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AN UPPER BOUND FOR THE FIRST NONZERO STEKLOV EIGENVALUE

Xiaolong Li[1], Kui Wang[2,*] and Haotian Wu[3]

Abstract. Let (M^n, g) be a complete simply connected n-dimensional Riemannian manifold with curvature bounds $\mathrm{Sect}_g \leq \kappa$ for $\kappa \leq 0$ and $\mathrm{Ric}_g \geq (n-1)Kg$ for $K \leq 0$. We prove that for any bounded domain $\Omega \subset M^n$ with diameter d and Lipschitz boundary, if Ω^* is a geodesic ball in the simply connected space form with constant sectional curvature κ enclosing the same volume as Ω, then $\sigma_1(\Omega) \leq C\sigma_1(\Omega^*)$, where $\sigma_1(\Omega)$ and $\sigma_1(\Omega^*)$ denote the first nonzero Steklov eigenvalues of Ω and Ω^* respectively, and $C = C(n, \kappa, K, d)$ is an explicit constant. When $\kappa = K$, we have $C = 1$ and recover the Brock–Weinstock inequality, asserting that geodesic balls uniquely maximize the first nonzero Steklov eigenvalue among domains of the same volume, in Euclidean space and the hyperbolic space.

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1. Introduction

Let (M^n, g) be a complete Riemannian manifold of dimension n and $\Omega \subset M^n$ be a bounded domain with Lipschitz boundary. The Steklov eigenvalue problem is to find a solution u of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

where Δ denotes the Laplace–Beltrami operator, ν denotes the outward unit normal to $\partial\Omega$, and σ is a real number. This problem was first introduced by Steklov [1] in 1902 for bounded domains in the plane. The set of eigenvalues for the Steklov problem is the same as that for the well-known Dirichlet-to-Neumann map, which maps $f \in L^2(\partial\Omega)$ to the normal derivative on the boundary of the harmonic extension of f inside Ω. Since the Dirichlet-to-Neumann map is a self-adjoint operator, it has a discrete spectrum given by

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to \infty.$$

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[1] Department of Mathematics, Statistics and Physics, Wichita State University, Wichita, KS 67260, USA.
[2] School of Mathematical Sciences, Soochow University, Suzhou 215006, PR China.
[3] School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia.

[*] Corresponding author: kuiwang@suda.edu.cn

The eigenfunctions of $\sigma_0(\Omega)$ are the constant functions. The first nonzero eigenvalue $\sigma_1(\Omega)$ is characterized by the following Rayleigh quotient

$$\sigma_1(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^2 \, d\mu_g}{\int_{\partial\Omega} u^2 \, dA_g} : u \in W^{1,2}(\Omega) \setminus \{0\}, \ \int_{\partial\Omega} u \, dA_g = 0 \right\}, \tag{1.1}$$

where $d\mu_g$ is the volume form of g and dA_g is the induced measure on $\partial\Omega$.

In 1954, Weinstock [2] showed that the round disk uniquely maximizes $\sigma_1(\Omega)$ among 'simply connected planar domains with prescribed perimeter. This result was generalized to arbitrary compact Riemannian surfaces by Fraser and Schoen [3] to obtain the upper bound $\sigma_1(\Omega)|\partial\Omega| \leq 2\pi(\gamma + k)$ for a surface of genus γ with k boundary components. In higher dimensions, Bucur, Ferone, Nitsch and Trombetti [4] proved that the ball uniquely maximizes $\sigma_1(\Omega)$ among bounded open convex sets in $\mathbb{R}^n$ with prescribed perimeter. The convexity assumption in the previous result is crucial. Indeed, for an annulus $B_1(0) \setminus B_\varepsilon(0)$ with ε sufficiently small, its first nonzero Steklov eigenvalue is strictly bigger than that of a ball with same volume, see [5]. Also, Fraser and Schoen [6] have shown that the ball does not maximize $\sigma_1(\Omega)$ among contractible domains in $\mathbb{R}^n$ with prescribed perimeter. Moreover, they have given an explicit upper bound on $\sigma_1(\Omega)$ for any smooth domain in $\mathbb{R}^n$ in terms of its boundary perimeter (*cf.* [6], Sect. 2).

When combined with the isoperimetric inequality, Weinstock's theorem implies that the round disk uniquely maximizes $\sigma_1(\Omega)$ among all simply connected planar domains with fixed area. In 2001, Brock [7] generalized Weinstock's result by removing any topological or dimensional restriction. As a result, we have the Brock–Weinstock inequality, which asserts that among domains in $\mathbb{R}^n$ with the same volume, the ball maximizes $\sigma_1(\Omega)$, and the equality occurs if and only if Ω is a ball. A sharp quantitative version of the Brock–Weinstock inequality has been proved by Brasco, De Philippis and Ruffini [8].

The Brock–Weinstock inequality is related to two classic spectral inequalities: the Faber–Krahn inequality, which asserts that the ball uniquely minimizes the first Dirichlet eigenvalue among domains with the same volume, and the Szegö-Weinberger inequality stating that among domains with the same volume, the ball uniquely maximizes the first nonzero Neumann eigenvalue. It is well-known that the Faber–Krahn inequality holds in any Riemannian manifold in which the isoperimetric inequality holds, see [9]. Also, the Szegö-Weinberger inequality holds for domains in the hemisphere and in the hyperbolic space [10]. Therefore, it is a natural question to extend the Brock–Weinstock inequality to space forms and more general Riemannian manifolds.

Concerning the previous question, only a few results are known. In 1999, Escobar [11] generalized Weinstock's theorem by proving that in a complete simply connected two-dimensional manifold with constant Gaussian curvature, geodesic balls maximize $\sigma_1(\Omega)$ among bounded simply connected domains with fixed area. In the same paper, the author obtained the more general eigenvalue comparison result: $\sigma_1(\Omega)$ of any bounded simply connected domain in a complete simply connected non-positively curved two-manifold is no larger than that of a ball in $\mathbb{R}^2$ with the same area, and equality holds only when the domain is isometric to the round disk. In 2014, Binoy and Santhanam [12] proved that in non-compact rank one symmetric spaces (including Euclidean space and hyperbolic space), geodesic balls maximize $\sigma_1(\Omega)$ among bounded domains of the same volume. Recently, a stability result for the theorem of Binoy and Santhanam has been proved by Castillon and Ruffini [13].

The main purpose of this paper is to give an upper bound for the first nonzero Steklov eigenvalue of a bounded domain in a simply connected Riemannian manifold (M^n, g) with non-positive sectional curvatures. Throughout the paper, the function sn_κ is defined by

$$\mathrm{sn}_\kappa(t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t), & \text{if } \kappa > 0, \\ t, & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t), & \text{if } \kappa < 0. \end{cases} \tag{1.2}$$

We denote by Sect_g and Ric_g the sectional curvature and the Ricci curvature of g respectively, and by $\mathrm{diam}(\Omega)$ the diameter of $\Omega \subset M^n$.

The main theorem of this paper states the following.

Theorem 1.1. *Let (M^n, g) be a complete simply connected Riemannian manifold of dimension n, and $\Omega \subset M^n$ be a bounded domain with Lipschitz boundary. Let M_κ be the n-dimensional simply connected space form of constant sectional curvature κ, and Ω^* be a geodesic ball in M_κ having the same volume as Ω. If $\mathrm{Sect}_g \le \kappa$ for $\kappa \le 0$, and $\mathrm{Ric}_g \ge (n-1)Kg$ for $K \le 0$, then*

$$\sigma_1(\Omega) \le \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)}\right)^{2n-2} \sigma_1(\Omega^*), \tag{1.3}$$

where $d = \mathrm{diam}(\Omega)$.

In Euclidean space or hyperbolic space, we have $k = K$ and the constant factor in (1.3) is 1. So Theorem 1.1 recovers the Brock–Weinstock inequality proved by Weinstock [2] and [7] for $\mathbb{R}^n$, and by Binoy and Santhanam [12] for $\mathbb{H}^n$.

Corollary 1.2. *In Euclidean space and hyperbolic space, geodesic balls uniquely maximize the first nonzero Steklov eigenvalue among bounded Lipschitz domains with the same volume.*

We note that Corollary 1.2 has been generalized to the Robin eigenvalues of the Laplacian in non-positively curved space forms by the authors [14].

On manifolds whose sectional curvatures are bounded from above by κ, where $\kappa \le 0$, Binoy and Santhanam obtained a result (*cf.* [12], Thm. 1.2) similar to Theorem 1.1. The constant in their inequality depends on the manifold and the space form in comparison, although in a rather non-transparent way. In contrast, the constant in our inequality (1.3) reveals the explicit dependency on the geometries.

When $\kappa = 0$, it is well-known that

$$\sigma_1(\Omega^*) = \left(\frac{\omega_n}{\mathrm{Vol}(\Omega)}\right)^{1/n},$$

where ω_n is volume of the unit ball in $\mathbb{R}^n$. Then Theorem 1.1 gives the following explicit estimate in a Cartan–Hadamard manifold, *i.e.*, a complete simply connected Riemannian manifold with non-positive sectional curvature.

Corollary 1.3. *Let (M^n, g) be a Cartan–Hadamard manifold of dimension n, and $\Omega \subset M^n$ be a bounded domain with Lipschitz boundary. If $\mathrm{Ric}_g \ge (n-1)Kg$ for $K \le 0$, then*

$$\sigma_1(\Omega)\,\mathrm{Vol}(\Omega)^{1/n} \le \omega_n^{1/n} \left(\frac{\mathrm{sn}_K(d)}{d}\right)^{2n-2}, \tag{1.4}$$

where $d = \mathrm{diam}(\Omega)$.

To conclude this section, we mention several other aspects of the first nonzero Steklov eigenvalue $\sigma_1(\Omega)$. First of all, the question of finding a metric on Ω maximizing $\sigma_1(\Omega)|\partial\Omega|$ has received considerable attention in recent years since the remarkable paper by Fraser and Schoen [15], in which the authors developed the theory of extremal metrics for Steklov eigenvalues *via* its connection to the free boundary minimal surfaces. Secondly, finding a lower bound for $\sigma_1(\Omega)$ in terms of the geometric data of Ω is also an interesting question. In this direction, Escobar [16] proved that for an n-dimensional ($n \ge 3$) compact smooth Riemannian manifold with boundary, which has non-negative Ricci curvature and the principal curvatures of the boundary bounded below by $c > 0$, the first nonzero Steklov eigenvalue is greater than or equal to $c/2$. Escobar then conjectured in [11] that the sharp lower bound is c with the equality being true only on isometrically Euclidean balls with radius $1/c$. Recently, Xia and Xiong [17] settled Escobar's conjecture under the stronger assumption of non-negative sectional curvature. Lastly, $\sigma_1(\Omega)$ is closely related to the first nonzero Laplace eigenvalue of $\partial\Omega$. We refer the reader to the papers by Wang and Xia [18], Karpukhin [19], Xiong [20], Xia and Xiong [17] for recent developments.

This paper is organized as follows. In Section 2, we set up the notation and recall some facts on the eigenfunctions for the first nonzero Steklov eigenvalue on space forms. Section 3 contains results on spherical symmetrizations and the comparison of isoperimetric profiles. We prove Theorem 1.1 in Section 4.

2. Preliminaries

Let (M^n, g) be a complete simply connected Riemannian manifold of dimension n. For any bounded Lipschitz domain $\Omega \subset M := M^n$, we denote by $|\Omega|$ the n-dimensional volume of Ω and by $|\partial\Omega|$ the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ respectively, both taken with respect to the Riemannian metric g on M. Let (M_κ, g_κ) denote the n-dimensional complete simply connected space form of constant sectional curvature κ. Fix any $q \in M_\kappa$, we define Ω_q^* to be a geodesic ball in M_κ centered at q and satisfying $|\Omega_q^*|_\kappa = |\Omega|$, where $|\Omega_q^*|_\kappa$ is the n-dimensional volume of Ω^* with respect to g_κ.

2.1. Steklov eigenfunctions on space forms

In this subsection, we collect some known facts on the Steklov eigenfunctions corresponding to $\sigma_1(\Omega_q^*)$ on space forms. Let R_0 be the radius of the geodesic ball Ω_q^* in M_κ, and (r, θ) be the polar coordinates centered at q. By separation of variables, the eigenfunctions on Ω_q^* corresponding to $\sigma_1(\Omega_q^*)$ are given by

$$u_i(r,\theta) = F(r)\psi_i(\theta), \qquad 1 \le i \le n,$$

where $\psi_i(\theta)$ are linear coordinate functions restricted to $\mathbb{S}^{n-1}$, satisfying

$$-\Delta_{\mathbb{S}^{n-1}}\psi_i(\theta) = (n-1)\psi_i(\theta),$$

and $F(r)$ solves the following ODE initial value problem

$$F''(r) + (n-1)\frac{\operatorname{sn}_\kappa'(r)}{\operatorname{sn}_\kappa(r)}F'(r) - \frac{n-1}{\operatorname{sn}_\kappa^2(r)}F(r) = 0, \quad r \in (0, R_0] \tag{2.1}$$

with $F(0) = 0$, and $\sigma_1(\Omega_q^*) = F'(R_0)/F(R_0)$ the minimal value of the quotient

$$Q(\varphi) = \frac{\int_0^{R_0}\left(\varphi'(r)^2 + \frac{n-1}{\operatorname{sn}_\kappa^{n-1}(r)}\varphi^2(r)\right)\operatorname{sn}_\kappa^{n-1}(r)\,dr}{\varphi^2(R_0)\operatorname{sn}_\kappa^{n-1}(R_0)} \quad \text{with } \varphi(0) = 0,$$

see [21], Lemma 3. Moreover, if $\kappa \le 0$, we can extend $F(r)$ for $r \ge R_0$ *via* ODE (2.1). Using equation (2.1), we calculate

$$\begin{aligned}
\left(\operatorname{sn}_\kappa^{n-1} F'\right)' &= \operatorname{sn}_\kappa^{n-1} F'' + (n-1)\operatorname{sn}_\kappa^{n-2}\operatorname{sn}_\kappa' F' \\
&= \operatorname{sn}_\kappa^{n-1}(n-1)\left(-\frac{\operatorname{sn}_\kappa' F'}{\operatorname{sn}_\kappa} + \frac{F}{\operatorname{sn}_\kappa^2} + \frac{\operatorname{sn}_\kappa' F'}{\operatorname{sn}_\kappa}\right) \\
&= (n-1)\operatorname{sn}_\kappa^{n-3} F.
\end{aligned}$$

From this, we see that if $F(r)$ is positive (or negative) near zero, then $F(r)$ is positive (or negative) in $(0, \infty)$. Thus the function F does not change sign. Without lost of generality, we assume that $F(r) \ge 0$, then $F'(r) \ge 0$, implying $F(r)>0$ or $F(r)\equiv 0$ in $(0,\infty)$. Thus we conclude that $F(r)>0$ and $F'(r)>0$ in $(0,\infty)$. In fact, equation (2.1) is equivalent to

$$\Big[\operatorname{sn}_\kappa^{-(n-1)}(r)\big(\operatorname{sn}_\kappa^{n-1}(r)F(r)\big)'\Big]'=0,$$

and then up to a constant multiple, one gets the exact expression of $F(r)$:

$$F(r)=\frac{\int_0^r(\operatorname{sn}_\kappa(t))^{n-1}\,dt}{(\operatorname{sn}_\kappa(r))^{n-1}}. \tag{2.2}$$

By calculating the first derivatives and using the differential equation (2.1), we have the following monotonicity results.

Proposition 2.1. *Let $F(r)$ be the function defined in equation (2.1). Define*

$$G(r):=(F^2(r))'+\frac{(n-1)\operatorname{sn}_\kappa'(r)}{\operatorname{sn}_\kappa(r)}F^2(r), \tag{2.3}$$

$$H(r):=F'(r)^2+\frac{n-1}{\operatorname{sn}_\kappa^2(r)}F^2(r). \tag{2.4}$$

Then G is non-negative and non-decreasing on $[0,\infty)$ for all $\kappa\in\mathbb{R}$, and H is non-negative and non-increasing on $[0,\infty)$ provided that $\kappa\le 0$.

Proof. The functions G and H are non-negative on $[0,\infty)$ since F is non-negative and increasing on $[0,\infty)$.

Using equation (2.1), we calculate on $(0,\infty)$ that

$$\begin{aligned}
G'(r)&=2FF''+2(F')^2+(n-1)\left(2\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa}FF'+\frac{\operatorname{sn}_\kappa''}{\operatorname{sn}_\kappa}F^2-\frac{(\operatorname{sn}_\kappa')^2}{\operatorname{sn}_\kappa^2}F^2\right)\\
&=2F\left(-(n-1)\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa}F'+\frac{n-1}{\operatorname{sn}_\kappa^2}F\right)+2(F')^2\\
&\quad+(n-1)\left(2\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa}FF'+\frac{\operatorname{sn}_\kappa''}{\operatorname{sn}_\kappa}F^2-\frac{(\operatorname{sn}_\kappa')^2}{\operatorname{sn}_\kappa^2}F^2\right)\\
&=2(F')^2+\frac{(n-1)F^2}{\operatorname{sn}_\kappa^2}\left(2+\operatorname{sn}_\kappa\operatorname{sn}_\kappa''-(\operatorname{sn}_\kappa')^2\right)\\
&=2(F')^2+\frac{(n-1)F^2}{\operatorname{sn}_\kappa^2}\\
&\ge 0,
\end{aligned}$$

where in the last equality we used the identity $\operatorname{sn}_\kappa\operatorname{sn}_\kappa''-(\operatorname{sn}_\kappa')^2=-1$ for all $\kappa\in\mathbb{R}$. Thus, G is non-decreasing on $(0,\infty)$.

Likewise, we have on $(0,\infty)$ that

$$\begin{aligned}
H'&=2F'F''-2(n-1)\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa^3}F^2+\frac{2(n-1)}{\operatorname{sn}_\kappa^2}FF'\\
&=2F'\left(-(n-1)\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa}F'+\frac{n-1}{\operatorname{sn}_\kappa^2}F\right)-2(n-1)\frac{\operatorname{sn}_\kappa'}{\operatorname{sn}_\kappa^3}F^2+\frac{2(n-1)}{\operatorname{sn}_\kappa^2}FF'\\
&=-\frac{2(n-1)}{\operatorname{sn}_\kappa^3}\left(\operatorname{sn}_\kappa'\operatorname{sn}_\kappa^2(F')^2-2\operatorname{sn}_\kappa FF'+\operatorname{sn}_\kappa'F^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{2(n-1)}{\mathrm{sn}_\kappa^3}\left(\mathrm{sn}_\kappa^2 (F')^2 - 2\,\mathrm{sn}_\kappa FF' + F^2\right)\\
&= -\frac{2(n-1)}{\mathrm{sn}_\kappa^3}\left(\mathrm{sn}_\kappa F' - F\right)^2\\
&\leq 0,
\end{aligned}
$$

where in the first inequality we used $\mathrm{sn}_\kappa'(r) \geq 1$ for $\kappa \leq 0$. Thus, H is non-increasing on $(0, \infty)$. □

3. Spherical symmetrizations and isoperimetric inequality

We recall the definitions of spherical symmetrizations. For any non-negative real-valued function f defined on a bounded domain $\Omega \subset M$, the measure of the super-level sets of f is defined by

$$\mu_f(t) := |\{x \in \Omega : f(x) > t\}|.$$

Let $r_q(x) = \mathrm{dist}_\kappa(q, x)$ be the distance function on the space form M_κ and $B_q(r)$ be the geodesic ball centered at q with radius r in M_κ.

Definition 3.1. Let $\Omega \subset M$ be a bounded domain and f be a non-negative integrable real-valued function defined on Ω. The spherical decreasing and increasing symmetrizations of f, denoted by $f^*(x)$ and $f_*(x)$ respectively, are radial functions defined on Ω_q^* by

$$f^*(x) := \sup\{t : \mu_f(t) \geq |B_q(r_q(x))|_\kappa\}$$

and

$$f_*(x) := \sup\left\{t : \mu_f(t) \geq |\Omega_q^*|_\kappa - |B_q(r_q(x))|_\kappa\right\},$$

where Ω_q^* is the geodesic ball in M_κ centered at q satisfying $|\Omega_q^*|_\kappa = |\Omega|$.

The L^s-norm ($s \geq 1$) is invariant under spherical symmetrizations.

Proposition 3.2. *For any $s \geq 1$, we have*

$$||f(x)||_{L^s(\Omega)} = ||f^*(x)||_{L^s(\Omega_q^*)} = ||f_*(x)||_{L^s(\Omega_q^*)}. \tag{3.1}$$

Proof. See [22], Proposition 2.2. □

For any $p \in M$, let $\eta_p : [0, \infty) \to [0, \infty)$ be the radial function defined by

$$|B_q(\eta_p(r))|_\kappa = |B_p(r)|. \tag{3.2}$$

Clearly, η_p is monotone non-decreasing in r. The volume comparison theorem for $\mathrm{Sect}_g \leq \kappa$ implies that $\eta_p(r) \geq r$.

The spherical symmetrizations of monotone radial functions have the following properties.

Lemma 3.3. *Assume $f(r)$ is a non-negative function on $[0, \infty)$.*

1. *If $f(r)$ is non-decreasing, then for $y \in \Omega_q^*$*

$$\left(f \circ \eta_p \circ r_p\right)_*(y) \geq f(r_q(y)). \tag{3.3}$$

2. If $f(r)$ is non-increasing, then for $y \in \Omega_q^*$

$$\left(f \circ \eta_p \circ r_p\right)^* (y) \leq f(r_q(y)). \tag{3.4}$$

Proof. It follows from the definitions of η_p and spherical symmetrizations that

$$\left(f \circ \eta_p \circ r_p\right)_* (y) = f(\eta_p(r_1)), \tag{3.5}$$

where r_1 satisfies $|B_q(r_q(y))|_\kappa = |B_p(r_1) \bigcap \Omega| \leq |B_q(\eta_p(r_1))|_\kappa$. So then

$$r_q(x) \leq \eta_p(r_1),$$

which implies (3.3) since f is non-decreasing.

The proof of (3.4) is similar as that of (3.3) and we omit the details. □

We now prove a comparison result for isoperimetric profiles.

Lemma 3.4. *Assume that (M^n, g) is complete simply connected with $\mathrm{Sect}_g \leq \kappa$ for $\kappa \leq 0$. For any fixed $p \in M^n$ and any fixed $q \in M_\kappa^n$, we define an isoperimetric profile $I_{M,p} : [0, \infty) \to \mathbb{R}^+$ by*

$$I_{M,p}(t) := \left|\partial B_p(r(t))\right|,$$

where $r(t)$ is so defined that $\left|B_{r(t)}(p)\right| = t$, and similarly define $I_{M_\kappa,q} : [0, \infty) \to \mathbb{R}^+$ by

$$I_{M_\kappa,q}(t) := \left|\partial B_q(\eta_p(r(t)))\right|_\kappa,$$

where $\left|B_q(\eta_p(r(t)))\right|_\kappa = t$, cf. (3.2). Then

$$I_{M,p}(t) \geq I_{M_\kappa,q}(t). \tag{3.6}$$

Proof. Let r_1 and r_2 satisfy

$$t = \int_{\mathbb{S}^{n-1}} \int_0^{r_1} J(r, \theta)\, dr\, d\theta = n\omega_n \int_0^{r_2} J_\kappa(r)\, dr.$$

Since $\mathrm{Sect}_g \leq \kappa$, we have the following comparisons

$$J(r, \theta) \geq J_\kappa(r) \quad \text{and} \quad \frac{J'(r, \theta)}{J(r, \theta)} \geq \frac{J_\kappa'(r)}{J_\kappa(r)}. \tag{3.7}$$

Then from the definitions of r_1 and r_2, we have $r_1 \leq r_2$. By direct calculation,

$$I_{M,p}'(t) = \frac{\int_{\mathbb{S}^{n-1}} J'(r_1, \theta)\, d\theta}{\int_{\mathbb{S}^{n-1}} J(r_1, \theta)\, d\theta} \geq \frac{\int_{\mathbb{S}^{n-1}} \frac{J_\kappa'(r_1)}{J_\kappa(r_1)} J(r_1, \theta)\, d\theta}{\int_{\mathbb{S}^{n-1}} J(r_1, \theta)\, d\theta} = \frac{J_\kappa'(r_1)}{J_\kappa(r_1)} \geq \frac{J_\kappa'(r_2)}{J_\kappa(r_2)},$$

where we used the comparison (3.7) in the first inequality and $r_1 \leq r_2$ in the last inequality. Similar calculation shows that

$$I_{M_\kappa,q}'(t) = \frac{J_\kappa'(r_2)}{J_\kappa(r_2)}.$$

Therefore, we have

$$I'_{M,p}(t) - I'_{M_\kappa,q}(t) \geq 0,$$

thus implying the lemma. □

The next lemma estimates the derivative of $\eta_p(r)$ in terms of the curvatures and the diameter of Ω.

Lemma 3.5. *Assume that (M^n, g) is complete simply connected with $\mathrm{Sect}_g \leq \kappa$ for $\kappa \leq 0$ and $\mathrm{Ric}_g \geq (n-1)Kg$ for $K \leq 0$. Fix any $p \in M^n$, then for all $r \in (0, d]$, where $d = \mathrm{diam}(\Omega)$, we have*

$$\eta'_p(r) \geq 1, \tag{3.8}$$

$$\max\left\{\eta'_p(r), \frac{\mathrm{sn}_\kappa\left(\eta_p(r)\right)}{\mathrm{sn}_\kappa(r)}\right\} \leq \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)}\right)^{n-1}. \tag{3.9}$$

Proof. We suppress p to simplify our notaiton, *e.g.*, we write η_p as η and $B_r(p)$ as B_r.

Since $\eta'(r) = \frac{d|B_r|}{dr}\frac{d\eta(r)}{d|B_r|}$, we have

$$\eta'(r) = \frac{|\partial B_r|}{m'_\kappa\left(m_\kappa^{-1}(|B_r|)\right)},$$

where $m_\kappa(r) = |B_r|_\kappa$. By the definition (3.2) of $\eta(r)$, we see that $\eta(r) = m_\kappa^{-1}(|B_r|)$. So then

$$m'_\kappa\left(m_\kappa^{-1}(|B_r|)\right) = m'_\kappa(\eta(r)) = |\partial B_{\eta(r)}|_\kappa,$$

which gives

$$\eta'(r) = \frac{|\partial B_r|}{|\partial B_{\eta(r)}|_\kappa}.$$

Since $\mathrm{Sect}_g \leq \kappa$, then from the isoperimetric inequality (3.6), we deduce

$$|\partial B_{\eta(r)}|_\kappa \leq |\partial B_r|,$$

thus proving (3.8).

Inequality (3.9) has been proven in [22], page 863. We give a different proof here. Since $\eta(r) \geq r$, we have

$$\eta'(r) = \frac{|\partial B_r|}{|\partial B_{\eta(r)}|_\kappa} \leq \frac{|\partial B_r|}{|\partial B_r|_\kappa} \leq \frac{|\partial B_r|_K}{|\partial B_r|_\kappa} \leq \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)}\right)^{n-1}, \tag{3.10}$$

where we have used the curvature condition $\mathrm{Ric}_g \geq (n-1)Kg$ and the fact that $\frac{\mathrm{sn}_K(r)}{\mathrm{sn}_\kappa(r)}$ is non-decreasing in r.

Using the isoperimetric inequality (3.6), we estimate that

$$\frac{\mathrm{sn}_\kappa\left(\eta(r)\right)}{\mathrm{sn}_\kappa(r)} = \left(\frac{|\partial B_{\eta(r)}|_\kappa}{|\partial B_r|_\kappa}\right)^{\frac{1}{n-1}} \leq \left(\frac{|\partial B_r|}{|\partial B_r|_\kappa}\right)^{\frac{1}{n-1}}.$$

Since $\mathrm{Ric}_g \geq (n-1)Kg$, we have $|\partial B_1| \leq |\partial B_r|_\kappa$. Therefore, we get

$$\frac{\mathrm{sn}_\kappa\left(\eta(r)\right)}{\mathrm{sn}_\kappa(r)} \leq \left(\frac{|\partial B_r|_K}{|\partial B_r|_\kappa}\right)^{\frac{1}{n-1}} = \frac{\mathrm{sn}_K(r)}{\mathrm{sn}_\kappa(r)} \leq \frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)} \tag{3.11}$$

where we have again used that $\frac{\mathrm{sn}_K(r)}{\mathrm{sn}_\kappa(r)}$ is non-decreasing in r. Then (3.9) follows from (3.10) and (3.11).

Therefore, the lemma is proved. □

4. Proof of Theorem 1.1

We first prove a center of mass result.

Lemma 4.1. *Assume that (M^n, g) is complete simply connected with $\mathrm{Sect}_g \leq \kappa$ for $\kappa \leq 0$, and $\Omega \subset M^n$ is any bounded domain. Then there exists a point $p \in \mathrm{hull}(\Omega)$, the closed geodesic convex hull of Ω, such that*

$$\int_{\partial\Omega} (F \circ \eta_p \circ r_p)(x) \frac{\exp_p^{-1}(x)}{r_p(x)}\, dA_g = 0,$$

where F is defined in equation (2.1), $r_p(x) = \mathrm{dist}_g(p, x)$, and $\exp_p^{-1}(x)$ denotes the inverse of the exponential map $\exp_p : T_pM^n \to M^n$.

Proof. The proof is similar to [22], Lemma 4.1. Define the vector field

$$X(p) = \int_{\partial\Omega} (F \circ \eta_p \circ r_p)(x) \frac{\exp_p^{-1}(x)}{r_p(x)}\, dA_g.$$

Then the integral curves of X defines a mapping from $\mathrm{hull}(\Omega)$ to itself. Since $\mathrm{hull}(\Omega)$ is convex and contained in the injectivity radius, $\mathrm{hull}(\Omega)$ is a topological ball and thus X must have a zero by the Brouwer fixed point theorem. □

We divide the proof of Theorem 1.1 into four propositions, each of which gives a different upper bound for $\sigma_1(\Omega)$ and might be of independent interest.

From here on, we fix $p \in \mathrm{hull}(\Omega)$ according to Lemma 4.1 so that

$$\int_{\partial\Omega} \frac{\exp_p^{-1}(x)}{r_p(x)} (F \circ \eta_p \circ r_p)(x)\, dA_g = 0. \tag{4.1}$$

We denote by (r, θ), where $\theta \in \mathbb{S}^{n-1}$, the polar coordinates centered at p and by $J(r,\theta)drd\theta$ the volume element at (r, θ). Then we have

$$\frac{\exp_p^{-1}(x)}{r_p(x)} = (\psi_1(\theta), \psi_2(\theta), \cdots, \psi_n(\theta)),$$

where $\psi_i(\theta)$'s are the restrictions of the linear coordinate functions on $\mathbb{S}^{n-1}$. We define

$$v_i := (F \circ \eta_p \circ r_p)\, \psi_i(\theta), \quad 1 \leq i \leq n.$$

Then (4.1) is equivalent to

$$\int_{\partial\Omega} v_i\, dA_g = 0, \quad 1 \leq i \leq n.$$

Using v_i's as test functions for $\sigma_1(\Omega)$, we obtain the following proposition.

Proposition 4.2. *Assuming the hypotheses of Theorem 1.1, then*

$$\sigma_1(\Omega) \le \frac{\int_\Omega \left(\left|F'(\eta_p(r_p))\eta_p'(r_p)\right|^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(r_p)} F^2(\eta_p(r_p)) \right) d\mu_g}{\int_{\partial\Omega} |F(\eta_p(r_p))|^2 \, dA_g}. \tag{4.2}$$

Proof. We write η_p and r_p as η and r for short.

We denote by $\nabla^{\mathbb{S}^{n-1}}$ the covariant derivative with respect to the standard metric on the unit sphere $\mathbb{S}^{n-1}$, and by ∇ the covariant derivative with respect to the metric $g = dr^2 + g_{ij}(r,\theta)d\theta^i d\theta^j$ on M. Using

$$\sum_{i=1}^n \psi_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^n |\nabla^{\mathbb{S}^{n-1}} \psi_i|^2 = n-1,$$

we compute that

$$\begin{aligned}
\sum_{i=1}^n \int_\Omega |\nabla v_i|^2 \, d\mu_g &= \sum_{i=1}^n \int_\Omega |\nabla \left(F(\eta(r))\psi_i\right)|^2 \, d\mu_g \\
&= \sum_{i=1}^n \int_\Omega \left\{ |F'(\eta(r))\eta'(r)|^2 \psi_i^2 + \frac{F^2(\eta(r))}{J^{\frac{2}{n-1}}(r,\theta)} |\nabla^{\mathbb{S}^{n-1}} \psi_i|^2 \right\} d\mu_g \\
&= \int_\Omega \left\{ |F'(\eta(r))\eta'(r)|^2 + F^2(\eta(r)) \frac{n-1}{J^{\frac{2}{n-1}}(r,\theta)} \right\} d\mu_g \\
&\le \int_\Omega \left\{ |F'(\eta(r))\eta'(r)|^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(r)} F^2(\eta(r)) \right\} d\mu_g,
\end{aligned} \tag{4.3}$$

where in the last step we used

$$J(r,\theta) \ge \mathrm{sn}_\kappa^{n-1}(r),$$

which follows from the Rauch comparison theorem. We also have

$$\sum_{i=1}^n \int_{\partial\Omega} v_i^2 \, dA_g = \sum_{i=1}^n \int_{\partial\Omega} |F(\eta(r))|^2 \psi_i^2 \, dA_g = \int_{\partial\Omega} |F(\eta(r))|^2 \, dA_g. \tag{4.4}$$

So using the averaging of Rayleigh quotients for v_i, (4.3) and (4.4), we obtain

$$\sigma_1(\Omega) \le \frac{\sum_{i=1}^n \int_\Omega |\nabla v_i|^2 \, d\mu_g}{\sum_{i=1}^n \int_{\partial\Omega} v_i^2 \, dA_g} \le \frac{\int_\Omega \left(|F'(\eta(r))\eta'(r)|^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(r)} F^2(\eta(r)) \right) d\mu_g}{\int_{\partial\Omega} |F(\eta(r))|^2 \, dA_g}.$$

This proves the proposition. ☐

Proposition 4.3. *Assuming the hypotheses of Theorem 1.1, then for functions G and H defined in Proposition 2.1, there holds*

$$\sigma_1(\Omega) \le \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)} \right)^{2n-2} \frac{\int_\Omega H(\eta_p(r_p)) \, d\mu_g}{\int_\Omega G(\eta_p(r_p)) \, d\mu_g}, \tag{4.5}$$

where $d = \mathrm{diam}(\Omega)$.

Proof. We write η_p and r_p as η and r for short.

It follows from the definition (2.4) of H and the estimate (3.9) in Lemma 3.5 that

$$\begin{aligned}
&|F'(\eta(r))|^2 \eta'(r)^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(r)} F^2(\eta(r)) \\
&\leq \max\left\{ \eta'(r)^2, \frac{\mathrm{sn}_\kappa^2\left(\eta(r)\right)}{\mathrm{sn}_\kappa^2(r)} \right\} \left(|F'(\eta(r))|^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(\eta(r))} F^2(\eta(r)) \right) \\
&\leq \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)} \right)^{2n-2} H(\eta(r)). \qquad (4.6)
\end{aligned}$$

We estimate the boundary integral.

$$\begin{aligned}
&\int_{\partial\Omega} F^2(\eta(r))\, dA_g \\
&\geq \int_{\partial\Omega} F^2(\eta(r)) \langle \nabla r, \nu \rangle \, dA_g \\
&= \int_\Omega \mathrm{div}\left(F^2(\eta(r)) \nabla r \right) \, d\mu_g \\
&= \int_\Omega \left\{ (F^2)'(\eta(r))\eta'(r) + F^2(\eta(r))\Delta r \right\} d\mu_g \\
&\geq \int_\Omega \left\{ (F^2)'(\eta(r))\eta'(r) + \frac{(n-1)\,\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)} F^2(\eta(r)) \right\} d\mu_g \\
&= \int_\Omega \left\{ (F^2)'(\eta(r))\eta'(r) + \frac{(n-1)\,\mathrm{sn}_\kappa'(\eta(r))}{\mathrm{sn}_\kappa(\eta(r))} F^2(\eta(r)) \frac{\mathrm{sn}_\kappa(\eta(r))}{\mathrm{sn}_\kappa'(\eta(r))} \frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)} \right\} d\mu_g \\
&\geq \int_\Omega \left\{ (F^2)'(\eta(r)) + \frac{(n-1)\,\mathrm{sn}_\kappa'(\eta(r))}{\mathrm{sn}_\kappa(\eta(r))} F^2(\eta(r)) \right\} d\mu_g \\
&= \int_\Omega G(\eta(r))\, d\mu_g, \qquad (4.7)
\end{aligned}$$

where the last equality follows from the definition (2.3) of G, in the first inequality we used $|\nabla r| = 1$, in the second inequality we used the Laplacian comparison theorem for the distance function, and in the last inequality we used $\eta'(r) \geq 1$ from Lemma 3.5 and

$$\frac{\mathrm{sn}_\kappa(\eta(r))}{\mathrm{sn}_\kappa'(\eta(r))} \frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)} \geq 1,$$

which follows from $\eta(r) \geq r$ and that $\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}$ is monotonically decreasing in r.

The proposition follows by substituting (4.6) and (4.7) into (4.2). ☐

Let $d\mu$ denote the volume form with respect to g_κ on the space form M_κ.

Proposition 4.4. *Assuming the hypotheses of Theorem 1.1, then for functions g and h defined in Proposition 2.1, there holds*

$$\sigma_1(\Omega) \leq \left(\frac{\mathrm{sn}_K(d)}{\mathrm{sn}_\kappa(d)} \right)^{2n-2} \frac{\int_{\Omega_q^*} H(r_q)\, d\mu}{\int_{\Omega_q^*} G(r_q)\, d\mu}. \qquad (4.8)$$

Proof. Lemma 3.3 applies to functions g and h defined in Proposition 2.1. Setting $f = G$ in inequality (3.3) and $f = H$ in inequality (3.4), and using Proposition 3.2, we obtain

$$\int_{\Omega} H \circ \eta_p \circ r_p \, d\mu_g = \int_{\Omega_q^*} (H \circ \eta_p \circ r_p)^* \, d\mu \leq \int_{\Omega_q^*} H(r_q) \, d\mu \tag{4.9}$$

and

$$\int_{\Omega} G \circ \eta_p \circ r_p \, d\mu_g = \int_{\Omega_q^*} (G \circ \eta_p \circ r_p)_* \, d\mu \geq \int_{\Omega_q^*} G(r_q) \, d\mu. \tag{4.10}$$

Assembling (4.5), (4.9) and (4.10) together, we conclude the proposition. □

Proposition 4.5. *Assuming the hypotheses of Theorem 1.1, then*

$$\sigma_1(\Omega_q^*) = \frac{\int_{\Omega_q^*} H(r_q) \, d\mu}{\int_{\Omega_q^*} G(r_q) \, d\mu}. \tag{4.11}$$

Proof. Recall that $F(r)\psi_i(\theta)$, $1 \leq i \leq n$, are the eigenfunctions for $\sigma_1(\Omega_q^*)$. It then follows that

$$\sigma_1(\Omega_q^*) = \frac{\int_{\Omega_q^*} \left(F'(r_q)^2 + \frac{n-1}{\mathrm{sn}_\kappa^2(r_q)} F^2(r_q) \right) d\mu}{\int_{\partial\Omega_q^*} F(^2 r_q) \, dA} = \frac{\int_{\Omega_q^*} H(r_q) \, d\mu}{\int_{\partial\Omega_q^*} F^2(r_q) \, dA},$$

where dA is the induced measure on $\partial\Omega_q^*$. Also recalling the definition (2.3) of G in Proposition 2.1, then we have

$$\begin{aligned}
\int_{\partial\Omega_q^*} F^2(r_q) \, dA &= \int_{\partial\Omega_q^*} \langle F^2(r_q) \nabla r_q, \nu \rangle \, dA \\
&= \int_{\Omega_q^*} \mathrm{div} \left(F^2(r_q) \nabla r_q \right) \, d\mu \\
&= \int_{\Omega_q^*} \left((F^2)' + F^2 \Delta r_q \right) \, d\mu \\
&= \int_{\Omega_q^*} \left((F^2)' + \frac{(n-1)\,\mathrm{sn}_\kappa'}{\mathrm{sn}_\kappa} F^2 \right) \, d\mu \\
&= \int_{\Omega_q^*} G(r_q) \, d\mu.
\end{aligned}$$

Therefore, we have proved the proposition. □

Proof of Theorem 1.1. Theorem 1.1 follows immediately from (4.8) and (4.11). □

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