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Extending the covariation framework: Connecting covariational reasoning to students' interpretation of rate of change

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ABSTRACT

Research on covariational reasoning has continued to evolve as researchers learn more about how students coordinate two (or more) quantities' values as covarying. In this study, I examine the connection between students' covariational reasoning and how they interpret the value of a rate of change. The findings suggest that attending to students' quantifications of a rate of change can provide insight into their covariational reasoning and how we might better support students in reasoning at higher levels. Additionally, this manuscript provides an update to the Carlson et al. (2002) Covariation Framework that includes two additional categories of student reasoning and an additional dimension that describes students' interpretation of a rate value at each level of the framework.

1. Introduction

National Council of Teachers of Mathematics (1998, 2000) has consistently recommended that students develop the ability to analyze patterns of change in various contexts. They suggest that students should understand how changes in quantities can be mathematically represented (2000). One aspect of this involves coordinating two variables' values as they change together, which mathematics education researchers call *covariational reasoning*. Covariational reasoning as a theoretical construct has been used to study an individual's mental actions when conceptualizing quantities as they vary together (Confrey, 1991, 1992; Confrey & Smith, 1994, 1995; Thompson, 1993, 1994a; Carlson et al., 2002; Thompson & Carlson, 2017). While the works of Carlson and Thompson focus on how someone reasons about quantities (Carlson et al., 2002; Thompson & Carlson, 2017), little work has been done on exploring how students interpret the value of a rate of change and how that impacts how they reason covariationally (e.g., Johnson, 2015). Thompson & Thompson's (1994b) (1996) study is one early example of how one student's conception of a rate of change affected how they conceptualized two quantities' values as covarying. They observed that their student conceived of speed as a length (e.g., a student would attempt to fit a number of speed-lengths into a total distance), and this conception prevented them from imagining time and distance as continuously covarying together. However, their study did not encompass all the ways students may reason about the value of a rate of change and the connection to their conception of two quantities' values as varying. Therefore, one contribution of this paper addresses this gap in the literature by extending the Carlson et al. (2002) covariation framework to unpack how students reason about a rate value in the context of coordinating changes between two quantities' values. The results of this study add knowledge to the field by providing further nuance into why students engage in particular levels of covariational reasoning. Additionally, this contribution provides insight into how to support students in reasoning at higher levels by addressing their quantitative understanding of rate of change.

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The research question this study investigated was: *How do students interpret the derivative at a specific input value in an instantaneous rate of change context? In this context, how do the students attend to quantities' values changing?*

2. Literature review

Whereas covariational reasoning refers to how an individual coordinates changes in two quantities' values, a rate of change refers to a quantification of the multiplicative relationship between two quantities' values as they vary together. It should be clear then that rate of change and covariational reasoning are related topics, yet few papers explicitly identify and address this connection (e.g., Kertil et al., 2019; Johnson, 2012, 2015).

2.1. Literature review on student conceptions of rate of change

Many studies have explored university students' ideas of rate of change (Carlson et al., 2002; Carlson, 1998; Orton, 1983; Yu, 2019; Monk & Nemirovsky, 1994). One common finding was that many students did not conceive of rate of change as a multiplicative comparison between changes in two quantities' values. For example, Byerley et al. (2012) investigated calculus students' understanding of division and found that students employed additive reasoning when interpreting the value of a rate. Similarly, Rasmussen and Marrongelle (2006) observed that some students interpreted a constant rate function of 2 as "you'd be saying that you only *added* 2 pounds of salt for the whole time" (p. 408). These students described rates of change as additive changes in the output quantity rather than expressing rates as a multiplicative comparison of changes in two quantities.

Studies investigating students' conceptions of rate of change illustrate clear connections to how they reasoned covariationally about a situation. For example, Thompson and Thompson (1994) noted that a student interpreted the value of a constant speed as traveling that amount of distance will produce an amount of time (e.g., a speed of 50 mph means that traveling 50 miles produces 1 h of time). In this example, the student is at the initial levels of reasoning covariationally since they noted that two quantities values have changed; however, they have not yet associated them as varying together simultaneously. Similarly, in the studies that found students

Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., *y* changes with changes in *x*)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Fig. 1. Carlson et al.'s Covariation Framework.

interpreting a rate of change in an additive fashion (Byerley et al., 2012; Rasmussen & Marrongelle, 2006), these students exhibited that they imagined two quantities varying in discrete amounts since they coordinated specific amounts of changes between the two quantities. It should be clear then that attending to a student's quantification of rate of change can provide insight into their covariational reasoning.

2.2. *Literature review on covariational reasoning*

Covariational Reasoning first appeared as a theoretical construct in the works of Confrey (1991, 1992) and Thompson (1993) to describe how an individual coordinates two quantities as varying simultaneously. The field's understanding of students' covariational reasoning has continued to evolve as researchers investigated the mental actions and reasoning students employ to make sense of how quantities change together (Carlson, 1998; Carlson et al., 2002; Jones, 2019), images of variational reasoning (Castillo-Garsow, 2010; Thompson & Carlson, 2017), and its relation to multivariational reasoning (Jones, 2018, 2022; Jones & Kuster, 2021). Other studies have investigated students' conceptions of functions (Carlson et al., 2002; Oehrtman et al., 2008) and graphs (Moore & Thompson, 2015; Moore et al., 2013) using covariational reasoning as an explanatory framework for student reasoning. Further, many studies have leveraged covariational reasoning as being central to their conception of mathematical concepts such as functional relationships (Ellis, 2011; Paoletti & Moore, 2018), limits as dynamic motion (Jones, 2015), and exponential relationships (Castillo, 2010; Confrey & Smith, 1995).

Covariational Reasoning is frequently discussed with the idea of "quantity" in mind (Thompson, 1994a; Ellis, 2011; Moore et al., 2013), which refers to a measurable attribute of an object (Thompson, 2011). In the works that address the connection between students' understanding of rate of change and their covariational reasoning (e.g., Kertil et al., 2019; Johnson, 2012, 2015), the idea of quantity continues to be leveraged to explain an individual's reasoning. Johnson's (2012, 2015) classification of students' quantification of ratio and rate explained why students would operate at a particular level of covariational reasoning. In particular, Johnson focused on the quantitative operations (*comparing* or *coordinating*) a student engaged with when quantifying a rate of change. She identified if a student compared two quantities in an additive fashion or if they coordinated the intensity of one change in conjunction with continuing changes in the other. Kertil et al.'s (2019) work with prospective teachers indicated that their way of thinking about quantities (*variables* in their paper) influenced their level of covariational reasoning. Compared to these works that focused on students' conceptions of quantities or the quantitative operations involved, this paper builds off these studies by providing additional insight into how students' quantification (assigning a numerical value to a quantity) is connected to their covariational reasoning.

While Thompson and Carlson's (2017) updated covariational reasoning framework includes Castillo-Garsow's (2010) chunky and continuous ways of variational reasoning, this study focuses on the mental actions a student engages with while interpreting the value of a rate of change instead of their image of variations in two quantities values. Therefore, I focus the discussion and leverage Carlson et al.'s (2002) Covariational Framework. In their framework (Fig. 1), the authors provide descriptions of the mental actions that students might evidence in coordinating how two quantities' values vary. At the top level of this framework, Mental Action 5 (MA5) describes someone coordinating the instantaneous rate of change of a function with continuous changes in the input variable. A student engaging in MA5 coordinates how two quantities change together, including an awareness that the instantaneous rate of change comes from choosing smaller and smaller intervals in calculating average rates of change around a particular input value. MA5 also describes how someone coordinates changes in two quantities over a function's domain. Thus, students exhibiting MA5 thinking can reason about inflection points and how/where the rate of change changes.

These mental actions are associated with Carlson et al.'s five developmental levels of covariational reasoning that describe the images of covariational reasoning, e.g., a student having an image of Level 5 Covariational Reasoning would support the types of actions consistent with MA5 reasoning. However, the authors also point out that students can demonstrate MA5 without utilizing Level 5 covariational reasoning. In this study, I highlight a similar phenomenon in that a student might have an image of Level 5 covariational reasoning, yet their mental actions are limited to MA3 due to their conception of rate of change.

One part absent in the Carlson et al. framework is how students interpret the value of a rate of change in the context of coordinating changes in two quantities' values. It is not abundantly apparent in Carlson et al.'s framework how a student would reason about what it means for a car's speedometer to read 43 miles per hour at a particular time (instantaneous rate of change). In Carlson et al.'s study, one of the tasks the researchers used was the bottle problem (Fig. 2), and a student exhibiting MA5 would reason that in the bottom-rounded half of the bottle, the rate at which the height changes with respect to changes in the volume is decreasing. However, there was no discussion on how a student with MA5 (or any of the other mental actions) interprets the rate at a particular volume of water in

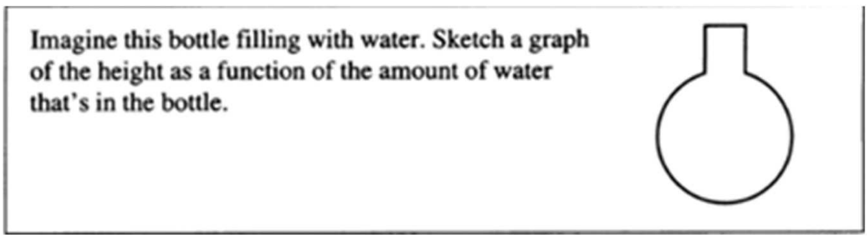

Fig. 2. The Bottle Problem.

this task. For example, in Fig. 3, a student engaging in MA5 might reason that the rate at V_1 is higher than the rate at V_2 which is higher than the rate at V_3, but it is unclear in the framework how someone thinks about quantities changing at the instance when the volume of the water is V_1. In this paper, I address this gap in the literature and discuss how students might coordinate changes in two varying quantities in the context of utilizing the value of a rate of change.

3. Conceptual framework

According to Thompson (1994b), understanding a rate of change involves imagining two quantities covarying such that the change in one quantity is proportional to the change in the other quantity. How a student imagines how two quantities covary (covariational reasoning) and how they quantify it are clearly interwoven. This aligns with Thompson and Carlson's claim that "for students to conceptualize rates of change requires they reason covariationally" (p.441) (along with many other ideas such as ratio, quotient, accumulation, and proportionality). While a student's conception of rate of change may not entirely explain how they reason covariationally, I assert that examining their quantification of a rate of change can provide insight into identifying how they reason covariationally.

For example, a student who interprets a rate of change additively might interpret "$\frac{1}{4}$ gallons of gas per dollar" as referring to 1 gallon of gas and 4 dollars and coordinate changes in two quantities' values by adding 1 gallon and 4 dollars simultaneously (e.g., A student goes from 1 gallon of gas and 4 dollars to 2 gallons of gas and 8 dollars) (Fig. 4). This type of thinking is what Johnson (2015) describes as someone making a *comparison* type of quantitative operation where an individual utilizes a single extensive quantity (1 Gallon) to quantify the change in another quantity (4 Dollars). Regarding covariational reasoning, this aligns with exhibiting MA3 reasoning since the student coordinates discrete amounts of changes in the quantities' values.

Comparatively, a student with reasoning at MA4 or MA5 for rate of change would interpret "$\frac{1}{4}$ gasoline gallons per dollar" as describing the constant ratio between the change in the number of gallons and the change in the number of dollars (the change in the number of gallons is $\frac{1}{4}$ times as large as the change in the number of dollars). Covariationally, a student with this conception can reason outside of 1-unit changes in the dependent quantity and has the potential to consider continuous changes in the independent quantity (Fig. 5). The differences in this way of thinking about a rate of change are similar to how Thompson (1994b) describes a student's conception of ratio as an *internalized ratio* versus an *interiorized ratio*, where the former describes a student's conception of a ratio in a specific situation while the latter describes a student conceiving of a whole class of situations that share a common proportion.

Table 1 describes each level of the modified covariation framework and how a student at various developmental levels will reason about the value of a rate of change. This modified framework combines Carlson et al.'s (2012) covariation framework with Thompson's (1994a) notion of quantity. One addition to the framework includes an extra column to explicitly connect a student's interpretation of a rate of change with a particular mental action. With the example of interpreting $f'(3) = 6$ as an instantaneous rate of change, an individual engaging in MA5 reasoning would imagine that if the independent variable were to vary a small amount from the input value of 3 (Δx), the variation in the output quantity would *essentially* be 6 times as large ($6 * \Delta x$). Of importance is this awareness that as the independent variable varies, the value of the rate of change would also likely vary (unless it is a constant rate of change), but for small enough variations, the difference would be imperceptible.

Additionally, I modify Carlson et al.'s (2002) covariation framework by incorporating MA0, where an individual imagines variation in only one quantity (in other words, they do not consider how two quantities vary together) when considering the value of a rate of change. Another addition to the framework is MA3 + , where students' intuitive understandings of quantities varying involve smooth and continuous changes but are limited in how they coordinate variations in each quantity's value because of their interpretation of a rate value (more on this in Section 5.5). These additions focus on an individual's coordination of two quantities as the covary and may not be as productive in describing students' multivariational reasoning (Jones, 2022).

4. A study on students' interpretation of an instantaneous rate of change

This study employs the Radical Constructivist stance (Thompson, 2000) and assumes that it is impossible to know another's

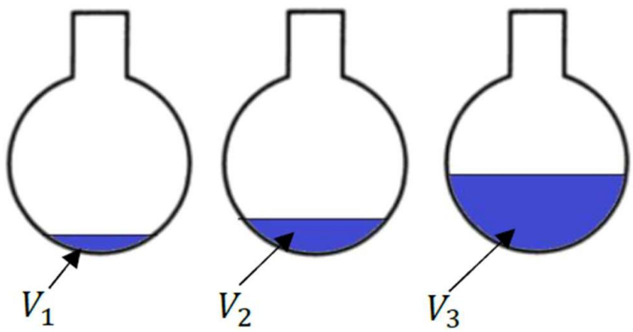

Fig. 3. The Bottle Problem Part 2.

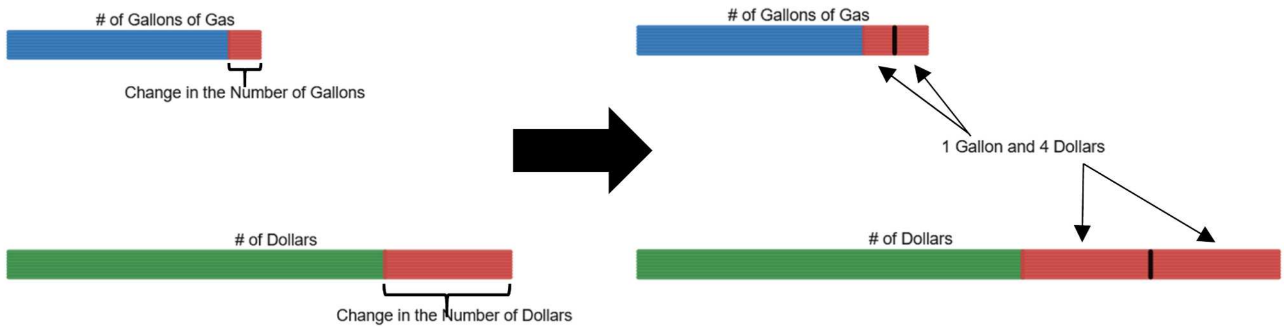

Fig. 4. Coordinating Changes in Two Quantities' Values with an Additive Conception of Rate.

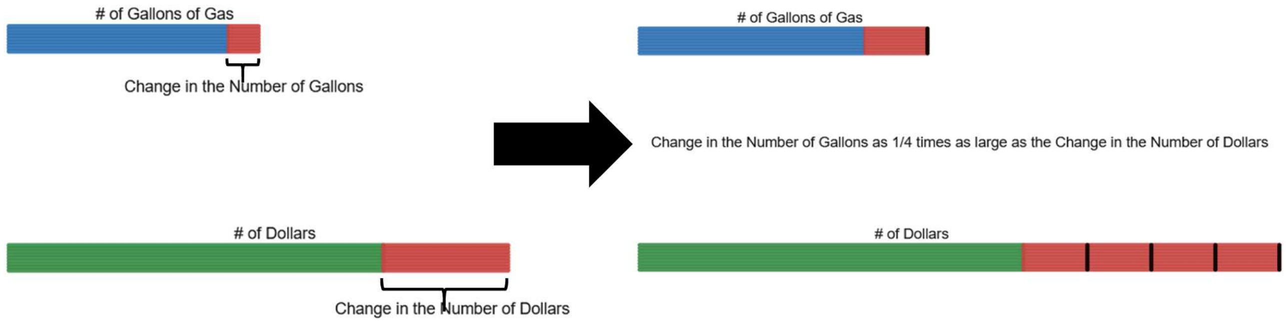

Fig. 5. Coordinating Changes in Two Quantities' Values with a Multiplicative Conception of Rate.

Table 1
A Modified Covariational Reasoning Framework.

Level	Description	Example of a student reasoning about $f'(3) = 6$
Mental Action 0 (MA0) - *No Coordination*	The student focuses on the variation in the value of one quantity only. The student has no image of quantities varying together.	The student may interpret the "6" as the output value of f changing or changed by 6. Alternatively, the student may interpret the "6" as the output value of f. In either case, there is no mention of the input quantity varying. Instead, the student conceptualizes the "6" as an extensive quantity (a particular amount or change in the amount of the output value).
Mental Action 1 (MA1) - *Coordinating Quantities*	The student coordinates the value of one quantity with changes in the other.	The student may believe that the value of the output quantity changed by 6 and then subsequently that the value of the input quantity changed from 3 to 4. (The student is not describing how the input and output quantities change together; instead, they observe that both quantities changed additively).
Mental Action 2 (MA2) - *Directional Coordination of Values*	The student conceptualizes that one quantity varies as another quantity varies but in a gross variation manner by not considering specific values.	The student interprets that the output value increases as the input increases. The 6 does not necessarily measure something; instead, it is like the reading on a speedometer. Quantitatively speaking, a student here has not yet associated a unit of measure with their conceptualized quantity.
Mental Action 3 (MA3) - *Coordination of Values*	The student coordinates the amount of change of one quantity with changes in the amount of the other quantity.	A student may consider the current input and output values $(3, f(3))$ and anticipate that a change in the input (usually a 1-unit change) results in new values for the input quantity and output quantity, e.g., $(4, f(4))$. For example, a student interprets the value of 6 as the change in the output value for a 1-unit change in the input value, e.g., $f(4) = f(3) + 6$. A student interprets the "6" by engaging in additive comparisons between the input and output quantities.
Mental Action 3 + (MA3 +) - *Coordination of Values+*	The student has an image of the value of the rate of change varying while coordinating the amount of change of one quantity with changes in the amount of the other quantity by assuming a constant rate of change.	A student verbalizes that the value of a rate of change should vary as the input quantity's value varies. However, they consider "6" as the additive change (or the approximated change) in the output quantity for a 1-unit change in the input quantity. For example: "If the rate of change stays constant, then the output value will change by 6 as the input value changes from 3 to 4."
Mental Action 4 (MA4) – *Coordinating Average Rates of Change*	The student coordinates the average rate of change of the function with uniform increments of change in the input variable.	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input, Δx, the output value will vary 6 times as much via a multiplicative comparison. However, the student does not verbalize an awareness that the value of the rate of change varies within this Δx interval.
Mental Action 5 (MA5) – *Coordinating Instantaneous Rate of Change*	The student coordinates the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.	A student may consider the current values of the input and output quantities $(3, f(3))$ and anticipate that for some change in the input, Δx, the output will vary 6 times as much. The student engages in a multiplicative comparison of changes between the input and output quantities. The student verbalizes an awareness that the value of the rate of change will vary in this Δx interval, but for small Δx values, the actual change in the output will be *essentially* 6 times as large. A student may consider continuous changes in the independent variable and anticipate that the values of the associated changes in the dependent variable will vary.

thinking. Therefore, investigating student thinking aims to build models of students' mathematics (Steffe et al., 2000) that may explain why students produce specific responses. In particular, this study aimed to construct these models to describe students' meanings for the derivative at a point. The word 'meaning' will be used in the way that Thompson (2013) uses it to describe a mathematical meaning. It is the organization of an individual's experiences with an idea that determines how the individual will act. Meanings are personal and might be incoherent, procedural, robust, or productive. However, these meanings are used by individuals to respond to mathematics tasks and make sense of and access mathematical ideas. For example, a person's meaning of derivative might only be associated with calculating the limit of the difference quotient. At the same time, another's meaning of derivative could involve the slope of a tangent line. Since meanings are personal, if one student writes a similar response to another student, we cannot assume they both have the same meaning.

4.1. Methodology

To address the research question, I report on the results of conducting a single clinical interview with 27 students (Clement, 2000). The interviews were semi-structured (Zazkis & Hazzan, 1998) in that the interview was planned in advance, but follow-up tasks would differ based on the interviewee's responses. The semi-structured interviews allowed for unplanned follow-up questions and variations on the prepared questions. Semi-structured interviews allow the interviewer to test their model of the students' thinking by presenting potential tasks based on how they respond during the interview. For example, if a student stated that they interpreted the derivative at a point as the slope of the tangent line, the interviewer might follow up by asking the student to draw a graph with the tangent line they are thinking of.

The main tasks for these interviews were as follows.

Task 0 (Fig. 6) aimed to elicit the students' spontaneous meaning for derivatives. Task 1 (Fig. 7) probed students' interpretation and use of a derivative value in an applied context. More specifically, the clinical interview focused on investigating the following questions:

1) Does a student associate a derivative with an "instantaneous rate of change"?
2) How does a student interpret the value of the derivative at a given input value?
3) How does a student utilize a derivative value to solve a linear approximation problem?
4) Does a student recognize that the linear approximation they performed in part b is an approximation because the value of the rate of change would likely change within that input interval?

As a note, even if a student did not verbalize the derivative as an instantaneous rate of change, I investigated how the student conceptualized the problem context and how they used their meaning for derivative to solve a linear approximation problem.

4.2. *Participant selection*

Twenty-seven student interviews were conducted over a period of 2.5 years, beginning in the Summer of 2017 and ending in the Fall of 2019. The subjects were students enrolled in a Calculus 1 or Calculus 2 course at a large southwestern university. Fourteen students were enrolled in Calculus 2, and thirteen were enrolled in a Calculus 1 course. There were four rounds of interviews, each conducted at the end of the semester over the duration of the study. Even though students were interviewed at different times throughout the study, the main tasks (Figs. 6 and 7) were the same for all 27 students.

4.3. *Data analysis*

Since the purpose of these clinical interviews was to generate new elements of a theoretical model in the form of mental actions and processes, the data analysis of the interviews was conducted through the lenses of quantitative reasoning (Smith & Thompson, 2007) and covariational reasoning. The analysis involved Open and Axial Coding (Strauss & Corbin, 1990) for moment-by-moment coding of students' responses and interpretations. Using the codes from each student, a thematic analysis (Clarke & Braun, 2013) was conducted across moments within each student's interviews and across different students' moments. This thematic analysis aimed to identify and analyze the patterns of student responses to model the types of thinking students were engaging in. As a clarification, the initial round of data analysis (Yu, 2019, 2020, 2021) did not leverage previous Covariational Reasoning frameworks. Instead, it became productive to conduct a follow-up analysis by utilizing existing constructs to explain the ways of thinking that emerged. This process is aligned with what Braun and Clarke (2012) call a mix of inductive and deductive data coding, where the analysis is driven by what is in the data, and then the data is brought to a series of ideas (existing theoretical constructs) used to interpret the data.

The follow-up analysis entailed comparing the codes and categories with the descriptions and Mental Actions described in the original Covariation framework (Carlson et al., 2012). From this, I observed several nuances and ways of thinking not sufficiently described in the original framework. These results led to extending the covariation framework by identifying a new level of reasoning and including descriptions of how students reason about the value of a rate of change and the connections to their covariational reasoning (Table 1). I hypothesized that attending to how students interpreted a rate's value would reveal potential mental obstacles preventing them from reasoning at higher covariational reasoning levels. For example, suppose students were reasoning about a rate of change as an amount to add. In that case, students' covariational reasoning would be limited to MA3 since they would coordinate amounts of change even if they had an image of continuous covariational reasoning (Level 5 in the Carlson et al. Framework).

5. Results

Table 2 summarizes the covariational reasoning the 27 students consistently exhibited in Task 1. As a clarification, these are not claims that a student only reasoned at a particular level or that they are solely that level reasoner. Further, some students may have exhibited higher or lower reasoning in some portions of the task. Table 2 counts each student based on the reasoning they mainly demonstrated throughout the task.

The following section provides examples of each level of covariational reasoning and examples of how students at each level

Task 0: What does the word 'derivative' mean to you?

Fig. 6. Task 0 - The Immediate Meaning.

Given that $P(t)$ represents the weight (in ounces) of a fish when it is t months old,

a.) Explain the meaning of $P'(3) = 6$

b.) If $P\ (3)\ =\ 15$ and $P'(3)\ =\ 6$ estimate the value of $P\ (3.05)$ and say what this value represents.

Fig. 7. Task 1 - The Fish Task.

Table 2
Summary of Mental Actions Exhibited by Students (n = 27).

Mental Action	MA0	MA1	MA2	MA3	MA3 +	MA4	MA5
Total Count	3	4	2	8	5	3	2

interpreted the value of a rate of change. The analysis focuses on describing and comparing MA3 versus MA3 + due to a significant portion of students reasoning at these levels.

5.1. *Mental action 0 (MA0) – No Coordination*

Researchers have indicated that many students confuse amount functions with rate of change functions (Flynn et al., 2018; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Ibrahim & Robello, 2012). One potential reason for this is that if students are not coordinating how two quantities' values covary. The updated framework utilizes Thompson and Carlson's (2017) construct of *No Coordination* (MA0) to classify this type of reasoning. A distinctive marker of this level of reasoning is conflating the value of a rate of change of a quantity with the amount of that quantity or how that quantity changed with no attention to the input quantity varying. It is important to note that this does not always mean that a student reasoning at MA0 does not think about the input quantity. Instead, they might think about the input quantity's value as a way to distinguish the instance in which the output quantity was measured. What characterizes MA0 reasoning is the lack of attention to the input quantity varying and its relation to how the output quantity varies.

5.1.1. *Examples of MA0 reasoning*

Gemma was a student who interpreted $P^{'}(3) = 6$ in the fish task as an amount of weight (Table 3). Throughout this entire task, Gemma only mentioned time once while explicating how she interpreted the value of 6. She appeared to have used a time value to tag a point in time instead of mentioning how time also varied [Line 3]. Additionally, Gemma discussed 6 as "the fish weight had changed by 6″ [Lines 5–6], which furthers the notion that Gemma was primarily tracking the value of the weight since she never articulated a reference point of where she measured from. In a follow-up task on interpreting a speedometer reading of 54 mph, she explained that 54 was "how many miles the car's distance had changed" and again, never explicitly discussed time as varying. Based on her responses, Gemma likely engaged in MA0 level reasoning because she interpreted the rate value as an amount of weight and her lack of attention to the input quantity varying in her explanations.

Similarly, Leah was a student who interpreted $P^{'}(3) = 6$ as an amount of weight gained by the fish (Table 4). While her initial writing of "from 0 to 3 months, the fish gained 6 ounces" might indicate MA3 reasoning (Fig. 8), her explanation revealed that she used the time values to distinguish between different measured instances of the fish's weight. Her choice of "then at 3 months" and "by that third month" supported the idea that she probably was not imagining time changing continuously [Lines 5&8]. She continued to reason in this manner after being asked to clarify whether 6 was the weight of the fish or how the weight had changed. She explained that she thought of the 6 as if she "looked at the fish at 0 months" and then "look(ed) at 3 months" [Lines 11–13]. Since Leah's explanation consistently used language that evidenced her thinking about two different points in time rather than over an interval, this corroborates the claim that she was not attending to variations in time. Instead, Leah utilized specific times to refer to which instance of

Table 3
Gemma's Explanation for Instantaneous Rate of Change.

1	Gem:	Cause if I know that if derivative is like at an instance… I
2		don't know, that's just the same as P(3). The fish is 15 ounces,
3		but at 3 months it's growing at 6 ounces.
4	Int:	Can you say a little more about what you mean by that?
5	Gem:	Yeah, like growing at 6 ounces is like the fish weight had
6		changed by 6…umm... like I know that the fish is 15 ounces
7		but like the 6 is like how the weight has changed.

Table 4
Leah's Explanation for Instantaneous Rate of Change.

Int:	So can you explain what you wrote and what that means to you?
Leah:	Yeah, like at 3 months the fish weighs 6 ounces, and that would be like at I'm guessing when the fish was born so like 0 ounces and then at 3 months the fish gained 6 ounces.
Int:	So 6 here is the fish's weight at 3 months?
Leah:	Umm yeah? Like it is also what the fish gained… the weight increased by 6 by that third month.
Int:	Wait so is 6 what the fish weighs at 3 months or how much weight the fish gained by then?
Leah:	I guess both? Like well if we looked at the fish at 0 months… the fish weighs 0? But like I now look at 3 months the fish weighs 6 ounces so the fish gained 6 ounces by the 3rd month.

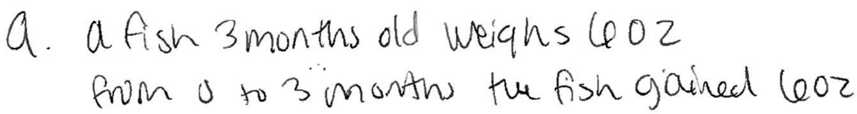

Fig. 8. Leah's Interpretation of $P'(3) = 6$.

weight she had in mind but never demonstrated that she was coordinating both weight and time as varying together.

5.2. Mental action 1 (MA1) – Coordinating Quantities

A student reasoning at MA1 notices variations in the values of two quantities but may not realize that these variations happen simultaneously. When a student engaging in MA1 considers the value of a rate of change, they will likely interpret the value as an amount of change in the output quantity and a subsequent change in the input quantity. Thompson and Thompson's (1994) construct of a speed-length is a prime example of MA1 reasoning where a student considered the value of a speed as an amount of distance for a given amount of time or that "traveling a distance at some constant speed will produce an amount of time" (p. 5).

5.2.1. Example of MA1 reasoning

Keenan initially wrote $P'(3) = 6$ as the "instantaneous weight at 3 months is 6 ounces" (Fig. 9), and while this may look similar to the MA0 examples, Keenan's explanation revealed that he noticed time passing (Table 5). However, as Keenan explained his interpretation, time did not seem to be the central focus of what a rate of change entailed to him. When discussing two measurements of the fish's weight, Keenan mentioned two different points in time: "I looked at the fish at 2 months then at 3 months the fish's weight gained 6 ounces" [Lines 5–6]. Keenan's language indicated that he primarily focused on the fish's weight changing since it "gained 6 ounces" and "changed by 6 ounces", and it was not until he paused for a moment (as indicated by the '…' in the transcript) that he noticed that time had changed as well [Lines 5–7]. Keenan primarily associated the value of a rate of change with the output quantity due to his consistent response of discussing the 6 as a number of ounces [Lines 2–3, 6, 10]. Even though Keenan eventually associated a month with the 6 ounces, he mainly coordinated the value of the fish's weight and later noticed time as elapsing; therefore, I classify his explanation as engaging in MA1.

5.3. Mental action 2 (MA2) – Directional Coordination of Values

MA2 marks the beginning of simultaneously coordinating the variations in two quantities' values. A student reasoning at MA2 recognizes that two quantities vary together, yet they will likely talk about non-specific amounts of change. They will probably interpret the value of a rate like a reading on a speedometer. This would mean that the value of the rate, for example, 6 ounces per

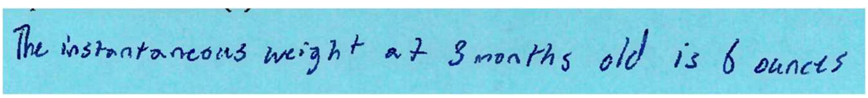

Fig. 9. Keenan's Interpretation of $P'(3) = 6$.

Table 5
Keenan's Explanation for Instantaneous Rate of Change.

1	Int:	So can you explain what you wrote down?
2	Kee:	Yes, so the instantaneous weight being 6 ounces is the
3		instantaneous change at 3 months is like 6 ounces.
4	Int:	So what are you imagining when you say this?
5	Kee:	Uh. Like if I looked at the fish at 2 months then the fish at 3
6		months the fish's weight gained 6 ounces, uh yeah changed by
7		6 ounces… like in that one month of time.
8	Int:	Okay so like the difference between the fish's weight at 2
9		months versus at 3 months would be 6 ounces?
10	Kee:	Yeah, like you know P(2) would be 6 less than P(3).

month, does not entail 6 of something; instead, the student utilizes the value to compare to other rates (e.g., 6 ounces per month is *slower* than 8 ounces per month). In terms of quantitative reasoning, a student engaging in MA2 may not have quantitatively developed a meaning for rate of change since they are not associating a rate of change with a unit of measure. In other words, they have yet to consider a quality of the situation to which they can measure and make sense of assigning a numerical value.

5.3.1. Examples of MA2 reasoning

Bob initially explained that he interpreted a rate as the weight increase over the third month (Fig. 10). As he continued to explain, he attended to both weight and time as varying, but his description lacked the specificity of what the 6 represented (Table 6). Bob coordinated both time and weight as changing as he verbalized that it would not be "like at between 2 and 3 months he's adding 6 pounds"; instead, he saw the 6 as "a number to throw out there" [Lines 3–5]. Later, when the interviewer probed him about his choice of units, Bob said that he chose 'ounces' because that was how the fish's weight was changing, but he also verbalized that when he "usually read these (rates), I kind of think of a unitless number." Throughout his explanation, Bob demonstrated that he attended to time and weight changing simultaneously and coordinated the variations in a unitless manner. Later in the interview, Bob was presented with a supplemental task (Fig. 11), where Bob was asked to explain the difference between three cars traveling at different speeds. Bob explained that one of the cars would travel faster, which meant that the car would travel further as time passed. Bob stated that "that car would obviously go farther than the other two cars, but like I can't really say exactly how much further it would travel." His statement revealed that he did not attribute the value of a speed as quantifying something. Instead, Bob only used the value to compare the distance traveled between each car in a gross variation manner. Bob's explanation of a rate as being unitless, and yet still as entailing how two quantities' values vary simultaneously, is consistent with MA2 reasoning.

5.4. Mental action 3 (MA3) – Coordination of Values

A student exhibiting MA3 coordinates specific amounts of variation between the values in two quantities. Students engaging at MA3 will likely interpret 6 ounces per month as the weight change for a 1-unit change in time. When approximating a future output value, students employing MA3 reasoning will likely refer to 6 as an amount of change and determine a corresponding amount proportional to the change in the input quantity. To use Johnson's (2012, 2015) words, students engaging in MA3 have likely associated a rate of change with a quantitative operation of comparing two quantities changes in an additive fashion. In this way of thinking, an individual has quantitatively considered a rate of change as describing a specific situation regarding 1 unit of change in the independent quantity and an associated change in the dependent quantity.

5.4.1. Examples of MA3 reasoning

Will was a student who interpreted $P'(3) = 6$ as "the instant rate of change of the fish's weight when it is 3 months old is 6 ounces" (Fig. 12). He explained that $P'(3) = 6$ as the change in weight for a 1-unit change in time (Table 7). Will described that "it's growing by 6 ounces" meant that "in that entire third month it gained 6 ounces" [Lines 3–4]. He later clarified his imagery by drawing a picture of a calendar and drawing an arrow through the dates (Fig. 13) to demonstrate his awareness of time passing and his coordination of the overall change in the weight of the fish [Line 6]. Will interpreted the rate value of 6 in an additive fashion since he coordinated discrete

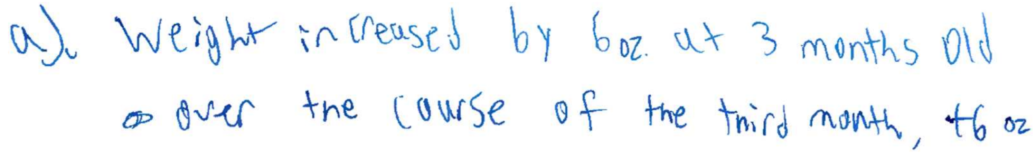

Fig. 10. Bob's Interpretation of $P'(3) = 6$.

Table 6
Bob's Explanation for Instantaneous Rate of Change.

Bob:	It would be like at the exact moment it is changing by 6. So I guess I mean I was thinking about this the other day. I mean I guess it wouldn't really just be like at between 2 and 3 months he's adding 6 pounds, I don't know how I think about that actually. I just kind of see it as a number to throw out there.

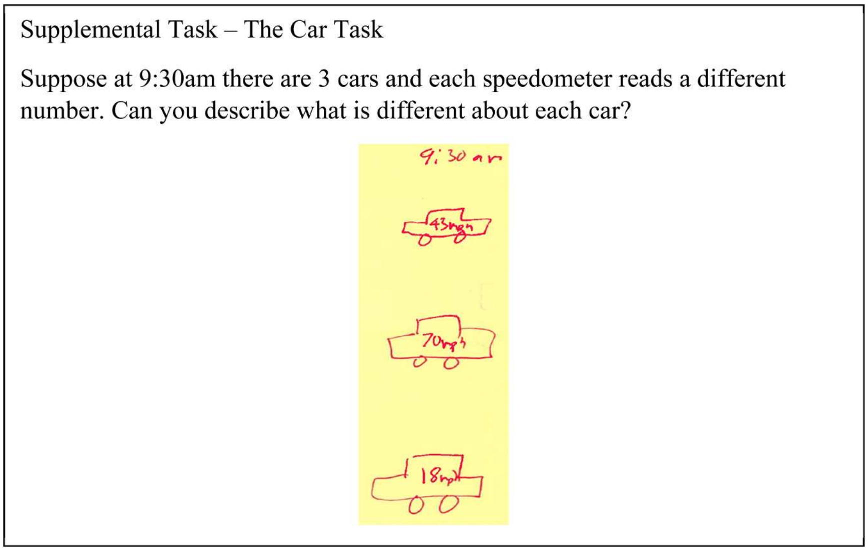

Fig. 11. Supplemental Task – The Car Task.

amounts of variations between weight and time (1 entire month and 6 ounces of weight); therefore, I classified Will as engaging in MA3 reasoning in this excerpt.

Lucy was another student who exhibited MA3 reasoning during the interview (Tables 8 and 9). Lucy wrote that $P'(3) = 6$ was "the instantaneous rate of the weight of a fish is 6 ounces when it is 3 months old" (Fig. 14) and initially explained it as the amount of change in weight over 3 months (Table 8). She eventually adjusted her explanation to say that it was a change in the weight for the next month [Lines 6, 10–11]. In both cases, her language indicated that she interpreted the 6 as an amount of weight to add since she used phrases such as "go up," "would be 6 more," and "go up 6 ounces" [Lines 3,5–6,10–11]. Lucy's explanation suggested that she was coordinating discrete amounts of change between time and weight together since she explained that it was "a certain rate over a period of time" [Lines 2–3], and she continually associated the change in the input value with a corresponding change in the output value [Lines 3, 5–6, 10–11]. Her explanation indicates that she considered a rate of change as entailing the quantitative operation of comparing two quantities' values (this is in contrast to a meaning that involves the relative size of one variation with the other) or what Johnson (2015) calls the association of extensive quantities.

In part b of the Fish Task, Lucy continued to reason that the value of a rate was an amount of change for a 1-unit change in the input quantity (Table 9). Lucy explained that she was trying to find "the 0.05 rate of change to add" to the initial value of 15 ounces [Line 7]. She then articulated that if the change in the number of months were one, she would add 6 and then deduced that since she had 0.05 of 1, the fish would gain 0.05 of 6 [Lines 7–9]. While Lucy employed proportionality in her explanation, it is important to highlight that to

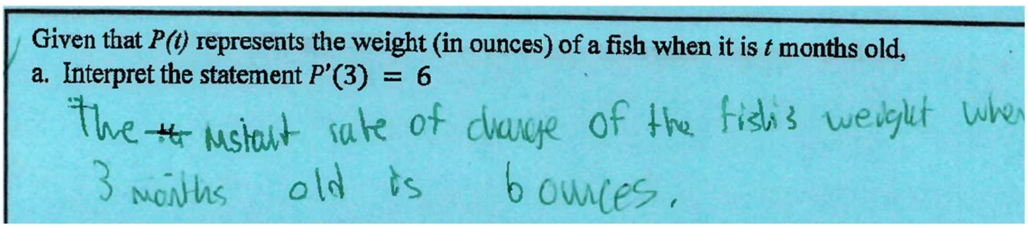

Fig. 12. Will's Interpretation of $P'(3) = 6$.

Table 7
Will's Explanation for Instantaneous Rate of Change.

1	Int:	So what does it mean to you that "the instant rate of change of the
2		fish's weight when it is 3 months old is 6 ounces?"
3	Will:	So it's at 3 months and to say that's it's growing by 6 ounces. So
4		like in that entire third month it gained 6 ounces.
5	Int:	Like that happened at the end of the month or something else?
6	Will:	No, that change was over the entire month. *Draws calendar*

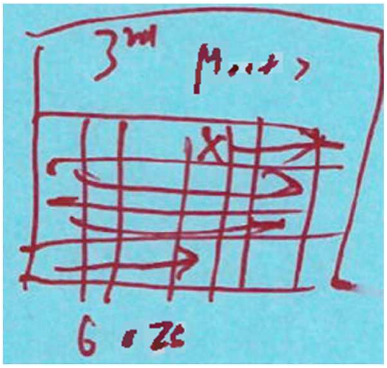

Fig. 13. Will's drawing of 6 ounces over the entire 3rd month.

Table 8
Lucy's Explanation for Instantaneous Rate of Change.

1	Int:	Can you explain what the means to you?
2	Lucy:	So the instantaneous rate is there's a certain rate over a period
3		of time so every like 3 months it is going to go up 6 ounces.
4	Int:	Like every 3 months the fish will gain 6 ounces?
5	Lucy:	Yeah like since $P(3) = 15$, then $P(6)$ would be 6 more, 21.
6		Oh wait, no it should be that $P(4)$ would be 6 more.
7	Int:	Oh? And why do you say that?
8	Lucy:	I like mixed it up with the time we are at.
9	Int:	Okay, so can you restate what you wrote down means?
10	Lucy:	Yeah, like the instantaneous rate here tells us that in one month
11		the fish's weight will go up 6 ounces.

Table 9
Lucy's Explanation for her calculation in part b.

1	Int:	So how did you get 15.25?
2	Lucy:	I was estimating because 3 would be 15 and you with that have
3		to divide and it is probably wrong.
4	Int:	So what did you want to do?
5	Lucy:	You would do 6 divided by 0.05 **writes it down in green**
6	Int:	And so why did you do that?
7	Lucy:	To get the 0.05 rate of change to add to this *points to 15*, like
8		if this like 4 it would be one more than this so it would be plus
9		6, but because it is 0.05 of 1, we want 0.05 of 6.

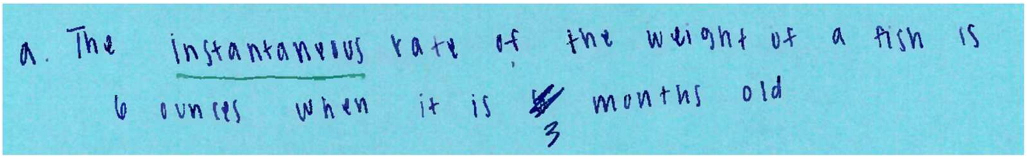

Fig. 14. Lucy's Interpretation of $P'(3) = 6$.

Lucy, 6 was not describing the multiplicative relationship between how the age and weight of the fish would covary. Instead, it was the change in weight for 1 month of time, and she wanted to find 0.05 of that 6-ounce change. Additionally, Lucy struggled to mathematically represent what she explained since she initially did not write a calculation and instead estimated it [Lines 2–3]. When prompted, she tried several incorrect calculations, which indicated a lack of procedural fluency in using the value of a rate of change (Work written in green in Fig. 15).

Other students also engaged in proportional correspondence in part b of the Fish Task by using 6 as the reference amount for a 1-month change in time and then setting up equivalent ratios to find a proportional amount of change. One student, April, explained how she solved part b by thinking of 6 as "how much it will change in a month" (Table 10). She later described that her calculation involved finding "0.05 of that" because she wanted 5% of 6 [Lines 7&10] (Fig. 16). While April appeared to have employed multiplicative reasoning, her explanation indicated that she interpreted a rate of change as a discrete amount of change and likely envisioned the two quantities as varying in completed amounts (Lines 6–7, 9–10).

Similarly, other students who explicated 6 as an amount of change in weight in a month also set up equivalent fractions (Fig. 16) because they looked for a proportional amount of change. For example, a student named Anu described her calculation as "like if I split the rate up into little pieces like 20ths." Anu's description revealed that she thought of 6 as an amount of change in the fish's weight and that she could subdivide the 6 into 20 equal pieces and could find the corresponding amount of change in the fish's weight for a 0.05 change in the fish's age. Many other students also described that 6 was an amount of change for a 1-month change in time and that their calculation found a portion of that change (Fig. 17). Similar to Anu, these students engaged in proportional correspondence by finding the corresponding amount of change in the weight that would maintain the ratio of 6 ounces to 1 month.

These students' actions and explanations for estimating the change in the fish's weight for a 0.05 change in the amount of time suggested that they imagined variations between weight and time using chunky reasoning (Castillo-Garsow, 2010). In other words, they imagined 6 as a discrete amount of change in the weight of the fish and took actions to find a smaller-sized chunk of change that maintained the 6:1 ratio. What was absent in the interpretations from all the students who exhibited MA3 reasoning was that the quantities of the fish's weight and the fish's age would vary continuously and smoothly. This is supported by the students' interpretation that 6 was a completed change in the number of ounces after some elapsed amount of time instead of a value that quantified the relative size of a varying amount of time and a varying fish weight (in ounces) since the fish hatched.

5.5. Mental action 3 + (MA3 +) – Coordination of Values+

MA3 + is similar to MA3, except that a student is cognizant that the value of the rate of change also varies. While verbalizing an awareness of how a function's instantaneous rate of change continually varies as the input variable varies is an indication of MA5 reasoning, MA3 + is different in that a student is limited to coordinating discrete amounts of changes between quantities instead of varying continuously and smoothly. I argue here that a student's meaning for the value of a rate of change is one of the potential obstacles that hinder them from reasoning at MA5. Suppose a student interprets the value of a rate of change additively. In that case, they will likely reason about variation happening in discrete chunks, which may be an obstacle to understanding what it means for a rate of change to vary. One explanation for this is that these students' conception of ratio is what Thompson (1994b) calls an *internalized* ratio, where the student associates the value of a ratio with particular amounts of two quantities (e.g., 23 miles per hour refers to 23 miles and 1 h). The idea of ratio is relevant to rate of change since, according to Thompson, a rate is a "reflectively abstracted constant ratio" (p. 192). So if a student quantifies a ratio/rate in this manner, their actions in coordinating two quantities values as they

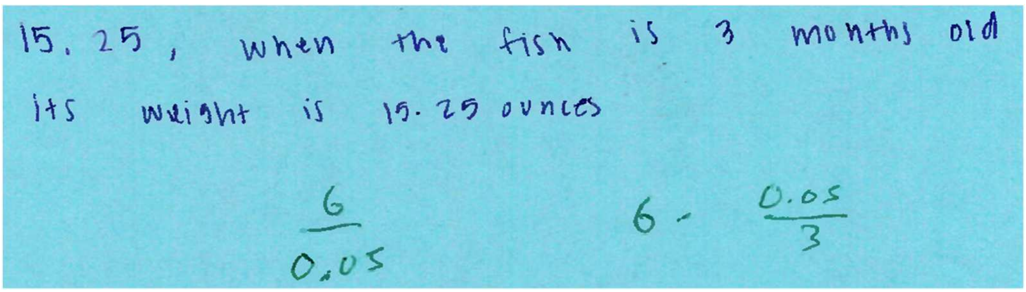

Fig. 15. Lucy's written work for part b of the Fish Task.

Table 10
April's Explanation for Instantaneous Rate of Change.

Int:	Can you tell me what this '15' represents to you?
Apr:	Uhh, yeah that was how much the fish weighed at 3 months.
Int:	Okay, so what does this '0.05 ∗ 6' mean?
Apr:	That was the change in the fish's weight
Int:	So why does '0.05 ∗ 6' represent that?
Apr:	Well like… 6 is how much it will change in a month and I wanted to find like 0.05 of that.
Int:	Sorry, can you say again what 0.05 meant to you?
Apr:	Yeah, like 0.05 was like how much of the change in 6 ounces I wanted to find, like…ummm like I wanted 5% of 6

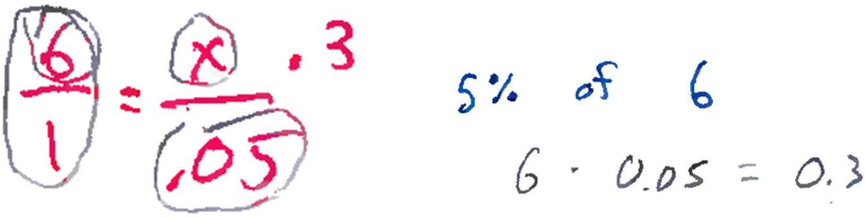

Fig. 16. Examples of MA3 reasoning.

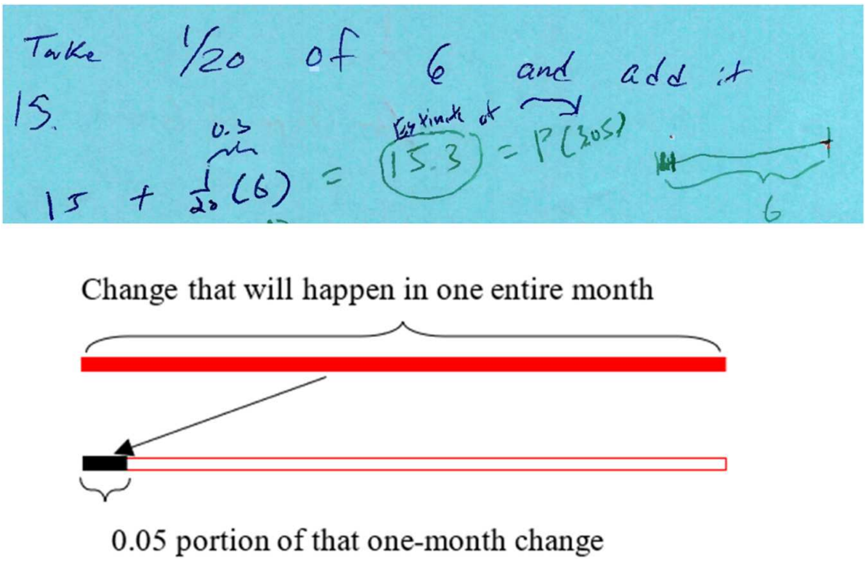

Fig. 17. Additional Example of MA3 Reasoning.

vary will be limited to MA3/3 + even if their image of quantities covarying is smooth and continuous.

A student reasoning at MA3 + experiences a dissonance between their intuitive understanding that quantities vary smoothly and continuously versus their interpretation that a rate refers to a fixed amount of change. This new classification of MA3 + is necessary since some students will demonstrate an awareness of how the instantaneous rate of change of a function continually varies as the input variable varies. However, their behaviors are limited to MA3 due to their conception of rate of change.

5.5.1. Examples of MA3 + reasoning

Max displayed MA3 + reasoning as he explained his interpretation of $P'(3) = 6$ as an amount of change in the fish's weight (Table 11). Similar to students who exhibited MA3 reasoning, Max also articulated that 6 was the amount the fish was "projected to grow" and that "in that month he should gain 6 ounces" [Lines 9–10]. Max coordinated an amount of change in the fish's weight with an amount of change in time; however, he also consistently qualified his language to indicate his awareness that the value of the rate of change would likely vary. Even though the interviewer asked if Max meant that "in the third month it gained 6 ounces", Max quickly denied that because he did "not have enough information" [Line 8]. His explanation included words such as "projected" and "should" to indicate that the 6 was not the exact amount of change in a month. Instead, it meant that if the rate stayed the same, the fish's weight would gain 6 ounces [Lines 6–11]. Although Max was cognizant that the rate at which the fish was growing was varying, it seemed that his meaning for rate of change as a "change in ounces" and being measured in ounces [Lines 14–15] prevented him from fully engaging in MA5 reasoning and instead limited him to coordinating amounts of variations between the two quantities (MA3).

Throughout the excerpt, we can see evidence that Max likely had an internalized ratio conception since he associated the value of the rate of change with specific amounts of change between time and weight [Lines 1,10–11, 14–15]. However, unlike the examples from MA3, Max also articulated that the value of the rate of change would also vary [Lines 8–11], which is indicative of MA5 reasoning. What likely is one primary source of this discrepancy between his coordination of two quantities varying and his image of two quantities varying is how he quantified the rate of change as describing the "change in ounces" with respect to 1 month of time [Lines 10–11, 14–15], and thus is distinctly different from MA3 reasoning.

Fred was another example of an MA3 + reasoner when he explained how he used the derivative value to estimate $P(3.05)$ (Table 12). Fred explained that 6 was the number of ounces the fish would grow "until the third month finishes" and repeated this later as "the entire third month [the fish] is going to grow 6 ounces" [Lines 3–4 & 9–10]. Like Max, Fred consistently justified his estimation with language such as "assuming the rate at which it grows is the same" and that he was "under the assumption that over the span of the third month they're growing at 6 ounces" [Lines 1–3 & 9–10]. His word choice demonstrated that he was aware that the value of the rate of change might not be constant, but using the value of an instantaneous rate of change involved making that assumption. Again, like Max, Fred's meaning for a rate of change entailed a change in the fish's weight of 6 ounces [Lines 4 &10], which led him to engage in coordinating specific amounts of variations since he wanted to find a "portion" of the 6 ounces for the "associated change for that time" [Lines 4–5 &10–11].

Other students exhibiting MA3 + reasoning explained their responses to Task 1 similarly to Max and Fred. The commonality between these students was their association of a rate of change with specific amounts of change (6 ounces and 1 month) and an awareness that the value of the rate of change would likely vary in this one month. For example, one student named Andrew explained $P'(3) = 6$ as "like when it has 3 months of age, the rate at which it is gaining weight is 6 ounces" and "if it stays at that rate of change, then yes it will gain 6 ounces, like that interval is like a month but the rate of change could change". These students all qualified their explanations with hypothetical language (e.g., "If the rate stays the same"). They interpreted the value of the instantaneous rate of change as a projected amount of change in the dependent quantity. Comparatively, the students employing MA3 reasoning utilized more definitive language, such as "the fish *will* grow 6 ounces in a month". This indicates a significant difference between how these students imagined two quantities as covarying. For the MA3 students, they were likely imagining quantities changing in discrete, additive chunks (consistent with Level 3 Covariational Reasoning). In contrast, the MA3 + students likely imagined these quantities changing continuously, but their conception of rate of change did not yet support them in articulating their Level 4/5 image of covariation. I claim that if these MA3 + students conceptualized rate of change as a multiplicative relationship between two varying quantities instead of a specific amount of variation, they would likely exhibit MA4 or MA5 covariational reasoning.

Table 11
Max's Explanation for $P'(3) = 6$.

1	Max:	It gained 6 ounces in weight when it was 3 months old
2	Int:	So are you saying that after 3 months, the change in weight is 6
3		ounces. So like from 0 months to 3 months the fish gained 6
4		ounces?
5	Max:	No… not really
6	Int:	So what are you trying to describe? Are you saying in the third
7		month it gained 6 ounces?
8	Max:	No because…I do not have enough information to give that
9		number. At 3 months he is in a sense projected to grow at that
10		rate that he is growing at. Like in that month he should gain 6
11		ounces.
12	Int:	Okay, so what are the units on this derivative? You wrote that
13		the units of $P(t)$ is ounces, what are the units for $P'(t)$?
14	Max:	Change in ounces?... Yeah cause change in ounces is still an
15		ounce so the unit is ounces.

Table 12
Fred's Explanation for his solution to part b.

Fred:	At $P(3.05)$… uh 15+0.3 ounces… assuming the rate at which it grows is the same or very close to the same oh okay… so this is under the assumption that over the span of the third month they're growing at 6 ounces so I just took a small portion of that.
Int:	Over the entire third month you said?
Fred:	Yeah until the third month finishes.
Int:	So the entire month finishes it's growing at a rate of 6?
Fred:	Yeah so I took a portion of that, like assuming the entire 3rd month is going to grow 6 ounces I took a portion of that like 0.05 and found the associated change for that time.

5.6. Mental action 4 (MA4) – Coordination of Average Rates of Change

Engaging in MA4 and higher requires recognizing that a rate of change entails a multiplicative relationship between the variations in the values of two quantities. In contrast to MA3, a student at MA4 would not utilize equivalent ratios or resize a one-unit change; instead, they conceptualize the value of a rate of change as describing how many times as large the variation in one quantity will be with respect to another.

5.6.1. Examples of MA4 reasoning

Randy's explanation of instantaneous rate of change was consistent with MA4 reasoning (Table 13). Randy described instantaneous rate of change as "how much it's (the fish's weight) changing by over a process of time," and as he said this, he slid his right hand away from his other hand to indicate the motion that went with his verbal description [Lines 1–3]. As Randy continued to explain, he articulated that the 6 described how the weight would change "from there to there it would keep changing by like 6 ounces per month" [Lines 2 & 6–7] and that they vary together because "it (the weight) is not changing if time isn't changing" [Lines 12–13]. Due to his gestures and how he attempted to describe weight and time changing together, Randy evidenced that he thought of a rate of change as describing how the quantities vary together smoothly and continuously.

Although Randy never explicitly described 6 as representing the relative size of the change in weight compared to the change in time, his actions suggested that this rate of change entailed the simultaneity of weight and time covarying together. Additionally, he verbalized that he was thinking about average rates of change over small intervals and that the weight would be changing at a rate of 6 [Lines 5–7 & 10–13]. His choice in picking differently sized time intervals suggested that he was not engaging in MA3 by thinking of 6 as a change in weight; instead, he attempted to articulate that the 6 described how fast the weight would change throughout those intervals. Randy never demonstrated an awareness that the rate of change would vary. In fact, he used more definitive language, such as "that's how much it is changing by" and "it would keep changing," which implied that he thought about the value of the rate as being constant over those small intervals [Lines 1–2 & 6–7]. Therefore, his interpretation and explanations are consistent with MA4 reasoning.

Another student, Winnie, exhibited MA4 reasoning as she explained her solution to part b of the Fish Task (Table 14). Winnie initially struggled to articulate why she multiplied 6 by 0.05, and only in the latter portion of the interview does she describe the 6 as being related to time: "the time only progresses after the interval…and the weight changes with it, so I multiplied those…" [Lines 13–15]. Similar to Randy, Winnie never demonstrated that she interpreted 6 as an amount of weight; instead, she explained that the 6 had something to do with how weight and time varied together [Lines 12–15]. Additionally, she utilized a hand gesture similar to Randy's when explaining her calculation. She slid one of her hands from her other stationary hand (Fig. 18) to describe what she imagined [Lines 8–10]. Her explanation for her calculation and gestures suggested that she imagined weight and time varying smoothly and continuously. Winnie's actions suggested that she interpreted a rate of change as entailing how two quantities would vary together. However, she did not communicate that she interpreted the 6 as a relative size measurement between variations in the fish's weight and the age of the fish. Lastly, Winnie never explicated an awareness that the value of the rate of change would vary, which would preclude her from being classified as MA5; therefore, I classify her as engaging in MA4 reasoning.

5.7. Mental action 5 (MA5) – Coordination of Instantaneous Rates of Change

MA5 includes all of MA4 with the added distinction of recognizing that the value of the rate of change varies as the input quantity varies. A student engaging at MA5 will consistently qualify the amount of change in a quantity with "if the rate stays the same…". This is further evidenced when a student anticipates that for some input, a, and for some change from the input, Δx, the output value will vary $f'(a)$ times as much; in other words $f'(a) * \Delta x \approx f(a + \Delta x) - f(a)$. The student also verbalizes an awareness that the rate of change will vary in this Δx interval, but for small Δx values, the change in the output will *essentially* be 6 times as large.

Table 13
Randy's Explanation for Instantaneous Rate of Change.

Ran:	Like the instantaneous rate of change, so like… that's how much it is changing by over a process of time. *Slides his hands to motion*
Int:	So like over the first three months it gained 6 ounces?
Ran:	No like…let's say like from…. like 2.9 to 3.1, like the average rate of change is like 6, like from there to there it would keep changing by like 6 ounces per month
Int:	So why did you pick 2.9 and 3.1? Does it have to be those numbers?
Ran:	Nah like that was just something close to 3, we could have picked like from 2.85 to 3.15 that average rate would be 6, I mean I'm just trying to explain it cause it (the weight) is not changing if time isn't changing.

Table 14
Winnie's Explanation for her solution to part b.

Win:	So I multiplied 6 by 0.05 for some reason
Int:	And what were you trying to represent?
Win:	I think it would represent the amount that it is changing in that small interval that it's defined up to 3.05… so we're using the fish's instantaneous rate of change from 3, yeah that's what I did.
Int:	So how did you get 0.05?
Win:	Because you know that $P(3)$ is 15… so I multiplied by 0.05 since that's what the amount that's after that. *Slides her right hand as she describes this* [Figure 18]
Int:	So why does 0.05 times 6 get a change in weight?
Win:	Uhh… because like the instantaneous rate of change is 6 and we know that the time only progresses after the interval for 0.05, and the weight changes with it so I multiplied those to get out the change.

Fig. 18. Depiction of Winnie's gesture as she explained her interpretation of $P'(3) = 6$.

5.7.1. Examples of MA5 reasoning

Leo interpreted $P'(3) = 6$ as "the weight of the fish is increasing at a rate of 6 ounces per month at 3 months" (Table 15). When asked what he meant by this, Leo initially described an average rate but qualified his statement with "it's not what it's always going to be," indicating an awareness that the average rates of change were not a constant value of 6 [Lines 3–5]. He then discussed an example by picking the interval 2.9–3.1 and that the average would be "about 6 per month." [Lines 7–8]. Later in the interview, Leo clarified that he was thinking of small intervals around 3 and drew a picture of a number line where he described that he could choose any interval close to 3 and that the average rate of change would still be around 6 (Fig. 19). He also noted that as he chose intervals closer to 3 (he

Table 15
Leo's explanation for Instantaneous Rate of Change.

Int:	So in this context what does that mean? What does it mean to be "increasing at a rate of 6 ounces per month at 3 months?"
Leo:	So when the fish is 3 months old it would be like the average rate would be 6 ounces per month at that time, like it's not what it's always going to be but if you took it at 3 it would be 6.
Int:	So what average are you looking at?
Leo:	Like the averages around 3, like at 2.9 and then 3.1 and how big it is, the fish then you could average out about 6 per month.
Int:	So are you looking at all the weights around 3 and the average of those weights would be 6?
Leo:	Not the averages of the weight, but how big of a difference it is increasing through time. And if you could expand it over a month it would be about 6.

drew in the tick marks in Fig. 19), the average rates of change would get "more precisely closer to 6." His description demonstrated that his understanding of rate of change did not entail 1-unit changes in the input quantity. Instead, he evidenced that a rate of change could involve any size change in the input quantity, even arbitrarily small ones.

Leo also explained that his interpretation of the average rate of change encompassed "how big of a difference it (the weight of the fish) is increasing through time" [Lines 11–12]. Leo's explanation exhibited his understanding that a rate of change involved "differences" or changes in the dependent quantity "through time." While his statement of "if you could expand it over a month it would be about 6″ could be interpreted as MA3 reasoning, it should be noted that he came up with this description after he explained his meaning for $P'(3) = 6$ [Lines 12–13]. His qualification of "if you could expand" evidenced that he went from considering the average rate in a small interval [Lines 7–8] and then imagined that if the fish grew at that rate for a month, then the change in the weight "would be about 6″ [Lines 12–13]. Compared to MA3 and MA3 + reasoners, Leo did not first indicate that he interpreted 6 as how much weight the fish would gain in a month. Instead, he thought about coordinating changes between the independent and dependent quantities around 3 by anticipating how the weight might change if he assumed a constant rate of change over an entire month. In this account, Leo never verbally stated that 6 represented multiplicative relationship between the change in the weight of the fish with respect to the change in the age of the fish. However, his actions and description of instantaneous rate of change indicated that he was engaging in MA5 reasoning. Leo's explanation for instantaneous rate of change entailed changes between two quantities' values [Lines 11–12] and that as he chose different-sized intervals, the (average) rate of change would also continually change in value (Fig. 19).

Cyrus also demonstrated MA5 reasoning as he explained his solution to part b of the Fish Task (Table 16). Throughout the entire interview, Cyrus never indicated that he interpreted 6 as an amount of change; instead, he always employed examples where he would use the 6 and multiply it by some amount of time. While Cyrus never explicitly stated he interpreted a rate as a ratio between changes in two quantities, he only utilized the 6 to employ multiplication to discuss how time and the fish's weight varied together [Lines 2 & 11–13]. Cyrus described the 6 as the fish was "changing at 6 ounces per month" and explained that as how the fish's weight was "changing" and not as an amount of change [Lines 6–9]. As Cyrus explained his calculation, he consistently verbalized that he assumed a constant rate since "it probably is not going to be changing very much faster or very much less" and that his estimation was "somewhere close, but I know that's not the correct value" [Lines 3&7–10]. This evidenced his awareness that the rate of change would vary even in the small interval between 3 and 3.05 and that even if the rate did vary, it would not change drastically unless it "hit a growth spurt right before or after," which meant that his estimation was close enough [Lines 7–10]. Altogether, Cyrus demonstrated that he was coordinating how time and the fish's weight covaried together smoothly and continuously as well as coordinating the instantaneous rate of change of the function with continuous changes in the independent variable.

6. Discussion

Based on the results of these clinical interviews, each student's explanation of the value of an instantaneous rate of change revealed how they might have conceptualized how two quantities' values covaried. In previous studies (Byerley et al., 2012; Castillo-Garsow,

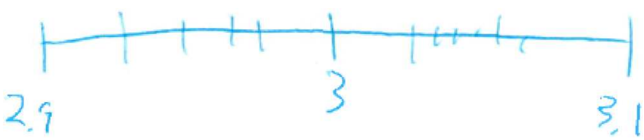

Fig. 19. Leo's Written Work.

Table 16
Cyrus' explanation for his solution to part b.

Int:	So what did you try doing here?
Cyr:	0.05 * 6 and got 0.3, so I estimated 15.3. I know it's somewhere close, but I know that's not the correct value.
Int:	So why did you do this part over here 0.05 * 6?
Cyr:	So that was the estimation, it was 3 and 0.05 months, at 3 months it was changing at 6 ounces per month, and at 3.05 months it probably is not going to be changing very much faster or very much less but that's an estimation you could have hit a growth spurt right before or after, that's why I did it, the rate of change probably won't change much between 3 and 3.05. So I multiplied the rate of change which was 6, times the value added on to 3 when the rate of change was 6 and I multiplied those two numbers together.

2010; Thompson & Thompson, 1994; Yu, 2020), researchers observed students' ideas about rate of change as being additive and the obstacles students with this conception may encounter in future mathematical learning (Flynn et al., 2018; Prince et al., 2012; Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006). The findings of this study further this area of research by describing various nuances in how students coordinate changes in two quantities' values with their additive or multiplicative conception of a rate of change. For example, in the updated Covariational Reasoning Framework (Table 1), students at MA0-MA4 may all exhibit additive reasoning when utilizing the value of a rate of change, however, how these students conceive of a situation differs between each level.

The findings also extend what is known about students' covariational reasoning. In Carlson et al.'s (2002) study, students could exhibit MA5 reasoning with the bottle problem if they coordinated that equal changes in water would result in decreasing (then increasing) changes in height. I hypothesize that some of these students leveraged their intuitive understanding but may have struggled to demonstrate MA5 reasoning if they had to attend to the values of a rate of change at a given volume. In this study, some of the students demonstrated an awareness that the instantaneous rate of change of the fish's weight varies as the age of the fish varies. However, it was apparent that their interpretation of a value of a rate of change limited them to coordinating specific amounts of change, which was demonstrative of MA3 reasoning.

To recap, I highlight two major insights from the results of this study.

1) Attending to how a student interprets the value of a rate of change can provide insight into how they reason covariationally. Further, it is likely that a student's meaning for rate of change impacts how and why they reason at a particular level of covariational reasoning.
2) New categories of MA0 and MA3 +, and an updated description of MA4 and MA5 to further describe several nuances in student thinking regarding covariational reasoning that was not originally described in the original Covariational Reasoning Framework as proposed by Carlson et al. (2002).

6.1. Conclusion

As the field continues to research and understand students' covariational reasoning, this study expands our understanding by examining how students quantify a rate of change. Compared to the works of Johnson (2012, 2015) and Kertil et al. (2019), whose contributions focused on the quantitative operations for the former and the identification of quantities for the latter, this study complements these works by examining how students assign and attribute meaning to the value of a rate of change. Many of the students in this study reasoned at MA3/3 + due to their conception of a rate of change as an amount to add to the function's output value for a one-unit change in the input. Students with an additive conception of rate of change took actions to suggest they were thinking about completed changes instead of quantities varying smoothly and continuously. Therefore, it stands to reason that supporting students in constructing a productive understanding of rate of change can benefit their understanding of derivative as instantaneous rate of change (Yu, 2023).

In the task, students engaging in additive reasoning could utilize proportional correspondence (Fig. 16) to answer linear approximation problems. However, this additive conception of rate of change will likely be an obstacle when trying to understand other key ideas of Calculus. For example, suppose a student conceives of rate of change as considering changes in one-unit chunks; how will they make sense of the limit definition of derivative or seeing a sliding secant line converge to a tangent line whose slope represents a quantity we call "instantaneous rate of change?" This perhaps explains why some students form disconnected or compartmentalized meanings for derivatives (Zandieh, 2000) because their meaning for rate of change is incompatible with the depictions of instantaneous rate of change in Calculus. Not only would supporting students in engaging in multiplicative reasoning benefit students' understanding of derivatives, but also in future mathematical learning, such as accumulation. Jones (2013), Sealey (2014), and Thompson and Harel (2021) all indicate that students will experience difficulties conceptualizing integrals when they do not conceive of quantities varying smoothly and covariationally. Thompson and Harel argue that to understand integrals as accumulation, a student "needs to envision variations happening within bits—at least smoothly and at best smoothly and continuously" (pg. 512). However, if a student only has the means of engaging in proportional correspondence (due to their additive conception), they can only consider chunks of variations instead of smooth variations.

While Thompson and Carlson (2017) state that conceptualizations of rate of change require understandings that go beyond covariational reasoning, such as ratio, quotient, accumulation, and proportionality (and therefore, we should not equate their understanding of rate of change with their covariational reasoning), this study provides an example on the reciprocal relationship between an individual's covariational reasoning and quantification of rate of change. The new category of MA3 + highlights the usefulness of examining a student's rate of change conceptualizations since these types of reasoners likely have an image of what Thompson and Carlson call *Smooth Continuous Covariation*, yet how they quantified a rate of change limits them to exhibiting a *Coordination of Values* level. Further, even the students who evidenced MA4 or MA5 reasoning could not articulate the underlying reason for employing multiplication when using the value of a rate of change. It stands to reason that supporting students in developing a multiplicative meaning for rate of change can help them engage in higher levels of covariational reasoning. While developing a robust understanding of rate of change should be seeded early on (e.g., Thompson & Thompson, 1994), the findings of this study suggest that instructors at the undergraduate level can support students in furthering their covariational reasoning by refining students meaning for rate of change into a multiplicative one.

Overall, this study extends what we know about the connection between covariational reasoning and rate of change reasoning. While reasoning about rate of change involves ideas beyond just covariational reasoning, this does not mean that the relationship between them goes solely in one direction. Instead, the findings of this study and the works of Johnson (2012, 2015) and Kertil et al. (2019) demonstrate a reciprocity between an individual's quantification of rate of change and how they coordinate two quantities as covarying. Therefore, further studies can continue to investigate this relationship and how developing one aspect can support the other.

6.2. Limitations

In considering the results of this study, it is important to keep in mind that the results are known to be true for the 27 students involved and may not necessarily explain all other thinking. Additionally, one limitation of the study is the usage of a context involving derivatives and instantaneous rate of change where we know that students have a variety of conceptions about the derivative concept (Zandieh, 2000). It is recommended that future studies examine a large sample of students using a variety of contexts, such as situations involving more than 2 variables that involve their quantification of rate of change. Despite these limitations, the study's findings support the idea that exploring a student's conception of rate of change can provide insight yet may not fully explicate or reveal the complete picture on how they reason covariationally.

CRediT authorship contribution statement

Yu Franklin: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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