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Abstract. Supersolvable hyperplane arrangements and matroids are known to give rise to certain Koszul algebras, namely their Orlik–Solomon algebras and graded Varchenko–Gel'fand algebras. We explore how this interacts with group actions, particularly for the braid arrangement and the action of the symmetric group, where the Hilbert functions of the algebras and their Koszul duals are given by Stirling numbers of the first and second kinds, respectively. The corresponding symmetric group representations exhibit branching rules that interpret Stirling number recurrences, which are shown to apply to all supersolvable arrangements. They also enjoy representation stability properties that follow from Koszul duality.

1. Introduction

This paper was motivated by a connection between *Stirling numbers* and *Koszul algebras.* The (signless) *Stirling numbers of the first kind* $c(n, k)$ and *Stirling numbers of the second kind* $S(n, k)$ are centuries-old answers to certain counting problems: $c(n, k)$ is the number of permutations $\{1, 2, \ldots, n\}$ with k cycles, while $S(n, k)$ is the number of set partitions of $\{1, 2, \ldots, n\}$ with k blocks. On the other hand, Koszul algebras A and their Koszul dual algebras $A^!$ originated in work of Priddy [72] and Fröberg [44] in the 1970s (see also Bărcănescu and Manolache [9, 10]), playing an important role in topology, and in homological and commutative algebra.

The connection stems from a particular Koszul dual pair of graded $\Bbbk$-algebras $A = \bigoplus_{d=0}^{\infty} A_d$ and $A^! = \bigoplus_{d=0}^{\infty} A^!_d$, described later, carrying actions of the symmetric group $\mathfrak{S}_n$.

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Their *Hilbert series*

$$\mathrm{Hilb}(A,t) := \sum_{d=0}^{\infty} \dim_{\Bbbk} A_d t^d = (1+t)(1+2t)\cdots(1+(n-1)t) \tag{1.1}$$

$$= \sum_{k=0}^{n-1} c(n, n-k) t^k, \tag{1.2}$$

$$\mathrm{Hilb}(A^!,t) := \sum_{d=0}^{\infty} \dim_{\Bbbk} A^!_d\, t^d = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} \tag{1.3}$$

$$= \sum_{k=0}^{\infty} S((n-1)+k, n-1)\, t^k \tag{1.4}$$

re-interpret the Stirling numbers $c(n,k), S(n,k)$.

In fact, there are two different well-studied algebras that can play the role of the algebra A above: the *Orlik–Solomon algebra* $\mathrm{OS}(\mathrm{Br}_n)$, or the *graded Varchenko–Gel'fand algebra* $\mathrm{VG}(\mathrm{Br}_n)$, associated to the matroid and oriented matroid Br_n for the *braid arrangement on n strands*, also known as the type *A reflection hyperplane arrangement*, or the *graphic arrangement* associated to the complete graph on n vertices. A great deal is known about the $\mathfrak{S}_n$-representations on the graded components A_d for either one of these algebras $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$, due to their importance in the topology of configuration spaces and in combinatorics. Their Koszul duals $A^!$ have seen less study from a combinatorial representation theory viewpoint, and were our original main interest.

A natural framework here turns out to be the combinatorial notion of *supersolvability*. Well-known results show that the algebras $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ for supersolvable matroids M and oriented matroids $\mathcal{M}$ have *quadratic Gröbner basis* presentations, which then implies their Koszulity.

Sections 2, 3, and 4 give background for this story. Section 2 is mainly a review of basic theory of Koszul algebras carrying group actions, although it contains one new observation on *branching rules* (Proposition 2.16). Section 3 recalls notions from *noncommutative Gröbner bases*, along with special features of commutative or anti-commutative rings, connecting quadratic Gröbner bases with Koszulity. Section 4 reviews *matroids*, *oriented matroids* and the notion of supersolvability.

Section 5 starts with a review of the well-studied anti-commutative Orlik–Solomon algebras $\mathrm{OS}(M)$ and their not quite as well-studied commutative counterparts, the graded Varchenko–Gel'fand rings $\mathrm{VG}(\mathcal{M})$. After recalling why both $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ are Koszul algebras whenever $M, \mathcal{M}$ are supersolvable, the first main result, Theorem 5.18, gives an explicit (noncommutative) quadratic Gröbner basis presentation for their Koszul duals $A^!$. In the case of $A = \mathrm{OS}(M)$, the presentation for $A^!$ is consistent with Kohno's presentation [53, 54] of the *holonomy Lie algebra* for the cohomology of the complement of a complex hyperplane arrangement; in the case of $A = \mathrm{VG}(\mathcal{M})$, the presentation for $A^!$ appears to be new. An application of the presentation, Corollary 5.22, gives a Koszul dual analogue of the fact that multiplication by the sum of the variables $\sum_i x_i$ endows $A = \mathrm{OS}(M)$ with an (equivariant) exact chain complex structure: in the supersolvable case, right-multiplication by the sum of the dual variables $\sum_i y_i$ within $A^! = \mathrm{OS}(M)^!$ gives an (equivariant) injective self-map of degree one.

Section 6 pauses to illustrate the foregoing theory on simple examples of supersolvable matroids, such as Boolean matroids and rank two matroids, including discussion of equivariant structure.

Section 7 proves the next main result, Theorem 7.1, giving branching rules for $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ and their Koszul duals $A^!$, in the form of short exact sequences that apply whenever $M, \mathcal{M}$ are supersolvable. For braid matroids Br_n, these short exact sequences re-interpret the two classical Stirling number recurrences:

$$\begin{aligned} c(n,k) &= (n-1)\cdot c(n-1,k) + c(n-1,k-1), \\ S(n,k) &= k \cdot S(n-1,k) + S(n-1,k-1). \end{aligned} \tag{1.5}$$

Sections 8, 9, and 10 review more general theory of Koszul algebras A, particularly when A is either anti-commutative (like $\mathrm{OS}(M)$) or commutative (like $\mathrm{VG}(\mathcal{M})$). Section 8 recalls why the Koszul dual $A^!$ is the universal enveloping algebra for its Lie (super-)algebra of primitive elements, also known as its *homotopy Lie algebra*, and why the latter coincides in this setting with its own *linear strand*, the *holonomy Lie algebra*. The Poincaré–Birkhoff–Witt Theorem for universal enveloping algebras then leads to equivariant versions of results such as the *lower central series formula* in the anti-commutative case, and the theory of *acyclic closures* and *deviations* in the commutative case. Section 9 briefly reviews the topological interpretations of Koszul duality, and the interpretation of $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ in terms of the cohomology of complements of subspace arrangements. Section 10 reviews Church and Farb's notion of representation stability for $\mathfrak{S}_n$-representations [25]. It then proves two results on its interaction with Koszul duality (Corollaries 10.6 and 10.10) showing that after fixing d, representation stability for the d^{th} graded components $\{A_d(n)\}_{n\geq 1}$ in a family of Koszul algebras implies the analogous representation stability for their Koszul duals $\{A^!_d(n)\}_{n\geq 1}$, along with a similar statement for their holonomy Lie algebras.

Finally, Section 11 returns to the motivating example of the braid arrangement matroids Br_n, examining the consequences of all the previous results for $\mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$, including the aforementioned branching rules re-interpreting the Stirling number recurrences, Corollary 11.5. One surprise here is Theorem 11.15, on the prevalence of *permutation* representations of $\mathfrak{S}_n$ among the homogeneous components $A^!_i$ of the Koszul dual $A^!$ when $A = \mathrm{OS}(\mathrm{Br}_n)$.

Section 12 collects some further remarks and questions. Appendix A includes tables of data for the characters of the Stirling representations of the first and second kind for $\mathrm{OS}(\mathrm{Br}_n)$ and $\mathrm{VG}(\mathrm{Br}_n)$ and the primitives of their corresponding holonomy Lie algebras. In addition, the code at [3] can also be used to generate more data.

Summary of main results. For the ease of the reader, we summarize below the main results and their applications to the type A braid arrangement Br_n.

- Theorem 5.18 provides an explicit noncommutative Gröbner basis for Koszul duals of Orlik–Solomon and Varchenko–Gel'fand rings of supersolvable matroids.
  - The discussion following Remark 11.2 explains the bijection between standard monomials for $\mathrm{OS}(\mathrm{Br}_n)$ and $\mathrm{VG}(\mathrm{Br}_n)$ and restricted growth functions.
- Corollary 5.22 shows that for a supersolvable matroid M or oriented matroid $\mathcal{M}$, right-multiplication by the sum of the dual variables $\sum_i y_i$ in $A^! = \mathrm{OS}(M)^!$ gives a degree one injective self-map, and the sum of the squares of the dual variables $\sum_i y_i^2$ in $A^! = \mathrm{VG}(\mathcal{M})^!$ gives a degree two injective self-map. These maps are equivariant with respect to any group G of automorphisms of $M, \mathcal{M}$.

  - We conjecture the existence of $\mathfrak{S}_n$-equivariant degree one injective self-maps for $\mathrm{VG}(\mathrm{Br}_n)$ in Conjecture 12.3.
- Proposition 2.16 shows that the graded pieces of an equivariant Koszul algebra A satisfy branching rules of a certain form if and only if the corresponding graded pieces for $A^!$ do. Theorem 7.1 gives short exact sequences for $\mathrm{OS}(M)$, $\mathrm{VG}(\mathcal{M})$ and their Koszul duals that lift such branching rules whenever one has supersolvable matroids.
  - Corollary 11.5 gives these branching rules for the Stirling representations, which lift the classical Stirling number recurrences (1.5).
- Theorem 8.6 gives a presentation for the holonomy Lie algebra of $\mathrm{VG}(\mathcal{M})$ for an arbitrary oriented matroid $\mathcal{M}$. In the supersolvable case, this presentation is consistent with the Gröbner basis for $\mathrm{VG}(\mathcal{M})^!$ from Theorem 5.18.
- Corollary 10.6 shows that if a family of Koszul algebras $A(n)$ with actions by $\mathfrak{S}_n$ are representation stable, then so are their Koszul duals.
  - Corollary 11.9 applies this to show representation stability for $\mathrm{OS}(\mathrm{Br}_n)^!$ and $\mathrm{VG}(\mathrm{Br}_n)^!$. Conjecture 11.11 conjectures that the bounds for the onset of stability given in Corollary 11.9 are tight.
- Corollary 10.10 shows that families of representation stable commutative or anti-commutative Koszul algebras $A(n)$ pass this representation stability to their holonomy Lie algebras $\mathcal{L}(n)$.
  - Corollary 11.12 states that this holds for the holonomy Lie algebras of $\mathrm{OS}(\mathrm{Br}_n)$ and $\mathrm{VG}(\mathrm{Br}_n)$. In Conjecture 11.14, we conjecture that the onset of stability is at $2i$ for high enough i.
- Theorem 11.15 summarizes several cases where $[\mathrm{OS}(\mathrm{Br}_n)^!_i]$ are permutation representations.

2. Koszul algebras

We review here the definitions and properties of Koszul algebras. Useful surveys and references are Berglund [12], Faber et al [40, Section 2], Fröberg [45], Mazorchuk, Ovsienko and Stroppel [61], McCullough and Peeva [62, Section 8], Polishchuk and Positselski [71], and Priddy [72].

2.1. Standard graded algebras and Koszul algebras.

Fix a field $\Bbbk$ throughout this discussion.

Definition 2.1 (Standard graded $\Bbbk$-algebras)**.** For V a $\Bbbk$-vector space with $\Bbbk$-basis $x_1, \ldots, x_n$, let

$$T^i(V) := V^{\otimes i} := \underbrace{V \otimes \cdots \otimes V}_{i \text{ tensor factors}},$$

and define the *tensor algebra* $T_\Bbbk(V) = \bigoplus_{i=0}^{\infty} T^i(V)$, with concatenation product. We identify it with

$$T_\Bbbk(V) \cong \Bbbk\langle x_1, \ldots, x_n\rangle,$$

the free associative $\Bbbk$-algebra on n letters. It is a *graded* $\Bbbk$-algebra, in which $T^i(V)$ is the i^{th} homogeneous component, and is generated as an algebra in degree 1 by V, the span of $x_1, \ldots, x_n$.

A *standard graded (associative)* $\Bbbk$*-algebra* is a graded quotient ring A of $T_\Bbbk(V)$, that is,

$$A = T_\Bbbk(V)/I \tag{2.1}$$

for some two-sided ideal $I \subset T_{\Bbbk}(V)$ which is *homogeneous*: $I = \bigoplus_{i=0}^{\infty} I_i$ where $I_i := I \cap T^i(V)$. We will generally assume that the images of $x_1, \ldots, x_n$ within A (which we still denote $x_1, \ldots, x_n$, abusing notation) are minimal generators for A as a $\Bbbk$-algebra, or equivalently, that $I = I_2 \oplus I_3 \oplus \cdots$.

Definition 2.2. (*Koszul algebras*) Given a standard graded $\Bbbk$-algebra A, let $A_+ := \bigoplus_{i=1}^{\infty} A_i$, and regard the field $\Bbbk = A/A_+$ as the *trivial (graded, left-)A-module*, generated in degree 0.

Call A a *Koszul algebra* if the surjection $A \twoheadrightarrow \Bbbk = A/A_+$ can be extended as the first step in a graded resolution of $\Bbbk$ by free left A-modules, which is *linear* in the sense that it has this form:

$$\begin{array}{ccccccccccccc} 0 \longleftarrow & \Bbbk & \longleftarrow & F_0 & \overset{d_1}{\longleftarrow} & F_1 & \overset{d_2}{\longleftarrow} & F_2 & \overset{d_3}{\longleftarrow} & F_3 & \longleftarrow \cdots \\ & & & \| & & \| & & \| & & \| & \\ & & & A & & A(-1)^{\beta_1} & & A(-2)^{\beta_2} & & A(-3)^{\beta_3} & \end{array} \tag{2.2}$$

Here $F_i = A(-i)^{\beta_i}$ is a graded free left A-module of rank β_i, all of whose A-basis elements have been shifted to degree i, that is $A(-i)_j := A_{j-i}$. Linearity of the above resolution is equivalent to saying that the matrices for the differentials $d_i : A_i \longrightarrow A_{i-1}$ in the resolution have only linear (degree one) entries, that is, all matrix entries lie in A_1.

Koszulity of A has strong consequences for its algebra presentation, and for the form of the resolution (2.2), related to the notion of *quadratic algebras* and their *quadratic duals*.

Definition 2.3. (*Quadratic algebras* and *quadratic duals*) Say that the standard graded $\Bbbk$-algebra A presented as in (2.1) is a *quadratic algebra* if I is generated as a two-sided ideal by

$$I_2 = I \cap T^2(V) = I \cap (V \otimes V).$$

For A any quadratic algebra, presented as in (2.1), one defines its *quadratic dual algebra* $A^!$ as follows. Let V^* have $\Bbbk$-dual basis $y_1, \ldots, y_n$ to the ordered $\Bbbk$-basis $x_1, \ldots, x_n$ for V, so that the bilinear pairing $V^* \times V \to \Bbbk$ has $(y_i, x_j) = \delta_{ij}$. Then $T^2(V^*)$ and $T^2(V)$ have dual $\Bbbk$-bases

$$\{y_i \otimes y_j\}_{1 \leq i, j \leq n} \text{ and } \{x_i \otimes x_j\}_{1 \leq i, j \leq n}$$

with respect to the bilinear pairing $T^2(V^*) \times T^2(V) \to \Bbbk$ defined by

$$(y \otimes y', x \otimes x') := (y, x) \cdot (y', x'). \tag{2.3}$$

Define $A^!$ as this quadratic algebra quotient of the free associative algebra $T_{\Bbbk}(V^*) = \Bbbk\langle y_1, \ldots, y_n \rangle$:

$$A^! := T_{\Bbbk}(V^*)/J$$

where J is the two-sided ideal generated by

$$J_2 = I_2^{\perp} = \left\{ p \in T^2(V^*) : (p, q) = 0 \text{ for all } q \in I_2 \right\}.$$

Note that this really is a duality, in the sense that $(A^!)^! \cong A$.

Example 2.4. A commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is a quadratic $\Bbbk$-algebra:

$$A = \mathrm{Sym}(V) = \Bbbk[x_1, \ldots, x_n] \cong \Bbbk\langle x_1, \ldots, x_n \rangle / I$$

where $I = (x_i x_j - x_j x_i)_{1 \leq i < j \leq n}$. Its quadratic dual $A^!$ is the anti-commutative exterior algebra

$$A^! = \wedge(V^*) = \wedge(y_1, \ldots, y_n) = \Bbbk\langle y_1, \ldots, y_n \rangle / J$$

where $J = (y_i y_j + y_j y_i)_{1 \leq i < j \leq n} + (y_i^2)_{1 \leq i \leq n}$.

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2.2. **Priddy's resolution and its consequences.** It is not hard to show that Koszul algebras A are always quadratic.[1] What is more remarkable is a result of Priddy [72], using $A^!$ to construct a simple, explicit linear A-resolution of $\Bbbk$ whenever A is Koszul. Before describing it, let us point out certain maps on A and on the *graded* $\Bbbk$*-dual* $(A^!)^*$. The latter is defined to be the following graded $\Bbbk$-vector subspace of the usual dual $\mathrm{Hom}_\Bbbk(A^!, \Bbbk)$:

$$(A^!)^* := \bigoplus_{i=0}^{\infty} (A^!_i)^*.$$

- For x in A_1, the map on A which right-multiplies by x, that is $a \mapsto ax$, gives a left A-module map $A \longrightarrow A$, raising degree by one.
- For y in $A^!_1$, the map precomposing φ in $(A^!)^*$ with left-multiplication[2] by y, that is $\varphi \mapsto \varphi.y$ where $(\varphi.y)(b) := \varphi(yb)$, gives a $\Bbbk$-linear map $(A^!)^* \to (A^!)^*$, lowering degree by one.
- Combining these, any $x \otimes y$ in $A_1 \otimes A^!_1 = V \otimes V^*$ gives rise to a (left A-module) map $A \otimes (A^!)^* \longrightarrow A \otimes (A^!)^*$ that sends $a \otimes \varphi \longmapsto (x \otimes y).(a \otimes \varphi) := ax \otimes \varphi.y$.

Theorem 2.5 (The Priddy resolution)**.** *When A is Koszul, the element $c := \sum_{j=1}^n x_j \otimes y_j$ in $A_1 \otimes A^!_1$ acting on $A \otimes_\Bbbk (A^!)^*$ as a left A-module map gives a linear resolution of $\Bbbk$ as in* (2.2)*,*

$$0 \longleftarrow \Bbbk \longleftarrow A \otimes_\Bbbk (A^!_0)^* \xleftarrow{d_1} A \otimes_\Bbbk (A^!_1)^* \xleftarrow{d_2} A \otimes_\Bbbk (A^!_2)^* \xleftarrow{d_3} \cdots$$

Its differential $d_i : A \otimes_\Bbbk (A^!_i)^* \xrightarrow{d_i} A \otimes_\Bbbk (A^!_{i-1})^*$ is given explicitly as follows:

$$a \otimes \varphi \longmapsto c.\,(a \otimes \varphi) = \sum_{j=1}^{n} ax_j \otimes \varphi.y_j. \tag{2.4}$$

Example 2.6. Continuing Example 2.4, one can check that the Priddy resolution for $\Bbbk$ over $A = \Bbbk[x_1, \ldots, x_n] = \mathrm{Sym}(V)$ becomes the usual *Koszul resolution*

$$0 \leftarrow \Bbbk \leftarrow \mathrm{Sym}(V) \otimes_\Bbbk \wedge^0(V) \leftarrow \mathrm{Sym}(V) \otimes_\Bbbk \wedge^1(V) \leftarrow \cdots \leftarrow \mathrm{Sym}(V) \otimes_\Bbbk \wedge^n(V) \leftarrow 0,$$

using that fact that $(A^!_i)^* = (\wedge^i(V^*))^* \cong \wedge^i(V)$.

We note some important consequences of Priddy's resolution. Taking graded $\Bbbk$-duals swaps the roles of A and $A^!$ in the resolution. Consequently, A is Koszul if and only if $A^!$ is Koszul. In this case, one calls $A^!$ the *Koszul dual algebra* of A. Priddy's resolution also has an important consequence for the *Hilbert series* of $A, A^!$:

$$\mathrm{Hilb}(A,t) := \sum_{i=0}^{\infty} \dim_\Bbbk A_i t^i,$$
$$\mathrm{Hilb}(A^!,t) := \sum_{i=0}^{\infty} \dim_\Bbbk A^!_i t^i = \sum_{i=0}^{\infty} \dim_\Bbbk (A^!_i)^* t^i = \mathrm{Hilb}\left((A^!)^*, t\right).$$

Corollary 2.7. *Whenever $A, A^!$ are Koszul, one has* $\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!, -t) = 1$.

[1] Quadraticity is equivalent to having a partial linear resolution $0 \leftarrow \Bbbk \leftarrow A \leftarrow F_1 \leftarrow F_2$ up to homological degree 2.

[2] This corrects a typo in the definition from [62], and agrees with [71, S 2.3, pp. 25-27], [61, Proposition 44].

Proof. For each degree $d \geq 1$, taking the coefficient of t^d on both sides in the corollary gives the identity

$$\sum_{i=0}^{d} (-1)^i \dim_{\Bbbk} A_{d-i} \cdot \dim_{\Bbbk} (A^!_i)^* = 0$$

asserting vanishing of Euler characteristic for the (exact) d^{th} graded component in Priddy's resolution

$$0 \to A_d \otimes_{\Bbbk} (A^!_0)^* \to A_{d-1} \otimes_{\Bbbk} (A^!_1)^* \to \cdots \to A_1 \otimes_{\Bbbk} (A^!_{d-1})^* \to A_0 \otimes_{\Bbbk} (A^!_d)^* \to 0. \quad (2.5)$$

□

Example 2.8. For the pair of Koszul dual algebras

$$A = \mathrm{Sym}(V) = \Bbbk[x_1, \ldots, x_n],$$
$$A^! = \wedge(V^*) = \wedge(y_1, \ldots, y_n),$$

one has these Hilbert series

$$\mathrm{Hilb}(A, t) = \frac{1}{(1-t)^n} \quad \text{with } \dim_{\Bbbk} A_i = \left(\!\binom{n}{i}\!\right) := \binom{n+i-1}{i},$$
$$\mathrm{Hilb}(A^!, t) = (1+t)^n \quad \text{with } \dim_{\Bbbk} A^!_i = \binom{n}{i}.$$

Example 2.9 (Noncommutative monomial Koszul algebras)**.** When a two-sided ideal I inside $T(V) = \Bbbk\langle x_1, \ldots, x_n\rangle$ is generated by a subset of noncommutative monomials, it is called a *monomial ideal.* It is called a *quadratic monomial ideal* if the generating monomials are quadratic, that is, they form a subset of the n^2 monomials $\{x_i x_j : (i,j) \subseteq [n] \times [n]\}$. Starting with any quadratic monomial ideal I, one can associate two complementary binary relations $D, D^c \subseteq [n] \times [n]$:

$$D := \{(i,j) \in [n] \times [n] : x_i x_j \notin I\},$$
$$D^c := \{(i,j) \in [n] \times [n] : x_i x_j \in I\}.$$

In this setting, denote the ideal I by I_D, and denote the quotient algebra $A_D := T(V)/I_D$. One can view D as a choice of a *directed graph* on vertex set $[n]$ having no repeated directed arcs $i \to j$, but allowing (single) copies of loops $i \to i$ and (single) pairs of antiparallel arcs $i \to j$ and $j \to i$. Then the d^{th} homogeneous component $(A_D)_d$ of A_D has a $\Bbbk$-basis indexed by the monomials $x_{i_1} x_{i_2} \cdots x_{i_d}$ whose subscripts $(i_1, i_2, \ldots, i_d)$ correspond to walks with $d-1$ steps along arcs $i_j \to i_{j+1}$ in the digraph D. Hence $\mathrm{Hilb}(A_D, t) = 1 + \sum_{d=1}^{\infty} a_D(d) t^d$, where $a_D(d)$ is the number of such walks.

It turns out that these (noncommutative) *quadratic monomial* $\Bbbk$*-algebras* are always Koszul. A linear resolution of $\Bbbk$ over A_D is a special case of a resolution constructed by Fröberg in [44], and was also described recursively by Bruns, Herzog and Vetter [22, Section 3]; we review the latter construction here. Note that the quadratic dual $A^!_D$ has the form

$$A^!_D = T(V^*)/J_{D^c} \quad \text{where} \quad J_{D^c} = (y_i y_j : (i,j) \in D) = \left((I_D)^{\perp}_2\right).$$

Letting $A := A_D$, the linear A-free resolution $0 \leftarrow \Bbbk \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ described recursively in [22] has F_d being a free left A-module whose A-basis elements $\{e_{(i_1, \ldots, i_d)}\}$

are indexed by all walks $(i_1, i_2, \ldots, i_d)$ taking $d-1$ steps along arcs $i_j \to i_{j+1}$ in the complement D^c. Unraveling their recursion, the resolution has these A-linear differentials:

$$e_{(i_1, i_2, \ldots, i_d)} \longmapsto x_{i_1} \, e_{(i_2, \ldots, i_d)}. \tag{2.6}$$

Note that one has an isomorphism of free A-modules

$$\begin{array}{rcl} F_d & \longrightarrow & A \otimes_{\Bbbk} (A^!_d)^* \\ a\, e_{(i_1, \ldots, i_d)} & \longmapsto & a \otimes [y_{i_1} \cdots y_{i_d}]^* \end{array} \tag{2.7}$$

where $[y_{i_1} \cdots y_{i_d}]^* \in (A^!_d)^*$ is the $\Bbbk$-linear functional $A^!_d \to \Bbbk$ sending $y_{i_1} \cdots y_{i_d}$ to 1 and sending all other degree d monomials to 0. One can check that the definitions preceding Theorem 2.5 imply

$$[y_{i_1} y_{i_2} \cdots y_{i_d}]^* . y_j = \begin{cases} [y_{i_2} \cdots y_{i_d}]^* & \text{if } j = i_1 \\ 0 & \text{otherwise.} \end{cases}$$

One therefore concludes that the differential in Priddy's resolution is the A-linear map sending

$$1 \otimes [y_{i_1} y_{i_2} \cdots y_{i_d}]^* \longmapsto \sum_{j=1}^{n} x_j \otimes ([y_{i_1} y_{i_2} \cdots y_{i_d}]^* . y_j) = x_{i_1} \otimes [y_{i_2} \cdots y_{i_d}]^*.$$

This agrees with the differential described by (2.6) after passing through the isomorphism (2.7).

Note that since A_D is Koszul, and $A^!_D \cong A_{D^c}$, one has $\mathrm{Hilb}(A_D, t) \cdot \mathrm{Hilb}(A_{D^c}, -t) = 1$, an identity which appeared earlier in work of Brenti [20, Section 7.5].

Our goal is to study Koszul algebras A together with symmetries coming from a finite group G of graded ring automorphisms. We will regard each graded component A_i and $A^!_i$ as representations of G, or equivalently, as $\Bbbk G$-modules. In order to work over arbitrary fields $\Bbbk$ where $\Bbbk G$ might not be semisimple, we introduce the *Grothendieck ring* $R_{\Bbbk}(G)$.

Definition 2.10 (Grothendieck ring)**.** As a $\mathbb{Z}$-module, the *Grothendieck group* of $\Bbbk G$-modules $R_{\Bbbk}(G)$ is a quotient of the free $\mathbb{Z}$-module whose basis is the set of isomorphism classes $[V]$ of finite-dimensional $\Bbbk G$-modules V, and where one mods out by the $\mathbb{Z}$-span of these relations:

$$\{[V] - ([U] + [W]) : \text{ for all } \Bbbk G\text{-module short exact sequences } 0 \to U \to V \to W \to 0\} \tag{2.8}$$

In particular, in $R_{\Bbbk}(G)$ one has $[U \oplus W] = [U] + [W]$. Multiplication in $R_{\Bbbk}(G)$ is induced by the rule $[V] \cdot [W] := [V \otimes_{\Bbbk} W]$, which one can check is consistent with the relations in (2.8).

We collect here a few standard facts about $R_{\Bbbk}(G)$, omitting the proofs.

Proposition 2.11. *For any finite group G, one has the following.*

(i) The relations in $R_{\Bbbk}(G)$ imply $\sum_{i=0}^{\ell} (-1)^i [V_i] = 0$ for longer exact sequences of $\Bbbk G$-modules

$$0 \leftarrow V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_\ell \leftarrow 0.$$

(ii) More generally, a finite $\Bbbk G$-module complex $0 \leftarrow C_0 \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} C_\ell \leftarrow 0$ with homology $\{H_*\}$ gives an Euler–Poincaré–Hopf–Lefschetz relation $\sum_{i=0}^{\ell} (-1)^i [C_i] = \sum_{i=0}^{\ell} (-1)^i [H_i]$ in $R_{\Bbbk}(G)$.

(iii) *Short exact sequences* $0 \to U \to V \to W \to 0$ *of* $\Bbbk G$*-modules lead to* dual/contragredient *exact sequences* $0 \to W^* \to V^* \to U^* \to 0$*, and also* $(U \otimes V)^* \cong U^* \otimes V^*$*. Hence the involution* $[U] \mapsto [U^*]$ *induces a well-defined involutive ring automorphism* $(-)^* : R_\Bbbk(G) \to R_\Bbbk(G)$*.*

(iv) *For subgroups* H *of* G*, the map* $[U] \mapsto [U \downarrow_H^G]$*, where* $U \downarrow_H^G$ *is the* restriction *of the* $\Bbbk G$*-module* U *to a* $\Bbbk H$*-module, induces a well-defined ring map* $(-)\downarrow : R_\Bbbk(G) \to R_\Bbbk(H)$*.*

(v) *Since* $(U^*) \downarrow_H^G \cong \left(U \downarrow_H^G\right)^*$ *as* $\Bbbk H$*-modules, the maps in (iii),(iv) commute.*

Remark 2.12. We explain here why a group G acting on a Koszul algebra A also acts on the Koszul dual $A^!$. When a standard graded $\Bbbk$-algebra $A = T(V)/I$ carries the action of a group G of graded $\Bbbk$-algebra automorphisms, the fact that G preserves $A_1 = V$, and A_1 generates A, implies that one can regard G as a subgroup of $GL(V)$, possibly replacing G by G/K if K is the kernel of its action on V. Then G also acts contragrediently on V^*, via $\varphi \mapsto \varphi \circ g^{-1}$. This gives the natural $\Bbbk$-bilinear pairing $V^* \otimes V \to \Bbbk$ defined by $\varphi \otimes v \mapsto \varphi(v)$ a certain G-invariance:

$$g(\varphi \otimes v) = \left(\varphi \circ g^{-1}\right) \otimes g(v) \mapsto \varphi\left(g^{-1}(g(v))\right) = \varphi(v).$$

The dual pairing (2.3) between $T^2(V^*)$ and $T^2(V)$ then inherits this same G-invariance.

Consequently, when $A = T(V)/I$ is a quadratic algebra with the action of a group G preserving the subspace $I_2 \subset T^2(V)$ that generates the ideal I, then G also preserves the subspace $J_2 = I_2^\perp$ that generates the ideal J defining the quadratic dual $A^! = T(V^*)/J$. Thus G also acts on $A^!$.

The following proposition should not be surprising.

Proposition 2.13. *When* $A, A^!$ *are Koszul, the Priddy resolution is* G*-equivariant for any group of graded* $\Bbbk$*-algebra automorphisms acting on* A *(and hence on* $A^!$*).*

Proof. This follows because the differential acts by $c = \sum_{j=1}^n x_j \otimes y_j$ in $A_1 \otimes A_1^! = V \otimes V^*$, and c is G-fixed: under the G-equivariant isomorphism $V \otimes V^* \cong \mathrm{End}_\Bbbk(V)$ that sends $v \otimes f$ to $\varphi : V \to V$ given by $\varphi(w) = f(w) \cdot v$, one can check that $c \mapsto 1_V$, which is a G-fixed element of $\mathrm{End}_\Bbbk(V)$. ☐

This gives a version of Corollary 2.7, regarding the *equivariant Hilbert series* in $R_\Bbbk(G)[[t]]$

$$\mathrm{Hilb}_{\mathrm{eq}}(A, t) := \sum_{i=0}^{\infty} [A_i] t^i. \tag{2.9}$$

Corollary 2.14 (cf. [51, Proposition 8.1])**.** *Let* $A, A^!$ *be Koszul dual algebras, both with the action of a group* G *of graded* $\Bbbk$*-algebra automorphisms. Then one has this identity in* $R_\Bbbk(G)[[t]]$*:*

$$\mathrm{Hilb}_{\mathrm{eq}}(A, t) \cdot \mathrm{Hilb}_{\mathrm{eq}}\left((A^!)^*, -t\right) = 1 \tag{2.10}$$

Equivalently, $[A_0] = [(A_0^!)^*] = [\mathbb{1}_G]$ *and one has these identities in* $R_\Bbbk(G)$ *for* $d \geq 1$*:*

$$\sum_{i=0}^{d} (-1)^i [A_{d-i}] \cdot \left[(A_i^!)^*\right] = 0 \tag{2.11}$$

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which can be rewritten as this recurrence for $[(A^!_d)^*]$*:*

$$[(A^!_d)^*] = \sum_{i=1}^{d} (-1)^{i-1}[A_i] \cdot \left[(A^!_{d-i})^*\right] \tag{2.12}$$

and this unraveled formula:

$$\left[(A^!_d)^*\right] = \sum_{\substack{\alpha=(\alpha_1,\,\ldots,\,\alpha_\ell):\\ \alpha_1+\cdots+\alpha_\ell=d}} (-1)^{d-\ell}[A_{\alpha_1}][A_{\alpha_2}]\cdots[A_{\alpha_\ell}]. \tag{2.13}$$

This last sum runs over all (strict) ordered compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ *of* d*, of any length* $\ell \geq 1$*, that is,* α_i *are positive integers summing to* d*.*

Proof. It suffices to prove (2.11), which follows from the G-equivariance and exactness of (2.5). ☐

Example 2.15. Continuing Examples 2.4, 2.6, 2.8, the Koszul algebras $A = \mathrm{Sym}(V)$, $A^! = \wedge(V^*)$ carry the action of $G = GL(V)$. There is a ring homomorphism from $R_{\Bbbk}(G)$ to the ring

$$\Lambda_{\Bbbk}(\mathbf{z}) := \Lambda_{\Bbbk}(z_1, \ldots, z_n) = \Bbbk[z_1, \ldots, z_n]^{\mathfrak{S}_n}$$

of symmetric polynomials in n variables with $\Bbbk$ coefficients, mapping the class $[U]$ of a $\Bbbk G$-module U to $\mathrm{trace}(g|_U)$ where $g = \mathrm{diag}(z_1, \ldots, z_n)$ in $GL(V)$ is the diagonal matrix in $GL(V)$ having $g(x_i) = z_i \cdot x_i$ in V for $i = 1, 2, \ldots, n$, so that $g(y_i) = z_i^{-1} \cdot y_i$ in V^*.

Applying this homomorphism to (2.10) gives a standard identity $H(t)E(-t) = 1$ in $\Lambda_{\Bbbk}(\mathbf{z})[[t]]$, where

$$H(t) := \sum_{k=0}^{\infty} h_k(z_1, \ldots, z_n)t^k = \prod_{j=1}^{n} \frac{1}{1 - z_j t},$$
$$E(t) := \sum_{k=0}^{n} e_k(z_1, \ldots, z_n)t^k = \prod_{j=1}^{n} (1 + z_j t)\,.$$

This can be viewed as the specialization of a well-known identity in the ring of symmetric functions in infinitely many variables $\Lambda := \Lambda_{\mathbb{Z}}(z_1, z_2, \ldots)$ with integer coefficients, relating the two sets of algebraically independent generators $\{h_1, h_2, \ldots\}$ and $\{e_1, e_2, \ldots\}$; see [58, Chapter 1, eq. (2.6)], [83, Theorem 7.6.1]. Rewritten as in (2.12), one has $e_0 = h_0 = 1$ and $e_d = \sum_{i=1}^{d}(-1)^{i-1} h_i \cdot e_{d-i}$ for all $d \geq 1$. Due to their algebraic independence, any symmetric function identities in Λ among $\{h_i\}, \{e_i\}$ lead to the same identities relating $\{[A_1], [A_2], \ldots\}, \{[(A^!_1)^*], [(A^!_2)^*]\}$ in $R_{\Bbbk}(G)$ for any Koszul algebra A over any field $\Bbbk$. For example, a special case of the *Jacobi–Trudi identity* [58, Chapter 1, eq. (3.4)], [83, Theorem 7.16.1] expresses the $\{e_k\}$ in terms of the $\{h_k\}$:

$$e_d = \det \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & & \\ 1 & h_1 & h_2 & \cdots & & \\ 0 & 1 & h_1 & \cdots & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & h_1 \end{bmatrix} = \sum_{\alpha=(\alpha_1,\,\ldots,\,\alpha_\ell)} (-1)^{d-\ell} h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell},$$

where α runs over all compositions of d. One now recovers the unraveled formula (2.13) for $[(A^!_d)^*]$.

2.3. **A Koszul branching relation.** We wish to lift several combinatorial recurrences to *branching rules* for Koszul algebras A and their Koszul duals $A^!$. Recall from Proposition 2.11 (iv) that for any subgroup H of a group G, the map $[U] \mapsto [U \downarrow_H^G]$ induces a ring map $(-)\downarrow\colon R_{\Bbbk}(G) \to R_{\Bbbk}(H)$.

Proposition 2.16. *Let A, B be two Koszul $\Bbbk$-algebras, with actions of groups G, H, where H is a subgroup of G, and let $\mathcal{X}$ be a $\Bbbk H$-module. Then in $R_{\Bbbk}(H)$, one has*

$$\begin{aligned} & [A_i \downarrow] = [B_i] + [\mathcal{X}] \cdot [B_{i-1}] \\ \textit{if and only if} \quad & \left[(A_i^!)^* \downarrow\right] = [(B_i^!)^*] + [\mathcal{X}] \cdot \left(\left[(A_{i-1}^!)^* \downarrow\right]\right) \\ \textit{if and only if} \quad & \left[A_i^! \downarrow\right] = [B_i^!] + [\mathcal{X}^*] \cdot \left(\left[A_{i-1}^! \downarrow\right]\right) \end{aligned}$$

Proof. The last equivalence uses the properties of the ring automorphism $(-)^* : R_{\Bbbk}(G) \to R_{\Bbbk}(G)$ from Proposition 2.11 (iii), (iv), (v). Hence it suffices to prove the first equivalence.

Introduce a few abbreviated notations

$$\begin{aligned} a_i &:= [A_i] \text{ and } a_i^{!*} := \left[(A_i^!)^*\right] \text{ in } R_{\Bbbk}(G), \\ b_i &:= [B_i] \text{ and } b_i^{!*} = \left[(B_i^!)^*\right] \text{ in } R_{\Bbbk}(H), \\ \bar{a}_i &:= [A_i \downarrow] \text{ and } \bar{a}_i^{!*} := \left[(A_i^!)^* \downarrow\right] \text{ in } R_{\Bbbk}(H), \\ x &:= [\mathcal{X}] \text{ in } R_{\Bbbk}(H) \end{aligned}$$

along with analogous generating functions in $R_{\Bbbk}(G)[[t]]$ and $R_{\Bbbk}(H)[[t]]$, such as $a(t) := \sum_i a_i t^i$, and similarly $b(t), a^{!*}(t), b^{!*}(t), \bar{a}(t)$. In this notation, the first equivalence of the proposition asserts

$$\bar{a}_i = b_i + x b_{i-1} \Leftrightarrow \bar{a}_i^{!*} = b_i^{!*} + x \bar{a}_{i-1}^{!*}.$$

Note that one has these three relations, coming from Corollary 2.14 for the Koszul algebras A, B, and applying the ring map $(-)\downarrow$ to the first relation:

$$\begin{aligned} a^{!*}(t) a(-t) &= 1 \\ b^{!*}(t) b(-t) &= 1 \\ \bar{a}^{!*}(t) \bar{a}(-t) &= 1 \end{aligned}$$

This lets one compute as follows:

$$\begin{aligned} \bar{a}_i = b_i + x b_{i-1} \Leftrightarrow \quad & \bar{a}(t) = (1 + xt) \cdot b(t) \\ \Leftrightarrow \quad & \frac{1}{\bar{a}(-t)} = \frac{1}{1 - xt} \cdot \frac{1}{b(-t)} \\ \Leftrightarrow \quad & \bar{a}^{!*}(t) = \frac{1}{1 - xt} \cdot b^{!*}(t) \\ \Leftrightarrow \quad & (1 - xt) \cdot \bar{a}^{!*}(t) = b^{!*}(t) \\ \Leftrightarrow \quad & \bar{a}_i^{!*} - x \bar{a}_{i-1}^{!*} = b_i^! \\ \Leftrightarrow \quad & \bar{a}_i^{!*} = b_i^{!*} + x \bar{a}_{i-1}^{!*}. \end{aligned}$$

□

Example 2.17. Continuing Example 2.8, the symmetric group $G = \mathfrak{S}_n$ acts on the Koszul dual algebras $A(n) := \Bbbk[x_1, \ldots, x_n] = \mathrm{Sym}(V)$ and $A(n)^! = \wedge(y_1, \ldots, y_n) = \wedge(V^*)$ by permuting variables. One can apply Proposition 2.16 with $B = A(n-1) = \Bbbk[x_1, \ldots, x_{n-1}], B^! = \wedge(y_1, \ldots, y_{n-1})$, which are both $\Bbbk H$-modules for $H = \mathfrak{S}_{n-1}$, and with $\mathcal{X} = \mathbb{1}_H$ the trivial $\Bbbk H$-module. Recalling the notation $\left(\!\binom{n}{i}\!\right) := \binom{n+i-1}{i}$, one then sees that the proposition lifts the equivalence of these two versions of the Pascal recurrence

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1},$$
$$\left(\!\binom{n}{i}\!\right) = \left(\!\binom{n-1}{i}\!\right) + \left(\!\binom{n}{i-1}\!\right),$$

to an equivalence of statements on restricting $A(n)_i, A(n)^!_i$ from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

$$\left[A(n)^!_i \downarrow\right] = \left[A(n-1)^!_i\right] + \left[A(n-1)^!_{i-1}\right],$$
$$[A(n)_i \downarrow] = [A(n-1)_i] + [A(n)_{i-1} \downarrow].$$

Both also follow from segregating the degree i monomials in $\Bbbk[x_1, \ldots, x_n]$ or $\wedge(y_1, \ldots, y_n)$, counted by the left sides, into monomials *not divisible* by the last variable x_n, y_n, versus those *divisible* by it.

3. Review of noncommutative, commutative, exterior Gröbner bases

We review here some of the theory of Gröbner bases for two-sided ideals I in noncommutative, commutative and exterior algebras over a field $\Bbbk$, emphasizing aspects that are special to the situation where I is homogeneous, and/or quadratic. Useful references for the

- commutative theory: Cox, Little and O'Shea [29], Adams and Loustaunau [1], Eisenbud [37, Ch. 15],
- exterior algebra theory: Aramova, Herzog and Hibi [4], Stokes [85],
- noncommutative theory: Bokut and Chen [18], Mora [64], Ufnarovskii [91, Section 2], Polishchuk and Positselski [71, Chapter 4], Shepler and Witherspoon [79, Section 3].

3.1. Monomial orders, initial forms, and initial ideals.

Fix a positive integer n, and abbreviate the free associative, commutative, and exterior algebras R in n variables $z_1, \ldots, z_n$ as follows:

$$\Bbbk\langle \mathbf{z} \rangle := \Bbbk\langle z_1, \ldots, z_n \rangle,$$
$$\Bbbk[\mathbf{z}] := \Bbbk[z_1, \ldots, z_n],$$
$$\wedge(\mathbf{z}) := \wedge(z_1, \ldots, z_n).$$

The set of monomials in each these rings R will be denoted

$$\mathrm{Mons}(\Bbbk\langle \mathbf{z} \rangle) := \left\{ z_{i_1} z_{i_2} \cdots z_{i_\ell} : \ell \geq 0 \text{ and } (i_1, \ldots, i_\ell) \in [n]^\ell \right\}$$
$$\mathrm{Mons}(\Bbbk[\mathbf{z}]) := \{ \mathbf{z}^{\mathbf{a}} = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} : \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \}$$
$$\mathrm{Mons}(\wedge(\mathbf{z})) := \{ \mathbf{z}_S = z_{i_1} \wedge z_{i_2} \wedge \cdots \wedge z_{i_\ell} : S = \{ i_1 < i_2 < \cdots < i_\ell \} \subseteq [n] \}.$$

Definition 3.1 (Monomial orders, initial forms, initial ideals)**.** A linear ordering $\prec$ on $\mathrm{Mons}(R)$ for any of the above three rings R is called a *monomial ordering* if

- it is a well-ordering: there are no infinite descending chains $m_1 \succ m_2 \succ m_3 \succ \cdots$, and
- whenever $m \prec m'$, then $m_1 m m_2 \prec m_1 m' m_2$ for any other monomials m_1, m_2.

Having fixed a monomial order $\prec$ on one of these rings $R = \Bbbk\langle \mathbf{z}\rangle, \Bbbk[\mathbf{z}], \wedge(\mathbf{z})$, write any ring element as a finite $\Bbbk$-linear sum of monomials m with nonzero coefficients c_m in $\Bbbk$

$$f = \sum_{m \in \mathrm{Mons}(R)} c_m m \ = c_{m_0} \cdot m_0 + \sum_{\substack{m \in \mathrm{Mons}(R): \\ m \prec m_0}} c_m m$$

and then define m_0 to be its unique *$\prec$-initial term* or *$\prec$-leading monomial*, denoted $\mathrm{in}_\prec(f) := m_0$. Given a (two-sided) ideal $I \subset R$, define its *$\prec$-initial ideal* to be this two-sided monomial ideal of R:

$$\mathrm{in}_\prec(I) := (\mathrm{in}_\prec(f) : f \in I)\,.$$

Definition 3.2. Given a monomial order $\prec$ on one of $R = \Bbbk\langle \mathbf{z}\rangle, \Bbbk[\mathbf{z}], \wedge(\mathbf{z})$, and a two-sided ideal $I \subset R$, one says that a subset $\mathcal{G} \subset I$ is a *Gröbner basis (GB) for I with respect to $\prec$* if

$$\mathrm{in}_\prec(I) = (\{\mathrm{in}_\prec(g) : g \in \mathcal{G}\}) =: (\mathrm{in}_\prec(\mathcal{G})).$$

Equivalently, every f in I has $\mathrm{in}_\prec(f) = m_0$ (left-right) divisible by at least one $\mathrm{in}_\prec(g) = m$ for some g in $\mathcal{G}$, meaning that $m_0 = m_1 m m_2$ for some m_1, m_2 in $\mathrm{Mons}(R)$. One calls a Gröbner basis $\mathcal{G}$ *reduced* if for each pair $g \neq g'$ in $\mathcal{G}$, none of the monomials m appearing in g with nonzero coefficient are divisible by $\mathrm{in}_\prec(g')$.

Gröbner bases for I exist, but may need to be infinite when working in $R = \Bbbk\langle \mathbf{z}\rangle$. For example, $\mathcal{G}_0 = I$ itself always gives a GB for I, but is infinite as long as $I \neq \{0\}$. The fact that a GB for an ideal always generates the ideal will follow from a certain *division algorithm*.

Definition 3.3. ($\mathcal{G}$-standard monomials and the division algorithm) Call a monomial m in $\mathrm{Mons}(R)$ a *$\mathcal{G}$-standard monomial with respect to $\prec$* if it is (left-right) divisible by *none* of $\{\mathrm{in}_\prec(g) : g \in \mathcal{G}\}$.

The *division algorithm on R with respect to $\mathcal{G}$ and $\prec$* starts with any f in R and produces a remainder r having $f \equiv r \bmod I$ (and written $f \to_{\mathcal{G}} r$) which is a $\Bbbk$-linear combination of $\mathcal{G}$-standard monomials, as follows. Assuming $f = \sum_m c_m m$ contains any monomials which are *not* $\mathcal{G}$-standard, pick the $\prec$-largest such monomial m, and write it as $m = m_1 m' m_2$ where $m' = \mathrm{in}_\prec(g)$ for some[3] g in $\mathcal{G}$. Then replace f by

$$f' := f - c_m \cdot m_1 \cdot g \cdot m_2$$

which has $f \equiv f' \bmod I$. Repeat the process with f'. One can show that, because $\prec$ is a well-ordering, this algorithm will eventually terminate with a remainder r that contains only $\mathcal{G}$-standard monomials. However, the remainder r may not be unique, due to choices of which element g in $\mathcal{G}$ has $m' = \mathrm{in}_\prec(g)$ dividing the non-$\mathcal{G}$-standard term m of f at each stage.

The following equivalent conditions defining Gröbner bases are standard verifications.

Proposition 3.4. *Fixing $\prec$ and the two-sided ideal $I \subset R$, the following are equivalent for $\mathcal{G} \subset I$:*

- *(i) $\mathcal{G}$ is a GB for I with respect to $\prec$.*
- *(ii) The division algorithm $f \to_{\mathcal{G}} r$ always gives the same remainder r for f.*
- *(iii) One has $f \in I$ if and only if $f \to_{\mathcal{G}} 0$, regardless of choices in the division algorithm. In particular, $\mathcal{G}$ generates I.*

[3]Without loss of generality, assume that all g in $\mathcal{G}$ are $\prec$-monic, meaning that $\mathrm{in}_\prec(g)$ has coefficient $+1$ in g.

(iv) The (images of the) $\mathcal{G}$-standard monomials with respect to $\prec$ give a $\Bbbk$-basis for R/I.

The GB condition has a useful rephrasing for *homogeneous* ideals I, meaning $I = \bigoplus_{d=0}^{\infty}(I \cap R_d)$.

Proposition 3.5. *For a homogeneous two-sided ideal $I \subset R$, a subset $\mathcal{G} \subset I$ forms a GB of I with respect to $\prec$ if and only if* $\mathrm{Hilb}(S/(\mathrm{in}_\prec(\mathcal{G})), t) = \mathrm{Hilb}(S/I, t)$.

Proof. By definition $\mathcal{G} \subset I$ is a GB if and only if the inclusion $(\mathrm{in}_\prec(\mathcal{G})) \subseteq \mathrm{in}_\prec(I)$ is an equality. This occurs if and only if the graded $\Bbbk$-algebra surjection $R/(\mathrm{in}_\prec(\mathcal{G})) \twoheadrightarrow R/\mathrm{in}_\prec(I)$ is a $\Bbbk$-vector space isomorphism in each degree. By dimension-counting, this occurs if and only if

$$\mathrm{Hilb}(S/(\mathrm{in}_\prec(\mathcal{G})), t) = \mathrm{Hilb}(S/\mathrm{in}_\prec(I), t)$$

However one also has $\mathrm{Hilb}(S/\mathrm{in}_\prec(I), t) = \mathrm{Hilb}(S/I, t)$, since the Gröbner basis $\mathcal{G}_0 := I$ itself has its $\mathcal{G}_0$-standard monomials giving a (homogeneous) $\Bbbk$-basis for both $S/\mathrm{in}_\prec(I)$ by definition, and for S/I by Proposition 3.4 (iv). ☐

There are some advantages to working with Gröbner bases in the commutative polynomial algebra $\Bbbk[\mathbf{z}]$ and exterior algebra $\wedge(\mathbf{z})$, where GBs for ideals are always finite, and can be computed via versions of *Buchberger's algorithm*. One can always view quotients $\Bbbk[\mathbf{z}]/I$ and $\wedge(\mathbf{z})/I$ as quotients of $\Bbbk\langle\mathbf{z}\rangle$ via the surjections

$$\Bbbk\langle\mathbf{z}\rangle \xrightarrow{\pi} \Bbbk[\mathbf{z}] \text{ with } \ker(\pi) = (z_i z_j - z_j z_i : 1 \le i < j \le n)$$
$$\Bbbk\langle\mathbf{z}\rangle \xrightarrow{\pi} \wedge(\mathbf{z}) \text{ with } \ker(\pi) = (z_i z_j + z_j z_i : 1 \le i < j \le n) + \left(z_i^2 : 1 \le i \le n\right)$$

In other words, $\Bbbk[\mathbf{z}]/I$ or $\wedge(\mathbf{z})/I$ is isomorphic to $\Bbbk\langle\mathbf{z}\rangle/\pi^{-1}(I)$. Note that since the

- *commutators* $[z_i, z_j]_+ := z_i z_j - z_j z_i$,
- *anti-commutators* $[z_i, z_j]_- := z_i z_j + z_j z_i$, and
- squares z_i^2

that generate $\ker(\pi)$ are homogeneous and quadratic, this means that if I is a homogeneous ideal of $\Bbbk[\mathbf{z}]$ or $\wedge(\mathbf{z})$, then $\pi^{-1}(I)$ will be a homogeneous two-sided ideal of $\Bbbk\langle\mathbf{z}\rangle$. Similarly, if I is a quadratic ideal, then the same holds for $\pi^{-1}(I)$, and $\Bbbk\langle\mathbf{z}\rangle/\pi^{-1}(I)$ will be a quadratic algebra.

This leads to one of the most common techniques for proving Koszulity.

Theorem 3.6. *Consider* (2-sided) *ideals I inside any of the rings $R = \Bbbk\langle\mathbf{z}\rangle, \Bbbk[\mathbf{z}], \wedge(\mathbf{z})$.*

(i) (Fröberg [44]) *The quotient R/I by any quadratic monomial ideal I is Koszul.*

(ii) [45, Section 4], [62, Theorem 8.14], [70, Section 3] If I has a quadratic Gröbner basis $\mathcal{G}$ with respect to some monomial order $\prec$ on R, then R/I is Koszul.

Proof. For assertion (i), Fröberg's main result in [44] proves Koszulity of a general class of algebras A, containing as special cases the quadratic monomial quotients R/I for any such R.

Assertion (ii) for the commutative case where $R = \Bbbk[\mathbf{x}]$ is credited in [39] to Fröberg's result (i) "and a deformation argument noticed by Kempf and others". This deformation argument is written down explicitly by Peeva in [70, Theorem 22.9 (3)], proving the following assertion. Given a graded A-module M, produce a free (left-)A-module resolution $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ which is *minimal* in the sense that the differentials have entries in A_+. Then define the *graded Betti number* $\beta_{ij}^A(M)$ to be the number of free summands of the form $A(-j)$ appearing in the i^{th} resolvent $F_i = \bigoplus_j A(-j)^{\beta_{ij}^A(M)}$. Thus Koszulity of A may be rephrased as $\beta_{ij}^A(\Bbbk) = 0$ for $j \neq i$. Then one has

$$\beta_{ij}^{\Bbbk[\mathbf{x}]/I}(\Bbbk) \leq \beta_{ij}^{\Bbbk[\mathbf{x}]/\mathrm{in}_\prec I}(\Bbbk) \tag{3.1}$$

for any monomial order $\prec$ on $\Bbbk[\mathbf{x}]$. Since I having a quadratic Gröbner basis with respect to $\prec$ implies that $\Bbbk[\mathbf{x}]/\mathrm{in}_\prec I$ is Koszul by assertion (i), this implies that $\Bbbk[\mathbf{x}]/I$ itself is Koszul.

Assertion (ii) for the anticommutative case where $R = \wedge(\mathbf{z})$ is asserted in Peeva [70, Section 3, p. 613], indicating that the exterior analogue of (3.1) can be proven by a similar deformation argument, using Gröbner basis theory over exterior algebras, and similar in spirit to [4, Proposition 1.8], [38, p. 4369]. This argument employs the exterior analogue of a (commutative) flat deformation result as in Eisenbud [37, Theorem 15.17], along the lines of Murai [66, Lemmas 2.1, 2.2].

Assertion (ii) for the noncommutative case where $R = \Bbbk\langle\mathbf{x}\rangle$ is asserted in [45, Section 4]. It is proven for certain kinds of noncommutative quadratic Gröbner bases called *PBW bases* in [72, Section 5] and [71, Chapter 4]. It is also proven for quadratic Gröbner bases with respect to *degree orderings* on monomials in $\Bbbk\langle\mathbf{z}\rangle$ in Jöllenbeck and Welker [50, Corollary 4.9]. A proof for general term orders $\prec$ on $\Bbbk\langle\mathbf{z}\rangle$ was written down recently in an unpublished preprint of Backelin [7]. □

4. Matroids, oriented matroids, and supersolvability

The Koszul algebras of interest to us are *Orlik–Solomon algebras* of matroids and *graded Varchenko–Gel'fand algebras* of oriented matroids, in the case where the matroids are supersolvable. We therefore review here the basics of matroids, oriented matroids, and supersolvability.

4.1. Matroid and oriented matroid review.

A useful reference for matroids is Oxley [69], and for oriented matroids is Björner, Las Vergnas, Sturmfels, White and Ziegler [16].

A matroid M (respectively, oriented matroid $\mathcal{M}$) on ground set $E = \{1, 2, \ldots, n\}$ is an abstraction of the linear dependence information about a list of vectors $v_1, v_2, \ldots, v_n$ in a vector space over a field $\Bbbk$ (respectively, $\Bbbk = \mathbb{R}$), forgetting the coordinates of the vectors themselves, but recording which subsets are linearly dependent (respectively, the $\pm$ signs in their linear dependences). One way to record this information is with the matroid or oriented matroid's *circuits*, abstracting the minimal dependences.

Definition 4.1. A *matroid* M on ground set $E = \{1, 2, \ldots, n\}$ is defined by its collection $\mathcal{C} \subset 2^E$ of *circuits*, satisfying these axioms:

- (C1.) $\varnothing \notin \mathcal{C}$
- (C2.) If C, C' in $\mathcal{C}$, and $C \subseteq C'$ then $C = C'$
- (C3.) If C, C' in $\mathcal{C}$, and $e \in C \cap C' \subsetneq C, C'$, then there exists $C'' \in \mathcal{C}$ with $C'' \subseteq C \cup C' \setminus \{e\}$.

An *oriented matroid* $\mathcal{M}$ on ground set $E = \{1, 2, \ldots, n\}$ is defined by its collection $\mathcal{C}^\pm = \{(C_+, C_-)\}$ of *signed circuits* which are pairs (C_+, C_-) of disjoint subsets $C_+ \sqcup C_- \subseteq E$, satisfying these axioms:

- (C1$^\pm$.) $(\varnothing, \varnothing) \notin \mathcal{C}^\pm$
- (C2$^\pm$.) If (C_+, C_-) in $\mathcal{C}^\pm$, then (C_-, C_+) in $\mathcal{C}^\pm$
- (C3$^\pm$.) If $(C_+, C_-), (C'_+, C'_-)$ in $\mathcal{C}^\pm$, and $C_+ \cup C_- \subseteq C'_+ \cup C'_-$ then $(C'_+, C'_-) = (C_+, C_-)$ or (C_-, C_+).

(C4$^\pm$.) If $(C_+, C_-), (C'_+, C'_-)$ in $\mathcal{C}^\pm$ and $e \in C_+ \cap C'_-$, then there exists $(C''_+, C''_-) \in \mathcal{C}^\pm$ with $C'' \subseteq C \cup C' \setminus \{e\}$ having $C''_+ \subseteq (C_+ \cup C'_+) \setminus \{e\}$, and $C''_- \subseteq (C_- \cup C'_-) \setminus \{e\}$.

One can check that every oriented matroid $\mathcal{M}$ with signed circuits $\mathcal{C}^\pm$ gives rise to a matroid M having circuits $\mathcal{C} := \{C_+ \cup C_- : (C_+, C_-) \in \mathcal{C}^\pm\}$; one calls the matroid M *orientable* whenever it comes from such an oriented matroid $\mathcal{M}$, and one calls $\mathcal{C}$ the *(matroid) circuits* of $\mathcal{M}$.

One calls M a *representable matroid* (over the field $\Bbbk$) if there exists a list of vectors $v_1, v_2, \ldots, v_n$ in a $\Bbbk$-vector space such that the subsets C in $\mathcal{C}$ index the *minimal dependent subsets* $\{v_j\}_{j \in C}$, that is, $\sum_{j \in C} c_j v_j = 0$ for some c_j in $\Bbbk$, but every proper subset of $\{v_j\}_{j \in C}$ is independent. Similarly, $\mathcal{M}$ is a *representable* oriented matroid if additionally $\Bbbk = \mathbb{R}$ and the pairs (C_+, C_-) in $\mathcal{C}^\pm$ give the subsets $C^+ = \{j : c_j > 0\}$, $C^- = \{j : c_j < 0\}$. for all such minimal dependent subsets of $v_1, \ldots, v_n$.

A matroid M on ground set E can also be specified by its collection of *flats* $\mathcal{F} = \{F\} \subseteq 2^E$, where $F \subseteq E$ is a flat if every circuit C in $\mathcal{C}$ with $|C \cap F| = |C| - 1$ has $C \subseteq F$. We will consider $\mathcal{F}$ as a poset ordered via inclusion. This poset turns out to always be a *geometric lattice*, meaning that

- any pair of flats F, F' have a *meet* (greatest lower bound) $F \wedge F' = F \cap F'$ and a *join* (least upper bound) $F \vee F'$,
- it is an *atomic lattice* in the sense that every flat F has
$$F = \bigvee_{\text{atoms } G \leq F} G,$$
where *atoms* are flats that cover the unique bottom element, and
- it is *upper semimodular*, meaning that there is a *rank function* $r : \mathcal{F} \to \{0, 1, 2, \ldots\}$ satisfying
$$r(F \vee F') \leq r(F) + r(F') - r(F \wedge F'). \tag{4.1}$$

The *rank* of the matroid M is defined to be $r(M) := r(E)$.

It will also be convenient later (in Definition 5.15 below) to note that every oriented matroid $\mathcal{M}$ on E of rank r can be specified via its *chirotope*. This is a function $\chi_{\mathcal{M}} : E^r \to \{0, \pm 1\}$ satisfying certain axioms; see [16, Section 1.9, 3.5]), and the values $\chi_{\mathcal{M}}(i_1, i_2, \ldots, i_r)$ are defined only up to an overall rescaling by ± 1. In the case where $\mathcal{M}$ is realized by vectors $v_1, v_2, \ldots, v_n$, then $\chi_{\mathcal{M}}(i_1, i_2, \ldots, i_r)$ is the $\{0, \pm 1\}$-valued sign of the determinant of the $r \times r$ matrix having $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ as its columns.

In studying Orlik–Solomon and Varchenko–Gel'fand rings, it will turn out (see Remark 5.3 below) that we lose no generality by restricting to matroids and oriented matroids which are *simple*, meaning that they have no *loops* (= singleton circuits $C = \{i\}$) and no *parallel elements* (= circuits $C = \{i, j\}$ of size two). Consequently, their matroid structure M is completely determined by the poset of flats $\mathcal{F}$ up to isomorphism, whose unique bottom element will be the empty flat $F = \varnothing$, and whose atoms at rank 1 are the singleton flats $F = \{1\}, \{2\}, \ldots, \{n\}$, identified with the ground set E.

4.2. **Supersolvability.** We will be focussing on matroids that satisfy the strong condition of supersolvability, reviewed here.

Definition 4.2. Say that a flat F in a matroid M is *modular* if one always has equality in (4.1):
$$r(F \vee F') = r(F) + r(F') - r(F \wedge F') \text{ for all } F' \in \mathcal{F}.$$

A matroid M is called *supersolvable* if the poset $\mathcal{F}$ contains a complete flag $\underline{F}$ of modular flats

$$\underline{F} := \left(\varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r(M)-1} \subsetneq F_{r(M)} = E\right).$$

We consider as examples the (strict) subset of supersolvable matroids among the *uniform matroids*, which we recall here.

Definition 4.3 (Uniform matroids)**.** The *uniform matroid* $M = U_{r,n}$ of rank r on ground set $E = \{1, 2, \ldots, n\}$ has circuits $\mathcal{C}$ equal to all $(r+1)$-element subsets of E. Its poset of flats $\mathcal{F}$ is obtained from 2^E, the *Boolean algebra* of rank n, by removing all subsets of cardinalities $r, r+1, \cdots, n-2, n-1$.

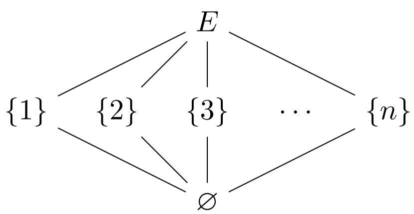
Remark 4.4. The uniform matroid $U_{r,n}$ is represented by any list of n vectors $v_1, v_2, \ldots, v_n$ in $\Bbbk^r$ that are sufficiently generic, in the sense that every r-element subset $\{v_{i_1}, \ldots, v_{i_r}\}$ is linearly independent. This imposes restrictions on the cardinality of the field $\Bbbk$, depending upon n and r, but means that $U_{r,n}$ is always representable over an infinite field, such as $\Bbbk = \mathbb{R}$, and hence is always orientable. Nevertheless, some of these orientations $\mathcal{M}$ of $M = U_{r,n}$ can behave differently, for example in their group of automorphisms $\mathrm{Aut}(\mathcal{M})$. In the examples of this section, we will consider only the unoriented matroid $M = U_{r,n}$.

It is not hard to see that the uniform matroid $M = U_{r,n}$ is

- simple if and only if $(r, n) = (0, 0)$ (the empty matroid), $(r, n) = (1, 1)$, or $r \geq 2$; and
- simple and supersolvable if and only if $(r, n) = (0, 0)$, $(r, n) = (1, 1)$, or $r = 2$ and $n \geq 2$.

Example 4.5. The *Boolean matroid* $M = U_{n,n}$ on ground set $E = \{1, 2, \ldots, n\}$ has no circuits, that is, $\mathcal{C} = \varnothing$, and poset of flats $\mathcal{F} = 2^E$. Every flat F is modular, so every complete flag $\underline{F}$ of flats is modular and M is supersolvable.

Example 4.6. Every rank two simple matroid is isomorphic to a uniform matroid $M = U_{2,n}$ on $E = \{1, 2, \ldots, n\}$, with this flat poset $\mathcal{F}$:



Again, every flat F is modular, and every complete flag $\varnothing \subset \{i\} \subset E$ shows that M is supersolvable.

Our original motivation came from *braid matroids*.

Example 4.7. (*Supersolvable graphic matroids* and *braid matroids*) Let G be a graph on vertex set $\{1, 2, \ldots, n\}$ with edge set $E \subseteq \{\{i, j\} : 1 \leq i < j \leq n\}$ which is *simple*, that is, G no self-loops and no parallel edges. Then G gives rise to a simple *graphic* matroid M (and oriented matroid $\mathcal{M}$) represented by the list of vectors $\{v_{ij} = e_i - e_j\}_{\{i,j\} \in E} \subset \mathbb{R}^n$, where $e_1, \ldots, e_n$ are standard basis vectors. The matroid circuits $\mathcal{C}$ are indexed by subsets $C \subseteq E$ of edges that form a cycle within G. Stanley showed [81, Proposition 2.8] that this graphic matroid is supersolvable if and only G is a *chordal* graph, meaning that for every minimal cycle of edges $u_1 - u_2 - \cdots - u_{\ell-1} - u_\ell - u_1$ in G having $\ell \geq 4$, there will be another edge $\{u_i, u_j\}$ of G with $i \not\equiv j \pm 1 \bmod \ell$ forming a chord.

In particular, the *complete graph* K_n on n vertices with all $\binom{n}{2}$ edges is a chordal graph, and its graphic matroid is called the *braid matroid* Br_n *on* n *strands*. Its poset of flats $\mathcal{F}$ is isomorphic to the lattice Π_n of all set partitions $\pi = (B_1, \ldots, B_\ell)$ of $\{1, 2, \ldots, n\} = \sqcup_{i=1}^{\ell} B_i$, with ordering by refinement: $\pi \leq \pi'$ if for every block B_i of π there exists some block $B'_{i'}$ of π' having $B_i \subseteq B'_{i'}$. The flat F corresponding to π contains all edges $\{i, j\}$ whose end vertices i, j lie in the same block B_k of π. The modular flats correspond to partitions π with at most one non-singleton block. For example, one modular complete flag $\underline{F}$ of flats corresponds to the set partitions $\pi_1 < \pi_2 < \cdots < \pi_n$ where

$$\pi_k := \{\{1, 2, \ldots, k\}, \{k+1\}, \{k+2\}, \ldots, \{n-1\}, \{n\}\}.$$

5. Orlik–Solomon and Varchenko–Gel'fand rings

We review here the Orlik–Solomon algebra of a matroid M and graded Varchenko–Gel'fand algebra[4] of an oriented matroid $\mathcal{M}$. Useful references for Orlik–Solomon algebras are Dimca [32, Ch. 3], Dimca and Yuzvinsky [33], Orlik and Terao [68, Ch. 3], Yuzvinsky [94]. Useful references for graded Varchenko–Gel'fand algebras are Brauner [19, Section 3.3, Section 5.2], Cordovil [28], Dorpalen-Barry [34], Dorpalen-Barry, Proudfoot and Wang [35], Moseley [65], Varchenko and Gel'fand [92].

For the remainder of this section, let $\Bbbk$ be any commutative ring with 1.

Definition 5.1 (*Orlik–Solomon algebra*)**.** For a simple matroid M on $E = \{1, 2, \ldots, n\}$, define its *Orlik–Solomon algebra* over $\Bbbk$ as an anti-commutative quotient

$$\mathrm{OS}(M) := \wedge(x_1, \ldots, x_n)/I_{\mathrm{OS}(M)}$$

where $\wedge(x_1, \ldots, x_n)$ is the exterior algebra over $\Bbbk$ on n generators. The Orlik-Solomon ideal

$$I_{\mathrm{OS}(M)} = (\partial(x_C) : C \in \mathcal{C}) \tag{5.1}$$

has one generator $\partial(x_C)$ for each circuit $C = \{c_1, c_2, \ldots, c_k\}$ in $\mathcal{C}$, with $\partial(x_C)$ defined by

$$\partial x_C := \sum_{j=1}^{k} (-1)^{j-1} x_{c_1} \wedge \cdots \wedge x_{c_{j-1}} \wedge \widehat{x_{c_j}} \wedge x_{c_{j+1}} \cdots \wedge x_{c_k}. \tag{5.2}$$

Definition 5.2 (Graded Varchenko–Gel'fand ring)**.** For a simple oriented matroid $\mathcal{M}$ on $E = \{1, 2, \ldots, n\}$, define its *graded Varchenko–Gel'fand ring* over $\Bbbk$ as the commutative quotient

$$\mathrm{VG}(\mathcal{M}) := \Bbbk[x_1, \ldots, x_n]/I_{\mathrm{VG}(\mathcal{M})}$$

where $\Bbbk[x_1, \ldots, x_n]$ is the polynomial algebra over $\Bbbk$. The graded Varchenko–Gel'fand ideal

$$I_{\mathrm{VG}(\mathcal{M})} = \left(x_1^2, \ldots, x_n^2\right) + (\partial^{\pm}(x_C) : C \in \mathcal{C}) \tag{5.3}$$

contains the squares $\{x_i^2\}_{i=1}^n$ along with one generator $\partial^{\pm}(x_C)$ for each circuit C in $\mathcal{C}$, with $\partial^{\pm}(x_C)$ defined by choosing one of the two signed circuits[5] (C_+, C_-) in $\mathcal{C}$ with $C = C_+ \cup C_-$, and setting

$$\partial^{\pm}(x_C) := \sum_{c_j \in C_+ \cup C_-} \mathrm{sgn}_{C,c_j} \cdot x_{c_1} \cdots x_{c_{j-1}} \widehat{x_{c_j}} x_{c_{j+1}} \cdots x_{c_k}. \tag{5.4}$$

[4] Also called the *Cordovil algebra* in [60].

[5] The choice is immaterial – making the other choice replaces $\partial^{\pm}(x_C)$ by its negative.

Here $\mathrm{sgn}_{C,c_j} = \pm 1$, namely $+1$ when $c_j \in C_+$ and -1 when $c_j \in C_-$.

Remark 5.3. Our assumption that $M, \mathcal{M}$ are simple really presents no restriction. In either case,

- a loop i in E would give a circuit $C = \{i\} \in \mathcal{C}$, causing the collapse $\mathrm{OS}(M) = 0 = \mathrm{VG}(\mathcal{M})$ since $I_{\mathrm{OS}(M)}$ or $I_{\mathrm{VG}(\mathcal{M})}$ contains the generator $\partial(x_C) = 1$ or $\partial^{\pm}(x_C) = 1$, and
- parallel elements i, j in E would give rise to a circuit $C = \{i, j\} \in \mathcal{C}$, making $x_i = \pm x_j$ in the rings $\mathrm{OS}(M)$ or $VG(\mathcal{M})$ because $I_{\mathrm{OS}(M)}$ or $I_{\mathrm{VG}(\mathcal{M})}$ contains a generator $\partial(x_C)$ or $\partial^{\pm}(x_C)$ of of the form $x_i \pm x_j$.

Thus our assumption in Section 2 that our standard graded $\Bbbk$-algebras are minimally generated by the variables $x_1, \ldots, x_n$ is consistent with assuming that $M, \mathcal{M}$ are simple matroids.

5.1. **Flat decomposition.** An important feature of both $\mathrm{OS}(M)$ and $\mathrm{VG}(\mathcal{M})$ is that their $\mathbb{N}$-grading is refined by a $\Bbbk$-vector space decomposition indexed by the matroid flats F in $\mathcal{F}$.

Definition 5.4. Given matroid M or oriented matroid $\mathcal{M}$ on $E = \{1, \ldots, n\}$ with flats $\mathcal{F}$, abbreviating the variable sets $\mathbf{x} = (x_1, \ldots, x_n)$, consider the $\Bbbk$-vector space decompositions

$$T(V) = \Bbbk\langle \mathbf{x} \rangle = \bigoplus_{F \in \mathcal{F}} \underbrace{T(V)_F}_{= \Bbbk\langle \mathbf{x} \rangle_F},$$

$$\mathrm{Sym}(V) = \Bbbk[\mathbf{x}] = \bigoplus_{F \in \mathcal{F}} \underbrace{\mathrm{Sym}(V)_F}_{= \Bbbk[\mathbf{x}]_F},$$

$$\wedge(V) = \wedge(\mathbf{x}) = \bigoplus_{X \in \mathcal{F}} \underbrace{\wedge(V)_F}_{= \wedge(\mathbf{x})_F},$$

where $\Bbbk\langle \mathbf{x} \rangle_F, \Bbbk[\mathbf{x}]_F, \wedge(\mathbf{x})_F$ are the $\Bbbk$-spans of monomials $x_{j_1} x_{j_2} \cdots x_{j_k}$ with $\{j_1\} \vee \cdots \vee \{j_k\} = F$.

Both $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ inherit these $\Bbbk$-vector space decompositions by flats; for $\mathrm{OS}(M)$, see [68, Theorem 3.26, Corollary 3.27], [33, Section 2.3], [94, Section 2.3], and for $\mathrm{VG}(\mathcal{M})$ see [19, Theorem 5.5].

Proposition 5.5. *For a matroid M or oriented matroid $\mathcal{M}$, the ideals $I_{\mathrm{OS}(M)}, I_{\mathrm{VG}(\mathcal{M})}$ are homogeneous with respect to the decomposition in Definition 5.4, that is,*

$$I_{\mathrm{OS}(M)} = \bigoplus_{F \in \mathcal{F}} \Bbbk(\mathbf{x})_F \cap I_{\mathrm{OS}(M)},$$

$$I_{\mathrm{VG}(\mathcal{M})} = \bigoplus_{F \in \mathcal{F}} \Bbbk[\mathbf{x}]_F \cap I_{\mathrm{VG}(\mathcal{M})}.$$

Hence they induce $\Bbbk$-vector space decompositions of the quotients $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$:

$$\mathrm{OS}(M) = \bigoplus_{F \in \mathcal{F}} \mathrm{OS}(M)_F, \tag{5.5}$$

$$\mathrm{VG}(\mathcal{M}) = \bigoplus_{F \in \mathcal{F}} \mathrm{VG}(\mathcal{M})_F. \tag{5.6}$$

We note here an implication for quadratic duals that will become important later, in Section 8.1. When considering $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ as quotients $A = \Bbbk\langle \mathbf{x}\rangle/I$ of the tensor algebra for a two-sided ideal I, the quadratic part $I_2 \subset T^2(V) = \Bbbk\langle \mathbf{x}\rangle_2$ inherits the flat decomposition $I_2 = \bigoplus_{F\in\mathcal{F}}[T^2(V)_F \cap I]$ from $T^2(V) = \bigoplus_{F\in\mathcal{F}} T^2(V)_F$. On the other hand, if one defines the analogous flat decomposition for the dual tensor algebra and its dual variables $\mathbf{y} = (y_1, \ldots, y_n)$

$$T(V^*) = \Bbbk\langle \mathbf{y}\rangle = \bigoplus_{F\in\mathcal{F}} \underbrace{T(V)_F}_{=\Bbbk\langle \mathbf{y}\rangle_F},$$

then the pairing $T^2(V^*) \times T^2(V) \to \Bbbk$ from (2.3) makes $T^2(V^*)_F$ and $T^2(V)_{F'}$ orthogonal for $F \neq F'$. This implies that the computation of $J_2 := I_2^{\perp}$ can be done flat-by-flat:

$$J_2 = \bigoplus_{F\in\mathcal{F}} \left[T^2(V^*)_F \cap J_2\right] \quad \text{where} \quad \left[T^2(V^*)_F \cap J_2\right] := \left[T^2(V)_F \cap I_2\right]^{\perp}. \tag{5.7}$$

In particular, whenever $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ are Koszul, or even just quadratic algebras $A = \Bbbk\langle \mathbf{x}\rangle/I$ with $I = (I_2)$, their quadratic duals $A^! = \Bbbk\langle \mathbf{x}\rangle/J$ where $J = (J_2) = (I_2^{\perp})$ inherit a flat decomposition:

$$A^! = \bigoplus_{F\in\mathcal{F}} A^!_F. \tag{5.8}$$

5.2. **Symmetry.** Symmetries of a matroid M or oriented matroid $\mathcal{M}$ lead to $\Bbbk$-algebra automorphisms of $\mathrm{OS}(M)$ or $\mathrm{VG}(\mathcal{M})$, as we explain next.

Definition 5.6. Let M be a matroid on $E = \{1, 2, \ldots, n\}$ with circuits $\mathcal{C}$. A permutation σ in the symmetric group $\mathfrak{S}_n$ is an *automorphism* of M, written $\sigma \in \mathrm{Aut}(M)$, if $\sigma(\mathcal{C}) = \mathcal{C}$, that is, for every C in $\mathcal{C}$, one has $\sigma(C) \in \mathcal{C}$.

One can then check that for any matroid M and σ in $\mathrm{Aut}(M)$, if σ acts on $\wedge(x_1, \ldots, x_n)$ by permuting subscripts of the variables, that is, $\sigma(x_i) := x_{\sigma(i)}$, then the generator $\partial(x_C)$ for the Orlik–Solomon ideal $I_{\mathrm{OS}(M)}$ has

$$\sigma(\partial(x_C)) = \pm\partial\left(x_{\sigma(C)}\right).$$

Consequently, σ preserves $I_{\mathrm{OS}(M)}$ and induces a graded $\Bbbk$-algebra automorphism of $\mathrm{OS}(M)$.

Definition 5.7. Let $\mathcal{M}$ be an oriented matroid on $E = \{1, 2, \ldots, n\}$. Its automorphism group $\mathrm{Aut}(\mathcal{M})$ will be a subgroup of the *hyperoctahedral group* $\mathfrak{S}_n^{\pm}$; this is the set of all *signed permutations* σ of $\{\pm 1, \pm 2, \ldots, \pm n\}$, meaning those permutations which commute with the involution $+i \leftrightarrow -i$, or in other words, $\sigma(\pm i) = -\sigma(\mp i)$. As notation, for $i, j \in \{1, 2, \ldots, n\}$, define

$$|\sigma(i)| := j \quad \text{if } \sigma(+i) \in \{\pm j\},$$
$$\epsilon(\sigma(i)) = \begin{cases} + & \text{if } \sigma(+i) = +j, \\ - & \text{if } \sigma(+i) = -j. \end{cases}$$

Then a signed permutation σ is an *automorphism* of $\mathcal{M}$ if for every signed circuit (C_+, C_-) in $\mathcal{C}^{\pm}$, the following pair (C'_+, C'_-) is also a signed circuit in $\mathcal{C}^{\pm}$, where

$$\begin{aligned} C'_+ &:= \{|\sigma(i)| : i \in C_+ \text{ and } \epsilon(\sigma(i)) = +\} \sqcup \{|\sigma(i)| : i \in C_- \text{ and } \epsilon(\sigma(i)) = -\}, \\ C'_- &:= \{|\sigma(i)| : i \in C_- \text{ and } \epsilon(\sigma(i)) = +\} \sqcup \{|\sigma(i)| : i \in C_+ \text{ and } \epsilon(\sigma(i)) = -\}. \end{aligned} \tag{5.9}$$

For σ in $\mathrm{Aut}(\mathcal{M})$, let σ act on $\Bbbk[x_1, \ldots, x_n]$ via

$$\sigma(x_i) := \epsilon(\sigma(i)) \cdot x_{|\sigma(i)|}.$$

One can then check that for signed circuits $(C_+, C_-), (C'_+, C'_-)$ related as in (5.9), if $C = C_+ \cup C_-$ and $C' = C'_+ \cup C'_-$, then the generator $\partial^{\pm}(x_C)$ for the ideal $I_{\mathrm{VG}(\mathcal{M})}$ has

$$\sigma\left(\partial^{\pm}(x_C)\right) = \pm\partial^{\pm}(x_{C'}).$$

Consequently, σ gives rise to a graded $\Bbbk$-algebra automorphism of $\mathrm{VG}(\mathcal{M})$.

In this way, when $M, \mathcal{M}$ have some group G of automorphisms, we consider $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ as graded $\Bbbk G$-modules, and study their equivariant Hilbert series as in (2.9). Similarly, when these algebras A are Koszul, we will study the equivariant Hilbert series for their Koszul dual $A^!$. Note that in the dual setting, the dual variables $y_1, \ldots, y_n$ that give a basis for V^* obey the same rules

$$\sigma(y_i) = y_{\sigma(i)} \text{ for } \mathrm{OS}(M)^!,$$
$$\sigma(y_i) = \epsilon(\sigma(i)) \cdot y_{|\sigma(i)|} \text{ for } \mathrm{VG}(\mathcal{M})^!.$$

This is because V^* carries the contragredient representation to V, where the matrix for the action of σ in the basis of $y_1, \ldots, y_n$ is the inverse transpose $(A^{-1})^t$ of the matrix A for its action on $x_1, \ldots, x_n$. However, signed (or unsigned) permutation matrices A are orthogonal: $(A^{-1})^t = A$.

5.3. **Gröbner bases and broken circuits.** It turns out that the above generators for the ideals presenting $\mathrm{OS}(M)$ and $\mathrm{VG}(\mathcal{M})$ are actually Gröbner bases, with easily-identified standard monomials.

Definition 5.8. Given a matroid M on $E = \{1, 2, \ldots, n\}$ and any circuit $C = \{c_1 < c_2 < \cdots < c_k\}$ in $\mathcal{C}$, the associated *broken circuit* is

$$C \setminus \{\min(C)\} = C \setminus \{c_1\} = \{c_2 < \cdots < c_k\}.$$

A subset $I \subset E$ is an *NBC (no-broken-circuit) set* if it contains none of the sets $\{C \setminus \{\min(C)\}\}_{C \in \mathcal{C}}$.

Theorem 5.9. *Fix a matroid M and oriented matroid $\mathcal{M}$ on $E = \{1, 2, \ldots, n\}$, with circuits $\mathcal{C}$. Choose any monomial orders $\prec$ on $\wedge(x_1, \ldots, x_n)$ and $\Bbbk[x_1, \ldots, x_n]$ having $x_1 \prec x_2 \prec \cdots \prec x_n$.*

- *(i)* [94, Thm 2.8] *The generators $\mathcal{G} = \{\partial(x_C)\}_{C \in \mathcal{C}}$ in* (5.1) *form a Gröbner basis for $I_{\mathrm{OS}(M)}$ with respect to $\prec$.*
- *(ii)* [34, Thm 1] *The generators $\mathcal{G} = \{x_i^2\}_{i=1}^n \cup \{\partial^{\pm}(x_C)\}_{C \in \mathcal{C}}$ in* (5.3) *form a Gröbner basis for $I_{\mathrm{VG}(\mathcal{M})}$ with respect to $\prec$.*

Furthermore, in both cases, if $C = \{c_1 < c_2 < \cdots < c_k\}$ in $\mathcal{C}$, then the $\prec$-initial term $\mathrm{in}_\prec(\partial(x_C))$ or $\mathrm{in}_\prec(\partial^{\pm}(x_C))$ is the monomial $x_{c_2} \cdots x_{c_k}$, supported on the broken circuit associated to C. Consequently, in either case, the $\mathcal{G}$-standard monomials are the NBC monomials

$$\{x_I = x_{i_1} \cdots x_{i_\ell} : \textit{NBC sets } I = \{i_1, \ldots, i_\ell\} \subseteq E\}.$$

In particular, $\mathrm{OS}(M)$ *and* $\mathrm{VG}(\mathcal{M})$ *have the same Hilbert series, given by*

$$\mathrm{Hilb}(\mathrm{OS}(M), t) = \mathrm{Hilb}(\mathrm{VG}(\mathcal{M}), t) = \sum_{\textit{NBC sets } I \subseteq E} t^{|I|}.$$

Remark 5.10. One can readily check that the NBC standard monomial bases for $\mathrm{OS}(M)$, $\mathrm{VG}(\mathcal{M})$ respect the flat decompositions (5.5), (5.6) in this sense: for each flat $F \in \mathcal{F}$, the components $\mathrm{OS}(M)_F, \mathrm{VG}(M)_F$ both have as $\Bbbk$-bases the monomials $\{x_I : I \text{ an NBC set with } \vee_{i \in I}\{i\} = F\}$.

For supersolvable M, one has *quadratic* Gröbner bases, making $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ Koszul, as we explain next. Björner, Edelman and Ziegler [14] gave a useful alternate characterization of the modular complete flags of flats witnessing supersolvability. To state it, recall that a flat F with $r(F) = r(M) - 1$ is called a *coatom* in $\mathcal{F}$. Also recall that for a matroid M on E and subset $A \subseteq E$, the *restriction* $M|_A$ is the matroid on ground set A defined with circuits $\{C \in \mathcal{C} : C \subseteq A\}$.

Proposition 5.11 ([14, Thm. 4.3])**.** *Let M be a simple matroid on ground set E.*

(i) For flats F which are coatoms, being a modular element is equivalent to the following condition: for any $j \neq k$ in $E \setminus F$, there exists i in F with $\{i\} \leq \{j\} \vee \{k\}$.
(ii) The flats in a complete flag $\underline{F} = (\varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r(M)-1} \subsetneq F_{r(M)} = E)$ are all modular if and only if F_{i-1} is a modular coatom within $M|_{F_i}$ for each $i = 1, 2, \ldots, r(M)$.

Björner and Ziegler [17] later elaborated on this, proving the following.

Proposition 5.12 ([17, Theorem 2.8])**.** *Let M be any simple matroid of rank r on ground set E. The following are equivalent:*

(i) M is supersolvable, say with a modular complete flag of flats $\underline{F} = (F_i)_{i=0,1,\ldots,r}$.
(ii) There exists an ordered set partition $\underline{E} = (E_1, E_2, \ldots, E_r)$ of $E = E_1 \sqcup \cdots \sqcup E_r$ such that if j, k in E_q with $j \neq k$, then there exists $p < q$ and i in E_p with $C = \{i, j, k\}$ in $\mathcal{C}$.
(iii) One can reindex/order $E = \{1 < 2 < \cdots < n\}$ so that the minimal broken circuits (with respect to inclusion) are all of size 2.

Furthermore, when these conditions hold,

(a) a modular flag $\underline{F}$ as in (i) gives an ordered set partition $\underline{E}$ as in (ii) via $E_i := F_i \setminus F_{i-1}$, and
(b) an ordered set partition $\underline{E}$ as in (ii) gives an ordering $\prec$ on E as in (iii) by extending the partial order that makes elements of E_p come $\prec$-earlier than elements of E_q when $p < q$,
(c) the minimal broken circuits with respect inclusion are all pairs of the form $\{j, k\}$ in some set E_q for $q = 1, 2, \ldots, r$; hence the NBC sets $I \subset E$ are the subsets containing at most one element from each E_p for $p = 1, 2, \ldots, r$.

Definition 5.13. For a supersolvable matroid M, with $\underline{F}, \underline{E}$ as in Proposition 5.12, denote by $\mathcal{C}_{\mathrm{BEZ}}(\underline{E}) \subseteq \mathcal{C}$ the circuits $C = \{i, j, k\}$ with $i \in E_p$ and $j \neq k \in E_q$ for $p < q$ from Proposition 5.12 (ii).

Corollary 5.14. *Let $M, \mathcal{M}$ be supersolvable simple matroids or oriented matroids on E, with $\underline{E}$ as in Proposition 5.12. Fix a field $\Bbbk$, and monomial orderings $\prec$ on $\wedge(x_1, \ldots, x_n)$ and $\Bbbk[x_1, \ldots, x_n]$ with $x_1 \prec x_2 \prec \cdots \prec x_n$.*

- [70], [94, Section 6.3] *$I_{\mathrm{OS}(M)}$ has quadratic Gröbner basis $\mathcal{G} = \{\partial(x_C)\}_{C \in \mathcal{C}_{\mathrm{BEZ}}}$, where*

$$\partial(x_C) = x_i \wedge x_j - x_i \wedge x_k + \underline{x_j \wedge x_k}. \tag{5.10}$$

- [34] $I_{\mathrm{VG}(\mathcal{M})}$ *has quadratic Gröbner basis* $\mathcal{G} = \{x_i^2\}_{i=1}^n \cup \{\partial^\pm(x_C)\}_{C \in \mathcal{C}_{\mathrm{BEZ}}}$, *where*
$$\partial^\pm(x_C) = \mathrm{sgn}_{C,k} \cdot x_i x_j + \mathrm{sgn}_{C,j} \cdot x_i x_k + \mathrm{sgn}_{C,i} \cdot \underline{x_j x_k}. \tag{5.11}$$

In both cases,

- *the* $\prec$*-initial terms of the elements of* $\mathcal{G}$ *are shown underlined above,*
- *the* $\mathcal{G}$*-standard monomial basis* $\{x_I\}$ *are indexed by the NBC sets* $I \subseteq E$*, which are exactly those sets containing at most one element from each* E_p *for* $p = 1, 2, \ldots, r$*,*
- $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ *are Koszul algebras,*
- *with the same Hilbert series*
$$\mathrm{Hilb}(\mathrm{OS}(M), t) = \mathrm{Hilb}(\mathrm{VG}(\mathcal{M}), t) = (1 + e_1 t)(1 + e_2 t) \cdots (1 + e_r t) \tag{5.12}$$
where $e_p = |E_p|$ *for* $p = 1, 2, \ldots, r$.

The integers $(e_1, e_2, \ldots, e_p)$ are often called the *exponents* of the supersolvable matroid M, due to their connection with the theory of *free hyperplane arrangements* and the exponents of *reflection arrangements*; see Orlik and Terao [68, §4.2].

5.4. **Quadratic Gröbner basis for the Koszul dual.** We next prove a counterpart to Corollary 5.14 for the Koszul duals $A^!$ of $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ in the supersolvable case. Since $A = \mathrm{OS}(M)$ or $\mathrm{VG}(\mathcal{M})$ are Koszul algebras, one can view them as noncommutative quotients $A = \Bbbk\langle x_1, \ldots, x_n\rangle/I$, and form their Koszul duals $A^! = \Bbbk\langle y_1, \ldots, y_n\rangle/J$, as in Section 2. Certain relations in $A^!$ will play a key role.

Definition 5.15. Let M be a simple matroid on $E = \{1, 2, \ldots, n\}$. For each rank two flat $F \subset E$ and each j in F, define an element of $\Bbbk\langle \mathbf{y}\rangle := \Bbbk\langle y_1, \ldots, y_n\rangle$ by
$$r(j, F) := \sum_{k \in F \setminus \{j\}} [y_j, y_k]_+ = \sum_{k \in F \setminus \{j\}} (y_j y_k - y_k y_j). \tag{5.13}$$
Let $\mathcal{M}$ be a simple oriented matroid on $E = \{1, 2, \ldots, n\}$. For each rank two flat $F \subset E$, pick one of the two chirotopes $\chi_{\mathcal{M}|_F} : F^2 \to \{0, \pm 1\}$ on the restriction $\mathcal{M}|_F$, which are the same up to the overall scaling by ± 1. Then for each j in F define an element of $\Bbbk\langle \mathbf{y}\rangle$ by
$$r^\pm(j, F) := \sum_{k \in F \setminus \{j\}} \chi_{\mathcal{M}|_F}(j, k) \cdot [y_j, y_k]_- = \sum_{k \in F \setminus \{j\}} \chi_{\mathcal{M}|_F}(j, k) \cdot (y_j y_k + y_k y_j). \tag{5.14}$$

The relations (5.13) appear in work of Kohno [53] presenting the *holonomy Lie algebra* for the complement of any complex hyperplane arrangement; see Section 8.1 for further discussion. As far as we know, relations (5.14) are new. Certain subsets of these relations in (5.13) or (5.14) play a distinguished role in the supersolvable case.

Definition 5.16. Let $M, \mathcal{M}$ be supersolvable simple matroids or oriented matroids, and $\underline{E} = (E_1, \ldots, E_r)$ a choice of an ordered partition of its ground set E as in Proposition 5.12. Call (j, i) in E^2 a *retrograde (ordered) pair* with respect to $\underline{E}$ if $i \in E_p$ and $j \in E_q$ with $p < q$.

For each retrograde pair (j, i), let $F := \{j\} \vee \{i\}$ be the rank two flat that they span, and denote by $r(j, i), r^\pm(j, i)$ the following two relations, equivalent to $r(j, F)$ from (5.13) and $r^\pm(j, F)$ from (5.14):
$$r(j, i) := \underline{y_j y_i} - y_i y_j + \sum_{k \in F \setminus \{i, j\}} [y_j, y_k]_+. \tag{5.15}$$
$$r^\pm(j, i) = \underline{y_j y_i} - y_i y_j + \chi_{\mathcal{M}|_F}(j, i) \sum_{k \in F \setminus \{i, j\}} \chi_{\mathcal{M}|_F}(j, k) \cdot [y_j, y_k]_-. \tag{5.16}$$

The following key point will be used in the proofs of Theorems 5.18 and 5.21.

Lemma 5.17. *In the context of Definition 5.16 of a retrograde pair (j, i) with $i \in E_p$ and $j \in E_q$ for $p < q$, the rank two flat $F := \{j\} \vee \{i\}$ has $F \setminus \{i, j\} \subset E_q$.*

Consequently, (5.15) *and* (5.16) *can be viewed as rewriting rules that replace the underlined term $y_j y_i$ by the term $y_i y_j$ together with a sum of monomials $y_j y_k, y_k y_j$ whose subscripts j, k both lie in E_q.*

Proof. Any $k \in F \setminus \{i, j\}$ leads to a circuit $C = \{i, j, k\}$ since M is a simple matroid and F has rank two. As $j > i$, one knows $j \neq \min C$, so the associated broken circuit $B \subset C$ is either $B = \{j, i\}$ or $B = \{j, k\}$. But assertion (c) in Proposition 5.12 implies B contains a pair lying in some set $E_{q'}$. This implies $q' = q$, and $\ell \neq i$ since $i \in E_p \neq E_q$. Thus $B = \{j, k\}$, and k lies in E_q. □

Theorem 5.18. *Let $M, \mathcal{M}$ be matroids and oriented matroids which are supersolvable, with ground set $E = \{1, 2, \ldots, n\}$ and $\underline{E}$ as in Proposition 5.12. Consider the Koszul algebras $A = \mathrm{OS}(M)$ or $\mathrm{VG}(\mathcal{M})$, and their Koszul dual $A^! = \Bbbk\langle y_1, \ldots, y_n\rangle / J$. Then there exist monomial orderings $\prec$ on $\Bbbk\langle y_1, \ldots, y_n\rangle$ with these properties.*

(i) $A^! = \mathrm{OS}(M)^! = \Bbbk\langle \mathbf{y}\rangle / J$ *has $\{r(j, F) : j \in F$ a rank two flat $\}$ as a Gröbner basis for J, and a reduced Gröbner basis*

$$\mathcal{G} := \{r(j, i) : \text{ retrograde pairs } (j, i)\}$$

with the $\prec$-initial term of $r(j, i)$ underlined in (5.15).

(ii) $A^! = \mathrm{VG}(\mathcal{M})^! = \Bbbk\langle \mathbf{y}\rangle / J$ *has $\{r^{\pm}(j, F) : j \in F$ a rank two flat $\}$ as a Gröbner basis for J, and a reduced Gröbner basis*

$$\mathcal{G} := \{r^{\pm}(j, i) : \text{ retrograde pairs } (j, i)\}$$

with the $\prec$-initial term of $r^{\pm}(j, i)$ underlined in (5.16).

In particular,

(iii) their ideals J share the same initial monomials $\{y_j y_i : \text{retrograde pairs } (i, j)\}$,

(iv) and hence the same $\mathcal{G}$-standard monomial $\Bbbk$-basis for $A^!$, of the form $\{m_1 \cdot m_2 \cdots m_{r-1} \cdot m_r\}$ where each m_p is any noncommutative monomial in the variable set $\{y_j\}_{j \in E_p}$,

(v) and they have the same Hilbert series

$$\mathrm{Hilb}\left(\mathrm{OS}(M)^!, t\right) = \mathrm{Hilb}\left(\mathrm{VG}(\mathcal{M})^!, t\right) = \frac{1}{(1 - e_1 t)(1 - e_2 t) \cdots (1 - e_r t)} \tag{5.17}$$

where $e_p = |E_p|$ are the exponents from Corollary 5.14.

Proof. First let us specify a monomial order $\prec$ on $\Bbbk\langle y_1, \ldots, y_n\rangle$ for which the underlined terms in (5.15), (5.16) are their $\prec$-initial terms. Recall that our indexing has $i < j$ for each retrograde pair (j, i). We claim that it suffices to let $\prec$ be a graded version of a lexicographic order having $y_1 \succ y_2 \succ \cdots \succ y_n$ that reads monomials from the right. More precisely, this means that for two unequal monomials

$$m = y_{i_1} \cdots y_{i_d},$$
$$m' = y_{j_1} \cdots y_{j_e},$$

one has $m \prec m'$ if either $\deg(m) = d < e = \deg(m')$, or if $d = e$ and there exists some $k \in \{1, 2, \ldots, d\}$ with $i_d = j_d, i_{d-1} = j_{d-1}, \ldots, i_{k+1} = j_{k+1}$ but $i_k > j_k$. It follows from Lemma 5.17 that for any retrograde pair (j, i) with $F = \overline{\{j, i\}}$, every k in $F \setminus \{i, j\}$ lies in

E_q, so that $k > i$ and $y_j y_k \prec y_j y_i$. Since also $j > i$, this makes $y_j y_i$ the $\prec$-initial term in either (5.15) or (5.16).

We next check that the relations $r(j,F), r^{\pm}(j,F)$ lie in $J_2 = I_2^{\perp} \subset V^* \otimes V^*$, with the pairing defined by $(y_i y_j, x_k x_\ell) = \delta_{(i,j),(k,\ell)}$. We do the check here for $r^{\pm}(j,F)$; the check for $r(j,F)$ is similar, but slightly easier. One must check that $r^{\pm}(j,F)$ is orthogonal to three types of generators of I in $\mathrm{VG}(\mathcal{M}) = \Bbbk\langle \mathbf{x}\rangle / I$:

$$x_k^2 \text{ for } k = 1, 2, \ldots, n, \tag{5.18}$$

$$x_k x_\ell - x_\ell x_k \text{ for } 1 \le k \le \ell \le n, \tag{5.19}$$

$$\partial^{\pm}(C) := \mathrm{sgn}_{C,m} x_k x_\ell + \mathrm{sgn}_{C,\ell} x_k x_m + \mathrm{sgn}_{C,k} x_\ell x_m \quad \text{for circuits } C = \{k,\ell,m\} \text{ of size three.} \tag{5.20}$$

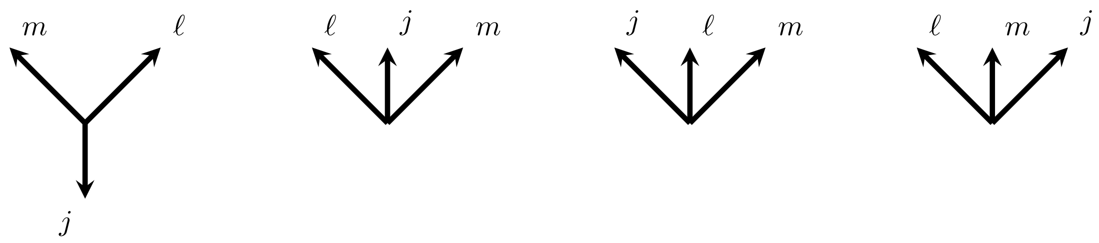
Note that $r^{\pm}(j,F)$ pairs to zero with any commutator in (5.19), because $r^{\pm}(j,F)$ is a sum of anti-commutators $[y_a, y_b]_- = y_a y_b + y_b y_a$. Note also that whenever quadratic monomials $f(\mathbf{y}), g(\mathbf{x})$ have disjoint E^2-support sets

$$\operatorname{supp} f(\mathbf{y}) := \left\{(i,j) \in E^2 : y_i y_j \text{ appears in } f \text{ with nonzero coefficient}\right\},$$
$$\operatorname{supp} g(\mathbf{x}) := \left\{(k,\ell) \in E^2 : x_k x_\ell \text{ appears in } f \text{ with nonzero coefficient}\right\},$$

then one will have $(f(\mathbf{y}), g(\mathbf{x})) = 0$. This already implies $r^{\pm}(j,F)$ pairs to zero with the x_k^2 in (5.18). It also shows that in order for $r^{\pm}(j,F)$ to have nonzero pairing with some $\partial^{\pm}(x_C)$ in (5.20), one must have that $C = \{k,\ell,m\}$ satisfies $F = \{k\} \vee \{\ell\} \vee \{m\}$, and furthermore one must have $j \in C$. In other words, without loss of generality, $C = \{j,\ell,m\} \subset F$. It remains to check that $r^{\pm}(j,F)$ still pairs to zero with $\partial^{\pm}(x_C)$ in this situation. Calculating the pairing, one finds

$$\begin{aligned}(\partial^{\pm}(x_C), r^{\pm}(j,F)) &\\ = \left(\mathrm{sgn}_{C,m} x_j x_\ell + \mathrm{sgn}_{C,\ell} x_j x_m + \mathrm{sgn}_{C,j} x_\ell x_m \right.&, \left.\sum_{h \in F\setminus\{j\}} \chi_{\mathcal{M}|_F}(j,h) \cdot [y_j, y_h]_- \right) \\ = \mathrm{sgn}_{C,m} \cdot \chi_{\mathcal{M}|_F}(j,\ell) + \mathrm{sgn}_{C,\ell} \cdot \chi_{\mathcal{M}|_F}(j,m).&\end{aligned} \tag{5.21}$$

Vanishing of the sum in (5.21) can be checked based on cases for the signed circuit $C = C_+ \sqcup C_-$ supported by $C = \{j,\ell,m\}$. One can relabel so that $|C_+| \ge |C_-|$, and hence $(|C_+|, |C_-|) = (3,0)$ or $(2,1)$. As the indices ℓ, m play a symmetric role in (5.21), one may assume without loss of generality that the oriented matroid $\mathcal{M}|_{\{j,\ell,m\}}$ matches that of one of these vector configurations in $\mathbb{R}^2$:



In each case, one can check that the sum in (5.21) vanishes.

Once one has checked that the elements of $\mathcal{G}$ lie in J_2, Proposition 3.5 together with the following Hilbert series calculations will show that they form a quadratic (noncommutative) GB for J. First note that the $\mathcal{G}$-standard monomials m in $y_1, \ldots, y_n$ are those that avoid all factors $y_j y_i$ in which (j,i) are a retrograde pair, and these are exactly the monomials described in (iv). Thus, denoting $e_p = |E_p|$, one has

$$\begin{aligned}
\mathrm{Hilb}(\Bbbk\langle \mathbf{y}\rangle/(\mathrm{in}_{\prec}(\mathcal{G})), t) &= \sum_{\substack{\mathcal{G}\text{-standard} \\ \text{monomials } m}} t^{\deg(m)} \\
&\overset{(a)}{=} \left(1 + e_1 t + e_1^2 t^2 + e_1^3 t^3 + \cdots\right) \cdots \left(1 + e_r t + e_r^2 t^2 + e_r^3 t^3 + \cdots\right) \\
&= \frac{1}{(1-e_1 t)\cdots(1-e_r t)} \\
&\overset{(b)}{=} \frac{1}{\mathrm{Hilb}(A, -t)} \\
&\overset{(c)}{=} \mathrm{Hilb}(A^!, t) = \mathrm{Hilb}(\Bbbk\langle \mathbf{y}\rangle/J, t).
\end{aligned}$$

where equalities (a), (b), (c) above are justified as follows. Equality (a) follows from the description in (iv) of $\mathcal{G}$-standard monomials as $m = m_1 \cdot m_2 \cdots m_r$ where m_p is any noncommutative monomial in the variable set $\{y_j\}_{y \in E_p}$. Equality (b) comes from (5.12), and equality (c) from Corollary 2.7.

Finally, to see that $\mathcal{G}$ is a *reduced* Gröbner basis, note that Lemma 5.17 implies that for each retrograde pair (j,i), the initial term $y_j y_i$ for the relations $r(j,i), r^{\pm}(j,i)$ cannot appear as a term in any of the other $r(k,\ell), r^{\pm}(k,\ell)$ with $(k,\ell) \neq (j,i)$. ☐

5.5. **Acyclicity and injectivity.** As an application of the Gröbner basis presentations for the algebras $A^! = \mathrm{OS}(M)^!, \mathrm{VG}(\mathcal{M})^!$ in Theorem 5.18, we explore a counterpart to an interesting fact about $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$: their Hilbert series contains a factor of $1+t$,

$$\mathrm{Hilb}(\mathrm{OS}(M), t) = \mathrm{Hilb}(\mathrm{VG}(\mathcal{M}), t) = (1+t) \cdot H(t) \tag{5.22}$$

and the remaining polynomial factor $H(t) \in \mathbb{Z}[t]$ always has *nonnegative* coefficients.

This fact has several explanations: combinatorial, topological, and algebraic. One algebraic explanation views the Orlik–Solomon algebra $A = \mathrm{OS}(M)$ as an algebraic cochain complex

$$0 \to A_0 \overset{d}{\to} A_1 \overset{d}{\to} \cdots \overset{d}{\to} A_{r-1} \overset{d}{\to} A_r \to 0 \tag{5.23}$$

whose differential d is given by multiplication by an element $x = \sum_{i=1}^n c_i x_i$ in A_1. The fact that A is a quotient of an exterior algebra implies that $x^2 = 0$ in A, so that indeed $d \circ d = 0$.

Theorem 5.19 ([94, Thm. 7.2])**.** *The cochain complex* (5.23) *on* $A = \mathrm{OS}(M)$ *is exact whenever* $x = \sum_{i=1}^n c_i x_i$ *has coefficients* c_i *satisfying the following* genericity *condition:* $\sum_{i \in F} c_i \neq 0$ *in* $\Bbbk$ *for all flats* F *whose restriction* $M|_F$ *is not a nontrivial direct sum.*

Thus whenever x is generic, multiplication by x on $A = OS(M)$ is "as injective as possible", given the constraint that $x^2 = 0$. This algebraically interprets the factor $H(t)$ in (5.22), since $tH(t)$ is the Hilbert series for the subspace of cocycles (= coboundaries) in the above cochain complex.

For $M, \mathcal{M}$ supersolvable, the Koszul duals $A^! = \mathrm{OS}(M)^!, \mathrm{VG}(\mathcal{M})^!$ inherit a similar factorization

$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A, -t)} = \frac{1}{1-t} \cdot \frac{1}{H(-t)} = \left(1 + t + t^2 + t^3 + \cdots\right) \cdot H(-t)^{-1}. \quad (5.24)$$

There is nothing that says, a priori, the rightmost factor $H(-t)^{-1}$ above should have nonnegative coefficients. However, this is a consequence of our next result.

Definition 5.20. Let $M, \mathcal{M}$ be supersolvable matroids or oriented matroids of rank r on the ground set $E = \{1, 2, \ldots, n\}$, with partition $\underline{E}$ as in Proposition 5.12. For a fixed $d \geq 1$, say that the power sum $p_d(\mathbf{y}) = \sum_{i=1}^n c_i y_i^d \in A^!_d \subset A^! = \mathrm{OS}(M)^!$ or $\mathrm{VG}(\mathcal{M})^!$ is *$\underline{E}$-generic* if for each $q = 1, 2, \ldots, r$, there exists $i \in E_q$ with the coefficient $c_i \neq 0$.

Theorem 5.21. *Let $M, \mathcal{M}$ be supersolvable matroids or oriented matroids of rank r on E, with partition $\underline{E}$ as in Proposition 5.12. Then for either $A^! = \mathrm{OS}(M)^!$ or $\mathrm{VG}(\mathcal{M})^!$, right-multiplication $a \longmapsto ay$ by any $\underline{E}$-generic element $p_d(\mathbf{y})$ in $A^!_d$ gives an injective map $A^! \longrightarrow A^!$. That is, every $\underline{E}$-generic y is a right-non-zero-divisor on A.*

Proof. Proceed by induction on the rank r. In the base case $r = 1$, the ring $A^! = \Bbbk\langle y\rangle \cong \Bbbk[y]$ is a univariate polynomial ring, and y^d is a nonzero element of $A^!_d$, so y^d is a nonzero divisor.

Preparing for the inductive step, segregate $E = F \sqcup E_r$ where $F := F_{r-1} = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_{r-1}$ is the modular coatom in the modular flag $\underline{F}$, and define the *early* and *late* variables:

$$\{y_1, \ldots, y_n\} = \underbrace{\{y_i\}_{i \in F}}_{\text{early}} \sqcup \underbrace{\{y_j\}_{j \in E_r}}_{\text{late}}.$$

Note that, by Theorem 5.11, the restriction $M|_F$ is a rank $r-1$ supersolvable matroid to which induction applies. Also, note that the presentations in Theorem 5.18 and the standard monomial bases show that the early variables generate a subalgebra of $A^!$ isomorphic to the Koszul dual $\mathrm{OS}(M|_F)^!$ or $\mathrm{VG}(\mathcal{M}|_F)^!$.

The standard monomial basis shows that every a in $A^!$ has a unique decomposition

$$a = \sum_m a(m) \cdot m \quad (5.25)$$

where m runs over all monomials in the late variables, and each $a(m)$ lies in the subalgebra generated by the early variables. Grouping this more coarsely via $\deg(m)$, one obtains a unique decomposition

$$a = \sum_{\ell=0}^{\infty} a^{(\ell)} \quad \text{where } a^{(\ell)} := \sum_{\substack{m: \\ \deg(m)=\ell}} a(m) \cdot m. \quad (5.26)$$

In particular, $p_d(\mathbf{y}) = \sum_{i=1}^n c_i y_i^d = y^{(0)} + y^{(d)}$.

Let $A^!_{(\ell)}$ denote the set of elements of the form $a^{(\ell)}$ above, so there is a $\Bbbk$-vector space decomposition

$$A^! = \bigoplus_{\ell=0}^{\infty} A^!_{(\ell)}$$

and also define

$$A^!_{(\geq \ell)} := \bigoplus_{p=\ell}^{\infty} A^!_{(p)} = A^!_{(\ell)} \oplus A^!_{(\geq \ell+1)}. \quad (5.27)$$

We will use these two facts, justified below:

$$A^!_{(0)} \cdot A^!_{(0)} \subseteq A^!_{(0)},$$
$$A^!_{(\geq p)} \cdot A^!_{(\geq q)} \subseteq A^!_{(\geq p+q)}.$$

These follow ultimately from Lemma 5.17, as we now explain. One can use the Gröbner basis relations $r(j,i), r^{\pm}(j,i)$ for retrograde pairs (j,i) appearing in Theorem 5.18 as rewriting rules, performing the division $f \to_{\mathcal{G}} r$ and rewriting f as a sum r of $\mathcal{G}$-standard monomials. Lemma 5.17 implies that at each division step, one is always replacing

- quadratic initial terms with no late variables by a sum of terms with no late variables,
- quadratic initial terms with one late variable by a sum of terms with one or two late variables.

Continuing the inductive step, assume $a \in A^!_q$ has $a \cdot p_d(\mathbf{y}) = 0$, and we want to conclude that $a = 0$. Writing $a = \sum_{\ell=0}^q a^{(\ell)}$ as in (5.26), we will show each $a^{(\ell)} = 0$ via an inner induction on ℓ.

In the inner induction base case $\ell = 0$, write

$$0 = a \cdot p_d(\mathbf{y}) = a^{(0)} \cdot y^{(0)} + \underbrace{a^{(0)} \cdot y^{(d)} + \sum_{\ell=1}^{q} a^{(\ell)} \cdot p_d(\mathbf{y})}_{\in A^!_{(\geq d)}}.$$

so that $0 \equiv a^{(0)} \cdot y^{(0)} \bmod A^!_{(\geq d)}$. By the direct sum decomposition (5.27), this means $a^{(0)} \cdot y^{(0)} = 0$. By induction on the rank applied to $M|_F$, since $y^{(0)}$ is still generic for $M|_F$, this implies $a^{(0)} = 0$.

In the inner inductive step, assume $a \cdot p_d(\mathbf{y}) = 0$ and that $a^{(0)} = a^{(1)} = \cdots = a^{(\ell-1)} = 0$, that is, a lies in $A^!_{(\geq \ell)}$. We wish to show that $a^{(\ell)} = 0$. Write

$$\begin{aligned} 0 = a \cdot y &= \Big(a^{(\ell)} + a^{(\ell+1)} + \cdots\Big) \cdot (y^{(0)} + y^{(1)}) \\ &= a^{(\ell)} \cdot y^{(0)} + \underbrace{a^{(\ell)} \cdot y^{(d)} + \Big(a^{(\ell+1)} + a^{(\ell+2)} + \dots\Big) \cdot p_d(\mathbf{y})}_{\in A^!_{(\geq \ell+1)}} \end{aligned}$$

so that $0 \equiv a^{(\ell)} \cdot y^{(0)} \bmod A^!_{(\geq \ell+d)}$.

Write $a^{(\ell)} = \sum_m a(m)m$ as in (5.26), so that m runs through all degree ℓ monomials in the late variables. Note that for any early variable y_i and any monomial m of degree ℓ in the late variables, the division algorithm $f \to_{\mathcal{G}} r$ and the form of the relations $r(j,i), r^{\pm}(j,i)$ in $\mathcal{G}$ (again using Lemma 5.17) will rewrite

$$m \cdot y_i^d \equiv y_i^d \cdot m \bmod A^!_{(\geq \ell+d)}.$$

Since $y^{(0)}$ is a sum of early variables, similarly $m \cdot y^{(0)} \equiv y^{(0)} \cdot m \bmod A^!_{(\geq \ell+d)}$, which implies

$$a^{(\ell)} \cdot y^{(0)} = \sum_m a(m) \cdot m \cdot y^{(0)} \equiv \sum_m a(m) \cdot y^{(0)} \cdot m \bmod A^!_{(\geq \ell+d)}$$

Hence one concludes that $0 \equiv \sum_m a(m) \cdot y^{(0)} \cdot m \bmod A^!_{(\geq \ell+d)}$. Since $\sum_m a(m) \cdot y^{(0)} \cdot m$ lies in $A^!_{(\ell)}$, by the direct sum decomposition (5.27), it must vanish. But by the uniqueness in (5.25), this implies each $a(m) \cdot y^{(0)} = 0$. Then by induction on r, each $a(m) = 0$. Hence $a^{(\ell)} = 0$, as desired. □

One has the following corollary to Theorems 5.19 and 5.21.

Corollary 5.22. *Let $M, \mathcal{M}$ be a matroid or oriented matroid, and G a group of automorphisms, that is, a subgroup of $\operatorname{Aut}(M)$ or $\operatorname{Aut}(\mathcal{M})$. Consider G as a group of $\Bbbk$-algebra automorphisms of $A := \mathrm{OS}(M)$ or $\mathrm{VG}(\mathcal{M})$.*

(i) Any $x \in A_1$ which is G-fixed and generic in the sense of Theorem 5.19 (e.g., $x = \sum_{i=1}^n x_i$ when $\Bbbk$ has characteristic zero) gives rise to a factorization in $R_\Bbbk(G)[[t]]$

$$\mathrm{Hilb}_{\mathrm{eq}}(\mathrm{OS}(M), t) = (1+t) \cdot H(t)$$

where $tH(t)$ is the equivariant Hilbert series for the cocycles (=coboundaries) of the $\Bbbk G$-module complex in (5.23).

(ii) Assuming that M is supersolvable with decomposition $\underline{E}$ as in Proposition 5.12, any $y \in A^!_1$ which is G-fixed and $\underline{E}$-generic (e.g., $y = \sum_{i=1}^n y_i$) gives a factorization in $R_\Bbbk(G)[[t]]$

$$\mathrm{Hilb}_{\mathrm{eq}}(\mathrm{OS}(M)^!, t) = \frac{1}{1-t} \cdot H^!(t)$$

where $H^!(t)$ is the equivariant Hilbert series for the quotient $\Bbbk G$-module $A^!/A^!y$.

(iii) Assuming that $\mathcal{M}$ is supersolvable with decomposition $\underline{E}$ as in Proposition 5.12, any $p_2(\mathbf{y}) \in A^!_2$ which is G-fixed and $\underline{E}$-generic[6] (e.g., $p_2(\mathbf{y}) = \sum_{i=1}^n y_i^2$) gives a factorization in $R_\Bbbk(G)[[t]]$

$$\mathrm{Hilb}_{\mathrm{eq}}(\mathrm{VG}(\mathcal{M})^!, t) = \frac{1}{1-t^2} \cdot H^!(t)$$

where $H^!(t)$ is the equivariant Hilbert series for the quotient $\Bbbk G$-module $A^!/A^!p_2(\mathbf{y})$.

Examples of the factorizations in the various parts of Corollary 5.22 appear later:

- Part (i) is illustrated by (6.3), (6.6), (6.12).
- Part (ii) is illustrated by (6.4), (6.7), (6.13).
- Part (iii) is illustrated by (6.8), (8.12).

6. Examples: Boolean matroids and matroids of low rank

Before developing further theory for supersolvable matroids and oriented matroids, we digress to discuss the action of symmetries in a few of our earlier examples, illustrating the results so far.

6.1. **Boolean matroids.** We return to Example 4.5 and the Boolean matroid $M = U_{n,n}$. Although $M = U_{n,n}$ is orientable, we will focus here on $\mathrm{OS}(M)$, where a bit more is known about the action of symmetries, rather than on $\mathrm{VG}(\mathcal{M})$. The discussion of $\mathrm{VG}(\mathcal{M})$ is deferred to Example 8.10 later.

The Boolean matroid M of of rank n has no circuits, so $A = \mathrm{OS}(M) = \wedge V = \wedge(x_1, \ldots, x_n)$, and $A^! = \mathrm{OS}(M)^! = \operatorname{Sym} V = \Bbbk[y_1, \ldots, y_n]$, swapping the roles of $A, A^!$ from Examples 2.4, 2.6, 2.8. Here $\operatorname{Aut}(M) = \mathfrak{S}_n$, and both V, V^* carry the defining representation of $\mathfrak{S}_n$ permuting the subscripts of the variables x_i or y_i.

[6] One cannot always find such G-fixed $\underline{E}$-generic elements in $A^!_1$, e.g., $\sum_{i=1}^n y_i$ is $\underline{E}$-generic, but not always G-fixed.

Thus V is the defining representation of $\mathfrak{S}_n$ by permutation matrices. Assuming that $\Bbbk$ has characteristic zero, V, V^* both decompose into irreducible $\Bbbk\mathfrak{S}_n$-modules as

$$V \cong V^* \cong \mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)}$$

where $\mathcal{S}^\lambda$ denotes the simple $\Bbbk\mathfrak{S}_n$-module indexed by a partition λ of n; here $\mathcal{S}^{(n)}$ is the trivial $\mathfrak{S}_n$-representation, while $\mathcal{S}^{(n-1,1)}$ is the irreducible reflection representation of $\mathfrak{S}_n$. Consequently, in this situation,

$$A = \wedge \left(\mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)}\right) = \wedge\mathcal{S}^{(n)} \otimes \wedge\mathcal{S}^{(n-1,1)} \tag{6.1}$$

$$A^! = \mathrm{Sym}\left(\mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)}\right) = \mathrm{Sym}\,\mathcal{S}^{(n)} \otimes \mathrm{Sym}\,\mathcal{S}^{(n-1,1)}, \tag{6.2}$$

and the factorizations in Corollary 5.22 become

$$\mathrm{Hilb}_{\mathrm{eq}}(A, t) = (1+t)\,\mathrm{Hilb}_{\mathrm{eq}}\left(\wedge\mathcal{S}^{(n-1,1)}, t\right) \tag{6.3}$$

$$\mathrm{Hilb}_{\mathrm{eq}}(A^!, t) = \frac{1}{1-t}\,\mathrm{Hilb}_{\mathrm{eq}}\left(\mathrm{Sym}\,\mathcal{S}^{(n-1,1)}, t\right). \tag{6.4}$$

Both (6.3) and (6.4) can be refined to explicit $\Bbbk\mathfrak{S}_n$-irreducible expansions. For (6.3), since it is known that $\wedge^i\mathcal{S}^{(n-1,1)} \cong \mathcal{S}^{(n-i,1^i)}$, one has

$$\mathrm{Hilb}_{\mathrm{eq}}\left(\wedge\mathcal{S}^{(n-1,1)}, t\right) = \sum_{i=0}^{n-1}\left[\mathcal{S}^{(n-i,1^i)}\right] t^i.$$

For (6.4), one can extend the tensor decomposition (6.2). The $\mathfrak{S}_n$-invariant subalgebra of $\Bbbk[\mathbf{y}]$ is $\Bbbk[\mathbf{y}]^{\mathfrak{S}_n} = \Bbbk[e_1, e_2, \ldots, e_n]$ where $e_k = e_k(\mathbf{y})$ is the k^{th} *elementary symmetric function* in the variables $\mathbf{y}$, and the theory of Cohen–Macaulay rings gives a graded $\Bbbk\mathfrak{S}_n$-module tensor product decomposition

$$\Bbbk[\mathbf{y}] \cong \Bbbk[e_1, e_2, \ldots, e_n] \;\otimes\; \Bbbk[\mathbf{y}]/(e_1, e_2, \ldots, e_n)$$

where $\Bbbk[\mathbf{y}]/(e_1, e_2, \ldots, e_n)$ is the type A *coinvariant algebra.* Hence one has

$$\begin{aligned}\mathrm{Hilb}_{\mathrm{eq}}(A^!, t) &= \mathrm{Hilb}(\Bbbk[e_1, e_2, \ldots, e_n], t) \cdot \mathrm{Hilb}_{\mathrm{eq}}(\Bbbk[\mathbf{y}]/(e_1, e_2, \ldots, e_n), t)\\ &= \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \cdot \sum_Q \left[\mathcal{S}^{\lambda(Q)}\right] t^{\mathrm{maj}(Q)}\end{aligned} \tag{6.5}$$

where the sum on the right, due to Lusztig and Stanley [82, Proposition 4.11]), has Q running over all *standard Young tableaux* with n cells, with $\lambda(Q)$ the partition shape of Q, and $\mathrm{maj}(Q)$ the sum of all entries i in Q for which $i+1$ appears weakly southwest of i (using English notation for tableaux).

We note for future reference in Section 11.6 that $\mathfrak{S}_n$ permutes the monomial basis $\{\mathbf{y}^{\mathbf{a}} = y_1^{a_1}\cdots y_n^{a_n} : \mathbf{a} \in \mathbb{N}^n\}$ of $A^! = \Bbbk[\mathbf{y}]$, making each graded component $A^!_i$ of $A^!$ a *permutation representation.*

6.2. **Rank one matroids.** A simple rank one matroid M has ground set $E = \{e\}$ of size one and no circuits. It is always orientable, and has

$$\begin{aligned}A &= \mathrm{OS}(M) \cong \mathrm{VG}(\mathcal{M}) = \Bbbk[x]/(x^2)\\ A^! &= \Bbbk[y].\end{aligned}$$

The only difference between $M, \mathcal{M}$ arises when one takes into account symmetries. The matroid M has no nontrivial automorphisms, while the oriented matroid $\mathcal{M}$ carries the action of the two-element group $G = \mathrm{Aut}(\mathcal{M}) = \mathfrak{S}_1^{\pm} \cong \mathbb{Z}/2\mathbb{Z}$. Assuming that the characteristic of $\Bbbk$ is not 2, then the generator of G negates both x, y when it acts on $A = \mathrm{VG}(\mathcal{M}) = \Bbbk[x]/(x^2)$ or $A^! = \mathrm{VG}(\mathcal{M})^! = \Bbbk[y]$. Denoting the class of this nontrivial 1-dimensional representation by ϵ in the Grothendieck ring

$$R_\Bbbk(G) \cong \mathbb{Z}[\epsilon]/\left(\epsilon^2 - 1\right)$$

then in the power series ring $R_\Bbbk(G)[[t]]$ one has

$$\begin{aligned} \mathrm{Hilb}_{\mathrm{eq}}(\mathrm{OS}(M), t) &= 1 + t, \\ \mathrm{Hilb}_{\mathrm{eq}}(\mathrm{VG}(\mathcal{M}), t) &= 1 + \epsilon t, \qquad (6.6)\\ \mathrm{Hilb}_{\mathrm{eq}}\left(\mathrm{OS}(M)^!, t\right) &= 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t} \qquad (6.7)\\ \mathrm{Hilb}_{\mathrm{eq}}\left(\mathrm{VG}(\mathcal{M})^!, t\right) &= 1 + \epsilon t + \epsilon^2 t^2 + \epsilon^3 t^3 + \cdots = \frac{1}{1-\epsilon t} \\ &= 1 + \epsilon t + t^2 + \epsilon t^3 + \cdots = \frac{1+\epsilon t}{1-t^2}. \qquad (6.8) \end{aligned}$$

6.3. **Rank two matroids.** As discussed in Example 4.6, a simple rank two matroid M on ground set $E = \{1, 2, \ldots, n\}$ is always orientable, and supersolvable. Any rank 1 flat, such as $F = \{1\}$ is a modular coatom, and one has the corresponding set partition decomposition $\underline{E} = (E_1, E_2) = (\{1\}, \{2, 3, \ldots, n\})$ with $(e_1, e_2) = (1, n-1)$. Therefore,

$$\begin{aligned} \mathrm{Hilb}(\mathrm{OS}(M), t) &= \mathrm{Hilb}(\mathrm{VG}(\mathcal{M}), t) = (1+t)(1+(n-1)t) = 1 + nt + (n-1)t^2, \\ \mathrm{Hilb}(\mathrm{OS}(M)^!!, t) &= \mathrm{Hilb}(\mathrm{VG}(\mathcal{M})^!, t) \\ &= \frac{1}{(1-t)(1-(n-1)t)} \qquad (6.9)\\ &= 1 + (1 + (n-1))t + (1 + (n-1) + (n-1)^2)t^2 + \cdots \\ &= \sum_{i=0}^{\infty} f(n,i) t^i \quad \text{where } f(n,i) := \sum_{j=0}^{i} (n-1)^i = \frac{(n-1)^{i+1} - 1}{n-2}. \end{aligned}$$

In considering symmetries, it is somewhat easier to compute with $\mathrm{OS}(M)$, rather than $\mathrm{VG}(\mathcal{M})$. The matroid M has as its symmetries the full symmetric group $G = \mathrm{Aut}(M) = \mathfrak{S}_n$, arbitrarily permuting $E = \{1, 2, \ldots, n\}$. It is also helpful to introduce a notation φ_λ for the class $[\Bbbk[\mathfrak{S}_n/\mathfrak{S}_\lambda]]$ within $R_\Bbbk(\mathfrak{S}_n)$ of the $\mathfrak{S}_n$-permutation representation on the cosets of the Young subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of $n = |\lambda| := \sum_{i=1}^{\ell} \lambda_i$. Hence, if $\Bbbk$ were a field of characteristic zero (which we do *not* assume here), then this class φ_λ corresponds to the product of complete homogeneous symmetric functions

$$h_\lambda := h_{\lambda_1} \cdots h_{\lambda_\ell}$$

under the *Frobenius characteristic isomorphism* $R_\Bbbk(\mathfrak{S}_n) \cong \Lambda_n$, where Λ_n are the degree n homogeneous symmetric functions $\Lambda(z_1, z_2, \ldots)_n$ in infinitely many variables.

One finds that $\mathrm{OS}(M)_1$ carries the defining permutation representation of $\mathfrak{S}_n$ permuting coordinates in $\Bbbk^n$, whose class in $R_\Bbbk(\mathfrak{S}_n)$ is $\varphi_{(n-1,1)}$. Introducing the $\Bbbk\mathfrak{S}_n$-submodule

$$\mathcal{S}^{(n-1,1)} := \{\mathbf{x} \in \Bbbk^n : x_1 + \cdots + x_n = 0\}, \qquad (6.10)$$

the quotient $\varphi_{(n-1,1)}/\mathcal{S}^{(n-1,1)}$ carries the trivial $\Bbbk\mathfrak{S}_n$-module $\mathcal{S}^{(n)}$, giving this identity in $R_\Bbbk(\mathfrak{S}_n)$:

$$[\mathrm{OS}(M)_1] = \varphi_{(n-1,1)} = \left[\mathcal{S}^{(n)}\right] + \left[\mathcal{S}^{(n-1,1)}\right] = 1 + \left[\mathcal{S}^{(n-1,1)}\right]. \qquad (6.11)$$

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The Hilbert series in (6.9) have the following equivariant lifts to $R_{\Bbbk}(\mathfrak{S}_n)[[t]]$:

$$\mathrm{Hilb}_{\mathrm{eq}}(\mathrm{OS}(M),t) = (1+t)\left(1+\left[\mathcal{S}^{(n-1,1)}\right]t\right) = 1+\varphi_{(n-1,1)}t+\left[\mathcal{S}^{(n-1,1)}\right]t^2, \tag{6.12}$$

$$\mathrm{Hilb}_{\mathrm{eq}}\left(\mathrm{OS}(M)^!,t\right) = \frac{1}{(1-t)\left(1-\left[\mathcal{S}^{(n-1,1)}\right]t\right)} \tag{6.13}$$

$$= 1+\left(1+\left[\mathcal{S}^{(n-1,1)}\right]\right)t+\left(1+\left[\mathcal{S}^{(n-1,1)}\right]+\left[\mathcal{S}^{(n-1,1)}\right]^2\right)t^2+\cdots$$

$$= \sum_{i=0}^{\infty} t^i\left(\sum_{k=0}^{i}\left[\mathcal{S}^{(n-1,1)}\right]^k\right) \tag{6.14}$$

We find the next proposition somewhat unexpected.

Proposition 6.1. *In $R_{\Bbbk}(\mathfrak{S}_n)$, the element $[\mathrm{OS}(M)^!_i] = \sum_{k=0}^{i}[\mathcal{S}^{(n-1,1)}]^k$ is the class of a permutation representation, expressible in the following form:*

$$\left[\mathrm{OS}(M)^!_i\right] = \begin{cases} \varphi_{(n)} + \sum_{d=2}^{i} a_{d,i}\,\varphi_{(n-d,1^d)}, & i \text{ even},\\ \varphi_{(n-1,1)} + \sum_{d=2}^{i} b_{d,i}\,\varphi_{(n-d,1^d)}, & i \text{ odd},\end{cases}$$

where $\{a_{d,i}\},\{b_{d,i}\}$ are positive integers, independent of n, given by sums of Stirling numbers:

$$a_{d,i} = \sum_{k=\lfloor \frac{d}{2}\rfloor}^{\frac{i}{2}} S(2k-1,d-1) \text{ for } i \text{ even},$$

$$b_{d,i} = \sum_{k=\lfloor \frac{d-1}{2}\rfloor}^{\frac{i-1}{2}} S(2k,d-1) \text{ for } i \text{ odd}.$$

Proof. The following identity is established in [89, Proposition 7.6, Theorem 7.7] for $j \geq 2$.

$$\left[\mathcal{S}^{(n-1,1)}\right]^j + \left[\mathcal{S}^{(n-1,1)}\right]^{j-1} = \sum_{d=2}^{j} S(j-1,d-1)\ \varphi_{(n-d,1^d)}. \tag{6.15}$$

Since the proofs in [89] are phrased in terms of symmetric functions, over a ground field of characteristic zero, we explain why this identity still holds in the Grothendieck ring $R_{\Bbbk}(\mathfrak{S}_n)$ for arbitrary fields $\Bbbk$. Sundaram constructs an explicit $\Bbbk\mathfrak{S}_n$-module realizing the jth tensor power of the $\mathfrak{S}_n$-permutation module $V_{1,n}$, whose class is $\varphi_{(n-1,1)}$, and decomposes it in terms of the coset permutation submodules

$$V_{d,n} = (\Bbbk\mathfrak{S}_d)\uparrow_{\mathfrak{S}_{n-d}\times\mathfrak{S}_1^d}^{\mathfrak{S}_n}$$

whose class is $\varphi_{(n-d,1^d)}$, obtaining [89, Eqn. 18, Lemma. 6.1])

$$V_{1,n}^{\otimes j} = \sum_{d=1}^{\min(n,j)} S(j,d)\,V_{d,n}. \tag{6.16}$$

In addition we will use the following three facts (1),(2),(3).

(1) $[\mathcal{S}^{(n-1,1)}\downarrow_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}^{\mathfrak{S}_n}] = [V_{1,n-1}]$. We show this as follows.

First, if $\mathcal{S}^{(n-1)}$ is the span of a fixed standard basis vector, and $V_{1,n-1}$ is the span of the $n-1$ non-fixed standard basis vectors, we clearly have $V_{1,n}\downarrow = \mathcal{S}^{(n-1)} \oplus V_{1,n-1}$, which in turn gives

$$[V_{1,n}\downarrow] = \left[\mathcal{S}^{(n-1)}\right] + [V_{1,n-1}]. \tag{6.17}$$

Now recall the definition in (6.10) of $\mathcal{S}^{(n-1,1)}$ and the Grothendieck group identity (6.11). The discussion around (6.10) and (6.11) in effect establishes the existence of a short exact sequence of $\Bbbk\mathfrak{S}_n$-modules

$$0 \to \mathcal{S}^{(n-1,1)} \to V_{1,n} \to \mathcal{S}^{(n)} \to 0,$$

which restricts to the same sequence as $(\mathfrak{S}_{n-1}\times\mathfrak{S}_1)$-modules. Hence we have

$$[V_{1,n}\downarrow] = \left[\mathcal{S}^{(n)}\downarrow\right] + \left[\mathcal{S}^{(n-1,1)}\downarrow\right] \tag{6.18}$$

Since $\mathcal{S}^{(n)}\downarrow = \mathcal{S}^{(n-1)}$, comparing (6.17) and (6.18) we obtain $[\mathcal{S}^{(n-1,1)}\downarrow] = [V_{1,n-1}]$.

(2) [93, Corollary 4.3.8, Part (2)] Transitivity of induction;
(3) [93, Corollary 4.3.8, Part (4)] For a finite group G and subgroup H, and $\Bbbk G$-module U, $\Bbbk H$-module V, over any field $\Bbbk$,

$$U \otimes \left(V\uparrow_H^G\right) \cong (U\downarrow_H \otimes V)\uparrow_H^G .$$

In the present situation we have $G = \mathfrak{S}_n$, $H = \mathfrak{S}_{n-1}\times\mathfrak{S}_1$, $V = 1_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}$, so that the class of $V\uparrow_H^G$ is $\varphi_{(n-1,1)}$, and $U = \mathcal{S}^{(n-1,1)^{\otimes j-1}}$. Following the proof of [89, Proposition 7.6]), we have

$$\begin{aligned}
\left[\mathcal{S}^{(n-1,1)}\right]^j + \left[\mathcal{S}^{(n-1,1)}\right]^{j-1} &= \left[\mathcal{S}^{(n-1,1)}\right]^{j-1}\left(\left[\mathcal{S}^{(n)}\right] + \left[\mathcal{S}^{(n-1,1)}\right]\right)\\
&= [U\otimes V_{1,n}] \quad \text{by (6.11) and definition of } U,\\
&= \left[U\otimes(\mathbb{1}_H)\uparrow_H^G\right] \quad \text{by definition of } H, G, V_{1,n},\\
&= \left[\left(U\downarrow_H^G\right)\uparrow_H^G\right] \quad \text{by item (3) above,}\\
&= \left[\left(\left(\mathcal{S}^{(n-1,1)}\right)^{\otimes(j-1)}\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}\right)\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}\right]\\
&= \left[V_{1,n-1}^{\otimes j-1}\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}\right] \text{ by item (1) above,}\\
&= \sum_{d'=1}^{\min(n-1,j-1)} S(j-1,d')\left[V_{d',n-1}\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right] \text{ using (6.16)}\\
&= \sum_{d'=1}^{\min(n-1,j-1)} S(j-1,d')\,[V_{d'+1,n}] \text{ by item (2) above,}\\
&= \sum_{d=2}^{\min(n,j)} S(j-1,d-1)\,[V_{d,n}].
\end{aligned}$$

Hence for $i \geq 2$, we have

$$\begin{aligned}
\left[\mathrm{OS}(M)^!_i\right] &= \sum_{k=0}^{i} \left[\mathcal{S}^{(n-1,1)}\right]^k \\
&= \begin{cases} \left[\mathcal{S}^{(n)}\right] + \sum_{k=1}^{i/2} \left(\left[\mathcal{S}^{(n-1,1)}\right]^{2k} + \left[\mathcal{S}^{(n-1,1)}\right]^{2k-1} \right), & i \text{ even}, \\ \left[\mathcal{S}^{(n)}\right] + \left[\mathcal{S}^{(n-1,1)}\right] + \sum_{k=1}^{(i-1)/2} \left(\left[\mathcal{S}^{(n-1,1)}\right]^{2k+1} + \left[\mathcal{S}^{(n-1,1)}\right]^{2k} \right), & i \text{ odd}, \end{cases} \\
&= \begin{cases} \varphi_{(n)} + \sum_{k=1}^{i/2} \left(\sum_{d=2}^{2k} S(2k-1, d-1)\, \varphi_{\left(n-d,1^d\right)} \right), & i \text{ even}, \\ \varphi_{(n-1,1)} + \sum_{k=1}^{(i-1)/2} \left(\sum_{d=2}^{2k+1} S(2k, d-1)\, \varphi_{\left(n-d,1^d\right)} \right), & i \text{ odd}. \end{cases}
\end{aligned}$$

Interchanging the order of summation then gives the assertion of the proposition for $i \geq 2$. For $i = 0, 1$ it is easy to check that $[\mathrm{OS}(M)^!_0] = \varphi_{(n)}$ and $[\mathrm{OS}(M)^!_1] = \varphi_{(n-1,1)}$. □

Remark 6.2. Rather than all of $\mathfrak{S}_n$, one might restrict the action on $\mathrm{OS}(M)$ to the *dihedral group*[7]

$$W = I_2(n) = \left\langle r, s : r^n = s^2 = 1, srs = r^{-1} \right\rangle$$

of order $2n$. Since $\mathfrak{S}_n$ acts on each $\mathrm{OS}(M)^!_i$ via permutation representations, the same must hold for W by restriction. One can check via character computations (omitted here) that $\mathrm{OS}(M)^!_i$ is always a nonnegative combination of these four permutation representations:

- the *trivial* representation,
- the *defining* representation ρ_{def} on $E = \{1, 2, \ldots, n\}$ with $r = (12 \cdots n)$, $s(i) = n + 1 - i$,
- the *regular representation* $\rho_{\mathrm{reg}} := \Bbbk W$, and
- when n is even, the *half-regular representation* $\rho_{\frac{1}{2}\mathrm{reg}} := \Bbbk[W/Z_W]$ where $Z_W := \{1, r^{\frac{n}{2}}\}$.

One has these expansions in $R_\Bbbk(W)$, where $f(n,i) := \dim_\Bbbk \mathrm{OS}^!_i = \frac{(n-1)^{i+1}-1}{n-2}$ as in (6.9):

$$\left[\mathrm{OS}(M)^!_i\right] = \begin{cases} \frac{1}{2n}\left[f(n,i) - 1\right] \cdot \rho_{\mathrm{reg}} + 1 & \text{if } n \text{ is odd, and } i \text{ is even}, \\ \frac{1}{2n}\left[f(n,i) - n\right] \cdot \rho_{\mathrm{reg}} + \rho_{\mathrm{def}} & \text{if } n \text{ is odd, and } i \text{ is odd}, \\ \\ \frac{1}{n}\left[f(n,i) - n \cdot \frac{i}{2} - 1\right] \cdot \rho_{\frac{1}{2}\mathrm{reg}} + \frac{i}{2} \cdot \rho_{\mathrm{def}} + 1 & \text{if } n \text{ is even, and } i \text{ is even}, \\ \frac{1}{n}\left[f(n,i) - n \cdot \frac{i+1}{2}\right] \cdot \rho_{\frac{1}{2}\mathrm{reg}} + \frac{i+1}{2} \cdot \rho_{\mathrm{def}} & \text{if } n \text{ is even, and } i \text{ is odd}. \end{cases}$$

Remark 6.3. It was observed earlier that uniform matroids $U_{r,n}$ are supersolvable if and only if $r \in \{1, 2, n\}$. This means that $M = U_{3,n}$ for $n \geq 4$ are *not supersolvable*, and in fact, $A = \mathrm{OS}(M)$ are not Koszul algebras, and not even quadratic. If one nevertheless tries to define virtual $\mathfrak{S}_n$-characters $\{[A^!_i]\}_{i \geq 0}$ in terms of the genunine characters $\{[A_i]\}_{i \geq 0}$ via the recurrence (2.12), then already $[A^!_3]$ are not genuine characters once $n \geq 4$.

[7] These are symmetries of the rank two *oriented* matroid $\mathcal{M}$, although we are ignoring the action on $\mathrm{VG}(\mathcal{M})$ here.

7. Branching rules for supersolvable matroids

Let $M, \mathcal{M}$ be supersolvable matroids or oriented matroids of rank r on ground set E, with a modular complete flag $\underline{F}$ and decomposition $\underline{E}$ as in Proposition 5.12. If $F = F_{r-1}$ denotes the modular coatom within the flag $\overline{F}$, then the restrictions $M|_F, \mathcal{M}|_F$ are again supersolvable. Furthermore, the formulas (5.12), (5.17) for the Hilbert series of the rings $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ and their Koszul duals $A^!$ show that they are closely related to the Hilbert series of the same rings $B, B^!$ for $M|_F, \mathcal{M}|_F$:

$$\mathrm{Hilb}(A,t) = (1+e_1 t)\cdots(1+e_{r-1}t)(1+e_r t) = \mathrm{Hilb}(B,t)\cdot(1+e_r t)$$
$$\mathrm{Hilb}(A^!,t) = \frac{1}{(1-e_1 t)\cdots(1-e_{r-1}t)(1-e_r t)} = \mathrm{Hilb}(B^!,t)\cdot\frac{1}{1-e_r t}$$

which one can rewrite suggestively as follows, for comparison with Proposition 2.16:

$$\mathrm{Hilb}(A,t) = \mathrm{Hilb}(B,t) + t\cdot e_r\cdot \mathrm{Hilb}(B,t) \tag{7.1}$$
$$\mathrm{Hilb}(A^!,t) = \mathrm{Hilb}(B^!,t) + t\cdot e_r\cdot \mathrm{Hilb}(A^!,t). \tag{7.2}$$

This suggests considering a group G of automorphisms of M or $\mathcal{M}$, and how its action on $A, A^!$ restricts to the setwise G-stabilizer subgroup of the modular coatom F

$$H := \{g \in G : g(F) = F\}.$$

Note that H also permutes the ground set elements $E_r := E \setminus F$; in the case where G acts on the oriented matroid $\mathcal{M}$, so that G acts via signed permutations in $\mathfrak{S}_n^{\pm}$ on E as in Definition 5.7, then H acts via signed permutations on E_r. This gives rise to either a permutation or signed permutation $\Bbbk H$-module $\mathcal{X} := \Bbbk[E_r]$, which in particular is self-contragredient. Our goal in the next two subsections is to prove Theorem 7.1 below, which not only lifts (7.1), (7.2) to these two branching relations in $R_{\Bbbk}(H)$

$$[A_i\downarrow] = [B_i] + [\mathcal{X}]\cdot[B_{i-1}] \tag{7.3}$$
$$[A^!_i\downarrow] = [B^!_i] + [\mathcal{X}]\cdot\Big([A^!_{i-1}\downarrow]\Big). \tag{7.4}$$

(equivalent by Proposition 2.16 as $\mathcal{X}^* \cong \mathcal{X}$), but also lifts them to short exact sequences.

Theorem 7.1. *With the above notations, and letting $\Bbbk$ be any field, one has the following short exact sequences of graded $\Bbbk H$-modules:*

(i) $0 \longrightarrow B \longrightarrow A\downarrow^G_H \longrightarrow \mathcal{X}\otimes B(-1) \longrightarrow 0,$

(ii) $0 \longrightarrow \mathcal{X}\otimes\Big(A^!\downarrow^G_H\Big)(-1) \longrightarrow A^!\downarrow^G_H \longrightarrow B^! \longrightarrow 0.$

7.1. Proof of Theorem 7.1 (i).

Proof. The injective maps $B \to A$ from Theorem 7.1 (i) are instances of injections of Orlik–Solomon and Varchenko-Gel'fand algebras coming from their flat decompositions

$$\mathrm{OS}(M) = \bigoplus_{F\in\mathcal{F}} \mathrm{OS}(M)_F,$$
$$\mathrm{VG}(\mathcal{M}) = \bigoplus_{F\in\mathcal{F}} \mathrm{OS}(M)_F$$

discussed in Section 5.1. The NBC monomial $\Bbbk$-bases from Remark 5.10 show that for any flat F in $\mathcal{F}$, one has $\Bbbk$-algebra inclusions (see [68, Proposition 3.30], [94, Proposition 2.5], and [19, Proposition 5.6])

$$\mathrm{OS}(M|_F) \cong \bigoplus_{F' \leq F} \mathrm{OS}(M)_{F'} \hookrightarrow \mathrm{OS}(M),$$
$$\mathrm{VG}(\mathcal{M}|_F) \cong \bigoplus_{F' \leq F} \mathrm{VG}(\mathcal{M})_{F'} \hookrightarrow \mathrm{VG}(\mathcal{M}).$$

Since $A = \mathrm{OS}(M)$ or $\mathrm{VG}(\mathcal{M})$ and $B = \mathrm{OS}(M|_F)$ or $\mathrm{VG}(\mathcal{M}|_F)$, this explains the injection $B \to A$ from the sequence Theorem 7.1 (i). In fact, the entire sequence actually holds in slightly more generality, and for Orlik–Solomon algebras was essentially observed by Orlik and Terao [68, Lemma 3.80]. One does not need to assume that $M, \mathcal{M}$ are supersolvable, only that F is a modular coatom within its lattice of flats $\mathcal{F}$. Keeping the same notations so that $A = \mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$, and $B = \mathrm{OS}(M|_F), \mathrm{VG}(\mathcal{M}|_F)$, with H the setwise G-stabilizer of F with a group of the autmorphisms G, one has the following.

Proposition 7.2. *Any modular coatom F gives rise to a $\Bbbk$-vector space direct sum decomposition*

$$A = B \oplus \left(\bigoplus_{j \in E \setminus F} B x_j \right),$$

which can also be viewed as a graded $\Bbbk H$-module isomorphism

$$A \downarrow_H^G \cong \; B \; \oplus \; \mathcal{X} \otimes B(-1),$$

where $\mathcal{X}$ is the permutation or signed permutation representation of H on $E \setminus F$ as before.

Proof. First note a consequence of Theorem 5.11: if one orders/indexes $E = \{1, 2, \ldots, n\}$ so that $i < j$ whenever $i \in F$ and $j \in E \setminus F$, then every pair $\{j, k\} \subseteq E \setminus F$ with $j \neq k$ is a broken-circuit, coming from the 3-element circuit $\{i, j, k\}$ with $\{i\} := F \cap (\{j\} \vee \{k\})$. This implies NBC subsets for M contain at most one element j of $E \setminus F$, so NBC monomials for M are either of the form

(a) $x_{i_1} \cdots x_{i_p}$ for NBC sets $\{i_1, \ldots, i_p\} \subseteq F$, or
(b) $x_{i_1} \cdots x_{i_p} x_j$ for NBC sets $\{i_1, \ldots, i_p\} \subseteq F$, and $j \in E \setminus F$.

Identifying B with $\bigoplus_{F' \subseteq F} \mathrm{OS}(M)_{F'}$ or $\bigoplus_{F' \subseteq F} \mathrm{VG}(\mathcal{M})_{F'}$ expresses A as a $\Bbbk$-vector space sum

$$A = B + \left(\sum_{j \in E \setminus F} B x_j \right).$$

However, these sums are *direct*, via dimension-counting: if $e := |E \setminus F|$, then one has

$$\dim_\Bbbk A = \dim_\Bbbk B (1 + e)$$

as the $t = 1$ specialization of the identity $\mathrm{Hilb}(A, t) = \mathrm{Hilb}(B, t)(1 + et)$ (cf. (7.1) above) which follows either from Stanley [80, Thm. 2] or from Orlik and Terao [68, Lem. 3.80].

Note that this dimension count also implies that the NBC monomials in (a),(b) above form $\Bbbk$-bases for B and $\bigoplus_{j \in E \setminus F} B x_j$, respectively. This lets one write a $\Bbbk$-vector space isomorphism

$$\mathcal{X} \otimes B \xrightarrow{f} \bigoplus_{j \in E \setminus F} B x_j$$

as follows: naming the $\Bbbk$-basis elements $\{t_j : j \in E \setminus F\}$ for the permutation or signed permutation representation $\mathcal{X}$ of H, let the isomorphism f map $t_j \otimes x_{i_1} \cdots x_{i_p} \longmapsto x_{i_1} \cdots x_{i_p} x_j$. Since this means that $f(t_j \otimes b) = b x_j$ for $b \in B$, the H-equivariance follows from this calculation: by definition, $g \in H$ has $g(t_j) = \pm t_k$ for $j, k \in E \setminus F$ if and only if $g(x_j) = \pm x_k$, with the same $\pm$ signs for both. ☐

7.2. **Proof of Theorem 7.1 (ii).** The surjective map $A^! \to B^!$ within the exact sequence of Theorem 7.1 (ii) is simple to define. As before, let $M, \mathcal{M}$ be supersolvable matroids or oriented matroids on ground set $E = \{1, 2, \ldots, n\}$ and let $\underline{E}$ be as in Proposition 5.12, with $F = F_{r-1} = E_1 \sqcup \cdots \sqcup E_{r-1}$ and $E_r = E \setminus F$. Let $A^! = \mathrm{OS}(M)^!$ or $\mathrm{VG}(\mathcal{M})^!$, and $B^! = \mathrm{OS}(M|_F)^!$ or $\mathrm{VG}(\mathcal{M}|_F)^!$.

Proposition 7.3. *The surjective* $\Bbbk$*-algebra map*

$$\begin{array}{rcll} \Bbbk\langle y_1, \ldots, y_n\rangle & \longrightarrow & \Bbbk\langle y_i\rangle_{i \notin E_r} & \\ y_i & \longmapsto & y_i & \text{if } i \notin E_r \\ y_j & \longmapsto & 0 & \text{if } j \in E_r, \end{array}$$

induces a surjective $\Bbbk$*-algebra map* $A^! \twoheadrightarrow B^!$.

Proof. Check that in the quadratic Gröbner basis presentation of Theoerem 5.18 for $A^!$, if a quadratic term is divisible by a variable y_j with $j \in E_r$, then every term is divisible by such a variable. ☐

It only remains to identify the kernel of the surjection in Proposition 7.3. Recall that H is the setwise G-stabilizer subgroup for the modular coatom F, and $\mathcal{X}$ is the permutation or signed permutation representation of H as it acts on E_r, with the $\Bbbk$-basis of $\mathcal{X}$ denoted $\{t_j\}_{j \in E_r}$.

Proposition 7.4. *The* $\Bbbk$*-linear map*

$$\begin{array}{rcl} \mathcal{X} \otimes A^! & \longrightarrow & A^! \\ t_j \otimes y_{i_1} \cdots y_{i_p} & \longmapsto & y_{i_1} \cdots y_{i_p} y_j \end{array}$$

is injective, with image equal to the kernel of the surjection $A^! \twoheadrightarrow B^!$ *in Proposition 7.3.*

Proof. Since the surjection $A^! \twoheadrightarrow B^!$ is induced by sending the variables $\{y_j\}_{j \in E_r}$ to zero, its kernel is the two-sided ideal $I = (y_j : j \in E_r) \subset A^!$ that they generate. As in the proof of Theorem 5.21, the presentation for $A^!$ described in Theorem 5.18 and its standard monomial $\Bbbk$-basis identify this ideal I as $A^!_{(\geq 1)}$, the span of standard monomials $m_1 m_2 \cdots m_{r-1} \cdot m_r$, with m_p in the variable set $\{y_j\}_{j \in E_p}$, that have $\deg(m_r) \geq 1$. Classifying such standard monomials according to their rightmost variable y_j shows that $I = A^!_{(\geq 1)}$ is the image of the map in the current proposition. The standard monomial basis for $A^!$ also shows that this map is injective. ☐

Noting that the maps in Propositions 7.3 and 7.4 are both H-equivariant proves Theorem 7.1 (ii).

8. Homotopy and holonomy Lie algebras

In Section 2, we defined a standard graded $\Bbbk$-algebra to be Koszul if it had a (left-)free resolution of $\Bbbk = A/A_+$ which is linear. It turns out (see [71, Section 2.1]) that this definition is equivalent to any of the following conditions:

(a) $\mathrm{Ext}^i_A(\Bbbk, \Bbbk)_j = 0$ for $i \neq j$,
(b) A is quadratic and $A^! \cong \mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$,
(c) A is quadratic and $(A^!_i)^* \cong \mathrm{Tor}^A_i(\Bbbk, \Bbbk)$,
(d) A is generated by A_1 and the algebra $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$ is generated by $\mathrm{Ext}^1_A(\Bbbk, \Bbbk)$.

The next proposition explains why quadratic algebras A that are either commutative or anti-commutative have quadratic duals $A^!$ which inherit a Hopf algebra structure from the tensor algebra, making $A^!$ the universal enveloping algebra of a graded Lie (super)-algebra. It can be considered an elaboration on [71, Section I.2 Examples 4,5].

Proposition 8.1. *When a quadratic algebra A is anti-commutative (resp. commutative), its quadratic dual $A^!$ is not just a $\Bbbk$-algebra, but actually a co-commutative Hopf algebra (resp. co-commutative signed Hopf algebra, in the sense of Cartier–Patras* [23, Section 3.9]*). Hence by the Cartier–Milnor–Moore Theorem, $A^!$ is the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of the graded Lie algebra (resp. Lie superalgebra) $\mathcal{L} \subset A^!$ which is its $\Bbbk$-subspace of primitive elements.*

Proof. Since $A^! := \Bbbk\langle \mathbf{y}\rangle/J$ for a two-sided (algebra) ideal J, it suffices to check that J is also a *co-ideal* for the co-product Δ on the Hopf algebra $H := \Bbbk\langle \mathbf{y}\rangle = T(V)$, that is, $\Delta(J) \subseteq H \otimes J + J \otimes H$. We give the argument for the case where A is commutative; the anti-commutative case is similar.

Since $J = HJ_2H$ is generated as a two-sided ideal by J_2, and since $\Delta : H \to H \otimes H$ is an algebra morphism, it suffices to check that $\Delta(J_2) \subseteq H \otimes J + J \otimes H$. Note that since the quadratic algebra $A = \Bbbk\langle \mathbf{x}\rangle/I$ is commutative, it must be that $I = (I_2)$ has I_2 containing the $\Bbbk$-span of all commutators $[x_i, x_j]_+$. Consequently, $J_2 = I_2^\perp$ lies in the perp space of the span of all such commutators, which is the $\Bbbk$-span of all anti-commutators $[y_i, y_j]_-$, allowing $i = j$ here.

Claim. Every anti-commutator $[y_i, y_j]_-$ is *primitive*, meaning $\Delta[y_i, y_j]_- = 1 \otimes [y_i, y_j]_- + [y_i, y_j]_- \otimes 1$.

Assuming the claim, every $j \in J_2$ is also primitive, so $\Delta j = 1\otimes j + j\otimes 1 \in H\otimes J + J\otimes H$, as desired. Checking the claim is a standard calculation: when $a = y_i$, and $b = y_j$ are both primitive, and of odd degree, then their anti-commutator is also primitive:

$$\begin{aligned}\Delta[a,b]_- = \Delta(ab+ba) &= (1\otimes a + a\otimes 1)(1\otimes b + b\otimes 1) + (1\otimes b + b\otimes 1)(1\otimes a + a\otimes 1)\\ &= 1\otimes ab - b\otimes a + a\otimes b + ab\otimes 1 + 1\otimes ba - a\otimes b + b\otimes a + ba\otimes 1\\ &= 1\otimes ab + ab\otimes 1 + 1\otimes ba + ba\otimes 1\\ &= 1\otimes(ab+ba) + (ab+ba)\otimes 1 = 1\otimes[a,b]_- + [a,b]_-\otimes 1.\end{aligned}$$

□

Remark 8.2. If a quadratic algebra A is neither commutative nor anti-commutative, then $A^!$ might not inherit a Hopf algebra structure from the tensor algebra. Consider the quadratic algebra

$$A^! = \Bbbk[x,y]/\left(x^2+y^2\right) = \Bbbk\langle x,y\rangle/\left(x^2+y^2, xy-yx\right).$$

This is the quadratic dual of $A = \Bbbk\langle x,y\rangle/(xy+yx, y^2-x^2)$, which is neither commutative nor anti-commutative. Notice that if the characteristic of $\Bbbk$ is not equal to 2, the ideal $J = (x^2+y^2, xy-yx)$ is *not* a co-ideal for the coproduct Δ on the tensor algebra $H = \Bbbk\langle x,y\rangle$:

$$\Delta(x^2+y^2) = \left(x^2+y^2\right)\otimes 1 + 2(x\otimes x + y\otimes y) + 1\otimes\left(x^2+y^2\right),$$

which one can check does not lie in $H\otimes J + J\otimes H$.

When $A = \bigoplus_{d=0}^{\infty} A_d$ is an associative standard graded $\Bbbk$-algebra, so generated by A_1, and is either commutative or anti-commutative, the Yoneda algebra $\operatorname{Ext}^\bullet_A(\Bbbk,\Bbbk)$ has a natural coproduct giving it the structure of a graded Hopf algebra. Therefore, $\operatorname{Ext}^\bullet_A(\Bbbk,\Bbbk)$ is also the universal enveloping algebra of a graded Lie (super-)algebra. See Avramov [6, Section 10.1] for a discussion when A is commutative, and Denham and Suciu [31, Section 1] for the case where A is anti-commutative.

Definition 8.3. In the above setting, the *homotopy Lie algebra* π_A is the graded Lie algebra or Lie superalgebra of primitive elements in the Yoneda algebra $\operatorname{Ext}^\bullet_A(\Bbbk,\Bbbk)$ of A, that is,

$$\mathcal{U}(\pi_A) \cong \operatorname{Ext}^\bullet_A(\Bbbk,\Bbbk).$$

8.1. **The holonomy Lie algebra.** Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an associative graded $\Bbbk$-algebra, with $\Bbbk$-basis $x_1, \ldots, x_n$ for $V = A_1$; for the moment we do not assume that A is generated by A_1. Then the *decomposable* elements of A_2 are defined to be those in the image of the multiplication map

$$\phi : A_1 \otimes A_1 \to A_2.$$

Letting $V = A_1^*$ have dual basis $y_1, \ldots, y_n$, if one considers the dual of this multiplication map

$$\phi^* : A_2^* \to (A_1 \otimes A_1)^* \cong A_1^* \otimes A_1^*,$$

then one has these identifications:

$$\begin{aligned} \operatorname{im}(\phi^*) &\cong \left(\frac{A_1 \otimes A_1}{\ker \phi}\right)^* \cong \{f \in A_1^* \otimes A_1^* : f(\ker\phi) = 0\}, \\ &\cong \left(\frac{T^2(V)}{I_2}\right)^* \cong I_2^\perp =: J_2. \end{aligned} \tag{8.1}$$

Here we consider $J_2 = I_2^\perp$ as a subspace of $T^2(V^*) = V^* \otimes V^*$, with pairing $T^2(V^*) \times T^2(V) \to \Bbbk$ just as in (2.3). Now just as the proof of Proposition 8.1, if one further assumes that A is commutative (resp. anti-commutative), then I_2 contains the $\Bbbk$-span of all commutators $[x_i, x_j]_+$ (resp. anti-commutators $[x_i, x_j]_-$). Consequently, $J_2 = I_2^\perp$ lies in the perp space of the span of all such commutators or anti-commutators, which is the $\Bbbk$-span of all anti-commutators $[y_i, y_j]_-$, allowing $i = j$ (resp. all commutators $[y_i, y_j]_+$). In other words, $\operatorname{im}(\phi^*) = J_2$ is identified with a subspace of $[A_1^*, A_1^*]_-$ or $[A_1^*, A_1^*]_+$ inside

$$\operatorname{Lie}(V^*) = \operatorname{Lie}(A_1^*) = \operatorname{Lie}(y_1, \ldots, y_n) \subset T^*(A_1^*)$$

where $\operatorname{Lie}(y_1, \ldots, y_n)$ denotes the *free Lie algebra* (resp. *free Lie superalgebra*) on $y_1, \ldots, y_n$ when A is anti-commutative (resp. commutative).

Definition 8.4. In the above context of an associative graded $\Bbbk$-algebra A which is either commutative or anti-commutative, define the *holonomy Lie algebra* $\mathfrak{h}_A$ via the quotient

$$\mathfrak{h}_A = \operatorname{Lie}(A_1^*)/\langle \operatorname{im}(\phi^*)\rangle = \operatorname{Lie}(y_1, \ldots, y_n)/\langle J_2\rangle. \tag{8.2}$$

Here $\operatorname{Lie}(A_1^*)$ is the free Lie algebra (resp. free Lie superalgebra) on the $\Bbbk$-basis $y_1, \ldots, y_n$ for V^* if A is anti-commutative (resp. commutative), and $\langle J_2\rangle = \langle I_2^\perp\rangle$ is the Lie ideal generated by $J_2 = I_2^\perp$.

The following result of Löfwall connects the holonomy and homotopy Lie algebras.

Lemma 8.5 ([56, Theorem 1.1]). *The universal enveloping algebra $\mathcal{U}(\mathfrak{h}_A)$ of the holonomy Lie algebra $\mathfrak{h}_A$ equals the linear strand of the Yoneda algebra $\mathrm{Ext}^\bullet_A(\Bbbk,\Bbbk)$. That is,*

$$\begin{aligned}\mathcal{U}(\mathfrak{h}_A) &\cong \bigoplus_{i\geq 0}\mathrm{Ext}^i_A(\Bbbk,\Bbbk)_i\\ &\subset \bigoplus_{i,j\geq 0}\mathrm{Ext}^i_A(\Bbbk,\Bbbk)_j = \mathrm{Ext}^\bullet_A(\Bbbk,\Bbbk).\end{aligned}$$

In particular, if A is a Koszul algebra, so $A^! = \mathrm{Ext}^\bullet_A(\Bbbk,\Bbbk)$ is equal to its own linear strand, one has

$$A^! = \mathcal{U}(\mathcal{L}) \text{ where } \mathcal{L} = \mathfrak{h}_A = \pi_A.$$

We next give a simple presentation for the holonomy Lie algebra $\mathfrak{h}_A$ when $A = \mathrm{OS}(M)$ or $A = \mathrm{VG}(\mathcal{M})$. In the case of $\mathrm{OS}(M)$, this is a well-known result of Kohno [53], but as far as the authors are aware, for $\mathrm{VG}(\mathcal{M})$ the presentation is new.

Theorem 8.6. *The holonomy Lie algebra of $\mathrm{OS}(M)$ (resp. $\mathrm{VG}(\mathcal{M})$) for any simple (oriented) matroid M (resp. $\mathcal{M}$) is generated by the relations (5.13) (resp. (5.14)).*

Proof. We give a proof for $\mathrm{VG}(\mathcal{M})$ analogous to Löfwall's proof [57] of Kohno's result for $\mathrm{OS}(M)$. By Equation (8.1), we can identify $\mathrm{im}(\phi^*)$ with $J_2 := I_2^\perp \subset \Bbbk\langle \mathbf{y}\rangle$, where $A = \Bbbk\langle x_1, \ldots, x_n\rangle/I$. There are three families of quadratic relations in the ideal I presenting $\mathrm{VG}(\mathcal{M})$ to consider:

$$x_k^2 \text{ for } k = 1, 2, \ldots, n, \tag{8.3}$$

$$x_k x_\ell - x_\ell x_k \text{ for } 1 \leq k \leq \ell \leq n, \tag{8.4}$$

$$\partial^\pm(C) := \mathrm{sgn}_{C,m} x_k x_\ell + \mathrm{sgn}_{C,\ell} x_k x_m + \mathrm{sgn}_{C,k} x_\ell x_m \quad \text{for circuits } C = \{k,\ell,m\} \text{ of size three.} \tag{8.5}$$

Recall from (5.7) in Proposition 5.5 that the pairing $T^2(V^*)\otimes T^2(V) \to \Bbbk$ makes $\Bbbk\langle\mathbf{x}\rangle_F$ and $\Bbbk\langle\mathbf{y}\rangle_{F'}$ pair to zero unless $F = F'$, so that one can compute $J_2 = I_2^\perp$ flat-by-flat, obtaining

$$[J_2 \cap \Bbbk\langle\mathbf{y}\rangle]_F = [I_2 \cap \Bbbk\langle\mathbf{x}\rangle]_F^\perp$$

for all rank 2 flats F in $\mathcal{F}$. From this one sees that it suffices to prove the result assuming $\mathcal{M} = \mathcal{M}|_F \cong U_{2,n}$, a uniform rank 2 matroid on $E = \{1, 2, \ldots, n\}$ with one rank 2 flat $F = E$.

The quadratic part I_2 contains these $n + \binom{n}{2} + \binom{n-1}{2} = n^2 - n + 1$ elements among (8.3), (8.4), (8.5):

- n of the form x_i^2,
- $\binom{n}{2}$ of the form $x_i x_j - x_j x_i$ for $i < j$, and
- $\binom{n-1}{2}$ of the form $\partial^\pm(C)$ for circuits $C = \{1, i, j\}$ with $1 < i < j \leq n$.

One can also easily check that they are $\Bbbk$-linearly independent inside $T^2(V)$. Consequently one has

$$\dim_\Bbbk J_2 = \dim_\Bbbk I_2^\perp = \dim_\Bbbk T^2(V^*) - \dim_\Bbbk I_2 \leq n^2 - \left(n^2 - n + 1\right) = n - 1.$$

On the other hand, the proof of Theorem 5.18 showed each of these elements from (5.14) lies in $I_2^\perp$:

$$r^\pm(j,E) := \sum_{\substack{1\leq k\leq n:\\ k\neq j}} \chi_{\mathcal{M}|_F}(j,k)\cdot [y_j, y_k]_- = \sum_{\substack{1\leq k\leq n:\\ k\neq j}} \chi_{\mathcal{M}|_F}(j,k)\cdot (y_j y_k + y_k y_j),$$

But the subset $\{r^{\pm}(j,E)\}_{1\leq j\leq n-1}$ gives $n-1$ such elements which are linearly independent in $T^2(V^*)$, and therefore they span $J_2 = I_2^{\perp}$. □

8.2. **PBW decomposition.** When the Koszul algebra A is commutative or anti-commutative, we can use variants of the Poincaré–Birkhoff–Witt (PBW) Theorem to relate the (G-equivariant) Hilbert series for $A^! = \mathcal{U}(\mathcal{L})$ and that of the graded Lie algebra $\mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d$.

Remark 8.7. We state the results in this section assuming that the characteristic of $\Bbbk$ is zero; however, these results can be extended to arbitrary characteristic by replacing the symmetric algebra $\mathrm{Sym}(V)$ or symmetric powers $\mathrm{Sym}^k(V)$ with the *divided power algebra* $D(V)$ or a divided power $D^k(V)$ in every place it appears.

8.2.1. *The anti-commutative case.* When A is *anti*-commutative, and $\Bbbk$ has characteristic zero, the PBW Theorem gives a graded $\Bbbk$-vector space isomorphism $A^! = \mathcal{U}(\mathcal{L}) \cong \mathrm{Sym}(\mathcal{L})$. Therefore, we have the Hilbert series relation

$$\mathrm{Hilb}\left(A^!, t\right) = \mathrm{Hilb}(\mathcal{U}(\mathcal{L}), t) = \mathrm{Hilb}(\mathrm{Sym}(\mathcal{L}), t) = \prod_{d=1}^{\infty} \frac{1}{(1-t^d)^{\varphi_d}}, \tag{8.6}$$

where $\varphi_d = \dim_{\Bbbk} \mathcal{L}_d$; see [71, Section 2.2, Example 2].

Remark 8.8. The *lower central series* (LCS) of a finitely-generated group G is a chain of normal subgroups $G = G_1 \geq G_2 \geq \dots$ defined recursively by $G_k = [G_{k-1}, G]$. Kohno [54] used the topological interpretation of the Orlik–Solomon algebra that we will discuss in Section 9 to investigate the LCS of the homotopy group of the complement of a complex hyperplane arrangement. By studying the holonomy Lie algebra of the Orlik-Solomon algebra of the braid arrangement, Kohno proved that the ranks φ_d of the successive quotients in the lower central series of the homotopy group of the complement of the braid arrangement satisfy equation (8.6). Falk and Randell [41] later showed that this formula also holds for supersolvable arrangements, and Shelton and Yuzvinsky [77] proved that an LCS formula of the form in equation (8.6) holds if and only if the Orlik–Solomon algebra of the arrangement is Koszul. Peeva [70] gave another proof that the LCS formula of this form holds for supersolvable arrangements using the fact that they have a quadratic Gröbner basis.

Any group G of graded $\Bbbk$-algebra symmetries of A, and therefore of $A^!$, will also act as graded Lie algebra symmetries of $\mathcal{L}$. The PBW Theorem then gives these equalities in $R_{\Bbbk}(G)[[t]]$:

$$\begin{aligned}\mathrm{Hilb}_{\mathrm{eq}}\left(A^!, t\right) = \mathrm{Hilb}_{\mathrm{eq}}(\mathcal{U}(\mathcal{L}), t) &= \mathrm{Hilb}_{\mathrm{eq}}(\mathrm{Sym}(\mathcal{L}), t) \\ &= \mathrm{Hilb}_{\mathrm{eq}}\left(\mathrm{Sym}\left(\bigoplus_{d=0}^{\infty} \mathcal{L}_d\right), t\right) \\ &= \sum_{\lambda=(1^{m_1}2^{m_2}\cdots)} t^{|\lambda|} \prod_{j\geq 1} \left[\mathrm{Sym}^{m_j} \mathcal{L}_j\right].\end{aligned} \tag{8.7}$$

In Section 10, we will use this description to investigate representation stability for $\mathcal{L}$ in the setting where A is anti-commutative.

8.2.2. *The commutative case.* Similarly, when A is *commutative*, Polishchuk and Positselski discuss in [71, Section 1.2, Example 4] how $A^! = \mathcal{U}(\mathcal{L})$ for the (graded) Lie superalgebra $\mathcal{L} = \bigoplus_{d=}^{\infty} \mathcal{L}_d$ over $\Bbbk$, in which the parity is induced by the grading, that is,

$$\mathcal{L}_{\text{even}} = \bigoplus_{d\equiv 0 \bmod 2} \mathcal{L}_d, \tag{8.8}$$

$$\mathcal{L}_{\text{odd}} = \bigoplus_{d\equiv 1 \bmod 2} \mathcal{L}_d. \tag{8.9}$$

The graded version of the PBW Theorem (see Milnor and Moore [63, Thm. 5.15], Ross [74, Thm. 2.1], Scheunert [75, Section 2.3 Thm. 1]), asserts that when $\Bbbk$ is a field of characteristic zero, one has a graded $\Bbbk$-vector space isomorphism

$$A^! = \mathcal{U}(\mathcal{L}) \cong \operatorname{Sym}_\pm(\mathcal{L}) := \operatorname{Sym}(\mathcal{L}_{\text{even}}) \otimes \wedge(\mathcal{L}_{\text{odd}})$$

and hence a Hilbert series relation

$$\operatorname{Hilb}\left(A^!, t\right) = \operatorname{Hilb}(\mathcal{U}(\mathcal{L}), t) = \operatorname{Hilb}(\operatorname{Sym}_\pm(\mathcal{L}), t) = \frac{\prod_{d \text{odd}} \left(1+t^d\right)^{\varphi_d}}{\prod_{d \text{even}, d\geq 2} (1-t^d)^{\varphi_d}}$$

where $\varphi_d = \dim_\Bbbk \mathcal{L}_d$.

Remark 8.9. For *any* formal power series $P(t) = 1 + \sum_{j\geq 1} b_j t^j$ with $b_j \in \mathbb{Z}$, there exist uniquely defined φ_d such that

$$P(t) = \frac{\prod_{d \text{ odd}} \left(1+t^d\right)^{\varphi_d}}{\prod_{d \text{ even}, d\geq 2} (1-t^d)^{\varphi_d}}.$$

If $P(t) = \sum_{d=0}^{\infty} \dim_\Bbbk \operatorname{Tor}_j^R(\Bbbk, \Bbbk)$ is the *Poincaré series* of a Noetherian commutative ring R in either

- the *local setting*, where $(R, \mathfrak{m})$ is a local ring with residue field $\Bbbk = R/\mathbf{m}$, or
- the *graded setting*, where $R = \bigoplus_{d=0}^{\infty}$ is an $\mathbb{N}$-graded commutative $\Bbbk$-algebra with $R_0 = \Bbbk$,

the exponent φ_d is called the d^{th} *deviation* of the ring R. This is because the nonvanishing of the φ_d measures whether R "deviates" from being a regular ring or a complete intersection in precise senses:

- R is *regular* if and only if $\varphi_2 = \varphi_3 = \cdots = 0$; see [6, 7.3.2]
- R is a *complete intersection* if and only if $\varphi_3 = \varphi_4 = \cdots = 0$; see [6, 7.3.3].

Moreover, in the local setting one can always resolve $\Bbbk$ over R via an *acyclic closure*; this was first proven in [47]. See [6, Section 6.3, Section 7, Section 10.2] for an in-depth discussion in the local setting; analogous results hold for commutative Noetherian graded $\Bbbk$-algebras. Informally, an acyclic closure is built by recursively adjoining formal variables to represent boundaries of any cycles that appear while computing an R-free resolution of $\Bbbk$. The number of formal variables that one must adjoin in homological degree d is exactly φ_d, which predicts the dimension of the d^{th} graded component of the indecomposables within $\operatorname{Tor}^R(\Bbbk, \Bbbk)$. Since the graded dual of $\operatorname{Tor}^R(\Bbbk, \Bbbk)$ is exactly $\operatorname{Ext}_R(\Bbbk, \Bbbk)$, the space of indecomposables of $\operatorname{Tor}^R(\Bbbk, \Bbbk)$ is the graded dual to the space of primitives in $\operatorname{Ext}_R(\Bbbk, \Bbbk)$, so that $\varphi_d = \dim_\Bbbk \mathcal{L}_d$.

Again, in the presence of a group G of graded $\Bbbk$-algebra symmetries acting on $A, A^!$, one also has these equalities in $R_\Bbbk(G)[[t]]$:

$$\begin{aligned}\mathrm{Hilb}_{\mathrm{eq}}\left(A^!, t\right) &= \mathrm{Hilb}_{\mathrm{eq}}(\mathcal{U}(\mathcal{L}), t)\\ &= \mathrm{Hilb}_{\mathrm{eq}}(\mathrm{Sym}_{\pm}(\mathcal{L}), t)\\ &= \mathrm{Hilb}_{\mathrm{eq}}\left(\wedge\left(\mathcal{L}_{\mathrm{odd}}\right) \otimes \mathrm{Sym}\left(\mathcal{L}_{\mathrm{even}}\right), t\right)\\ &= \sum_{\lambda=(1^{m_1}2^{m_2}\cdots)} t^{|\lambda|} \prod_{j \text{ odd}} \left[\wedge^{m_j}\mathcal{L}_j\right] \cdot \prod_{j \text{ even}, j\geq 2} \left[\mathrm{Sym}^{m_j}\mathcal{L}_j\right].\end{aligned} \tag{8.10}$$

We will use this description in Section 10 to investigate representation stability for $\mathcal{L}$ in the setting where A is commutative.

Example 8.10. Let us return to the Boolean matroid $U_{n,n}$ discussed in Example 4.5 and Section 6.1, but now considered as an *oriented matroid* represented by the standard basis vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$. Since the $\{v_i\}$ are linearly independent, there are no circuits, and the graded Varchenko–Gel'fand ring $A = \mathrm{VG}(U_{n,n})$ and its Koszul dual $A^!$ have these descriptions:

$$\begin{aligned}A &= \Bbbk[x_1, \ldots, x_n]/\left(x_1^2, \ldots, x_n^2\right)\\ &= \Bbbk\langle x_1, \ldots, x_n\rangle/\left(x_k^2, x_ix_j - x_jx_i\right)_{1\leq k\leq n,\, 1\leq i<j\leq n}\end{aligned}$$

$$A^! = \Bbbk\langle y_1, \ldots, y_n\rangle/(y_iy_j + y_jy_i)_{1\leq i<j\leq n}$$

The oriented matroid automorphisms is $\mathrm{Aut}(\mathcal{M})$ are the full hyperoctahedral group $\mathfrak{S}_n^{\pm}$, in which a signed permutation w with $w(v_i) = \pm v_j$ acts on the variables via $w(x_i) = \pm x_j, w(y_i) = \pm y_j$.

We next analyze the graded $\Bbbk\mathfrak{S}_n^{\pm}$-modules $A, A^!$ when $\Bbbk$ has characteristic zero. To do this, first recall (e.g., from Geissinger and Kinch [46], Macdonald [58, Chap. 1., App. B]) that irreducible $\Bbbk\mathfrak{S}_n^{\pm}$-modules $\mathcal{S}^{(\lambda^+,\lambda^-)}$ are indexed by ordered pairs of partitions (λ^+, λ^-) where $|\lambda^+| = n_+, |\lambda| = n_-$ with $n_+ + n_- = n$. One can construct $\mathcal{S}^{(\lambda^+,\lambda^-)}$ using the irreducible $\Bbbk\mathfrak{S}_n$-modules $\{\mathcal{S}^\mu\}$ as building blocks as follows. Introduce the operation of *inflation* $U \longmapsto U \Uparrow$ of a $\Bbbk\mathfrak{S}_n$-module U to a $\Bbbk\mathfrak{S}_n^{\pm}$-module by precomposing with the group surjection $\pi : \mathfrak{S}_n^{\pm} \longrightarrow \mathfrak{S}_n$ that ignores the $\pm$ signs in a signed permutation. Also introduce the one-dimensional character $\chi_{\pm} : \mathfrak{S}_n^{\pm} \to \{\pm 1\}$ sending a signed permutation w to the product of its ± 1 signs, that is, $\chi_{\pm}(w) := \det(w)/\det(\pi(w))$. Then starting with irreducible $\Bbbk\mathfrak{S}_n$-modules $\mathcal{S}^\lambda$, one builds $\mathcal{S}^{(\lambda^+,\lambda^-)}$ as follows:

$$\mathcal{S}^{(\lambda^+,\lambda^-)} := \left(\mathcal{S}^{\lambda^+}\Uparrow \quad \otimes \quad \left(\chi_{\pm} \otimes (\mathcal{S}^{\lambda^-}\Uparrow)\right)\right)\uparrow_{\mathfrak{S}_{n_+}^{\pm}\times\mathfrak{S}_{n_-}^{\pm}}^{\mathfrak{S}_n^{\pm}}.$$

For example, this identifies the graded component A_i of $A = \Bbbk[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ as the irreducible $\Bbbk\mathfrak{S}_n^{\pm}$-module $\mathcal{S}^{((n-i),(i))}$. This is because it is a direct sum of the $\binom{n}{i}$ lines which are the $\mathfrak{S}_n^{\pm}$-images of the line $L := \Bbbk \cdot x_1x_2\cdots x_i$. This line L is stabilized setwise by the subgroup $\mathfrak{S}_{n-i}^{\pm} \times \mathfrak{S}_i^{\pm}$, with the $\mathfrak{S}_{n-i}^{\pm}$ factor acting trivially, and the $\mathfrak{S}_i^{\pm}$ factor acting via $\chi_{\pm}$. Hence one has

$$\mathrm{Hilb}_{\mathrm{eq}}(A, t) = \sum_{i=0}^{n} t^i \cdot \left[\mathcal{S}^{((n-i),(i))}\right].$$

We next analyze $A^!$ as a $\Bbbk\mathfrak{S}_n^{\pm}$-module. Since $(x_1^2, \dots, x_n^2)$ is a regular sequence in $\Bbbk[\mathbf{x}]$, the quotient A is a complete intersection, and $\mathcal{L}_1 = V^* = \operatorname{span}_{\Bbbk}\{y_1, \dots, y_n\}$ and $\mathcal{L}_2 = \operatorname{span}_{\Bbbk}\{y_1^2, \dots, y_n^2\}$. This gives a $\Bbbk\mathfrak{S}_n^{\pm}$-module isomorphism

$$\begin{aligned} A^! &\cong \wedge_{\Bbbk}\mathcal{L}_1 \otimes_{\Bbbk} \operatorname{Sym}\mathcal{L}_2 \\ &= \wedge_{\Bbbk}(y_1, \dots, y_n) \otimes_{\Bbbk} \Bbbk\left[y_1^2, \dots, y_n^2\right]. \end{aligned} \tag{8.11}$$

One can analyze each tensor factor in (8.11) separately. An analysis similar to the one for A_i gives an $\Bbbk\mathfrak{S}_n^{\pm}$-module isomorphism

$$\wedge_{\Bbbk}^i(y_1, \dots, y_n) \cong \mathcal{S}^{\left((n-i),(1^i)\right)}.$$

In the other tensor factor of (8.11), the action of $\mathfrak{S}_n^{\pm}$ on $\Bbbk[y_1^2, \dots, y_n^2]$ is inflated through the surjection $\pi : \mathfrak{S}_n^{\pm} \longrightarrow \mathfrak{S}_n$, letting one compute its $\mathfrak{S}_n^{\pm}$-equivariant Hilbert series from the one for $\mathfrak{S}_n$ on $\Bbbk[y_1, \dots, y_n]$ given in (6.5), and doubling the grading. The upshot is this equivariant Hilbert series:

$$\begin{aligned} &\operatorname{Hilb}_{\mathrm{eq}}\left(A^!, t\right) \\ &= \operatorname{Hilb}_{\mathrm{eq}}\left(\Bbbk\left[y_1^2, \dots, y_n^2\right], t\right) \cdot \operatorname{Hilb}_{\mathrm{eq}}(\wedge\{y_1, \dots, y_n\}, t) \\ &= \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2n})} \left(\sum_Q t^{2\operatorname{maj}(Q)} \cdot \left[\mathcal{S}^{(\lambda(Q),\varnothing)}\right]\right) \left(\sum_{i=0}^n t^i \cdot \left[\mathcal{S}^{\left((n-i),(1^i)\right)}\right]\right) \\ &= \frac{\sum_{i=0}^n \sum_Q t^{2\operatorname{maj}(Q)+i} \cdot \left[\mathcal{S}^{(\lambda(Q),\varnothing)}\right] \cdot \left[\mathcal{S}^{\left((n-i),(1^i)\right)}\right]}{(1-t^2)(1-t^4)\cdots(1-t^{2n})} \end{aligned} \tag{8.12}$$

where in the sums above, Q ranges over all standard Young tableaux with n cells.

9. Topological interpretations of $\mathrm{OS}(M), \mathrm{VG}(\mathcal{M})$ and Koszul duality

Orlik–Solomon algebras $\mathrm{OS}(M)$ have their origins in the following result.

Theorem 9.1 ([67]). *For an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ of linear hyperplanes in $\mathbb{C}^r$ with normal vectors $v_1, \dots, v_n$ representing a matroid M, the cohomology ring of their complement $X := \mathbb{C}^r \setminus \bigcup_{H \in \mathcal{A}} H$ has presentation (using any coefficient ring $\Bbbk$) as*

$$H^*(X, \Bbbk) \cong \mathrm{OS}(M).$$

An analogue for $\mathrm{VG}(\mathcal{M})$ was given by de Longueville and Schultz [30] and later Moseley [65].

Theorem 9.2 ([30, Corollary 5.6]). *Moseley* [65] *For an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ of linear hyperplanes in $\mathbb{R}^r$ with normal vectors $v_1, \dots, v_n$ representing an oriented matroid $\mathcal{M}$, the cohomology ring of their "$\mathbb{R}^3$-thickened" complement $X_{\mathbb{R}^3} := t(\mathbb{R}^r \otimes_{\mathbb{R}} \mathbb{R}^3) \setminus \bigcup_{H \in \mathcal{A}}(H \otimes_{\mathbb{R}} \mathbb{R}^3)$ has presentation (using any coefficient ring $\Bbbk$) as*

$$H^*\left(X_{\mathbb{R}^3}, \Bbbk\right) \cong \mathrm{VG}(\mathcal{M}),$$

with the cohomology concentrated in even degrees, so that the isomorphism halves the grading.

Remark 9.3. The result of de Longueville and Schultz [30, Corollary 5.6] proves more generally that, for any $d \geq 2$, under the same assumptions on $\mathcal{A} \subset \mathbb{R}^r$, the "$\mathbb{R}^d$-thickened" complement $X_{\mathbb{R}^d} := (\mathbb{R}^r \otimes_{\mathbb{R}} \mathbb{R}^d) \setminus \bigcup_{H \in \mathcal{A}} (H \otimes_{\mathbb{R}} \mathbb{R}^d)$ has presentation (using any coefficient ring $\Bbbk$) as

$$H^*\left(X_{\mathbb{R}^d}, \Bbbk\right) \cong \begin{cases} \mathrm{OS}(M) & \text{for } d = 2, 4, 6, \ldots, \text{ even}, \\ \mathrm{VG}(\mathcal{M}) & \text{for } d = 3, 5, 7, \ldots, \text{ odd}, \end{cases}$$

with cohomology concentrated in degrees divisible by d_1, so the isomorphism divides the grading by $d - 1$. Here $M, \mathcal{M}$ are the matroid, oriented matroid associated to the normal vectors $v_1, \ldots, v_n$.

Remark 9.4. The type A_n and B_n reflection arrangements are both supersolvable and realizable over $\mathbb{R}$ (and therefore $\mathbb{C}$) and therefore we can apply this topological interpretation of the Orlik–Solomon and Varchenko-Gel'fand rings of these families of arrangements. In fact, for the type A reflection arrangements, we can also view the Orlik–Solomon and Varchenko–Gel'fand rings as cohomology rings of configuration spaces of points in $\mathbb{R}^d$; this perspective will be discussed further in Section 11.1.

In general, not every arrangement realizable over $\mathbb{C}$ is realizable over $\mathbb{R}$, and there exist matroids (including supersolvable ones) that are represented only in positive characteristics, and some not representable over any field. See for example, some of the matroids discussed in Sections 12.1 and 12.2. One may visualize some of the implications as follows:

$$\begin{array}{ccc} \text{matroid } M & \Leftarrow & \text{oriented matroid } \mathcal{M} \\ \Uparrow & & \Uparrow \\ \mathcal{A} \text{ realized over } \mathbb{C} & \Leftarrow & \mathcal{A} \text{ realized over } \mathbb{R} \end{array}$$

If the cohomology ring $H^*(X, \Bbbk)$ of a simply connected topological space X is a Koszul $\Bbbk$-algebra (as in the case of the Orlik–Solomon and Varchenko–Gel'fand rings for supersolvable arrangements), then the Koszul dual $H^*(X, \Bbbk)^!$ can be interpreted as the homology ring $H_*(\Omega X, \Bbbk)$ of the based loop space ΩX.

Proposition 9.5 (See [12, 13])**.** *Let X be a simply connected topological space such that its cohomology ring $A := H^*(X, \Bbbk)$ is a Koszul $\Bbbk$-algebra over a field $\Bbbk$. Then*

$$A^! = H^*(X, \Bbbk)^! = H_*(\Omega X, \Bbbk)$$

where ΩX is the basepointed loop space of X.

Proof. The authors thank Craig Westerland for communicating the following proof to them. Observe that these spaces participate in the path-loop fibration

$$\Omega X \to PX \to X,$$

where $PX := \{f : I \to X : f(0) = * \text{ and } f \text{ continuous}\}$ is the space of based maps from an interval into X; note that PX is contractible. In general, for a Serre fibration $F \to E \to B$ having B simply connected, the Eilenberg–Moore spectral sequence for cohomology is

$$E_2^{*,*} = \mathrm{Tor}_{H^*(B)}(\Bbbk, H^*(E)) \Rightarrow H^*(F)$$

where $H^*(E)$ is a $H^*(B)$-module by the map in the fibration, and $\Bbbk$ is our base field (or ring, if everything is suitably flat over $\Bbbk$). In the case of the path-loop fibration, since PX is contractible,

$$\mathrm{Tor}_{H^*(X)}(\Bbbk, \Bbbk) \Rightarrow H^*(\Omega X).$$

If $\Bbbk$ is a field, we can apply $\mathrm{Hom}_{\Bbbk}(-,\Bbbk)$ to get

$$\mathrm{Ext}_{H^*(X)}(\Bbbk,\Bbbk) \Rightarrow H_*(\Omega X).$$

If $H^*(X)$ is Koszul, then $\mathrm{Ext}_{H^*(X)}(\Bbbk,\Bbbk) \cong H^*(X)^!$. Further, as $\mathrm{Ext}_{H^*(X)}(\Bbbk,\Bbbk)$ is concentrated in diagonal bidegrees, its differentials are zero, so the spectral sequence collapses at E_2, giving

$$H_*(\Omega X,\Bbbk) = H^*(X,\Bbbk)^!. \qquad \square$$

Remark 9.6. Under the hypotheses of Proposition 9.5, the terminology from Definition 8.3 of *homotopy Lie algebra* for the $\Bbbk$-subspace of primitives $\mathcal{L} \subset A^! = \mathrm{Ext}^\bullet_A(\Bbbk,\Bbbk)$ is consistent with the same terminology in *rational homotopy theory*, where a simply connected space X has *homotopy Lie algebra* defined as the $\Bbbk$-subspace of primitives $\mathcal{L} \cong \pi_*(\Omega X) \otimes \Bbbk$ inside the Hopf algebra $H_*(\Omega X,\Bbbk)$; see, e.g., Félix, Halperin and Thomas [42, Section 21 (d), Theorem 21.5].

10. Representation stability and Koszul algebras

We wish to show how sequences of Koszul algebras $\{A(n)\}_{n\geq 1}$ with $\mathfrak{S}_n$-actions that satisfy representation stability in the sense of Church and Farb [25] lead to representation stability for their Koszul duals $\{A(n)^!\}_{n\geq 1}$, and for the primitive parts $\{\mathcal{L}(n)\}_{n\geq 1}$ of the duals. Useful references on representation stability are Church and Farb [25], Church, Ellenberg and Farb [24] and Matherne, Miyata, Proudfoot and Ramos [60].

In this section, $\Bbbk$ is a field of characteristic zero. Recall this definition from [25].

Definition 10.1. For a partition μ of k, recall that $\mathcal{S}^\mu$ denotes the irreducible $\Bbbk\mathfrak{S}_k$-module indexed by μ. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $n \geq |\lambda| + \lambda_1$, define a partition of n by

$$\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_\ell).$$

Say that a sequence $\{V_n\}_{n\geq 1}$ of $\Bbbk\mathfrak{S}_n$-modules[8] is *representation stable* if there is a list of (not necessarily distinct) partitions $\{\lambda^{(i)}\}_{i=1}^t$ and an integer N such that for $n \geq N$, one has

$$V_n = \bigoplus_{i=1}^t \mathcal{S}^{\lambda^{(i)}[n]}.$$

Say that $\{V_n\}_{n\geq 1}$ is *representation stable past* N when the above equality holds for $n \geq N$.

The following easy observations are recorded in [60, Section 3].

Proposition 10.2. *When $\{V_n\}, \{W_n\}$ are representation stable sequences, then so is $\{V_n \oplus W_n\}$. On the other hand, if the virtual modules $[U_n] = [V_n] - [W_n]$ come from genuine $\Bbbk\mathfrak{S}_n$-modules $\{U_n\}$, then $\{U_n\}$ is also a representation stable sequence.*

It is less obvious what happens for tensor products. The following precise version of a result of Murnaghan was proven by Briand, Orellana, Rosas [21, Theorem 1.2].

Theorem 10.3. *The sequence $\{\mathcal{S}^{\alpha[n]} \otimes \mathcal{S}^{\beta[n]}\}$ is representation stable past $|\alpha|+|\beta|+\alpha_1+\beta_1$.*

This consequence was noted by Matherne, Miyata, Proudfoot and Ramos [60, Theorem 3.3].

Lemma 10.4. *If the $\{V_n\}, \{W_n\}$ are representation stable past A, B, respectively, then $\{V_n \otimes W_n\}$ is representation stable past $A + B$.*

[8] Such a sequence of $\Bbbk\mathfrak{S}_n$-modules is equivalent to what is called an *FB-module* in [60].

The following observation is occasionally useful for pinpointing the onset of representation stability.

Lemma 10.5 ([48, Lemma 2.2]). *Let $\{V_n\}_{n\geq N}$ be $\Bbbk\mathfrak{S}_n$-modules defined via a finite direct sum*

$$V_n \cong \bigoplus_{\mu} \left(\mathcal{S}^{\mu} \uparrow_{\mathfrak{S}_{|\mu|}\times\mathfrak{S}_{n-|\mu|}}^{\mathfrak{S}_n}\right)^{\oplus c_\mu}$$

with $|\mu| \geq N$, and integers $c_\mu \geq 1$. Then $\{V_n\}$ is representation stable, stabilizing exactly at

$$n = \max_{\mu}\{|\mu| + \mu_1\}.$$

We next use some of the foregoing observations to show how representation stability of families of Koszul algebras $\{A(n)\}$ passes to their Koszul duals $\{A(n)^!\}$.

Corollary 10.6. *Let $\{A(n)\}_{n\geq 1}$ be a sequence of Koszul algebras, with Koszul duals $\{A(n)^!\}$.*

- *(i) If for each fixed $i \geq 0$, the sequence $\{A(n)_i\}$ is representation stable, then so is each $\{A(n)^!_i\}$.*
- *(ii) If furthermore there exists a constant c (independent of i) such that each sequence $\{A(n)_i\}$ is representation stable past ci, then each $\{A(n)^!_i\}$ is also representation stable past ci.*

Proof. We prove (ii); the proof for (i) is the same, ignoring the bounds involving ci. But (ii) is immediate using equation (2.13) that appeared in Corollary 2.14, along with Lemma 10.4, since each factor $A_{\alpha_p} = A_{\alpha_p}(n)$ in the right-hand side of (2.13), now in characteristic zero, has the sequence $\{A_{\alpha_p}(n)\}$ representation stable past $c\alpha_p$. □

Example 10.7. Rank two matroids $U_{2,n}$ were discussed in Example 4.6 and Section 6.3. Their group of matroid automorphisms is $\mathrm{Aut}(U_{2,n}) = \mathfrak{S}_n$, and (6.12) showed that as $\Bbbk\mathfrak{S}_n$-modules, one has

$$\begin{aligned}
[\mathrm{OS}(U_{2,n})_0] &\cong \left[\mathcal{S}^{(n)}\right],\\
[\mathrm{OS}(U_{2,n})_1] &\cong \left[\mathcal{S}^{(n)}\right] + \left[\mathcal{S}^{(n-1,1)}\right],\\
[\mathrm{OS}(U_{2,n})_2] &\cong \left[\mathcal{S}^{(n-1,1)}\right],
\end{aligned}$$

which are representation stable past $n = 2$. Consequently, applying Corollary 10.6 (ii) with $c = 2$ implies that the Koszul duals $\{\mathrm{OS}(U_{2,n})^!_i\}$, which are the $\mathfrak{S}_n$-permutation representations discussed in Proposition 6.1, should be representation stable past $n = 2i$. In fact, one has the following.

Proposition 10.8. *For $i \geq 0$, representation stability of $\{\mathrm{OS}(U_{2,n})^!_i\}$ starts* exactly *at $n = 2i$.*

Proof. Recall that Proposition 6.1 expresses $[\mathrm{OS}(U_{2,n})^!_i]$ as a nonnegative combination of classes $\varphi_{(n-d,1^d)}$ for various d in the range $2 \leq d \leq i$, where φ_λ is the class of the coset representation $\Bbbk[\mathfrak{S}_n/\mathfrak{S}_\lambda]$. Furthermore, the coefficient on $\varphi_{(n-i,1^i)}$ is 1. Since one can write

$$\varphi_{(n-d,1^d)} = \left[(\Bbbk\mathfrak{S}_d)\uparrow_{\mathfrak{S}_d\times\mathfrak{S}_{n-d}}^{\mathfrak{S}_n}\right] \quad \text{where} \quad \Bbbk\mathfrak{S}_d \cong \bigoplus_{\mu:|\mu|=d} (\mathcal{S}^{\mu})^{\oplus \dim \mathcal{S}^{\mu}},$$

Lemma 10.5 shows $\{\varphi_{(n-d,1^d)}\}_n$ stabilizes exactly at $n = 2d$, and $\{[\mathrm{OS}(U_{2,n})^!_i]\}$ exactly at $n = 2i$. □

Recall from Section 8 that when A is Koszul and commutative or anti-commutative, then $A^! = \mathcal{U}(\mathcal{L})$ is the universal enveloping algebra for a graded Lie algebra or superalgebra $\mathcal{L}$. We next show how representation stability of families of Koszul algebras $\{A(n)\}$ passes to the Lie (super-)algebras $\{\mathcal{L}(n)\}$. The following lemma will be useful for this purpose.

Lemma 10.9. *For any representation stable sequence $\{V_n\}$ and any partition μ giving rise to a Schur functor $\mathbb{S}^\mu(-)$, the sequence $\{\mathbb{S}^\mu(V_n)\}$ is also representation stable. In particular, for each fixed $m = 0, 1, 2, \ldots$, the sequences $\{\wedge^m(V_n)\}, \{\mathrm{Sym}^m(V_n)\}$ are representation stable.*

Proof. Express the representation stable sequence $\{V_n\}$ for $n \gg 0$ as $V_n = \bigoplus_{i=1}^t \mathcal{S}^{\lambda^{(i)}[n]}$, and proceed by induction on t to show $\{\mathbb{S}^\mu(V_n)\}$ is representation stable for all μ.

The case $t = 1$ was proven by Church, Ellenberg, Farb [24, Proposition 3.4.5] who showed that for any partitions λ, μ, the sequence $\{\mathbb{S}^\mu(\mathcal{S}^{\lambda[n]})\}$ is representation stable. In the inductive step, write $V_n = U_n \oplus \mathcal{S}^{\lambda^{(i)}[n]}$ for $n \gg 0$, where $U_n := \bigoplus_{i=1}^{t-1} \mathcal{S}^{\lambda^{(i)}[n]}$, so that induction applies to the representation stable sequence $\{U_n\}$. Using the general isomorphism (see, e.g., [2, Theorem II.4.11])

$$\mathbb{S}(X \oplus Y) \cong \bigoplus_{\nu \subseteq \mu} \mathbb{S}^\nu(X) \otimes \mathbb{S}^{\mu/\nu}(Y)$$

one concludes that, for $n \gg 0$,

$$\begin{aligned}\mathbb{S}^\mu(V_n) = \mathbb{S}^\mu\left(U_n \oplus \mathcal{S}^{\lambda[n]}\right) &\cong \bigoplus_{\nu \subseteq \mu} \mathbb{S}^\nu(U_n) \otimes \mathbb{S}^{\mu/\nu}\left(\mathcal{S}^{\lambda[n]}\right)\\ &\cong \bigoplus_{\nu,\mu,\rho} \left(\mathbb{S}^\nu(U_n) \otimes \mathbb{S}^\rho\left(\mathcal{S}^{\lambda[n]}\right)\right)^{\oplus c_\rho^{\mu/\nu}}\end{aligned}$$

for some nonnegative integer (Littlewood–Richardson) coefficients $c_\rho^{\mu/\nu}$. By induction on t, the sequences $\{\mathbb{S}^\nu(U_n)\}$ are representation stable, and by the $t = 1$ case, the same holds for $\{\mathbb{S}^\rho(\mathcal{S}^{\lambda[n]})\}$. Hence by Theorem 10.3, each summand $\{\mathbb{S}^\nu(U_n) \otimes \mathbb{S}^\rho(\mathcal{S}^{\lambda[n]})\}$ on the right side is a representation stable sequence, and the same holds for the entire direct sum. ☐

We now apply this to the sequences of Lie (super-)algebras $\{\mathcal{L}(n)\}$.

Corollary 10.10. *Let $\{A(n)\}$ be a family of Koszul algebras, all commutative (resp. anti-commutative), with $\{\mathcal{L}(n)\}$ defined by $A(n)^! = \mathcal{U}(\mathcal{L}(n))$. If for each fixed $i = 0, 1, 2, \ldots$, the sequence $\{A(n)_i\}$ is representation stable, then for each fixed $i = 1, 2, \ldots$, the sequence $\{\mathcal{L}(n)_i\}$ is also representation stable.*

Proof. In either case where $\{A(n)\}$ are commutative or anti-commutative, use induction on i. In the base case $i = 1$, one has this string of equalities, justified below:

$$[\mathcal{L}(n)_1] \overset{(a)}{=} \left[A(n)^!_1\right] \overset{(b)}{=} [(A(n)_1)^*] \overset{(c)}{=} [A(n)_1].$$

Equality (a) comes from comparing coefficients of t^1 on either side of (8.10) or (8.7), equality (b) comes from (2.12), and equality (c) comes from the fact that $\Bbbk\mathfrak{S}_n$-modules are all self-contragredient. Since the right sides $\{A(n)_1\}$ are representation stable, so are the left sides $\{\mathcal{L}(n)_1\}$.

In the induction step where $i \geq 2$, rewrite the equalities that come from comparing the coefficient of t^i on either side of (8.10) or (8.7), isolating the summand $[\mathcal{L}(n)_i]$ on the right corresponding to $\lambda = (i)$. For fixed n, this expresses $\mathcal{L}(n)_i$ recursively in terms of $A(n)^!_i$ and $\mathcal{L}(n)_1, \mathcal{L}(n)_2, \ldots, \mathcal{L}(n)_{i-1}$:

$$[\mathcal{L}(n)_i] = \left[A(n)^!_i\right] - \begin{cases} \sum_{\substack{\lambda\vdash i:\\ \lambda=(1^{m_1}2^{m_2}\cdots i^{m_i})\\ \lambda\neq(i)}} \prod_{1\leq j<i} [\mathrm{Sym}^{m_j}(\mathcal{L}(n)_j)] & \text{for } A(n) \text{ anti-commutative.}\\ \sum_{\substack{\lambda\vdash i:\\ \lambda=(1^{m_1}2^{m_2}\cdots i^{m_i})\\ \lambda\neq(i)}} \prod_{\substack{1\leq j<i\\ j \text{ odd}}} [\wedge^{m_j}(\mathcal{L}(n)_j)] \prod_{\substack{2\leq j<i\\ j \text{ even}}} [\mathrm{Sym}^{m_j}(\mathcal{L}(n)_j)] & \\ & \text{for } A(n) \text{ commutative.}\end{cases}$$

Now use Corollary 10.6 asserting that each sequence $\{A(n)^!_i\}$ is representation stable. Induction on i shows each sequence $\{\mathcal{L}(n)_j\}$ for $j \leq i-1$ appearing on the right is representation stable. Lemma 10.9, then implies the same for all sequences $\{\wedge^{m_j}(\mathcal{L}(n)_j\}$, $\{\mathrm{Sym}^{m_j}(\mathcal{L}(n)_j\}$ appearing on the right. Then Theorem 10.3 gives the same for their tensor products. Thus every summand on the right is a representation stable sequence in n, and hence so is $\{\mathcal{L}(n)_i\}$. $\square$

Remark 10.11. The above proof shows the following statement for a sequence of graded Lie (super-)algebras and $\Bbbk\mathfrak{S}_n$-modules $\mathcal{L}(n)$, with universal enveloping algebras $\mathcal{U}(\mathcal{L}(n))$: one has for all $i \geq 1$ that $\{\mathcal{L}(n)_i\}$ is representation stable if and only if one has for all $i \geq 0$ that $\{\mathcal{U}(\mathcal{L}(n))_i\}$ is representation stable.

Remark 10.12. Unlike Corollary 10.6, we have not seriously tried to bound the onset of stability for the sequences $\{\mathcal{L}(n)_i\}$, in terms of a given bound for the onset of stability in $\{A(n)_i\}$. However, `Sage` computations up to $i = 10$ suggest the following for uniform matroids $U_{2,n}$ of rank 2.

Conjecture 10.13. Defining $\mathcal{L}(n)_i$ by $\mathrm{OS}(U_{2,n})^! = \mathcal{U}(\mathcal{L}(n))$, the sequence $\{\mathcal{L}(n)_i\}$ is representation stable past $n = 2i-1$ for fixed $i \geq 3$.

Remark 10.14. Although $[A(n)^!_i]$ is a permutation representation when $A(n) = \mathrm{OS}(U_{2,n})$ by Proposition 6.1, the primitives $[\mathcal{L}(n)_i]$ are generally *not* classes of permutation representations. This fails immediately for $n = 3$ and $i = 2$, where $[\mathcal{L}(3)_2]$ is the sign representation. Also for braid matroids, if $A(n) = \mathrm{OS}(\mathrm{Br}_n)$, and $A(n)^! = \mathrm{OS}(\mathrm{Br}_n)^! = \mathcal{U}(\mathcal{L}(n))$, one can check

$$\mathcal{L}(n)_2 = \left(\mathrm{sgn}_{\mathfrak{S}_3} \otimes \mathbb{1}_{\mathfrak{S}_{n-3}}\right) \uparrow_{\mathfrak{S}_3\times\mathfrak{S}_{n-3}}^{\mathfrak{S}_n}$$

which is again not a permutation module.

11. The motivating example: braid matroids and Stirling representations

As mentioned prior to Example 4.7, our motivation came from the *braid matroids* $M = \mathrm{Br}_n$, which are also known as the graphic matroids for complete graphs K_n. They are also known as the matroids represented by the vectors $\{v_{ij}\}_{1\leq i<j\leq n}$ with $v_{ij} := e_i - e_j$ in $\mathbb{R}^n$ which are the (positive) roots in the type A_{n-1} root system, whose normal hyperplanes $H_{ij} = \{\mathbf{x} \in \mathbb{R}^n : x_i = x_j\}$ are the reflecting hyperplanes for the transposition (i, j) in the symmetric group $\mathfrak{S}_n$ when it acts on $V = \mathbb{R}^n$ by permuting coordinates. Thus $M = \mathrm{Br}_n$ is orientable, and abusing notation slightly, we will also denote by $\mathcal{M} = \mathrm{Br}_n$ the oriented matroid on the ground set $E = \{\{i, j\} : 1 \leq i < j \leq n\}$ represented by these vectors $\{v_{ij}\}$.

11.1. **Comparison with cohomology of configuration spaces.** It turns out that the algebras $\mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$ had been studied historically earlier as the cohomology rings of certain *configuration spaces* of n ordered (labeled) points in a space X

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\}.$$

The arrangement of hyperplanes H_{ij} in $V = \mathbb{R}^n$ described above allow one to identify

$$\mathrm{Conf}_n(\mathbb{R}^d) = V \otimes_{\mathbb{R}} \mathbb{R}^d \setminus \bigcup_{1 \leq i < j \leq n} H_{ij} \otimes_{\mathbb{R}} \mathbb{R}^d,$$

that is, as the complement of subspace arrangements coming from the reflection hyperplane arrangements "thickened" by tensoring with $\mathbb{R}^d$ as in Theorems 9.1, 9.2 and Remark 9.3. In fact, the special cases of those results for the braid arrangements, along with quadratic presentations for the associated algebras, were known to Arnol'd [5] for $\mathrm{OS}(\mathrm{Br}_n)$ and Cohen [27] for $\mathrm{VG}(\mathrm{Br}_n)$:

$$\begin{aligned}\mathrm{OS}(\mathrm{Br}_n) &\cong \wedge_{\Bbbk}(x_{ij}) / \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}\right)\\ \mathrm{VG}(\mathrm{Br}_n) &\cong \Bbbk[x_{ij}] / \ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk},\ x_{ij}^2\ \right)\end{aligned}$$

Here permutations σ in $\mathfrak{S}_n$ act on the variables by permuting subscripts, that is, $\sigma(x_{ij}) = x_{\sigma(i),\sigma(j)}$, but with the convention that $x_{ji} = x_{ij}$ in $\mathrm{OS}(\mathrm{Br}_n)$, and $x_{ji} = -x_{ij}$ in $\mathrm{VG}(\mathrm{Br}_n)$.

Note that that these presentations are consistent with the general presentation coming from supersolvable matroids in Corollary 5.14, using the modular complete flag $\underline{F}$ of flats chosen in Example 4.7: one checks that the corresponding decomposition $\underline{E} = (E_1, E_2, \ldots, E_{n-1})$ of $E = \{\{i,j\}\}_{1 \leq i < j \leq n}$ has

$$\begin{aligned} E_1 &= \{\{1,2\}\},\\ E_2 &= \{\{1,3\},\{2,3\}\},\\ E_3 &= \{\{1,4\},\{2,4\},\{3,4\}\},\\ &\vdots\\ E_{n-1} &= \{\{1,n\},\{2,n\}, \ldots, \{n-2,n\},\{n-1,n\}\},\end{aligned} \tag{11.1}$$

and the subset of circuits $\mathcal{C}_{\mathrm{BEZ}}(\underline{E}) = \{\{i,j\},\{i,k\},\{j,k\}\}_{1 \leq i < j < k \leq n}$. Here the NBC monomial basis for either $\mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$ are the products of x_{ij} that choose at most one $\{i,j\}$ from each set E_p with $p = 1, 2, \ldots, n-1$ above; Barcelo [11, Theorem 2.1] calls this *picking at most one finger* x_{ij} *from each hand* E_p. Since the exponents $e_p = |E_p|$ here are $(e_1, \ldots, e_r) = (1, 2, \ldots, n-1)$, one has these Hilbert series

$$\begin{aligned}\mathrm{Hilb}(\mathrm{OS}(\mathrm{Br}_n), t) &= \mathrm{Hilb}(\mathrm{VG}(\mathrm{Br}_n), t)\\ &= (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{i=0}^{n-1} t^i\, c(n, n-i)\\ \mathrm{Hilb}\left(\mathrm{OS}(\mathrm{Br}_n)^!, t\right) &= \mathrm{Hilb}\left(\mathrm{VG}(\mathrm{Br}_n)^!, t\right)\\ &= \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} t^i\, S((n-1)+i, n-1),\end{aligned}$$

where the coefficients $c(n,k), S(n,k)$ appearing here are the *(signless) Stirling numbers of the first kind* $c(n,k)$, counting permutations in $\mathfrak{S}_n$ with k cycles, and the *Stirling numbers of the 2nd kind* $S(n,k)$, counting partitions of the set $\{1, 2, \ldots, n\}$ into k blocks.

Comparing coefficients on powers of t, one has for either $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ or $\mathrm{VG}(\mathrm{Br}_n)$ that

$$\begin{aligned} \dim_{\Bbbk} A(n)_i &= c(n, n-i), \\ c(n,k) &= \dim_{\Bbbk} A(n)_{n-k}, \end{aligned}$$

$$\begin{aligned} \dim_{\Bbbk} A(n)^!_i &= S((n-1)+i, n-1), \\ S(n,k) &= \dim_k A(k+1)^!_{n-k}. \end{aligned}$$

Definition 11.1 (Stirling representations)**.** For either $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ or $A(n) = \mathrm{VG}(\mathrm{Br}_n)$, call $A(n)_i$ the *Stirling representations of the first kind*, and call $A(n)^!_i$ the *Stirling representations of the second kind.* When emphasizing their dimensions as representations, we will abbreviate them as

$$\begin{aligned} \mathfrak{c}_{\mathrm{OS}}(n,k) &:= \mathrm{OS}(\mathrm{Br}_n)_{n-k}, \\ \mathfrak{c}_{\mathrm{VG}}(n,k) &:= \mathrm{VG}(\mathrm{Br}_n)_{n-k}, \text{ both } \Bbbk\mathfrak{S}_n\text{-modules}, \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{\mathrm{OS}}(n,k) &:= \mathrm{OS}(\mathrm{Br}_{k+1})^!_{n-k}, \\ \mathcal{S}_{\mathrm{VG}}(n,k) &:= \mathrm{VG}(\mathrm{Br}_{k+1})^!_{n-k}, \text{ both } \Bbbk\mathfrak{S}_{k+1}\text{-modules}. \end{aligned}$$

Remark 11.2. The coincidence between $\dim_{\Bbbk} A(k+1)^!_{n-k}$ and $S(n,k)$, counting set partitions of $\{1,2,\ldots,n\}$ into k blocks, is closely related to a well-known combinatorial encoding of set partitions via *restricted growth functions*, as we explain here; see also Stanton and White [84, Section 1.5].

Given any k-block set partition $\pi = \{B_1,\ldots,B_k\}$ of $\{1,2,\ldots,n\}$, re-index the blocks so that $\min B_1 < \min B_2 < \cdots < \min B_k$. Then the *restricted growth function (rgf)* encoding π is the sequence $(i_1, i_2, \ldots, i_n)$ defined by $i_j = \ell$ if $j \in B_\ell$ for $j = 1,2,\ldots,n$. By definition, $i_1 := 1$ and $i_j \le 1 + \max\{i_0, i_1, \ldots, i_{j-1}\}$; it is not hard to check that these two properties characterize the rgf's. For example, with $n = 15$ and $k = 3$, this set partition

$$\pi = \{\underbrace{\{\mathbf{1},2,3,5,8,10,15\}}_{B_1}, \underbrace{\{\mathbf{4},6,7,12\}}_{B_2}, \underbrace{\{\mathbf{9},11,13,14\}}_{B_3}\}$$

corresponds to this rgf $(i_1, i_2, \ldots, i_{15})$:

$$\begin{array}{ccccccccccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 & i_9 & i_{10} & i_{11} & i_{12} & i_{13} & i_{14} & i_{15} \\ 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 3 & 1 & 3 & 2 & 3 & 3 & 1 \end{array}$$

We claim that these rgf's correspond bijectively to the standard monomial $\Bbbk$-basis for $A(k+1)^!_{n-k}$ given in Corollary 5.18. To explain this bijection, underline the first (leftmost) occurrence of each value $p = 1,2,\ldots,k$ among the i_j, and append an extra (underlined) $i_{n+1} := k+1$ on the right, as a convention. One then associates to $(i_1,\ldots,i_n)$ the product $m_2 \cdot m_3 \cdots m_k \cdot m_{k+1}$ where m_p is the noncommutative monomial in variables $\{y_{ip}\}_{i=1}^{p-1}$ obtained by replacing each non-underlined value i_j above with the variable $x_{i_j,p}$ if p is the next underlined value to the right of i_j:

$$\begin{array}{cccccccccccccccc} \underline{1} & 1 & 1 & \underline{2} & 1 & 2 & 2 & 1 & \underline{3} & 1 & 3 & 2 & 3 & 3 & 1 & \underline{4} \\ & y_{12} & y_{12} & \cdot & y_{13} & y_{23} & y_{23} & y_{13} & \cdot & y_{14} & y_{34} & y_{24} & y_{34} & y_{34} & y_{14} & \end{array}$$

Since the number of non-underlined values i_j is $n-k$, this is a standard monomial in $A(k+1)^!_{n-k}$.

Remark 11.3. The presentations of $\mathrm{OS}(\mathrm{Br}_n)^!, \mathrm{VG}(\mathrm{Br}_n)^!$ in Theorem 5.18 are equivalent to what Cohen and Gitler [26] called *graded infinitesimal braid relations* in their presentation of the loop space homology algebra $H_*(\Omega X, \Bbbk)$ where $X = \mathrm{Conf}_n(\mathbb{R}^d)$; see also Berglund [12, Example 5.5]. For the case of $\mathrm{OS}(\mathrm{Br}_n)^!$, considered as a universal enveloping algebra $\mathrm{OS}(\mathrm{Br}_n)^! = \mathcal{U}(\mathcal{L})$, see also the discussion by Fresse [43, Chapter 10] referring to $\mathcal{L}$ as the *Drinfeld–Kohno Lie algebra* and $\mathcal{U}(\mathcal{L})$ as the *algebra of chord diagrams.*

11.2. **Stirling representations of the first kind: generating functions.** The $\Bbbk\mathfrak{S}_n$-module structure for either $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ or $\mathrm{VG}(\mathrm{Br}_n)$ are well-studied. Explicit irreducible decompositions are not known, but can be computed reasonably efficiently through symmetric function formulas involving plethysm and generating functions, given in work of Sundaram and Welker [90, Theorem 4.4 (iii)] and reviewed here; see also the summary in Hersh and Reiner [48, Section 2].

Let $\Bbbk$ be a field of characteristic zero. The *Frobenius characteristic isomorphism* $R_\Bbbk(\mathfrak{S}_n) \cong \Lambda_n$, where Λ_n are the degree n homogeneous symmetric functions $\Lambda(z_1, z_2, \ldots)_n$, mentioned in Section 6.3 above, can be compiled for all n to give a ring isomorphism

$$\bigoplus_{n=0}^{\infty} R_\Bbbk(\mathfrak{S}_n) \xrightarrow{\mathrm{ch}} \bigoplus_{n=0}^{\infty} \Lambda_n = \Lambda.$$

Here the product on the left is the *external* or *induction product*

$$([U],[V]) \longmapsto \left[(U \otimes_\Bbbk V)\uparrow_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}}\right],$$

while the product on the right simply multiplies symmetric functions. If one defines the power sum symmetric function $p_r := z_1^r + z_2^r + \cdots$, and the $\Bbbk$-basis $\{p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots\}$ indexed by partitions λ of n for Λ_n, then for each $\Bbbk\mathfrak{S}_n$-module U with character χ_U, the Frobenius isomorphism maps $[U] \overset{\mathrm{ch}}{\longmapsto} \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \chi_U(\sigma)\, p_{\lambda(\sigma)}$ where $\lambda(\sigma)$ is the cycle type partition of σ.

Let Lie_n denote the n^{th} *Lie representation*: the $\mathfrak{S}_n$-representation on the multilinear component of the free Lie algebra on n variables. It has a formula due to Klyachko [52] as $\mathrm{Lie}_n = \zeta \uparrow_{C_n}^{\mathfrak{S}_n}$, where ζ is the one-dimensional representation of the cyclic group C_n inside $\mathfrak{S}_n$ generated by an n-cycle c, that sends $c \mapsto e^{\frac{2\pi i}{n}}$. Defining symmetric functions

$$\begin{aligned} \ell_n &:= \mathrm{ch}(\mathrm{Lie}_n), \\ \pi_n &:= \mathrm{ch}(\mathrm{sgn}_n \otimes \mathrm{Lie}_n), \end{aligned}$$

and letting $(f,g) \mapsto f[g]$ denote *plethystic composition* of symmetric functions [58, Section I.8], one has the following plethystic expressions and product generating functions (see Sundaram [87, Theorem 1.8, and p. 249], Sundaram and Welker [90, Theorem 4.4 (iii)] and Hersh and Reiner [48, Section 2, Theorem 2.17]):

$$1+\sum_{n=1}^{\infty} u^n \sum_{k=1}^{n} \mathrm{ch}\left([\mathrm{VG}(\mathrm{Br}_n)_{n-k}]\right) t^k = \sum_{\lambda=1^{m_1}2^{m_2}\cdots} u^{|\lambda|} t^{\ell(\lambda)} \prod_{j\geq 1} h_{m_j}[\ell_j] \tag{11.2}$$

$$= \prod_{m=1}^{\infty} \left(1-u^m p_m\right)^{-a_m(t)}, \tag{11.3}$$

$$1+\sum_{n=1}^{\infty} u^n \sum_{k=1}^{n} \mathrm{ch}\left([\mathrm{OS}(\mathrm{Br}_n)_{n-k}]\right) t^k = \sum_{\lambda=1^{m_1}2^{m_2}\cdots} u^{|\lambda|} t^{\ell(\lambda)} \prod_{j \text{odd}} h_{m_j}[\pi_j] \prod_{\substack{j \text{ even}\\ j\geq 2}} e_{m_j}[\pi_j], \tag{11.4}$$

$$= \prod_{m=1}^{\infty} \left(1+(-u)^m p_m\right)^{a_m(-t)}, \tag{11.5}$$

where here $a_m(t) := \frac{1}{m}\sum_{d|m}\mu(d)t^{\frac{m}{d}}$, with $\mu(d)$ the number-theoretic Möbius function. Equivalently, define for a partition $\lambda = 1^{m_1}2^{m_2}\cdots$ of n (written $\lambda \vdash n$) with m_i parts equal to i, the $\mathfrak{S}_n$-representations $\mathrm{OS}_\lambda, \mathrm{VG}_\lambda$ whose Frobenius characteristics are the products appearing above. Then

$$\begin{aligned} \mathrm{ch}\,\mathrm{VG}_\lambda &:= \prod_i h_{m_i}[\ell_i], \quad \text{so that } \mathrm{ch}(\mathrm{VG}(\mathrm{Br}_n)_{n-k}) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \mathrm{VG}_\lambda, \\ \mathrm{ch}\,\mathrm{OS}_\lambda &:= \prod_{i \text{ odd}} h_{m_i}[\pi_i] \prod_{i \text{ even}} e_{m_i}[\pi_i], \quad \text{so that } \mathrm{ch}(\mathrm{OS}(\mathrm{Br}_n)_{n-k}) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \mathrm{OS}_\lambda. \end{aligned} \tag{11.6}$$

Thus $\mathrm{VG}_{(n)}$ is the *Lie representation* with $\mathrm{chVG}_{(n)} = \ell_n$ mentioned above. Similarly, $\{\mathrm{VG}_\lambda\}$ are called *higher Lie characters*; see Schocker [76]. Also, the last equality in (11.6) implies that $\mathrm{OS}(\mathrm{Br}_n)_{n-k}$ coincides with the $\mathfrak{S}_n$-representation on the $(n-k)^{\text{th}}$ *Whitney homology* of the partition lattice, $1 \leq k \leq n$; see Lehrer and Solomon [55, Theorem 4.5], Sundaram [87, Theorem 1.8].

11.3. **Data on Stirling representations of the second kind.** In contrast to the above $\Bbbk\mathfrak{S}_n$-descriptions of $A(n)_i$ when $A(n) = \mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$, for the Koszul duals $A(n)^!_i$, we currently lack formulas of this nature, although we can tabulate $A(n)^!_i$ recursively from the $A(n)_i$ using (2.12).

Question 11.4. *Are there formulas like* (11.2), (11.3), (11.4), (11.5) *for the duals*

$$\mathrm{VG}(\mathrm{Br}_n)^!, \mathrm{OS}(\mathrm{Br}_n)^!\,?$$

11.4. **Branching rules for both kinds of Stirling representations.** Stirling numbers of both kinds satisfy well-known recurrences, mentioned in the Introduction:

$$c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1) \tag{11.7}$$

$$S(n,k) = k\cdot S(n-1,k) + S(n-1,k-1) \tag{11.8}$$

Theorem 7.1 will allow us to lift these to branching rules for the Stirling representations of both kinds. We consider here the action of $G = \mathfrak{S}_n$ on the matroid and oriented matroid Br_n. In this case, the setwise $\mathfrak{S}_n$-stabilizer for the modular coatom $F = E_{n-2}$ in (11.1) is the subgroup $H = \mathfrak{S}_{n-1}$. Furthermore, the permutation action $\mathcal{X}$ of $\mathfrak{S}_{n-1}$ on the set

$$E_{n-1} = E \setminus F = \{\{1,n\},\{2,n\},\ldots,\{n-1,n\}\}$$

and its signed permtuation action on the real vectors representing E_{n-1} in the oriented matroid Br_n

$$\{v_{1n}, v_{2n}, \ldots, v_{n-1,n}\} = \{e_1 - e_n, e_2 - e_n, \cdots, e_{n-1} - e_n\},$$

are both equivalent to the defining $\mathfrak{S}_{n-1}$-permutation representation $\chi_{\mathrm{def}}^{(n-1)}$ via $(n-1) \times (n-1)$ permutation matrices. Translating Theorem 7.1 then immediately implies the following.

Corollary 11.5. *For any field* $\Bbbk$*, the recurrences* (11.7)*,* (11.8) *lift equivariantly as follows.*

(i) Letting $\mathfrak{c}(n,k)$ *denote either* $\mathrm{OS}(\mathrm{Br}_n)_{n-k}$ *or* $\mathrm{VG}(\mathrm{Br}_n)_{n-k}$ *as* $\Bbbk\mathfrak{S}_n$*-module, the recurrence* (11.7) *lifts to two recurrences in* $R_\Bbbk(\mathfrak{S}_{n-1})$

$$\left[\mathfrak{c}(n,k)\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right] = \left[\chi_{\mathrm{def}}^{(n-1)}\right] \cdot [\mathfrak{c}(n-1,k)] + [\mathfrak{c}(n-1,k-1)], \tag{11.9}$$

reflecting two $\Bbbk\mathfrak{S}_{n-1}$*-module exact sequences*

$$0 \longrightarrow \mathfrak{c}(n-1,k-1) \longrightarrow \mathfrak{c}(n,k)\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \longrightarrow \chi_{\mathrm{def}}^{(n-1)} \otimes \mathfrak{c}(n-1,k) \longrightarrow 0. \tag{11.10}$$

(ii) Letting $\mathcal{S}(n,k)$ *denote either* $\mathrm{OS}(\mathrm{Br}_{k+1})^!_{n-k}$ *or* $\mathrm{VG}(\mathrm{Br}_{k+1})^!_{n-k}$ *as* $\Bbbk\mathfrak{S}_{k+1}$*-module, the recurrence* (11.8) *lifts to two relations in* $R_\Bbbk(\mathfrak{S}_k)$

$$\left[\mathcal{S}(n,k)\downarrow^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k}\right] = \left[\chi_{\mathrm{def}}^{(k)}\right] \cdot \left[\mathcal{S}(n-1,k)\downarrow^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k}\right] + [\mathcal{S}(n-1,k-1)], \tag{11.11}$$

reflecting two $\Bbbk\mathfrak{S}_k$*-module exact sequences*

$$0 \longrightarrow \chi_{\mathrm{def}}^{(k)} \otimes \mathcal{S}(n-1,k)\downarrow^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \longrightarrow \mathcal{S}(n,k)\downarrow^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \longrightarrow \mathcal{S}(n-1,k-1) \longrightarrow 0. \tag{11.12}$$

Remark 11.6. Proposition 2.16 implies the two versions of (11.9) are equivalent to those of (11.11).

Remark 11.7. All of the assertions Corollary 11.5 are new, as far as we know, when working over fields $\Bbbk$ of positive characteristic, and (11.10),(11.11), (11.12) are new even when $\Bbbk$ has characteristic zero. However, when $\Bbbk$ has characteristic zero, it turns out that (11.9) also follows from work of Sundaram in [87, 88]. For example, the relation (11.9) for $\mathfrak{c}(n,k) = \mathrm{OS}(\mathrm{Br}_n)_{n-k}$ can be deduced by combining [87, Theorem 2.2, Part (2) and Proposition 1.9]; we omit the details here. Also it turns out that both cases of (11.9), when either $\mathfrak{c}(n,k) = \mathrm{OS}(\mathrm{Br}_n)_{n-k}$ or $\mathrm{VG}(\mathrm{Br}_n)_{n-k}$, follow from the symmetric function branching result [88, Theorem 4.10]. In the notation there, choosing $F = \sum_{n\geq 1} \ell_n$, one takes $G_n^j = h_j[F]|_{\deg n}$ to deduce (11.9) for $\mathfrak{c}(n,k) = \mathrm{VG}(\mathrm{Br}_n)_{n-k}$, and one takes $G_n^j = e_j[F]|_{\deg n}$ to deduce (11.9) for $\mathfrak{c}(n,k) = \mathrm{OS}(\mathrm{Br}_n)_{n-k}$. We again omit the details here.

11.5. **Braid matroids and representation stability.** Here we wish to apply the representation stability results of Section 10 to the braid matroids Br_n. A special case of the main result of Church and Farb [25] shows, in our language, that for each fixed $i = 0, 1, 2 \ldots$, both sequences $\{A(n)_i\}$ where $A(n) = \mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$ are representation stable. Hersh and Reiner [48, Theorem 1.1] pinned down the onset of this representation stability.

Theorem 11.8. *For each fixed* $i \geq 1$*, both sequences* $\{A(n)_i\}$ *where* $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ *and* $\mathrm{VG}(\mathrm{Br}_n)$ *are representation stable, past* $3i$ *for* $\mathrm{VG}(\mathrm{Br}_n)$ *and past* $3i+1$ *for* $\mathrm{OS}(\mathrm{Br}_n)$*.*

One then deduces the following representation stability for their Koszul duals.

Corollary 11.9. *For each fixed* $i \geq 1$*, both sequences* $\{A(n)^!_i\}$ *where* $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ *and* $\mathrm{VG}(\mathrm{Br}_n)$ *are representation stable, past* $3i$ *for* $\{\mathrm{VG}(\mathrm{Br}_n)^!_i\}$ *and past* $4i$ *for* $\{\mathrm{OS}(\mathrm{Br}_n)^!_i\}$*.*

Proof. Theorem 11.8 gives the necessary hypotheses to apply Corollary 10.6, using the constant $c = 3$ for $\{\mathrm{VG}(\mathrm{Br}_n)_i\}$ and using the constant $c = 4$ (since $3i + 1 \leq 4i$) for $\{\mathrm{OS}(\mathrm{Br}_n)_i\}$. □

Remark 11.10. The bounds in Corollary 11.9 happen to be tight for $\mathrm{OS}(\mathrm{Br}_n)^!_i, \mathrm{VG}(\mathrm{Br}_n)^!_i$ when $i = 0, 1, 2$. To see this, one can apply Lemma 10.5 to Propositions B.1, B.2 and Remark B.3 below (specifically, see equations (B.7), (B.9)) to deduce that for $i = 0, 1, 2$, the sequences $\{\mathrm{OS}(\mathrm{Br}_n)^!_i\}$ start to stabilize exactly when $n \geq 4i$, and the sequences $\{\mathrm{VG}(\mathrm{Br}_n)^!_i\}$ start to stabilize exactly when $n \geq 3i$. This suggests the following conjecture, confirmed by `Sage/Cocalc` for $\mathrm{OS}(\mathrm{Br}_n)^!_i$ up to $i = 5$, and for $\mathrm{VG}(\mathrm{Br}_n)^!_i$ up to $i = 7$.

Conjecture 11.11. The bounds for onset of stability in Corollary 11.9 are tight: for $i \geq 0$, the sequences $\{\mathrm{OS}(\mathrm{Br}_n)^!_i\}$ and $\{\mathrm{VG}(\mathrm{Br}_n)^!_i\}$ start to stabilize exactly when $n = 4i$ and $n = 3i$, respectively.

Since $A(n) = \mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$ are anti-commutative and commutative, respectively, Corollary 10.10 immediately implies the following.

Corollary 11.12. *Letting $A(n) = \mathrm{OS}(\mathrm{Br}_n), \mathrm{VG}(\mathrm{Br}_n)$, and defining $\mathcal{L}(n)_i$ by $A(n)^! = \mathcal{U}(\mathcal{L}(n))$, for each fixed $i = 1, 2, \ldots$, the sequence $\{\mathcal{L}(n)_i\}$ is representation stable.*

Remark 11.13. The case of Corollary 11.12 for $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ also follows from work of Church, Ellenberg and Farb [24, Theorem 7.3.4]. They consider instead of $\mathcal{L}(n)$ the *Malcev Lie algebra* $\mathfrak{p}_n$ associated to the fundamental group $\pi_1(X)$ for the configuration space $X = \mathrm{Conf}_n(\mathbb{R}^2) = \mathrm{Conf}_n(\mathbb{C})$ considered in Section 11.1; alternatively, X is the complement of the complex braid arrangement $\mathcal{A}$ as in Theorem 9.1. These two Lie algebras $\mathfrak{p}_n$ and $\mathcal{L}(n)$ coincide due to the 1-formality of complements of complex algebraic hypersurfaces; see, e.g., Suciu and Wang [86, Section 6,7].

Computations in `Sage/Cocalc` through $i = 8$ suggest the following conjecture.

Conjecture 11.14. Defining $\{\mathcal{L}(n)_i\}$ by $A(n)^! = \mathcal{U}(\mathcal{L}(n))$, its onset of representation stability is:

- $n = 2i$ for a fixed $i \geq 1$ when $A(n) = \mathrm{OS}(\mathrm{Br}_n)$,
- $n = 2i$ for a fixed $i \geq 3$ when $A(n) = \mathrm{VG}(\mathrm{Br}_n)$.

11.6. **Near-boundary cases for Stirling representations of the second kind.** Stirling numbers $S(n, k)$ of the second kind have more explicit formulas when either k or $n-k$ is small. We similarly present here more explicit formulas, in the language of symmetric functions for the Stirling representations

$$\mathrm{OS}(\mathrm{Br}_n)_i = \mathcal{S}_{\mathrm{OS}}((n-1) + i, n-1),$$
$$\mathrm{VG}(\mathrm{Br}_n)_i = \mathcal{S}_{\mathrm{VG}}((n-1) + i, n-1),$$

as $\Bbbk\mathfrak{S}_n$-modules, when either i or n is small.

Part of our motivation comes from the following observations about when $\mathrm{OS}(M)_i$, $\mathrm{VG}(\mathcal{M})_i$ and their Koszul duals $\mathrm{OS}(M)^!_i, \mathrm{VG}(\mathcal{M})^!_i$ turn out to be *permutation representations* of their automorphism groups $G = \mathrm{Aut}(M)$ or $\mathrm{Aut}(\mathcal{M})$. The discussion of Boolean matroids in Section 6.1 and low rank matroids in Section 6.2 and Proposition 6.1 showed that

- $\mathrm{OS}(M)_i, \mathrm{VG}(\mathcal{M})_i$ are *rarely* permutation representations,
- $\mathrm{VG}(\mathcal{M})^!_i$ is *not always* a permutation representation,

- but $\mathrm{OS}(M)^!_i$ was *always* a G-permutation representation in these previous examples.

For the braid matroids Br_n, it is *not always* true that $\mathrm{OS}(\mathrm{Br}_n)_i$ is a permutation representation, but the next result shows that it happens in many cases where i or n is small.

Theorem 11.15. *For $\Bbbk$ of characteristic zero, the $\Bbbk\mathfrak{S}_n$-modules $\mathrm{OS}(\mathrm{Br}_n)^!_i = \mathcal{S}_{\mathrm{OS}}((n-1)+i, n-1)$*

(i) are permutation modules for $i = 0, 1$,
(ii) are half-*permutation modules for $i = 2$, meaning that $2 \cdot [\mathrm{OS}(\mathrm{Br}_n)^!_2]$ is the class of a permutation module in $R_\Bbbk(\mathfrak{S}_n)$,*
(iii) are permutation modules[9] for $n = 1, 2, 3, 4, 5$.

However, both

$$\mathcal{S}_{\mathrm{OS}}(10,5) = \mathrm{OS}(\mathrm{Br}_6)^!_5,$$
$$\mathcal{S}_{\mathrm{OS}}(11,6) = \mathrm{OS}(\mathrm{Br}_7)^!_5$$

fail to be permutation modules, even after scaling them by positive integers, since they can be shown to have negative character values.[10]

Table 11.1. When are $[\mathrm{OS}(\mathrm{Br}_n)^!_i] = \mathcal{S}_{\mathrm{OS}}((n-1)+i, n-1)$ permutation modules or "fractions" thereof?

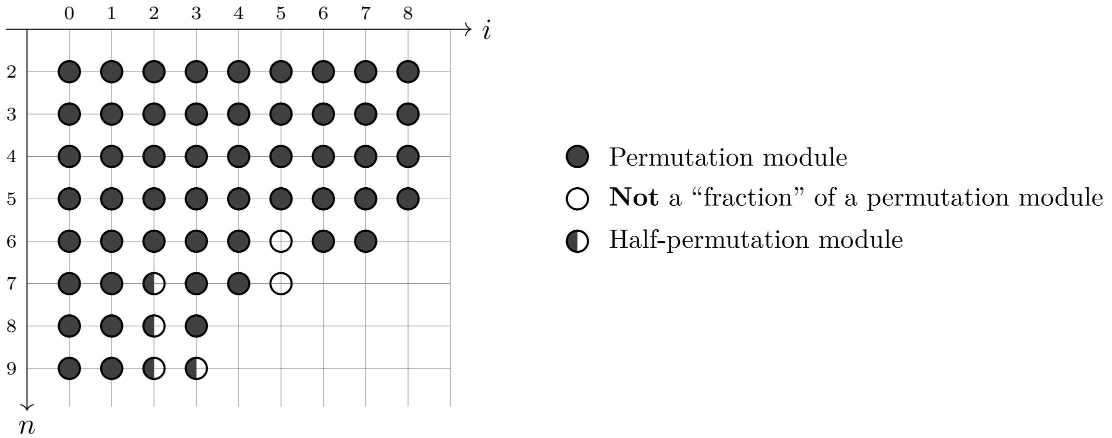

Table 11.1 summarizes the results of Theorem 11.15; an outline of the proof appears in Appendix B.

[9] And $[\mathrm{OS}(\mathrm{Br}_n)_i]$ are even *h-positive* permutation modules when $n = 1, 2, 3$, overlapping with the discussion in Section 6.3 on rank two matroids, as $U_{2,3} = \mathrm{Br}_3$.

[10] Trevor Karn's Burnside Solver further shows that $\mathcal{S}_{\mathrm{OS}}(5+i, 5)$ is a permutation module for $i \le 4$ and $i = 6, 7, 10$; it fails to be one at $i = 8, 9$.

12. Further remarks and questions

We remark here on some further directions which could merit exploration.

12.1. **Projective geometries over finite fields.** The Boolean matroids $U_{n,n}$ discussed in Example 4.5 and Section 6.1 have a well-studied "q-analogue": the *projective geometries* $PG(n,q)$, associated with the finite vector spaces $\mathbb{F}_q^n$. These $PG(n,q)$ are non-orientable simple matroids whose ground set $E = \mathbb{P}(\mathbb{F}_q^n) = \mathbb{P}_{\mathbb{F}_q}^{n-1}$ is the set of points in a finite projective space, so $|E| = [n]_q := 1 + q + q^2 + \cdots + q^{n-1}$, with poset of flats $\mathcal{F}$ given by the lattice of all subspaces in $\mathbb{F}_q^n$; see Oxley [69, Section 6.1] and Orlik and Terao [68, Example 4.33]. The lattices $\mathcal{F}$ are *modular*, meaning that every flat is a modular element, so that every complete flag $\underline{F}$ is a modular complete flag. Hence the matroids $PG(n,q)$ are supersolvable, with exponents $(e_1, \ldots, e_n) = (1, q, q^2, \ldots, q^{n-1})$. Consequently, the family of $\Bbbk$-algebras $A(n) := \mathrm{OS}(PG(n,q))$ is Koszul, satisfying

$$\mathrm{Hilb}(A(n),t) = (1+t)(1+qt)(1+q^2t)\cdots(1+q^{n-1}t) \quad \text{with } \dim_{\Bbbk} A(n)^!_i = q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q$$

$$\mathrm{Hilb}(A(n)^!,t) = \frac{1}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^{n-1}t)}$$

$$\text{with } \dim_{\Bbbk} A(n)^!_i = \begin{bmatrix} n+i-1 \\ i \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$ with $[n]!_q := [n]_q[n-1]_q\cdots[2]_q[1]_q$; see Macdonald [58, Example I.2.2].

Problem 12.1. *Study $A(n) = \mathrm{OS}(PG(n,q))$ and $A(n)^! = \mathrm{OS}(PG(n,q))^!$ as $GL_n(\mathbb{F}_q)$-representations.*

For example, the *q-Pascal recurrences* for $A(n)_i = \begin{bmatrix} n \\ i \end{bmatrix}_q$ and $A(n)^!_i = \begin{bmatrix} n+i-1 \\ i \end{bmatrix}$ will have lifts to branching rules via Proposition 2.16 and Theorem 7.1. There is also an appropriate analogue here of *representation stability for $GL_n(\mathbb{F}_q)$-representations* developed by Putman and Sam [73].

12.2. **Type B, wreath products, and Dowling geometries.** As mentioned in Section 11, the braid matroids Br_n are represented by the root systems of type A_{n-1}, accounting for the action of the reflection group $\mathfrak{S}_n$ on them as symmetries.

There are other real and complex reflection groups giving rise to matroids with large symmetry, but relatively few of these matroids are supersolvable; see Hoge and Röhrle [49] for their classification. They include the dihedral reflection groups giving rise to the rank two matroids already discussed in Example 4.6 and Section 6.3. They also include the *reflection groups of type B_n or C_n*, isomorphic to the *hyperoctahedral group* or *signed permutation group* $\mathfrak{S}_n^{\pm}$ that appeared in Section 5.2. Their root systems can be realized over $\mathbb{R}$, giving rise to an oriented matroid from the positive roots

$$\Phi^+_{B_n} := \{+e_i \pm e_j\}_{1 \leq i < j \leq n} \ \sqcup \ \{e_i\}_{1 \leq i \leq n}. \tag{12.1}$$

More generally, one has the complex reflection groups $\mathfrak{S}_n[\mathbb{Z}/m\mathbb{Z}] = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$ for $m \geq 2$, also known as the groups $G(m,1,n)$ within Shephard and Todd's classification [78] of irreducible complex reflection groups. Letting $\zeta_m := e^{\frac{2\pi i}{m}}$, their associated matroids can be represented by this list of vectors in $\mathbb{C}^n$:

$$\{e_i - \zeta^k e_j\}_{\substack{1 \leq i < j \leq n \\ 0 \leq k \leq m-1}} \sqcup \{e_i\}_{1 \leq i \leq n}. \tag{12.2}$$

These matroids are not realizable over $\mathbb{R}$ (and not orientable) unless $m = 2$ where they recover the type B_n/C_n reflection groups.

Motivated by these examples, Dowling [36] introduced a more general class of matroids, now known as the *Dowling geometries* $Q_n(G)$; see Oxley [69, Section 6.10] for definitions and discussion. Here G is *any* finite group, and the matroid automorphisms of $Q_n(G)$ contain the wreath product $\mathfrak{S}_n[G] = G \wr \mathfrak{S}_n$. Interestingly, Dowling proved that the matroid $Q_n(G)$ is representable over a field $\mathbb{F}$ *if and only if* the finite group G is a subgroup of $\mathbb{F}^\times$; in particular, this forces G to be cyclic, as in the complex reflection groups $\mathfrak{S}_n[\mathbb{Z}/m\mathbb{Z}]$ mentioned above.

Dowling also showed that $Q_n(G)$ is supersolvable for any finite group G. Consequently, their Orlik–Solomon algebras $\mathrm{OS}(Q_n(G))$ are always Koszul, and when $|G| = 2$, the same holds for the Varchenko–Gel'fand ring $\mathrm{VG}(\mathcal{M}(B_n))$, e.g., if $\mathcal{M}(B_n)$ is realized by the vectors in (12.1) above.

Problem 12.2. *Study these families of Koszul algebras* $A(n) = \mathrm{OS}(Q_n(G))$ *and* $\mathrm{VG}(\mathcal{M}(B_n))$*, along with their Koszul duals* $A(n)^!$*, as* $\mathfrak{S}_n[G]$*-representations.*

If $m := |G|$, then the exponents for the supersolvable matroids $Q_n(G)$ turn out to be

$$(e_1, e_2, \ldots, e_n) = (1, m+1, 2m+1, \ldots, (n-1)m+1).$$

Combining this with Dowling's formulas [36, Section 4], for the rank sizes[11] in the poset of flats of $Q_n(G)$, one encounters a similar coincidence to the equality $\dim_{\Bbbk} \mathrm{OS}(\mathrm{Br}_n)_i = S((n-1)+i, n-1)$ discussed in Remark 11.2: the dimension of $\mathrm{OS}(Q_n(G))^!_i$ is the size of the $(n-1)^{st}$ rank in the flat poset of $Q_{(n-1)+i}(G)$. This again reflects a bijection between the standard monomial $\Bbbk$-basis for $\mathrm{OS}(Q_n(G))^!_i$ from Theorem 5.18 and an encoding of flats in $Q_n(G)$ generalizing restricted growth functions, similar to work of Komatsu, Bagno, and Garber [8, Section 2.3]. We omit the details here.

12.3. **Equivariant degree one injections.** Recall the following consequences of Theorem 5.21: By Part (ii) of Corollary 5.22, for the *matroid* automorphism group $G = \mathrm{Aut}(M)$, there are G-equivariant degree one injections

$$\left[\mathrm{OS}(M)^!_i\right] \hookrightarrow \left[\mathrm{OS}(M)^!_{i+1}\right], \text{ for all } i \geq 0 \tag{12.3}$$

while Part (iii) of Corollary 5.22 asserts that for the full *oriented* matroid automorphism group $G = \mathrm{Aut}(\mathcal{M})$, there are G-equivariant degree *two* injections

$$\left[\mathrm{VG}(\mathcal{M})^!_i\right] \hookrightarrow \left[\mathrm{VG}(\mathcal{M})^!_{i+2}\right], \text{ for all } \ i \geq 0.$$

The latter injections arise from right-multiplication by a degree two G-invariant $\underline{E}$-generic power sum $p_2(\mathbf{y})$, such as $p_2(\mathbf{y}) = \sum_i y_i^2$. Unfortunately, for some oriented matroids $\mathcal{M}$, there are no degree one $\underline{E}$-generic element power sums $p_1(\mathbf{y})$ in $A^!_1$ that are also G-invariant. For example, $p_1(\mathbf{y}) = \sum_i y_i$ is *not* always G-invariant. In fact, the calculation for rank one oriented matroids $\mathcal{M} = U_{1,1}$ in (6.8) shows that in that case, there are no G-equivariant injections $\mathrm{VG}(\mathcal{M})^!_i \hookrightarrow \mathrm{VG}(\mathcal{M})^!_{i+1}$ for any i.

Nonetheless, for the braid matroids $\mathcal{M} = \mathrm{Br}_n$, `Sage` calculations for $n \leq 10$ and $1 \leq i \leq 9$ support the following conjecture.

[11] Also called the *Whitney numbers of the second kind for the poset.*

Conjecture 12.3. For the braid oriented matroid $\mathcal{M} = \mathrm{Br}_n$, there exist equivariant injections

$$[\mathcal{S}_{\mathrm{VG}}((n-1)+i, n-1)] = \left[\mathrm{VG}(\mathrm{Br}_n)^!_i\right]$$
$$\hookrightarrow [\mathcal{S}_{\mathrm{VG}}((n-1)+i+1, n-1)] = \left[\mathrm{VG}(\mathrm{Br}_n)^!_{i+1}\right], \text{ for all } i \geq 1.$$

Propositions B.4 and B.8 establish Conjecture 12.3 for $n \leq 4$ in characteristic zero.

We close with some observations on a consequence of the G-equivariant injections in (12.3): they imply that the following alternating sum in $R_{\Bbbk}(G)$ is always the class of a genuine $\Bbbk G$-module:

$$\left[\mathrm{OS}^!(M)_i\right] - \left[\mathrm{OS}^!(M)_{i-1}\right] + \ldots + (-1)^{i-1}\left[\mathrm{OS}^!(M)_0\right] \tag{12.4}$$

Problem 12.4. *Investigate the genuine* $\Bbbk G$*-modules* (12.4). *Do they have interesting descriptions?*

For example, for the braid matroid $M = \mathrm{Br}_n$, the dimension of the genuine module (12.4) is

$$S(n-1+i, n-1) - S(n-2+i, n-1) + \cdots + (-1)^i S(n-1, n-1). \tag{12.5}$$

This has an interpretation via a result of Mansour and Munagi [59, Corollary 11]: it is the number of set partitions of $\{1, 2, \ldots, n+i\}$ into n blocks, where no block contains a pair $j, j+1$ modulo $n+i$ for $1 \leq j \leq n+i$. We know of no accompanying $\Bbbk\mathfrak{S}_n$-module built from these objects.

We remark that for any matroid M, the alternating sum analogous to (12.4) for $\mathrm{OS}(M)$, namely

$$[\mathrm{OS}(M)_i] - [\mathrm{OS}(M)_{i-1}] + \ldots + (-1)^{i-1}[\mathrm{OS}(M)_0] \tag{12.6}$$

is always a genuine $\Bbbk G$-module for $G = \mathrm{Aut}(M)$, isomorphic to the top homology of a rank-selected subposet of the lattice of flats. We quickly sketch how this follows from combining these two results:

- [67] exhibits a $\Bbbk G$-module isomorphism $\mathrm{OS}(M) \cong \mathbf{Whit}(\mathcal{L}_M)$, where $\mathbf{Whit}(\mathcal{L}_M)$ is the Whitney homology of the lattice of flats $\mathcal{L}_M$ of M, and
- [87] If $G = \mathrm{Aut}(P)$ for a Cohen-Macaulay poset P, then the alternating sum in $\mathbb{R}_{\Bbbk}(G)$

  $$[\mathbf{Whit}_i(P)] - [\mathbf{Whit}_{i-1}(P)] + \ldots + (-1)^{i-1}[\mathbf{Whit}_0(P)]$$

  is $\Bbbk G$-isomorphic to the top homology of the rank-selected subposet of P consisting of the bottom i nonzero ranks. The Hopf trace argument in [87, Lemma 1.1], written for characteristic zero, can be replaced by applying, for an arbitrary field $\Bbbk$, Proposition 2.11 (ii) to Baclawski's complex. Similarly the arguments of Baclawski and Björner as cited in [87, Theorem 1.2] can be adapted for any field $\Bbbk$, by appealing to the isomorphism in [15, p. 262, Theorem 7.9.6]. Finally, the equivariant isomorphism with $\mathrm{OS}(M)$ follows from [15, Theorem 7.10.2], extending the argument of [67, Theorem 4.3] to the whole Orlik–Solomon algebra.

Appendix A. Tables of irreducibles for Stirling representations

This section consists of several tables for the decomposition into irreducible modules for $A(n)^!_i$ and its primitives $\mathcal{L}(A(n))_i$ when $A(n) = \mathrm{OS}(\mathrm{Br}_n)$ or $\mathrm{VG}(\mathrm{Br}_n)$. For each table, the observed onset of representation stability in each column is shaded in blue. The data is presented in terms of the Frobenius characteristics of the modules, expanded in the Schur basis. All data was generated using SAGE code which is publicly available at [3].

TABLE A.1. Irreducibles for $[\mathrm{OS}(\mathrm{Br}_n)^!_i] = [\mathcal{S}_{\mathrm{OS}}((n-1)+i, n-1)]$. Note that $\mathrm{ch}[\mathrm{OS}(\mathrm{Br}_1)^!_i] = s_2$ for $i \geq 0$ and $\mathrm{ch}[\mathrm{OS}(\mathrm{Br}_n)^!_0] = s_n$ for $n \geq 2$.

n \ i	1	2	3
3	$s_{2,1}+s_3$	$s_{1,1,1}+2s_{2,1}+2s_3$	$2s_{1,1,1}+5s_{2,1}+3s_3$
4	$s_{2,2}+s_{3,1}+s_4$	$s_{1,1,1,1}+2s_{2,1,1}$ $+3s_{2,2}+3s_{3,1}+3s_4$	$4s_{1,1,1,1}+9s_{2,1,1}$ $+10s_{2,2}+11s_{3,1}+6s_4$
5	$s_{3,2}+s_{4,1}+s_5$	$s_{2,1,1,1}+2s_{2,2,1}+2s_{3,1,1}$ $+4s_{3,2}+4s_{4,1}+3s_5$	$2s_{1,1,1,1,1}+8s_{2,1,1,1}+13s_{2,2,1}$ $+15s_{3,1,1}+18s_{3,2}+16s_{4,1}+7s_5$
6	$s_{4,2}+s_{5,1}+s_6$	$s_{2,2,2}+s_{3,1,1,1}$ $+2s_{3,2,1}+s_{3,3}+2s_{4,1,1}$ $+5s_{4,2}+4s_{5,1}+3s_6$	$3s_{2,1,1,1,1}+5s_{2,2,1,1}+7s_{2,2,2}$ $+10s_{3,1,1,1}+21s_{3,2,1}+8s_{3,3}$ $+17s_{4,1,1}+24s_{4,2}+17s_{5,1}+8s_6$
7	$s_{5,2}+s_{6,1}+s_7$	$s_{3,2,2}+s_{4,1,1,1}$ $+2s_{4,2,1}+2s_{4,3}+2s_{5,1,1}$ $+5s_{5,2}+4s_{6,1}+3s_7$	$s_{2,2,1,1,1}+2s_{2,2,2,1}$ $+3s_{3,1,1,1,1}+7s_{3,2,1,1}+9s_{3,2,2}$ $+8s_{3,3,1}+10s_{4,1,1,1}+24s_{4,2,1}+14s_{4,3}$ $+17s_{5,1,1}+25s_{5,2}+18s_{6,1}+8s_7$
8	$s_{6,2}+s_{7,1}+s_8$	$s_{4,2,2}+s_{4,4}+s_{5,1,1,1}$ $+2s_{5,2,1}+2s_{5,3}+2s_{6,1,1}$ $+5s_{6,2}+4s_{7,1}+3s_8$	$s_{2,2,2,2}+s_{3,2,1,1,1}+2s_{3,2,2,1}+2s_{3,3,1,1}$ $+2s_{3,3,2}+3s_{4,1,1,1,1}+7s_{4,2,1,1}+10s_{4,2,2}$ $+11s_{4,3,1}+6s_{4,4}+10s_{5,1,1,1}+24s_{5,2,1}$ $+15s_{5,3}+17s_{6,1,1}+26s_{6,2}+18s_{7,1}+8s_8$
9	$s_{7,2}+s_{8,1}+s_9$	$s_{5,2,2}+s_{5,4}+s_{6,1,1,1}$ $+2s_{6,2,1}+2s_{6,3}+2s_{7,1,1}$ $+5s_{7,2}+4s_{8,1}+3s_9$	$s_{3,2,2,2}+s_{4,2,1,1,1}+2s_{4,2,2,1}$ $+2s_{4,3,1,1}+3s_{4,3,2}+3s_{4,4,1}$ $+3s_{5,1,1,1,1}+7s_{5,2,1,1}+10s_{5,2,2}$ $+11s_{5,3,1}+7s_{5,4}+10s_{6,1,1,1}+24s_{6,2,1}$ $+16s_{6,3}+17s_{7,1,1}+26s_{7,2}+18s_{8,1}+8s_9$
10	$s_{8,2}+s_{9,1}+s_{10}$	$s_{6,2,2}+s_{6,4}+s_{7,1,1,1}$ $+2s_{7,2,1}+2s_{7,3}+2s_{8,1,1}$ $+5s_{8,2}+4s_{9,1}+3s_{10}$	$s_{4,2,2,2}+s_{4,4,2}+s_{5,2,1,1,1}+2s_{5,2,2,1}$ $+2s_{5,3,1,1}+3s_{5,3,2}+3s_{5,4,1}+s_{5,5}$ $+3s_{6,1,1,1,1}+7s_{6,2,1,1}+10s_{6,2,2}+11s_{6,3,1}$ $+8s_{6,4}+10s_{7,1,1,1}+24s_{7,2,1}+16s_{7,3}$ $+17s_{8,1,1}+26s_{8,2}+18s_{9,1}+8s_{10}$
11	$s_{9,2}+s_{10,1}+s_{11}$	$s_{7,2,2}+s_{7,4}+s_{8,1,1,1}$ $+2s_{8,2,1}+2s_{8,3}+2s_{9,1,1}$ $+5s_{9,2}+4s_{10,1}+3s_{11}$	$s_{5,2,2,2}+s_{5,4,2}+s_{6,2,1,1,1}+2s_{6,2,2,1}$ $+2s_{6,3,1,1}+3s_{6,3,2}+3s_{6,4,1}+2s_{6,5}$ $+3s_{7,1,1,1,1}+7s_{7,2,1,1}+10s_{7,2,2}+11s_{7,3,1}$ $+8s_{7,4}+10s_{8,1,1,1}+24s_{8,2,1}+16s_{8,3}$ $+17s_{9,1,1}+26s_{9,2}+18s_{10,1}+8s_{11}$
12	$s_{10,2}+s_{11,1}+s_{12}$	$s_{8,2,2}+s_{8,4}+s_{9,1,1,1}$ $+2s_{9,2,1}+2s_{9,3}+2s_{10,1,1}$ $+5s_{10,2}+4s_{11,1}+3s_{12}$	$s_{6,2,2,2}+s_{6,4,2}+s_{6,6}+s_{7,2,1,1,1}$ $+2s_{7,2,2,1}+2s_{7,3,1,1}+3s_{7,3,2}$ $+3s_{7,4,1}+2s_{7,5}+3s_{8,1,1,1,1}$ $+7s_{8,2,1,1}+10s_{8,2,2}+11s_{8,3,1}$ $+8s_{8,4}+10s_{9,1,1,1}+24s_{9,2,1}+16s_{9,3}$ $+17s_{10,1,1}+26s_{10,2}+18s_{11,1}+8s_{12}$

TABLE A.2. Irreducibles for $[\mathrm{VG}(\mathrm{Br}_n)^!_i] = [\mathcal{S}_{\mathrm{VG}}((n-1)+i, n-1)]$.

$n \backslash i$	0	1	2	3
2	s_2	$s_{1,1}$	s_2	$s_{1,1}$
3	s_3	$s_{1,1,1} + s_{2,1}$	$s_{1,1,1} + 2s_{2,1} + 2s_3$	$3s_{1,1,1} + 5s_{2,1} + 2s_3$
4	s_4	$s_{2,1,1} + s_{3,1}$	$s_{1,1,1,1} + 3s_{2,1,1}$ $+2s_{2,2} + 3s_{3,1} + 2s_4$	$4s_{1,1,1,1} + 11s_{2,1,1} + 8s_{2,2}$ $+11s_{3,1} + 4s_4$
5	s_5	$s_{3,1,1} + s_{4,1}$	$2s_{2,1,1,1} + 2s_{2,2,1} + 3s_{3,1,1}$ $+3s_{3,2} + 3s_{4,1} + 2s_5$	$3s_{1,1,1,1,1} + 10s_{2,1,1,1} + 14s_{2,2,1}$ $+17s_{3,1,1} + 15s_{3,2} + 14s_{4,1} + 4s_5$
6	s_6	$s_{4,1,1} + s_{5,1}$	$s_{2,2,1,1} + 2s_{3,1,1,1}$ $+2s_{3,2,1} + s_{3,3} + 3s_{4,1,1}$ $+3s_{4,2} + 3s_{5,1} + 2s_6$	$s_{1,1,1,1,1,1} + 5s_{2,1,1,1,1} + 8s_{2,2,1,1}$ $+7s_{2,2,2} + 13s_{3,1,1,1} + 21s_{3,2,1} + 7s_{3,3}$ $+18s_{4,1,1} + 18s_{4,2} + 14s_{5,1} + 4s_6$
7	s_7	$s_{5,1,1} + s_{6,1}$	$s_{3,2,1,1} + 2s_{4,1,1,1}$ $+2s_{4,2,1} + s_{4,3} + 3s_{5,1,1}$ $+3s_{5,2} + 3s_{6,1} + 2s_7$	$s_{2,1,1,1,1,1} + 2s_{2,2,1,1,1} + 3s_{2,2,2,1}$ $+6s_{3,1,1,1,1} + 11s_{3,2,1,1} + 8s_{3,2,2}$ $+7s_{3,3,1} + 13s_{4,1,1,1} + 22s_{4,2,1} + 10s_{4,3}$ $+18s_{5,1,1} + 18s_{5,2} + 14s_{6,1} + 4s_7$
8	s_8	$s_{6,1,1} + s_{7,1}$	$s_{4,2,1,1} + 2s_{5,1,1,1}$ $+2s_{5,2,1} + s_{5,3}$ $+3s_{6,1,1} + 3s_{6,2}$ $+3s_{7,1} + 2s_8$	$s_{2,2,2,2} + s_{3,1,1,1,1,1} + 3s_{3,2,1,1,1}$ $+3s_{3,2,2,1} + 3s_{3,3,1,1} + s_{3,3,2}$ $+6s_{4,1,1,1,1} + 11s_{4,2,1,1} + 8s_{4,2,2}$ $+8s_{4,3,1} + 3s_{4,4} + 13s_{5,1,1,1}$ $+22s_{5,2,1} + 10s_{5,3} + 18s_{6,1,1}$ $+18s_{6,2} + 14s_{7,1} + 4s_8$
9	s_9	$s_{7,1,1} + s_{8,1}$	$s_{5,2,1,1} + 2s_{6,1,1,1}$ $+2s_{6,2,1} + s_{6,3}$ $+3s_{7,1,1} + 3s_{7,2}$ $+3s_{8,1} + 2s_9$	$s_{3,2,2,2} + s_{3,3,1,1,1} + s_{4,1,1,1,1,1}$ $+3s_{4,2,1,1,1} + 3s_{4,2,2,1} + 3s_{4,3,1,1}$ $+s_{4,3,2} + s_{4,4,1} + 6s_{5,1,1,1,1}$ $+11s_{5,2,1,1} + 8s_{5,2,2} + 8s_{5,3,1}$ $+3s_{5,4} + 13s_{6,1,1,1} + 22s_{6,2,1} + 10s_{6,3}$ $+18s_{7,1,1} + 18s_{7,2} + 14s_{8,1} + 4s_9$
10	s_{10}	$s_{8,1,1} + s_{9,1}$	$s_{6,2,1,1} + 2s_{7,1,1,1}$ $+2s_{7,2,1} + s_{7,3}$ $+3s_{8,1,1} + 3s_{8,2}$ $+3s_{9,1} + 2s_{10}$	$s_{4,2,2,2} + s_{4,3,1,1,1} + s_{5,1,1,1,1,1}$ $+3s_{5,2,1,1,1} + 3s_{5,2,2,1} + 3s_{5,3,1,1}$ $+s_{5,3,2} + s_{5,4,1} + 6s_{6,1,1,1,1}$ $+11s_{6,2,1,1} + 8s_{6,2,2} + 8s_{6,3,1}$ $+3s_{6,4} + 13s_{7,1,1,1} + 22s_{7,2,1} + 10s_{7,3}$ $+18s_{8,1,1} + 18s_{8,2} + 14s_{9,1} + 4s_{10}$

Table A.3. Irreducibles for $[\mathcal{L}(n)_i]$ when $A = \mathrm{OS}(\mathrm{Br}_n)$.

$n \backslash i$	1	2	3	4	5
2	s_2	0	0	0	0
3	$s_{2,1}+s_3$	$s_{1,1,1}$	$s_{2,1}$	$s_{1,1,1}+s_{2,1}$	$s_{1,1,1}+2s_{2,1}+s_3$
4	$s_{2,2}+s_{3,1}+s_4$	$s_{1,1,1,1}+s_{2,1,1}$	$s_{2,1,1}+2s_{2,2}+s_{3,1}$	$2s_{1,1,1,1}+3s_{2,1,1}+2s_{2,2}+2s_{3,1}$	$3s_{1,1,1,1}+6s_{2,1,1}+6s_{2,2}+6s_{3,1}+3s_4$
5	$s_{3,2}+s_{4,1}+s_5$	$s_{2,1,1,1}+s_{3,1,1}$	$2s_{2,2,1}+s_{3,1,1}$ $+2s_{3,2}+s_{4,1}$	$s_{1,1,1,1,1}+3s_{2,1,1,1}+3s_{2,2,1}$ $+5s_{3,1,1}+3s_{3,2}+2s_{4,1}$	$2s_{1,1,1,1,1}+8s_{2,1,1,1}+11s_{2,2,1}$ $+11s_{3,1,1}+12s_{3,2}+10s_{4,1}+3s_5$
6	$s_{4,2}+s_{5,1}+s_6$	$s_{3,1,1,1}+s_{4,1,1}$	$s_{2,2,2}+2s_{3,2,1}$ $+s_{4,1,1}+2s_{4,2}+s_{5,1}$	$s_{2,1,1,1,1}+s_{2,2,1,1}+s_{2,2,2}$ $+4s_{3,1,1,1}+5s_{3,2,1}+s_{3,3}$ $+5s_{4,1,1}+3s_{4,2}+2s_{5,1}$	$4s_{2,1,1,1,1}+7s_{2,2,1,1}+7s_{2,2,2}$ $+11s_{3,1,1,1}+18s_{3,2,1}+6s_{3,3}$ $+13s_{4,1,1}+17s_{4,2}+10s_{5,1}+3s_6$
7	$s_{5,2}+s_{6,1}+s_7$	$s_{4,1,1,1}+s_{5,1,1}$	$s_{3,2,2}+2s_{4,2,1}$ $+s_{5,1,1}+2s_{5,2}+s_{6,1}$	$s_{3,1,1,1,1}+2s_{3,2,1,1}+s_{3,2,2}$ $+2s_{3,3,1}+4s_{4,1,1,1}+5s_{4,2,1}$ $+s_{4,3}+5s_{5,1,1}+3s_{5,2}+2s_{6,1}$	$2s_{2,2,1,1,1}+3s_{2,2,2,1}+5s_{3,1,1,1,1}$ $+10s_{3,2,1,1}+9s_{3,2,2}+7s_{3,3,1}$ $+11s_{4,1,1,1}+21s_{4,2,1}+11s_{4,3}$ $+13s_{5,1,1}+17s_{5,2}+10s_{6,1}+3s_7$
8	$s_{6,2}+s_{7,1}+s_8$	$s_{5,1,1,1}+s_{6,1,1}$	$s_{4,2,2}+2s_{5,2,1}$ $+s_{6,1,1}+2s_{6,2}+s_{7,1}$	$s_{3,3,1,1}+s_{4,1,1,1,1}$ $+2s_{4,2,1,1}+s_{4,2,2}+2s_{4,3,1}$ $+4s_{5,1,1,1}+5s_{5,2,1}+s_{5,3}$ $+5s_{6,1,1}+3s_{6,2}+2s_{7,1}$	$s_{2,2,2,2}+3s_{3,2,1,1,1}+3s_{3,2,2,1}$ $+3s_{3,3,1,1}+2s_{3,3,2}+5s_{4,1,1,1,1}$ $+10s_{4,2,1,1}+10s_{4,2,2}+10s_{4,3,1}$ $+5s_{4,4}+11s_{5,1,1,1}+21s_{5,2,1}+11s_{5,3}$ $+13s_{6,1,1}+17s_{6,2}+10s_{7,1}+3s_8$
9	$s_{7,2}+s_{8,1}+s_9$	$s_{6,1,1,1}+s_{7,1,1}$	$s_{5,2,2}+2s_{6,2,1}$ $+s_{7,1,1}+2s_{7,2}+s_{8,1}$	$s_{4,3,1,1}+s_{5,1,1,1,1}$ $+2s_{5,2,1,1}+s_{5,2,2}+2s_{5,3,1}$ $+4s_{6,1,1,1}+5s_{6,2,1}+s_{6,3}$ $+5s_{7,1,1}+3s_{7,2}+2s_{8,1}$	$s_{3,2,2,2}+s_{3,3,1,1,1}+3s_{4,2,1,1,1}$ $+3s_{4,2,2,1}+3s_{4,3,1,1}+3s_{4,3,2}$ $+3s_{4,4,1}+5s_{5,1,1,1,1}+10s_{5,2,1,1}$ $+10s_{5,2,2}+10s_{5,3,1}+5s_{5,4}$ $+11s_{6,1,1,1}+21s_{6,2,1}+11s_{6,3}$ $+13s_{7,1,1}+17s_{7,2}+10s_{8,1}+3s_9$
10	$s_{8,2}+s_{9,1}+s_{10}$	$s_{7,1,1,1}+s_{8,1,1}$	$s_{6,2,2}+2s_{7,2,1}$ $+s_{8,1,1}+2s_{8,2}+s_{9,1}$	$s_{5,3,1,1}+s_{6,1,1,1,1}$ $+2s_{6,2,1,1}+s_{6,2,2}+2s_{6,3,1}$ $+4s_{7,1,1,1}+5s_{7,2,1}+s_{7,3}$ $+5s_{8,1,1}+3s_{8,2}+2s_{9,1}$	$s_{4,2,2,2}+s_{4,3,1,1,1}+s_{4,4,2}$ $+3s_{5,2,1,1,1}+3s_{5,2,2,1}+3s_{5,3,1,1}$ $+3s_{5,3,2}+3s_{5,4,1}+5s_{6,1,1,1,1}$ $+10s_{6,2,1,1}+10s_{6,2,2}+10s_{6,3,1}$ $+5s_{6,4}+11s_{7,1,1,1}+21s_{7,2,1}+11s_{7,3}$ $+13s_{8,1,1}+17s_{8,2}+10s_{9,1}+3s_{10}$

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Table A.4. Irreducibles for $[\mathcal{L}(n)_i]$ when $A = \mathrm{VG}(\mathrm{Br}_n)$.

$n \backslash i$	1	2	3	4	5
2	$s_{1,1}$	s_2	0	0	0
3	$s_{1,1,1} + s_{2,1}$	$s_{2,1} + 2s_3$	$s_{2,1}$	$s_{1,1,1} + s_{2,1}$	$s_{1,1,1} + 2s_{2,1} + s_3$
4	$s_{2,1,1} + s_{3,1}$	$s_{2,2} + 2s_{3,1} + 2s_4$	$s_{2,1,1} + 2s_{2,2} + s_{3,1}$	$s_{1,1,1,1} + 4s_{2,1,1} + s_{2,2} + 2s_{3,1}$	$2s_{1,1,1,1} + 7s_{2,1,1} + 4s_{2,2} + 7s_{3,1} + 2s_4$
5	$s_{3,1,1} + s_{4,1}$	$2s_{3,2} + 2s_{4,1} + 2s_5$	$2s_{2,2,1} + s_{3,1,1}$ $+2s_{3,2} + s_{4,1}$	$3s_{2,1,1,1} + 3s_{2,2,1}$ $+6s_{3,1,1} + 2s_{3,2} + 2s_{4,1}$	$s_{1,1,1,1,1} + 8s_{2,1,1,1} + 10s_{2,2,1}$ $+13s_{3,1,1} + 11s_{3,2} + 10s_{4,1} + 2s_5$
6	$s_{4,1,1} + s_{5,1}$	$s_{3,3} + 2s_{4,2}$ $+2s_{5,1} + 2s_6$	$s_{2,2,2} + 2s_{3,2,1}$ $+s_{4,1,1} + 2s_{4,2} + s_{5,1}$	$2s_{2,2,1,1} + 4s_{3,1,1,1}$ $+5s_{3,2,1} + s_{3,3}$ $+6s_{4,1,1} + 2s_{4,2} + 2s_{5,1}$	$3s_{2,1,1,1,1} + 9s_{2,2,1,1} + 4s_{2,2,2}$ $+10s_{3,1,1,1} + 18s_{3,2,1} + 8s_{3,3}$ $+16s_{4,1,1} + 14s_{4,2} + 10s_{5,1} + 2s_6$
7	$s_{5,1,1} + s_{6,1}$	$s_{4,3} + 2s_{5,2}$ $+2s_{6,1} + 2s_7$	$s_{3,2,2} + 2s_{4,2,1}$ $+s_{5,1,1} + 2s_{5,2} + s_{6,1}$	$3s_{3,2,1,1} + 2s_{3,3,1}$ $+4s_{4,1,1,1} + 5s_{4,2,1} + s_{4,3}$ $+6s_{5,1,1} + 2s_{5,2} + 2s_{6,1}$	$3s_{2,2,1,1,1} + 3s_{2,2,2,1} + 3s_{3,1,1,1,1}$ $+11s_{3,2,1,1} + 6s_{3,2,2} + 9s_{3,3,1}$ $+11s_{4,1,1,1} + 21s_{4,2,1} + 11s_{4,3}$ $+16s_{5,1,1} + 14s_{5,2} + 10s_{6,1} + 2s_7$
8	$s_{6,1,1} + s_{7,1}$	$s_{5,3} + 2s_{6,2}$ $+2s_{7,1} + 2s_8$	$s_{4,2,2} + 2s_{5,2,1}$ $+s_{6,1,1} + 2s_{6,2} + s_{7,1}$	$s_{3,3,1,1} + 3s_{4,2,1,1}$ $+2s_{4,3,1} + 4s_{5,1,1,1}$ $+5s_{5,2,1} + s_{5,3}$ $+6s_{6,1,1} + 2s_{6,2} + 2s_{7,1}$	$s_{2,2,2,1,1} + 3s_{3,2,1,1,1} + 3s_{3,2,2,1}$ $+2s_{3,3,1,1} + 3s_{3,3,2} + 3s_{4,1,1,1,1}$ $+12s_{4,2,1,1} + 6s_{4,2,2} + 12s_{4,3,1}$ $+3s_{4,4} + 11s_{5,1,1,1} + 21s_{5,2,1} + 11s_{5,3}$ $+16s_{6,1,1} + 14s_{6,2} + 10s_{7,1} + 2s_8$
9	$s_{7,1,1} + s_{8,1}$	$s_{6,3} + 2s_{7,2}$ $+2s_{8,1} + 2s_9$	$s_{5,2,2} + 2s_{6,2,1}$ $+s_{7,1,1} + 2s_{7,2} + s_{8,1}$	$s_{4,3,1,1} + 3s_{5,2,1,1}$ $+2s_{5,3,1} + 4s_{6,1,1,1}$ $+5s_{6,2,1} + s_{6,3}$ $+6s_{7,1,1} + 2s_{7,2} + 2s_{8,1}$	$s_{3,2,2,1,1} + s_{3,3,3} + 3s_{4,2,1,1,1}$ $+3s_{4,2,2,1} + 3s_{4,3,1,1} + 3s_{4,3,2}$ $+3s_{4,4,1} + 3s_{5,1,1,1,1} + 12s_{5,2,1,1}$ $+6s_{5,2,2} + 12s_{5,3,1} + 3s_{5,4}$ $+11s_{6,1,1,1} + 21s_{6,2,1} + 11s_{6,3}$ $+16s_{7,1,1} + 14s_{7,2} + 10s_{8,1} + 2s_9$
10	$s_{8,1,1} + s_{9,1}$	$s_{7,3} + 2s_{8,2}$ $+2s_{9,1} + 2s_{10}$	$s_{6,2,2} + 2s_{7,2,1}$ $+s_{8,1,1} + 2s_{8,2} + s_{9,1}$	$s_{5,3,1,1} + 3s_{6,2,1,1}$ $+2s_{6,3,1} + 4s_{7,1,1,1}$ $+5s_{7,2,1} + s_{7,3}$ $+6s_{8,1,1} + 2s_{8,2} + 2s_{9,1}$	$s_{4,2,2,1,1} + s_{4,3,3} + s_{4,4,1,1}$ $+3s_{5,2,1,1,1} + 3s_{5,2,2,1} + 3s_{5,3,1,1}$ $+3s_{5,3,2} + 3s_{5,4,1} + 3s_{6,1,1,1,1}$ $+12s_{6,2,1,1} + 6s_{6,2,2} + 12s_{6,3,1}$ $+3s_{6,4} + 11s_{7,1,1,1} + 21s_{7,2,1} + 11s_{7,3}$ $+16s_{8,1,1} + 14s_{8,2} + 10s_{9,1} + 2s_{10}$

Appendix B. Proof of Theorem 11.15

The proof of Theorem 11.15 and calculation of near-boundary cases for $\mathrm{OS}(\mathrm{Br}_n)^!_i$, $\mathrm{VG}(\mathrm{Br}_n)^!_i$ employ a brute-force strategy, which we outline here, giving only brief sketches of the arguments.

Note that since all $\Bbbk\mathfrak{S}_n$-modules U are self-contragredient, one has $[U^*] = [U]$ in $R_\Bbbk(\mathfrak{S}_n)$, and so the defining recurrence (2.12) for Koszul modules simplifies to this:

$$[A^!_d] = \sum_{i=1}^{d} (-1)^{i-1}[A_i] \cdot [(A^!_{d-i})] \tag{B.1}$$

This means that if one defines

$$\begin{aligned} g_i &= \operatorname{ch} \mathrm{OS}(n)_i \text{ for } 0 \leq i \leq n-1, \\ f_i &:= \operatorname{ch} \mathrm{OS}(n)^!_i = \mathcal{S}_{\mathrm{OS}}(n-1+i, n-1) \text{ for } i = 0, 1, 2, \ldots, \end{aligned}$$

then, with $*$ below denoting the internal (Kronecker) product of symmetric functions, (11.6) lets one sometimes compute explicit formulas for the g_i in terms of the homogeneous symmetric functions $\{h_\lambda\}$, and (B.1) gives a recurrence for f_i in terms of $f_0, f_1, \ldots, f_{i-1}$:

$$f_i = \sum_{i=1}^{d} (-1)^{i-1} g_i * f_{d-i} \tag{B.2}$$

In each of the cases below, we identify a small subset T of partitions of n such that the linear span of $\{h_\lambda : \lambda \in T\}$ contains the f_i. Further manipulation then gives the results described in Theorem 11.15, and the precise formulas below.

B.1. Proof of Theorem 11.15 (i).

Corresponding to the Stirling number formulas

$$\begin{aligned} S(n-1, n-1) &= 1, \\ S(n, n-1) &= \binom{n}{2}, \end{aligned}$$

one has the following result, implying Theorem 11.15 (i).

Proposition B.1. *For the cases $i = 0, 1$, one has*

$$\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(n-1, n-1) = \operatorname{ch} \mathcal{S}_{\mathrm{VG}}(n-1, n-1) = h_n \qquad \textit{for } n \geq 1, \tag{B.3}$$

$$\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(n, n-1) = h_2 h_{n-2} \qquad \textit{for } n \geq 2, \tag{B.4}$$

$$\operatorname{ch} \mathcal{S}_{\mathrm{VG}}(n, n-1) = e_2 h_{n-2} \qquad \textit{for } n \geq 2. \tag{B.5}$$

Proof. Equation (B.3) follows since $\mathrm{OS}(\mathrm{Br}_n)_0 = \mathrm{VG}(\mathrm{Br}_n)_0 = \Bbbk$, carrying the trivial $\mathfrak{S}_n$ representation in either case. For (B.4), (B.5), note that (B.1) and (11.6) imply

$$\begin{aligned} \left[\mathrm{OS}(\mathrm{Br}_n)^!_1\right] &= [\mathrm{OS}(\mathrm{Br}_n)_1] = [\mathrm{OS}(\mathrm{Br}_n)_{(21^{n-2})}] = h_1[\pi_2] \cdot h_{n-2}[\pi_1] = h_2 h_{n-2},, \\ \left[\mathrm{VG}(\mathrm{Br}_n)^!_1\right] &= [\mathrm{VG}(\mathrm{Br}_n)_1] = [\mathrm{VG}(\mathrm{Br}_n)_{(21^{n-2})}] = h_1[\ell_2] \cdot h_{n-2}[\ell_1] = e_2 h_{n-2}. \end{aligned}$$

□

B.2. **Proof of Theorem 11.15** (ii). Here we prove the curious fact that for $n \geq 7$, $\mathrm{OS}(\mathrm{Br}_n)^!_2 = \mathcal{S}_{\mathrm{OS}}(n+1, n-1)$ is in fact *half* of a permutation module.

For $n = 7, 8, 9, 10$, Sage computation with the Burnside ring shows that $\mathcal{S}_{\mathrm{OS}}(n+1, n-1)$ is NOT a permutation module. By running his Burnside solver on the first formula in Proposition B.2 below with rational coefficients, Trevor Karn noticed positive half-integers in the data and conjectured that two copies of $\mathcal{S}_{\mathrm{OS}}(n+1, n-1)$ together constitute a permutation module.

Proposition B.2. *One has the following decompositions as permutation modules for* $n \leq 6$*:*

$$\begin{aligned}
\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(3,1) &= h_2,\\
\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(4,2) &= h_3 + h_1^3,\\
\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(5,3) &= h_2[h_1^2] + h_1^2 h_2 + h_4,\\
\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(6,4) &= h_1 \cdot h_2[h_2] + h_2(h_3 + e_3) + h_2^2 h_1,\\
\operatorname{ch} \mathcal{S}_{\mathrm{OS}}(7,5) &= h_2[h_2 h_1] + h_3^2 + h_4 h_1^2,
\end{aligned}$$

and then for $n \geq 4$ *one has*

$$\begin{aligned}
&\operatorname{ch} \mathrm{OS}(\mathrm{Br}_n)^!_2\\
&\quad = \operatorname{ch} \mathcal{S}_{\mathrm{OS}}(n+1, n-1)\\
&\quad = h_{n-2}h_2 + h_{n-3}h_1^3 + h_{n-3}h_3 + h_{n-4}h_2^2 + h_{n-4}h_4 - \mathbf{h_{n-3}h_2h_1} - \mathbf{h_{n-4}h_3h_1} \qquad \text{(B.6)}\\
&\quad = h_{n-2}\, s_{(2)} + h_{n-3} \left(s_{(1^3)} + s_{(2,1)} + s_{(3)} \right) + h_{n-4} \left(s_{(2,2)} + s_{(4)} \right) \qquad \text{(B.7)}\\
&\quad = h_{n-2}h_2 + \frac{1}{2} h_{n-3} h_1^3 + \frac{1}{2} h_{n-3}(h_3 + e_3) + h_{n-4} \cdot h_2[h_2]. \qquad \text{(B.8)}
\end{aligned}$$

So two copies of $\mathrm{OS}(\mathrm{Br}_n)^!_2 = \mathcal{S}_{\mathrm{OS}}(n+1, n-1)$ *together form a permutation module, with orbit stabilisers*

$$\{\mathfrak{S}_{n-2} \times S_2,\ \mathfrak{S}_{n-3},\ \mathfrak{S}_{n-3} \times C_3,\ \mathfrak{S}_{n-4} \times I_2(4)\}$$

where C_n *is the cyclic group of order* n *(generated by the* n*-cycle* $(1, 2, \ldots, n)$ *in* $\mathfrak{S}_n$*) and* $I_2(n)$ *is the dihedral group of order* $2n$ *inside* $\mathfrak{S}_n$ *containing that same* n*-cycle.*

Sketch of proof. The expansion for $n = 2$ is clear. For $n \geq 3$, writing f_i, g_i as in (B.2), one finds that

$$\begin{aligned}
f_1 &= g_1,\\
f_2 &= f_1 * g_1 - g_2.
\end{aligned}$$

Using (11.6) and writing $\delta(S) \in \{0, 1\}$ depending on whether statement S is false or true, one has

$$\begin{aligned}
f_1 = g_1 = \operatorname{ch} A(n)_1 = \operatorname{ch} \mathrm{OS}_{(2,1^{n-2})} = h_{n-2}\pi_2 = h_{n-2}h_2,\\
g_2 = h_{n-3}(h_2h_1 - h_3) \cdot \delta_{n\geq 3} + h_{n-4}(h_3h_1 - h_4) \cdot \delta_{n\geq 4}.
\end{aligned}$$

Using the standard fact that $U \otimes (V{\uparrow}_H^G) \cong (U{\downarrow}_H \otimes V){\uparrow}_H^G$, for the Young subgroup $H = \mathfrak{S}_2 \times \mathfrak{S}_{n-2}$, and the *skewing operators* $s_{(2)}^{\perp}$ and $s_{(1^2)}^{\perp}$ as defined in Macdonald

[58, Ex. I.5.3], the expression (B.6) follows by routine manipulation. Then to establish (B.8), we use these facts:

$$h_1^3 + 2h_3 - 2h_2h_1 = h_3 + e_3 = \mathrm{ch}\, 1\!\uparrow_{C_3}^{\mathfrak{S}_3},$$
$$h_2[h_2] = h_4 + s_{(2,2)} = h_4 + h_2^2 - h_3h_1 = \mathrm{ch}\, 1\!\uparrow_{I_2(4)}^{\mathfrak{S}_4}.$$ □

Remark B.3. A similar analysis gives the following for $\mathrm{VG}(\mathrm{Br}_n^!)_2$:

$$\begin{aligned}\mathrm{ch}\, &\mathcal{S}_{\mathrm{VG}}(n+1, n-1)\\ &= h_{n-2}h_2 + h_{n-3}\left(h_3 - h_2h_1 + h_1^3\right) + h_{n-4}\left(h_4 + h_2h_1^2 - h_3h_1 - h_2^2\right),\ n \geq 4\\ &= h_{n-2}\, s_{(2)} + h_{n-3}\left(s_{(1^3)} + s_{(2,1)} + s_{(3)}\right) + h_{n-4}\, s_{(2,1,1)}.\end{aligned} \tag{B.9}$$

Also $\mathrm{ch}\,\mathcal{S}_{\mathrm{VG}}(n+1, n-1) = h_1^n + h_n$ for $n = 3, 4$, and hence $\mathcal{S}_{\mathrm{VG}}(n+1, n-1)$ is a permutation module for $n \leq 4$; it is half a permutation module for $n = 5$. However for $n = 6, 7$, the Burnside solver shows that it is not a permutation module (even after scaling), even though all character values are nonnegative. At $n = 8$ there are negative character values, so even scaling will not result in a permutation module.

B.3. **Proof of Theorem 11.15** (iii).

Proof. For fixed small k, the general Stirling number formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n \tag{B.10}$$

gives fairly simple explicit formulas for $S(n,k)$ as a function of n, e.g., for $k = 1, 2, 3, 4, 5$:

$$S(n,1) = 1 \tag{B.11}$$
$$S(n,2) = \frac{1}{2}(2^n - 2\cdot 1^n) = 2^{n-1} - 1 = 1 + 2 + 2^2 + \cdots + 2^{n-2}, \tag{B.12}$$
$$S(n,3) = \frac{1}{6}(3^n - 3\cdot 2^n + 3) \tag{B.13}$$
$$S(n,4) = \frac{1}{24}(4^n - 4\cdot 3^n + 6\cdot 2^n - 4) \tag{B.14}$$
$$S(n,5) = \frac{1}{120}(5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5) \tag{B.15}$$

We give here analogous descriptions of the $\Bbbk\mathfrak{S}_n$-modules $\mathrm{OS}(\mathrm{Br}_n)_i^!, \mathrm{VG}(\mathrm{Br}_n)_i^!$, having dimension $S(n-1+i, n-1)$, starting[12] with $n = 2, 3$.

Proposition B.4. *The Frobenius characteristics of the* $\Bbbk\mathfrak{S}_n$*-modules*

$$\mathrm{OS}(\mathrm{Br}_n)_i^! = \mathcal{S}_{\mathrm{OS}}(n-1+i, n-1),$$
$$\mathrm{VG}(\mathrm{Br}_n)_i^! = \mathcal{S}_{\mathrm{VG}}(n-1+i, n-1)$$

for $n = 2, 3$ *have these formulas:*

$n = 2$: $\quad \mathrm{ch}\,\mathcal{S}_{\mathrm{OS}}(i+1, 1) = h_2,$

$$\mathrm{ch}\,\mathcal{S}_{\mathrm{VG}}(i+1,1) = \begin{cases} h_2, & i \text{ even},\\ e_2, & i \text{ odd}.\end{cases}$$

[12]There is little to say for $n = 1$, as $\mathfrak{S}_1$ is the trivial group, and $\mathrm{OS}(\mathrm{Br}_1) = \mathrm{VG}(\mathrm{Br}_1) = \Bbbk = \mathrm{OS}(\mathrm{Br}_1)^! = \mathrm{VG}(\mathrm{Br}_1)^!$, and $\mathrm{ch}\,\mathcal{S}_{\mathrm{OS}}(0,0) = \mathrm{ch}\,\mathcal{S}_{\mathrm{VG}}(0,0) = h_1$.

$n = 3:$ $\quad \mathrm{ch}\, \mathcal{S}_{OS}(i+2,2) = \mathrm{ch}\, \mathcal{S}_{VG}(i+2,2) = \frac{2^i-1}{3} h_1^3 + h_3,$ *if i even,*

$\mathrm{ch}\, \mathcal{S}_{OS}(i+2,2) = \frac{2(2^{i-1}-1)}{3} h_1^3 + h_1 h_2,$ *if i odd,*

$\mathrm{ch}\, \mathcal{S}_{VG}(i+2,2) = \frac{2(2^{i-1}-1)}{3} h_1^3 + h_1 e_2,$ *if i odd.*

In particular,

- $\mathrm{OS}(\mathrm{Br}_2)^!_i, \mathrm{OS}(\mathrm{Br}_3)^!_i,$ *are permutation modules, while*
- $\mathrm{VG}(\mathrm{Br}_2)^!_i, \mathrm{VG}(\mathrm{Br}_3)^!_i$ *are permutation modules for all* even i, *and*
- *When $n = 2, 3$ one has*

$$\mathrm{OS}(\mathrm{Br}_n)^!_i \cong \begin{cases} \mathrm{VG}(\mathrm{Br}_n)^!_i & \text{if } i \text{ is even,} \\ \mathrm{sgn}_n \otimes \mathrm{VG}(\mathrm{Br}_n)^!_i & \text{if } i \text{ is odd.} \end{cases}$$

Sketch of proof. These all follow by induction on i via the recurrences (B.1) and (11.6). □

Remark B.5. Note the expressions for $n = 2$ are consistent with the formula $S(i+1,1) = 1$ coming from (B.11). We claim that the expressions for $n = 3$ are also consistent with the formulas

$$S(i+2,2) = 2^{i+1} - 1 \tag{B.16}$$

$$= 1 + 2 + 2^2 + \cdots + 2^i \tag{B.17}$$

coming from (B.12), which we illustrate here for $\mathrm{OS}(\mathrm{Br}_3)_i = \mathcal{S}(i+2,2)$. One can rewrite it as

$$\mathrm{ch}\ \mathrm{OS}(\mathrm{Br}_3)_i = \mathrm{ch}\, \mathcal{S}_{\mathrm{OS}}(i+2,2) = \underbrace{\left(\frac{2^{i+1} + (-1)^i}{6} - \frac{1}{2} \right)}_{\text{call this } c_i} \cdot h_1^3 + \begin{cases} h_3 & \text{if } i \text{ is even,} \\ h_2 h_1 & \text{if } i \text{ is odd} \end{cases}$$

$$= \frac{2^{i+1}}{6} \cdot h_1^3 + \begin{cases} h_3 - \frac{1}{3} h_1^3 & \text{if } i \text{ is even,} \\ h_2 h_1 - \frac{2}{3} h_1^3 & \text{if } i \text{ is odd.} \end{cases}$$

Since $h_1^3, h_2 h_1, h_3$ correspond to $\Bbbk\mathfrak{S}_3$-modules of dimensions $6, 3, 1$, one can check that this last formula lifts (B.16). Interestingly, the number c_i of copies of the regular representation here (that is, the coefficient of h_1^3) gives a sequence $0, 0, 1, 2, 5, 10, 21, 42, 85, \ldots$ for which every other term $0, 1, 5, 21, 85, 341, \ldots$ appears in the Online Encyclopedia of Integer Sequences as `OEIS A002450`.

Expressions lifting (B.17) arise when one uses the recurrences (B.1) and (11.6), without trying to rewrite things in terms of h_λ. Recalling that $\mathcal{S}^\lambda$ is the irreducible $\Bbbk\mathfrak{S}_n$-module indexed by λ, with ch $\mathcal{S}^\lambda = s_\lambda$, a *Schur function.* One can check that these recurrences give

$$\mathrm{ch}\, \mathcal{S}_{\mathrm{OS}}(i+2,2) = h_3 + s_{(2,1)} + s_{(2,1)}^{*2} + \cdots + s_{(2,1)}^{*i}. \tag{B.18}$$

$$\mathrm{ch}\, \mathcal{S}_{\mathrm{VG}}(i+2,2) = \omega^i(h_3) + s_{(2,1)} + s_{(2,1)}^{*2} + \cdots + s_{(2,1)}^{*i}, \tag{B.19}$$

where $\omega : \Lambda \to \Lambda$ is the involution on symmetric functions swapping $h_n \leftrightarrow e_n$ for $n \geq 1$, corresponding to tensoring $\Bbbk\mathfrak{S}_n$-modules by the sign character sgn_n. Bearing in mind that h_3, e_3 correspond to 1-dimensional modules, while $s_{(2,1)}$ corresponds to the 2-dimensional reflection representation $\mathcal{S}^{(2,1)}$ of $\mathfrak{S}_3$, one sees that (B.18), (B.19) lift (B.17). Note also that (B.18) is consistent with the $n = 3$ case of (6.14), since one has a matroid isomorphism $\mathrm{Br}_3 \cong U_{2,3}$.

B.4. **The cases** $\mathrm{OS}(\mathrm{Br}_4)^!$ **and** $\mathrm{VG}(\mathrm{Br}_4)^!$**.** Here we show that the $\mathfrak{S}_4$-modules $\mathcal{S}_{\mathrm{OS}}(n+3,3)$ are permutation modules. One observes a periodicity in the initial expressions for $f_n = \mathrm{ch}\,\mathcal{S}_{\mathrm{OS}}(n+3,3)$ below.

$$\begin{aligned}
f_0 &= h_4 \\
f_1 &= h_2^2 \\
f_2 &= h_1^4 - \mathbf{h_2 h_1^2} + 2h_2^2 + h_4 && = h_2[h_1^2] + h_1^2 h_2 + h_4 \\
f_3 &= 4h_1^4 - \mathbf{3h_2 h_1^2} + 5h_2^2 && = 2h_1^4 + 2h_2[h_1^2] + h_1^2 h_2 + h_2^2 \\
f_4 &= 14h_1^4 - \mathbf{8h_2 h_1^2} + 10h_2^2 + h_4 && = 10h_1^4 + 4h_2[h_1^2] + 2h_2^2 + h_4 \\
f_5 &= 44h_1^4 - \mathbf{18h_2 h_1^2} + 21h_2^2 && = 35h_1^4 + 9h_2[h_1^2] + 3h_2^2 \\
f_6 &= 135h_1^4 - \mathbf{39h_2 h_1^2} + 42h_2^2 + h_4 && = 115h_1^4 + 20h_2[h_1^2] + 2h_2^2 + h_2h_1^2 + h_4 \\
f_7 &= 408h_1^4 - \mathbf{81h_2 h_1^2} + 85h_2^2 && = 367h_1^4 + 41h_2[h_1^2] + 3h_2^2 + h_2h_1^2
\end{aligned} \tag{B.20}$$

Proposition B.6. *The Frobenius characteristic* $\mathrm{ch}\,\mathcal{S}_{OS}(n+3,3) = \mathrm{ch}\,\mathrm{OS}^!(\mathrm{Br}_4)_n$ *is an integer combination of* $\{h_1^4, h_1^2h_2, h_2^2, h_4\}$.

Let $\mathrm{ch}\,\mathcal{S}_{\mathbf{OS}}(\mathbf{n+3,3}) = \mathbf{a_n h_1^4} + \mathbf{b_n h_1^2 h_2} + \mathbf{c'_n h_2^2} + \mathbf{d_n h_4}$, $n \geq 0$. *Let* G_2 *be the subgroup of order 2 generated by* $(12)(34)$*. Then* $\mathcal{S}_{OS}(n+3,3)$ *is a permutation module with orbit stabilisers consisting of the wreath product* $\mathfrak{S}_2[G_2]$*, as well as a subset of the Young subgroups* $\mathfrak{S}_\lambda$, $\lambda \in \{(1^4), (2,1^2), (2^2), (4)\}$*. We have, for* $a_n, c'_n, d_n \geq 0$ *and* $b_n < 0$,

$$\begin{aligned}
&\mathrm{ch}\,\mathcal{S}_{OS}(n+3,3) \\
&\quad = \left(a_n + \frac{b_n}{2}\right) h_1^4 - \frac{b_n}{2} h_2[h_1^2] + (c'_n + b_n)h_2^2 + d_n h_4,\ n \equiv 0,1 \bmod 4, &\text{(B.21)}\\
&\quad = \left(a_n + \frac{b_n - 1}{2}\right) h_1^4 - \frac{b_n - 1}{2} h_2[h_1^2] + h_1^2 h_2 + (c'_n + b_n - 1)h_2^2 + d_n h_4,\ n \\
&\qquad \equiv 2,3 \bmod 4. &\text{(B.22)}
\end{aligned}$$

The coefficients a_n, b_n, c'_n, d_n *are determined below.*

(1) The coefficients of h_4 *are the sequence* $d_n = \frac{1+(-1)^n}{2}$, $n \geq 0$.

(2) The coefficients of h_2^2 *are* $\{0, 0, 1, 2, 5, 10, 21, \ldots\}$, *i.e. the numbers* c_n *from Remark B.5. More precisely,* $c'_n = c_{n+3} = \frac{2^{n+2}-3+(-1)^{n+3}}{6}$, $n \geq 0$.

(3) $b_n - b_{n-1} = -c_{n+2} = -\frac{2^{n+1}-3+(-1)^n}{6}$, $n \geq 1$, *with* $b_0 = 0 = b_1$*. In particular,* $b_n < 0$ *for* $n \geq 2$*. We have*

$$b_n = -\frac{2}{3}(2^n - 1) + \frac{n}{2} + \frac{1}{12}(1 - (-1)^n) = -\frac{2^{n+1}}{3} + \frac{n}{2} + \frac{3}{4} - \frac{1}{12}(-1)^n,\ n \geq 0.$$

Thus $\mathcal{S}_{OS}(n+3,3)$ *is NOT h-positive for* $n \geq 2$.

(4) The coefficients a_n *are all nonnegative, and strictly positive if* $n \geq 2$*. We have* $a_0 = a_1 = 0$ *and for* $n \geq 2$,

$$a_n = \frac{3^{n+1}}{16} - \frac{n}{4} - \frac{4 - (-1)^n}{16}$$

In particular, $a_n + \frac{b_n}{2}$, *and* $a_n + \frac{b_n - 1}{2}$ *are positive integers for* $n \geq 3$.

Also, a_n *is the multiplicity of the sign representation.*

Remark B.7. Computing dimensions shows that the h-expansion of $\mathrm{ch}\,\mathcal{S}(n+3,3)$ lifts the formula (B.13) for the Stirling number $S(n+3,3)$.

The coefficient a_n of h_1^4 gives the sequence $0, 0, 1, 4, 14, 44, 135, 408, 1228, \ldots$, appearing as `OEIS A097137`. (One checks that $a_n - a_{n-2} = (3^{n-1} - 1)/2$.) Also the negative of the coefficient b_n of $h_1^2 h_2$ gives $0, 0, 1, 3, 8, 18, 39, 81, 166, 336, 677, \ldots$, which is `OEIS A011377` or `OEIS A178420`.

A similar analysis for $\mathrm{VG}(\mathrm{Br}_4)^!_i$ shows that its Frobenius characteristic $\mathrm{ch}\, \mathcal{S}_{\mathrm{VG}}(3+i, 3)$ is also an integer combination of $\{h_1^4, h_1^2 h_2, h_2^2, h_4\}$, in fact of $\{h_1^4, h_2 e_2, h_4\}$.

Here is the data for $f_i = \mathrm{ch}\, \mathrm{VG}^!(4)_i$ with $0 \le i \le 11$:

$$
\begin{aligned}
f_0 &= h_4 & f_1 &= h_2 e_2 \\
f_2 &= h_1^4 + h_4 & f_3 &= 4\, h_1^4 - h_2 e_2 \\
f_4 &= 12\, h_1^4 + 2\, h_2 e_2 + h_4 & f_5 &= 40\, h_1^4 + h_2 e_2 \\
f_6 &= 127\, h_1^4 - 4\, h_2 e_2 + h_4 & f_7 &= 388\, h_1^4 + 3\, h_2 e_2 \\
f_8 &= 1186\, h_1^4 + 6\, h_2 e_2 + h_4 & f_9 &= 3608\, h_1^4 - 11\, h_2 e_2 \\
f_{10} &= 10901\, h_1^4 + h_4 & f_{11} &= 32868\, h_1^4 + 23\, h_2 e_2
\end{aligned}
$$

Observe that the set $\{h_1^4, h_2 e_2, h_4\}$ is linearly independent. One then has the following more precise statement:

Proposition B.8. *Write f_n for $\mathrm{ch}\, \mathcal{S}_{VG}(n+3, 3) = \mathrm{ch}\, \mathrm{VG}^!(\mathrm{Br}_4)_n$. Then $f_{2n-1}, f_{2n} - h_4 \in \mathbb{Z}[h_1^4, h_2 e_2]$ and hence for $n \ge 0$, both the representation $\mathcal{S}_{VG}(2n+2, 3)$ and the quotient representation $\mathcal{S}_{VG}(2n+3, 3)/\mathbb{1}_{\mathfrak{S}_4}$ are fixed under tensoring with the sign representation* sgn *of $\mathfrak{S}_4$.*

Let $f_n = a_n h_1^4 + b_n h_2 e_2 + d_n h_4$. Then, with initial values $a_0 = a_1 = 0, a_2 = 1, a_3 = 4$, $b_0 = 0, b_1 = 1, b_2 = 0, b_3 = -1$, one has that $d_n = \frac{1+(-1)^n}{2}$, $a_n \ge 0$ for all $n \ge 0$, and for $n \ge 3$:

$$
\begin{aligned}
a_n &= 6a_{n-1} - 11a_{n-2} + 6a_{n-3} + 2(b_{n-1} - b_{n-2} + b_{n-3}) - d_{n-2}, \\
b_n &= -b_{n-2} + 2b_{n-3}.
\end{aligned}
$$

The sequence $\{b_n\}_{n \ge 0}$ appears in `OEIS A077912`*, with generating function $\frac{x}{1+x^2-2x^3}$.*

Moreover $\mathcal{S}_{VG}(n+3, 3)$ is a permutation module if and only if $b_n = 0$ or $b_n \le -2$. Write $-b_n = 2\alpha_n + 3\beta_n$ for nonnegative integers α_n, β_n. Then $a_n - (\alpha_n + \beta_n)$ is nonnegative and

$$
f_n = (a_n - (\alpha_n + \beta_n)) h_1^4 + \alpha_n\, \mathrm{ch}\left(1\!\uparrow_{G_2}^{\mathfrak{S}_4}\right) + \beta_n\, \mathrm{ch}\left(1\!\uparrow_{V_4}^{\mathfrak{S}_4}\right) + d_n h_4
$$

is the Frobenius characteristic of a permutation module, where the orbit stabilisers are $\mathfrak{S}_1$, $\mathfrak{S}_4$ and the subgroups $G_2 = \langle (12)(34) \rangle$ and $V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$ of $\mathfrak{S}_4$.

B.5. **The case** $\mathrm{OS}(\mathrm{Br}_5)^!$**.** In this section we show that the $\mathfrak{S}_5$-modules $\mathcal{S}_{\mathrm{OS}}(n+4, 4)$ are also permutation modules. We also show that the h-expansions exhibit a curious periodicity modulo 4.

The initial expressions for $f_n = \operatorname{ch} \mathcal{S}_{\mathrm{OS}}(n+4,4)$ are as follows.

$$\begin{aligned}
&f_0 = h_5,\ f_1 = h_3h_2,\\
&f_2 = h_4h_1 + h_2h_1^3 + 2h_3h_2 - \mathbf{h_3h_1^2} = h_1h_2[h_2] + h_2(h_3+e_3) + h_2^2h_1,\\
&f_3 = 2h_1^5 + 3h_2^2h_1 + 2h_3h_2,\\
&f_4 = 12h_1^5 + 8h_2^2h_1 + 2h_3h_2 + h_5\\
&f_5 = 60h_1^5 + 18h_2^2h_1 + 3h_3h_2\\
&f_6 = 274h_1^5 + 38h_2^2h_1 + h_2h_1^3 - \mathbf{h_3h_1^2} + 4h_3h_2 + h_4h_1\\
&\quad = \left(274h_1^5 + 38h_2h_1^2 + 2h_3h_2\right) + f_2\\
&f_7 = 1194h_1^5 + 81h_2^2h_1 + 4h_3h_2.
\end{aligned} \tag{B.23}$$

Proposition B.9. *The $\mathfrak{S}_5$-module $\mathcal{S}_{\mathrm{OS}}(n+4,4) = \operatorname{ch} \mathrm{OS}^!(\mathrm{Br}_5)_n$ is a permutation module for all $n \geq 0$, with orbit stabilisers given by*

- *the Young subgroups $\mathfrak{S}_\lambda$ for $\lambda \in \{(1^5),(2^2,1),(3,2)\}$ if* $\mathbf{n} \equiv \mathbf{1},\mathbf{3} \bmod \mathbf{4}$.
- *the Young subgroup $\mathfrak{S}_{(2^2,1)}$, as well as the subgroups $\mathfrak{S}_1 \times I_2(4), A_3 \times \mathfrak{S}_2$ if* $\mathbf{n} \equiv \mathbf{2} \bmod \mathbf{4}$.

  Here $A_3 \times \mathfrak{S}_2$ is the subgroup of the Young subgroup $\mathfrak{S}_3 \times \mathfrak{S}_2$, for the alternating subgroup A_3 of $\mathfrak{S}_3$.
- *the Young subgroups $\mathfrak{S}_\lambda$ for $\lambda \in \{(1^5),(2^2,1),(3,2),(5)\}$ if* $\mathbf{n} \equiv \mathbf{0} \bmod \mathbf{4}$.

Let $J = \{h_1^5, h_2^2h_1, h_3h_2\}$. Let $f_n = \operatorname{ch} \mathcal{S}_{OS}(n+4,4) = \operatorname{ch} A^!(5)_n$. Then

(1) f_n is a nonnegative integer combination of the set J if $n \equiv 1,3 \bmod 4$.
(2) $f_n - f_2$ is a nonnegative integer combination of J if $n \equiv 2 \bmod 4$.
(3) $f_n - f_0$ is a nonnegative integer combination of J if $n \equiv 0 \bmod 4$.

The following explicit decomposition holds for $f_{n+4} - f_n$:

$$f_{n+4} - f_n = a_nh_1^5 + b_nh_1h_2^2 + 2h_2h_3,\ n \geq 0, \tag{B.24}$$

where $b_0 = 8$, $b_n = 10(2^n) - 2$, $n \geq 1$, and

$$a_n = \frac{1}{3}\left(1 + 17\cdot 4^{n+1} - 3\cdot 2^{n+1} - 3^{n+3}\right). \tag{B.25}$$

Let $0 \leq i \leq 3$ and $k \geq 0$. Then

$$\begin{aligned}
&f_{4k+4+i} - f_i = \alpha_{k,i}h_1^5 + \beta_{k,i}h_2^2h_1 + 2(k+1)h_2h_3\\
&\text{where}\\
&\alpha_{k,i} = \frac{k+1}{3} + 4^{i+1}\frac{256^{k+1}-1}{45} - 3^{i+2}\frac{81^{k+1}-1}{80} - 2^{i+1}\frac{16^{k+1}-1}{15},\\
&\beta_{k,i} = 2^{i+1}\frac{16^{k+1}-1}{3} - 2(k+1).
\end{aligned} \tag{B.26}$$

The multiplicity of the sign representation in f_n is

$$\begin{cases} \alpha_{k,i}, & n = 4(k+1)+i \text{ and } k \geq 0,\\ 2, & n = 3,\\ 0, & n < 3. \end{cases}$$

Remark B.10 (The restriction of $\mathcal{S}_{\mathrm{OS}}(n+1,n-1)$ and $\mathcal{S}_{\mathrm{VG}}(n+1,n-1)$ to $\mathfrak{S}_{n-1}$)**.** Observe that in each of the cases $\mathrm{OS}(\mathrm{Br}_n)$, $1 \le n \le 5$, the restriction of the $\mathfrak{S}_n$-module to $\mathfrak{S}_{n-1}$ is always an h-positive permutation module. The restriction is not h-positive for $\mathcal{S}(n+1,n-1)$ when $n \ge 5$, although the following formula shows that it is a permutation module.

$$\begin{aligned}&\operatorname{ch}\mathcal{S}(n+1,n-1)\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\\&\quad=\Big(h_{n-2}h_1+h_{n-3}h_2+2h_{n-3}h_1^2+h_{n-4}h_1^3+h_{n-4}h_3\Big)\,\delta_{n\ge 4}+h_{n-5}\operatorname{ch}\mathbb{1}\uparrow^{\mathfrak{S}_4}_{I_2(4)}\delta_{n\ge 5}.\end{aligned}$$

Here $I_2(4)$ is the dihedral group of order 8.

Remark B.11 (The restriction of $\mathcal{S}_{\mathrm{OS}}(n+3,3)$ and $\mathcal{S}_{\mathrm{VG}}(n+3,3)$)**.** With the coefficients defined in Proposition B.6, the restriction of $\mathcal{S}_{OS}(n+3,3)$ to $\mathfrak{S}_3$ has Frobenius characteristic

$$(4a_n+b_n)h_1^3+2(b_n+c'_n)h_1h_2+d_nh_3,$$

and is thus h-positive. In particular $\mathcal{S}_{\mathrm{OS}}(n+3,3)\big\downarrow_{\mathfrak{S}_3}$ is a permutation module whose point stabilisers are Young subgroups.

Proposition B.8 shows that a similar statement holds for $\mathcal{S}_{\mathrm{VG}}(n+3,3)\big\downarrow_{\mathfrak{S}_3}$; here the orbit stabilisers are $\mathfrak{S}_1$ and $\mathfrak{S}_3$.

Remark B.12 (The restriction of $\mathcal{S}_{\mathrm{OS}}(n+4,4)$)**.** The restriction of $f_n = \operatorname{ch}\mathcal{S}_{\mathrm{OS}}(n+4,4)$ to $\mathfrak{S}_4$ is h-positive, supported on the set $\{h_1^4, h_1^2h_2, h_2^2\}$ if $n \equiv 1,3 \bmod 4$, the set $\{h_1^4, h_1^2h_2, h_2^2, h_4\}$ if $n \equiv 0 \bmod 4$, and finally the set $\{h_1^4, h_1^2h_2, h_2^2, h_3h_1, h_4\}$ if $n \equiv 2 \bmod 4$. In particular $\mathcal{S}_{\mathrm{OS}}(n+4,4)\big\downarrow_{\mathfrak{S}_4}$ is a permutation module whose point stabilisers are Young subgroups.

Remark B.13 (*The multiplicity of the trivial representation*)**.** Here we collect formulas for the multiplicity of the trivial representation:

For $\mathcal{S}_{\mathrm{OS}}(n+1,n-1)$, the multiplicity of the trivial representation is 3 for $n \ge 4$, and the multiplicity of the sign representation is 0 for $n \ne 3,4$, and 1 otherwise.

For $\mathcal{S}_{OS}(n+2,2)$, the multiplicity is

$$\frac{2^{n+1}}{6}+\frac{3+(-1)^n}{6}.$$

For $\mathcal{S}_{\mathrm{OS}}(n+3,3)$, the multiplicity of the trivial representation is

$$\frac{3^{n+1}}{16}+\frac{n}{4}+\frac{8+5(-1)^n}{16},$$

giving $\{1,1,3,6,17,47,139,412,\ldots\}$.

For $\mathcal{S}_{\mathrm{OS}}(n+4,4)$, (with definitions as in Proposition B.9), the multiplicity of the trivial representation is

$$\frac{k+1}{3}+4^{i+1}\frac{256^{k+1}-1}{45}-3^{i+2}\frac{81^{k+1}-1}{80}+2^{i+3}\frac{16^{k+1}-1}{5}+\langle f_i,h_5\rangle. \qquad \square$$

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References

[1] William W. Adams and Philippe Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics, vol. 3, American Mathematical Society, 1994.

[2] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman, *Schur functors and Schur complexes*, Adv. Math. **44** (1982), no. 3, 207–278.

[3] Ayah Almousa, *Stirling Representations*, 2024, version 1.0, BSD-3-Clause licence, `https://github.com/aalmousa/StirlingRepresentations.git`.

[4] Annetta Aramova, Jürgen Herzog, and Takayuki Hibi, *Gotzmann theorems for exterior algebras and combinatorics*, J. Algebra **191** (1997), no. 1, 174–211.

[5] Vladimir I. Arnol'd, *The cohomology ring of the group of dyed braids*, Mat. Zametki **5** (1969), 227–231.

[6] Luchezar L. Avramov, *Infinite free resolutions*, in Six lectures on commutative algebra, Modern Birkhäuser Classics, Birkhäuser, 2010, pp. 1–118.

[7] Joergen Backelin, *Low degrees in a Gröbner basis may force the Koszul property*, 2024, `https://staff.math.su.se/joeb/papers/list.shtml`.

[8] Eli Bagno, David Garber, and Takao Komatsu, *A q,r-analogue for the Stirling numbers of the second kind of Coxeter groups of type B*, PU.M.A., Pure Math. Appl. **30** (2022), no. 1, 8–16.

[9] Şerban Bărcănescu and Nicolae Manolache, *Nombres de Betti d'une singularité de Segre–Veronese*, C. R. Acad. Sci., Paris, Sér. A **288** (1979), no. 4, A237–A239.

[10] ———, *Betti numbers of Segre–Veronese singularities*, Rev. Roum. Math. Pures Appl. **26** (1981), no. 4, 549–565.

[11] Hélène Marie-Louise Barcelo, *On the action of the symmetric group on the Free Lie Algebra and on the homology and cohomology of the partition lattice*, Ph.D. thesis, University of California, San Diego, USA, 1988, `https://www.proquest.com/docview/303690441`.

[12] Alexander Berglund, *Koszul spaces*, Trans. Am. Math. Soc. **366** (2014), no. 9, 4551–4569.

[13] Alexander Berglund and Kaj Börjeson, *Free loop space homology of highly connected manifolds*, Forum Math. **29** (2017), no. 1, 201–228.

[14] Anders Björner, Paul H. Edelman, and Günter M. Ziegler, *Hyperplane arrangements with a lattice of regions*, Discrete Comput. Geom. **5** (1990), no. 3, 263–288.

[15] Anders Björner et al., *The homology and shellability of matroids and geometric lattices*, in Matroid applications, Encyclopedia of Mathematics and Its Applications, vol. 40, Cambridge University Press, 1992, pp. 226–283.

[16] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, second ed., Encyclopedia of Mathematics and Its Applications, vol. 46, Cambridge University Press, 1999.

[17] Anders Björner and Günter M. Ziegler, *Broken circuit complexes: factorizations and generalizations*, J. Comb. Theory, Ser. B **51** (1991), no. 1, 96–126.

[18] Leonid A. Bokut and Yuqun Chen, *Gröbner–Shirshov bases and their calculation*, Bull. Math. Sci. **4** (2014), no. 3, 325–395.

[19] Sarah Brauner, *Eulerian representations for real reflection groups*, J. Lond. Math. Soc. (2) **105** (2022), no. 1, 412–444.

[20] Francesco Brenti, *Unimodal, log-concave and Pólya frequency sequences in combinatorics*, Memoirs of the American Mathematical Society, vol. 413, American Mathematical Society, 1989.

[21] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas, *The stability of the Kronecker product of Schur functions*, J. Algebra **331** (2011), 11–27.

[22] Winfried Bruns, Jürgen Herzog, and Udo Vetter, *Syzygies and walks*, in Commutative algebra (Trieste, 1992), World Scientific, 1994, pp. 36–57.

[23] Pierre Cartier and Frédéric Patras, *Classical Hopf algebras and their applications*, Algebra and Applications, vol. 29, Springer, 2021.

[24] Thomas Church, Jordan S. Ellenberg, and Benson Farb, *Fi-modules and stability for representations of symmetric groups*, Duke Math. J. **164** (2015), no. 9, 1833–1910.

[25] Thomas Church and Benson Farb, *Representation theory and homological stability*, Adv. Math. **245** (2013), 250–314.

[26] Frederick R. Cohen and Samuel Gitler, *On loop spaces of configuration spaces*, Trans. Am. Math. Soc. **354** (2002), no. 5, 1705–1748.

[27] Frederick R. Cohen, Thomas J. Lada, and J. Peter May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, vol. 533, Springer, 1976.

[28] Raul Cordovil, *A commutative algebra for oriented matroids*, Discrete Comput. Geom. **27** (2002), 73–84.

[29] David A. Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms. an introduction to computational algebraic geometry and commutative algebra*, fourth ed., Undergraduate Texts in Mathematics, Springer, 2015.

[30] Mark de Longueville and Carsten A. Schultz, *The cohomology rings of complements of subspace arrangements*, Math. Ann. **319** (2001), no. 4, 625–646.

[31] Graham Denham and Alexander I. Suciu, *On the homotopy Lie algebra of an arrangement*, Mich. Math. J. **54** (2006), no. 2, 319–340.

[32] Alexandru Dimca, *Hyperplane arrangements. An introduction*, Universitext, Springer, 2017.

[33] Alexandru Dimca and Sergey Yuzvinsky, *Lectures on Orlik–Solomon algebras*, in Arrangements, local systems and singularities, Progress in Mathematics, vol. 283, Birkhäuser, 2010, pp. 83–110.

[34] Galen Dorpalen-Barry, *The Varchenko–Gel'fand ring of a cone*, J. Algebra **617** (2023), 500–521.

[35] Galen Dorpalen-Barry, Nicholas Proudfoot, and Jidong Wang, *Equivariant cohomology and conditional oriented matroids*, 2022, `https://arxiv.org/abs/2208.04855`.

[36] Thomas A. Dowling, *A class of geometric lattices based on finite groups*, J. Comb. Theory, Ser. B **14** (1973), 61–86.

[37] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer, 1995, With a view toward algebraic geometry.

[38] David Eisenbud, Sorin Popescu, and Sergey Yuzvinsky, *Hyperplane arrangement cohomology and monomials in the exterior algebra*, Trans. Am. Math. Soc. **355** (2003), no. 11, 4365–4383.

[39] David Eisenbud, Alyson Reeves, and Burt Totaro, *Initial ideals, Veronese subrings, and rates of algebras*, Adv. Math. **109** (1994), no. 2, 168–187.

[40] Eleonore Faber, Martina Juhnke-Kubitzke, Haydee Lindo, Claudia Miller, Rebecca Rebhuhn-Glanz, and Alexandra Seceleanu, *Canonical resolutions over Koszul algebras*, in Women in commutative algebra, Association for Women in Mathematics Series, vol. 29, Springer, 2022, pp. 281–301.

[41] Michael Falk and Richard Randell, *The lower central series of a fiber-type arrangement*, Invent. Math. **82** (1985), no. 1, 77–88.

[42] Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer, 2001.

[43] Benoit Fresse, *Homotopy of operads and Grothendieck–Teichmüller groups. Part 1*, Mathematical Surveys and Monographs, vol. 217, American Mathematical Society, 2017, The algebraic theory and its topological background.

[44] Ralf Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), no. 1, 29–39.

[45] ———, *Koszul algebras*, in Advances in commutative ring theory (Fez, 1997), Lecture Notes in Pure and Applied Mathematics, vol. 205, Marcel Dekker, 1999, pp. 337–350.

[46] Ladnor Geissinger and D. Kinch, *Representations of the hyperoctahedral groups*, J. Algebra **53** (1978), no. 1, 1–20.

[47] Tor H. Gulliksen and Gerson Levin, *Homology of local rings*, Queen's Papers in Pure and Applied Mathematics, vol. 20, Queen's University, Kingston, Ontario, 1969.

[48] Patricia Hersh and Victor Reiner, *Representation stability for cohomology of configuration spaces in* $\mathbb{R}^d$, Int. Math. Res. Not. **2017** (2017), no. 5, 1433–1486, With an appendix written jointly with Steven Sam.

[49] Torsten Hoge and Gerhard Röhrle, *Supersolvable reflection arrangements*, Proc. Am. Math. Soc. **142** (2014), no. 11, 3787–3799.

[50] Michael Jöllenbeck and Volkmar Welker, *Minimal resolutions via algebraic discrete Morse theory*, Memoirs of the American Mathematical Society, vol. 923, American Mathematical Society, 2009.

[51] Matthieu Josuat-Vergès and Philippe Nadeau, *Koszulity of dual braid monoid algebras via cluster complexes*, 2021, `https://arxiv.org/abs/2107.13442`.

[52] Aleksandr A. Klyachko, *Lie elements in a tensor algebra*, Sib. Mat. Zh. **15** (1974), 1296–1304.

[53] Toshitake Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, Nagoya Math. J. **92** (1983), 21–37.

[54] ———, *Série de Poincaré–Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985), no. 1, 57–75.

[55] Gustav I. Lehrer and Louis Solomon, *On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes*, J. Algebra **104** (1986), no. 2, 410–424.

[56] Clas Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, in Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Mathematics, vol. 1183, Springer, 1986, pp. 291–338.

[57] ———, *The holonomy Lie algebra of a matroid*, 2020, `https://arxiv.org/abs/2012.12044`.

[58] Ian G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, Clarendon Press; Oxford University Press, 1995, with contributions by A. Zelevinsky, Oxford Science Publications.

[59] Toufik Mansour and Augustine O. Munagi, *Set partitions with circular successions*, Eur. J. Comb. **42** (2014), 207–216.

[60] Jacob P. Matherne, Dane Miyata, Nicholas Proudfoot, and Eric Ramos, *Equivariant log concavity and representation stability*, 2021, `https://arxiv.org/abs/2104.00715`.

[61] Volodymyr Mazorchuk, Serge Ovsienko, and Catharina Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Am. Math. Soc. **361** (2009), no. 3, 1129–1172.

[62] Jason McCullough and Irena Peeva, *Infinite graded free resolutions*, in Commutative algebra and noncommutative algebraic geometry. Volume I. Expository articles, Mathematical Sciences Research Institute Publications, vol. 67, Cambridge University Press, 2015, pp. 215–257.

[63] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. Math. (2) **81** (1965), 211–264.

[64] Teo Mora, *An introduction to commutative and noncommutative Gröbner bases*, Theor. Comput. Sci. **134** (1994), no. 1, 131–173.

[65] Daniel Moseley, *Equivariant cohomology and the Varchenko-Gelfand filtration*, J. Algebra **472** (2017), 95–114.

[66] Satoshi Murai, *Generic initial ideals and exterior algebraic shifting of the join of simplicial complexes*, Ark. Mat. **45** (2007), no. 2, 327–336.

[67] Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189.

[68] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer, 1992.

[69] James G. Oxley, *Matroid theory*, Oxford Graduate Texts in Mathematics, vol. 3, Oxford Science Publications; Oxford University Press, 1992.

[70] Irena Peeva, *Hyperplane arrangements and linear strands in resolutions*, Trans. Am. Math. Soc. **355** (2003), no. 2, 609–618.

[71] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, 2005.

[72] Stewart B. Priddy, *Koszul resolutions*, Trans. Am. Math. Soc. **152** (1970), 39–60.

[73] Andrew Putman and Steven V. Sam, *Representation stability and finite linear groups*, Duke Math. J. **166** (2017), no. 13, 2521–2598.

[74] Leonard E. Ross, *Representations of graded Lie algebras*, Trans. Am. Math. Soc. **120** (1965), 17–23.

[75] Manfred Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics, vol. 716, Springer, 1979, An introduction.

[76] Manfred Schocker, *Multiplicities of higher Lie characters*, J. Aust. Math. Soc. **75** (2003), no. 1, 9–21.

[77] Brad Shelton and Sergey Yuzvinsky, *Koszul algebras from graphs and hyperplane arrangements*, J. Lond. Math. Soc. **56** (1997), no. 3, 477–490.

[78] Geoffrey C. Shephard and John A. Todd, *Finite unitary reflection groups*, Can. J. Math. **6** (1954), 274–304.

[79] Anne V. Shepler and Sarah Witherspoon, *Poincaré–Birkhoff–Witt theorems*, in Commutative algebra and noncommutative algebraic geometry. Vol. I, Mathematical Sciences Research Institute Publications, vol. 67, Cambridge University Press, 2015, pp. 259–290.

[80] Richard P. Stanley, *Modular elements of geometric lattices*, Algebra Univers. **1** (1971), 214–217.

[81] ———, *Supersolvable lattices*, Algebra Univers. **2** (1972), no. 1, 197–217.

[82] ———, *Invariants of finite groups and their applications to combinatorics*, Bull. Am. Math. Soc. **1** (1979), no. 3, 475–511.
[83] ———, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, 1999, with a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
[84] Dennis Stanton and Dennis White, *Constructive combinatorics*, Undergraduate Texts in Mathematics, Springer, 1986.
[85] Timothy Stokes, *Gröbner bases in exterior algebra*, J. Autom. Reasoning **6** (1990), no. 3, 233–250.
[86] Alexander I. Suciu and He Wang, *Formality properties of finitely generated groups and Lie algebras*, Forum Math. **31** (2019), no. 4, 867–905.
[87] Sheila Sundaram, *The homology representations of the symmetric group on Cohen–Macaulay subposets of the partition lattice*, Adv. Math. **104** (1994), no. 2, 225–296.
[88] ———, *On a curious variant of the S_n-module* Lie_n, Algebr. Comb. **3** (2020), no. 4, 985–1009.
[89] ———, *The reflection representation in the homology of subword order*, Algebr. Comb. **4** (2021), no. 5, 879–907.
[90] Sheila Sundaram and Volkmar Welker, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Am. Math. Soc. **349** (1997), no. 4, 1389–1420.
[91] Victor A. Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, in Algebra, VI, Encyclopaedia of Mathematical Sciences, vol. 57, Springer, 1995, pp. 1–196.
[92] Alexander N. Varchenko and Israil' M. Gel'fand, *Heaviside functions of a configuration of hyperplanes*, Funkts. Anal. Prilozh. **21** (1987), no. 4, 1–18.
[93] Peter Webb, *A course in finite group representation theory*, Cambridge Studies in Advanced Mathematics, vol. 161, Cambridge University Press, 2016.
[94] Sergey Yuzvinsky, *Orlik–Solomon algebras in algebra and topology*, Usp. Mat. Nauk **56** (2001), no. 2(338), 87–166.

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