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The Hodge structure on the singularity category of a complex hypersurface

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Given a complex affine hypersurface with isolated singularity determined by a homogeneous polynomial, we identify the noncommutative Hodge structure on the periodic cyclic homology of its singularity category with the classical Hodge structure on the primitive cohomology of the associated projective hypersurface. As a consequence, we show that the Hodge conjecture for the projective hypersurface is equivalent to a dg-categorical analogue of the Hodge conjecture for the singularity category.

1. Introduction

Katzarkov, Kontsevich and Pantev conjecture in [Katzarkov et al. 2008] that the periodic cyclic homology of any smooth and proper $\mathbb{C}$-linear differential $\mathbb{Z}$-graded category $\mathcal{C}$ may be equipped with a "noncommutative (nc) Hodge structure", generalizing the pure Hodge structure on the cohomology of a smooth and proper complex variety. More precisely, the proposed nc Hodge structure on the 0-th periodic cyclic homology of $\mathcal{C}$, denoted $HP_0(\mathcal{C})$, is given by analogues of the Hodge filtration and rational structure on the cohomology of a smooth and proper complex variety: the former is the filtration of $HP_0(\mathcal{C})$ arising from the negative cyclic homology of $\mathcal{C}$, and the latter is the image of the rationalized topological Chern character map $K_0^{\text{top}}(\mathcal{C})_{\mathbb{Q}} \to HP_0(\mathcal{C})$. The statement of the classical Hodge conjecture generalizes to dg-categories equipped with an nc Hodge structure: we say such a dg-category $\mathcal{C}$ satisfies the "nc Hodge condition" provided the image of the rationalized algebraic Chern character map $K_0^{\text{alg}}(\mathcal{C})_{\mathbb{Q}} \to HP_0(\mathcal{C})$ coincides with the space of Hodge classes. We refer the reader to Section 2.2 for more details (see also [Perry 2022, Section 5.2]). It is known that the Hodge conjecture holds

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for a smooth projective complex variety Y if and only if the Hodge condition holds for the dg-enhancement of its derived category $\mathrm{D}^{\mathrm{b}}(Y)$; see Example 2.16 or [Lin 2023].

The past two decades have seen a flurry of work focused on developing the Hodge theory of singularity categories of hypersurfaces (i.e., matrix factorization categories); see, e.g., [Brown and Dyckerhoff 2020; Ballard et al. 2014a; 2014b; Beraldo and Pippi 2025; Blanc et al. 2018; Brown and Walker 2020a; 2020b; 2022; Căldăraru and Tu 2013; Dyckerhoff 2011; Efimov 2018; Halpern-Leistner and Pomerleano 2020; Kim and Polishchuk 2022; Kim and Kim 2024; Pippi 2022; Polishchuk and Vaintrob 2012; Segal 2013; Shklyarov 2014; 2016]. In this paper, we show that the singularity category of a complex hypersurface with isolated singularity determined by a homogeneous polynomial may be equipped with an nc Hodge structure, and we describe it in terms of invariants arising in classical Hodge theory. More specifically, our main goal is to prove Theorem 1.2 below (see Theorem 2.23 for a more precise statement). Before stating it, we fix some notation that will be used throughout the paper:

Notation 1.1. Let $f \in \mathbb{C}[x_0, \dots, x_{n+1}]$ be a nonzero homogeneous polynomial, R the associated affine hypersurface $\mathbb{C}[x_0, \dots, x_{n+1}]/(f)$ of dimension $n+1$, and $\mathfrak{m} := (x_0, \dots, x_{n+1})$ its homogeneous maximal ideal. Set $X := \mathrm{Proj}(R) \subseteq \mathbb{P}^{n+1}$, a projective hypersurface of dimension n. We assume n is even; otherwise, X has no interesting Hodge theory. We assume also that R has an isolated singularity, i.e., that X is smooth. Let $\mathrm{D}^{\mathrm{sg}}(R)$ denote a dg-enhancement of the singularity category of R (we specify in Section 2 which dg-enhancement we use). We write $H^n_{\mathrm{prim}}(X)$ for the n-th primitive cohomology of X with rational coefficients. That is, for $n \geq 2$,

$$H^n_{\mathrm{prim}}(X) = \ker\big(H^n(X;\mathbb{Q}) \xrightarrow{L} H^{n+2}(X;\mathbb{Q})\big),$$

where L is the Lefschetz operator; and for $n = 0$, $H^0_{\mathrm{prim}}(X) := \widetilde{H}^0(X;\mathbb{Q})$, the 0-th reduced rational cohomology group of X. Since X is a smooth hypersurface, $H^n_{\mathrm{prim}}(X)$ may be identified with $\mathrm{coker}(H^n(\mathbb{P}^{n+1};\mathbb{Q}) \to H^n(X;\mathbb{Q}))$; see Example 2.5 for more details.

Theorem 1.2. *There is an isomorphism*

$$HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \cong H^n_{\mathrm{prim}}(X;\mathbb{C})$$

that identifies the noncommutative Hodge structure associated to $\mathrm{D}^{\mathrm{sg}}(R)$ *with the pure Hodge structure on* $H^n_{\mathrm{prim}}(X)$*. This isomorphism is compatible with the Chern character maps from topological K-theory, and hence the classical Hodge conjecture holds for* X *if and only if* $\mathrm{D}^{\mathrm{sg}}(R)$ *satisfies the nc Hodge condition* (Definition 2.15).

We note that it is a consequence of [Orlov 2009, Theorem 2.5] that the Hodge conjecture for X is also equivalent to the nc Hodge condition for the *graded* singularity category of R; see Remark 2.26 for more details.

The dg-category $D^{sg}(R)$ is smooth over $\mathbb{C}$; this follows by combining [Lunts 2010, Theorem 6.3] and [Keller 2011, Proposition 3.10(c)]. However, it is not proper as a differential $\mathbb{Z}$-graded category. By results of Buchweitz and Eisenbud, $D^{sg}(R)$ is quasi-equivalent to the category $\mathrm{mf}(f)$ of matrix factorizations of f, and so it may be equipped canonically with the structure of a proper differential $\mathbb{Z}/2$-graded category (see Section 3.1); but in this theorem, $HP_0(D^{sg}(R))$ refers to the periodic cyclic homology of $D^{sg}(R)$ viewed as a $\mathbb{Z}$-graded category. Nevertheless, as we prove below, the Hochschild invariants of $D^{sg}(R)$ have the necessary features (see Properties 2.8) to make sense of an nc Hodge structure on $HP_0(D^{sg}(R))$ and to formulate an nc Hodge condition.

The Hodge conjecture is known to hold for a projective hypersurface X in the following cases [Shioda 1983, §2]:

(1) $\dim(X)$ odd, trivially.

(2) $\dim(X) = 2$ (by the Lefschetz 1-1 theorem).

(3) $\dim(X) = 4$ and $\deg(X) \leq 5$ [Zucker 1977; Murre 1977; Conte and Murre 1978].

(4) X a Fermat hypersurface, under certain arithmetic conditions on the dimension and degree of X [Ran 1980; Shioda 1979].

We therefore conclude that $D^{sg}(R)$ satisfies the nc Hodge condition in all of the above cases. In Example 6.4, we explicitly compute the Hodge classes for $D^{sg}(R)$ when X is the 2-dimensional Fermat hypersurface of degree 3.

As mentioned above, since R is a hypersurface, we may replace $D^{sg}(R)$ by the quasi-equivalent dg-category $\mathrm{mf}(f)$ of matrix factorizations of f; see Section 3.1 for the definition. Thus, our main theorem may be recast as an isomorphism

$$HP_0(\mathrm{mf}(f)) \cong H^n_{\mathrm{prim}}(X; \mathbb{C}) \tag{1.3}$$

that preserves Hodge structures and is compatible with Chern character maps.

Let us give an overview of the paper. We collect in Section 2 the necessary background and terminology in order to state our main result precisely; see Theorem 2.23. This includes constructing an explicit map from $H^n_{\mathrm{prim}}(X)$ to $HP_0(D^{sg}(R)) \cong HP_0(\mathrm{mf}(f))$; see (2.22). In Section 3, we recall (and extend) several results necessary for the proof of Theorem 1.2. More precisely, we describe the quasi-equivalence relating $D^{sg}(R)$ and $\mathrm{mf}(f)$, and we establish important details regarding "de Rham models" for the Hochschild, negative cyclic, and periodic homology complex of $\mathrm{mf}(f)$ and related dg-categories. We also recall from [Brown and Walker 2024] an explicit description of the boundary map in a certain dévissage long exact sequence; this map plays a crucial role in relating the nc Hodge filtration on $HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \cong HP_0(\mathrm{mf}(f))$ with the classical Hodge filtration on $H^n_{\mathrm{prim}}(X)$.

In Section 4, we establish the following analogue of the Wang exact sequence of a fibration over a circle (see Theorem 4.1 below for a more precise statement, and see Remark 4.2 for an explanation of how this result relates to the Wang exact sequence):

Theorem 1.4. *Let k be a field of characteristic 0, Q a smooth k-algebra, and $f \in Q$ a non-zero-divisor. There is a distinguished triangle*

$$HN(\mathrm{mf}(f)) \to HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \to HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \to$$

of complexes of k-vector spaces, where $HN(\mathrm{mf}(f))$ (resp. $HN^{\mathbb{Z}/2}(\mathrm{mf}(f))$) denotes the negative cyclic complex of $\mathrm{mf}(f)$ considered as a differential $\mathbb{Z}$-graded (resp. $\mathbb{Z}/2$-graded) category.

The map $HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \to HN^{\mathbb{Z}/2}(\mathrm{mf}(f))$ in Theorem 1.4 is an analogue of the endomorphism $T - \mathrm{id}$ on the cohomology of the Milnor fiber, where T is induced by monodromy (see Remark 4.2). Theorem 1.4 thus closely resembles a result of Blanc, Robalo, Toën and Vezzosi [Blanc et al. 2018, Main Theorem] concerning the ℓ-adic realization of singularity categories.

Section 5 contains the proof of Theorem 1.2. A summary of the content of Sections 2–4 that is necessary for the proof of Theorem 1.2 is provided in Theorem 5.1; readers who are already familiar with noncommutative Hodge theory and singularity categories may wish to skip directly to Theorem 5.1 and refer back to Sections 2–4 as needed. The most technical aspect of the proof of our main result is verifying that the isomorphism $HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \cong H^n_{\mathrm{prim}}(X)$ identifies the nc Hodge filtration on $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ with the classical Hodge filtration on $H^n_{\mathrm{prim}}(X)$. Our approach is to identify both with an intermediate object: the "polar filtration" on $H^n(U)$, where U is the complement of X in $\mathbb{P}^{n+1}$. In Section 6, we discuss some examples in the setting of Fermat hypersurfaces, applying [Shioda 1979].

2. Background

2.1. *Hodge structures.* The following definition is nonstandard, but it will be useful in this paper.

Definition 2.1. A *pre-Hodge structure* is a pair $V = (V_{\mathbb{Q}}, F^\bullet V_{\mathbb{C}})$, where $V_{\mathbb{Q}}$ is a finite-dimensional $\mathbb{Q}$-vector space, and $F^\bullet V_{\mathbb{C}}$ is a filtration of $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ that is decreasing, complete, and exhaustive: $F^p V_{\mathbb{C}} \subseteq F^{p-1} V_{\mathbb{C}}$ for all p, $F^p V_{\mathbb{C}} = 0$ for $p \gg 0$, and $F^p V_{\mathbb{C}} = V_{\mathbb{C}}$ for $p \ll 0$. A *morphism of pre-Hodge structures* $V \to V'$ is a $\mathbb{Q}$-linear map $V_{\mathbb{Q}} \to V'_{\mathbb{Q}}$ whose complexification respects the filtrations.

Remark 2.2. An isomorphism of pre-Hodge structures V and V' is determined by an isomorphism $\alpha : V_{\mathbb{C}} \xrightarrow{\cong} V'_{\mathbb{C}}$ such that $\alpha(F^p V_{\mathbb{C}}) = F^p V'_{\mathbb{C}}$ for all p, and $\alpha(V_{\mathbb{Q}}) = V'_{\mathbb{Q}}$.

The notion of a pre-Hodge structure is a weakening of the classical notion of a pure Hodge structure, whose definition we now recall:

Definition 2.3. Let $n \in \mathbb{Z}$. A *pure Hodge structure of weight* n is a pre-Hodge structure V with the property that, for all p, q with $p + q = n + 1$, we have $F^p V_{\mathbb{C}} \oplus \overline{F^q V_{\mathbb{C}}} = V_{\mathbb{C}}$, where the overline denotes complex conjugation. A *morphism* of pure Hodge structures is a morphism of the underlying pre-Hodge structures.

Given a pre-Hodge structure V and $m \in \mathbb{Z}$, we write $V(m)$ for its *m-th Tate twist*, which is defined by setting $V(m)_{\mathbb{Q}} = V_{\mathbb{Q}}$, $V(m)_{\mathbb{C}} = V_{\mathbb{C}}$ and $F^p V(m)_{\mathbb{C}} = F^{p+m} V_{\mathbb{C}}$. If V is pure of weight n, then $V(m)$ is pure of weight $n - 2m$.

Example 2.4. Let X be a smooth, proper complex variety, and let $V_{\mathbb{Q}} = H^j(X; \mathbb{Q})$, the singular cohomology of X with rational coefficients. Equip $V_{\mathbb{C}} = H^j(X; \mathbb{C})$ with the filtration given by

$$F^p V_{\mathbb{C}} := \operatorname{im}\bigl(H^j(X, \tau^{\geq p} \Omega^{\bullet}_{X/\mathbb{C}}) \to H^j_{\mathrm{dR}}(X; \mathbb{C}) \cong H^j(X; \mathbb{C})\bigr),$$

where $\tau^{\geq p} \Omega^{\bullet}_{X/\mathbb{C}}$ denotes the brutal truncation of the de Rham complex in cohomological degrees $\geq p$, and $H^j_{\mathrm{dR}}(X; \mathbb{C})$ denotes the j-th hypercohomology of $\Omega^{\bullet}_{X/\mathbb{C}}$. It is a classical result that $(V_{\mathbb{Q}}, F^{\bullet} V_{\mathbb{C}})$ is a pure Hodge structure of weight j. The m-th Tate twist of this Hodge structure is written as $H^j(X; \mathbb{Q}(m))$.

Example 2.5. Let X be a smooth, projective complete intersection of codimension c in $\mathbb{P}^{n+c}$. That is, $X = \operatorname{Proj}(R)$, where $R = \mathbb{C}[x_0, \dots, x_{n+c}]/(f_1, \dots, f_c)$ for a regular sequence of homogeneous polynomials $f_1, \dots, f_c$ such that R has an isolated singularity. Assume also that n is even. As with any smooth, projective variety, the primitive cohomology of X may be equipped with a pure Hodge structure; let us recall the definition of primitive cohomology. In the case where $n = 0$ (so that X is a collection of points), we define $H^0_{\mathrm{prim}}(X) = \widetilde{H}^0(X; \mathbb{Q})$. Assume $n \geq 2$. We let

$$L : H^*(X; \mathbb{Q}) \to H^{*+2}(X; \mathbb{Q}(1))$$

denote the *Lefschetz operator*, i.e., the map given by multiplication by the class in $H^2(X; \mathbb{Q}(1))$ of a generic hyperplane section of X. It is a morphism of pure Hodge structures. For $0 \leq j \leq n$, the j-th *primitive cohomology* of X is

$$H^j_{\mathrm{prim}}(X) := \ker\bigl(L^{n+1-j} : H^j(X; \mathbb{Q}) \to H^{2n+2-j}(X; \mathbb{Q}(n+1-j))\bigr).$$

As L is a morphism of pure Hodge structures, $H^j_{\mathrm{prim}}(X)$ acquires a pure Hodge structure, and it is pure of weight j.

Since we assume that X is a smooth complete intersection of even dimension, the hard Lefschetz theorem gives

$$H^j(X;\mathbb{Q}) = \begin{cases} 0, & j \text{ odd}; \\ L^{j/2}\cdot H^0(X;\mathbb{Q}(-j/2)), & j \text{ even}, \ j \neq n. \end{cases}$$

In other words, the map $i^* : H^j(\mathbb{P}^{n+c};\mathbb{Q}) \to H^j(X;\mathbb{Q})$ induced by the canonical embedding $i : X \hookrightarrow \mathbb{P}^{n+c}$ is an isomorphism for all $j \neq n$; in particular, $H^j_{\text{prim}}(X) = 0$ unless $j = n$. So, the only "interesting" cohomology lies in degree n, and in that degree we have a canonical decomposition of Hodge structures

$$H^n(X;\mathbb{Q}) = H^n_{\text{prim}}(X;\mathbb{Q}) \oplus L^{n/2}\cdot H^0(X;\mathbb{Q}(-n/2)).$$

Since the summand $L^{n/2}\cdot H^0(X;\mathbb{Q}(-n/2))$ equals the image of the map

$$L^{n/2}\cdot H^0\big(\mathbb{P}^{n+c};\mathbb{Q}(-n/2) = H^n(\mathbb{P}^{n+c};\mathbb{Q}) \xrightarrow{i^*} H^n(X;\mathbb{Q})\big),$$

we have a canonical isomorphism

$$H^n_{\text{prim}}(X;\mathbb{Q}) \cong \operatorname{coker}(H^n(\mathbb{P}^{n+c};\mathbb{Q}) \xrightarrow{i^*} H^n(X;\mathbb{Q}))$$

of pure Hodge structures.

2.2. *Hodge structures associated to dg-categories.* Let $\mathcal{C}$ be a $\mathbb{C}$-linear differential $\mathbb{Z}$-graded category (or dg-category, for short). As discussed in the introduction, it follows from [Katzarkov et al. 2008] that one may associate a pre-Hodge structure to $\mathcal{C}$ whenever $\mathcal{C}$ enjoys certain properties that resemble features of the bounded derived category of a smooth, proper complex variety. Let us now explain this in detail.

We write $HH_*(\mathcal{C})$, $HN_*(\mathcal{C})$, and $HP_*(\mathcal{C})$ for the Hochschild, negative cyclic, and periodic cyclic homology of $\mathcal{C}$; we refer the reader to, e.g., [Brown and Walker 2020a, Section 3] for the definitions of these invariants. We recall that $HN_*(\mathcal{C})$ is a $\mathbb{C}[u]$-module with u an indeterminate of homological degree -2, determined by the identification $HN_*(\mathbb{C}) = \mathbb{C}[u]$, and $HP_*(\mathcal{C}) = HN_*(\mathcal{C}) \otimes_{\mathbb{C}[u]} \mathbb{C}[u,u^{-1}]$. There is a notion of topological K-theory for dg-categories, developed by Blanc [2016, Definition 4.13]; let $K^{\text{top}}_*(\mathcal{C})$ denote the topological K-theory groups of $\mathcal{C}$, and set $K^{\text{top}}_*(\mathcal{C})_{\mathbb{Q}} := K^{\text{top}}_*(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Topological K-theory and periodic cyclic homology are related via the *topological Chern character map* $\text{ch}^{\text{top}} : K^{\text{top}}_*(\mathcal{C})_{\mathbb{Q}} \to HP_*(\mathcal{C})$ [Blanc 2016, Section 4.4].

Notation 2.6. Given a noetherian $\mathbb{C}$-scheme Y with enough locally free sheaves, let $\mathrm{D}^{\mathrm{b}}(Y)$ and $\operatorname{Perf}(Y)$ denote dg-enhancements of the bounded derived category of Y and category of perfect complexes on Y, respectively; all such dg-enhancements are unique up to a sequence of quasi-equivalences [Lunts and Orlov 2010]. We

write
$$K_*^{\mathrm{top}}(Y) := K_*^{\mathrm{top}}(\mathrm{Perf}(Y)), \quad HP_*(Y) := HP_*(\mathrm{Perf}(Y)),$$
$$HN_*(Y) := HN_*(\mathrm{Perf}(Y)), \quad HH_*(Y) := HH_*(\mathrm{Perf}(Y)).$$

We adopt the widely used notation $G_*(Y)$ and $K_*(Y)$ for the (nonconnective) algebraic K-theory groups of $\mathrm{D}^{\mathrm{b}}(Y)$ and $\mathrm{Perf}(Y)$; see [Schlichting 2011, Section 3.2.32] for background on algebraic K-theory of dg-categories. We also write
$$G_*^{\mathrm{top}}(Y) := K_*^{\mathrm{top}}(\mathrm{D}^{\mathrm{b}}(Y)), \quad HP_*^{\mathrm{BM}}(Y) := HP_*(\mathrm{D}^{\mathrm{b}}(Y)),$$
$$HN_*^{\mathrm{BM}}(Y) := HN_*(\mathrm{D}^{\mathrm{b}}(Y)), \quad HH_*^{\mathrm{BM}}(Y) := HH_*(\mathrm{D}^{\mathrm{b}}(Y));$$
here, "BM" stands for "Borel–Moore". Given a commutative ring A, we write $G_*(A) := G_*(\mathrm{Spec}(A))$, and similarly for the other invariants discussed here.

We recall the notions of smoothness and properness for dg-categories:

Definition 2.7. The dg-category $\mathcal{C}$ is *smooth* if $\mathcal{C}$ is perfect as a $\mathcal{C}$-$\mathcal{C}$-bimodule, and it is *proper* if $\dim_{\mathbb{C}} H^*\mathrm{Hom}_{\mathcal{C}}(C, C') < \infty$ for all objects C, C' of $\mathcal{C}$.

When X is a separated scheme of finite type over $\mathbb{C}$, it follows from (the proof of) [Orlov 2016, Proposition 3.31] that X is smooth (resp. proper) if and only if the dg-category $\mathrm{Perf}(X)$ of perfect complexes of $\mathcal{O}_X$-modules is smooth (resp. proper).

In this paper, the dg-categories we consider will not always be smooth and proper. We will be interested in dg-categories that satisfy the following conditions, which are exactly what one needs to equip a dg-category with a pre-Hodge structure.

Properties 2.8. For a dg-category $\mathcal{C}$, we consider the following properties:

(1) $\dim_{\mathbb{C}} HP_0(\mathcal{C}) < \infty$.

(2) The filtration of $HP_0(\mathcal{C})$ given by
$$F_{\mathrm{nc}}^p HP_0(\mathcal{C}) = \mathrm{im}\big(HN_{2p}(\mathcal{C}) \xrightarrow{\mathrm{can}} HP_{2p}(\mathcal{C}) \xrightarrow{u^p} HP_0(\mathcal{C})\big)$$
satisfies $F_{\mathrm{nc}}^p HP_0(\mathcal{C}) = 0$ for $p \gg 0$ and $F_{\mathrm{nc}}^p HP_0(\mathcal{C}) = HP_0(\mathcal{C})$ for $p \ll 0$.

(3) $\mathrm{im}(\mathrm{ch}^{\mathrm{top}}) \otimes_{\mathbb{Q}} \mathbb{C} = HP_0(\mathcal{C})$.

Proposition 2.9. *Properties* (1) *and* (2) *hold for any dg-category that is smooth and proper over* $\mathbb{C}$.

Proof. By [Kontsevich and Soibelman 2009, Proposition 8.2.3], we have
$$\dim_{\mathbb{C}} HH_*(\mathcal{C}) < \infty.$$
We also have noncanonical isomorphisms
$$HN_*(\mathcal{C}) \cong HH_*(\mathcal{C})[u] \quad \text{and} \quad HP_*(\mathcal{C}) \cong HH_*(\mathcal{C})[u, u^{-1}];$$
these follow from Kaledin's noncommutative Hodge-to-de Rham degeneration theorem [Kaledin 2017]. Properties (1) and (2) follow immediately. □

Property (3) is conjectured by Blanc to hold for any smooth and proper dg-category:

Conjecture 2.10 (the lattice conjecture [Blanc 2016, Conjecture 1.7]). *If $\mathcal{C}$ is smooth and proper over $\mathbb{C}$, then the map $K_*^{\mathrm{top}}(\mathcal{C})_{\mathbb{C}} \to HP_*(\mathcal{C})$ induced by $\mathrm{ch}^{\mathrm{top}}$ is an isomorphism.*

In fact, the lattice conjecture is known to hold for many dg-categories that are not smooth or proper: we refer the reader to [Brown and Sridhar 2025, Section 5] for a list of known cases of the lattice conjecture.

Definition 2.11. Assume the dg-category $\mathcal{C}$ satisfies Properties 2.8. The *nc pre-Hodge structure* for $\mathcal{C}$, written $\mathrm{pHS}(\mathcal{C})$, is the pair $(V_{\mathbb{Q}}, F^\bullet V_{\mathbb{C}})$, where

$$V_{\mathbb{Q}} := \mathrm{im}(\mathrm{ch}^{\mathrm{top}} : K_0^{\mathrm{top}}(\mathcal{C})_{\mathbb{Q}} \to HP_0(\mathcal{C})) \quad \text{and} \quad F^p V_{\mathbb{C}} := F^p_{\mathrm{nc}} HP_0(\mathcal{C}).$$

The filtration $F^\bullet_{\mathrm{nc}} HP_0(\mathcal{C})$ is called the *noncommutative Hodge filtration*, or *nc Hodge filtration* for short.

Example 2.12. Let X be a smooth, proper complex variety, and take $\mathcal{C} = \mathrm{D}^{\mathrm{b}}(X)$. Since $\mathrm{D}^{\mathrm{b}}(X)$ is smooth and proper, properties (1) and (2) hold for $\mathcal{C}$. Property (3) also holds for $\mathcal{C}$, since the lattice conjecture holds in this case [Blanc 2016]. We have a commutative diagram

$$\begin{array}{ccc}
HN_{2p}(X) & \xrightarrow{\cong} & \bigoplus_{j\in\mathbb{Z}} H^{2j}(X, \tau^{\geq j+p}\Omega^\bullet_{X/\mathbb{C}}) \\
\downarrow{\scriptstyle u^p} & & \downarrow \\
HN_0(X) & \xrightarrow{\cong} & \bigoplus_{j\in\mathbb{Z}} H^{2j}(X, \tau^{\geq j}\Omega^\bullet_{X/\mathbb{C}}) \\
\downarrow & & \downarrow \\
HP_0(X) & \xrightarrow{\cong} & \bigoplus_{j\in\mathbb{Z}} H^{2j}_{\mathrm{dR}}(X;\mathbb{C}) \\
{\scriptstyle \mathrm{ch}^{\mathrm{top}}}\uparrow & & {\scriptstyle \mathrm{ch}^{\mathrm{top}}}\uparrow \\
K_*^{\mathrm{top}}(X) & \xrightarrow{\cong} & KU^*(X)
\end{array}$$

where $KU^*(X)$ denotes the topological K-theory of X. The first three horizontal isomorphisms are given by combining theorems of Keller [2005, Section 5.2] and Weibel [1997, Theorem 3.3], and the bottom isomorphism is due to Blanc [2016, Theorem 1.1(b)]. A straightforward calculation shows that the top and middle squares commute, and the bottom square commutes by [Blanc 2016, Proposition 4.32]. We conclude that there is a natural isomorphism of pre-Hodge structures

$$\mathrm{pHS}(\mathrm{D}^{\mathrm{b}}(X)) \cong \bigoplus_{j\in\mathbb{Z}} H^{2j}(X, \mathbb{Q}(j)).$$

In particular, $\mathrm{pHS}(\mathrm{D}^{\mathrm{b}}(X))$ is a pure Hodge structure of weight 0. See also [Tu 2024], where a more detailed comparison of the classical Hodge structure on the cohomology of X and the nc Hodge structure on $\mathrm{D}^{\mathrm{b}}(X)$ is carried out.

2.3. *The nc Hodge condition for a dg-category.* We begin by recalling the statement of the Hodge conjecture. Let X be a smooth, projective complex variety and $K_*(X)_{\mathbb{Q}} := K_*(X) \otimes \mathbb{Q}$ the rationalized algebraic K-theory groups of X. The classical Hodge conjecture proposes a description of the image of the Chern character map

$$\mathrm{ch} : K_0(X)_{\mathbb{Q}} \to \bigoplus_{p\in\mathbb{Z}} H^{2p}(X;\mathbb{C}). \tag{2.13}$$

We set

$$\mathrm{Hdg}^{2p}(X) := H^{2p}(X;\mathbb{Q}) \cap F^p H^{2p}(X;\mathbb{C}) = H^{2p}(X;\mathbb{Q}(p)) \cap F^0 H^{2p}(X,\mathbb{C}(p)),$$

and write $\mathrm{Hdg}(X) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Hdg}^{2p}(X)$. Elements of $\mathrm{Hdg}(X)$ are called *Hodge classes*. It is well-known that the Chern character map (2.13) takes values in $\mathrm{Hdg}(X)$; the Hodge conjecture predicts that the image of the Chern character map (2.13) is precisely $\mathrm{Hdg}(X)$.

The statement of the Hodge conjecture can be extended to any dg-category $\mathcal{C}$ enjoying Properties 2.8. Let $K_0(\mathcal{C})$ denote the Grothendieck group of $\mathcal{C}$ and $\mathrm{ch}_{HN} : K_0(\mathcal{C}) \to HN_0(\mathcal{C})$ the associated Chern character map; see, e.g., [Brown and Walker 2020a, Section 4] for the definition of ch_{HN}. Composing with the natural map $HN_0(\mathcal{C}) \to HP_0(\mathcal{C})$, one also obtains

$$\mathrm{ch}_{HP} : K_0(\mathcal{C}) \to HP_0(\mathcal{C}).$$

As above, let $K_0(\mathcal{C})_{\mathbb{Q}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$. By [Blanc 2016, Theorem 1.1(d)], the maps ch_{HN} and $\mathrm{ch}^{\mathrm{top}}$ are related by a commutative square

$$\begin{array}{ccc} K_0(\mathcal{C})_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}_{HN}} & HN_0(\mathcal{C}) \\ \downarrow & & \downarrow \\ K_0^{\mathrm{top}}(\mathcal{C})_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}^{\mathrm{top}}} & HP_0(\mathcal{C}) \end{array} \tag{2.14}$$

where the vertical maps are the canonical ones.

Definition 2.15. For a dg-category $\mathcal{C}$ that satisfies Properties 2.8, the subspace $\mathrm{Hdg}(\mathcal{C}) \subseteq HP_0(\mathcal{C})$ of *Hodge classes of* $\mathcal{C}$ is defined to be

$$\mathrm{Hdg}(\mathcal{C}) := \mathrm{im}(\mathrm{ch}^{\mathrm{top}} : K_0^{\mathrm{top}}(\mathcal{C})_{\mathbb{Q}} \to HP_0(\mathcal{C})) \cap F^0_{\mathrm{nc}} HP_0(\mathcal{C}).$$

In other words, $\mathrm{Hdg}(\mathcal{C}) = \mathrm{Hdg}(\mathrm{pHS}(\mathcal{C}))$, where for any pre-Hodge structure V, we set $\mathrm{Hdg}(V) = V_{\mathbb{Q}} \cap F^0 V_{\mathbb{C}}$. By the commutativity of (2.14), the Chern character map ch_{HP}, which is given by composing the top and rightmost maps in (2.14), takes values in $\mathrm{Hdg}(\mathcal{C})$. We say $\mathcal{C}$ satisfies the *nc Hodge condition* if $\mathrm{im}(\mathrm{ch}_{HP}) = \mathrm{Hdg}(\mathcal{C})$.

Example 2.16. When X is a smooth, projective complex variety, the isomorphism of pure Hodge structures in Example 2.12 yields a natural isomorphism $\mathrm{Hdg}(\mathrm{D^b}(X)) \cong \mathrm{Hdg}(X)$. Moreover, this isomorphism makes the triangle

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}} & \mathrm{Hdg}(X) \\ & \searrow^{\mathrm{ch}_{HP}} & \downarrow \cong \\ & & \mathrm{Hdg}(\mathrm{D^b}(X)) \end{array}$$

commute. It follows that the Hodge conjecture holds for X if and only if $\mathrm{D^b}(X)$ satisfies the nc Hodge condition; this was also recently proven by Lin [2023].

2.4. ***Statement of the main theorem.*** To state our main result (Theorem 2.23), we need the following two technical results.

Proposition 2.17. *With R and $\mathfrak{m}$ as in Notation 1.1, set $W := \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. The canonical maps*

$$K_0(W) \to G_0(W), \quad K_0^{\mathrm{top}}(W) \to G_0^{\mathrm{top}}(W) \quad \textit{and} \quad HP_0(W) \to HP_0^{\mathrm{BM}}(W)$$

are isomorphisms, and so are the maps

$$G_0(R) \to G_0(W), \quad G_0^{\mathrm{top}}(R) \to G_0^{\mathrm{top}}(W) \quad \textit{and} \quad HP_0^{\mathrm{BM}}(R) \to HP_0^{\mathrm{BM}}(W)$$

induced by pullback along the natural map $W \hookrightarrow \mathrm{Spec}(R)$.

Proof. The first batch of isomorphisms hold since W is regular, by assumption. Since G-theory satisfies dévissage, we have a right exact sequence

$$G_0(R/\mathfrak{m}) \to G_0(R) \to G_0(W) \to 0.$$

By [Yoshino 1990, Lemma 13.4], the pushforward map $G_0(R/\mathfrak{m}) \to G_0(R)$ is 0; this proves the result for G-theory.

We now address HP^{BM}; the proof involving G^{top} is nearly identical. By [Khan 2023, Theorem A.2] (see also [Brown and Walker 2024, Example 4.8]), there is a dévissage quasi-isomorphism $HP^{\mathrm{BM}}(R/\mathfrak{m}) \to HP(\mathrm{D}^{\mathrm{b},\{\mathfrak{m}\}}(R))$, where $\mathrm{D}^{\mathrm{b},\{\mathfrak{m}\}}(R)$ denotes the subcategory of $\mathrm{D^b}(R)$ given by objects with support contained in $\{\mathfrak{m}\}$. (Dévissage also holds for topological K-theory, as observed in [Halpern-Leistner and Pomerleano 2020, Example 2.3].) There is thus a localization exact triangle

$$HP^{\mathrm{BM}}(R/\mathfrak{m}) \to HP^{\mathrm{BM}}(R) \to HP^{\mathrm{BM}}(W) \to;$$

see, e.g., [Brown and Walker 2024, Lemma 2.8]. Since $HP_{-1}^{\mathrm{BM}}(R/\mathfrak{m}) \cong HP_{-1}(\mathbb{C}) = 0$, it suffices to show that the pushforward map $HP_0(R/\mathfrak{m}) \to HP_0^{\mathrm{BM}}(R)$ is 0. To prove this, we use that the Chern character map is natural for dg-functors, so that we have a commutative square

$$\begin{array}{ccc} G_0(R/\mathfrak{m}) & \longrightarrow & G_0(R) \\ \big\downarrow{\scriptstyle \mathrm{ch}_{HP}} & & \big\downarrow{\scriptstyle \mathrm{ch}_{HP}} \\ HP_0^{\mathrm{BM}}(R/\mathfrak{m}) & \longrightarrow & HP_0^{\mathrm{BM}}(R) \end{array}$$

The left-hand map is isomorphic to $\mathrm{ch}_{HP} : K_0(\mathbb{C}) \to HP_0(\mathbb{C})$, which induces an isomorphism $K_0(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C} \cong HP_0(\mathbb{C})$. Once again applying [Yoshino 1990, Lemma 13.4], the top arrow in this square is the zero map, and so the bottom arrow must be zero as well. □

Let R be as in Notation 1.1. The *singularity category* of the ring R is the dg-quotient $\mathrm{D}^{\mathrm{sg}}(R) := \mathrm{D}^{\mathrm{b}}(R)/\operatorname{Perf}(R)$. We note that the (triangulated) homotopy category of $\mathrm{D}^{\mathrm{sg}}(R)$ need not have a unique dg-enhancement; see, e.g., [Antieau 2018, Example 8.24].

Proposition 2.18. *There are short exact sequences*

$$0 \to HP_0(\mathbb{C}) \to HP_0^{\mathrm{BM}}(R) \to HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \to 0$$

and

$$0 \to K_0^{\mathrm{top}}(\mathbb{C}) \to G_0^{\mathrm{top}}(R) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R)) \to 0,$$

where the maps are induced by the extension of scalars functor $\mathrm{D}^{\mathrm{b}}(\mathbb{C}) \to \mathrm{D}^{\mathrm{b}}(R)$ *and the canonical functor* $\mathrm{D}^{\mathrm{b}}(R) \to \mathrm{D}^{\mathrm{sg}}(R)$.

Proof. Let E denote either HP or K^{top}. In both cases, E is a localizing $\mathbb{A}^1$-homotopy invariant such that $E_{-1}(\mathbb{C}) = 0$. Since R is $\mathbb{Z}_{\geq 0}$-graded, extension of scalars along $\mathbb{C} \to R$ induces an isomorphism $E_*(\mathbb{C}) \xrightarrow{\cong} E_*(R)$. Using that $E_{-1}(\mathbb{C}) = 0$, the exact triangle

$$E(R) \to E(\mathrm{D}^{\mathrm{b}}(R)) \to E(\mathrm{D}^{\mathrm{sg}}(R)) \to$$

induces an exact sequence $E_0(\mathbb{C}) \to E_0(\mathrm{D}^{\mathrm{b}}(R)) \to E_0(\mathrm{D}^{\mathrm{sg}}(R)) \to 0$.

Finally, we observe that $E_0(\mathbb{C}) \to E_0(\mathrm{D}^{\mathrm{b}}(R))$ is split injective. To see this, choose a smooth closed point $x = V(\mathfrak{m}) \in \operatorname{Spec}(R)$; extension of scalars along the map $R \to R/\mathfrak{m} \cong \mathbb{C}$ determines a functor $\mathrm{D}^{\mathrm{b}}(R) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C})$, yielding the desired splitting. □

We have a composition

$$K_0(X) \to K_0(W) \xrightarrow{\cong} G_0(R) \to K_0(\mathrm{D}^{\mathrm{sg}}(R)), \tag{2.19}$$

where the first map is induced by the fibration $W \to X$ with fiber $\mathbb{C}^*$, the second is the (inverse of the) isomorphism from Proposition 2.17, and the last is induced by the canonical map $\mathrm{D}^{\mathrm{b}}(R) \to \mathrm{D}^{\mathrm{sg}}(R)$. The composition (2.19) admits the following simpler description: given a vector bundle $\mathcal{F}$ on X, write $\mathcal{F} = \widetilde{M}$ for some graded R-module M. Since M and $\mathcal{F}$ pull back to the same sheaf on W, the composition (2.19) sends the class $[\mathcal{F}]$ to $[M] \in K_0(\mathrm{D}^{\mathrm{sg}}(R))$.

We have compositions

$$KU^0(X) \cong K_0^{\mathrm{top}}(X) \to K_0^{\mathrm{top}}(W) \xrightarrow{\cong} G_0^{\mathrm{top}}(R) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R)) \tag{2.20}$$

and

$$H^{\mathrm{even}}(X;\mathbb{C}) \cong HP_0(X) \to HP_0(W) \xrightarrow{\cong} HP_0^{\mathrm{BM}}(R) \to HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \tag{2.21}$$

that are defined in the same way, except the first isomorphism in (2.20) is given by Blanc's comparison isomorphism [2016, Theorem 1.1(b)], and the first map in (2.21) is induced by the HKR isomorphism [Loday 1998, Theorem 3.4.4]. By Proposition 2.18, (2.21) induces a map on reduced cohomology; restricting to primitive cohomology, we arrive at the map

$$\alpha : H^n_{\mathrm{prim}}(X;\mathbb{C}) \to HP_0(\mathrm{D}^{\mathrm{sg}}(R)). \tag{2.22}$$

We may now precisely formulate our main result (Theorem 1.2) as follows:

Theorem 2.23. *Let $R = \mathbb{C}[x_0, \dots, x_{n+1}]/(f)$, where f is a homogeneous polynomial such that $X = \mathrm{Proj}(R)$ is smooth. Assume n is even.*

(1) *The dg-category $\mathrm{D}^{\mathrm{sg}}(R)$ enjoys Properties 2.8. In particular, we may associate a pre-Hodge structure to $\mathrm{D}^{\mathrm{sg}}(R)$.*

(2) *The diagram*

$$\begin{array}{ccccc} K_0(X) & \xrightarrow{\text{can}} & KU^0(X) & \xrightarrow{\mathrm{ch}^{\mathrm{top}}} & H^{\mathrm{even}}(X;\mathbb{C}) \\ \downarrow{\scriptstyle (2.19)} & & \downarrow{\scriptstyle (2.20)} & & \downarrow{\scriptstyle (2.21)} \\ K_0(\mathrm{D}^{\mathrm{sg}}(R)) & \xrightarrow{\text{can}} & K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R)) & \xrightarrow{\mathrm{ch}^{\mathrm{top}}} & HP_0(\mathrm{D}^{\mathrm{sg}}(R)) \end{array} \tag{2.24}$$

commutes (where the maps denoted can *are the canonical maps), the middle and rightmost vertical maps are surjective as indicated, and the images of $K_0(X)$ and $K_0(\mathrm{D}^{\mathrm{sg}}(R))$ in $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ coincide.*

(3) *The map α defined in (2.22) is an isomorphism of complex vector spaces that induces an isomorphism*

$$H^n_{\mathrm{prim}}\big(X, \mathbb{Q}\big(\tfrac{n}{2}\big)\big) \xrightarrow{\cong} \mathrm{pHS}(\mathrm{D}^{\mathrm{sg}}(R))$$

of pre-Hodge structures (see Remark 2.2). In particular, the pre-Hodge structure $\mathrm{pHS}(\mathrm{D}^{\mathrm{sg}}(R))$ is pure of weight 0.

As a consequence of Theorem 2.23, we have:

Corollary 2.25. *The dg-category $\mathrm{D}^{\mathrm{sg}}(R)$ satisfies the nc Hodge condition if and only if the Hodge conjecture holds for X.*

Proof. Theorem 2.23 gives the commutative square

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{Hdg}\big(H^n_{\mathrm{prim}}\big(X, \mathbb{Q}\big(\tfrac{n}{2}\big)\big)\big) \\ \downarrow & & \downarrow \cong \\ K_0(\mathrm{D}^{\mathrm{sg}}(R))_{\mathbb{Q}} & \longrightarrow & \mathrm{Hdg}(HP_0(\mathrm{D}^{\mathrm{sg}}(R))) \end{array}$$

where the right vertical map is an isomorphism. The Hodge conjecture for X (resp. nc Hodge condition for $\mathrm{D}^{\mathrm{sg}}(R)$) is the assertion that the top (resp. bottom) horizontal map in this square is onto. Clearly, the Hodge conjecture for X implies the nc Hodge condition for $\mathrm{D}^{\mathrm{sg}}(R)$. The converse holds since Theorem 2.23 also gives that the images of $K_0(X)_{\mathbb{Q}}$ and $K_0(\mathrm{D}^{\mathrm{sg}}(R))_{\mathbb{Q}}$ in $\mathrm{Hdg}(HP_0(\mathrm{D}^{\mathrm{sg}}(R))) \subseteq HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ coincide. □

Remark 2.26. The Hodge conjecture for X is also equivalent to the nc Hodge condition for the *graded* singularity category of R, i.e., the dg-quotient $\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)$ of the bounded derived category of $\mathbb{Z}$-graded R-modules by its subcategory of perfect complexes. Indeed, this is nearly immediate from [Orlov 2009, Theorem 2.5]. Orlov's theorem also implies that, when X is Calabi–Yau, there is an equivalence of categories $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)$ and hence an isomorphism $H^{\mathrm{even}}(X;\mathbb{C}) \cong HP_0(\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R))$ that preserves Hodge structures. However, while the categories $\mathrm{D}^{\mathrm{sg}}(R)$ and $\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)$ are closely related, we do not see a way to deduce Theorem 2.23 from these results concerning $\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)$. In a bit more detail, there is a canonical functor $\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R) \to \mathrm{D}^{\mathrm{sg}}(R)$ given by forgetting the grading, and in fact, by [Tabuada 2015, Theorem 1.5; Keller et al. 2011, Proposition A.8], there is a distinguished triangle

$$E(\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)) \to E(\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)) \to E(\mathrm{D}^{\mathrm{sg}}(R)) \to \tag{2.27}$$

for any localizing, $\mathbb{A}^1$-homotopy invariant E of dg-categories taking values in a triangulated category. The middle map in (2.27) is the canonical functor, and the first map is induced by the endofunctor $T - \mathrm{id}$ of $\mathrm{D}^{\mathrm{sg}}_{\mathrm{gr}}(R)$, where T denotes the grading twist by 1. However, the Hodge structure on a dg-category involves negative cyclic homology, which is not an $\mathbb{A}^1$-homotopy invariant; the triangle (2.27) is therefore ultimately not useful for studying the Hodge structure of $\mathrm{D}^{\mathrm{sg}}(R)$.

3. Some intermediate results

Before we embark on the proof of Theorem 2.23, we need some intermediate results of a technical nature. Throughout this section, we let k be a field of characteristic 0, Q a smooth k-algebra, and R a hypersurface ring of the form $Q/(f)$ for a non-zero-divisor $f \in Q$. We recall in this section the interpretation of $\mathrm{D}^{\mathrm{sg}}(R)$ as a dg-category of matrix factorizations of f, and also the "de Rham models" for HH, *HN* and *HP* of the latter. We also recall an explicit description of a certain boundary map occurring in a long exact dévissage sequence for *HP*.

3.1. ***Matrix factorizations.*** A *matrix factorization of* f is a finitely generated, $\mathbb{Z}/2$-graded projective Q-module $F = F_{\bar{0}} \oplus F_{\bar{1}}$ equipped with an odd degree endomorphism ∂ such that ∂^2 coincides with multiplication by f. Matrix factorizations were introduced by Eisenbud [1980] in his study of maximal Cohen–Macaulay modules over hypersurface rings. Matrix factorizations of f form the objects of a differential $\mathbb{Z}/2$-graded category $\mathrm{mf}(f)$, whose morphisms are the $\mathbb{Z}/2$-graded complexes $\mathrm{Hom}_{\mathrm{mf}}((F,\partial),(F',\partial')) := \mathrm{Hom}_Q(F,F')$ with differential sending a homogeneous map α of degree $\bar{i} \in \mathbb{Z}/2$ to $\partial'\alpha - (-1)^i \alpha\partial$. If F is free, then $F_{\bar{0}}$ and $F_{\bar{1}}$ necessarily have the same rank, so we may view ∂ as a block matrix $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$, where A and B are square matrices such that $AB = BA = f \cdot \mathrm{id}$. In this case, we sometimes denote matrix factorizations as pairs (A, B).

By "unfolding" the $\mathbb{Z}/2$-grading, it is also possible to interpret $\mathrm{mf}(f)$ as a classical differential $\mathbb{Z}$-graded category, and we use both points of view in this paper. To keep this straight, it is useful to introduce a formal indeterminate t of homological degree -2, and to identify $\mathbb{Z}/2$-graded vector spaces with $\mathbb{Z}$-graded modules over $k[t,t^{-1}]$: given a $\mathbb{Z}/2$-graded k-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, the associated graded $k[t,t^{-1}]$-module is

$$\cdots \oplus V_{\bar{0}}t^{-1} \oplus V_{\bar{1}}t^{-1} \oplus V_{\bar{0}} \oplus V_{\bar{1}} \oplus V_{\bar{0}}t \oplus V_{\bar{1}}t \oplus \cdots,$$

and the inverse procedure is given by setting $t = 1$ and taking degrees modulo 2. Using this identification, the *unfolding* of a $\mathbb{Z}/2$-graded vector space is restriction of scalars along $k \to k[t,t^{-1}]$. (There is also a "folding" procedure, given by extension of scalars along this map, but it does not arise in this paper.)

From this point of view, an object of $\mathrm{mf}(f)$ becomes a finitely generated $\mathbb{Z}$-graded projective module over the graded ring $Q[t,t^{-1}]$, equipped with a $Q[t,t^{-1}]$-linear differential of degree -1 whose square equals ft. When we think of $\mathrm{mf}(f)$ as a $\mathbb{Z}/2$-graded dg-category, we are regarding it as a $k[t,t^{-1}]$-linear dg-category, and its unfolding amounts to regarding it as merely a k-linear dg-category.

A key result used throughout this paper is that there is a quasi-equivalence of Q-linear ($\mathbb{Z}$-graded) dg-categories

$$\mathrm{mf}(f) \xrightarrow{\simeq} \mathrm{D}^{\mathrm{sg}}(R). \tag{3.1}$$

This follows from a combination of [Buchweitz 2021, Theorem 4.4.1] and [Eisenbud 1980, Corollary 6.3]. Unlike $\mathrm{mf}(f)$, the dg-category $\mathrm{D}^{\mathrm{sg}}(R)$ cannot directly be realized as the unfolding of a $\mathbb{Z}/2$-graded dg-category.

Notation 3.2. Let $\mathcal{C}$ be a $k[t,t^{-1}]$-linear dg-category, where t has degree -2 (i.e., a k-linear differential $\mathbb{Z}/2$-graded category). We write $HN(\mathcal{C})$ (resp. $HN^{\mathbb{Z}/2}(\mathcal{C})$)

for its Hochschild homology relative to k (resp. $k[t, t^{-1}]$), and similarly for HP and HH.

3.2. *De Rham models for Hochschild, negative cyclic, and periodic cyclic homology.* We make use of explicit de Rham-type models for Hochschild, negative cyclic, and periodic cyclic homology of $\mathrm{mf}(f)$ and related categories. We begin with a technical point:

3.2.1. *Adjoining a power series variable to a graded ring.* For a (homologically) $\mathbb{Z}$-graded vector space W, let u be an indeterminate of degree -2, so that $W[u]$, and hence also $W[u]/u^m$, is $\mathbb{Z}$-graded. We set

$$W[\![u]\!] := \lim_{m} W[u]/u^m,$$

where, importantly, the inverse limit is taken in the category of $\mathbb{Z}$-graded vector spaces. Thus, $W[\![u]\!]$ is graded, and for each d, its degree d part is the subspace

$$W[\![u]\!]_d = \left\{ \sum_{i \geq 0} v_i u^i : v_i \in W_{2i+d} \right\}$$

of the collection of all power series with W coefficients. Note that if W is concentrated in degree 0 (or, more generally, if $W_m = 0$ for $m \gg 0$), then $W[\![u]\!]$ is really just a polynomial ring with W coefficients. For instance, if we regard k itself as being $\mathbb{Z}$-graded but concentrated in degree 0, then the above definition of $k[\![u]\!]$ yields $k[u]$. We stick with the traditional notation $k[u]$ in this case.

The following is easily verified:

Lemma 3.3. *Suppose W is a $\mathbb{Z}$-graded k-vector space with only a finite number of nonzero degree components, and let t be a degree -2 indeterminate. We have an identity*

$$W[t, t^{-1}][\![u]\!] = W[\![v]\!][t, t^{-1}],$$

where $v := ut^{-1}$ and $W[\![v]\!]$ denotes all power series with W coefficients in the degree 0 variable v.

Example 3.4. In particular, we have an identity

$$k[t, t^{-1}][\![u]\!] = k[\![v]\!][t, t^{-1}],$$

with $v := ut^{-1}$ and $k[\![v]\!]$ the usual ring of power series. In other words, $k[t, t^{-1}][\![u]\!]$ is a $\mathbb{Z}/2$-graded ring that is 0 in odd degree and a power series ring in even degree. We use this identity frequently in this paper.

Remark 3.5. The ring $k[t][\![u]\!]$ may be identified with the $\mathbb{Z}$-graded polynomial ring $k[u, t]$, concentrated in negative even degrees.

3.2.2. *De Rham models.* Let A be a smooth k-algebra equipped with a $\mathbb{Z}$-grading (written homologically) such that $A_i = 0$ for all odd i, and suppose $w \in A_{-2}$ is an element of degree -2. We call such a pair (A, w) a *smooth curved algebra*, and we define its *de Rham HN complex* to be

$$HN^{\mathrm{deR}}(A, w) := (\Omega^{\bullet}_{A/k}[\![u]\!], ud + \lambda_{dw}).$$

Here, u is an indeterminate of degree -2, and $\Omega^{\bullet}_A = \bigoplus_p \Omega^p_{A/k}$ is homologically graded by declaring $|a_0 da_1 \cdots da_p| := p + \sum_i |a_i|$, where $|-|$ refers to the degree of a homogenous element. In the formula for the differential, $ud + \lambda_{dw}$, the d refers to the de Rham differential $d : \Omega^p_A \to \Omega^{p+1}_A$ (observe that it has homological degree $+1$, so that ud has degree -1) and λ_{dw} refers to left multiplication by the degree -1 element $dw \in \Omega^1_A$.

We define the *de Rham HP complex* for the pair (A, w) to be

$$HP^{\mathrm{deR}}(A, w) := HN^{\mathrm{deR}}(A, w)[u^{-1}] = (\Omega^{\bullet}_{A/k}((u)), ud + \lambda_{dw}),$$

where, in general, $W((u))$ is shorthand for $W[\![u]\!][u^{-1}]$. The *de Rham HH complex* of (A, w) is

$$HH^{\mathrm{deR}}(A, w) := \frac{HN^{\mathrm{deR}}(A, w)}{u \cdot HN^{\mathrm{deR}}(A, w)} = (\Omega^{\bullet}_A, \lambda_{dw}).$$

If A is a smooth $k[t, t^{-1}]$-algebra for a degree -2 indeterminate t (i.e., a $\mathbb{Z}/2$-graded algebra), we set

$$HN^{\mathrm{deR}, \mathbb{Z}/2}(A, w) := (\Omega^{\bullet}_{A/k[t,t^{-1}]}[\![u]\!], ud + \lambda_{dw}).$$

We also define $HP^{\mathrm{deR}, \mathbb{Z}/2}(A, w)$ and $HH^{\mathrm{deR}, \mathbb{Z}/2}(A, w)$ just as above. Observe that the complex $HN^{\mathrm{deR}, \mathbb{Z}/2}(A, w)$ is a dg-module over $k[t, t^{-1}][\![u]\!] = k[\![v]\!][t, t^{-1}]$ (see Example 3.4); that is, it is a differential $\mathbb{Z}/2$-graded module over the power series ring $k[\![v]\!]$. When A is a $\mathbb{Z}/2$-graded algebra, we can, and sometimes will, ignore $k[t, t^{-1}]$-linearity and consider the invariants $HN^{\mathrm{deR}}(A, w)$, $HP^{\mathrm{deR}}(A, w)$, and $HH^{\mathrm{deR}}(A, w)$ defined above.

If $A = A_0$ and $w = 0$, we write $HN^{\mathrm{deR}}(A, w)$ as just $HN^{\mathrm{deR}}(A)$, and if $Y = \mathrm{Spec}(A)$, we also write $HN^{\mathrm{deR}}(Y) := HN^{\mathrm{deR}}(A)$; we use the analogous notation for *HH* and *HP* as well. In this case, the classical Hochschild–Kostant–Rosenberg (HKR) isomorphism [Loday 1998, Theorem 3.4.4] induces quasi-isomorphisms

$$HN(A) \xrightarrow{\simeq} HN^{\mathrm{deR}}(A), \quad HP(A) \xrightarrow{\simeq} HP^{\mathrm{deR}}(A) \quad \text{and} \quad HH(A) \xrightarrow{\simeq} HH^{\mathrm{deR}}(A),$$

thus justifying the notation. More generally, for a smooth (ungraded) k-algebra Q and non-zero-divisor $f \in Q$, we may form the smooth curved algebras $(Q[t], ft)$ and $(Q[t, t^{-1}], ft)$. For these, we have the following HKR-type isomorphisms, which build on results in [Căldăraru and Tu 2013; Polishchuk and Positselski 2012; Segal 2013]:

Theorem 3.6. *For a field k of characteristic* 0, *a smooth k-algebra Q, and a non-zero-divisor $f \in Q$, we have the following isomorphisms in the derived category of dg-$Q[u]$-modules* (in parts (2) and (3), we use Notation 3.2):

(1) $HN^{\mathrm{BM}}(Q/f) \cong HN^{\mathrm{deR}}(Q[t], ft)$,

(2) $HN(\mathrm{D}^{\mathrm{sg}}(Q/f)) \cong HN(\mathrm{mf}(f)) \cong HN^{\mathrm{deR}}(Q[t,t^{-1}], ft)$, *and*

(3) $HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \cong HN^{\mathrm{deR},\mathbb{Z}/2}(Q[t,t^{-1}], ft)$.

Remark 3.7. The third isomorphism is a map of graded $Q[t,t^{-1}][\![u]\!] = Q[\![v]\!][t,t^{-1}]$-modules, or, equivalently, of $\mathbb{Z}/2$-graded $Q[\![v]\!]$-modules.

Remark 3.8. The isomorphisms in this theorem imply the analogous results involving both Hochschild and periodic cyclic homology upon modding out by and inverting u, respectively.

Proof of Theorem 3.6. Part (1) follows from [Brown and Walker 2024, Proposition 2.16 and Theorem 2.17]. Parts (2) and (3) follow from [Brown and Walker 2020a, Proposition 3.25 and Theorem 3.31] and [Briggs and Walker 2024], respectively. □

3.3. ***An explicit calculation of a boundary map.*** Suppose $Z \hookrightarrow X$ is a closed immersion of schemes of finite type over $\mathbb{C}$. Let $\mathrm{D}^{\mathrm{b},Z}(X)$ denote the full dg-subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ consisting of objects whose supports are contained in Z. We set $HP^{\mathrm{BM},Z}(X) := HP(\mathrm{D}^{\mathrm{b},Z}(X))$. As already noted in the proof of Proposition 2.17, [Khan 2023, Theorem A.2] (see also [Brown and Walker 2024, Theorem 1.2]) implies that the induced map

$$HP^{\mathrm{BM}}(Z) \to HP^{\mathrm{BM},Z}(X)$$

is a quasi-isomorphism: that is, HP^{BM} satisfies the dévissage property. By a result of Keller [1999], we have a distinguished triangle

$$HP^{\mathrm{BM},Z}(X) \to HP^{\mathrm{BM}}(X) \to HP^{\mathrm{BM}}(X \setminus Z) \to$$

of dg-$k[u,u^{-1}]$-modules; see [Brown and Walker 2024, Section 2] for details. Combining this with the dévissage property, we obtain a distinguished triangle

$$HP^{\mathrm{BM}}(Z) \to HP^{\mathrm{BM}}(X) \to HP^{\mathrm{BM}}(X \setminus Z) \to . \tag{3.9}$$

Remark 3.10. By a result of Blanc, topological K-theory of dg-categories is a localizing invariant (see, e.g., [Brown and Walker 2024, Theorem 2.6]). Moreover, topological K-theory satisfies dévissage [Halpern-Leistner and Pomerleano 2020, Example 2.3]. Thus, we have a distinguished triangle

$$G^{\mathrm{top}}(Z) \to G^{\mathrm{top}}(X) \to G^{\mathrm{top}}(X \setminus Z) \to \tag{3.11}$$

of spectra analogous to (3.9). We will make use of this in the proof of Theorem 2.23.

We apply the distinguished triangle (3.9) in the following special case: for a smooth k-algebra Q and a non-zero-divisor $f \in Q$, we obtain a long exact sequence

$$\cdots \to HP_1(Q) \to HP_1(Q[1/f]) \xrightarrow{\partial} HP_0^{\mathrm{BM}}(Q/f) \to HP_0(Q) \to \cdots. \quad (3.12)$$

Using [Brown and Walker 2024, Theorem 5.5], one may explicitly describe the boundary map ∂ in terms of the de Rham models for periodic cyclic homology discussed in Section 3.2.2. Before recalling this result, we note that every element of $HP_1^{\mathrm{deR}}(Q[1/f])$ is represented by a finite sum of classes of the form $(\alpha/f^s)u^{(p-1)/2}$ for some integer $s \geq 1$ and some class $\alpha \in \Omega_Q^p$, with p odd, such that $f d\alpha = s df \alpha$ [Brown and Walker 2024, Lemma 5.3].

Theorem 3.13 [Brown and Walker 2024, Theorem 5.5]. *The boundary map*

$$\partial : HP_1(Q[1/f]) \to HP_0^{\mathrm{BM}}(Q/f)$$

in (3.12) corresponds, via the isomorphisms relating its source and target to their de Rham models, to the map

$$\partial^{\mathrm{deR}} : HP_1^{\mathrm{deR}}(Q[1/f]) \to HP_0^{\mathrm{deR}}(Q[t], ft)$$

given by

$$\partial^{\mathrm{deR}}\left(\frac{\alpha}{f^s}u^{\frac{1}{2}(p-1)}\right) = \frac{(-1)^s}{s!} d(\alpha t^s) u^{\frac{1}{2}(p+1)-s}.$$

3.4. *Some calculations of Hochschild invariants of matrix factorization categories.* Formulas for the Hochschild, negative cyclic, and periodic cyclic homology of $\mathrm{mf}(f)$ relative to $k[t, t^{-1}]$ (as opposed to k) are well-known, due to [Dyckerhoff 2011]:

Theorem 3.14. *Let k be a field of characteristic 0, Q a smooth k-algebra, and $f \in Q$ a non-zero-divisor. Assume that the morphism $f : \mathrm{Spec}(Q) \to \mathbb{A}_k^1$ determined by f has only one singular point $\mathfrak{m} \in \mathrm{Spec}(Q)$, and it lies over the origin (i.e., $f \in \mathfrak{m}$). Set $d = \dim(Q_{\mathfrak{m}})$, and let Ω_f be the finite-dimensional vector space*

$$\frac{\Omega_Q^d}{df \cdot \Omega_Q^{d-1}}.$$

We use Notation 3.2.

(1) *There is an isomorphism of $\mathbb{Z}/2$-graded k-vector spaces*

$$\Sigma^d \Omega_f \xrightarrow{\cong} HH_*^{\mathbb{Z}/2}(\mathrm{mf}(f)),$$

where d is considered mod 2. Moreover, under the identification of $HH_*^{\mathbb{Z}/2}(\mathrm{mf}(f))$ with $HH_*^{\mathrm{deR},\mathbb{Z}/2}(\mathrm{mf}(f))$ given by Theorem 3.6, the isomorphism is induced by the canonical inclusion $\Omega_Q^d \subseteq \Omega_Q^\bullet$.

(2) *$HN^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$ is a free, finite-rank $\mathbb{Z}/2$-graded $k[\![v]\!]$-module, $HP^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space over $k(\!(v)\!)$, and both are concentrated in degree $d \pmod 2$. (Recall that $v = t^{-1}u$, and $k(\!(v)\!) := k[\![v]\!][v^{-1}]$.)*

Remark 3.15. Rephrasing in terms of $k[t,t^{-1}]$-modules, Theorem 3.14(1) means that we have an isomorphism

$$\Sigma^d \Omega_f[t,t^{-1}] \cong HH^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$$

of graded $k[t,t^{-1}]$-modules induced by the inclusion $\Omega^d_Q[t,t^{-1}] \hookrightarrow \Omega^\bullet_Q[t,t^{-1}]$, and Theorem 3.14(2) says that $HN^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$ is a graded free module of finite rank over the ring $k[t,t^{-1}][\![u]\!] = k[\![v]\!][t,t^{-1}]$ (see Example 3.4), and similarly for HP. In particular, Theorem 3.14(2) implies that $HN^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$ is v-torsion free, and thus its quotient by v may be identified with $HH^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$. A choice of $k[t,t^{-1}]$-linear splitting of $HN^{\mathbb{Z}/2}_*(\mathrm{mf}(f)) \twoheadrightarrow HH^{\mathbb{Z}/2}_*(\mathrm{mf}(f))$ determines an isomorphism $HN^{\mathbb{Z}/2}_*(\mathrm{mf}(f)) \cong \Sigma^d \Omega_f[\![v]\!][t,t^{-1}]$ of $k[\![v]\!][t,t^{-1}]$-modules, but such an isomorphism is not canonical.

Proof. Part (1) is [Dyckerhoff 2011, Theorem 6.6]; it also follows from Theorem 3.6 (and Remark 3.8). It is a consequence of (1) that the Hodge-to-de Rham spectral sequence degenerates [Dyckerhoff 2011, Section 7]. The statement in (2) concerning negative cyclic homology follows: see, e.g., [Shklyarov 2016, proof of Proposition 9]. The statement in (2) about periodic cyclic homology is then clear. ☐

4. A Wang-type exact sequence

Let k, Q, f, and R be as in Section 3. The goal of this section is to leverage the calculations in Theorem 3.14 to compute the negative cyclic and periodic cyclic homology of $\mathrm{mf}(f) \simeq \mathrm{D}^{\mathrm{sg}}(R)$ relative to k rather than $k[t,t^{-1}]$.

4.1. ***The distinguished triangle.*** A key tool is the following distinguished triangle, whose associated long exact sequence on periodic cyclic homology is reminiscent of the Wang exact sequence of a circle fibration (see Remark 4.2(3)). This result implies Theorem 1.4 from the introduction.

Theorem 4.1. *Let k be a field of characteristic 0, Q a smooth k-algebra, and $f \in Q$ a non-zero-divisor.*

(1) *There is a distinguished triangle*

$$HN(\mathrm{mf}(f)) \to HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \xrightarrow{h} HN^{\mathbb{Z}/2}(\mathrm{mf}(f)) \to$$

in the derived category of dg-$k[u]$-modules, where h is a map satisfying

$$h(g(t)\cdot\alpha) = ug'(t)\alpha + g(t)h(\alpha)$$

on the level of homology for any cycle α *and* $g \in k[t,t^{-1}]$. *In more detail, in terms of the equivalent de Rham models,* h *is given by the endomorphism* $u\frac{\partial}{\partial t} + \lambda_f$ *of* $\Omega^\bullet_Q[t,t^{-1}][\![u]\!]$.

(2) *Assume that the morphism* $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$ *determined by* f *has only one singular point,* $\mathfrak{m} \in \operatorname{Spec}(Q)$, *and it lies over the origin (i.e.,* $f \in \mathfrak{m}$*). We have, for all* $j \in \mathbb{Z}$ *such that* $j \equiv \dim(Q_\mathfrak{m}) \pmod 2$, *an exact sequence*

$$0 \to HN_j(\mathrm{mf}(f)) \to HN_j^{\mathbb{Z}/2}(\mathrm{mf}(f)) \xrightarrow{h} HN_j^{\mathbb{Z}/2}(\mathrm{mf}(f)) \to HN_{j-1}(\mathrm{mf}(f)) \to 0.$$

(3) *Parts* (1) *and* (2) *hold with HN replaced with HP throughout.*

Proof. We have a canonical isomorphism

$$\Omega^\bullet_Q[t,t^{-1}] \oplus \Omega^\bullet_Q[t,t^{-1}] \xrightarrow[\cong]{(\alpha,\beta)\mapsto\alpha+\beta dt} \Omega^\bullet_{Q[t,t^{-1}]},$$

under which the differential for $HN^{\mathrm{deR}}(Q[t,t^{-1}], ft)$ corresponds to

$$u\begin{bmatrix} d_Q & 0 \\ \frac{\partial}{\partial t} & d_Q \end{bmatrix} + \begin{bmatrix} \lambda_{tdf} & 0 \\ \lambda_f & \lambda_{tdf} \end{bmatrix};$$

here, d_Q denotes the de Rham differential on $\Omega^\bullet_Q$, extended $k[t,t^{-1}]$-linearly to $\Omega^\bullet_Q[t,t^{-1}]$. It is thus immediate that $HN^{\mathrm{deR}}(Q[t,t^{-1}], ft)$ is isomorphic, as a dg-$k[\![u]\!]$-module, to the homotopy fiber of the map

$$HN^{\mathrm{deR},\mathbb{Z}/2}(Q[t,t^{-1}], ft) \xrightarrow{u\frac{\partial}{\partial t}+\lambda_f} HN^{\mathrm{deR},\mathbb{Z}/2}(Q[t,t^{-1}], ft).$$

Letting $h := u\frac{\partial}{\partial t} + \lambda_f$, part (1) therefore follows from (2) and (3) of Theorem 3.6. Part (2) is immediate from (1) and Theorem 3.14(2), and part (3) follows by the exactness of inverting u. □

Remark 4.2. (1) As is evident from the relation $h(g(t)\cdot\alpha) = ug'(t)\alpha + g(t)h(\alpha)$, the map h is not $k[t,t^{-1}]$-linear, although its source and target are both complexes of $k[t,t^{-1}]$-modules. Put differently, $HN^{\mathbb{Z}/2}(\mathrm{mf}(f))$ is two-periodic, but the map h is not.

(2) Suppose $Q = \mathbb{C}[x_0,\dots,x_{n+1}]$ and that the hypersurface $Q/(f)$ has an isolated singularity at the origin. Denote by $\varphi : E \to S^1$ the *Milnor fibration* associated to f; see [Dimca 1992, Chapter 3] for background on the Milnor fibration. Milnor [1968, Theorem 6.5] proves that the fiber F of this fibration is homotopy equivalent to a wedge sum of copies of S^{n+1}; F is called the *Milnor fiber*. The Serre spectral sequence associated to the Milnor fibration collapses to a long exact sequence, called the *Wang exact sequence*, of the form

$$\cdots \to H^i(E) \to H^i(F) \xrightarrow{T-\mathrm{id}} H^i(F) \to H^{i+1}(E) \to \cdots, \tag{4.3}$$

where the maps $H^i(E) \to H^i(F)$ are the natural ones, and T denotes the automorphism of $H^*(F)$ induced by monodromy.

On the other hand, by [Efimov 2020, Theorem 1.1], there is a canonical isomorphism

$$HP_j^{\mathbb{Z}/2}(\mathrm{mf}(f)) \cong \bigoplus_{i \in \mathbb{Z}} H^{j+2i+1}(F; \mathbb{C})$$

for all j (see [Beraldo and Pippi 2025] for an ℓ-adic version of this result); we note that [Efimov 2020, Theorem 1.1] identifies periodic cyclic homology with vanishing cohomology, but the latter is isomorphic to the cohomology of the Milnor fiber since the singularity of $Q/(f)$ is isolated. It is therefore reasonable to consider the long exact sequence in Theorem 4.1(3) as an analogue of the Wang exact sequence. Theorem 4.1 also suggests that one should consider $HP_*(\mathrm{mf}(f)) \cong HP_*(\mathrm{D}^{\mathrm{sg}}(R))$ as an analogue of the cohomology of the total space E of the Milnor fibration.

4.2. ***The homogeneous case.*** Now let Q be the polynomial ring $k[x_0, \dots, x_{n+1}]$, equipped with its standard internal grading given by $\deg(x_i) = 1$ for all i. Assume also that f is homogeneous, say of internal degree $\deg(f) = e$. Henceforth, we will use the notation $|-|$ for homological degree and $\deg(-)$ for internal degree. The main goal of this subsection is to prove Corollary 4.7, which provides an important link between the $\mathbb{Z}$-graded and $\mathbb{Z}/2$-graded negative cyclic homology of the singularity category of a homogeneous isolated hypersurface singularity.

Recall that $\Omega_Q^\bullet[t, t^{-1}]$ has a homological grading, where $|t| = -2$ and $|\Omega_Q^p| = p$; we may also equip $\Omega_Q^p[t, t^{-1}]$ with an internal grading by declaring $\deg(t) = -e$ and $\deg(g_0 dg_1 \cdots dg_j) = \sum_{i=0}^j \deg(g_i)$. With these conventions, we have $|ft| = -2$ and $\deg(ft) = 0$, and the de Rham differential d_Q is an operator of homological degree $+1$ and internal degree 0.

Remark 4.4. It is tempting to extend the internal grading to $\Omega_Q^\bullet[t, t^{-1}][\![u]\!]$ by declaring $\deg(u) = 0$, so that, for instance, the differential on $HN^{\mathrm{deR}, \mathbb{Z}/2}(Q[t, t^{-1}], ft)$, namely $ud_Q + t\lambda_{df}$, has internal degree 0. But, this does not entirely make sense, for observe that this would require setting $\deg(v) = \deg(ut^{-1}) = e$, and, since $\Omega_Q^\bullet[t, t^{-1}][\![u]\!] = \Omega_Q^\bullet[\![v]\!][t, t^{-1}]$, this would mean we are "grading" the power series ring $k[\![v]\!]$ with $\deg(v) \neq 0$, which is nonsensical.

Remark 4.4 notwithstanding, we may define a grading-like operator Γ on $\Omega_Q^\bullet[\![v]\!][t, t^{-1}] = \Omega_Q^\bullet[t, t^{-1}][\![u]\!]$ by setting $\Gamma(\omega t^i u^j) = (\deg(\omega) - i \cdot e)\omega t^i u^j$ for $\omega \in \Omega_Q^\bullet$ and extending along infinite sums. Equivalently, and more explicitly, we define the operator Γ on $\Omega_Q^\bullet[\![v]\!][t, t^{-1}]$ by declaring $\Gamma(v) = ev$ and extending along infinite sums as follows: a typical element of $\Omega_Q^\bullet[\![v]\!][t, t^{-1}]$ has the form $\alpha = \sum_j \sum_m \sum_p \omega_{j,m,p} v^m t^j$, where j ranges over a finite subset of $\mathbb{Z}$, m ranges over all nonnegative integers, p ranges over a finite subset of $\mathbb{N}$, and $\omega_{j,m,p}$ is a homogeneous element of $\Omega^\bullet_Q$ of internal degree p. We set

$$\Gamma(\alpha) = \sum_{j,m,p} (p + em - ej)\omega_{j,m,p} v^m t^j.$$

Since $\deg(u) = 0$, the operator Γ is a $k[u]$-linear derivation: $\Gamma(u\alpha) = u\Gamma(\alpha)$, and $\Gamma(\alpha\beta) = \Gamma(\alpha)\beta + \alpha\Gamma(\beta)$. Note that the homological grading is ignored, and there are no signs.

Notation 4.5. For any integer i and operator Γ on a k-vector space W, define $\Gamma_i W := \ker(\Gamma - i)$, the eigenspace of Γ with eigenvalue i.

For an honest $\mathbb{Z}$-graded vector space W, with Γ taken to be the grading operator, $\Gamma_i(W)$ coincides with the i-th graded piece of W, which we typically write as $[W]_i$; we thus have $W = \bigoplus_i \Gamma_i(W) = \bigoplus_i [W]_i$. Such a decomposition does not hold for $W = \Omega^\bullet_Q[\![v]\!][t, t^{-1}]$, even if we take $Q = k$. For instance, $\Gamma_j k[\![v]\!][t, t^{-1}] = 0$ when $e \nmid j$, and $\Gamma_{i \cdot e} k[\![v]\!][t, t^{-1}] = k[u]t^{-i}$. We therefore have

$$\bigoplus_j \Gamma_j k[\![v]\!][t, t^{-1}] = k[v][t, t^{-1}] \subsetneq k[\![v]\!][t, t^{-1}].$$

More generally, for each integer i,

$$\Gamma_i \Omega^\bullet_Q[\![v]\!][t, t^{-1}] = \bigoplus_{j \in \mathbb{Z}} \bigoplus_{m \geq 0} \Gamma_{i-em+ej} \Omega^\bullet_Q v^m t^j = \left[\Omega^\bullet_Q[v, t, t^{-1}]\right]_i.$$

The differential $tdf + ud_Q$ on $HN^{\mathrm{deR}}(Q[t, t^{-1}], tf)$ commutes with Γ (loosely speaking, the differential "has degree zero"); it follows that Γ induces an operator on $HN_*(\mathrm{mf}^{\mathbb{Z}/2}(f))$, which we also write as Γ. As with $\Omega^\bullet_Q[\![v]\!][t, t^{-1}]$, Γ does not induce a $\mathbb{Z}$-grading on $HN_*(\mathrm{mf}^{\mathbb{Z}/2}(f))$, but $\Gamma_0 HN_*(\mathrm{mf}^{\mathbb{Z}/2}(f))$ is a module over $k[u] = \Gamma_0 k[t, t^{-1}][\![u]\!]$.

Theorem 4.6. *Let $Q = k[x_0, \dots, x_{n+1}]$, equipped with the internal grading given by $\deg(x_i) = 1$, and let f be a homogeneous form of degree $e \geq 1$. The $k[u]$-linear endomorphism $h = u\frac{\partial}{\partial t} + \lambda_f$ of $HN^{\mathrm{deR}, \mathbb{Z}/2}(Q[t, t^{-1}], ft)$ is homotopic to the endomorphism $-\frac{v}{e}\Gamma$.*

Proof. Define $\varepsilon_Q : \Omega^\bullet_Q \to \Omega^\bullet_Q$ to be the map induced by the *Euler derivation* on Q sending $g \in Q$ to $\deg(g)g$; i.e., ε_Q maps Ω^p_Q to Ω^{p-1}_Q by the formula

$$\varepsilon_Q(g_0 dg_1 \cdots dg_p) := \sum_{i=1}^{p} (-1)^{i-1} \deg(g_i) g_0 g_i dg_1 \cdots \widehat{dg_i} \cdots dg_p.$$

The map ε_Q has homological degree -1 and internal degree 0. We also denote by ε_Q its $k[t, t^{-1}][\![u]\!]$-linear extension (i.e., its $k[\![v]\!][t, t^{-1}]$-linear extension) to $\Omega^\bullet_Q[t,t^{-1}][\![u]\!] = \Omega^\bullet_Q[\![v]\!][t,t^{-1}]$. We have relations

$$[\varepsilon_Q, t\lambda_{df}] = \deg(df)t\lambda_f = et\lambda_f \quad \text{and} \quad [\varepsilon_Q, ud_Q] = u\Gamma_Q,$$

where $[\alpha,\beta] := \alpha\circ\beta + \beta\circ\alpha$ for operators of odd homological degree, and

$$\Gamma_Q : \Omega^\bullet_Q[\![v]\!][t,t^{-1}] \to \Omega^\bullet_Q[\![v]\!][t,t^{-1}]$$

denotes the $k[\![v]\!][t,t^{-1}]$-linear map determined by $\Gamma_Q(\omega) = \deg(\omega)\omega$ for $\omega \in \Omega^\bullet_Q$. Since the differential on $HN^{\mathrm{deR}}(\mathrm{mf}^{\mathbb{Z}/2}(f)) = \Omega^\bullet_Q[\![v]\!][t,t^{-1}]$ is $ud_Q + t\lambda_{df}$, we may interpret ε_Q/et as a homotopy exhibiting that λ_f and $-\frac{v}{e}\Gamma_Q$ are homotopic endomorphisms of $HN^{\mathrm{deR}}(\mathrm{mf}^{\mathbb{Z}/2}(f))$, and hence that h and $vt\frac{\partial}{\partial t} - \frac{v}{e}\Gamma_Q$ are homotopic.

It remains to show $vt\frac{\partial}{\partial t} - \frac{v}{e}\Gamma_Q = -\frac{v}{e}\Gamma$. We first note that, since $v = \frac{u}{t}$, we have

$$\frac{\partial v^j}{\partial t} = -\frac{jv^j}{t}.$$

Thus, for $\alpha = \omega v^j t^i$, we have

$$\begin{aligned}\Big(vt\frac{\partial}{\partial t} - \frac{v}{e}\Gamma_Q\Big)(\alpha) &= \Big(i - j + \frac{\deg(\omega)}{e}\Big)\omega v^{j+1}t^i \\ &= -\frac{v}{e}(\deg(\omega) + (j-i)e)\omega v^j t^i = -\frac{v}{e}\Gamma(\alpha),\end{aligned}$$

and this extends along all infinite sums. □

Corollary 4.7. *Let Q and f be as in Theorem 4.6. Assume also that n is even, and that $\mathrm{Proj}(Q/f) \subseteq \mathbb{P}^{n+1}$ is smooth. The canonical map*

$$HN^{\mathrm{deR}}(Q[t,t^{-1}], ft) \to HN^{\mathrm{deR},\mathbb{Z}/2}(Q[t,t^{-1}], ft),$$

induced by the surjection $\Omega^\bullet_{Q[t,t^{-1}]/k} \twoheadrightarrow \Omega^\bullet_{Q[t,t^{-1}]/k[t,t^{-1}]}$ that sets $dt = 0$, induces a $k[u]$-linear isomorphism

$$HN^{\mathrm{deR}}_{\mathrm{even}}(Q[t,t^{-1}], tf) \cong \Gamma_0 HN^{\mathrm{deR},\mathbb{Z}/2}_*(Q[t,t^{-1}], tf).$$

Proof. By Theorem 3.14(2), we have $HN_i^{\mathbb{Z}/2}(\mathrm{mf}(f)) = 0$ for i odd. It therefore follows from Theorem 4.6 that, for each $j \in \mathbb{Z}$, there is an exact sequence of k-vector spaces

$$\begin{aligned}0 \to HN^{\mathrm{deR}}_{2j}(Q[t,t^{-1}], ft) \to HN^{\mathrm{deR},\mathbb{Z}/2}_{2j}&(Q[t,t^{-1}], ft) \\ &\xrightarrow{-v\frac{\Gamma}{e}} HN^{\mathrm{deR},\mathbb{Z}/2}_{2j}(Q[t,t^{-1}], ft).\end{aligned}$$

Again by Theorem 3.14(2), the $k[\![v]\!]$-module $HN^{\mathrm{deR},\mathbb{Z}/2}_*(Q[t,t^{-1}], ft)$ is v-torsion free, and hence $\ker\big(-v\frac{\Gamma}{e}\big) = \ker(\Gamma)$. □

5. Proof of Theorem 2.23

We start with the following theorem, which is a consequence of the results in Sections 3 and 4; it encapsulates exactly what we will need from these sections for the proof of Theorem 2.23:

Theorem 5.1. *Let Q be the standard graded polynomial ring $k[x_0, \dots, x_{n+1}]$ with n even, and let f be a nonzero, homogeneous element of degree $e \geq 1$. Assume that* $\mathrm{Proj}(Q/f) \subseteq \mathbb{P}^{n+1}$ *is smooth. Define Ω_f to be the graded k-vector space*

$$\Omega_f := \frac{\Omega_Q^{n+2}}{df \cdot \Omega_Q^{n+1}} \cong \frac{k[x_0, \dots, x_{n+1}]}{(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_{n+1}})} dx_0 \cdots dx_{n+1}.$$

(1) *$\bigoplus_m HN_{2m}(\mathrm{mf}(f))$ is a free (homologically) graded $k[u]$-module of finite rank. In particular, $HN_{2m}(\mathrm{mf}(f)) = 0$ for $m \gg 0$, and multiplication by u determines an isomorphism*

$$HN_{2m+2}(\mathrm{mf}(f)) \xrightarrow{\cong} HN_{2m}(\mathrm{mf}(f))$$

for $m \ll 0$.

(2) *The grading operator Γ on the de Rham HN complex $HN^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft)$ induces an operator on $HH_*^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft)$, and, for all $m \in \mathbb{Z}$, there is an isomorphism*

$$[\Omega_f]_{((n/2)+1-m)\cdot e} \xrightarrow{\cong} \Gamma_0 HH_{2m}^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft)$$

induced by sending $\omega \in [\Omega_Q^{n+2}]_{((n/2)+1-m)\cdot e}$ to the class $\omega t^{(n/2)+1-m}$.

(3) *For each m, we have a short exact sequence*

$$0 \to HN(\mathrm{mf}(f))_{2m+2} \xrightarrow{u} HN_{2m}(\mathrm{mf}(f)) \xrightarrow{q} [\Omega_f]_{((n/2)+1-m)\cdot e} \to 0; \tag{5.2}$$

here, the map q is the composition

$$\begin{aligned} HN_{2m}(\mathrm{mf}(f)) \xrightarrow{\cong} HN_{2m}^{\mathrm{dR}}(Q[t, t^{-1}], ft) &\xrightarrow{dt \mapsto 0} \Gamma_0 HN_{2m}^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft) \\ &\xrightarrow{u \mapsto 0} \Gamma_0 HH_{2m}^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft) \\ &\xrightarrow{\cong} [\Omega_f]_{((n/2)+1-m)\cdot e}, \end{aligned}$$

where the first isomorphism is from part (2) of Theorem 3.6, and the last is (the inverse of) the isomorphism in (2).

Proof. Part (1) follows from Theorem 3.14(2) and Corollary 4.7, using that $\Gamma_0(k[\![v]\!][t, t^{-1}]) = k[u]$. Let us now prove (2) and (3). To ease notation, we write $V = HN^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft)$ and $W = HH^{\mathrm{dR},\mathbb{Z}/2}(Q[t, t^{-1}], ft)$. Since the homologies of both V and W vanish in odd degrees, the distinguished triangle

$$V[2] \xrightarrow{u\cdot} V \xrightarrow{u \mapsto 0} W \to$$

(where $V[2]$ denotes the shift of V by 2 in homological degree) yields a short exact sequence

$$0 \to V_{2m+2} \xrightarrow{u\cdot} V_{2m} \xrightarrow{u \mapsto 0} W_{2m} \to 0.$$

Since $\deg(u) = 0$, the map $V \xrightarrow{u\cdot} V$ commutes with Γ; it follows that Γ induces an operator on W. Moreover, this operator commutes with the differential on W and therefore induces an operator on its homology. The isomorphism $\Sigma^d \Omega_f \cong HH^{\mathrm{dR},\mathbb{Z}/2}(Q[t,t^{-1}], ft)$ arising from Theorem 3.14(1) induces the desired isomorphism

$$[\Omega_f]_{((n/2)+1-m)\cdot e} \xrightarrow{\cong} \Gamma_0 HH^{\mathrm{deR},\mathbb{Z}/2}_{2m}(\mathrm{mf}(f)),$$

which proves (2).

We evidently have an exact sequence

$$0 \to \Gamma_0 V_{2m+2} \xrightarrow{u\cdot} \Gamma_0 V_{2m} \xrightarrow{u\mapsto 0} \Gamma_0 W_{2m}.$$

It is straightforward to check that $\Gamma_0 V_{2m} \xrightarrow{u \mapsto 0} \Gamma_0 W_{2m}$ is surjective, so in fact we have a short exact sequence

$$0 \to \Gamma_0 V_{2m+2} \xrightarrow{u\cdot} \Gamma_0 V_{2m} \xrightarrow{u\mapsto 0} \Gamma_0 W_{2m} \to 0$$

of vector spaces. Applying Corollary 4.7 again, we obtain the short exact sequence

$$0 \to HN^{\mathrm{dR}}_{2m+2}(Q[t,t^{-1}], ft) \xrightarrow{u\cdot} HN^{\mathrm{dR}}_{2m}(Q[t,t^{-1}], ft) \xrightarrow{u\mapsto 0} \Gamma_0 HH^{\mathrm{dR},\mathbb{Z}/2}_{2m}(\mathrm{mf}(f)) \to 0.$$

The square

$$\begin{array}{ccc} HN_{2m+2}(\mathrm{mf}(f)) & \xrightarrow{u\cdot} & HN_{2m}(\mathrm{mf}(f)) \\ \downarrow \cong & & \downarrow \cong \\ HN^{\mathrm{dR}}_{2m+2}(Q[t,t^{-1}], ft) & \xrightarrow{u\cdot} & HN^{\mathrm{dR}}_{2m}(Q[t,t^{-1}], ft) \end{array}$$

evidently commutes, where the vertical isomorphisms arise from Theorem 3.6. We therefore arrive at the short exact sequence

$$0 \to HN_{2m+2}(mf(f)) \xrightarrow{u\cdot} HN_{2m}(mf(f)) \xrightarrow{u\mapsto 0} \Gamma_0 HH^{\mathrm{dR},\mathbb{Z}/2}_{2m}(\mathrm{mf}(f)) \to 0.$$

The exactness of (5.2) therefore follows from the commutativity of the triangle

$$\begin{array}{ccc} HN_{2m}(mf(f)) & \xrightarrow{u\mapsto 0} & \Gamma_0 HH^{\mathrm{dR},\mathbb{Z}/2}_{2m}(\mathrm{mf}(f)) \\ & \searrow^{q} & \uparrow \cong \\ & & [\Omega_f]_{((n/2)+1-m)\cdot e} \end{array}$$

where the vertical isomorphism is given by (2). ☐

Corollary 5.3. *In the setting of Theorem 5.1, the k-vector space $HP_0(\mathrm{mf}(f))$ has dimension equal to $\dim_{\mathbb{C}}[\Omega_f]_{\mathbb{Z}\cdot e}$. Moreover, setting $F^p_{\mathrm{nc}} = F^p_{\mathrm{nc}} HP_0(\mathrm{mf}(f))$, we have canonical isomorphisms*

$$\frac{F^p_{\mathrm{nc}}}{F^{p+1}_{\mathrm{nc}}} \cong [\Omega_f]_{((n/2)+1-p)\cdot e}$$

for each integer p.

Remark 5.4. Note that we are not claiming that there is a *canonical* isomorphism $HP_0(\mathrm{mf}(f)) \cong [\Omega_f]_{\mathbb{Z}\cdot e}$.

Remark 5.5. It follows from the definition of a Hodge structure of weight 0 that the intersection of $\mathrm{Hdg}(HP_0(\mathrm{mf}(f)))$ with F^1_{nc} is 0. Thus, the composition

$$\mathrm{Hdg}(HP_0(\mathrm{mf}(f))) \hookrightarrow F^0_{\mathrm{nc}} \xrightarrow{\mathrm{can}} F^0_{\mathrm{nc}}/F^1_{\mathrm{nc}} \cong [\Omega_f]_{((n+2)/2)\cdot e}$$

is injective; in particular, the Hodge classes of $HP_0(\mathrm{mf}(f))$ may be identified with a rational subspace of $[\Omega_f]_{((n+2)/2)\cdot e}$. As a consequence, we see that there is no information lost when passing from the Chern character map $\mathrm{ch}_{HN} : K_0(\mathrm{mf}(f)) \to HN_0(\mathrm{mf}(f))$ taking values in negative cyclic homology to the a priori coarser map $\mathrm{ch}_{HH} : K_0(\mathrm{mf}(f)) \to HH_0(\mathrm{mf}(f))$ given by the composition

$$K_0(\mathrm{mf}(f)) \xrightarrow{\mathrm{ch}_{HN}} HN_0(\mathrm{mf}(f)) \xrightarrow{u \mapsto 0} HH_0(\mathrm{mf}(f)).$$

5.1. *The commutative diagram.*

We now prove Theorem 2.23(2). This follows from the existence and properties of the diagram

$$\begin{array}{ccccc}
K_0(X) & \xrightarrow{=} & K_0(X) & \longrightarrow & K_0(\mathrm{D}^{\mathrm{sg}}(R)) \\
\downarrow & & \downarrow & & \downarrow \\
KU^0(X) & \xrightarrow{\cong} & K_0^{\mathrm{top}}(X) & \twoheadrightarrow & K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R)) \\
\downarrow{\scriptstyle \mathrm{ch}^{\mathrm{top}}} & & \downarrow{\scriptstyle \mathrm{ch}^{\mathrm{top}}} & & \downarrow{\scriptstyle \mathrm{ch}^{\mathrm{top}}} \\
H^{\mathrm{even}}(X;\mathbb{C}) & \xrightarrow{\cong} & HP_0(X) & \twoheadrightarrow & HP_0(\mathrm{D}^{\mathrm{sg}}(R))
\end{array} \tag{5.6}$$

The top vertical maps are the canonical ones. The map $KU^0(X) \to K_0^{\mathrm{top}}(X)$ is Blanc's comparison isomorphism, and the bottom horizontal map on the left is the HKR isomorphism. Letting E denote K, K^{top} or HP, the horizontal maps on the right side are defined by the sequence maps

$$E_0(X) \overset{p^*}{\twoheadrightarrow} E_0(W) \xleftarrow{\cong} E_0(\mathrm{D}^{\mathrm{b}}(R)) \xrightarrow{\mathrm{can}} E_0(\mathrm{D}^{\mathrm{sg}}(R)), \tag{5.7}$$

where $W = \mathrm{Spec}(R)$, $p : W \to X$ is the map given by modding out the $\mathbb{C}^*$ action, and the isomorphism is given by Proposition 2.17. (The fact that p^* is indeed surjective, as indicated, will be justified below.)

In particular, diagram (2.24) is the "boundary" of diagram (5.6). It therefore suffices to prove (5.6) commutes, the two maps $K_0^{\mathrm{top}}(X) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$ and $HP_0(X) \to HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ are surjective as indicated, and the images of the maps $K_0(X) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$ and $K_0(\mathrm{D}^{\mathrm{sg}}(R)) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$ coincide.

The commutativity of the top left square of (5.6) is a consequence of the construction of Blanc's map, and the bottom left square commutes by [Blanc 2016, Proposition 4.32]. The right side of this diagram commutes by the naturality of the map from algebraic to topological K-theory and the topological Chern character map. Let us now justify that $p^* : E_0(X) \to E_0(W)$ is onto for each of $E = K$, K^{top} or HP. Toward this goal, let Y be the blow-up of $\mathrm{Spec}(R)$ at $\mathfrak{m}$. The fiber of this blow-up is X, and the inclusion $i : X \hookrightarrow Y$ is the zero section of a map $\pi : Y \to X$ making Y into a line bundle over X. Moreover, we may identify W with $Y \setminus X$; let $j : W \hookrightarrow Y$ denote the canonical open immersion. Then we have $p = \pi \circ j$, and since π is a line bundle over a smooth base, $\pi^* : E_0(X) \xrightarrow{\cong} E_0(Y)$ is an isomorphism. Since Y, W and X are all smooth, dévissage gives that j^* fits into the long exact sequence

$$\cdots \to E_0(Y) \xrightarrow{j^*} E_0(W) \to E_{-1}(X) \to \cdots .$$

For each of these functors, we have $E_{-1}(X) = 0$, and thus p^* is surjective, as claimed. Now assume $E = K^{\mathrm{top}}$ or $E = HP$. Proposition 2.18 gives that $E_0(\mathrm{D}^{\mathrm{b}}(R)) \to E_0(\mathrm{D}^{\mathrm{sg}}(R))$ is onto, and thus the rightmost map in (5.7) is also surjective in these two cases. This proves the lower two horizontal maps on the right side of (5.6) are surjections as indicated.

To complete the proof, it suffices to show the image of the map

$$K_0(\mathrm{D}^{\mathrm{sg}}(R)) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$$

and the image of the composition

$$G_0(R) = K_0(\mathrm{D}^{\mathrm{b}}(R)) \to K_0(\mathrm{D}^{\mathrm{sg}}(R)) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$$

coincide.

Remark 5.8. The map $K_0(\mathrm{D}^{\mathrm{b}}(R)) \to K_0(\mathrm{D}^{\mathrm{sg}}(R))$ itself need not be onto, due to the fact that $K_{-1}(R)$ is typically nonzero; see [Cortiñas et al. 2013].

However, the map $K_0(\mathrm{D}^{\mathrm{b}}(R)) \to K_0(\mathrm{D}^{\mathrm{sg}}(R))$ is onto "up to $\mathbb{A}^1$-homotopy". In detail, consider the diagram

$$\begin{array}{ccccccc}
K_0(\mathrm{D}^{\mathrm{b}}(R[x])) & \longrightarrow & K_0(\mathrm{D}^{\mathrm{sg}}(R[x])) & \longrightarrow & K_{-1}(\mathrm{Perf}(R[x])) & \longrightarrow & 0 \\
\downarrow{\scriptstyle i_0^* - i_1^*} & & \downarrow{\scriptstyle i_0^* - i_1^*} & & \downarrow{\scriptstyle i_0^* - i_1^*} & & \\
K_0(\mathrm{D}^{\mathrm{b}}(R)) & \longrightarrow & K_0(\mathrm{D}^{\mathrm{sg}}(R)) & \longrightarrow & K_{-1}(\mathrm{Perf}(R)) & \longrightarrow & 0
\end{array}$$

with exact rows and in which the vertical maps are given by the difference of the two maps induced by setting x equal to 0 and 1. (We may model D^{b} as bounded below complexes of finitely generated projective modules with bounded homology, and with this model it is clear that setting x equal to any constant determines a dg-functor. This restricts to a dg-functor on Perf and hence on D^{sg}.) Let us write $\overline{K_0(-)}$ for the cokernels of the columns of this diagram, so that we have a right exact sequence

$$\overline{K_0(\mathrm{D}^{\mathrm{b}}(R))} \to \overline{K_0(\mathrm{D}^{\mathrm{sg}}(R))} \to \overline{K_{-1}(\mathrm{Perf}(R))} \to 0. \tag{5.9}$$

The result we seek follows directly from the following two claims:

(1) the map $K_0(\mathrm{D}^{\mathrm{sg}}(R)) \to K_0^{\mathrm{top}}(\mathrm{D}^{\mathrm{sg}}(R))$ factors through the canonical surjection $K_0(\mathrm{D}^{\mathrm{sg}}(R)) \twoheadrightarrow \overline{K_0(\mathrm{D}^{\mathrm{sg}}(R))}$, and

(2) the map $\overline{K_0(\mathrm{D}^{\mathrm{b}}(R))} \to \overline{K_0(\mathrm{D}^{\mathrm{sg}}(R))}$ is onto.

The first claim follows from the fact that K^{top} is $\mathbb{A}^1$-homotopy invariant. For the second claim, since the functor $\overline{K_{-1}(-)}$ is $\mathbb{A}^1$-homotopy invariant and R is standard graded, we have $\overline{K_{-1}(R)} \cong K_{-1}(k) = 0$. The second claim therefore follows from (5.9).

5.2. ***An alternative description of the map α.*** We next establish in Lemma 5.13 an alternative description of the map $\alpha : H^n_{\mathrm{prim}}(X) \to HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ defined in (2.22). This description will be used to show that it is an isomorphism that preserves Hodge filtrations.

We begin with some setup. Let U denote the affine variety $\mathbb{P}^{n+1} \setminus X$. Applying the distinguished triangle (3.9), we obtain a dévissage long exact sequence

$$\cdots \to HP_1(\mathbb{P}^{n+1}) \to HP_1(U) \xrightarrow{\partial_{U,X}} HP_0(X) \to HP_0(\mathbb{P}^{n+1}) \to \cdots \tag{5.10}$$

(the subscript U, X on the boundary map is included to distinguish it from the other boundary maps we consider). We set

$$HP_0^{\mathrm{prim}}(X) := \ker(HP_0(X) \to HP_0(\mathbb{P}^{n+1})).$$

Since X is a smooth hypersurface in $\mathbb{P}^{n+1}$ of even dimension, we have

$$H^n_{\mathrm{prim}}(X; \mathbb{C}) \cong HP_0^{\mathrm{prim}}(X).$$

As $HP_1(\mathbb{P}^{n+1}) = 0$, it follows that there is an isomorphism

$$\partial_{U,X} : HP_1(U) \xrightarrow{\cong} HP_0^{\mathrm{prim}}(X). \tag{5.11}$$

Let $V = \mathrm{Spec}(Q[1/f])$ be the open complement of $\mathrm{Spec}(R) = \mathrm{Spec}(Q/f)$ in $\mathbb{A}^{n+2} = \mathrm{Spec}(Q)$. There is a canonical surjection $p : V \twoheadrightarrow U$ given by modding out the action of $\mathbb{C}^*$, and it induces a map

$$p^* : HP_1(U) \to HP_1(V).$$

We also have the dévissage long exact sequence

$$\cdots \to HP_1(\mathbb{A}^{n+2}) \to HP_1(V) \xrightarrow{\partial_{V,R}} HP_0^{\mathrm{BM}}(R) \to HP_0(\mathbb{A}^{n+2}) \to HP_0(V) \to \cdots . \tag{5.12}$$

Since $HP_1(\mathbb{A}^{n+2}) = 0$, and $HP_0(\mathbb{C}) \cong HP_0(\mathbb{A}^{n+2}) \to HP_0(V)$ is injective, the boundary map $\partial_{V,R}$ is an isomorphism.

Lemma 5.13. *The composition*

$$H^n_{\mathrm{prim}}(X,\mathbb{C}) \cong HP_0^{\mathrm{prim}}(X) \xrightarrow[\cong]{\partial_{U,X}^{-1}} HP_1(U) \xrightarrow{p^*} HP_1(V) \xrightarrow[\cong]{\partial_{V,R}} HP_0^{\mathrm{BM}}(R) \twoheadrightarrow HP_0(\mathrm{D}^{\mathrm{sg}}(R))$$

coincides with the map α defined in (2.22).

Proof. This is a diagram chase involving the long exact sequences (5.10) and (5.12), as well as the dévissage long exact sequence

$$\cdots \to HP_1(V) \xrightarrow{\partial_{V,W}} HP_0(W) \to HP_0(\mathbb{A}^{n+2} \setminus \{0\}) \to HP_0(V) \to \cdots , \tag{5.14}$$

where, as above, $W = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. In a bit more detail, the naturality of these dévissage sequences, along with the fact that (5.14) maps to both (5.10) and (5.12), yields the commutative diagrams

$$\begin{array}{ccc} HP_1(U) & \xrightarrow{p^*} & HP_1(V) \\ \downarrow{\scriptstyle \partial_{U,X}} & & \downarrow{\scriptstyle \partial_{V,X}} \\ HP_0(X) & \xrightarrow{p^*} & HP_0(W) \end{array} \quad \text{and} \quad \begin{array}{ccc} HP_1(V) & & \\ \downarrow{\scriptstyle \partial_{V,W}} & \searrow{\scriptstyle \partial_{V,R}} & \\ HP_0(W) & \xleftarrow[\cong]{} & HP_0^{\mathrm{BM}}(R) \end{array}$$

The statement follows. □

Proposition 5.15. *The map α is an isomorphism.*

Proof. By Theorem 3.6 and Lemma 5.13, it suffices to prove that the composition

$$HP_1^{\mathrm{deR}}(U) \xrightarrow{p^*} HP_1^{\mathrm{deR}}(V) \xrightarrow[\cong]{\partial^{\mathrm{deR}}_{V,R}} HP_0^{\mathrm{deR}}(Q[t], ft) \twoheadrightarrow HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft) \tag{5.16}$$

is an isomorphism, where $\partial^{\mathrm{deR}}_{V,R}$ is the de Rham version of the boundary map $\partial_{V,R}$ in (5.12). By Proposition 2.18, the last map in this composition is surjective, and its kernel is given by the image of $\mathbb{C} \xrightarrow{\cong} HP_0^{\mathrm{deR}}(R) \to HP_0^{\mathrm{deR}}(Q[t], ft)$; we begin by identifying this image. We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow[1\mapsto[R]]{\cong} & K_0(R) & \longrightarrow & G_0(R) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ch}_{HP}} & & \downarrow{\scriptstyle \mathrm{ch}_{HP}} \\ \mathbb{C} & \xrightarrow{\cong} & HP_0^{\mathrm{deR}}(R) & \longrightarrow & HP_0^{\mathrm{deR}}(Q[t], tf) \end{array}$$

where the rightmost horizontal maps are induced by the inclusion $\mathrm{Perf}(R) \hookrightarrow \mathrm{D^b}(R)$, and the leftmost vertical map is the inclusion. The image of $1 \in \mathbb{C}$ under the composition $\mathbb{C} \xrightarrow{\cong} HP_0^{\mathrm{deR}}(R) \to HP_0^{\mathrm{deR}}(Q[t], ft)$ is therefore $\mathrm{ch}_{HP}([R])$, which is equal to $[df\,dt]$ by [Brown and Walker 2020a, Example 6.4].

The formula for $\partial_{V,R}^{\mathrm{deR}}$ given in Theorem 3.13 implies that $\partial_{V,R}^{\mathrm{deR}}([df/f]) = [df\,dt]$. We thus need only show that the map

$$HP_1^{\mathrm{deR}}(U) \oplus \mathbb{C} \xrightarrow{(p^*,[df/f])} HP_1^{\mathrm{deR}}(V), \tag{5.17}$$

where $[df/f] \in HP_1^{\mathrm{deR}}(V)$, is an isomorphism. This appears to be well-known (see, e.g., [Dimca 1992, Chapter 6, Section 1]), but we sketch a proof.

Set $A := \mathbb{C}[x_0, \dots, x_{n+1}][1/f]$, and recall that $V = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(A_0)$. The Euler derivation gives a contracting homotopy on the internal degree j part of the de Rham complex (Ω_A^*, d) for all $j \neq 0$, and thus we may identify the de Rham cohomology of V with the cohomology of the complex $([\Omega_A^*]_0, d)$. Moreover, we have an isomorphism

$$\Omega_{A_0}^* \oplus \Omega_{A_0}^{*-1} \xrightarrow{\cong} [\Omega_A^*]_0 \tag{5.18}$$

given by $(\alpha, \beta) \mapsto \alpha + (df/f)\beta$. This gives an isomorphism

$$(p^*, [df/f]p^*) : H_{\mathrm{deR}}^m(U) \oplus H_{\mathrm{deR}}^{m-1}(U) \xrightarrow{\cong} H_{\mathrm{deR}}^m(V)$$

for each m. The isomorphism (5.17) thus follows from the HKR isomorphisms $HP_1^{\mathrm{deR}}(V) \cong H_{\mathrm{deR}}^{\mathrm{odd}}(V)$ and $HP_0^{\mathrm{deR}}(U) \cong H_{\mathrm{deR}}^{\mathrm{even}}(U)$, along with the fact that, since U is the complement of a smooth projective hypersurface of even dimension, we have $H_{\mathrm{deR}}^{\mathrm{even}}(U) = H_{\mathrm{deR}}^0(U) = \mathbb{C}$. □

5.3. *Spanning set for $HN_{2m}^{\mathrm{deR}}(Q[t, t^{-1}], ft)$.* Fix $m \in \mathbb{Z}$. We next exhibit an explicit spanning set for $HN_{2m}^{\mathrm{deR}}(Q[t, t^{-1}], ft)$ as a complex vector space. This is the content of Lemma 5.24, which plays a key role in the identification of the "polar filtration" on $HP_1(U)$ and the nc Hodge filtration on $HP_0(\mathrm{D^{sg}}(R))$ (Lemma 5.27). If $j \leq \frac{n+2}{2} - m$, then given $\omega \in [\Omega_Q^{n+2}]_{j \cdot \deg(f)}$, we define

$$\psi_{m,j}(\omega) := \frac{(-1)^j}{j!} d(\varepsilon_Q(\omega) t^j) u^{\frac{1}{2}(n+2)-m-j} \in HN^{\mathrm{deR}}(Q[t, t^{-1}], ft), \tag{5.19}$$

where ε_Q is as defined in the proof of Theorem 4.6. Observe that $\psi_{m,j}(\omega)$ has homological degree $2m$ and internal degree 0. We have

$$d_Q(\varepsilon_Q(\omega)) = d_Q(\varepsilon_Q(\omega)) + \varepsilon_Q(d_Q(\omega)) = \deg(\omega)\omega = j\deg(f)\omega,$$

where the first equality holds since $d_Q(\omega) = 0$, and the second follows from the proof of Theorem 4.6. We may therefore equivalently write

$$\psi_{m,j}(\omega) = \frac{(-1)^j \deg(f)}{(j-1)!}\left(\omega t^j - \frac{\varepsilon_Q(\omega)}{\deg(f)} t^{j-1} dt\right) u^{\frac{1}{2}(n+2)-m-j}. \qquad (5.20)$$

Using (5.20) along with Euler's formula

$$\deg(f)\cdot f = \sum_{i=0}^{n+1} \frac{\partial f}{\partial x_i} x_i,$$

one sees that $\psi_{m,j}(\omega)$ is a cycle, and so it determines a class in $HN_{2m}^{\mathrm{deR}}(Q[t,t^{-1}], ft)$. We write ψ_m for the induced map

$$\psi_m : \bigoplus_{j \le \frac{1}{2}(n+2)-m} [\Omega_Q^{n+2}]_{j\cdot\deg(f)} \to HN_{2m}^{\mathrm{deR}}(Q[t,t^{-1}], ft). \qquad (5.21)$$

Setting $m = 0$, and replacing HN with HP, we obtain the map

$$\psi : [\Omega_Q^{n+2}]_{\mathbb{Z}\cdot\deg(f)} := \bigoplus_{j\in\mathbb{Z}} [\Omega_Q^{n+2}]_{j\cdot\deg(f)} \to HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft) \qquad (5.22)$$

given by the same formula: if $\omega \in [\Omega_Q^{n+2}]_{j\cdot\deg(f)}$, then $\psi(\omega)$ is the class of

$$\frac{(-1)^j}{j!} d(\varepsilon_Q(\omega)t^j) u^{\frac{1}{2}(n+2)-j}.$$

Remark 5.23. The composition of ψ_m with

$$HN_{2m}^{\mathrm{deR}}(Q[t,t^{-1}], ft) \xrightarrow{\mathrm{can}} HP_{2m}^{\mathrm{deR}}(Q[t,t^{-1}], ft) \xrightarrow{u^m} HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft)$$

coincides with the restriction of ψ to

$$\bigoplus_{j \le \frac{1}{2}(n+2)-m} [\Omega_Q^{n+2}]_{j\cdot\deg(f)}.$$

Lemma 5.24. *The map ψ_m in (5.21) is a surjection for all m, and the map ψ in (5.22) is a surjection.*

Proof. To ease notation, we write HN_* for $HN_*^{\mathrm{deR}}(Q[t,t^{-1}], ft)$. Let $e = \deg(f)$ and $p = \frac{n+2}{2}$, and consider the following diagram, in which the right column is given by Theorem 5.1(2), and the map γ will be defined shortly:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{0\le j\le p-m-1} [\Omega_Q^{n+2}]_{j\cdot e} & \xrightarrow{\psi_{m+1}} & HN_{2m+2} \\
\downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle u\cdot} \\
\bigoplus_{0\le j\le p-m} [\Omega_Q^{n+2}]_{j\cdot e} & \xrightarrow{\psi_m} & HN_{2m} \\
\downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle q} \\
[\Omega_Q^{n+2}]_{(p-m)\cdot e} & \overset{\gamma}{\twoheadrightarrow} & [\Omega_f]_{(p-m)\cdot e} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The formula for ψ_m and the description of q in Theorem 5.1 imply that

$$
q(\psi_m(\omega)) = \begin{cases} \dfrac{(-1)^{p-m}\deg(f)}{(p-m-1)!}\overline{\omega} & \text{if } |\omega| = (p-m)\deg(f), \\ 0 & \text{otherwise.} \end{cases}
$$

Setting

$$
\gamma(\omega) = \frac{(-1)^{p-m}\deg(f)}{(p-m-1)!}\overline{\omega}
$$

thus makes the diagram commute, and it is clear that γ is a surjection. Since $HN_{2m} = 0$ for $m \gg 0$ by Theorem 5.1(1), it follows by descending induction that ψ_m is surjective for all m. It follows from Remark 5.23 that the map ψ is therefore also surjective. □

5.4. *Relating filtrations.* The goal of this section is to relate the "polar filtration" on $HP_1(U)$ (defined below) with the nc Hodge filtration on $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$; this leads quickly to a proof that the isomorphism α preserves filtrations. Recall that

$$
X = \mathrm{Proj}(Q/f) \subseteq \mathbb{P}^{n+1},
$$

and U denotes the affine variety $\mathbb{P}^{n+1} \setminus X$. As above, we also set $V = \mathrm{Spec}(Q[1/f])$. The map $p : V \to U$ given by modding out by the $\mathbb{C}^*$-action on V induces a map $\Omega_U^\bullet \to \Omega_V^\bullet$ via pullback. As explained in [Dimca 1992, Chapter 6, Section 1], p^* induces a chain isomorphism from $\Omega_U^\bullet$ to a subcomplex of $\Omega_V^\bullet$; in more detail, we have

$$
\Omega_U^j \xrightarrow{\cong} \left\{ \frac{\varepsilon_Q(\alpha)}{f^s} \in \Omega_V^j : s \ge 0,\ \alpha \in [\Omega_Q^{j+1}]_{s\deg(f)} \right\} \tag{5.25}
$$

for all j, where ε_Q is as defined in the proof of Theorem 4.6. Given $\omega \in \Omega^j_U$, we let $\mathrm{ord}(\omega)$ denote the minimum s such that there is a representation of ω of the form $\varepsilon_Q(\alpha)/f^s$ as above.

Definition 5.26 [Dimca 1992, Chapter 6, Definition 1.28]. The *polar filtration* on $\Omega^\bullet_U$ is given by

$$P^s\Omega^i_U := \begin{cases} \{\omega \in \Omega^i_U : \mathrm{ord}(\omega) \le i - s + 1\}, & i - s + 1 \ge 0, \\ 0, & i - s + 1 < 0. \end{cases}$$

The polar filtration induces a filtration $P^\bullet HP^{\mathrm{deR}}_*(U)$ on homology in the evident way.

Let ϕ denote the composition

$$[\Omega^{n+2}_Q]_{\mathbb{Z}\cdot\deg(f)} \to HP^{\mathrm{deR}}_1(V) \to HP^{\mathrm{deR}}_1(U)$$

given by sending $\omega \in [\Omega^{n+2}_Q]_{j\cdot\deg(f)}$ to $(\varepsilon_Q(\omega)/f^j)u^{n/2} \in HP^{\mathrm{deR}}_1(V)$ and then applying the isomorphism (5.25). Define

$$\beta : HP^{\mathrm{deR}}_1(U) \to HP^{\mathrm{deR}}_0(\mathrm{D^b}(R))$$

to be the composition

$$HP^{\mathrm{deR}}_1(U) \xrightarrow{p^*} HP^{\mathrm{deR}}_1(V) \xrightarrow{\partial^{\mathrm{deR}}_{V,R}} HP^{\mathrm{deR}}_0(Q[t], ft) \xrightarrow{\mathrm{can}} HP^{\mathrm{deR}}_0(Q[t, t^{-1}], ft),$$

where $\partial^{\mathrm{deR}}_{V,R}$ is the de Rham version of the boundary map $\partial_{V,R}$ in the dévissage long exact sequence (5.12), and can is the canonical map.

Lemma 5.27. *The diagram*

$$\begin{array}{ccc} & [\Omega^{n+2}_Q]_{\mathbb{Z}\cdot\deg(f)} & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\psi} \\ HP^{\mathrm{deR}}_1(U) & \xrightarrow[\cong]{\beta} & HP^{\mathrm{deR}}_0(Q[t, t^{-1}], ft) \end{array}$$

commutes, ϕ and ψ are surjective, and β is an isomorphism. Moreover, β induces an isomorphism

$$P^{s+(n/2)+1}HP^{\mathrm{deR}}_1(U) \xrightarrow{\cong} F^s_{\mathrm{nc}}HP^{\mathrm{deR}}_0(Q[t, t^{-1}], ft).$$

Remark 5.28. It follows from [Griffiths 1969, (8.6)] (see also [Dimca 1992, Chapter 6, Section 1]) that ϕ is surjective; our argument in the proof of Lemma 5.27 gives a new proof of this fact.

Proof. Let $\omega \in \Omega^{n+1}_Q$, where $|\omega| = j \cdot \deg(f)$. We have

$$(\beta \circ \phi)(\omega) = (\mathrm{can} \circ \partial^{\mathrm{deR}}_{V,R})\left(\frac{\varepsilon(\omega)}{f^j}u^{n/2}\right) = \frac{(-1)^j}{j!}d(\varepsilon(\omega)t^j)u^{(n/2)+1-j} = \psi(\omega),$$

where the first and third equalities follow immediately from the definitions of ϕ and ψ, and the second is a consequence of Theorem 3.13. Thus, the diagram commutes. It follows from Lemma 5.13 and Proposition 5.15 that β is an isomorphism. The map ψ is surjective by Lemma 5.24, and so ϕ is surjective as well. Finally, suppose $y \in P^{s+(n/2)+1}HP_1^{\mathrm{deR}}(U)$, so that $\mathrm{ord}(y) \le \frac{n}{2} + 1 - s$. Choose $\omega \in [\Omega_Q^{n+2}]_{\mathrm{ord}(y)\cdot\deg(f)}$ such that $\phi(\omega) = y$. We have

$$\beta(y) = \psi(\omega) \in F_{\mathrm{nc}}^{(n/2)+1-\mathrm{ord}(y)}(HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft)) \subseteq F_{\mathrm{nc}}^s HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft).$$

This shows that β maps

$$P^{s+(n/2)+1}HP_1^{\mathrm{deR}}(U) \mapsto F_{\mathrm{nc}}^s HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft),$$

and a similar argument shows that β^{-1} maps

$$F_{\mathrm{nc}}^s HP_0^{\mathrm{deR}}(Q[t,t^{-1}], ft) \mapsto P^{s+(n/2)+1}HP_1^{\mathrm{deR}}(U). \qquad \square$$

5.5. *Completion of the proof.*

Proof of Theorem 2.23. The first two conditions in Properties 2.8 follow from part (3) of the theorem, and the third is a consequence of [Khan 2023, Theorem B] (see also [Brown and Walker 2024, Theorem 1.4]). We proved (2) in Section 5.1; it therefore remains to prove (3). By Proposition 5.15, the map α is an isomorphism. The commutativity of (2.24) and the surjectivity of (2.20) imply that α identifies rational structures, so we need only show that α induces an isomorphism

$$F^s H_{\mathrm{prim}}^n\Big(X; \mathbb{C}\Big(\frac{n}{2}\Big)\Big) \xrightarrow{\cong} F_{\mathrm{nc}}^s HP_0(\mathrm{D}^{\mathrm{sg}}(R))$$

for all $s \in \mathbb{Z}$. By [Griffiths 1969, (8.6)] (see also [Dimca 1992, Chapter 6, Section 1]), there is an isomorphism

$$P^s H^i(U; \mathbb{C}) \cong F^s H^i(U; \mathbb{C})$$

for all $i, s \in \mathbb{Z}$. We therefore have a chain of isomorphisms

$$F^s H_{\mathrm{prim}}^n\Big(X; \mathbb{C}\Big(\frac{n}{2}\Big)\Big) \xleftarrow{\cong} F^s H^{n+1}\Big(U; \mathbb{C}\Big(\frac{n}{2}+1\Big)\Big) \xrightarrow{\cong} P^s H^{n+1}\Big(U; \mathbb{C}\Big(\frac{n}{2}+1\Big)\Big) \xrightarrow{\cong} P^{s+(n/2)+1}HP_1^{\mathrm{deR}}(U);$$

the first is the boundary map in the evident long exact sequence, and the third is the identification of singular and de Rham cohomology. Applying Lemma 5.27 therefore finishes the proof. $\square$

6. Examples

Let $R = \mathbb{C}[x_0, \dots, x_{n+1}]/(f)$, where f is a homogeneous polynomial such that the projective hypersurface $X = \operatorname{Proj}(R) \subseteq \mathbb{P}^{n+1}$ is smooth, and assume n is even. By Theorem 2.23(3), the dg-category $\mathrm{D}^{\mathrm{sg}}(R)$ satisfies the nc Hodge condition if and only if the Hodge conjecture holds for X. In this section, we study the Hodge classes in $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ in several cases in which the Hodge conjecture holds for X. Let us start with the simplest example:

Example 6.1 (the $n = 0$ case). In this case, X is a collection of points, and so the Hodge conjecture clearly holds for X. The complexified Chern character map $K_0(X)_{\mathbb{C}} \to H^{\mathrm{even}}(X; \mathbb{C})$ is surjective, and so the same is true of

$$\mathrm{ch}_{HP} : K_0(\mathrm{D}^{\mathrm{sg}}(R))_{\mathbb{C}} \to HP_0(\mathrm{D}^{\mathrm{sg}}(R));$$

in other words, $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ is spanned by Hodge classes. Write $f = \ell_1 \cdots \ell_d$, with each ℓ_i homogeneous of degree 1, and let M_i be the R-module $\mathbb{C}[x_0, x_1]/(\ell_i)$. It is not hard to check that $K_0(\mathrm{D}^{\mathrm{sg}}(R))$ is generated by $[M_1], \dots, [M_d]$ modulo the relation $\sum_{i=1}^{d} [M_i] = 0$. It follows that $HP_0(\mathrm{D}^{\mathrm{sg}}(R))$ is generated by $\mathrm{ch}_{HP}[M_1], \dots, \mathrm{ch}_{HP}[M_d]$ modulo the analogous relation.

Before we consider more complicated examples, we must discuss some background on Chern characters of matrix factorizations.

6.1. *Chern characters of matrix factorizations.* It follows from the calculations in [Brown and Walker 2020a, Example 6.1] that the Chern character map

$$\mathrm{ch}_{HP} : K_0(\mathrm{mf}(f)) \to HP_0(\mathrm{mf}(f)) \cong HP_0^{\mathrm{dR}}(Q[t, t^{-1}], tf)$$

sends a class of the form[1] $[(A, B)] \in K_0(\mathrm{mf}(f))$ to the class

$$\frac{2t^{\frac{1}{2}(n+2)}}{(n+2)!} \operatorname{tr}\big((dA\, dB)^{\frac{1}{2}(n+2)}\big) \in HP_0^{\mathrm{dR}}(Q[t, t^{-1}], tf),$$

where dA and dB denote the square matrices with entries in Ω^1_Q obtained by applying the de Rham differential d to the entries of A and B. We note that [Brown and Walker 2020a, Example 6.1] concerns Chern characters taking values in negative cyclic homology relative to $k[t, t^{-1}]$ rather than k, but the exact same calculations yield the above formula in our setting.

The Chern character map is compatible with tensor products of matrix factorizations; let us explain what we mean by this. Suppose A and A' are k-algebras, $g \in A$,

[1] We recall that $K_0(\mathrm{mf}(f))$ is, by definition, the free abelian group generated by isomorphism classes of objects in the *idempotent completion* of $\mathrm{mf}(f)$ modulo relations arising from exact triangles. In particular, not every class in $K_0(\mathrm{mf}(f))$ is necessarily of the form $[(A, B)]$, where (A, B) is a matrix factorization of f.

and $g' \in A'$. If $F \in \mathrm{mf}(g)$, and $F' \in \mathrm{mf}(g')$, then we may form the tensor product $F \otimes_k F' \in \mathrm{mf}(f \otimes 1 + 1 \otimes f')$; see, e.g., [Yoshino 1998] for details. Assume now that $A = k[y_0, \dots, y_{m+1}]$ and $A' = k[y'_0, \dots, y'_{m'+1}]$, g and g' are homogeneous, and the hypersurfaces $A/(g)$ and $A'/(g')$ both have isolated singularities. In this case, we identify $g \otimes 1 + 1 \otimes g'$ with $g + g' \in k[y_0, \dots, y_{m+1}, y'_0, \dots, y'_{m'+1}]$. The tensor product functor induces a map

$$HP_0^{\mathrm{dR}}(\mathrm{mf}(g)) \otimes_k HP_0^{\mathrm{dR}}(\mathrm{mf}(g')) \to HP_0^{\mathrm{dR}}(\mathrm{mf}(g + g'))$$

given by multiplication, which we denote by $\gamma \otimes \gamma' \mapsto \gamma \cdot \gamma'$. A straightforward calculation shows that $\mathrm{ch}_{HP}(F) \cdot \mathrm{ch}_{HP}(F') = \mathrm{ch}_{HP}(F \otimes F')$.

6.2. ***Hodge classes of Fermat hypersurfaces.*** Assume now that $n \geq 2$, and suppose $f = x_0^m + \cdots + x_{n+1}^m$, so that X is a Fermat hypersurface. For the remainder of this section, we will write X as X_m^n. It follows from [Shioda 1979] that the Hodge classes in $H^n_{\mathrm{prim}}(X; \mathbb{C})$ can be explicitly described in the following way. Let μ_m denote the group of m-th roots of unity and G the quotient of μ_m^{n+2} by the diagonal subgroup. Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$; we identify $\widehat{G}$ with the group

$$\left\{(a_0, \dots, a_{n+1}) \in (\mathbb{Z}/m)^{n+2} : \sum_{i=0}^{n+1} a_i = 0\right\}$$

via the isomorphism described in [Shioda 1979, Section 1]. We fix the following notation:

- $U = \{(a_0, \dots, a_{n+1}) \in \widehat{G} : a_i \neq 0 \text{ for all } i\}$.
- For $a \in \mathbb{Z}/m$, we let $\langle a \rangle$ denote the unique representative of a between 0 and $m - 1$.
- For $\alpha = (a_0, \dots, a_{n+1}) \in U_m^n$, we set $|\alpha| = \sum_{i=0}^{n+1} \langle a_i \rangle / m$.
- $B = \left\{\alpha \in U : |t\alpha| = \frac{n}{2} + 1 \text{ for all } t \in (\mathbb{Z}/m)^\times\right\}$.

The group G acts on X_m^n by scaling the variables, and so G acts on $H^n_{\mathrm{prim}}(X; \mathbb{C})$ as well. Given $\alpha \in \widehat{G}$, define

$$V(\alpha) = \{\xi \in H^n_{\mathrm{prim}}(X_m^n) : g^*(\xi) = \alpha(g)\xi \text{ for all } g \in G\}.$$

The following calculation of the Hodge classes of X_m^n is due to Shioda:

Theorem 6.2 [Shioda 1979, Theorem I]. *Given $\alpha \in \widehat{G}$, we have $\dim_{\mathbb{C}} V(\alpha) = 0$ or 1, and $\dim_{\mathbb{C}} V(\alpha) = 1$ if and only if $\alpha \in U$. Moreover, the complexified Hodge classes of X_m^n may be described as follows*:

$$\mathrm{Hdg}(X_m^n) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\alpha \in B} V(\alpha).$$

In particular, $\dim_{\mathbb{Q}} \mathrm{Hdg}(X_m^n) = |B|$.

Shioda [1979] applies this result to confirm a family of cases of the Hodge conjecture for Fermat hypersurfaces; see also [da Silva 2021].

Example 6.3. If $m = 2$, then $B = \{(1, \dots, 1)\}$, and so $\dim_{\mathbb{Q}} \mathrm{Hdg}(X_2^n) = 1$. On the other hand, by Knörrer periodicity, the dg-category $\mathrm{D}^{\mathrm{sg}}(R)$ has exactly one indecomposable object up to homotopy equivalence, namely the tensor product of the matrix factorization $(x + iy, x - iy)$ with itself $\frac{n+2}{2}$ times. The Chern character of this matrix factorization is $(-2i)^{(n+2)/2} dx_0 \cdots dx_{n+1}$, and so this class gives a basis for $\mathrm{Hdg}(\mathrm{D}^{\mathrm{sg}}(R))$.

Example 6.4. Now suppose $m = 3$ and $n = 2$, so that $f = x_0^3 + x_1^3 + x_2^3 + x_3^3$. In this case, we have

$$B = \{(1,1,2,2), (1,2,1,2), (1,2,2,1), (2,1,1,2), (2,1,2,1), (2,2,1,1)\},$$

and so $\dim_{\mathbb{Q}} \mathrm{Hdg}(X_3^2) = 6$. As discussed in the introduction, the Hodge conjecture is known to hold for all surfaces, and so $\mathrm{D}^{\mathrm{sg}}(R)$ satisfies the nc Hodge condition. It follows that there are six classes in $K_0(\mathrm{D}^{\mathrm{sg}}(R))_{\mathbb{Q}}$ whose Chern characters form a basis of $\mathrm{Hdg}(X_3^2)$: let us now describe these classes in terms of matrix factorizations of $x_0^3 + x_1^3 + x_2^3 + x_3^3$.

Let $\alpha = e^{2\pi i/3}$. Taking tensor products of the two matrix factorizations

$$E_1(x_0, x_1) = (x_0 + x_1, (x_0 + \alpha x_1)(x_0 + \alpha^2 x_1)),$$
$$E_2(x_0, x_1) = (x_0 + \alpha x_1, (x_0 + x_1)(x_0 + \alpha^2 x_1))$$

of $x_0^3 + x_1^3$ yields the following six matrix factorizations of $x_0^3 + x_1^3 + x_2^3 + x_3^3$:

$$E_1(x_0,x_1) \otimes E_1(x_2,x_3), \quad E_2(x_0,x_1) \otimes E_1(x_2,x_3), \quad E_1(x_0,x_1) \otimes E_2(x_2,x_3),$$
$$E_2(x_0,x_1) \otimes E_2(x_2,x_3), \quad E_1(x_0,x_2) \otimes E_1(x_1,x_3), \quad E_1(x_0,x_2) \otimes E_2(x_1,x_3).$$

Let us compute the Chern characters of these objects. We have

$$\mathrm{ch}_{HP}(E_1(x_0, x_1)) = 3(x_1 - x_0)\, dx_0\, dx_1,$$
$$\mathrm{ch}_{HP}(E_2(x_0, x_1)) = 3\alpha(\alpha x_1 - x_0)\, dx_0\, dx_1.$$

Thus, letting $\omega := dx_0\, dx_1\, dx_2\, dx_3$, the Chern characters of our six matrix factorizations of $x_0^3 + x_1^3 + x_2^3 + x_3^3$ are

$$9(x_1 - x_0)(x_3 - x_2)\omega, \quad 9\alpha(\alpha x_1 - x_0)(x_3 - x_2)\omega, \quad 9\alpha(x_1 - x_0)(\alpha x_3 - x_2)\omega,$$
$$9\alpha^2(\alpha x_1 - x_0)(\alpha x_3 - x_2)\omega, \quad 9(x_2 - x_0)(x_3 - x_1)\omega, \quad 9\alpha(x_2 - x_0)(\alpha x_3 - x_1)\omega.$$

A straightforward calculation shows that these classes are $\mathbb{Q}$-linearly independent and therefore form a basis of $\mathrm{Hdg}(\mathrm{D}^{\mathrm{sg}}(R))$.

Example 6.4 shows that every Hodge class of $\mathrm{D}^{\mathrm{sg}}(\mathbb{C}[x_0,\dots,x_3]/(x_0^3+\cdots+x_3^3))$ can be built out of products of Hodge classes of $\mathrm{D}^{\mathrm{sg}}(\mathbb{C}[x_0,x_1]/(x_0^3+x_1^3))$. The next example shows that this isn't always the case for Fermat hypersurfaces, even in four variables.

Example 6.5. We now take $m=6$ and $n=2$. We have $(2,2,3,5)\in B$ in this case. Notice that $(2,2,3,5)$ is not the concatenation of elements of $(\mathbb{Z}/6)^2$ corresponding to Hodge classes of X_6^0. This implies that the Hodge class corresponding to $(2,2,3,5)$ cannot arise as the product of Hodge classes of $\mathrm{D}^{\mathrm{sg}}(\mathbb{C}[x_0,x_1]/(x_0^3+x_1^3))$. Indeed, we do not know how to explicitly express this Hodge class as a $\mathbb{C}$-linear combination of Chern characters of matrix factorizations, even though, since the Hodge conjecture holds for X_6^2, this must be possible.

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