ON EXTREMIZING SEQUENCES FOR ADJOINT FOURIER RESTRICTION TO THE SPHERE

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Abstract. In this article, we develop a linear profile decomposition for the $L^p \to L^q$ adjoint Fourier restriction operator associated to the sphere, valid for exponent pairs $p < q$ for which this operator is bounded. Such theorems are new when $p \neq 2$. We apply these methods to prove new results regarding the existence of extremizers and the behavior of extremizing sequences for the spherical extension operator. Namely, assuming boundedness, extremizers exist if $q > \max\{p, \frac{d+2}{d}p'\}$, or if $q = \frac{d+2}{d}p'$ and the operator norm exceeds a certain constant times the operator norm of the parabolic extension operator.

1. Introduction and statement of results

We consider the Fourier restriction and extension operators associated to the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$, which are given by

$$\mathcal{R}g(\omega) := \int_{\mathbb{R}^{d+1}} e^{-ix\omega} g(x)\, dx,\ \omega \in \mathbb{S}^d, \qquad \mathcal{E}f(x) = \int_{\mathbb{S}^d} e^{ix\omega} f(\omega)\, d\sigma(\omega),$$

for g in the Schwartz class, $\mathcal{S}(\mathbb{R}^{d+1})$, and $f \in C^\infty(\mathbb{S}^d)$. Here σ denotes d-dimensional Hausdorff measure on $\mathbb{S}^d$. Despite decades of study, the precise conditions on exponents p and q for which (say) $\mathcal{E}$ extends as a bounded linear operator from $L^p(\mathbb{S}^d)$ to $L^q(\mathbb{R}^{1+d})$ are not fully resolved for any $d \geq 2$. (For an introductory discussion of these operators and their history, one may see, e.g., [15].)

We do not seek to directly address such questions. Rather, we ask, under the assumption of $L^p \to L^q$ boundedness of $\mathcal{E}$, what is the behavior of bounded sequences $\{f_n\}$ whose extensions $\{\mathcal{E}f_n\}$ do not converge to zero in norm. This will lead us to develop a qualitative description of such sequences called a profile decomposition. A particular scenario of interest is when the sequence $\{f_n\}$ is both L^p-normalized ($\|f_n\|_{L^p(\mathbb{S}^d;d\sigma)} \equiv 1$) and extremizing ($\|\mathcal{E}f_n\|_{L^q(\mathbb{R}^{1+d})} \to \|\mathcal{E}\|_{L^p(\mathbb{S}^d;d\sigma)\to L^q(\mathbb{R}^{1+d})}$), in which case our profile decompositions provide quite a bit of information (at least when $q > p$).

In order to state our results, we will need some notation and terminology. Noting that $\mathcal{R}$ and $\mathcal{E}$ are dual to one another, we denote their (common) operator norm by

$$S_{p\to q} := \sup_{\|f\|_{L^p(\mathbb{S}^d;d\sigma)}=1} \|\mathcal{E}f\|_{L^q(\mathbb{R}^{d+1})} = \sup_{\|g\|_{L^{q'}(\mathbb{R}^{d+1})}=1} \|\mathcal{R}g\|_{L^{p'}(\mathbb{S}^d;d\sigma)},$$

where the suprema are taken over (e.g.) smooth, compactly supported functions. These operator norms are conjectured to be finite whenever $q \geq \frac{d+2}{d}p'$ and $q > \frac{2(d+1)}{d}$ both hold, and these conditions are known to be necessary.

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We are interested in the questions of whether there exist nonzero functions for which equality holds in the restriction/extension inequalities

$$\|\mathcal{R}g\|_{L^{p'}(\mathbb{S}^d;d\sigma)} \leq S_{p\to q}\|g\|_{L^{q'}(\mathbb{R}^{d+1})}, \qquad \|\mathcal{E}f\|_{L^q(\mathbb{R}^{d+1})} \leq S_{p\to q}\|f\|_{L^p(\mathbb{S}^d;d\sigma)}, \quad (1.1)$$

in cases where the operator norms are finite, and whether L^p-normalized extremizing sequences are convergent in some sense. We are further interested in connections between these operators and the restriction/extension operators associated to the paraboloid, $\mathbb{P} := \{(\frac{1}{2}|\xi|^2, \xi) : \xi \in \mathbb{R}^d\}$:

$$\mathcal{R}_{\mathbb{P}}g(\xi) := \int_{\mathbb{R}^{d+1}} e^{-i(\frac{1}{2}|\xi|^2,\xi)x} g(x)\, dx,\ \xi \in \mathbb{R}^d, \qquad \mathcal{E}_{\mathbb{P}}f(x) := \int_{\mathbb{R}^d} e^{ix(\frac{1}{2}|\xi|^2,\xi)} f(\xi)\, d\xi,$$

whose common operator norms we denote by

$$P_{p\to q} := \sup_{\|f\|_{L^p(\mathbb{R}^d)}=1} \|\mathcal{E}_{\mathbb{P}}f\|_{L^q(\mathbb{R}^{d+1})} = \sup_{\|g\|_{L^{q'}(\mathbb{R}^{d+1})}=1} \|\mathcal{R}_{\mathbb{P}}g\|_{L^{p'}(\mathbb{R}^d)}. \quad (1.2)$$

More generally, we extend the profile decomposition methods of Fanelli–Visciglia–Vega [16] for the case $p = 2$, $q > \frac{d+2}{d}p'$ and Frank–Lieb–Sabin in [21] for the case $p = 2$, $q = \frac{d+2}{d}p'$, to the region $p < q \leq \frac{d+2}{d}p'$, $p \neq 1$. (A more complete history is given in the next section.)

Before stating our results, we set some basic terminology. Fix a pair of exponents (p, q). A sequence (f_n) in $L^p(\mathbb{S}^d)$ is L^p *normalized* if $\|f_n\|_p = 1$ for all n and is *extremizing* if $\lim_{n\to\infty} \|\mathcal{E}f_n\|_q/\|f_n\|_p = S_{p\to q}$. (We note that normalized extremizing sequences exist for every exponent pair (p, q); whether they converge and what are their properties are more subtle questions.)

Many of our results are partly conditional on progress toward the (adjoint) restriction conjecture for the sphere. We adopt the following convention, which will make for cleaner statements later on: We say that the extension conjecture holds at $(p, q) \in [1,\infty]^2$ if $S_{p\to q} < \infty$, if $q \leq \frac{2(d+1)}{d}$, or if $q < \frac{d+2}{d}p'$. (Of course, in the latter two cases, the extension operator is known to be unbounded.) In the non-vacuous range, the conjecture has been verified for all (p, q) when $d = 1$ [17, 41], and, in higher dimensions, has been verified on a neighborhood of the region $q \geq \frac{2(d+3)}{d+1}$, $q \geq \frac{d+2}{d}p'$ (see [1, 22, 23, 24, 36, 37, 40] for more precision regarding the current status).

Our results are cleanest off of the parabolic scaling line, wherein Hölder's inequality rules out the possibility that extremizing sequences might concentrate.

Theorem 1.1. *Assume that $q > \max\{p, \frac{d+2}{d}p'\}$ and that the extension conjecture holds on a neighborhood of (p, q). Then every L^p-normalized extremizing sequence for the inequality $\|\mathcal{E}f\|_q \leq S_{p\to q}\|f\|_p$ is precompact in L^p after the application of an appropriate sequence of spacetime modulations. In particular, extremizers exist for this inequality.*

In the case $p = 2$, this result is due to Fanelli–Visciglia–Vega [16]. The hypothesis $q > p$ is likely an artifact of our proof, which uses a convexity argument. Indeed, in certain special cases, such as when $p = \infty$ and q is an even integer, one can use other means to prove the existence of extremizers [8].

For inequalities with (p^{-1}, q^{-1}) on the parabolic scaling line $\{q = \frac{d+2}{d}p'\}$, we cannot (yet) rule out the possibility of concentration.

Theorem 1.2. *Let $1 < p < \frac{2(d+1)}{d}$ and $q = \frac{d+2}{d}p'$, and assume that the extension conjecture for the sphere holds on a neighborhood of (p, q). Let (f_n) be an $L^p(\mathbb{S}^d)$*

normalized extremizing sequence of (1.1)*. After passing to a subsequence, either:*
(i) There exists a sequence $\{x_n\} \subseteq \mathbb{R}^{d+1}$ *such that* $e^{ix_n\omega} f_n(\omega)$ *converges in* $L^p(\mathbb{S}^d)$ *to an extremizer* f *of* (1.1)*,*
or
(ii) There exist sequences $\{x_n\} \subseteq \mathbb{R}^{1+d}$*, orthogonal transformations* $\{R_n\} \subseteq O(d)$*, positive numbers* $\lambda_n \searrow 0$*, and functions* $\phi^{\pm} \in L^p(\mathbb{R}^d)$ *such that*

$$\lim_{n\to\infty} \big\| e^{ix_n\omega} f_n(R_n\omega) - \lambda_n^{-d/p} \sum_{\pm} \phi^{\pm}(\tfrac{\omega'}{\lambda_n})\chi_{\{\pm\omega_1 > \frac{1}{2}\}} \big\|_p = 0. \tag{1.3}$$

Two remarks are in order. First, this result is known in the case $p = 2$, [21]. (We note that Theorem 1.5 below is a more precise generalization of the results of [21].) Second, when $p = 1$, existence of extremizers and noncompactness modulo symmetries of normalized extremizing sequences (or even, sequences of normalized extremizers) are both elementary to prove, as any nonnegative L^1 function is extremal.

We say that (along a subsequence) (f_n) converges modulo the modulation symmetry in Conclusion (i) and that (f_n) concentrates antipodally and converges modulo translations, dilations (a nonsymmetry), and rotations in Conclusion (ii).

We can improve upon Theorem 1.2 by estimating the operator norm in the case of concentration, generalizing the main results of [21] and [14] (therein carried out in the $p = 2$ case). This will require some further notation.

For $1 \le p < q = \frac{d+2}{d}p'$, we define

$$\alpha_{p\to q} := \max_{t\in[0,1]} \frac{\|1 + te^{i\theta}\|_{L^q([0,2\pi],d\theta/2\pi)}}{(1 + t^p)^{1/p}}. \tag{1.4}$$

The parameter t will arise as the ratio between the norms of the extensions of two antipodally concentrating profiles. Considering such pairs will lead us to a lower bound for $S_{p\to q}$.

Proposition 1.3. *Let* $1 \le p < \frac{2(d+1)}{d}$ *and set* $q := \frac{d+2}{d}p'$*. Then*

$$S_{p\to q} \ge \alpha_{p\to q} P_{p\to q}. \tag{1.5}$$

The quantity

$$\beta_{p\to q} := 2^{\frac{1}{r'}} \left(\frac{\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{q+2}{2})} \right)^{\frac{1}{q}}, \qquad r := \max\{p, 2\},$$

seems somewhat easier to understand than $\alpha_{p\to q}$, and we note the following relationship between the two.

Proposition 1.4. *For* $p \ge 2$*,* $\alpha_{p\to q} = \beta_{p\to q}$*; while for* $p < 2$*,* $\alpha_{p\to q} < \beta_{p\to q}$*.*

The transition at $p = 2$ in Proposition 1.4 is connected with a bifurcation of our results along the parabolic scaling line into the cases $1 < p < 2$ and $2 < p < \frac{2(d+1)}{d}$. We begin with the latter case, in which our results are stronger.

Theorem 1.5. *Let* $2 \le p < \frac{2(d+1)}{d}$ *and* $q = \frac{d+2}{d}p'$*, and assume that the extension conjecture for the sphere holds on a neighborhood of* (p, q)*. If* $S_{p\to q} > \beta_{p\to q} P_{p\to q}$*, then extremizers exist for the extension operator in* (1.1) *and all normalized extremizing sequences possess subsequences that converge in* L^p*, after modulation.*

Otherwise, $S_{p\to q} = \beta_{p\to q} P_{p\to q}$, and concentrating, extremizing sequences (f_n) exist; after passing to a subsequence and normalizing, they must obey (1.3), for some $\phi^\pm$ extremal for $\mathcal{E}_{\mathbb{P}} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ and obeying

$$|\mathcal{E}\phi^+(x_1, x')| = |\mathcal{E}\phi^-(-x_1, x')|.$$

When $p < 2$ the gap between $\alpha_{p\to q}$ and $\beta_{p\to q}$ seen in Proposition 1.4 leaves some room for improvement in the following theorem, as discussed at the end of Section 8. (We note that in subsequent work, [5], a different approach was taken for restriction to odd monomial curves, leading to a partial analogue of Theorem 1.5 that holds in the full range of admissible p.)

Theorem 1.6. *Let $1 < p < 2$ and $q = \frac{d+2}{d} p'$, and assume that the extension conjecture for the sphere holds on a neighborhood of (p, q). If $S_{p\to q} \geq \beta_{p\to q} P_{p\to q}$ then extremizers exist for the extension operator in (1.1) and all normalized extremizing sequences possess subsequences that converge in L^p, modulo spatial translations of the extension. If $S_{p\to q} = \alpha_{p\to q} P_{p\to q}$, $\phi^\pm$ is extremal for (1.2), and*

$$|\mathcal{E}\phi^+(x_1, x')| = t|\mathcal{E}\phi^-(-x_1, x')|, \qquad x \in \mathbb{R}^{1+d},$$

where t is an argument of the maximum on the right hand side of (1.4), then any sequence (f_n) obeying (1.3) is extremizing.

Inequality (1.5) is known to hold with strict inequality for the case $p = 2$ in dimensions $d = 1, 2$, and is conjectured to be a strict inequality for $p = 2$ in all dimensions (see [18], [19], [21], and [25]). Since both sides of (1.5) are continuous in p along the line $q = \frac{d+2}{d} p'$ (for the operator norms, this follows from complex interpolation), the inequality continues to be strict in a small neighborhood of $p = 2$, allowing a modest extension of the range of p for which extremizers were previously known to exist for the $L^p(\mathbb{S}^d) \to L^{\frac{d+2}{d} p'}(\mathbb{R}^{1+d})$ extension problem.

Corollary 1.7. *In dimensions $d = 1, 2$, for $|p-2|$ sufficiently small and $q = \frac{d+2}{d} p'$, extremizers exist for the extension operator in (1.1), and extremizing sequences are precompact modulo symmetries.*

If true, the conjecture that the extremizers of the Stein–Tomas inequality for the paraboloid are Gaussians in all dimensions would imply that Corollary 1.7 holds in all dimensions [21].

The question of what are these extremizers is, of course, extremely interesting, but it is beyond the scope of this article. In [13], Christ–Quilodrán proved that Gaussian functions are not extremal for (1.2) (unless, possibly, $p = 2$), by proving that Gaussians do not satisfy the corresponding Euler–Lagrange equation unless $p = 2$. In the case of the sphere, however, symmetry makes it elementary to verify that constants do satisfy the analogous Euler–Lagrange equations for all (p, q), as was noted in [13], but this is insufficient to verify that constants are extremizers.

When $p = 2$, Theorems 1.2 and 1.5, Proposition 1.3, and Corollary 1.7 are due to Frank–Lieb–Sabin, [21].

As can be seen from the comparison with prior results, the main advantage of our approach is that it allows us to consider restriction inequalities with $p \neq 2$, for which the loss of the Hilbert space structure and Plancherel substantially reduces our available tools [32]. We achieve our results by adapting the approach laid out in [35], wherein it was proved that all valid, nonendpoint parabolic extension estimates possess extremizers and have precompact (modulo symmetries) extremizing sequences. The major difference between the spherical case and the parabolic one is the defect in compactness due to the lack of a scaling symmetry. In particular, from the perspective of a concentrating sequence, the sphere begins to resemble a rotated paraboloid. In this spirit, we treat scaling as an almost-symmetry, analogously with prior works such as [21, 26]. Relative to [26], the existence of distinct points on the sphere with parallel normal vectors presents an additional complication, which we address by adapting the approach of [21].

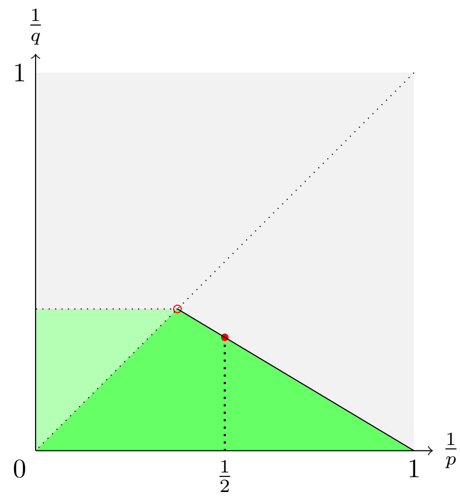

FIGURE 1. The extension operator is conjectured to be bounded in the green quadrilateral. We consider the subset of the darker green triangle on which the adjoint restriction conjecture holds, including the parabolic scaling line on the right, but excluding the diagonal $p = q$ on the left. We have indicated the $p = 2$ case, which featured prominently in prior work, with a dotted line.

Outline of paper. In the next section, we will give an in-depth overview of some of the recent history of related questions, placing our work in context. Our strongest result, from which the others all follow, is an L^p-profile decomposition for bounded sequences on the sphere with nonnegligible extensions. This result is somewhat complex, and will occupy three theorems in Section 3. The first of these three results gives a frequency decomposition. Roughly, if $\{f_n\}$ is bounded in $L^p(\mathbb{S}^d)$ and $\{\mathcal{E}f_n\}$ does not tend to zero, then (along a subsequence) each f_n decomposes as a finite sum of pieces with good frequency localization properties, plus a small error; this result is proved in Section 4, with bilinear restriction as a primary tool. Though the summands arising in the first decomposition are bounded by sequences that are (after a rotation) either precompact in $L^p(\mathbb{S}^d)$ or correspond to sequences precompact in $L^p(\mathbb{R}^d)$ (after scaling), they are not themselves precompact, as their extensions may not be well-localized in space. The second profile decomposition establishes good spatial localization for those sequences that are (nearly) pointwise bounded (i.e., non-concentrating), and is proved in Section 5. The third and final profile decomposition establishes good spatial localization (after rescaling) for concentrating sequences, and is proved in Section 6. In Section 7, we prove Theorem 1.1, that extremizers exist for exponent pairs $p < q$ lying off of the parabolic scaling line $q = \frac{d+2}{d}p'$. Section 8 provides an analysis of the behavior of antipodally concentrating profiles, which is then applied in Section 9 to deduce properties of concentrating extremizing sequences (supposing that they exist).

Notation. We will use throughout the standard notation $A \lesssim B$ to mean $A \leq CB$ for C an *admissible* constant that will be allowed to change from line to line. Admissible constants may depend on the dimension d, the exponents p, q, and (in cases where our results are conditional) on bounds for the spherical restriction/extension operators for exponents in a small neighborhood of p, q. Occasionally we will decorate the '$\lesssim$' symbol with subscripts to indicate additional dependencies.

Though we will use Lebesgue norms on three different spaces (the sphere, $\mathbb{R}^{d+1}$, and $\mathbb{R}^d$); when the meaning is clear and space is limited, we will abbreviate these norms by using only the exponent as a subscript.

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2. Prior results

An excellent survey on sharp Fourier restriction results is given in [20]. As this is an active area, we highlight a few more recent results as well as the prior results most relevant to our analysis. For the sake of completeness, we will also state and prove an elementary result that we have not been able to find written elsewhere.

Existence results for extremizers of L^p-L^q inequalities for the sphere have largely[1] involved one or both of the hypotheses that $p = 2$ or q is an even integer. These cases are special because of the Hilbert space structure available in $p = 2$, on the one hand, and an explicit formula for the L^{2k} norm of spherical extensions as the L^2 norm of a k-fold convolution, on the other. In addition to the previously discussed results of [21] when $p = 2$ and $q = \frac{d+2}{d}p'$, existence of extremizers has been established in the cases that $p = 2$, $q = 4$, $d = 2$ [14]; $p = 2$, $q > \frac{d+2}{d}p'$, $d \geq 1$ [16]; $p = 2$, $q = 6$, $d = 1$ [33]; $p \geq 2$, $q = 4$, $d \in \{2, 3, 4, 5, 6\}$; $p \geq 4$, $q = 4$, $d \geq 7$; $p \geq q$, $q = 2k$, $q \geq 6$, $d \geq 1$ (the last three results are all in [8]). We note that the $p \geq q$ condition in [8] includes $p = \infty$ and is precisely the reverse of our $q > p$ condition.

In some of these cases extremizers are known to be modulations of constants. Namely, when $p = 2$, $q = 4$, and $d = 2$, this result is due to [19]; for $p \geq 2$, $q = 4$, and $d \in \{2, 3, 4, 5, 6\}$, $p \geq 4$, $q = 4$, and $d \geq 7$, $p \geq q$, $q = 2k$, $q \geq 6$, and $d \geq 1$, [8]; and for $p = 2$, $q = 2k$, when $d \in \{2, 3, 4, 5, 6\}$ [31]. Stability of these results is investigated in [9] where they show that in $d \in \{2, 3, 4, 5, 6\}$, for $p = 2$, when $L^4(\mathbb{R}^d)$ is replaced by a weighted L^4 with a radial weight which is a small perturbation of the unweighted case, the only extremizers are constants.

Our results build on the profile decomposition approach of [16] (and, implicitly, [21]), extending these methods to address the absence of Hilbert space structure

[1] A partial exception is [10] in which $p = 2$ and $L^q(\mathbb{R}^{d+1})$ is replaced by the mixed norm space $L^q_{rad}L^2_{ang}(\mathbb{R}^{d+1})$, and the analysis is based upon a careful study of Bessel functions.

when $p \neq 2$. The methods for this adaptation originate in the study of sharp restriction for non-compact manifolds, specifically [35], which proves that all valid, nonendpoint parabolic extension estimates possess extremizers and have precompact (modulo symmetries) extremizing sequences. The ideas are further developed in [4], [38], and [39] which consider the precompactness of extremizing sequences for adjoint Fourier restriction to other non-compact manifolds. Recently, L^2 based concentration compactness methods have also been used to investigate convergence of extremizing sequences on the hyperbola in [11] and [12].

Finally, for completeness, we prove an elementary result, which is surely known to experts, but which we haven't found in the literature.

Proposition 2.1. *For all $1 \leq p \leq \infty$, nonzero modulated constants, $e^{ix_0\omega}\lambda$ are maximizers of the $L^p \to L^\infty$ extension inequality. When $p > 1$, such functions are the unique maximizers. When $1 < p < \infty$, after possible modulation and multiplication by unimodular constants, every normalized extremizing sequence in L^p converges in L^p to the constant function $\lambda_p := \sigma(\mathbb{S}^d)^{-\frac{1}{p}}$.*

Proof. Sufficiency follows from Hölder's inequality,

$$\|\mathcal{E}f\|_{L^\infty(\mathbb{R}^{d+1})} \leq \|f\|_{L^1(\mathbb{S}^d)} \leq \sigma(\mathbb{S}^d)^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{S}^d)}, \tag{2.1}$$

and for all p, equality holds for the modulated constants. For necessity of the constants, we observe that for all p the first inequality in (2.1) is equality if and only if $e^{i\theta}e^{ix_0\omega}f$ is nonnegative for some θ, x_0, while for $p > 1$, the second inequality is equality if and only if $|f|$ is constant.

Finally, let $1 < p < \infty$ and let $\{f_n\}$ be a normalized extremizing sequence in $L^p(\mathbb{S}^d)$. By modulating and multiplying the f_n by unimodular constants, we may assume that $\mathcal{E}f_n(0) = \|\mathcal{E}f_n\|_{L^\infty(\mathbb{R}^{d+1})}$ for all n. It suffices to prove that every subsequence of $\{f_n\}$ has a further subsequence convergent to λ_p. Therefore, since $\{f_n\}$ was arbitrary, it suffices, by Banach–Alaoglu, to prove that $f_n \to \lambda_p$ in $L^p(\mathbb{S}^d)$, under the additional hypothesis that f_n converges weakly to some $f \in L^p(\mathbb{S}^d)$.

By construction and our above computation of the operator norm,

$$\mathcal{E}f(0) = \lim \mathcal{E}f_n(0) = \lim \|\mathcal{E}f_n\|_{L^\infty(\mathbb{R}^{d+1})} = \|\mathcal{E}\|_{L^p(\mathbb{S}^d)\to L^\infty(\mathbb{R}^{d+1})} = \sigma(\mathbb{S}^d)^{\frac{1}{p'}}.$$

Since $\|f\|_{L^p(\mathbb{S}^d)} \leq 1$, while $\|\mathcal{E}f\|_{L^\infty(\mathbb{R}^{d+1})} \geq \|\mathcal{E}\|_{L^p\to L^\infty}$, the uniqueness portion of the proposition (already proved) implies that $f \equiv \lambda_p$. Since $f_n \rightharpoonup \lambda_p$ and $\|f_n\|_{L^p(\mathbb{S}^d)} \equiv 1 = \|\lambda_p\|_{L^p(\mathbb{S}^d)}$, Theorem 2.11 of [29] implies that $f_n \to \lambda_p$ in $L^p(\mathbb{S}^d)$. □

Thus our setting introduces some key differences relative to what has come before. Namely, as opposed to the vast majority of published results, we impose very few conditions on (p, q), requiring only that $q > p$, $S_{p\to q} < \infty$, and that the extension conjecture is valid on a neighborhood of (p, q). Further, due to compactness of the sphere, we are able to consider an even wider range of exponent pairs than [35] (which was limited to the scaling line). Additionally, as already observed in [21], the sphere lacks some simplifications available for other surfaces (e.g. the paraboloid or hyperboloid), since it lacks a scaling symmetry and possesses antipodal points; consideration of these features without the condition $p = 2$ presents some new complications. Finally, our results go further than those of [4, 35, 38, 39] by establishing a full profile decomposition for bounded L^p sequences, rather than exclusively focusing on the extremal case. In particular, one may apply the profile decomposition to (e.g.) a norm-one sequence of functions whose extensions attain at least half the operator norm to deduce that, after passing to a subsequence, the terms can be decomposed into a structured part, composed of a few precompact (modulo symmetries), asymptotically orthogonal pieces responsible for most of the extension, and a random part, whose extension is small.

3. A weak L^p profile decomposition

In this section, we will introduce our main tool, a weak L^p profile decomposition. A profile decomposition associated to an operator $T : X \to Y$ is a means of decomposing bounded sequences in X as the sum of a structured part, which has good compactness properties modulo symmetries, and a "random" part, which is small after an application of T. The method was introduced by Lions [30] and has found extensive application in PDE. For Fourier extension operators, the L^2-based theory of profile decompositions is comparatively well-developed, both because of the role that L^2-based inequalities play in the study of dispersive equations and also because more tools, namely, Plancherel and the Hilbert space structure, are available. In particular, all of the essential ingredients for the L^2 profile decomposition are given in [21], though the full profile decomposition was never explicitly stated in that article.

In [35], a profile decomposition of extremizing, frequency localized L^p sequences was used to prove that the extension operator associated to the paraboloid possesses extremizers. Here, we give a more quantitative result, providing a decomposition of more general sequences. Our profile decomposition is weak in the sense that it gives poor control over the remainder terms, which, despite having small extension, may blow up in L^p. This blowup of the L^p norm results from our use of weak limits, and does not affect the L^2 theory because of elementary Hilbert space manipulations. (When we are not in a Hilbert space, subtracting a weak limit from a sequence does not necessarily decrease the limit of the norms [32].) An alternative, stronger profile decomposition for L^p sequences and operators satisfying certain conditions is developed in [34]; it is based on Δ-limits, rather than weak limits. A significant advantage of using Δ-limits, rather than weak limits, is that the remainder terms in the Δ profile decomposition are bounded in L^p, in addition to having small extensions. A disadvantage is that Δ-limits do not seem to yield sufficiently sharp inequalities to control the number of profiles of an extremizing sequence and thereby prove the existence of extremizers. (I.e., we will rely on inequalities involving the relation '$\leq$,' rather than '$\lesssim$.')

Our results include the possible case of concentration at antipodal points. For this reason, it is convenient to use the real projective space $\mathbb{RP}^d = \mathbb{S}^d/\{\omega \sim -\omega\}$, whose elements we denote by $[\omega] := \{\omega, -\omega\}$, $\omega \in \mathbb{S}^d$. We observe that

$$\mathrm{dist}([\omega], [\omega']) = \min\{|\omega - \omega'|, |\omega + \omega'|\}.$$

To produce statements that are somewhat easier to parse, we have broken our profile decomposition into three parts. We begin with a decomposition of the frequency space $\mathbb{S}^d$, distinguishing between the critical and subcritical regime.

Theorem 3.1 (Frequency decomposition)**.** *Let $1 < p < q = \frac{d+2}{d}p'$, and assume that the restriction conjecture for $\mathcal{E}$ holds on a neighborhood of (p, q). Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{S}^d)$. After passing to a subsequence, there exist $\{\lambda_n^j\}_{j,n\in\mathbb{N}} \subseteq (0, 1]$, $\{[\omega_n^j]\}_{j,n\in\mathbb{N}} \subseteq \mathbb{RP}^d$, and a sequence of decompositions $f_n = \sum_{j=1}^J F_n^j + R_n^J$,*

$J \in \mathbb{N}$, such that:
(i) For each j, either $\lambda_n^j \to 0$, or $\lambda_n^j \equiv 1$.
(ii) For $j \neq j'$, either $|\log \frac{\lambda_n^j}{\lambda_n^{j'}}| \to \infty$ or $\lambda_n^j \equiv \lambda_n^{j'}$, and, in the latter case, if $\lambda_n^j \to 0$, then $(\lambda_n^j)^{-1}\operatorname{dist}([\omega_n^j],[\omega_n^{j'}]) \to \infty$.
(iii) For each $n, J \in \mathbb{N}$, $\|f_n\|_{L^p(\mathbb{S}^d)}^p = \sum_{j=1}^J \|F_n^j\|_{L^p(\mathbb{S}^d)}^p + \|R_n^J\|_{L^p(\mathbb{S}^d)}^p$,
(iv) For all $J \in \mathbb{N}$, $\lim_{n\to\infty} \|\mathcal{E}f_n\|_q^q - \sum_{j=1}^J \|\mathcal{E}F_n^j\|_q^q - \|\mathcal{E}R_n^J\|_q^q = 0$,
(v) The remainders have small extension: $\lim_{J\to\infty}\limsup_{n\to\infty}\|\mathcal{E}R_n^J\|_{L^q(\mathbb{R}^{d+1})} = 0$
(vi) The F_n^j are adapted to antipodal caps of radius λ_n^j with centers on $[\omega_n^j]$ in the sense that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\sup_{E\subseteq E_n^{j,M}} \|\mathcal{E}F_n^j\chi_E\|_{L^q(\mathbb{R}^{d+1})} = 0, \text{ where}$$
$$E_n^{j,M} := \{|F_n^j| > M(\lambda_n^j)^{-d/p}\} \cup \{\operatorname{dist}(\omega,[\omega_n^j]) > M\lambda_n^j\}.$$

We observe that without (vi), the result follows trivially by taking $F_n^1 \equiv f_n$.

In the subcritical regime $q > \frac{d+2}{d}p'$, the frequency "decomposition" is much simpler (and an elementary consequence of Hölder's inequality, as we will see).

Proposition 3.2. *Let $q > \max\{\frac{d+2}{d}p', \frac{2(d+1)}{d}\}$, and assume that the restriction conjecture for $\mathcal{E}$ holds on a neighborhood of (p,q). Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{S}^d)$. Then if $E_n^M := \{|f_n| > M\}$, then*

$$\lim_{M\to\infty}\sup_n \sup_{E\subseteq E_n^M} \|\mathcal{E}f_n\chi_E\|_{L^q(\mathbb{R}^{d+1})} = 0.$$

Our next two results provide a spatial decomposition of functions obeying the frequency localization property described in part (vi) of Theorem 3.1 (and the conclusion of Proposition 3.2), in the cases of nonconcentration and antipodal concentration, respectively. In both, we will use the notation

$$\tilde{p} := \max\{p, p'\}.$$

Theorem 3.3 (Scale 1 spatial decomposition)**.** *Let $1 < p < q < \infty$ with $q \geq \frac{d+2}{d}p'$ and $q > \frac{2(d+1)}{d}$, and assume that the restriction conjecture for $\mathcal{E}$ holds on a neighborhood of (p,q). Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{S}^d)$ satisfying the condition*

$$\lim_{M\to\infty}\limsup_{n\to\infty}\|\mathcal{E}f_n\chi_{\{|f_n|>M\}}\|_{L^q(\mathbb{R}^{d+1})} = 0.$$

After passing to a subsequence, there exist $\{x_n^j\}_{j,n\in\mathbb{N}} \subseteq \mathbb{R}^{d+1}$ obeying

$$\lim_{n\to\infty}|x_n^j - x_n^{j'}| = \infty, \text{ for } j \neq j', \tag{3.1}$$

and weak limits, $\phi^j = \text{wk-lim}\, e^{-ix_n^j\omega}f_n$, such that for $J \in \mathbb{N}$,

$$\begin{gathered}\Big(\sum_j \|\phi^j\|_{L^p(\mathbb{S}^d)}^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}} \leq \liminf \|f_n\|_{L^p(\mathbb{S}^d)}, \\ \limsup \|\sum_{j=1}^J e^{ix_n^j\omega}\phi^j\|_{L^p(\mathbb{S}^d)} \leq \Big(\sum_{j=1}^J \|\phi^j\|_{L^p(\mathbb{S}^d)}^{\tilde{p}'}\Big)^{\frac{1}{\tilde{p}'}},\end{gathered} \tag{3.2}$$

and the remainders $r_n^J := f_n - \sum_{j=1}^J e^{ix_n^j\omega}\phi^j$ *satisfy*

$$\lim_{n\to\infty} \|\mathcal{E}f_n\|_{L^q(\mathbb{R}^{d+1})}^q - \sum_{j=1}^J \|\mathcal{E}\phi^j\|_{L^q(\mathbb{R}^{d+1})}^q - \|\mathcal{E}r_n^J\|_{L^q(\mathbb{R}^{d+1})}^q = 0, \quad J \in \mathbb{N}, \tag{3.3}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\mathcal{E}r_n^J\|_{L^q(\mathbb{R}^{d+1})} = 0. \tag{3.4}$$

In what follows, we write $\omega = (\omega_1, \omega')$.

Theorem 3.4 (Large scale spatial decomposition)**.** *Let* $1 < p < \frac{2(d+1)}{d}$ *and* $q = \frac{d+2}{d}p'$. *Assume that the restriction conjecture for* $\mathcal{E}$ *holds on a neighborhood of* (p,q). *Let* $\{f_n\} \subset L^p(\mathbb{S}^d)$ *be a bounded sequence and let* $\lambda_n \searrow 0$. *Assume that*

$$\begin{gathered}\lim_{M\to\infty} \limsup_{n\to\infty} \sup_{E\subseteq E_n^M} \|\mathcal{E}f_n\chi_E\|_{L^q(\mathbb{R}^{d+1})} = 0, \text{ where} \\ E_n^M := \{|f_n| > M(\lambda_n)^{-d/p}\} \cup \{\mathrm{dist}(\omega, [e_1]) > M\lambda_n\}\end{gathered} \tag{3.5}$$

After passing to a subsequence, there exist $\{x_n^j\}_{j,n\in\mathbb{N}} \subset \mathbb{R}^{d+1}$ *with*

$$\lim_{n\to\infty} \left(\lambda_n^2|(x_n^j - x_n^{j'})_1| + \lambda_n|(x_n^j - x_n^{j'})'|\right) = \infty, \text{ for } j \neq j',$$

and weak limits $\phi^{j,\bullet} \in L^p(\mathbb{R}^d)$, $\bullet = +, -$, *given by*

$$\phi^{j,\pm} = \text{wk-lim}\, \lambda_n^{d/p} e^{-ix_n^j(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} f_n(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|<\frac{1}{2}\lambda_n^{-1}\}}, \tag{3.6}$$

such that the following conclusions hold. Setting

$$g_n^{j,\pm}(\omega) := \lambda_n^{-d/p}\phi^{j,\pm}(\lambda_n^{-1}\omega')\chi_{\{\pm\omega_1>0\}}\chi_{\{|\omega'|<\frac{1}{2}\}}, \qquad g_n^j := \sum_\pm g_n^{j,\pm}, \tag{3.7}$$

and

$$r_n^J := f_n - \sum_{j=1}^J e^{ix_n^j\omega} g_n^j,$$

then,

(i) $\lim_{n\to\infty} \|\mathcal{E}g_n^j - \sum_\pm \lambda_n^{\frac{d+2}{q}} e^{\pm ix_1}\mathcal{E}_{\mathbb{P}}\phi^{j,\pm}(\mp\lambda_n^2 x_1, \lambda_n x')\|_{L^q(\mathbb{R}^{d+1})} = 0.$
(ii) $\left[\sum_\pm\left(\sum_j \|\phi^{j,\pm}\|_{L^p(\mathbb{R}^d)}^{\tilde{p}}\right)^{p/\tilde{p}}\right]^{1/\tilde{p}} \leq \liminf \|f_n\|_{L^p(\mathbb{S}^d)}$,
(iii) $\limsup_{n\to\infty} \|\sum_{j=1}^J e^{ix_n^j\omega}g_n^j\|_{L^p(\mathbb{S}^d)} \leq \left[\sum_\pm\left(\sum_j \|\phi^{j,\pm}\|_{L^p(\mathbb{R}^d)}^{\tilde{p}'}\right)^{p/\tilde{p}'}\right]^{1/\tilde{p}'}$, $J \in \mathbb{N}$,
(iv) $\lim_{n\to\infty} \|\mathcal{E}f_n\|_{L^q(\mathbb{R}^{d+1})}^q - \sum_{j=1}^J \|\mathcal{E}g_n^j\|_{L^q(\mathbb{R}^{d+1})}^q - \|\mathcal{E}r_n^J\|_{L^q(\mathbb{R}^{d+1})}^q = 0$, $J \in \mathbb{N}$
(v) $\lim_{J\to\infty} \limsup_{n\to\infty} \|\mathcal{E}r_n^J\|_{L^q(\mathbb{R}^{d+1})} = 0.$

4. Proof of the frequency decomposition

In this section, we will prove Theorem 3.1 and Proposition 3.2, which decompose the functions from bounded sequences into pieces with good frequency localization.

We begin with the essentially elementary proof of Proposition 3.2, in which (p,q) lies off of the parabolic scaling line.

Proof of Proposition 3.2. By assumption, $\mathcal{E}$ extends as a bounded operator from $L^r(\mathbb{S}^d)$ to $L^s(\mathbb{R}^{d+1})$ for (r,s) in a neighborhood of (p,q). In particular, $\mathcal{E}$ maps $L^r(\mathbb{S}^d)$ into $L^q(\mathbb{R}^{d+1})$ for some $r < p$. For $f \in L^p(\mathbb{S}^d)$, $M > 0$, and $E \subseteq E^M = \{|f| > M\}$ which is a measurable set,

$$\|\mathcal{E}(f\chi_E)\|_{L^q(\mathbb{R}^{d+1})} \lesssim \|f\chi_E\|_{L^r(\mathbb{S}^d)} \leq M^{-(\frac{p}{r}-1)}\|f\|_{L^p(\mathbb{S}^d)}^{\frac{p}{r}}.$$

Hence for a bounded sequence $\{f_n\}$, the stated conclusion holds. $\square$

The frequency localization is more involved for exponents on the scaling line. The heart of the argument is the bilinear extension theorem of Tao [36] and the bilinear-to-linear argument of Tao–Vargas–Vega [37], with inspiration from [3, 6, 7].

Lemma 4.1 ([36]). *Let $1 < p < \frac{2(d+1)}{d}$ and $q = \frac{d+2}{d}p'$, and assume that the spherical extension conjecture holds on a neighborhood of (p,q). Then there exists $s < p$ such that for all $(\bar{s},\bar{q})$ in a neighborhood of (s,q), all $0 < r \ll 1$, and $f \in L^{\bar{s}}$, we have the bilinear inequality*

$$\|\mathcal{E}(f\chi_\tau)\mathcal{E}(f\chi_{\tau'})\|_{\frac{\bar{q}}{2}} \lesssim r^{-2d(\frac{d+2}{d\bar{q}}-\frac{1}{\bar{s}'})}\|f\chi_\tau\|_{L^{\bar{s}}(\mathbb{S}^d)}\|f\chi_{\tau'}\|_{L^{\bar{s}}(\mathbb{S}^d)}, \tag{4.1}$$

whenever τ,τ' are defined by

$$\tau := \{\nu \in \mathbb{S}^d : \mathrm{dist}([\nu],[\omega]) \ll r\}, \qquad \tau' := \{\nu \in \mathbb{S}^d : \mathrm{dist}([\nu],[\omega']) \ll r\},$$

for some $\omega,\omega' \in \mathbb{S}^d$ with $\mathrm{dist}([\omega],[\omega']) \sim r$. The implicit constant is independent of f,τ,τ',r.

We give the details of the deduction from the remarks in Section 9 of [36].

Proof. Since exponents $\bar{q}$ lying sufficiently close to q have similar properties (namely, finiteness of $S_{\bar{p}\to\bar{q}}$ when $\bar{p} = (\frac{2q}{d+2})'$), it suffices prove that such an estimate holds at (s,q), for s lying in some open subinterval of $(1,p)$.

After a rotation, we may assume that $\omega = e_1$. We decompose $\tau = \tau_0 \cup (-\tau_0)$, $\tau' = \tau_0' \cup (-\tau_0')$, where $\mathrm{diam}(\tau_0) \sim \mathrm{diam}(\tau_0') \ll r \sim \mathrm{dist}(\tau_0,\tau_0')$. Using the triangle inequality and taking conjugates, it suffices to bound

$$\|\mathcal{E}(f\chi_{\tau_0})\,\mathcal{E}(f\chi_{\tau_0'})\|_{L^{\frac{q}{2}}(\mathbb{R}^{d+1})}.$$

By assumption, for $\bar{q}$ sufficiently near q and $\bar{p} := (\frac{d\bar{q}}{d+2})'$, we have the linear estimate $S_{\bar{p}\to\bar{q}} < \infty$. Hence by Cauchy–Schwarz, for any f_1, f_2:

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{L^{\frac{\bar{q}}{2}}} \leq \|\mathcal{E}f_1\|_{L^{\bar{q}}}\|\mathcal{E}f_2\|_{L^{\bar{q}}} \leq S^2_{\bar{p}\to\bar{q}}\|f_1\|_{L^{\bar{p}}(\mathbb{S}^d)}\|f_2\|_{L^{\bar{p}}(\mathbb{S}^d)}. \tag{4.2}$$

Further Tao's bilinear estimate [36] gives that for any $t > \frac{2(d+3)}{d+1}$ and any pair of test functions f_1, f_2 supported on spherical caps whose width and separation are comparable to some sufficiently small dimensional constant,

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{L^{\frac{t}{2}}(\mathbb{R}^{d+1})} \leq C\|f_1\|_{L^2(\mathbb{S}^d)}\|f_2\|_{L^2(\mathbb{S}^d)}. \tag{4.3}$$

Hence by interpolating (4.3) for some $\frac{2(d+3)}{d+1} < t < \frac{2(d+2)}{d}$ with (4.2), we see that (4.1) holds in the case $r \sim 1$. Inequality (4.1) in the case $r \ll 1$ follows by parabolic rescaling; we omit the details. $\square$

For technical reasons, we will use a slightly non-typical definition of caps. By a *(square) cap* in $\mathbb{S}^d$, we mean the intersection of $\mathbb{S}^d$ with with two axis-parallel cubes having diameter at most $\frac{1}{4}$, with centers given by two antipodal points contained in $\mathbb{S}^d$. By the sidelength of a cap, we mean the sidelength of the corresponding cube. For each $j \in \{1, 2 \dots, d\}$ and each $k \in \mathbb{N}$ satisfying $k \geq C$ for some sufficiently large

C, we fix a non-overlapping covering $\mathcal{D}_j^k$ of the nonequatorial region $W_j := \{\omega \in \mathbb{S}^d : |\omega_j| \geq \frac{1}{2\sqrt{d}}\}$ by caps of sidelength 2^{-k}. We denote various unions of the $\mathcal{D}_j^k$ by

$$\mathcal{D}^k := \bigcup_{j=1}^d \mathcal{D}_j^k, \qquad \mathcal{D}_j := \bigcup_{k\geq C} \mathcal{D}_j^k, \qquad \mathcal{D} := \bigcup_{j=1}^d \mathcal{D}_j.$$

We say that two caps $\tau, \tau' \in \mathcal{D}_j^k$ are related, $\tau \sim \tau'$, if $2^{-k+C} \leq \operatorname{dist}(\tau,\tau') \leq 2^{-k+2C}$. It is well-known (see [37, 3]) that when C is sufficiently large, all of the possible sumsets $\tau+\tau'$ for related $\tau,\tau' \in \mathcal{D}^k, k \geq C$ are contained in finitely overlapping parallelepipeds. (These parallelepipeds have much smaller volume than the cubes whose intersection with $\mathbb{S}^d$ equals τ,τ'.) We use f_τ to denote the product $f\cdot\chi_\tau$.

Lemma 4.2. *Let $q > \frac{2(d+1)}{d}$ and $p = (\frac{qd}{d+2})'$, and assume that the spherical extension conjecture holds on a neighborhood of (p,q). There exists $s < p$ such that for sufficiently small $0 < \nu < 1$,*

$$\|\mathcal{E}f\|_q \lesssim (\sup_{\tau\in\mathcal{D}} |\tau|^{-\frac{1}{p'}} \|\mathcal{E}f_\tau\|_\infty)^\nu \big(\sum_{\tau\in\mathcal{D}} |\tau|^{-2t(1-\nu)(\frac1s-\frac1p)} \|f_\tau\|_{L^s(\mathbb{S}^d)}^{2t(1-\nu)}\big)^{\frac{1}{2t}}. \tag{4.4}$$

Here $t := \min\{\frac{q}{2}, (\frac{q}{2})'\}$.

Proof. By Lemma 4.1, there exists $s < p$ such that for sufficiently small $0 < \nu < 1$ and $\bar{q} := (1-\nu)q$, (4.1) holds with exponent $\frac{\bar{q}}{2}$ on the left side and exponent s on the right.

By the triangle inequality, there exists j such that $\|\mathcal{E}f\|_q \lesssim \|\mathcal{E}(f\chi_{W_j})\|_q$, and after a rotation, we may assume that $j = 1$. Employing a Whitney decomposition of $W_1 \times W_1 \setminus \{(\omega,\omega) : \omega \in \mathbb{S}^d\}$, followed by almost orthogonality (Lemma 6.1 of [37]), Hölder's inequality, and finally our bilinear extension estimate, the arithmetic-geometric mean inequality, and some reindexing,

$$\begin{aligned}\|\mathcal{E}(f\chi_{W_1})\|_q^2 &= \|\sum_{\tau\sim\tau'\in\mathcal{D}_1} c_{\tau,\tau'}\mathcal{E}f_\tau\mathcal{E}f_{\tau'}\|_{\frac{q}{2}} \lesssim \big(\sum_{\tau\sim\tau'\in\mathcal{D}} \|\mathcal{E}f_\tau\mathcal{E}f_{\tau'}\|_{\frac{q}{2}}^t\big)^{\frac1t} \\ &\leq \big(\sum_{\tau\sim\tau'\in\mathcal{D}} (|\tau|^{-\frac{2}{p'}}\|\mathcal{E}f_\tau\mathcal{E}f_{\tau'}\|_\infty)^{t\nu} (|\tau|^{\frac{2\nu}{(1-\nu)p'}}\|\mathcal{E}f_\tau\mathcal{E}f_{\tau'}\|_{\frac{\bar{q}}{2}})^{t(1-\nu)}\big)^{\frac1t} \\ &\lesssim (\sup_{\tau\in\mathcal{D}} |\tau|^{-\frac{1}{p'}}\|\mathcal{E}f_\tau\|_\infty)^{2\nu} \big(\sum_{\tau\in\mathcal{D}} (|\tau|^{-(\frac1s-\frac1p)}\|f_\tau\|_{L^s(\mathbb{S}^d)})^{2t(1-\nu)}\big)^{\frac1t}.\end{aligned}$$

Here the $c_{\tau,\tau'}$ are constants with $|c_{\tau,\tau'}| \lesssim 1$. ☐

Lemma 4.3. *Let $q = \frac{d+2}{d}p' > p > 1$, and assume that the spherical extension conjecture holds on a neighborhood of (p,q). Then there exist $c_0 > 0$ and $0 < \theta < 1$ such that*

$$\|\mathcal{E}f\|_q \lesssim (\sup_{\tau\in\mathcal{D}} |\tau|^{-\frac{1}{p'}}\|\mathcal{E}f_\tau\|_\infty)^\theta \|f\|_p^{1-\theta} \lesssim \sup_{\tau\in\mathcal{D}}\sup_{n\geq 0} 2^{-c_0 n}\|f_{\tau,n}\|_p^\theta \|f\|_p^{1-\theta}, \tag{4.5}$$

where $f_{\tau,n}$ equals f multiplied by the characteristic function of $\tau\cap\{|f| \leq 2^n\|f\|_p|\tau|^{-\frac1p}\}$.

Proof. We will prove the superficially stronger bound wherein $f_{\tau,n}$ is replaced by f_τ^n on the right hand side, where $f_\tau^0 := f_{\tau,0}$ and $f_\tau^n := f_{\tau,n} - f_{\tau,n-1}$ for $n \geq 1$. We observe that for $n \geq 1$,

$$|f_\tau^n| \sim 2^n|\tau|^{-\frac1p}\|f\|_{L^p(\mathbb{S}^d)}\chi_{\{f_\tau^n\neq 0\}}. \tag{4.6}$$

We begin by showing the second inequality in (4.5). We apply Hölder's inequality and decompose into the f_τ^n to see that

$$|\tau|^{-\frac{1}{p'}}\|\mathcal{E}f_\tau\|_\infty \leq \sum_{n=0}^{\infty}|\tau|^{-\frac{1}{p'}}\int_{\mathbb{S}^d}|f_\tau^n|\,d\sigma. \tag{4.7}$$

By Hölder's inequality, $|\tau|^{-\frac{1}{p}}\|f_\tau^0\|_{L^1} \lesssim \|f_\tau^0\|_{L^p}$, and by basic arithmetic and (4.6),

$$|\tau|^{-\frac{1}{p'}}\int_{\mathbb{S}^d}|f_\tau^n|\,d\sigma \lesssim 2^{-n(p-1)}\|f\|_p^{-(p-1)}\|f_\tau^n\|_p^p \leq 2^{-n(p-1)}\|f_\tau^n\|_p.$$

Inserting these estimates into (4.7), using Hölder's inequality, and summing a geometric series,

$$|\tau|^{-\frac{1}{p'}}\|\mathcal{E}f_\tau\|_\infty \lesssim \sup_{n\geq 0} 2^{-c_0 n}\|f_\tau^n\|_p,$$

whenever $c_0 < p-1$.

Now we turn to the first inequality in (4.5). By (4.4), it suffices to prove that

$$\sum_{\tau\in\mathcal{D}}|\tau|^{-2t(1-\nu)(\frac{1}{s}-\frac{1}{p})}\|f_\tau\|_s^{2t(1-\nu)} \lesssim \|f\|_p^{2t(1-\nu)}.$$

Since $2t>p$, for ν sufficiently small, $2t(1-\nu)>p>s$. Hence by the triangle inequality and Hölder's inequality,

$$\begin{aligned}\sum_{\tau\in\mathcal{D}}|\tau|^{-2t(1-\nu)(\frac{1}{s}-\frac{1}{p})}\|f_\tau\|_s^{2t(1-\nu)} &\lesssim \sum_{\tau\in\mathcal{D}}|\tau|^{-2t(1-\nu)(\frac{1}{2t(1-\nu)}-\frac{1}{p})}\|f_\tau^0\|_{2t(1-\nu)}^{2t(1-\nu)} \\ &\quad + \Big(\sum_{\tau\in\mathcal{D}}|\tau|^{-2t(1-\nu)(\frac{1}{s}-\frac{1}{p})}\|f_\tau - f_\tau^0\|_s^s\Big)^{\frac{2t(1-\nu)}{s}}.\end{aligned} \tag{4.8}$$

We recall that $\mathcal{D}=\bigcup_{k\geq 0}\mathcal{D}_k$ and that each $\mathcal{D}_k$ is a finitely overlapping cover of $\mathbb{S}^d$ by caps τ of measure 2^{-kd}. Again using the fact that $s<2t(1-\nu)$, we may bound the right hand side of (4.8) by a constant multiple of

$$\begin{aligned}&\sum_k 2^{-2kdt(1-\nu)(\frac{1}{p}-\frac{1}{2t(1-\nu)})}\int_{\{|f|\lesssim 2^{\frac{kd}{p}}\|f\|_p\}}|f|^{2t(1-\nu)}\,d\sigma \\ &\quad + \Big(\sum_k 2^{kds(\frac{1}{s}-\frac{1}{p})}\int_{\{|f|\gtrsim 2^{\frac{kd}{p}}\|f\|_p\}}|f|^s d\sigma\Big)^{\frac{2t(1-\nu)}{s}}.\end{aligned}$$

By Fubini, the preceding sum equals

$$\begin{aligned}&\int |f|^{2t(1-\nu)}\Big(\sum_{k:|f|\lesssim 2^{\frac{kd}{p}}\|f\|_p} 2^{-2kdt(1-\nu)(\frac{1}{p}-\frac{1}{2t(1-\nu)})}\Big)\,d\sigma \\ &\quad + \left(\int |f|^s\Big(\sum_{k:|f|\gtrsim 2^{\frac{kd}{p}}\|f\|_p} 2^{kds(\frac{1}{s}-\frac{1}{p})}\Big)\,d\sigma\right)^{\frac{2t(1-\nu)}{s}},\end{aligned}$$

and we conclude by summing the geometric series. □

Proof of Theorem 3.1 for $q=\frac{d+2}{d}p'$. Multiplying by a constant and passing to a subsequence, we may assume that $\|f_n\|_p\to 1$ and that $\|f_n\|_p\leq 2$ for all n. We begin by decomposing the f_n into *chips* with good frequency/L^p orthogonality (but whose extensions are not orthogonal in space/L^q). Namely, we set $r_n^0 := f_n$ and

$$f_n = \sum_{j=1}^{J} h_n^j + r_n^J, \qquad 0 \le J < \infty,$$

with $h_n^j = r_n^{j-1}\chi_{\tau_n^j}\chi_{\{|f_n|\le 2^j|\tau_n^j|^{-\frac{1}{p}}\}}$. Here τ_n^j is a dyadic spherical cap, with center ν_n^j and diameter ρ_n^j, that is chosen to maximize $\|h_n^j\|_p$. An application of the dominated convergence theorem shows that there is indeed such a maximal cap for each j, n and that for each n, $\sum_j h_n^j$ converges to f_n in $L^p(\mathbb{S}^d)$.

The next lemma is the key step to establish conclusion (vi) of the theorem.

Lemma 4.4. *Under the hypotheses of Theorem 3.1, remainders in the decomposition above satisfy*

$$\lim_{J\to\infty} \sup_{n\in\mathbb{N}} \sup_{|R_n^J|=|r_n^J|\chi_E} \|\mathcal{E}R_n^J\|_q = 0,$$

where the supremum is taken over measurable functions R_n^J whose absolute value equals a characteristic function times $|r_n^J|$.

Proof of Lemma 4.4. Let $J \ge 1$, $n \in \mathbb{N}$, and $|R_n^J| = |r_n^J|\chi_E$. We claim that for any dyadic spherical cap τ and $j < J$,

$$\|R_n^J\chi_\tau\chi_{\{|R_n^J|\le 2^j\|R_n^J\|_p|\tau|^{-\frac{1}{p}}\}}\|_{L^p(\mathbb{S}^d)} \le \tfrac{2}{(J-j)^{\frac{1}{p}}}. \tag{4.9}$$

Indeed, if (4.9) holds, then taking $j \sim \frac{J}{2}$ and applying Lemma 4.3 completes the proof of the lemma.

Now we turn to (4.9). By the construction of each r_n^i from r_n^{i-1}, there exist measurable sets $E_n^0 \supseteq \cdots \supseteq \ldots \supseteq E_n^J = E$, such that $|R_n^J| = |r_n^i|\chi_{E_n^i}$, $i = 0, \ldots, J$. We also recall that $r_n^0 = f_n$. We then have

$$|R_n^J|\chi_\tau\chi_{\{|R_n^J|\le 2^j\|R_n^J\|_p|\tau|^{-\frac{1}{p}}\}} \le |r_n^{i-1}|\chi_\tau\chi_{\{|f_n|\le 2^i\|f_n\|_p|\tau|^{-\frac{1}{p}}\}}, \qquad i = j, \ldots, J+1.$$

Since each τ_n^i is chosen to maximize $\|h_n^i\|_p$, if (4.9) were to fail, we would also have $\|h_n^i\|_p > \frac{2}{(J-j)^{\frac{1}{p}}}$ for $i = j, \ldots, J+1$, whence $\|f_n\|_p^p \ge \sum_{i=j}^{J+1}\|h_n^i\|_p^p > 2$, a contradiction. This completes the proof of (4.9), and hence the lemma. $\square$

Our next step is to organize the chips into clumps, which also possess good spatial orthogonality. After passing to a successively passing to a subsequence for each j and then choosing a diagonal subsequence, we may assume the existence of all of the limits arising below. We form a partition $\mathbb{N} = \bigcup_{i=1}^\infty \mathcal{J}^i$ (the $\mathcal{J}^i$ may be empty for large i) so that j and j' lie in the same $\mathcal{J}^i$ if and only if the conditions $\lim_{n\to\infty}\frac{\rho_n^j}{\rho_n^{j'}} \in (0,\infty)$ and $\{(\rho_n^j)^{-1}\operatorname{dist}([\nu_n^j],[\nu_n^{j'}])\}_{n\in\mathbb{N}}$ is bounded, both hold. Thus for $i \ne i'$, $j \in \mathcal{J}^i, j' \in \mathcal{J}^{i'}$ implies

$$\lim_{n\to\infty}\frac{\rho_n^j}{\rho_n^{j'}} \in \{0,\infty\}; \text{ or } \lim_{n\to\infty}\frac{\rho_n^j}{\rho_n^{j'}} \in (0,\infty), \text{ and } \lim_{n\to\infty}(\rho_n^j)^{-1}\operatorname{dist}([\nu_n^j],[\nu_n^j]) = \infty.$$

Let $F_n^i := \sum_{j\in\mathcal{J}^i} h_n^j$ and $R_n^I := f_n - \sum_{i=1}^I F_n^i$; for each n, these sums converge in L^p by the dominated convergence theorem. Passing to a further subsequence, we may associate to each $\mathcal{J}^i$ sequences $\{\lambda_n^i\}$, $\{[\omega_n^i]\}$ satisfying, for each i: $\lambda_n^i \to 0$ or

$\lambda_n^i \equiv 1$, and $[\omega_n^i] \equiv [e_1]$ if $\lambda_n^i \equiv 1$; and $\lim \frac{\rho_n^j}{\lambda_n^i} \in (0,\infty)$, and $(\lambda_n^i)^{-1}\operatorname{dist}([\omega_n^i],[\nu_n^j])$ is bounded, for $j \in \mathcal{J}^i$.

Conditions (i-iii) of Theorem 3.1 are automatic, while (v-vi) are simple consequences of the construction and Lemma 4.4. It remains to prove (iv), which asserts that the extensions of the F_n^i have good L^q orthogonality.

We define

$$f_n^{\le J} := f_n - r_n^J, \qquad F_n^{i,\le J} := \sum_{j\in\mathcal{J}^i, j\le J} h_n^j, \qquad R_n^{I,\le J} := f_n^{\le J} - \sum_{i=1}^I F_n^{i,\le J}.$$

By Lemma 4.4, for each I,

$$\lim_{n\to\infty} \|\mathcal{E}f_n\|_q^q - \sum_{i=1}^I \|\mathcal{E}F_n^i\|_q^q - \|\mathcal{E}R_n^I\|_q^q = \lim_{J\to\infty}\lim_{n\to\infty} \|\mathcal{E}f_n^{\le J}\|_q^q - \sum_{i=1}^I \|\mathcal{E}F_n^{i,\le J}\|_q^q - \|\mathcal{E}R_n^{I,\le J}\|_q^q,$$

so to prove (iv), it suffices to prove that for all I and J,

$$\lim_{n\to\infty} \|\mathcal{E}f_n^{\le J}\|_q^q - \sum_{i=1}^I \|\mathcal{E}F_n^{i,\le J}\|_q^q - \|\mathcal{E}R_n^{I,\le J}\|_q^q = 0.$$

The elementary inequality

$$\left| |\sum_{i=1}^I x_i|^q - \sum_{i=1}^I |x_i|^q \right| \le C_{q,I}\left(\sup_{i\ne j} |x_i||x_j|^{q-1}\right), \qquad q \ge 2,$$

Hölder's inequality, boundedness of the $\|f_n\|_p$, and the triangle inequality imply that

$$\left| \|\mathcal{E}f_n^{\le J}\|_q^q - \sum_{i=1}^I \|\mathcal{E}F_n^{i,\le J}\|_q^q - \|\mathcal{E}R_n^{I,\le J}\|_q^q \right| \le C_{I,J,q} \sup_{i\ne i', j\in\mathcal{J}^i, j'\in\mathcal{J}^{i'}} \|\mathcal{E}h_n^j \mathcal{E}h_n^{j'}\|_{\frac{q}{2}}.$$

Thus it remains to prove that for $i \ne i'$, $j \in \mathcal{J}^i$, and $j' \in \mathcal{J}^{i'}$,

$$\lim_{n\to\infty} \|\mathcal{E}h_n^j \mathcal{E}h_n^{j'}\|_{\frac{q}{2}} = 0. \tag{4.10}$$

Suppose first that $\frac{\rho_n^j}{\rho_n^{j'}} \to \infty$. By assumption, there exist $q_0 < q < q_1$, $p_0 > p > p_1$ with

$$\tfrac{1}{q} = \tfrac{1}{2q_0} + \tfrac{1}{2q_1}, \qquad q_i = \tfrac{d+2}{d} p_i',$$

such that $\mathcal{E}$ extends as a bounded linear operator from L^{p_i} to L^{q_i}, $i = 0,1$. By Hölder's inequality, boundedness of $\mathcal{E}$, and the support conditions and pointwise boundedness of $h_n^j, h_n^{j'}$,

$$\begin{aligned}\|\mathcal{E}h_n^j \mathcal{E}h_n^{j'}\|_{\frac{q}{2}} &\le \|\mathcal{E}h_n^j\|_{q_0}\|\mathcal{E}h_n^{j'}\|_{q_1} \lesssim \|h_n^j\|_{p_0}\|h_n^{j'}\|_{p_1} \\ &\lesssim 2^{j+j'}(\rho_n^j)^{\frac{d}{p_0}-\frac{d}{p}}(\rho_n^{j'})^{\frac{d}{p_1}-\frac{d}{p}} = 2^{j+j'}\big(\tfrac{\rho_n^{j'}}{\rho_n^j}\big)^{\frac{d}{p_1}-\frac{d}{p}} \to 0.\end{aligned}$$

By symmetry in j, j', it remains to consider the case when $\lim \frac{\rho_n^j}{\rho_n^{j'}} \in (0,\infty)$ and $\lim_{n\to\infty}(\rho_n^j)^{-1}\operatorname{dist}([\nu_n^j],[\nu_n^{j'}]) = \infty$. Of course, this implies that $\rho_n^j, \rho_n^{j'} \to 0$. Set $r_n = \frac{1}{100}\operatorname{dist}([\nu_n^j],[\nu_n^{j'}])$. Then $\frac{\rho_n^j}{r_n} \to 0$, and hence, for large n, $\tau_n^j \subseteq B([\nu_n^j], r_n)$, $\tau_n^{j'} \subseteq B([\nu_n^{j'}], r_n)$. By Lemma 4.1, for some $s < p$,

$$\|\mathcal{E}h_n^j \mathcal{E}h_n^{j'}\|_{\frac{q}{2}} \lesssim r_n^{-(\frac{2d}{s}-\frac{2d}{p})}\|h_n^j\|_s\|h_n^{j'}\|_s \le \big(\tfrac{\rho_n^j}{r_n}\big)^{d(\frac{1}{s}-\frac{1}{p})}\big(\tfrac{\rho_n^{j'}}{r_n}\big)^{d(\frac{1}{s}-\frac{1}{p})} \to 0.$$

We have thus established (4.10), completing the proof of (iv) in Theorem 3.1. $\square$

5. Scale one spatial decomposition: Proof of Theorem 3.3

In this section, we establish a finer decomposition for sequences that are (almost) pointwise bounded, in the sense that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\|\mathcal{E}f_n\chi_{\{|f_n|>M\}}\|_q = 0.$$

Lemma 5.1. *If, under the hypotheses of Theorem 3.3, we have in addition that*

$$\limsup_{n\to\infty}\|f_n\|_p \le A < \infty \text{ and } \liminf_{n\to\infty}\|\mathcal{E}f_n\|_q \ge B > 0,$$

then there exists $\{x_n\}\subseteq\mathbb{R}^{d+1}$ such that after passing to a subsequence,

$$e^{-ix_n\omega}f_n \rightharpoonup \phi, \text{ weakly in } L^p, \text{ with } \|\mathcal{E}\phi\|_q \gtrsim B(\tfrac{B}{A})^C. \tag{5.1}$$

Furthermore, along this subsequence,

$$\lim_{n\to\infty}\|\mathcal{E}f_n\|_q^q - \|\mathcal{E}(f_n - e^{ix_n\omega}\phi)\|_q^q - \|\mathcal{E}\phi\|_q^q = 0. \tag{5.2}$$

The proof is simplest if we divide into two cases.

Proof when $q > \frac{d+2}{d}p'$. It follows from our hypotheses that the $\mathcal{E}f_n$ are bounded, continuous functions. After a modulation and multiplication by unimodular constants, we may assume that $\mathcal{E}f_n(0) = \|\mathcal{E}f_n\|_\infty$. Under this normalization, we will prove (5.1) with $x_n \equiv 0$. After passing to a subsequence, there exists $\phi\in L^p$ such that $f_n \rightharpoonup \phi$, weakly in L^p. Therefore

$$\|\mathcal{E}\phi\|_\infty \ge |\mathcal{E}\phi(0)| = \lim|\mathcal{E}f_n(0)| = \lim\|\mathcal{E}f_n\|_\infty.$$

By hypothesis, there exists $s<q$ such that $\mathcal{E}$ extends as a bounded linear operator from L^p to L^s. By Hölder's inequality and our hypotheses,

$$B \le \limsup\|\mathcal{E}f_n\|_q \le \limsup\|\mathcal{E}f_n\|_s^{1-\frac{s}{q}}\|\mathcal{E}f_n\|_\infty^{\frac{s}{q}} \lesssim A^{1-\frac{s}{q}}\lim\|\mathcal{E}f_n\|_\infty^{\frac{s}{q}}.$$

Therefore $\|\mathcal{E}\phi\|_\infty \gtrsim B(\frac{B}{A})^{\frac{q}{s}-1}$. On the other hand, $\mathcal{E}\phi = (\mathcal{E}\phi)*h$ whenever $\widehat{h}\equiv 1$ on $\mathbb{S}^d$, so by Young's inequality, $\|\mathcal{E}f_n\|_\infty \lesssim \|\mathcal{E}f_n\|_q$, and (5.1) follows.

The final conclusion, (5.2), follows from the Brezis–Lieb lemma, since $\mathcal{E}e^{-ix_n\omega}f_n \to \mathcal{E}\phi$ pointwise, by virtue of the weak convergence. $\square$

Proof when $q = \frac{d+2}{d}p'$. In this argument, we will use the fact that a cap is the intersection of a union of antipodal cubes with $\mathbb{S}^d$. For $\tau\in\mathcal{D}$, we denote by Q_τ a union of two rather smaller (relative to the cubes) antipodal parallelepipeds whose intersection with $\mathbb{S}^d$ also gives τ, but whose volume is comparable to that of the convex hull of τ. More precisely, due to the curvature of the sphere, $|Q_\tau|\sim|\tau|^{\frac{q}{p'}}$.

Now let $0<\varepsilon<\frac{B}{2}$ be sufficiently small for later purposes. By hypothesis, there exists $M>0$ (over which we have no control) such that

$$\limsup_{n\to\infty}\|\mathcal{E}(f_n\chi_{\{|f_n|>M\}})\|_q < \varepsilon.$$

By (4.5) and our hypotheses,

$$B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}} \lesssim \liminf_{n\to\infty}\sup_{\tau_n\in\mathcal{D}}|\tau_n|^{-\frac{1}{p'}}\|\mathcal{E}(f_n\chi_{\{|f_n|\le M\}})_{\tau_n}\|_\infty.$$

By Hölder's inequality, for each n, the above supremum may be taken over caps whose volumes are bounded below by a constant depending on M. As there are only a finite number of such caps, after passing to a subsequence, there is a single τ_M that realizes the supremum for all sufficiently large n. In other words,

$$B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}} \lesssim \liminf_{n\to\infty} |\tau_M|^{-\frac{1}{p'}} \|\mathcal{E}(f_n\chi_{\{|f_n|<M\}})_{\tau_M}\|_\infty. \tag{5.3}$$

We may assume, after modulation, that $\mathcal{E}(f_n\chi_{\{|f_n|<M\}})_{\tau_M}(0) = \|\mathcal{E}(f_n\chi_{\{|f_n|<M\}})_{\tau_M}\|_\infty$.

Passing to a subsequence, the weak limits

$$\phi^g := \text{wk-lim}\, f_n\chi_{\{|f_n|\le M\}}, \qquad \phi^b := \text{wk-lim}\, f_n\chi_{\{|f_n|>M\}}$$

exist in L^p. In particular,

$$\lim \mathcal{E}(f_n\chi_{\{|f_n|\le M\}})_{\tau_M}(x) = \mathcal{E}(\phi^g)_{\tau_M}(x), \qquad \lim \mathcal{E}(f_n\chi_{\{|f_n|>M\}})(x) = \mathcal{E}\phi^b(x),$$

for all $x\in\mathbb{R}^{1+d}$. By Fatou,

$$\|\mathcal{E}\phi^b\|_q \le \liminf_{n\to\infty} \|\mathcal{E}(f_n\chi_{\{|f_n|>M\}})\|_q < \varepsilon.$$

On the other hand, by Young's inequality and $q'>1$,

$$|\tau_M|^{-\frac{1}{p'}}\|\mathcal{E}\phi^b_{\tau_M}\|_\infty \le |\tau_M|^{-\frac{1}{p'}}\|\widehat{\chi_{Q_{\tau_M}}}\|_{q'}\|\mathcal{E}\phi^b\|_q \lesssim \|\mathcal{E}\phi^b\|_q.$$

Hence by the triangle inequality, (5.3), Young's inequality, and $q'>1$,

$$B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}} \lesssim |\tau_M|^{-\frac{1}{p'}}\|\mathcal{E}\phi^g_{\tau_M}\|_\infty \le |\tau_M|^{-\frac{1}{p'}}\|\widehat{\chi_{Q_{\tau_M}}}\|_{q'}\|\mathcal{E}\phi^g\|_q \lesssim \|\mathcal{E}\phi^g\|_q.$$

Setting $\phi := \phi^g + \phi^b = \text{wk-lim}\, f_n$, the lower bound on the extension of ϕ^g and upper bound on the extension of ϕ^b give

$$B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}} \lesssim \|\mathcal{E}\phi\|_q,$$

provided $\varepsilon \ll B(\frac{B}{A})^{\frac{1-\theta}{\theta}}$.

The final inequality, (5.2), follows as above from the Brezis–Lieb lemma and pointwise convergence $\mathcal{E}e^{-ix_n\omega}f_n \to \mathcal{E}\phi$. □

In the special case $p=2$, iteratively applying Lemma 5.1 and using elementary Hilbert space identities yields the following L^2 profile decomposition.

Lemma 5.2. *If $q \ge \frac{2(d+2)}{d}$ and $\{f_n\}$ is a bounded sequence in $L^2(\mathbb{S}^d)$ satisfying*

$$\lim_{M\to\infty}\limsup_{n\to\infty} \|\mathcal{E}f_n\chi_{\{|f_n|>M\}}\|_q = 0, \tag{5.4}$$

then, after passing to a subsequence, there exist $\{x_n^j\}_{j,n\in\mathbb{N}} \subseteq \mathbb{R}^{d+1}$ obeying

$$\lim_{n\to\infty} |x_n^j - x_n^{j'}| = \infty, \text{ for } j\ne j', \tag{5.5}$$

and weak limits $\phi^j = \text{wk-lim}_{n\to\infty} e^{-ix_n^j\omega}f_n \in L^2$, such that for every $J\in\mathbb{N}$,

$$\lim_{n\to\infty} \|f_n\|_2^2 - \sum_{j=1}^J \|\phi^j\|_2^2 - \|r_n^J\|_2^2 = 0 \tag{5.6}$$

$$\lim_{n\to\infty} \|\mathcal{E}f_n\|_q^q - \sum_{j=1}^J \|\mathcal{E}\phi^j\|_q^q - \|\mathcal{E}r_n^J\|_q^q = 0, \tag{5.7}$$

and, moreover,

$$\lim_{J\to\infty}\lim_{n\to\infty} \|\mathcal{E}r_n^J\|_q^q = 0. \tag{5.8}$$

We briefly sketch the proof of the lemma, which follows a well-known outline (see, for instance, [27]). Only minor modifications to the familiar argument are needed to use the condition (5.4).

Sketch proof of Lemma 5.2. We set $r_n^0 := f_n$. Given some r_n^J, we are done (setting $\phi^{J'} = 0$ for $J' > J$) if $\lim_{n\to\infty} \|\mathcal{E} r_n^J\|_q = 0$. Otherwise, we apply Lemma 5.1, which produces a nonzero weak limit ϕ^{J+1} and sequence $\{x_n^{J+1}\} \subseteq \mathbb{R}^{d+1}$; we then set $r_n^{J+1} := r_n^J - e^{ix_n^{J+1}\omega}\phi^{J+1}$. If (5.5) failed for some $j < j'$ but held with j' replaced by any $j < i < j'$, then $x_n^j - x_n^{j'}$ would converge along a subsequence, and, after a bit of algebraic manipulation, we can deduce that $\phi^{j'} = 0$. Thus (5.5) holds. The hypothesis (5.4) continues to hold with r_n^{J+1} in place of f_n, as a finite number of L^p functions have been subtracted from the f_n. Equation (5.6) follows from basic Hilbert space manipulations, and (5.7) may be proved inductively using Brézis–Lieb. Finally, (5.8) holds because (5.1) and (5.6) give a lower bound for $\|r_n^J\|_2^2 - \|r_n^{J+1}\|_2^2$ whenever $\lim_{n\to\infty} \|\mathcal{E} r_n^J\|_q^q \not\to 0$. □

For $p \neq 2$, L^p is not a Hilbert space, and the direct analogue of (5.6) may fail. Instead, we will prove the L^p almost orthogonality estimates (3.2) by defining and bounding a family of vector-valued operators. Fix a nonnegative, smooth, radial function ψ on $\mathbb{R}^{d+1}$ with compact support in the unit ball and $\int_{\mathbb{R}^d} \psi(\xi', 0)\, d\xi' = 1$. (Since ψ is radial, it thus has integral 1 on every hyperplane through the origin.) For $0 < r \leq 1$, define $\psi_r(\zeta) := r^{-d}\psi(r^{-1}\zeta)$. We define

$$(\pi_r)_n^j f(\omega) := \int \psi_r(\omega - \nu) e^{-ix_n^j \nu} f(\nu)\, d\sigma(\nu), \qquad (\Pi_r)_n^J f := ((\pi_r)_n^j f)_{j=1}^J.$$

We recall $\tilde{p} := \max\{p, p'\}$.

Lemma 5.3. *Let $1 < p \leq \infty$. Assume that the sequences $\{x_n^j\}_{j,n\in\mathbb{N}}$ obey $\lim_{n\to\infty} |x_n^j - x_n^{j'}| = \infty$ for all $j \neq j'$. Then the $(\Pi_r)_n^J$ map L^p boundedly into $\ell^{\tilde{p}}(L^p)$, with operator norms bounded uniformly in r, n. Moreover,*

$$\lim_{r\to\infty} \lim_{n\to\infty} \|(\Pi_r)_n^J\|_{L^p \to \ell^{\tilde{p}}(L^p)} = 1. \tag{5.9}$$

Finally, given sequences of functions $f_n = \sum_{j=1}^J e^{ix_n^j\omega}\phi^j + r_n^J$, with $\{f_n\}$ bounded in L^p, satisfying $\phi^j = \text{wk-lim}\, e^{-ix_n^j\omega} f_n$, for each $j \in \mathbb{N}$, we have

$$\lim_{r\to 0} \lim_{n\to\infty} \|(\pi_r)_n^j f_n - \phi^j\|_{L^p} = 0 \tag{5.10}$$

$$\lim_{r\to 0} \lim_{n\to\infty} \|[(\pi_r)_n^j]^* \phi^j - e^{ix_n^j\omega}\phi^j\|_{L^p} = 0. \tag{5.11}$$

Proof of Lemma 5.3. We will be brief. Verification that the ψ_r approximate the identity is routine; boundedness of the $(\Pi_r)_n^J$ and the limit (5.11) immediately follow. By our weak limit hypothesis on the ϕ^j and the dominated convergence theorem (with constant dominating function), for each r,

$$(\pi_r)_n^j f_n(\omega) \to \int \psi_r(\omega - \nu)\phi^j(\nu)\, d\sigma(\nu), \text{ in } L^p, \text{ as } n \to \infty.$$

Equation (5.10) follows immediately.

We will verify the dual form of (5.9), namely, that

$$\lim_{r\to 0} \lim_{n\to\infty} \|[(\Pi_r)_n^J]^*\|_{\ell^{\tilde{p}'}(L^p) \to L^p} \to 1, \qquad 1 \leq p \leq \infty.$$

For the convenience of the reader, we record

$$[(\Pi_r)_n^J]^*(\phi^j)(\nu) = \sum_{j=1}^J e^{ix_n^j \nu} \int \psi_r(\nu - \omega)\phi^j(\omega)\, d\sigma(\omega).$$

That the limit of the operator norms is bounded below by 1 is elementary, as can be seen by applying the $[(\Pi_r)_n^J]^*$ to the vector-valued constant function $(1, 0, \dots, 0)$. In the cases $p = 1, \infty$, the upper bound is a direct consequence of the triangle inequality, Hölder's inequality, and

$$\int \psi_r(\omega - \nu)\, d\sigma(\nu) = \int \psi_r(e_1 - \nu)\, d\sigma(\nu) \to 1, \text{ for any } \omega \in \mathbb{S}^d.$$

By interpolation, it remains to verify the $p = 2$ case, for which it suffices to prove that $\|(\Pi_r)_n^J[(\Pi_r)_n^J]^*\|_{\ell^2(L^2)\to\ell^2(L^2)} \to 1$. We expand

$$\|(\Pi_r)_n^J[(\Pi_r)_n^J]^*(\phi^j)_{j\le J}\|_{\ell^2(L^2)}^2 = \sum_j \|\sum_{k=1}^J \int \phi^k(\vartheta)(K_r)_n^{jk}(\vartheta,\omega)\, d\sigma(\vartheta)\|_{L^2_\omega}^2,$$

where

$$(K_r)_n^{jk}(\vartheta,\omega) = \int \psi_r(\omega-\nu)\psi_r(\nu-\vartheta)e^{i(-x_n^j + x_n^k)\nu} d\sigma(\nu).$$

Let

$$(A_r)_n^{jk} := \|(K_r)_n^{jk}\|_{L^\infty_\vartheta L^1_\omega}\|(K_r)_n^{jk}\|_{L^\infty_\omega L^1_\vartheta}.$$

When $j \neq k$ and $r > 0$, $(K_r)_n^{jk} \to 0$ uniformly as $n \to \infty$ by stationary phase and $|x_n^j - x_n^k| \to \infty$, so the off-diagonal terms satisfy $(A_r)_n^{jk} \to 0$ as $n \to \infty$. By construction of ψ_r, $(A_r)_n^{jj}$ (which is independent of n) tends to 1 as $r \to 0$. By the elementary inequality

$$|\sum_{j=1}^J x_j|^2 \le (1+\varepsilon)x_1^2 + C_{\varepsilon,J}\sum_{j=2}^J x_j^2$$

and Schur's test,

$$\|\sum_{k=1}^J \int \phi^k(\vartheta)(K_r)_n^{jk}(\vartheta,\omega)\, d\sigma(\vartheta)\|_{L^2_\omega}^2 \le (1+\varepsilon)(A_r)_n^{jj}\|\phi^j\|_2^2 + C_{\varepsilon,J}\sum_{k\neq j}(A_r)_n^{jk}\|\phi^k\|_2^2,$$

and (5.9) follows. □

Lemma 5.4. *Under the hypotheses of Theorem 3.3, suppose that we are given sequences* $\{x_n^j\}_{j,n\in\mathbb{N}}$ *with* $|x_n^j - x_n^{j'}| \to \infty$ *for* $j \neq j'$ *and* $\{f_n\}$ *such that the weak limits*

$$\phi^j := \text{wk-lim}\, e^{-ix_n^j\omega} f_n$$

exist. Define

$$r_n^J := f_n - \sum_{j=1}^J e^{ix_n^j\omega} f_n.$$

Then (3.2) and (3.3) both hold.

Proof of Lemma 5.4. The inequalities in (3.2) follow directly from Lemma 5.3 and the definition of the r_n^J. We will use the Brezis–Lieb lemma to prove (3.3). Set $r_n^0 := f_n$. By hypothesis, for $j \neq j'$, $\text{wk-lim}\, e^{i(x_n^j - x_n^{j'})}\phi^j = 0$. Therefore $\phi^j = \text{wk-lim}\, e^{-ix_n^j} r_n^{j-1}$, for $j \geq 1$. Applying the extension, $\mathcal{E}(e^{-ix_n^j\omega} r_n^{j-1} - \phi^j) \to 0$, pointwise, and so by the Brezis–Lieb lemma,

$$\lim_{n\to\infty} \|\mathcal{E}(e^{-ix_n^j\omega} r_n^{j-1})\|_q^q - \|\mathcal{E}\phi^j\|_q^q - \|\mathcal{E}(e^{-ix_n^j\omega} r_n^{j-1} - \phi^j)\|_q^q = 0.$$

Summing the preceding identity over $j = 1, \ldots, J$ and using $r_n^j = r_n^{j-1} - e^{ix_n^j\omega}\phi^j$ establishes (3.3). $\square$

With the above lemmas in place, we are now ready to complete the proof of Theorem 3.3.

Proof of Theorem 3.3. We may assume that $\limsup \|f_n\|_p = 1$. When $p = 2$, the conclusions of Lemma 5.2 are stronger than we need, so it suffices to consider pairs (p, q) meeting the hypotheses of our theorem in the case $p \neq 2$. In light of Lemma 5.4, it suffices to prove that there exist $\{x_n^j\}_{j,n\in\mathbb{N}}$ obeying (3.1) such that the resulting remainder terms r_n^J have small extension, i.e., that (3.4) holds.

Given $M \in \mathbb{N}$, we set $f_n^M := f_n \chi_{\{|f_n| \leq M\}}$. Let $\varepsilon > 0$ and take M_ε sufficiently large that

$$\limsup_{n\to\infty} \|\mathcal{E}(f_n - f_n^M)\|_q < \varepsilon,$$

when $M \geq M_\varepsilon$.

The advantage of working with the truncation $\{f_n^M\}$ is that it forms a bounded sequence in every Lebesgue space (albeit with a bad, M-dependent, bound), putting us in a position to apply Lemma 5.2. To this end, set $q_1 := \frac{2(d+2)}{d}$ and choose an exponent pair (p_0, q_0) meeting the hypotheses on (p, q) from Theorem 3.3, as well as the condition

$$(\tfrac{1}{p}, \tfrac{1}{q}) = (1-\theta)(\tfrac{1}{p_0}, \tfrac{1}{q_0}) + \theta(\tfrac{1}{2}, \tfrac{1}{q_1}),$$

for some $0 < \theta < 1$.

By Lemma 5.2, after passage to subsequence (independent of M by standard diagonalization arguments), there exist points $\{x_n^{M,j}\}_{j,n\in\mathbb{N}}$ and weak limits $\phi^{M,j}$ such that (5.6), (5.7), and (5.8) all hold, with the superscript M inserted where appropriate. Since $e^{-ix_n^{M,j}\omega} f_n^M \rightharpoonup \phi^{M,j}$ weakly in both L^p and in L^{p_0}, we may also apply Lemma 5.4 with exponents (p, q) and (p_0, q_0). By (3.3),

$$\limsup_{n\to\infty} \|\mathcal{E}r_n^{M,J}\|_{q_0} \leq \limsup_{n\to\infty} \|\mathcal{E}f_n^M\|_{q_0} \lesssim \limsup_{n\to\infty} \|f_n^M\|_{p_0} \lesssim_M 1,$$

for all M, J. Therefore, by Hölder's inequality and (5.8),

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\mathcal{E}r_n^{M,J}\|_q = 0, \text{ for all } M. \tag{5.12}$$

To conclude, we need to remove the dependence on M in (5.12). We begin by showing that non-negligible profiles $\mathcal{E}e^{ix_n^{j,M}\omega}\phi^{j,M}$ cannot wander around too much as M varies.

After passing to a subsequence, we may assume that for any M, j and M', j', either $\lim_{n\to\infty} |x_n^{M,j} - x_n^{M',j'}| = \infty$ or $x_n^{M,j} - x_n^{M',j'}$ converges in $\mathbb{R}^{d+1}$, as $n \to \infty$. In fact, in the latter case, we may assume that $x_n^{M,j} \equiv x_n^{M',j'}$, simply by modulating our $\phi^{M,j}$ as needed.

Now, let C be a sufficiently large constant, and suppose that (after reordering the $\phi^{j,M}$ and perhaps inserting some zero profiles) we had distinct sequences $\{x_n^j\}$, $1 \leq j \leq J_\varepsilon := C\varepsilon^{-q}$ such that $x_n^{M,j} \equiv x_n^j$ for some $M \geq M_\varepsilon$ with $\|\mathcal{E}\phi^{M,j}\|_q > 2\varepsilon$. By construction, $|x_n^j - x_n^{j'}| \to \infty$ whenever $j \neq j'$. Passing to a subsequence, we have weak limits $e^{-ix_n^j\omega} f_n \rightharpoonup \phi^j$, weakly in L^p, for each $1 \leq j \leq J_\varepsilon$. By Fatou,

$$\|\mathcal{E}\phi^j - \mathcal{E}\phi^{M,j}\|_q \leq \limsup_{n\to\infty} \|\mathcal{E}f_n - \mathcal{E}f_n^M\|_q < \varepsilon.$$

Therefore $\|\mathcal{E}\phi^j\|_q > \varepsilon$, $1 \leq j \leq J_\varepsilon$. By Lemma 5.4 (namely, inequality (3.3)),

$$J_\varepsilon \varepsilon^q < \sum_{j=1}^{J_\varepsilon} \|\mathcal{E}\phi^j\|_q^q \leq \lim_{n\to\infty} \|\mathcal{E}f_n\|_q^q \lesssim 1,$$

a contradiction. Thus, after reordering, $x_n^{M,j} \equiv x_n^j$ whenever $M \geq M_\varepsilon$, $j \leq J_\varepsilon$ and

$$\|\mathcal{E}\phi^{M,j}\|_q < \varepsilon, \text{ whenever } M \geq M_\varepsilon \text{ and } j > J_\varepsilon. \tag{5.13}$$

Inequality (5.13) will give us uniform control on the extensions of the r_n^{M,J_ε}. Recalling that $\tilde{p} = \max\{p, p'\} < q$, choose, for each $M \geq M_\varepsilon$, some $J_{\varepsilon,M}$ sufficiently large that $\|\mathcal{E}r_n^{M,J_{\varepsilon,M}}\|_q < \varepsilon^{q-\tilde{p}}$. We will show that $\|\mathcal{E}(r_n^{M,J_\varepsilon} - r_n^{M,J_{\varepsilon,M}})\|_q \lesssim \varepsilon^{q-\tilde{p}}$. Noting that

$$r_n^{M,J_\varepsilon} - r_n^{M,J_{\varepsilon,M}} = \sum_{J_\varepsilon < j < J_{\varepsilon,M}} e^{ix_n^{M,j}\omega} \phi^{M,j},$$

we apply Lemma 5.4 and Hölder's inequality to obtain

$$\begin{aligned}
\limsup_{n\to\infty} \Big\| \sum_{J_\varepsilon < j < J_{\varepsilon,M}} \mathcal{E}e^{ix_n^{M,j}\omega}\phi^{M,j} \Big\|_q &\leq \sum_{J_\varepsilon < j < J_{\varepsilon,M}} \|\mathcal{E}\phi^{M,j}\|_q^q \\
&\leq \varepsilon^{q-\tilde{p}} \sum_{J_\varepsilon < j < J_{\varepsilon,M}} \|\mathcal{E}\phi^{M,j}\|_q^{\tilde{p}} \leq \varepsilon^{q-\tilde{p}} A_{p\to q}^{\tilde{p}} \sum_{J_\varepsilon < j < J_{\varepsilon,M}} \|\phi^{M,j}\|_p^{\tilde{p}} \\
&\leq \varepsilon^{q-\tilde{p}} A_{p\to q}^{\tilde{p}} \limsup \|f_n\|_p^{\tilde{p}} \lesssim \varepsilon^{q-\tilde{p}}.
\end{aligned}$$

It remains to transfer the bound $\|\mathcal{E}r_n^{M,J_\varepsilon}\|_q \lesssim \varepsilon^{q-\tilde{p}}$ to $\mathcal{E}r_n^{J_\varepsilon}$. Let $1 \leq j \leq J_\varepsilon$. By Fatou and our assumption,

$$\lim_{M\to\infty} \|\mathcal{E}(\phi^j - \phi^{M,j})\|_q \leq \lim_{M\to\infty} \limsup_{n\to\infty} \|\mathcal{E}(f_n - f_n^M)\|_q = 0.$$

Hence by the triangle inequality,

$$\lim_{M\to\infty} \|\mathcal{E}(r_n^{J_\varepsilon} - r_n^{M,J_\varepsilon})\|_q = 0,$$

and so we have the desired inequality $\|\mathcal{E}r_n^{J_\varepsilon}\|_q < \varepsilon$, completing the proof of Theorem 3.3. □

6. Large scale spatial decomposition: Proof of Theorem 3.4

We begin by recording the connection between the spherical and parabolic extension operators at small frequency scales.

Lemma 6.1. *Let $1 < p < \frac{2(d+1)}{d}$ and $q = \frac{d+2}{d}p'$. Assume that the restriction conjecture for $\mathcal{E}$ holds on a neighborhood of (p, q). Let $\lambda_n \searrow 0$ and $\phi \in L^p(\mathbb{R}^d)$. Define*

$$g_n(\omega) := \lambda_n^{-d/p} \phi(\lambda_n^{-1}\omega') \chi_{\{\omega_1 > 0\}} \chi_{\{|\omega'| < \frac{1}{2}\}}. \tag{6.1}$$

Then

$$\lim_{n\to\infty} \|\mathcal{E}g_n - \lambda_n^{\frac{d+2}{q}} e^{ix_1}\mathcal{E}_{\mathbb{P}}\phi(-\lambda_n^2 x_1, \lambda_n x')\|_{L^q} = 0. \tag{6.2}$$

Lemma 6.1 is proved in the case $p = 2$ in [21]; we make the simple adaptation here to the case of general p (for which we make no *a priori* assumption of boundedness of $\mathcal{E}_{\mathbb{P}}$) for the convenience of the reader.

Proof of Lemma 6.1. It suffices to prove that

$$\lim_{n\to\infty} \|\lambda_n^{-\frac{d+2}{q}} e^{i\lambda_n^{-2}x_1}\mathcal{E}g_n(-\lambda_n^{-2}x_1, \lambda_n^{-1}x') - \mathcal{E}_{\mathbb{P}}\phi\|_{L^q} = 0,$$

and we assume initially that $\phi \in C^\infty_{cpct}$; therefore, the L^q norms in the above limit are finite for each n by stationary phase. Set

$$G_n(x) := \lambda_n^{-\frac{d+2}{q}} e^{i\lambda_n^{-2}x_1}\mathcal{E}g_n(-\lambda_n^{-2}x_1, \lambda_n^{-1}x').$$

After a change of variables, we see that for sufficiently large n,

$$G_n(x) = \int e^{i(-x_1,x')(\lambda_n^{-2}[\sqrt{1-|\lambda_n\xi|^2}-1],\xi)}\phi(\xi)\, d\xi.$$

Examining the phase function, $G_n \to \mathcal{E}_{\mathbb{P}}\phi$, pointwise. Moreover, by stationary phase, $|G_n(x)| \lesssim_\phi \langle x\rangle^{-\frac{d}{2}}$. Therefore by dominated convergence, $G_n \to \mathcal{E}\phi$ in L^q.

From our assumption, having proved the lemma in the case of C^∞_{cpct} functions implies, in addition, that $\|\mathcal{E}_{\mathbb{P}}\phi\|_q \lesssim \|\phi\|_p$ for $\phi \in C^\infty_{cpct}$. Therefore $\mathcal{E}_{\mathbb{P}}$ extends as a bounded linear operator from L^p to L^q, and we may conclude that the lemma also holds for general L^p functions by standard approximation arguments. ☐

Next, we isolate a nonzero weak limit in bounded, concentrating sequences with nonnegligible extensions.

Lemma 6.2. *Let $1 < p < \frac{2(d+1)}{d}$ and $q = \frac{d+2}{d}p'$. Assume that the restriction conjecture for $\mathcal{E}$ holds on a neighborhood of (p, q). Let $\lambda_n \searrow 0$ and assume that*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \|\mathcal{E}f_n\chi_{\{|f_n|>M\lambda_n^{-d/p}\}\cup\{|\omega-e_1|>M\lambda_n\}}\|_q = 0, \tag{6.3}$$

for some sequence $\{f_n\} \subseteq L^p(\mathbb{S}^d)$, with $\limsup \|f_n\|_p \leq A$ and $\liminf \|\mathcal{E}f_n\|_q \geq B > 0$. After passing to a subsequence, there exists $\{x_n\} \subseteq \mathbb{R}^{d+1}$ such that

$$\lambda_n^{d/p}e^{-ix_n(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi)}f_n(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|<\frac{1}{2}\lambda_n^{-1}\}} \rightharpoonup \phi,$$

weakly in $L^p(\mathbb{R}^d)$, with $\|\mathcal{E}_{\mathbb{P}}\phi\|_q \gtrsim B(\frac{B}{A})^C$.

Proof of Lemma 6.2. Given $M \in \mathbb{N}$, we set

$$f_n^{>M} := f_n\chi_{\{|f_n|>M\lambda_n^{-d/p}\}\cup\{|\omega-e_1|>M\lambda_n\}}, \text{ and } f_n^M := f_n - f_n^{>M}.$$

Let $\varepsilon > 0$ sufficiently small for later purposes. By hypothesis, there exists $M := M_\varepsilon \in \mathbb{N}$ such that, after passing to a subsequence,

$$\|\mathcal{E}f_n^{>M}\|_q < \varepsilon, \text{ for all } n. \tag{6.4}$$

As long as $\varepsilon < \frac{B}{2}$, after passing to a further subsequence,

$$\|f_n^M\|_p \leq A, \text{ and } \|\mathcal{E}f_n^M\|_q \geq \tfrac{B}{2}, \text{ for all } n.$$

By (4.5), there exists a sequence $\{\tau_n\} \subseteq \mathcal{D}$ such that for all n,

$$B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}} \lesssim |\tau_n|^{-\frac{1}{p'}}\|\mathcal{E}(f_n^M)_{\tau_n}\|_\infty.$$

We will show that after rescaling by λ_n, the τ_n have a convergent subsequence. On the one hand, by (6.4), for $\varepsilon \ll B(\frac{B}{A})^{\frac{1-\theta}{\theta}}$, each τ_n must intersect $\{|\xi - e_1| < M\lambda_n\}$. On the other hand, by Hölder's inequality,

$$\begin{aligned}|\tau_n|^{-\frac{1}{p'}}\|\mathcal{E}(f_n^M)_{\tau_n}\|_\infty &\leq |\tau_n|^{-\frac{1}{p'}}\|f_n^M\|_\infty|\operatorname{supp} f_n^M|\\ &\lesssim_M \min\{|\tau_n|^{-1/p'}\lambda_n^{d/p'}, |\tau_n|^{1/p}\lambda_n^{-d/p}\}\end{aligned}$$

where $|\operatorname{supp} f_n^M|$ denotes the measure of the support of f_n^M. Therefore $1 \lesssim_{A,B,M} \min\{|\tau_n|^{-1/p'}\lambda_n^{d/p'}, |\tau_n|^{1/p}\lambda_n^{-d/p}\}$, which implies $|\tau_n| \sim_{A,B,M} \lambda_n^d$. Therefore, after passing to a subsequence, $\chi_{\tau_n}(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi) \to \chi_\tau(\xi)$, for some cube $\tau \subseteq \mathbb{R}^d$, pointwise, a.e.

We may assume, after modulation and multiplication by a constant that

$$\mathcal{E}(f_n^M)_{\tau_n}(0) = \|\mathcal{E}(f_n^M)_{\tau_n}\|_\infty.$$

Passing to a subsequence, the weak limits

$$\begin{aligned}\phi^g(\xi) &:= \text{wk-lim}\, \lambda_n^{d/p} f_n^M(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\\ \phi(\xi) &:= \text{wk-lim}\, \lambda_n^{d/p} f_n(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|<\frac{1}{2}\lambda_n^{-1}\}}\end{aligned}$$

exist; we set $\phi^b := \phi - \phi^g$. By the dominated convergence theorem and the observation of the previous paragraph,

$$(\phi^g(\xi))_\tau = \text{wk-lim}\, \lambda_n^{d/p}(f_n^M)_{\tau_n}(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi).$$

Standard convergence arguments give,

$$\mathcal{E}_{\mathbb{P}}\phi(x) = \lim_{n\to\infty} \lambda_n^{-\frac{d+2}{q}} e^{-i\lambda_n^{-2}x_1}\mathcal{E}(f_n)_{\tau_n}(\lambda_n^{-2}x_1, \lambda_n^{-1}x'),$$

for all x, and analogous relations hold for ϕ^g and the f_n^M (and hence for ϕ^b and the $f_n^{>M}$). Therefore

$$|\tau|^{-\frac{1}{p'}}\|\mathcal{E}_{\mathbb{P}}(\phi^g)_\tau\|_\infty \gtrsim B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}}.$$

By Young's convolution inequality and the observation that the measure of the convex hull of τ satisfies $|\operatorname{ch}\tau| \sim |\tau|^{\frac{d+2}{d}}$,

$$\|\mathcal{E}_{\mathbb{P}}(\phi^g)_\tau\|_q \gtrsim B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}}.$$

On the other hand, by Fatou,

$$\|\mathcal{E}_{\mathbb{P}}(\phi^b)_\tau\|_q < \varepsilon,$$

so by the triangle inequality,

$$\|\mathcal{E}_{\mathbb{P}}(\phi)_\tau\|_q \gtrsim B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}}.$$

Hence by L^q boundedness of Fourier multiplication by $\chi_{\tilde\tau}$,

$$\|\mathcal{E}_{\mathbb{P}}\phi\|_q \gtrsim B(\tfrac{B}{A})^{\frac{1-\theta}{\theta}}.$$

□

With Lemma 6.2 in place, we are ready for the L^2-based profile decomposition.

Lemma 6.3. *Theorem 3.4 holds when $p = 2$. Moreover, with assumptions and notation as in the statement of that result,*
(ii'-iii') $\lim_{n\to\infty} \|f_n\|_2^2 - \sum_{j=1}^J(\|\phi^{j,+}\|_2^2 + \|\phi^{j,-}\|_2^2) - \|r_n^J\|_2^2 = 0$, *for all $J \in \mathbb{N}$.*

Essentially all of the key ingredients needed for this lemma were already established in [21]; we provide details both for the convenience of the reader and because a key step, an improved Brézis–Lieb Lemma, will be used in later arguments as well.

Proof. We initially treat the two hemispheres separately, setting $f_n^{\pm} := f_n \chi_{\{\pm\omega_1>0\}}$. To decompose f_n^+, we set $r_n^{0,+} := f_n^+$, and apply the following iterative process. Given a bounded sequence of remainders $\{r_n^{J,+}\} \subseteq L^2(\mathbb{S}^d)$ obeying (6.3), we stop if $\lim \|\mathcal{E} r_n^{J,+}\|_{q_2} = 0$. If this limit is nonzero, we apply Lemma 6.2, obtaining a subsequence of $\{f_n\}$, points $\{x_n^{J+1,+}\} \subseteq \mathbb{R}^{d+1}$, and a weak limit $\phi^{J+1,+} \in L^2(\mathbb{R}^d)$. We set

$$r_n^{J+1,+} := r_n^{J,+} - e^{ix_n^{J+1,+}\omega} g_n^{J+1,+},$$

with $g_n^{J+1,+}$ defined as in (6.1).

That the $\{x_n^{j,+}\}$ move apart after parabolic rescaling follows from familiar arguments. Namely, we suppose that there is some minimal superscript j for which there exists some (minimal) superscript $j' > j$ with $\phi^{j,+}, \phi^{j',+} \not\equiv 0$ and

$$\lambda_n^2 |(x_n^{j,+} - x_n^{j',+})_1| + \lambda_n |(x_n^{j,+} - x_n^{j',+})'| \not\to \infty.$$

Passing to a subsequence,

$$(\lambda_n^2 (x_n^{j,+} - x_n^{j',+})_1, \lambda_n (x_n^{j,+} - x_n^{j',+})') \to y^{jj'}.$$

Passing to a further subsequence,

$$e^{i(x_n^{j,+} - x_n^{j',+})(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n \xi)} \to e^{i\theta^{jj'}} e^{iy^{jj'}(-\frac{1}{2}|\xi|^2, \xi)},$$

locally uniformly, for some $\theta^{jj'} \in [0, 2\pi)$. On the other hand,

$$e^{i(x_n^k - x_n^{j'})(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} \chi_{\{|\xi| \le R\}} \rightharpoonup 0,$$

weakly in L^p for all R. Noting that $r_n^{j'-1,+} = r_n^{j-1,+} - \sum_{k=j}^{j'-1} e^{ix_n^{k,+}\omega} g_n^{k,+}$,

$$\begin{aligned} \phi^{j',+} &= \text{wk-lim}\, \lambda_n^{\frac{d}{p}} e^{-ix_n^{j',+}(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} g_n^{k,+} \\ &= e^{i\theta^{jj'}} e^{iy^{jj'}(-\frac{1}{2}|\xi|^2, \xi)} \phi^{j,+} - \sum_{k=j}^{j'-1} \text{wk-lim}\, e^{i(x_n^{k,+} - x_n^{j',+})(\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} \phi^{k,+} \\ &= 0, \end{aligned}$$

a contradiction.

Taking the complex conjugate and applying the preceding argument (along our new subsequence), we obtain decompositions for the lower hemisphere as well,

$$f_n^- = \sum_{j=1}^J e^{ix_n^{j,-}\omega} g_n^{j,-} + r_n^{J,-}.$$

Passing to a subsequence, for all j, j',

$$(\lambda_n^2 (x_n^{j,+} - x_n^{j',-})_1, \lambda_n (x_n^{j,+} - x_n^{j',-})')$$

either converges or tends to ∞ in norm. In the former case, changing the $\phi^{j,-}$ if needed, we may assume that for all j, j', either $x_n^{j,+} \equiv x_n^{j',-}$ or

$$\lambda_n^2 |(x_n^{j,+} - x_n^{j',-})_1| + \lambda_n |(x_n^{j,+} - x_n^{j',-})'| \to \infty.$$

Reordering and inserting 0's in place of $\phi^{j,+}$ or $\phi^{j,-}$ where needed, we may assume that $x_n^{j,+} \equiv x_n^{j,-}$ for all j.

We thus obtain decompositions

$$f_n = \sum_{j=1}^{J} e^{ix_n^j \omega} g_n^j + r_n^J, \qquad J \in \mathbb{N},$$

as in the statement of Theorem 3.4. It remains to verify (i), (ii-iii'), (iv) and (v).

The approximation (i) of $\mathcal{E}g_n^j$ by a rescaling and modulation of $\mathcal{E}_{\mathbb{P}}\phi^j$ follows from (6.2) and the triangle inequality. The L^2 orthogonality, conclusion (ii-iii'), follows on each hemisphere separately by the weak limit condition; we put the pieces together via $\|f_n\|_2^2 = \|f_n^+\|_2^2 + \|f_n^-\|_2^2$.

The L^q-orthogonality, conclusion (v), follows by iterating the generalized Brézis–Lieb lemma, Lemma 3.1 of [21] (cf. [2]); because the lemma was developed to address precisely this situation, we will be brief in showing how that lemma applies here. In the notation of that lemma, given J and M, we set

$$\begin{aligned}
\alpha_n^M(x) &:= \lambda_n^{-\frac{d+2}{q}} \mathcal{E}(e^{-ix_n^J\omega} r_n^{J-1} \chi_{\{|\omega'| \le M\lambda_n\}})(\lambda_n^{-2}x_1, \lambda_n^{-1}x') \\
\pi_n^M(x) &:= \sum_{\pm} e^{\pm i\lambda_n^{-1}x_1} \mathcal{E}_{\mathbb{P}}(\phi^{j,\pm}\chi_{|\xi|\le M\}})(\mp x_1, x') \\
\rho_n^M(x) &:= \mathcal{E}(\lambda_n^{-\frac{d+2}{q}} e^{-ix_n^J\omega} r_n^J \chi_{\{|\omega'|\le M\lambda_n\}})(\lambda_n^{-2}x_1, \lambda_n^{-1}x') \\
\sigma_n^M(x) &:= \lambda_n^{-\frac{d+2}{q}} \mathcal{E}(g_n^J \chi_{\{|\omega'|\le M\lambda_n\}})(\lambda_n^{-2}x_1, \lambda_n^{-1}x') - \pi_n^M(x),
\end{aligned}$$

and let $\alpha_n, \pi_n, \rho_n, \sigma_n$ denote the corresponding functions with no truncation in the frequency variables. Then $\alpha_n^M = \pi_n^M + \rho_n^M + \sigma_n^M$. We immediately see that $|\pi_n^M|$ is bounded by a fixed L^q function. Observing that

$$\begin{aligned}
\rho_n^M(x) = \sum_{\pm} e^{\pm i\lambda_n^{-2}x_1} \int & e^{i(x_1,x')(\lambda_n^{-2}(\pm\sqrt{1-|\lambda_n\xi|^2}\mp 1),\xi)} e^{-ix_n^J(\lambda_n^{-2}(\pm\sqrt{1-|\lambda_n\xi|^2}\mp 1),\xi)} \\
&\times r_n^J(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|\le M\}} \, \tfrac{d\xi}{\sqrt{1-|\lambda_n\xi|^2}},
\end{aligned}$$

we see that $\rho_n^J \to 0$ pointwise. Finally,

$$\begin{aligned}
\sigma_n^M = \sum_{\pm} e^{\pm i\lambda_n^{-2}x_1} \int & \big(e^{i(x_1,x')(\lambda_n^{-2}(\pm\sqrt{1-|\lambda_n\xi|^2}\mp 1),\xi)} \tfrac{1}{\sqrt{1-|\lambda_n\xi|^2}} - e^{i(x_1,x')(\mp\frac{1}{2}|\xi|^2,\xi)}\big) \\
&\times \phi^{J,\pm}(\xi)\, \chi_{\{|\xi|\le M\}} \, d\xi.
\end{aligned}$$

If $\phi^{J,\pm}(\xi)\,\chi_{\{|\xi|\le M\}}$ are assumed to be smooth, $\sigma_n^M \to 0$ in L^q by stationary phase and the dominated convergence theorem; for general $\phi^{J,\pm}$, convergence to 0 follows from boundedness of $\mathcal{E}$ and $\mathcal{E}_{\mathbb{P}}$ from L^p to L^q and density arguments.

By the generalized Brézis–Lieb lemma,

$$\lim_{n\to\infty} \|\alpha_n^M\|_q^q - \|\pi_n^M + \sigma_n^M\|_q^q - \|\rho_n^M\|_q^q = 0.$$

By hypothesis (3.5) and $L^p \to L^q$ boundedness of $\mathcal{E}$ and $\mathcal{E}_{\mathbb{P}}$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} \|\alpha_n^M - \alpha_n\|_q + \|\pi_n^M - \pi_n\|_q + \|\rho_n^M - \rho_n\|_q + \|\sigma_n^M - \sigma_n\|_q = 0.$$

Therefore

$$\lim_{n\to\infty} \|\alpha_n\|_q^q - \|\pi_n + \sigma_n\|_q^q - \|\rho_n\|_q^q = 0,$$

i.e.,

$$\lim_{n\to\infty} \|\mathcal{E}r_n^{J-1}\|_q^q - \|\mathcal{E}g_n^J\|_q^q - \|\mathcal{E}r_n^J\|_q^q = 0.$$

The L^q orthogonality, (iv) follows by induction.

Finally, smallness of the errors follows from (ii-iii') and Lemma 6.2: a nonnegligible remainder term yields a weak limit with L^2 norm bounded below, and that reduces the L^2 norm of the subsequent remainder by a nonnegligible amount. □

Now we turn to the analogue of Lemma 5.3 for the case of antipodal frequency concentration. Let $\psi, \rho \in C_c^\infty(\mathbb{R}^d; [0,1])$ with $\psi(0) = 1$ and $\int \rho = 1$. For $r > 0$, we define $\psi_r(\xi) := \psi(r\xi)$ and $\rho^r(\xi) = r^{-d}\rho(\frac{\xi}{r})$. Given a doubly indexed sequence $\{x_n^j\}_{j,n\in\mathbb{N}}$, and $\lambda_n \searrow 0$, we define operators on integrable functions f on $\mathbb{S}^d$ by

$$(\pi_r)_n^{j,\pm} f(\xi) := \rho^r *_\eta (\psi_r(\eta) e^{-ix_n^j(\pm\sqrt{1-|\lambda_n\eta|^2}, \lambda_n\eta)} \lambda_n^{d/p} f(\pm\sqrt{1-|\lambda_n\eta|^2}, \lambda_n\eta).$$

We also define vector-valued operators

$$(\Pi_r)_n^J f := (((\pi_r)_n^{j,\bullet} f)_{j=1}^J)_{\bullet\in\{\pm\}}, \qquad J \in \mathbb{N}.$$

We recall the notation $\tilde{p} := \max\{p, p'\}$. Thus $\tilde{p} \geq p$.

Lemma 6.4. *Let $1 < p < \infty$. Assume that the sequences $\{y_n^j\}$ obey $\lim_{n\to\infty} |y_n^j - y_n^{j'}| = \infty$ for all $j \neq j'$, where $y_n^j := (\lambda_n^2(x_n^j)_1, \lambda_n(x_n^j)')$. Then the $(\Pi_r)_n^J$ map $L^p(\mathbb{S}^d)$ boundedly into $\ell_\bullet^p(\ell_j^{\tilde{p}}(L^p(\mathbb{R}^d)))$, with operator norms bounded uniformly in r, n. Moreover,*

$$\lim_{r\to 0} \limsup_{n\to\infty} \|(\Pi_r)_n^J\|_{L^p \to \ell_\bullet^p(\ell_j^{\tilde{p}}(L^p(\mathbb{R}^d)))} = 1. \tag{6.5}$$

Finally, let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{S}^d)$, supported in $\{\omega \in \mathbb{S}^d : |\omega'| < \frac{1}{2}\}$, for which the weak limits $\phi^{j,\pm}$ in (3.6) exist, and define g_n^j as in (3.7). Then

$$\lim_{r\to 0} \lim_{n\to\infty} \|(\pi_r)_n^{j,\pm} f_n - \phi^{j,\pm}\|_p = 0 \tag{6.6}$$

$$\lim_{r\to 0} \lim_{n\to\infty} \|[(\pi_r)_n^{j,\pm}]^* \phi^{j,\pm} - e^{ix_n^j\omega} g_n^{j,\pm}\|_p = 0. \tag{6.7}$$

Proof. Since $f \mapsto (f\chi_{\{\omega_1>0\}}, f\chi_{\{\omega_1<0\}})$ maps $L^p(\mathbb{S}^d)$ boundedly into $\ell^p(L^p(\mathbb{S}^d) \times L^p(\mathbb{S}^d))$, with operator norm 1, (6.5) would follow from

$$\lim_{r\to 0} \limsup_{n\to\infty} \|(\Pi_r)_n^{J,\bullet}\|_{L^p \to \ell_j^{\tilde{p}}(L^p(\mathbb{R}^d))} = 1, \quad \bullet = +, -, \tag{6.8}$$

and using reflection across the hyperplane $\{0\} \times \mathbb{R}^d$, we may choose the positive sign in (6.8), (6.6), and (6.7). To keep equations within lines, we will omit the superscript $+$ from the operators from the remainder of the proof.

It is elementary to show that $\|(\pi_r)_n^j f\|_p \lesssim \|f\|_p$, with implicit constant independent of f, r, n, j. Moreover, for any j,

$$\lim_{r\to 0} \limsup_{n\to\infty} \|(\pi_r)_n^j\|_{L^p \to L^p} = 1.$$

Indeed, the upper bound uses the compact support of ψ, and the lower bound is obtained by considering a shrinking profile $e^{ix_n^j\omega} g_n^{j,+}$. In particular, (6.8) holds when $p = 1, \infty$.

The upper bound in (6.5) will thus follow from that in the case $p = 2$ by complex interpolation.

We turn now to the proof that

$$\limsup_{n\to\infty} \|(\Pi_r)_n^J\|_{L^2\to\ell^2(\ell^2(L^2))} = 1, \tag{6.9}$$

for all r. We bound

$$\begin{aligned} &\|(\Pi_r)_n^J[(\Pi_r)_n^J]^*(\phi^j)_{j=1}^J\|_{\ell^2(L^2)}^2 \\ &\quad\le (1+\varepsilon)\sum_{j=1}^J \|(\pi_r)_n^j[(\pi_r)_n^j]^*\phi^j\|_{L^2}^2 + C_{J,\varepsilon}\sum_{j\ne k}\|(\pi_r)_n^j[(\pi_r)_n^k]^*\phi^k\|_{L^2}^2. \end{aligned} \tag{6.10}$$

We expand (for $\xi\in\mathbb{R}^d$)

$$(\pi_r)_n^j[(\pi_r)_n^k]^*\phi(\xi) = \int \phi(\zeta)(K_r)_n^{jk}(\zeta,\xi)\,d\xi,$$

where

$$(K_r)_n^{jk}(\zeta,\xi) := \int \psi_r(\eta)^2\rho_r(\xi-\eta)\rho_r(\zeta-\eta)e^{i(x_n^k-x_n^j)(\sqrt{1-|\lambda_n\eta|^2},\lambda_n\eta)}\sqrt{1-|\lambda_n\eta|^2}\,d\eta.$$

A straightforward computation (using our hypotheses on ρ,ψ) gives

$$\|(K_r)_n^{jk}\|_{L^\infty_\zeta L^1_\xi}, \|(K_r)_n^{jk}\|_{L^\infty_\xi L^1_\zeta} \le 1.$$

For $j\ne k$, stationary phase gives $\|(K_r)_n^{jk}\|_{L^\infty} \lesssim \langle y_n^k - y_n^j\rangle^{-\frac{d}{2}}$. Since the $(K_r)_n^{jk}$ have supports contained in a fixed compact set (for fixed r),

$$\|(K_r)_n^{jk}\|_{L^\infty_\zeta L^1_\xi}, \|(K_r)_n^{jk}\|_{L^\infty_\xi L^1_\zeta} \to 0,$$

as $n\to\infty$. Hence by (6.10),

$$\limsup_{n\to\infty}\|(\Pi_r)_n^j[(\Pi_r)_n^k]^*\|_{\ell^2(L^2)\to\ell^2(L^2)} \le 1+\varepsilon,$$

for all $\varepsilon>0$. Sending $\varepsilon\to 0$, we obtain (6.9).

It remains to prove (6.6) and (6.7). By boundedness of the $(\pi_r)_n^k$ on L^p, we may assume the ϕ^k are compactly supported. Modulating $f_n \rightsquigarrow e^{-x_n^j\omega}f_n$ if needed, it suffices to consider the case $x_n^j=0$, for all n.

Since (6.7) is essentially elementary, we turn to (6.6). Noting that $|y_n^k|\to\infty$ when $k\ne j$,

$$\lambda_n^{\frac{d}{p}} e^{ix_n^k(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi)} g_n^{k,+}(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi) = e^{i(x_n^k)_1}e^{iy_n^k(\lambda_n^{-2}\sqrt{1-|\lambda_n\xi|^2},\xi)}\phi^k(\xi)\rightharpoonup 0,$$

weakly when $k\ne j$. Hence, by construction,

$$\lambda_n^{\frac{d}{p}} f(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi)\rightharpoonup\phi^j,$$

weakly, as $n\to\infty$. Therefore $(\pi_r)_n^j f_n(\xi)\to\int\rho_r(\xi-\eta)\psi_r(\eta)\phi^j(\eta)\,d\eta$ pointwise in ξ, as $n\to\infty$. In fact, the convergence is in L^p by the dominated convergence theorem, and (6.6) follows. □

We are finally ready to conclude the proof of the L^p profile decomposition in the case of antipodal concentration, Theorem 3.4.

Proof of Theorem 3.4. By Lemma 6.3, we may assume that $p_1 := p \ne 2$. We fix an exponent $p_0 < \frac{2(d+1)}{d}$ such that p_1 lies between p_0 and $2 =: p_2$ and the extension conjecture holds at (p_0,q_0), with $q_0 := \frac{d+2}{d}p_0'$.

It suffices to prove the theorem under the additional assumptions that $\|f_n\|_p \leq 1$, $\lim \|f_n\|_p = 1$, and f_n is supported in $\{|\omega'| < \frac{1}{2}\}$, for all n. (Indeed, we may simply add $f_n \chi_{\{|\omega'| > \frac{1}{2}\}}$ to the error terms r_n^J and then multiply by a constant.) We define

$$f_n^M := f_n \chi_{\{|f_n| \leq M \lambda_n^{-d/p}\}} \chi_{\operatorname{dist}(\omega, [e_1]) < M\lambda_n\}}.$$

Setting $\alpha_i := \frac{d}{p} - \frac{d}{p_i}$ for $i \in \{0, 1, 2\}$, $\{\lambda_n^{\alpha_2} f_n\}$ obeys the hypotheses of Theorem 3.4 with $p = 2$. Therefore, by Lemma 6.3, after passing to a subsequence (which does not depend on $M \in \mathbb{N}$), for all M, there exist $\{x_n^{j,M}\} \subseteq \mathbb{R}^{d+1}$ with

$$\lim_{n\to\infty} (\lambda_n^2 |(x_n^{j,M} - x_n^{j,M})_1| + \lambda_n |(x_n^{j,M} - x_n^{j,M})'|) = \infty, \qquad j \neq j',$$

and weak limits

$$L^2(\mathbb{R}^d) \ni \phi^{j,M,\pm} := \text{wk-lim}\, \lambda_n^{d/p} e^{-i x_n^{j,M}(\pm\sqrt{1-|\lambda_n \xi|^2}, \lambda_n \xi)} f_n^M(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n \xi),$$

such that, with

$$g_n^{j,M}(\omega) := \sum_{\pm} \lambda_n^{-d/p} \phi^{j,M,\pm}(\lambda_n^{-1}\omega') \chi_{\{\pm\omega_1 > 0\}} \chi_{\{|\omega'| < \frac{1}{2}\}},$$

$$r_n^{J,M} := f_n^M - \sum_{j=1}^{J} e^{i x_n^{j,M} \omega} g_n^{j,M},$$

we have

$$\lim_{n\to\infty} \|\lambda_n^{\alpha_2} f_n^M\|_2 - \sum_{j=1}^{J} \sum_{\pm} \|\phi^{j,M,\pm}\|_2^2 - \|\lambda_n^{\alpha_2} r_n^{J,M}\|_2^2 = 0,$$

$$\lim_{n\to\infty} \|\lambda_n^{\alpha_2} \mathcal{E} f_n^M\|_{q_2}^{q_2} - \sum_{j=1}^{J} \|\lambda_n^{\alpha_2} \mathcal{E} g_n^{j,M}\|_{q_2}^{q_2} - \|\lambda_n^{\alpha_2} \mathcal{E} r_n^{J,M}\|_{q_2}^{q_2} = 0, \tag{6.11}$$

$$\lim_{J\to\infty} \lim_{n\to\infty} \|\lambda_n^{\alpha_2} \mathcal{E} r_n^{J,M}\|_{q_2} = 0, \tag{6.12}$$

for all J. In fact, the generalized Brézis–Lieb lemma applied in the proof of Lemma 6.3 implies that (6.11) holds with α_i, q_i in place of α_2, q_2, when $i = 0, 1$ as well. In particular, $\{\lambda_n^{\alpha_0} \mathcal{E} r_n^{J,M}\}$ is bounded in L^{q_0}, uniformly in J, as $n \to \infty$, and hence by Hölder's inequality, (6.12) also holds with α_1, q_1 in place of α_2, q_2, i.e. (since $\alpha_1 = 0$),

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\mathcal{E} r_n^{M,J}\|_{q_1} = 0, \text{ for all } M. \tag{6.13}$$

After passing to a subsequence (and modulating the $\phi^{j,M,\pm}$ and multiplying by unimodular constants if needed), we may assume that for any M, j and M', j', either $x_n^{M,j} \equiv x_n^{M',j'}$, or

$$\lim_{n\to\infty} \lambda_n^2 |(x_n^{M,j} - x_n^{M',j'})_1| + \lambda_n |(x_n^{M,j} - x_n^{M',j'})'| = \infty,$$

for all $j, j', M, M' \in \mathbb{N}$. By reordering the profiles and inserting zero profiles as needed (using yet another diagonal argument), we may write $x_n^{j,M} = x_n^j$, for all M. Passing to a further subsequence, the weak limits $\phi^{j,\pm}$ defined in (3.6) all exist.

By Lemma 6.4, for every $\varepsilon > 0$, and any choice of $\bullet \in \{\pm\}$, $\|\phi^{j,\bullet}\|_p \geq \varepsilon$ for at most $\varepsilon^{-\tilde{p}}$ values of j. Indeed,

$$\sum_{j=1}^{J} \|\phi^{j,\bullet}\|_p^{\tilde{p}} = \lim_{r\to 0} \lim_{n\to\infty} \|(\Pi_r)_n^J f_n\|_{\ell^{\tilde{p}}(L^p)}^{\tilde{p}} \leq 1. \tag{6.14}$$

In particular, we may reorder the profiles so that $\|\phi^{j,+}\|_p^p + \|\phi^{j,-}\|_p^p$ is decreasing. In the notation of Theorem 3.4, it remains to prove (ii), (iii), (iv), and (v).

Inequality (ii), which was one of the L^p almost orthogonality conditions, follows from Lemma 6.4. Indeed,

$$\sum_{\pm}\Big(\sum_{j=1}^{J}\|\phi^{j,\pm}\|_p^{\tilde p}\Big)^{p/\tilde p} = \lim_{r\to\infty}\lim_{n\to\infty}\|(\Pi_r)_n^J f_n\|_{\ell^p(\ell^{\tilde p}(L^p))} \le 1.$$

Inequality (iii), the other L^p almost orthogonality estimate, follows from Lemma 6.4 as well, since

$$\limsup_{n\to\infty}\|\sum_{j=1}^{J} e^{ix_n^j\omega} g_n^j\|_p = \lim_{r\to\infty}\lim_{n\to\infty}\|[(\Pi_r)_n^J]^*((\phi^{j,\bullet})_{j=1}^J)_{\bullet\in\{\pm\}}\|_p \le \|((\phi^{j,\bullet})_{j=1}^J)_{\bullet\in\{\pm\}}\|_{\ell^p(\ell^{\tilde p'}(L^p))}.$$

The L^q-orthogonality condition (iv) follows from the generalized Brézis–Lieb argument given in the proof of Lemma 6.3 (indeed, the condition $p=2$ played no role in that part of the argument).

We are left to establish (v), smallness of the extensions of the remainder terms. We wish to remove the dependence on M in (6.13), and, as in the previous section, we begin by showing that the rate of convergence to 0 in (6.13) as $J\to\infty$ is independent of M.

Given $M\in\mathbb{N}$ and $(j,\bullet)\in\mathbb{N}\times\{\pm\}$, we define $\phi_M^{j,\bullet} := \phi^{j,\bullet}\chi_{\{|\xi|<M\}}$ and

$$(g_M)_n^{j,\bullet}(\omega) := \lambda_n^{-d/p}\phi^{j,\bullet}(\lambda_n^{-1}\omega')\chi_{\{\bullet\omega_1>0\}}\chi_{\{|\omega'|<\frac12\}}.$$

Lemma 6.5. *For every $M,j\in\mathbb{N}$ and $\bullet\in\{\pm\}$,*

$$\lim_{n\to\infty}\|\mathcal{E}((g_M)_n^{j,\bullet} - g_n^{M,j,\bullet})\|_q \le \limsup_{n\to\infty}\sup_{E\subseteq E_n^M}\|\mathcal{E}f_n\chi_E\|_q, \tag{6.15}$$

where E_n^M was defined in (3.5).

Proof of Lemma 6.5. We give the proof when $\bullet = +$; the other case follows by taking conjugates. To simplify notation, we omit the j and $+$ from our superscripts.

By Lemma 6.1,

$$\lim_{n\to\infty}\|\mathcal{E}((g_M)_n - g_n^M)\|_q = \|\mathcal{E}_{\mathbb{P}}(\phi_M - \phi^M)\|_q.$$

Since

$$\phi_M(\xi) = \operatorname*{wk-lim}_{n\to\infty}\lambda_n^{d/p} f_n(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi)\chi_{\{|\xi|<M\}},$$

we have

$$\begin{aligned}
&\mathcal{E}_{\mathbb{P}}(\phi_M - \phi^M)(-x_1,x')\\
&\quad= \lim_{n\to\infty}\int e^{ix(-\frac12|\xi|^2,\xi)}\lambda_n^{d/p} f_n(\sqrt{1-|\lambda_n\xi|^2},\lambda_n\xi)\,\chi_{\{|\xi|<M\}}\chi_{\{|f_n|>M\lambda_n^{-d/p}\}}\,d\xi\\
&\quad= \lim_{n\to\infty} e^{-i\lambda_n^{-2}x_1}\int e^{i(\lambda_n^{-2}x_1,\lambda_n^{-1}x')(1-\frac12|\xi|^2,\xi)}\lambda_n^{-d/p'} f_n(\sqrt{1-|\xi|^2},\xi)\\
&\qquad\qquad\times\chi_{\{|\xi|<M\lambda_n\}}\chi_{\{|f_n|>M\lambda_n^{-d/p}\}}\,d\xi.
\end{aligned}\tag{6.16}$$

By Hölder's inequality,

$$\|\lambda_n^{-d/p'} f_n(\sqrt{1-|\xi|^2},\xi)\chi_{\{|\xi|<M\lambda_n\}}\|_{L^1_\xi} \le M^{-d/p'}.$$

For $|\xi| < M\lambda_n$,

$$|1-\sqrt{1-|\xi|^2}| \lesssim M^2\lambda_n^2, \qquad |(1-\tfrac{1}{2}|\xi|^2)-\sqrt{1-|\xi|^2}| \lesssim M^4\lambda_n^4.$$

Thus by Hölder's inequality, (6.16), and $\frac{d}{p'} = \frac{d+2}{q}$,

$$\begin{aligned}&\mathcal{E}_{\mathbb{P}}(\phi_M-\phi^M)(-x_1,x')\\&\quad=\lim_{n\to\infty}\lambda_n^{-(d+2)/q}e^{i\lambda_n^{-2}x_1}\mathcal{E}(f_n\chi_{\{|\omega'|<M\lambda_n,|f_n|>M\lambda_n^{-d/p},\omega_1>0\}})(\lambda_n^{-2}x_1,\lambda_n^{-1}x'),\end{aligned}$$

for all x. Finally, by Fatou and a change of variables,

$$\|\mathcal{E}_{\mathbb{P}}(\phi_M-\phi^M)\|_q \le \|\mathcal{E}(f_n\chi_{\{|\omega'|<M\lambda_n,|f_n|>M\lambda_n^{-d/p},\omega_1>0\}})\|_q.$$

□

Let $\varepsilon>0$. By (6.14), $\lim_{j\to\infty}\|\phi^{j,\bullet}\|_p=0$, so by boundedness of $\mathcal{E}$, $\lim_{j\to\infty}\lim_{n\to\infty}\|\mathcal{E}g_n^j\|_q=0$. Hence, by (3.5) and (6.15), there exist $M_\varepsilon, J_\varepsilon$ such that $\lim_{n\to\infty}\|\mathcal{E}g_n^{M,j}\|_q<\varepsilon$ for all $M\ge M_\varepsilon$ and $j>J_\varepsilon$. Let $M\ge M_\varepsilon$. By (6.13), there exists $J_{M,\varepsilon}$ such that $\lim_{n\to\infty}\|\mathcal{E}r_n^{M,J}\|_q<\varepsilon$, for all $J\ge J_{M,\varepsilon}$.

By the definition of the remainder terms, the generalized Brézis–Lieb lemma, boundedness of $\mathcal{E}$, and the L^p-almost orthogonality condition (ii),

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{E}(r_n^{M,J_\varepsilon}-r_n^{M,J_{M,\varepsilon}})\|_q^q &= \lim_{n\to\infty}\|\mathcal{E}(\sum_{j=J_\varepsilon+1}^{J_{M,\varepsilon}}e^{ix_n^j\omega}g_n^{M,j})\|_q^q\\&=\sum_{j=J_\varepsilon+1}^{J_{M,\varepsilon}}\lim_{n\to\infty}\|\mathcal{E}g_n^{M,j}\|_q^q \le S_{p\to q}^{q-\tilde p}\varepsilon^{q-\tilde p}\sum_{j=J_\varepsilon+1}^{J_{M,\varepsilon}}(\sum_{\bullet\in\{\pm\}}\|\phi^{M,j,\bullet}\|_p^p)^{\tilde p/p}\lesssim\varepsilon^{q-\tilde p}.\end{aligned}$$

Since $q>\tilde p$, after changing the value of ε, we may assume that

$$\|\mathcal{E}r_n^{M,J}\|_q<\varepsilon, \tag{6.17}$$

for all $M>M_\varepsilon$ and $J>J_\varepsilon$.

Finally, we transfer (6.17) to the $\mathcal{E}r_n^J$. The L^q-orthogonality (iv) applied with some $r_n^{J_0}$ in place of f_n implies that $\lim_{n\to\infty}\|\mathcal{E}r_n^J\|_q$ is non-increasing in J. Thus it suffices to bound $\lim_{n\to\infty}\|\mathcal{E}r_n^{J_\varepsilon}\|_q$.

By the triangle inequality, the definition of the remainder terms, and (6.17); then (3.5), $\phi_M^{j,\bullet}\to\phi^{j,\bullet}$ in L^p, and (6.15),

$$\lim_{n\to\infty}\|\mathcal{E}r_n^{J_\varepsilon}\|_q \le O(\varepsilon)+\lim_{M\to\infty}\lim_{n\to\infty}\|\mathcal{E}(f_n-f_n^M)\|_q+\lim_{M\to\infty}\sum_{j=1}^{J_\varepsilon}\|\mathcal{E}(g_n^j-g_n^{j,M})\|_q=O(\varepsilon).$$

As we have confirmed conclusion (v), the proof of Theorem 3.4 is complete.

□

7. Proof of Theorem 1.1

Proof of Theorem 1.1. Under the hypotheses of Theorem 1.1, let $\{f_n\}$ be an L^p-normalized extremizing sequence. By Proposition 3.2, the conditions of Theorem 3.3 apply after passing to a subsequence. Letting $\{\phi^j\}$ denote the profiles in the conclusion of that theorem,

$$S_{p\to q}=\lim_{n\to\infty}\|\mathcal{E}f_n\|_q=(\sum_{j=1}^\infty\|\mathcal{E}\phi^j\|_q^q)^{\frac{1}{q}}\le S_{p\to q}(\sum_{j=1}^\infty\|\phi^j\|_p^q)^{\frac{1}{q}}.$$

By Theorem 3.3 the profiles satisfy (3.2), which for an L^p-normalized sequence gives $\sum_j \|\phi^j\|_p^{\tilde{p}} \leq 1$. Further, we have assumed that $q > \max\{p, \frac{d+2}{d}p'\} \geq \tilde{p}$. Therefore,

$$S_{p\to q} \leq S_{p\to q} \sup_j \|\phi^j\|_p^{1-\frac{\tilde{p}}{q}} \big(\sum_{j=1}^{\infty} \|\phi^j\|_p^{\tilde{p}}\big)^{\frac{1}{q}} \leq S_{p\to q} \sup_j \|\phi^j\|_p^{1-\frac{\tilde{p}}{q}} \leq S_{p\to q}.$$

Equality must hold at each step, and thus, after reordering the profiles, $\|\phi^1\|_p = 1$, $\|\mathcal{E}\phi^1\|_q = S_{p\to q}$, and $\phi^j \equiv 0$ for $j \neq 1$. Hence by Theorem 2.11 of [29], $e^{-ix_n^1} f_n \to \phi^1$, an extremizer, (strongly) in L^p. □

8. Antipodally concentrating profiles

In the next section, we will prove Theorems 1.2, 1.5, and 1.6, which we recall concern extremizing sequences along the parabolic scaling line $q = \frac{d+2}{d}p'$. In this section, we prove a preliminary lemma that addresses the interactions between the extensions of pairs of sequences of functions that concentrate antipodally, building on the connection between spherical and parabolic extension previously discussed in Lemma 6.1. This will lead naturally into the proofs of Propositions 1.3 and 1.4, which are also contained in this section.

We recall that

$$\beta_{p\to q} := 2^{\frac{1}{r'}} \big(\tfrac{\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\Gamma(\frac{q+2}{2})}\big)^{\frac{1}{q}},$$

where $r := \max\{p, 2\}$.

Lemma 8.1. *Let $\lambda_n \searrow 0$, let ϕ^+, ϕ^- be L^p functions on $\mathbb{R}^d$, and define g_n as in (3.7). Then*

$$\lim_{n\to\infty} \|g_n\|_p = \big(\|\phi^+\|_p^p + \|\phi^-\|_p^p\big)^{\frac{1}{p}}, \tag{8.1}$$

$$\lim_{n\to\infty} \|\mathcal{E}g_n\|_q \leq \beta_{p\to q} P_{p\to q} \big(\|\phi^+\|_p^p + \|\phi^-\|_p^p\big)^{\frac{1}{p}}, \tag{8.2}$$

and equality in (8.2) occurs if and only if either $\phi^+ \equiv \phi^- \equiv 0$, or $p \geq 2$, $|\mathcal{E}\phi^+(-x_1, x')| \equiv |\mathcal{E}\phi^-(x_1, x')|$, and the $\phi^\pm$ are both extremizers for $\mathcal{E}_{\mathbb{P}}$.

Proof. The identity (8.1) follows by parametrizing the upper and lower hemispheres, rescaling, and applying the dominated convergence theorem.

Now we turn to inequality (8.2). Identity (3.7), our Lemma 6.1, and Lemma 6.1 of [21] imply that

$$\begin{aligned}
\lim_{n\to\infty} \|\mathcal{E}g_n\|_q &= \lim_{\alpha\to\infty} \big(\|e^{i\alpha x_1}\mathcal{E}_{\mathbb{P}}\phi^+(-x_1, x') - \mathcal{E}_{\mathbb{P}}\phi^-(x_1, x')\|_q^q\big)^{\frac{1}{q}} \\
&= \big(\tfrac{1}{2\pi}\int_0^{2\pi} \|e^{i\theta}\mathcal{E}_{\mathbb{P}}\phi^+(-x_1, x') - \mathcal{E}_{\mathbb{P}}\phi^-(x_1, x')\|_q^q \, d\theta\big)^{\frac{1}{q}} \\
&\leq 2^{\frac{1}{2}} \big(\tfrac{\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\Gamma(\frac{q+2}{2})}\big)^{\frac{1}{q}} P_{p\to q} \big(\|\phi^+\|_p^2 + \|\phi^-\|_p^2\big)^{\frac{1}{2}},
\end{aligned}$$

with equality if and only if either $|\mathcal{E}_{\mathbb{P}}\phi^+(-x_1, x')| = |\mathcal{E}_{\mathbb{P}}\phi^-(x_1, x')|$ a.e. and $\phi^\pm$ are both extremizers for (1.2), or $\phi^\pm \equiv 0$.

If $p \geq 2$,

$$\big(\|\phi^+\|_p^2 + \|\phi^-\|_p^2\big)^{\frac{1}{2}} \leq 2^{\frac{1}{2}-\frac{1}{p}} \big(\|\phi^+\|_p^p + \|\phi^-\|_p^p\big)^{\frac{1}{p}},$$

with equality if and only if $\|\phi^+\|_p = \|\phi^-\|_p$. If $p < 2$,

$$\big(\|\phi^+\|_p^2 + \|\phi^-\|_p^2\big)^{\frac{1}{2}} \leq \big(\|\phi^+\|_p^p + \|\phi^-\|_p^p\big)^{\frac{1}{p}},$$

with equality if and only if $\phi^+ \equiv 0$ or $\phi^- \equiv 0$. This proves (8.2), and the remark following on cases of equality. $\square$

Adapting the construction of the g_n, yields the lower bound on the extension operator norm from Proposition 1.3.

Proof of Proposition 1.3. If $P_{p\to q}$ is infinite, then $S_{p\to q}$ is as well, so in this case there is nothing to prove. Let $1 \le p < \frac{2(d+1)}{d}$ and set $q := \frac{d+2}{d}p'$ and suppose $P_{p\to q} < \infty$. Further take $t \in [0,1]$, and suppose that $\phi^\pm$ are chosen such that each is an extremizer for $\mathcal{E}_{\mathbb{P}}$, $\|\phi^+\|_p = t\|\phi^-\|_p$ and $|\mathcal{E}_{\mathbb{P}}\phi^+(-x_1,x')| \equiv t|\mathcal{E}_{\mathbb{P}}\phi^-(x_1,x')|$. We may always construct such a pair. Indeed, an extremizer for the parabolic extension problem, ϕ^+, exists by [35], and after setting $\phi^-(x) = t\overline{\phi^+(-x)}$ a direct computation shows that these conditions are satisfied. Let $\lambda_n \to 0$. Analogously to (3.7) set

$$g_n^\pm(\omega) := \lambda_n^{-d/p}\phi^\pm(\lambda_n^{-1}\omega')\chi_{\{\pm\omega_1>0\}}\chi_{\{|\omega'|<\frac{1}{2}\}}, \qquad g_n := g_n^+ + g_n^-. \tag{8.3}$$

An argument similar to (8.1), shows that, as $\|\phi^+\|_p = t\|\phi^-\|_p$ and ϕ^+ is extremizing,

$$\begin{aligned}\lim_{n\to\infty}\|g_n\|_p =&\|\phi^+\|_p\big(1+t^p\big)^{\frac{1}{p}}\\ =&P_{p\to q}^{-1}\|\mathcal{E}_{\mathbb{P}}\phi^+\|_q\big(1+t^p\big)^{\frac{1}{p}}.\end{aligned}$$

Next we compute

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{E}g_n\|_q &= \lim_{\alpha\to\infty}\big(\|\mathcal{E}_{\mathbb{P}}\phi^+(-x_1,x') - t\,e^{i\alpha x_1}\mathcal{E}_{\mathbb{P}}\phi^-(x_1,x')\|_q^q\big)^{\frac{1}{q}}\\ &= \big(\tfrac{1}{2\pi}\int_0^{2\pi}\|\mathcal{E}_{\mathbb{P}}\phi^+(-x_1,x') - t\,e^{i\theta}\mathcal{E}_{\mathbb{P}}\phi^-(x_1,x')\|_q^q\,d\theta\big)^{\frac{1}{q}}\\ &= \|\mathcal{E}_{\mathbb{P}}\phi^+\|_q\big(\tfrac{1}{2\pi}\int_0^{2\pi}|1+t\,e^{i\theta}|^q\,d\theta\big)^{\frac{1}{q}},\end{aligned}$$

using that $|\mathcal{E}\phi^+(-x_1,x')| \equiv t|\mathcal{E}\phi^-(x_1,x')|$.

Thus,

$$\lim_{n\to\infty}\frac{\|\mathcal{E}g_n\|_q}{\|g_n\|_p} = P_{p\to q}\frac{\big(\frac{1}{2\pi}\int_0^{2\pi}|1+t\,e^{i\theta}|^q\,d\theta\big)^{\frac{1}{q}}}{\big(1+t^p\big)^{\frac{1}{p}}}.$$

Whence the maximum of this quantity for $t \in [0,1]$ is a lower bound for $S_{p\to q}$, the operator norm of $\mathcal{E}$. Note that $[0,1]$ is the natural domain for t as it represents the ratio of the smaller L^p-norm to the larger, for two concentrating profiles. $\square$

Proposition 1.4 is an immediate corollary of Proposition 1.3 and Lemma 8.1, which characterizes concentrating extremizing sequences when $p \ge 2$.

An aside on α's and β's. Let $\bar{S}_{p\to q}$ denote the supremum over all antipodally concentrating sequences g_n (of the form (3.7)) of the quantity

$$\lim_{n\to\infty}\|\mathcal{E}g_n\|_q/\|g_n\|_p.$$

In this section, we have shown that $\alpha_{p\to q}P_{p\to q} \le \bar{S}_{p\to q} \le \beta_{p\to q}P_{p\to q}$, for p and q along the scaling line, that both inequalities are equalities when $p \ge 2$, and that the second inequality is strict when $1 < p < 2$. In the latter range, however, a bit more information might lead to a sharper version of Theorem 1.6. Namely, two questions that seem interesting are whether $\bar{S}_{p\to q}$ might equal $\alpha_{p\to q}P_{p\to q}$, and which value of

t maximizes the right hand side of (1.4). These questions may be modified slightly into a more general framework, and we ask what is the value of the quantity

$$\sup_{0\not\equiv F,G\in L^q} \frac{\left(\frac{1}{2\pi}\int_0^{2\pi}\|e^{i\theta}F+G\|_q^q\,d\theta\right)^{1/q}}{\left(\|F\|_q^p+\|G\|_q^p\right)^{1/p}},$$

do maximizers exist, and, if so, what are their properties, in the case $1<p<2<q$?

Numerical computations (unpublished) due to Arthur DressenWall, an undergraduate student at Macalester, lead us to ask whether the right hand side of (1.4) might be attained when t is either 0 or 1, in which case we would have

$$\alpha_{p\to q} = \max\left\{2^{\frac{1}{p'}}\Big(\frac{\Gamma(\frac{q+1}{2})}{\sqrt{\pi}\Gamma(\frac{q+2}{2})}\Big)^{\frac{1}{q}},1\right\},$$

but we are not quite bold enough to formulate this as a conjecture.

9. Proof of Theorem 1.2

The proof of Theorem 1.2 follows a similar outline to the proof of Theorem 1.1, with the added complication of handling the profiles from the case of concentration.

Proof of Theorem 1.2. Under the hypotheses of Theorem 1.2, we let $\{f_n\}$ be an L^p-normalized extremizing sequence of (1.1). Applying one stage of the frequency decomposition in Theorem 3.1 (*à la* [28]),

$$\begin{aligned}(S_{p\to q})^q &= \lim_{n\to\infty}\|\mathcal{E}f_n\|_q^q = \lim_{n\to\infty}\|\mathcal{E}F_n^1\|_q^q+\|\mathcal{E}R_n^1\|_q^q\\ &\le \lim_{n\to\infty}\max\{\|\mathcal{E}F_n^1\|_q^{q-p},\|\mathcal{E}R_n^1\|_q^{q-p}\}(\|\mathcal{E}F_n^1\|_q^p+\|\mathcal{E}R_n^1\|_q^p)\\ &\le \lim_{n\to\infty}(S_{p\to q})^q(\|F_n^1\|_p^p+\|R_n^1\|_p^p) = (S_{p\to q})^q.\end{aligned}\tag{9.1}$$

Passing to a subsequence, we may assume that all of the norms in (9.1) converge. By reordering, we may assume that $\lim_{n\to\infty}\|\mathcal{E}F_n^1\|_q\neq 0$. As all inequalities in (9.1) must be equalities, $\mathcal{E}R_n^1\to 0$ in L^q (first inequality), and F_n^1 is extremizing and $R_n^1\to 0$ in L^p (second inequality). In other words, f_n obeys the hypotheses of either Theorem 3.3 or, after applying a sequence of rotations, of Theorem 3.4.

If we are in the case of nonconcentration, described in the hypotheses of Theorem 3.3, we may follow the proof of Theorem 1.1 from Section 7 to see that f_n converges in L^p to an extremizer.

Thus, it remains to consider the case of antipodal concentration, in which, by neglecting the role of rotations, we may apply Theorem 3.4. In the notation of that theorem, we have

$$\begin{aligned}(S_{p\to q})^q &= \sum_{j=1}^\infty \limsup_{n\to\infty}\|\mathcal{E}g_n^j\|_q^q \le (S_{p\to q})^q\sum_{j=1}^\infty\limsup_{n\to\infty}\|g_n^j\|_p^q\\ &= (S_{p\to q})^q\sum_{j=1}^\infty(\|\phi^{j,+}\|_p^p+\|\phi^{j,-}\|_p^p)^{q/p}\\ &\le (S_{p\to q})^q\sup_j(\|\phi^{j,+}\|_p^p+\|\phi^{j,-}\|_p^p)^{\frac{q-\tilde p}{p}}\lim_{n\to\infty}\|f_n\|_p^{\tilde p}\\ &= (S_{p\to q})^q\sup_j(\|\phi^{j,+}\|_p^p+\|\phi^{j,-}\|_p^p)^{\frac{q-\tilde p}{p}}\le (S_{p\to q})^q\cdot 1,\end{aligned}\tag{9.2}$$

and equality holds at each step in the above argument. In particular, after reordering,

$$\|\phi^{1,+}\|_p^p + \|\phi^{1,-}\|_p^p = 1,$$

and $\phi^{j,\pm} \equiv 0$, for $j \neq 1$.

Since

$$\phi^{j,\pm} = \text{wk-lim}\, \lambda_n^{d/p} e^{-ix_n^j(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} f_n(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|<\frac{1}{2}\lambda_n^{-1}\}},$$

weak lower semi-continuity of L^p norms gives

$$\liminf_{n\to\infty} \|f_n\chi_{\{\pm\xi_1>0\}}\|_p \geq \|\phi^{1,\pm}\|_p.$$

As the sets $\{\pm\xi_1 > 0\}$ are disjoint,

$$1 = \liminf_{n\to\infty} \sum_{\bullet=+,-} \|f_n\chi_{\{\bullet\xi_1>0\}}\|_p^p \geq \|\phi_1^{1,+}\|_p^p + \|\phi_1^{1,-}\|_p^p \geq 1.$$

Therefore,

$$\|\phi^{1,\pm}\|_p = \lim_{n\to\infty} \|f_n\chi_{\{\pm\xi_1>0\}}\|_p.$$

Weak convergence plus convergence of norms implies strong convergence, i.e.,

$$\lambda_n^{d/p} e^{-ix_n^1(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)} f_n(\pm\sqrt{1-|\lambda_n\xi|^2}, \lambda_n\xi)\chi_{\{|\xi|<\frac{1}{2}\lambda_n^{-1}\}} \to \phi_j^{k,\pm}, \text{ in } L^p,$$

which completes the proof of Theorem 1.2. □

We also have a bit more information. After inserting the equation $\lim \|g_n^1\|_p^q = 1$, equality in (9.2) becomes

$$\lim_{n\to\infty} \|\mathcal{E}g_n^1\|_q = S_{p\to q}\|g_n^1\|_p = S_{p\to q}\big(\|\phi^+\|_p^p + \|\phi^-\|_p^p\big)^{\frac{1}{p}} \tag{9.3}$$

Depending on the values of $p, S_{p\to q}$, and $P_{p\to q}$, this equality has different implications.

Proof of Theorems 1.5 and 1.6. If $S_{p\to q} > \beta_{p\to q}P_{p\to q}$, then (9.3) and (8.2) create a contradiction, ruling out the possibility of concentration. In this case, extremizers exist, and extremizing sequences possess convergent (modulo symmetries) subsequences.

If $p \geq 2$ and $S_{p\to q} = \alpha_{p\to q}P_{p\to q} = \beta_{p\to q}P_{p\to q}$, then (9.3) implies that the equality case of Lemma 8.1 holds, which prescribes the manner of concentration.

If $p < 2$ and $S_{p\to q} = \alpha_{p\to q}P_{p\to q}$, then the construction from the proof of Proposition 1.3 gives a possible case of equality. □

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