. If the set $\mathcal{N}$ of stationary points of $(\mathcal{H}, S(t))$ is also bounded in $\mathcal{H}$, then $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$.*

Remark 6.7. The asymptotic smoothness guarantees the compact property of trajectories, and the existence of strict Lyapunnov functional ensures the disspative property of the dynamical system.

Let X and Y be two reflexive Banach spaces with X compactly being embedded into Y and put $\mathcal{H} = X \times Y$. We consider that $(\mathcal{H}, S(t))$ satisfies

$$S(t)U_0 = U(t) = (u, u_t) \qquad U_0 = (u_0, u_1) \in \mathcal{H}, \tag{136}$$

where the function u has the regularity

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y). \tag{137}$$

To obtain the notion of quasi-stability, we introduce a seminorm $n_X(\cdot)$ which is compact if whenever a sequence $x_j \rightharpoonup 0$ weakly in X, one has $n_X(x_j) \to 0$.

Proposition 6.8. *([3, 4, 5, 7, 10, 11, 12, 27, 28]) If there exist a compact semi-norm n_X on X and two locally bounded nonnegative functions $a(t)$ and $c(t)$ satisfying*

$$b(t) \in L^1(\mathbb{R}^+) \quad \text{with} \quad \lim_{t\to\infty} b(t) = 0, \tag{138}$$

$$\|S(t)U^1 - S(t)U^2\|^2_{\mathcal{H}} \le a(t)\|U^1 - U^2\|^2_{\mathcal{H}} \tag{139}$$

and

$$\|S(t)U^1 - S(t)U^2\|^2_{\mathcal{H}} \le b(t)\|U^1 - U^2\|^2_{\mathcal{H}} + c(t) \sup_{0<s<t} [n_X(u^1(s) - u^2(s))]^2 \tag{140}$$

for any $U^1(t) = (u^1, u^1_t)$ and $U^2(t) = (u^2, u^2_t) \in \mathcal{B} \subset \mathcal{H}$, then $(\mathcal{H}, S(t))$ is called to be quasi-stable on $\mathcal{B}$.

The so-called stabilizability inequality has been given by (140).

Lemma 6.9. *([5, 11]) Let (X, d) be a complete metric space and M be a bounded closed set in X. Assume that there exists a mapping $V : M \mapsto X$ such that (i) $M \subseteq VM$;*
(ii) there exist a compact pseudometric ϱ on M and a number $0 < \eta < 1$ such that

$$d(Vv_1, Vv_2) \le \eta \cdot d(v_1, v_2) + \varrho(v_1, v_2), v_1, v_2 \in M. \tag{141}$$

Then M is a compact set in X with the fractal dimension

Lemma 6.10. *([3, 4, 5, 7, 10, 11, 12, 27, 28]) Let $(\mathcal{H}, S(t))$ be a dynamical system satisfying (136). If $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$, then the attractor $\mathcal{A}$ has finite fractal dimension.*

Lemma 6.11. *Let $(\mathcal{H}, S(t))$ be a dynamical system satisfying (136). Assume that $(\mathcal{H}, S(t))$ is dissipative and quasi-stable on some bounded absorbing set $\mathcal{B}$. Assume also that there exsits an extended space $\tilde{\mathcal{H}} \supset \mathcal{H}$ such that*

$$\|S(t_1)y - S(t_2)\|_{\tilde{\mathcal{H}}} \le C_{\mathcal{B},T}|t_1 - t_2|^\delta, \quad \forall t_1, t_2 \in [0, T]$$

where $C_{\mathcal{B},T} > 0$ and $\delta \in [0, 1)$ are constants. Then the dynamical system possesses a generalized exponential attractor $\mathcal{A}_e \subset \mathcal{H}$ whose dimension is finite in the space $\tilde{\mathcal{H}}$.

Acknowledgments. The work of I. Lasiecka has been partially supported by the NSF Grant DMS -2205508 and National Science Center (NCN) Grant nr UMO-2023/49/B/ST1/04261. The work of Y. Qin has been partially supported by the NNSF of China with contract No. 12171082, the fundamental research funds for the central universities with contract numbers 2232023G-13, 2232024G-3. The work of X. Han has been supported by China Scholarship Council with contract No.202406630049.

REFERENCES

[1] J. M. Ball, Global attractors for damped semilinear wave equations, *Discret. Contin. Dyn. Syst.*, **10** (2004), 31-52.

[2] V. Belleri and V. Pata, Attractors for semilinear strongly damped wave equations on $\mathbb{R}^3$, *Discrete Contin. Dynam. Systems*, **7** (2001), 719-735.

[3] I. Chueshov, *Dynamics of Quasi-stable Dissipative Systems*, Springer, 2015.

[4] I. Chueshov, M. Eller and I. Lasiecka, On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation, *Commun. Partial Differ. Equ.*, **27** (2002), 1901-1951.

[5] I. Chueshov, M. Eller and I. Lasiecka, *Von Karman Evolution Equations: Well-Posedness and Long Time Dynamics*, Research Monographs, Springer, New York, 2002.

[6] I. Chueshov, M. Eller and I. Lasiecka, Finite dimensionality of the attractor for a semilinear wave equation with nonlinear boundary dissipation, *Commun. Part. Diff. Equ.*, **29** (2004), 1847-1876.

[7] I. Chueshov and I. Lasiecka, Attractors for second order evolution equations with a nonlinear damping, *J. Dyn. Differ. Equ.*, **16** (2004), 469-512.

[8] I. Chueshov and I. Lasiecka, Long-time dynamics of semilinear wave equation with nonlinear interior boundary damping and sources of critical exponents, in *Control Methods in PDE-Dynamical Systems*, Snowbird, 2005, 153-192.

[9] I. Chueshov and I. Lasiecka, Long-time dynamics of von Karman semi-flows with nonlinear boundary/interior damping, *J. Diff. Equ.*, **233** (2007), 42-86.

[10] I. Chueshov and I. Lasiecka, Attractors and long-time behavior of von Karman thermoelastic plates, *Appl. Math. Optim.*, **58** (2008), 195-241.

[11] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Memoirs of the American Mathematical Society, **195** (2008).

[12] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations. Well-posedness and Long Time Dynamics*, Springer, New York, 2010.

[13] I. Chueshov, I. Lasiecka and D. Toundykov, Long-term dynamics of semilinear wave equation with nonlinear localized interior damping and a source term of critical exponent, *Discr. Contin. Dyn. Syst.*, **20** (2008), 459-509.

[14] I. Chueshov, I. Lasiecka and D. Toundykov, Global attractor for a wave equation with nonlinear localized boundary damping and a source term of critical exponent, *J. Dyn. Differ. Equ.*, **21** (2009), 269-314.

[15] A. Eden, A. J. Milani and B. Nicolaenko, Finite dimensional attractor for semilinear wave equations with damping, *J. Math. Anal. Appl.*, **169** (1992), 408-409.

[16] E. Feireisl, Finite dimensional asymptotic behavior of some semilinear damped hyperbolic problems, *J. Dyn. Differ. Equ.*, **6** (1994), 23-35.

[17] J. M. Ghidaglia and R. Temam, Attractors for damped nonlinear hyperbolic equations, *J. Math. Pures Appl.*, **66** (1987), 273-319.

[18] A. Haraux, *Semi-Linear Hyperbolic Problems in Bounded Domains*, Mathematical Repprts, 1987.

[19] L. Hormander, *Partial Differential Equations*, Springer Verlag, New York, 1965.

[20] A. Kh. Khanmamedov, Finite dimensionality of the global attractors for von Karman equations with nonlinear interior dissipation, *Nonlinear Anal.*, **66** (2007), 204-213.

[21] I. N. Kostin, Attractor for a semilinear wave equation with boundary damping, *J. Math. Sci.*, **98** (2000), 753-764.

[22] I. Lasiecka and D. Tataru, Uniform boundary stabilization of a semilinear wave equation with nonlinear boundary damping, *Differ. Inte. Equ.*, **6** (1993), 507-533.

[23] I. Lasiecka and R. Triggiani, Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions, *Appl. Math. Optim.*, **25** (1992), 189-224.

[24] I. Lasiecka and R. Triggiani, Carleman estimates and exact boundary controllability for a system of coupled nonconservative second-order hyperbolic equations, in *Partial Differential Equation Methods in Control and Shape Analysis*, Pisa. Marcel Dekker, New York, 1997, 215-243.

[25] I. Lasiecka, R. Triggiani and X. Zhang, Nonconservative wave equations with unobserved Neumann B.C., global uniqueness and observability in one shot, in *Differential Geometric Methods in the Control of Partial Differential Equations*, Boulder, 1999, 227-325.

[26] D. Prazak, On finite fractal dimension of the global attractor for the wave equation with nonlinear damping, *J. Dyn. Diff. Equ.*, **14** (2002), 763-776.
[27] Y. Qin and X. Han, Quasi-stability and upper semicontinuity for coupled wave equations with fractional damping, *Appl. Math. Optim.*, **89** (2024), Paper No. 26, 34 pp.
[28] Y. Qin, X. Han and K. Wang, Quasi-stability and continuity for wave equation with a fractional damping and a small delay, *Math. Methods Appl. Sci.*, **47** (2024), 14936-14959.
[29] Y. Qin and K. Su, *Attractors for Nonlinear Autonomous Dynamical Systems*, EDP Sciences (France), Science Press (China), 2022.
[30] J. H. Rodrigues and M. Roy, Existence of global attractors for a semilinear wave equation with nonlinear boundary dissipation and nonlinear interior and boundary sources with critical exponents, *Appl. Math. Optim.*, **86** (2022), Paper No. 35, 39 pp.
[31] J. L. Shomberg, Attractors for damped semilinear wave equations with a robin-acoustic boundary perturbation, *Nonlinear Anal.*, **189** (2019), 111582, 36 pp.
[32] D. Tataru, The X_θ^s spaces and unique continuation for solutions to the semilinear wave equation, *Comm Partial Differ. Equ.*, **21** (1996), 841-887.
[33] J. Vancostenobla and P. Martinez, Optimality of energy estimates for the wave equation with nonlinear boundary velocity damping, *SIAM J. Control Optim.*, **39** (2000), 776-797.
[34] S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, *Discret. Contin. Dyn. Syst.*, **11** (2004), 351-392.

Received November 2024; revised February 2025; early access February 2025.