On the time constant of high dimensional first passage percolation, revisited[*]

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Abstract

In [2], it was claimed that the time constant $\mu_d(e_1)$ for the first-passage percolation model on $\mathbb{Z}^d$ is $\mu_d(e_1) \sim \log d/(2ad)$ as $d \to \infty$, if the passage times $(\tau_e)_{e \in \mathbb{E}^d}$ are i.i.d., with a common c.d.f. F satisfying $\left| \frac{F(x)}{x} - a \right| \leq \frac{C}{|\log x|}$ for some constants a, C and sufficiently small x.

However, the proof of the upper bound, namely, Equation (2.1) in [2]

$$\limsup_{d \to \infty} \frac{\mu_d(e_1)ad}{\log d} \leq \frac{1}{2} \tag{0.1}$$

is incorrect. In this article, we provide a different approach that establishes (0.1). As a side product of this new method, we also show that the variance of the non-backtracking passage time to the first hyperplane is of order $o\big((\log d/d)^2\big)$ as $d \to \infty$ in the case of the when the edge weights are exponentially distributed.

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1 Introduction

In this paper we study first passage percolation on $\mathbb{Z}^d$ which is defined as follows. At each nearest-neighbor edge in $\mathbb{Z}^d$, we attach a non-negative random variable τ_e, known as the passage time of the edge e. These random variables (τ_e) are independent and identically distributed with common distribution F. We will also assume that F satisfies the following conditions:

$$\lim_{x \to 0} \frac{F(x)}{x} = a, \text{ for some } a > 0 \tag{1.1}$$

$$\text{and } \int x \, dF(x) < \infty. \tag{1.2}$$

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A *path* γ is a sequence of nearest-neighbor edges in $\mathbb{Z}^d$ such the starting point of each edge coincides with the endpoint of the previous edge. For any finite path γ we define the passage time of γ to be

$$T(\gamma) = \sum_{e \in \gamma} \tau_e.$$

Given two points $x, y \in \mathbb{Z}^d$ we define

$$T(x, y) = \inf_\gamma T(\gamma),$$

where the infimum is over all finite paths γ that start at the point x and end at y. For a review, we invite the readers to see the book [1] or the classical paper of Kesten [6].

If $\mathbb{E}\tau_e < \infty$, the following limit exists

$$\mu_d(e_1) := \lim_{n \to \infty} \frac{T(0, ne_1)}{n} \quad \text{a.s. and in } L^1.$$

and $\mu_d(e_1)$ is called the time constant. See [1, Theorem 2.1] and the discussion therein.

In this paper, we prove the following limit for $\mu_d(e_1)$, as the dimension $d \to \infty$.

Theorem 1.1. *Assume* (1.1) *and* (1.2). *Then*

$$\lim_{d \to \infty} \frac{\mu_d(e_1)d}{\log d} = \frac{1}{2a}.$$

This result was first claimed in [2], under an additional assumption on the convergence rate in (1.1). However, the proof for the upper bound there, namely,

$$\limsup_{d \to \infty} \frac{\mu_d(e_1)ad}{\log d} \le \frac{1}{2}$$

contains an error. Specifically, for the choices of p, n and x in [2, Equation (3.1)], the error term is not negligible compared to the main order $(\log d/d)$ as $d \to \infty$. Here, we fix this error by presenting a new method that also has consequences to point-to-hyperplane passage times.

The main result in this paper is the following.

Theorem 1.2. *Assume* (1.1) *and* (1.2). *The following bound holds:*

$$\limsup_{d \to \infty} \frac{\mu_d(e_1)d}{\log d} \le \frac{1}{2a}.$$

The lower bound was correctly established in [2], which we state below.

Proposition 1.3 ([2, Proposition 4.1]). *Assume* (1.1) *and* (1.2) *for the passage times, then*

$$\liminf_{d \to \infty} \frac{\mu_d(e_1)ad}{\log d} \ge \frac{1}{2}.$$

Proof of Theorem 1.1. It follows from the combination of Theorem 1.2 and Proposition 1.3. $\qquad\square$

As an outcome of the new method we introduce to prove Theorem 1.2, we also obtain the following theorem. Let

$$\tilde{s}_{0,1} := \inf \left\{ T(\gamma) : \begin{array}{l} \gamma : \mathbf{0} \to \mathbb{H}_1 \text{ such that except for the end point,} \\ \text{all other vertices on } \gamma \text{ are contained in } \mathbb{H}_0 \end{array} \right\},$$

where $\mathbb{H}_n := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_1 = n\}$ is the n-th hyperplane orthogonal to e_1.

Theorem 1.4. *Assume* (1.1) *and* (1.2). *Then, as* $d \to \infty$

$$\frac{2ad\tilde{s}_{0,1}}{\log d} \to 1 \text{ in probability and in } L^1.$$

Let us make a few comments about our strategy and how the rest of the paper is organized. Our approach will first consider (the edge version of) the Eden model [5], i.e., the FPP model where the passage times are i.i.d. Exponential(a). Under the Exponential setting, we will make use of Dhar's exploration idea (see [4]), and then combine with an appropriate coupling between the F-distribution and the Exponential(a) distribution (a similar coupling was also used in [8, Section 6]).

More precise, Dhar used in [4] a cluster exploration process to predict that for the Exponential(a) FFP model

$$\limsup_{d \to \infty} \frac{2ad}{\log d} \mathbb{E}\tilde{s}_{0,1} \leq 1. \tag{1.3}$$

By a standard subadditivity argument (see, e.g., [7, Theorem 4.2.5], [6, pp.246] or [9, Lemma 5.2]), we know that $\mu_d(e_1)$ is no larger than $\mathbb{E}\tilde{s}_{0,1}$. So Equation (1.3) implies an upper bound for $\mu_d(e_1)$ in the Exponential(a) case. Proposition 1.3 then implies that

$$\mathbb{E}\frac{2ad}{\log d}\tilde{s}_{0,1} \to 1.$$

The main obstacle that prevents us from directly generalizing this result to other distributions is that the exploration process will only provide convergence in expectation for $\tilde{s}_{0,1}$, which is not enough for our purposes. Thus, we first establish a convergence-in-probability result $\frac{2ad}{\log d}\tilde{s}_{0,1} \xrightarrow{P} 1$ for the Exponential(a)-weighted case by showing

$$\mathrm{Var}(\tilde{s}_{0,1}) = o\left(\left(\frac{\log d}{d}\right)^2\right), \quad \text{as } d \to \infty,$$

expanding on Dhar's cluster exploration idea (see Sections 2 and 3).

In Section 4, we derive a coupling between F-distributed FPP and the Eden model that preserves the convergence in probability. Finally, in Section 5, we show that the collection of random variables $\{(2ad/\log d)\,\tilde{s}_{0,1}\}_{d \geq 1}$ is uniformly integrable, which completes a proof of the L_1-convergence of $\frac{2ad}{\log d}\tilde{s}_{0,1} \to 1$ in the F-distributed case.

2   A recap of Dhar's exploration idea in the Exponential case

In this section, we recap Dhar's cluster exploration idea, where we introduce the notations and fill in some technical details that were omitted in [4]. Without loss of generality, we assume $a = 1$ and consider the standard Exponential case. Note that the exploration idea works nicely because in this case, the first-passage percolation model is Markovian: given the configuration of already-infected sites at any time t, the time until next infection is independent of the history before time t.

Consider any infected cluster $C \subset \mathbb{H}_0$ that contains i vertices and S perimeter edges within $\mathbb{H}_0$. Let $T(C)$ denote the waiting time until infection reaches $\mathbb{H}_1$ using a non-backtracking path (i.e., one of those paths in the definition of $\tilde{s}_{0,1}$). There are $i + S$ possible edges to cross for the next infection to happen, where i of them are along the e_1 direction and leading to $\mathbb{H}_1$ (denoted by $f_1, f_2, \ldots, f_i$) and S of them remain in the hyperplane $\mathbb{H}_0$ (denoted by $f_{i+1}, \ldots, f_{i+S}$). The passage times of these $i + S$ edges are i.i.d. Exponential(1) random variables. Let Y denote the edge being crossed when next infection occurs (i.e., one of the boundary edges of C with the smallest passage time),

and we have

$$\mathbb{E}[T(C)] = \mathbb{E}\left[\mathbb{E}[T(C)|Y]\right] = \sum_{j=1}^{i+S} \mathbb{E}[T(C)|Y = f_j]P(Y = f_j)$$

$$= \sum_{j=1}^{i+S} \mathbb{E}[T(C)|Y = f_j] \cdot \frac{1}{i+S}.$$

Given that $Y = f_1, \ldots, f_i$, then the conditional distribution of $T(C)$ is the same as the minimum of $(i + S)$ i.i.d. Exponential(1) random variables and thus

$$\mathbb{E}[T(C)|Y = f_j] = \frac{1}{i+S} \quad \text{for } j = 1, 2, \ldots, i.$$

Given that $Y = f_{i+1}, \ldots, f_{i+S}$, then the next infection will cross a perimeter edge of C within $\mathbb{H}_0$ and expand the infected cluster to a larger cluster C' of $(i + 1)$ vertices. In this case, the conditional distribution of $T(C)$ is the same as

$$\text{Exponential}(i + S) + T(C \cup f_j)$$

where the Exponential random variable is independent of $T(C \cup f_j)$, due to the Markov property. In this case, we have

$$\mathbb{E}[T(C)|Y = f_j] = \frac{1}{i+S} + \mathbb{E}[T(C \cup f_j)] \quad \text{for } j = i+1, i+2, \ldots, i+S.$$

Combining these together, we obtain

$$\mathbb{E}[T(C)] = \left\{ \frac{i}{i+S} + \sum_{j=i+1}^{i+S} \left[\frac{1}{i+S} + \mathbb{E}[T(C \cup f_j)] \right] \right\} \cdot \frac{1}{i+S}$$

$$= \frac{1}{i+S}\left[1 + \sum_{C'} \mathbb{E}[T(C')]\right],$$

where the sum is over all clusters $C' \subset \mathbb{H}_0$ of $(i + 1)$ infected vertices that can be obtained from cluster C by infecting one additional healthy vertex adjacent to C. For each $i = 1, 2, 3, \ldots$, define

$$x_i := \max_{C : C \subset \mathbb{H}_0, |C|=i} \mathbb{E}T(C).$$

Note that $x_1 = \mathbb{E}\tilde{s}_{0,1}$ is the quantity of interest here. Taking the maximum over all clusters C', we obtain

$$\mathbb{E}[T(C)] \le \frac{1}{i+S}[1 + Sx_{i+1}] = \frac{1}{i+S} + \frac{1}{i/S+1}x_{i+1}. \tag{2.1}$$

For any cluster $C \subset \mathbb{H}_0$ of i vertices, the number of its perimeter edges in $\mathbb{H}_0$ is bounded above by i times $2(d-1)$, the maximum number of edges in $\mathbb{H}_0$ adjacent to a given vertex. Meanwhile, by the edge-isoperimetric inequality on $\mathbb{Z}^d$ [3, Theorem 8], one has for any set in a box $\{0, \ldots, \ell\}^{d-1} \subset \mathbb{Z}^{d-1}$ of cardinality $i \le \ell^{d-1}/2$ the number of perimeter edges is bounded below by

$$\min_{1 \le k \le d-1} \left\{ 2ki^{1-\frac{1}{k}} \ell^{\frac{d-1}{k}-1} \right\}. \tag{2.2}$$

Choosing $\ell = i$, with $i > 3$, the minimum of (2.2) is achieved at $k = d - 1$ and thus we obtain the bounds

$$s_i := 2(d-1)i^{\frac{d-2}{d-1}} \le S \le (2d-1)i. \tag{2.3}$$

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Here, we also used the fact that the lower bound in (2.3) for $i = 1, 2, 3$ can be easily checked case by case as follows:

$$i = 1, \quad S = 2(d-1) \geq s_1,$$
$$i = 2, \quad S = 4(d-1) - 2 \geq s_2,$$
$$i = 3, \quad S = 6(d-1) - 4 \geq s_3,$$

as long as d is large enough. Therefore, using (2.3), we have

$$\frac{1}{i+S} \leq \frac{1}{i+s_i} \quad \text{and} \quad \frac{1}{i/S+1} \leq \frac{1}{1/(2d-1)+1} = 1 - \frac{1}{2d}. \tag{2.4}$$

Thus, we bound the right side of (2.1) by

$$\mathbb{E}[T(C)] \leq \frac{1}{i+s_i} + \left(1 - \frac{1}{2d}\right) x_{i+1}.$$

Write $A = 1 - \frac{1}{2d}$ for notation simplicity. Now taking the maximum over all possible cluster $C \subset \mathbb{H}_0$ with i vertices yields an iterative inequality

$$x_i \leq \frac{1}{i+s_i} + A x_{i+1}. \tag{2.5}$$

Iterating the relation, we get an upper bound for $\mathbb{E}\tilde{s}_{0,1}$

$$\mathbb{E}\tilde{s}_{0,1} = x_1 \leq \frac{1}{1+s_1} + A x_2 \leq \frac{1}{1+s_1} + A\left[\frac{1}{2+s_2} + A x_3\right]$$
$$\leq \cdots \leq \frac{1}{1+s_1} + \frac{A}{2+s_2} + \frac{A^2}{3+s_3} + \cdots \leq \frac{1}{1+s_1} + \sum_{n=2}^{\infty} \frac{A^{n-1}}{s_n}. \tag{2.6}$$

The reason to single out the first term in the summation is to avoid integrating from 0 when bounding the infinite sum by an integral in the next step. Plugging in the expression for s_i, we get

$$\mathbb{E}\tilde{s}_{0,1} \leq \frac{1}{2d-1} + \frac{1}{2(d-1)} \sum_{n=2}^{\infty} A^{n-1} \frac{1}{n^{\frac{d-2}{d-1}}}$$
$$\leq \frac{1}{2d-1} + \frac{1}{2(d-1)} \int_1^{\infty} A^{x-1} \frac{1}{x^{\frac{d-2}{d-1}}} dx. \tag{2.7}$$

We break the integral above into two parts: $\int_1^{2d}$ and $\int_{2d}^{\infty}$. For the first integral, we have

$$\int_1^{2d} A^{x-1} \frac{1}{x^{\frac{d-2}{d-1}}} dx = \int_1^{2d} A^{x-1} \frac{x^{\frac{1}{d-1}}}{x} dx \leq \frac{(2d)^{\frac{1}{d-1}}}{A} \int_1^{\infty} \frac{A^x}{x} dx.$$

Notice that as $d \to \infty$, $(2d)^{\frac{1}{d-1}} \to 1$, $A \to 1$ and $\int_1^{\infty} \frac{A^x}{x} dx \sim \log d$. To see the last asymptotic, we apply L'Hopital's rule and get

$$\lim_{d\to\infty} \frac{\int_1^{\infty} A^x \frac{1}{x} dx}{\log d} = \lim_{d\to\infty} \frac{\frac{1}{2d^2 A} \int_1^{\infty} A^x dx}{1/d}$$
$$= \lim_{d\to\infty} \frac{1}{2dA} \cdot \frac{(1-\frac{1}{2d})^x\big|_{x=1}^{\infty}}{\log(1-1/(2d))} = \lim_{d\to\infty} \frac{1}{2d} \cdot \frac{\frac{1}{2d}-1}{-\frac{1}{2d}} = 1. \tag{2.8}$$

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For the second integral, we get

$$\int_{2d}^{\infty}\left(1-\frac{1}{2d}\right)^{x-1}\frac{1}{x^{\frac{d-2}{d-1}}}dx \le \frac{1}{(2d)^{\frac{d-2}{d-1}}}\int_{2d}^{\infty}\left(1-\frac{1}{2d}\right)^{x-1}dx$$

$$= \frac{(2d)^{\frac{1}{d-1}}}{2d}\cdot\frac{(1-\frac{1}{2d})^x\big|_{x=2d}^{\infty}}{\log(1-1/(2d))}$$

$$= (2d)^{\frac{1}{d-1}}\cdot\frac{-(1-\frac{1}{2d})^{2d}}{2d\log(1-1/(2d))}\to e^{-1}.$$

Thus, combining the two integrals, we have as $d\to\infty$,

$$\int_{1}^{\infty}\left(1-\frac{1}{2d}\right)^{x-1}\frac{1}{x^{\frac{d-2}{d-1}}}dx \sim \log d.$$

Plugging this back to (2.7), we prove $\mathbb{E}\tilde{s}_{0,1}\le\frac{\log d}{2d}(1+o(1))$ for d large. Combining with Proposition 1.3, we conclude

$$\mathbb{E}\tilde{s}_{0,1}\sim\frac{\log d}{2d}$$

as $d\to\infty$ in the i.i.d. Exponentially-weighted case.

3 Iteration of the second moment in the Exponential case

In this section, we prove that $\mathbb{E}\tilde{s}_{0,1}^2\le\left(\frac{\log d}{2d}\right)^2(1+o(1))$ as $d\to\infty$ in the Exponential case. Thus, combining with the first moment result, $\mathbb{E}\tilde{s}_{0,1}\sim\frac{\log d}{2d}$, we achieve $\mathrm{Var}(\tilde{s}_{0,1})=o\left(\left(\frac{\log d}{d}\right)^2\right)$ as desired. Like before, consider any already-infected cluster $C\subset\mathbb{H}_0$ with i infected vertices and S perimeter edges within $\mathbb{H}_0$, and define $T(C)$ and Y in the same way. For each $i=1,2,3,\ldots$, define

$$y_i := \max_{C:C\subset\mathbb{H}_0,|C|=i}\mathbb{E}T^2(C).$$

We now derive an iterative inequality that relates y_i with y_{i+1} (and also x_{i+1}, see below). Again, using law of total expectation, we get

$$\mathbb{E}[T^2(C)]=\mathbb{E}\left[\mathbb{E}[T^2(C)|Y]\right]=\sum_{j=1}^{i+S}\mathbb{E}[T^2(C)|Y=f_j]P(Y=f_j)$$

$$=\sum_{j=1}^{i+S}\mathbb{E}[T^2(C)|Y=f_j]\cdot\frac{1}{i+S}.$$

Like before, given $Y=f_j$ for $j=1,\ldots,i$, then $T(C)$ follows an Exponential$(i+S)$ distribution and

$$\mathbb{E}[T^2(C)|Y=f_j]=\frac{2}{(i+S)^2},\ \ j=1,2,\ldots,i,$$

whereas given $Y=f_j$ for $j=i+1,\ldots,i+S$, $T(C)$ has the same distribution as Exponential$(i+S)+T(C\cup f_j)$ distribution. Thus, recalling that $x_i=\max_{C:C\subset\mathbb{H}_0,|C|=i}\mathbb{E}T(C)$, we have

$$\mathbb{E}[T^2(C)|Y=f_j]=\frac{2}{(i+S)^2}+\mathbb{E}[T^2(C\cup f_j)]+2\cdot\frac{1}{i+S}\cdot\mathbb{E}[T(C\cup f_j)]$$

$$\le\frac{2}{(i+S)^2}+y_{i+1}+\frac{2}{i+S}x_{i+1},\ \ j=i+1,\ldots,i+S.$$

Therefore, we get

$$\mathbb{E}[T^2(C)] \leq \left\{ \frac{2i}{(i+S)^2} + \frac{2S}{(i+S)^2} + Sy_{i+1} + \frac{2S}{i+S}x_{i+1} \right\} \cdot \frac{1}{i+S}$$

$$= \frac{2}{(i+S)^2}(1 + Sx_{i+1}) + \frac{S}{i+S}y_{i+1}$$

$$\leq \frac{2}{(i+s_i)^2} + \frac{2}{i+s_i}\left(1 - \frac{1}{2d}\right)x_{i+1} + \left(1 - \frac{1}{2d}\right)y_{i+1}$$

where, in the last step, we use (2.4) again to make sure that the right hand side does not depend on the boundary size S of the cluster C. Again, we write $A = 1 - \frac{1}{2d}$. Now taking the maximum over all clusters of size i on the left hand side, we get an iterative inequality

$$y_i \leq \frac{2}{(i+s_i)^2} + \frac{2A}{i+s_i}x_{i+1} + Ay_{i+1}. \tag{3.1}$$

Keeping iterating this relation together with (2.5) and noticing that $\mathbb{E}\tilde{s}_{0,1}^2 = y_1$, we achieve an upper bound for $\mathbb{E}\tilde{s}_{0,1}^2$, which is of the following form

$$\mathbb{E}\tilde{s}_{0,1}^2 = y_1 \leq \frac{2}{(1+s_1)^2} + \frac{2A}{1+s_1}x_2 + Ay_2$$

$$\leq \frac{2}{(1+s_1)^2} + \frac{2A}{1+s_1}\left[\frac{1}{2+s_2} + Ax_3\right] + A\left[\frac{2}{(2+s_2)^2} + \frac{2A}{2+s_2}x_3 + Ay_3\right]$$

$$= \frac{2}{(1+s_1)^2} + \frac{2A}{2+s_2}\left[\frac{1}{1+s_1} + \frac{1}{2+s_2}\right] + 2A^2\left[\frac{1}{1+s_1} + \frac{1}{2+s_2}\right]x_3 + A^2y_3$$

$$\leq \frac{2}{(1+s_1)^2} + \frac{2A}{2+s_2}\left[\frac{1}{1+s_1} + \frac{1}{2+s_2}\right] + \frac{2A^2}{3+s_3}\left[\frac{1}{1+s_1} + \frac{1}{2+s_2} + \frac{1}{3+s_3}\right]$$

$$+ 2A^3\left[\frac{1}{1+s_1} + \frac{1}{2+s_2} + \frac{1}{3+s_3}\right]x_4 + A^3y_4 \leq \cdots$$

$$\leq 2\sum_{n=1}^{\infty}\frac{A^{n-1}}{n+s_n}\sum_{k=1}^{n}\frac{1}{k+s_k} = 2\sum_{k=1}^{\infty}\frac{1}{k+s_k}\sum_{n=k}^{\infty}\frac{A^{n-1}}{n+s_n}.$$

where each inequality follows plugging-in Equations (2.5) and (3.1), and the equalities are simple expansions of the brackets. Again, we use integral to bound the double sum from above. First, for each k fixed, $\frac{A^{n-1}}{n+s_n}$ decreases in n, and thus the inner sum over n is no more than

$$\sum_{n=k}^{\infty}\frac{A^{n-1}}{n+s_n} \leq \int_{k-1}^{\infty}\frac{A^{y-1}}{y+s_y}dy.$$

Moreover, since $\frac{1}{k+s_k} \leq \frac{1}{s_k}$ and $\frac{1}{s_k}\int_{k-1}^{\infty}\frac{A^{y-1}}{s_y}dy$ decreases in k, we have

$$\mathbb{E}\tilde{s}_{0,1}^2 \leq 2\sum_{k=1}^{\infty}\frac{1}{k+s_k}\sum_{n=k}^{\infty}\frac{A^{n-1}}{n+s_n}$$

$$\leq \frac{2}{1+s_1}\sum_{n=1}^{\infty}\frac{A^{n-1}}{n+s_n} + \frac{2}{2+s_2}\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n} + 2\sum_{k=3}^{\infty}\frac{1}{s_k}\sum_{n=k}^{\infty}\frac{A^{n-1}}{s_n}$$

$$\leq \frac{2}{1+s_1}\sum_{n=1}^{\infty}\frac{A^{n-1}}{n+s_n} + \frac{2}{2+s_2}\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n} + 2\int_{2}^{\infty}\frac{1}{s_x}\int_{x-1}^{\infty}\frac{A^{y-1}}{s_y}dydx.$$

Here, the reason to single out the first two terms is again to avoid integrating from 0 when applying an integral approximation in the next step. We note that $s_1 = 2(d-1)$ and

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$s_2 \geq d$, thus the first two terms are smaller order of $(\log d/d)^2$, i.e.,

$$\frac{2}{1+s_1}\sum_{n=1}^{\infty}\frac{A^{n-1}}{n+s_n} \leq \frac{2}{(1+s_1)^2} + \frac{2}{1+s_1}\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n}$$

$$\leq \frac{2}{(2d-1)^2} + \frac{2}{2d-1}\cdot\frac{C\log d}{d} = o\big((\log d/d)^2\big), \text{ and}$$

$$\frac{1}{2+s_2}\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n} \leq \frac{1}{d}\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n} \leq \frac{C\log d}{d^2} = o\big((\log d/d)^2\big),$$

where we used the fact that $\sum_{n=2}^{\infty}\frac{A^{n-1}}{s_n} \leq \frac{C\log d}{d}$; see the last term in (2.6). It remains to show that for d sufficiently large

$$2\int_2^{\infty}\frac{1}{s_x}\left(\int_{x-1}^{\infty}\frac{A^{y-1}}{s_y}dy\right)dx \leq \left(\frac{\log d}{2d}\right)^2(1+o(1)).$$

We break the integral into three parts:

$$\text{(I)} := 2\int_2^{2d+1}\frac{1}{s_x}\left(\int_{x-1}^{2d}\frac{A^{y-1}}{s_y}dy\right)dx,$$

$$\text{(II)} := 2\int_2^{2d+1}\frac{1}{s_x}\left(\int_{2d}^{\infty}\frac{A^{y-1}}{s_y}dy\right)dx,$$

$$\text{(III)} := 2\int_{2d+1}^{\infty}\frac{1}{s_x}\left(\int_{x-1}^{\infty}\frac{A^{y-1}}{s_y}dy\right)dx.$$

For (III), when $x \geq 2d+1$ and $y \geq x-1 \geq 2d$, both s_x and s_y are greater than $s_{2d} = 2(d-1)(2d)^{\frac{d-2}{d-1}}$. Thus, we replace them by s_{2d} and evaluate the integrals to get

$$\text{(III)} \leq \frac{2}{s_{2d}^2}\int_{2d+1}^{\infty}\int_{x-1}^{\infty}A^{y-1}dydx = \frac{-2}{As_{2d}^2\cdot\log A}\int_{2d+1}^{\infty}A^{x-1}dx$$

$$= \frac{-2}{As_{2d}^2\cdot\log A}\int_{2d}^{\infty}A^x dx = \frac{2A^{2d}}{As_{2d}^2\cdot(\log A)^2}.$$

Note that $2A^{2d} \to 2e^{-1}$ and $(\log A)^2 \sim (-\frac{1}{2d})^2 = \frac{1}{4d^2}$, thus we get for all d sufficiently large,

$$\text{(III)} \leq \frac{C_1\cdot 4d^2}{[2(d-1)(2d)^{\frac{d-2}{d-1}}]^2} \sim \frac{C_2 d^{\frac{2}{d-1}}}{d^2} \sim \frac{C_3}{d^2} = o\big((\log d/d)^2\big),$$

where C_1, C_2 and C_3 are absolute constants that do not depend on d and may vary from line to line. For (II), again since $s_y \geq s_{2d}$ when $y \geq 2d$ and $s_x = (2d-1)x^{\frac{d-2}{d-1}}$, we have

$$\text{(II)} \leq \frac{2}{s_{2d}}\int_2^{2d+1}\frac{1}{s_x}\int_{x-1}^{\infty}A^{y-1}dydx = \frac{-2}{A^2 s_{2d}\cdot\log A}\int_2^{2d+1}\frac{A^x}{s_x}dx$$

$$\leq \frac{-2}{2(d-1)A^2 s_{2d}\cdot\log A}\int_2^{2d+1}\frac{A^x x^{\frac{1}{d-1}}}{x}dx$$

$$\leq \frac{-2(2d+1)^{\frac{1}{d-1}}}{2(d-1)A^2 s_{2d}\cdot\log A}\int_2^{2d+1}\frac{A^x}{x}dx$$

$$\leq \frac{-2(2d+1)^{\frac{1}{d-1}}}{2(d-1)A^2 s_{2d}\cdot\log A}\int_1^{\infty}\frac{A^x}{x}dx.$$

The rightmost integral is of order $\log d$ as $d \to \infty$, following (2.8). Thus for all sufficiently large d,

$$\text{(II)} \leq \frac{-C_1(2d+1)^{\frac{1}{d-1}}}{2(d-1)A^2 s_{2d} \cdot \log A} \log d.$$

As $d \to \infty$, $(2d+1)^{\frac{1}{d-1}} \to 1$, $2(d-1)\log A \to -1$ and $A \to 1$, thus for all d large,

$$\text{(II)} \leq \frac{C_2 \log d}{s_{2d}} = \frac{C_3 d^{\frac{1}{d-1}} \log d}{d^2} = O(\log d/d^2) = o\big((\log d/d)^2\big).$$

Finally, for (I), we plug in the expression for s_x and s_y and use the fact that x, y are both less than $2d+1$, and

$$\text{(I)} \leq \frac{(2d+1)^{\frac{2}{d-1}}}{4A(d-1)^2} \int_1^\infty \frac{2}{x} \int_{x-1}^\infty \frac{A^y}{y} dy dx.$$

The double integral is $\sim (\log d)^2$ as $d \to \infty$. To see this, we apply L'Hopital's rule and take the derivative with respect to d:

$$\lim_{d\to\infty} \frac{\int_1^\infty \frac{2}{x} \int_{x-1}^\infty \frac{A^y}{y} dy dx}{(\log d)^2} = \lim_{d\to\infty} \frac{\frac{1}{2d^2} \int_1^\infty \frac{1}{x} \int_{x-1}^\infty A^{y-1} dy dx}{(\log d) \cdot \frac{1}{d}}$$

$$= \lim_{d\to\infty} \frac{\frac{-1}{2dA\log A} \int_1^\infty \frac{A^{x-1}}{x} dx}{\log d} = \lim_{d\to\infty} \frac{\frac{-1}{2dA^2\log A} \int_1^\infty \frac{A^x}{x} dx}{\log d} = 1,$$

where the last equality follows from (2.8), $2d \log A \to -1$, and $A \to 1$ as $d \to \infty$. Plugging this back to the upper bound of (I), we obtain that for all sufficiently large d,

$$\text{(I)} \leq \frac{(2d+1)^{\frac{2}{d-1}}}{4A(d-1)^2} (\log d)^2 (1+o(1)) = \left(\frac{\log d}{2d}\right)^2 (1+o(1)),$$

which concludes the proof for $\mathbb{E}\tilde{s}_{0,1}^2 \leq \left(\frac{\log d}{2d}\right)^2 (1+o(1))$ in the Exponentially-weighted case.

4 Generalization to edge-weight distribution F

Recall that if X is an Exponential(1) random variable, then X/a follows an Exponential(a) distribution. Thus if the passage times are i.i.d. Exponential(a) distributed, we have

$$\mathbb{E}\tilde{s}_{0,1} \sim \frac{\log d}{2ad}, \quad \mathbb{E}\tilde{s}_{0,1}^2 \leq \left(\frac{\log d}{2ad}\right)^2 (1+o(1)),$$

which implies $\text{Var}(\tilde{s}_{0,1}) = o\big((\log d/d)^2\big)$ and $\frac{2ad}{\log d}\tilde{s}_{0,1} \xrightarrow{P} 1$ as $d \to \infty$. In this section, we generalize this in-probability convergence to other distributions F satisfying (1.1).

Since we are working with two different probability distributions in this section, we will always write a superscript F when we work with F-distributed passage times; the superscript is omitted when the passage times are Exponential(a) distributed. We couple the distribution F with an Exponential(a) distribution using the left-continuous inverse function $F^* : [0,1] \to \mathbb{R}$ of F, i.e., for any $y \in [0,1]$

$$F^*(y) := \inf\{x : F(x) \geq y\}.$$

It follows that if $(\tau_e)_{e\in\mathbb{E}}$ are i.i.d. Exponential(a)-distributed edge weights in a first-passage percolation model, then $(\tau_e^F)_{e\in\mathbb{E}} := (h(\tau_e))_{e\in\mathbb{E}}$ are i.i.d. F-distributed, where

$$h(t) := F^*(1 - e^{-at}) = \inf\{x \geq 0 : F(x) \geq 1 - e^{-at}\}.$$

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Note that h is a monotonically increasing function. Also, since $F(x) \sim ax$ as $x \to 0$, then $F^*(y) \sim y/a$ as $y \to 0$. This implies $\lim_{t \to 0} h(t)/t = 1$, i.e., for any $\epsilon > 0$, there is $\delta = \delta_\epsilon > 0$ such that

$$(1 - \epsilon)t \leq h(t) \leq (1 + \epsilon)t, \quad \text{for all } t \in [0, \delta]. \tag{4.1}$$

Denote by $\tilde{s}_{0,1}^F$ in the same way as $\tilde{s}_{0,1}$ but for the case when the passage times are i.i.d. F-distributed. Let Γ be one path that realizes $\tilde{s}_{0,1}$ in the Exponential(a) case. Then, for any $\delta > 0$,

$$P(\tau_e \leq \delta \text{ for all } e \in \Gamma) \to 1. \tag{4.2}$$

This is because the complement of the event satisfies

$$P(\tau_e > \delta \text{ for some } e \in \Gamma) \leq P(\tilde{s}_{0,1} > \delta) = P(2ad\tilde{s}_{0,1}/\log d > 2\delta ad/\log d) \to 0,$$

as we have already showed that $\frac{2ad}{\log d}\tilde{s}_{0,1} \xrightarrow{P} 1$ as $d \to \infty$.

The main result of this section is the following.

Proposition 4.1. *Assume* (1.1). *As d goes to infinity, we have*

$$\frac{2ad}{\log d}\tilde{s}_{0,1}^F \xrightarrow{P} 1.$$

Proof of Proposition 4.1. We start with an upper bound for $\tilde{s}_{0,1}^F$. Consider any $\eta > 0$. Choose $\epsilon \in (0, \eta)$ and $\delta > 0$ according to ϵ such that (4.1) is satisfied. Then

$$
\begin{aligned}
P\left(\frac{2ad}{\log d}\tilde{s}_{0,1}^F \leq 1 + \eta\right) &\geq P\left(\frac{2ad}{\log d}\tilde{s}_{0,1}^F \leq 1 + \eta, \tau_e \leq \delta \text{ for all } e \in \Gamma\right) \\
&\geq P\left(\frac{2ad}{\log d}T^F(\Gamma) \leq 1 + \eta, \tau_e \leq \delta \text{ for all } e \in \Gamma\right),
\end{aligned}
$$

where $T^F(\Gamma)$ denotes the passage time along the path Γ with edge weights $(\tau_e^F)_{e \in \mathbb{E}} := (h(\tau_e))_{e \in \mathbb{E}}$

$$T^F(\Gamma) := \sum_{e \in \Gamma} \tau_e^F = \sum_{e \in \Gamma} h(\tau_e).$$

On the event that $\{\tau_e \leq \delta \text{ for all } e \in \Gamma\}$, we have $h(\tau_e) \leq (1 + \epsilon)\tau_e$, and thus $\sum_{e \in \Gamma} h(\tau_e) \leq (1 + \epsilon) \sum_{e \in \Gamma} \tau_e = (1 + \epsilon)\tilde{s}_{0,1}$. Thus, the probability above is bounded from below by

$$P\left(\frac{2ad}{\log d}\tilde{s}_{0,1}^F \leq 1 + \eta\right) \geq P\left(\frac{2ad}{\log d}\tilde{s}_{0,1} \leq \frac{1 + \eta}{1 + \epsilon}, \tau_e \leq \delta \text{ for all } e \in \Gamma\right) \to 1,$$

due to $\frac{1+\eta}{1+\epsilon} > 1$, $\frac{2ad}{\log d}\tilde{s}_{0,1} \xrightarrow{P} 1$ and (4.2). The lower bound is similar. Let Γ^F be one path that realizes $\tilde{s}_{0,1}^F$ in the case that edge weights are i.i.d. F-distributed. Take any $\eta > 0$ and $\epsilon \in (0, \eta)$, and choose $\delta = \delta_\epsilon$ according to ϵ such that (4.1) is satisfied. Then

$$\tilde{s}_{0,1}^F = \sum_{e \in \Gamma^F} \tau_e^F = \sum_{e \in \Gamma^F} h(\tau_e)$$

$$
\begin{aligned}
P\left(\frac{2ad}{\log d}\tilde{s}_{0,1}^F \le 1-\eta\right) &= P\left(\frac{2ad}{\log d}\sum_{e \in \Gamma^F} h(\tau_e) \le 1-\eta\right) \\
&= P\left(\frac{2ad}{\log d}\sum_{e \in \Gamma^F} h(\tau_e) \le 1-\eta,\ \tau_e \le \delta \text{ for all } e \in \Gamma^F\right) \\
&\quad + P\left(\frac{2ad}{\log d}\sum_{e \in \Gamma^F} h(\tau_e) \le 1-\eta,\ \tau_e > \delta \text{ for some } e \in \Gamma^F\right) \\
&\le P\left(\frac{2ad}{\log d}\sum_{e \in \Gamma^F} (1-\epsilon)\tau_e \le 1-\eta,\ \tau_e \le \delta \text{ for all } e \in \Gamma^F\right) \\
&\quad + P\left(\frac{2ad}{\log d} h(\delta) \le 1-\eta,\ \tau_e > \delta \text{ for some } e \in \Gamma^F\right) \\
&\le P\left(\frac{2ad}{\log d}\tilde{s}_{0,1} \le \frac{1-\eta}{1-\epsilon}\right) + P\left(\frac{2ad}{\log d}(1-\epsilon)\delta \le 1-\eta\right).
\end{aligned}
$$

The first term vanishes as $d \to \infty$ because $\frac{2ad}{\log d}\tilde{s}_{0,1} \xrightarrow{P} 1$ and $\frac{1-\eta}{1-\epsilon} < 1$; and the second term is 0 when d is sufficiently large. This complete the proof for $\frac{2ad}{\log d}\tilde{s}_{0,1}^F \xrightarrow{P} 1$. $\qquad\square$

5 Proofs of Theorems 1.2 and 1.4

In this section, we prove Theorem 1.2 and Theorem 1.4. For this, we show the following proposition.

Proposition 5.1. *Assume* (1.1) *and* (1.2). *Then, as* $d \to \infty$,

$$\frac{2ad}{\log d}\mathbb{E}\tilde{s}_{0,1}^F \to 1.$$

Proof of Proposition 5.1. In view of Proposition 4.1, it suffices to show that the collection of random variables $\{X_d\}_{d \ge 1} := \{\frac{2ad}{\log d}\tilde{s}_{0,1}^F\}_{d \ge 1}$ is uniformly integrable, i.e.,

$$\lim_{M \to \infty} \sup_d \mathbb{E}\left[X_d \mathbb{1}_{\{X_d \ge M\}}\right] = 0. \tag{5.1}$$

To do this, we adopt the "search-and-cross" strategy that were used in [6] and [2] when estimating $\mathbb{E}\tilde{s}_{0,1}^F$, which we briefly describe below. In order to get to $\mathbb{H}_1$ from $\mathbf{0}$ quickly, one can first make a move in one of the $e_{p+2}, \dots, e_d$ directions and then search for a fast path γ (of length n) in a subspace of $\mathbb{Z}^d$ spanned by $\{\pm e_2, \dots, \pm e_{p+1}\}$ that ends with the last step in the e_1 direction leading to $\mathbb{H}_1$. For $j = p+2, \dots, d$, if the first step has passage time $\tau_{e_j}^F \le y$ and the path $T^F(\gamma) \le x$ (denote this event by F_j), follow e_j and then this path γ to get to $\mathbb{H}_1$; otherwise move directly from $\mathbf{0}$ to $\mathbb{H}_1$ using the edge in the e_1 direction. Thus $\tilde{s}_{0,1}^F$ is bounded from above by

$$\tilde{s}_{0,1}^F \le (x+y)\mathbb{1}_{\cup_{j=p+2,\dots,d}^{\infty} F_j} + \mathbb{1}_{\cap_{j=p+2,\dots,d}^{\infty} F_j^c}\tau_{e_1}^F$$

Note that the choices for x, y, the length n of the fast path γ, and the dimension p of the search space are different in [6] and [2]. For example, in [6], Kesten chose

$$p = \left\lfloor\frac{d}{2}\right\rfloor, \quad n = \left\lfloor\frac{3}{4}\log d\right\rfloor, \quad x = \frac{9\log d}{4ad},$$

and with these choices, he showed that the probability of finding such a fast path γ is at least $\frac{1}{4}$, based on a second-moment method. These arguments rely on a rate $O(|\log x|^{-1})$

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for the convergence in (1.1), which ensured that there were sufficiently many fast paths as $d \to \infty$; see [6, pp.245] for the details on the moment computation. Thus we first assume that the convergence in (1.1) has a logarithmic rate, that is, for some $C, \epsilon_0 > 0$, $|F(x)/x - a| < C/|\log x|$ for all $x \in [0, \epsilon_0]$. We also adopt Kesten's choices for p, n and x, but other choices can also work. Moreover, for small enough y, the probability $P(\tau_{e_j}^F \le y) \sim ay$ due to (1.1). Thus, by choosing $y = 32 \log d/(ad)$, we have that for all d sufficiently large (say, $d > d_0$),

$$P(\tau_{e_j}^F \le y) \ge \frac{ay}{2} = \frac{16 \log d}{d},$$

which leads to

$$P(F_j) \ge \frac{1}{4} \cdot P(\tau_{e_j}^F \le y) \ge \frac{4 \log d}{d}$$

because for $j = p+2, \ldots, d$, the random variable $\tau_{e_j}^F$ is independent of the edge weights along the fast path γ. It then follows that

$$\mathbb{E}\left[X_d \mathbb{1}_{\{X_d \ge M\}}\right] \le \frac{2ad}{\log d} \mathbb{E}\left[(x+y)\mathbb{1}_{\cup_{j=p+2,\ldots,d}^{\infty} F_j} \mathbb{1}_{\{\tilde{s}_{0,1}^F \ge M \log d/(2ad)\}}\right]$$

$$+ \frac{2ad}{\log d} \mathbb{E}\left[\mathbb{1}_{\cap_{j=p+2,\ldots,d}^{\infty} F_j^c} \mathbb{1}_{\{\tilde{s}_{0,1}^F \ge M \log d/(2ad)\}}\right] \mathbb{E}(\tau_{e_1}^F).$$

If we choose $M \ge 100$, then the event $\tilde{s}_{0,1}^F > \frac{M \log d}{2ad} > x+y$ implies that none of the F_j events would happen, and the first term above is zero. Thus, for $M \ge 100$, we have

$$\mathbb{E}\left[X_d \mathbb{1}_{\{X_d \ge M\}}\right] \le \frac{2ad}{\log d} \mathbb{E}\left[\mathbb{1}_{\cap_{j=p+2,\ldots,d}^{\infty} F_j^c}\right] \mathbb{E}(\tau_{e_1}^F)$$

$$\le \frac{2ad}{\log d} \left(1 - \frac{4 \log d}{d}\right)^{d-p-1} \mathbb{E}(\tau_{e_1}^F)$$

$$\le \frac{2ad}{\log d} \left(1 - \frac{4 \log d}{d}\right)^{\frac{d}{3}} \mathbb{E}(\tau_{e_1}^F) \le \frac{2ad}{\log d} \cdot Cd^{-\frac{4}{3}} \to 0, \quad \text{as } d \to \infty.$$

Therefore, for any $\epsilon > 0$, choose d_1 such that $\mathbb{E}\left[X_d \mathbb{1}_{\{X_d \ge 100\}}\right] \le \epsilon/2$ for all $d > d_1$. Then for $M \ge 100$,

$$\sup_d \mathbb{E}[X_d \mathbb{1}_{\{X_d \ge M\}}] \le \sup_{d \le d_1} \mathbb{E}[X_d \mathbb{1}_{\{X_d \ge M\}}] + \sup_{d > d_1} \mathbb{E}[X_d \mathbb{1}_{\{X_d \ge 100\}}]$$

$$\le \sup_{d \le d_1} \mathbb{E}[X_d \mathbb{1}_{\{X_d \ge M\}}] + \frac{\epsilon}{2}.$$

The first term vanishes by sending $M \to \infty$. This finishes the proof of uniform integrability (5.1) under an additional logarithmic convergence rate assumption for (1.1). Now consider an arbitrary edge-weight distribution F satisfying (1.1) and (1.2), and define $\tilde{F}(x) := \min\{F(x), 1 - e^{-ax/2}\}$. Then $\tilde{F}$ satisfies $|\tilde{F}(x)/x - a/2| < C/|\log x|$ for all x small (1.2). Thus $\{\frac{ad}{\log d} \tilde{s}_{0,1}^{\tilde{F}}\}_{d \ge 1}$ is uniformly integrable, as we already shown above. Note that $\tilde{F}$ stochastically dominates F, which implies the stochastic dominance of $\tilde{s}_{0,1}^{\tilde{F}}$ over $\tilde{s}_{0,1}^F$, by a standard coupling argument. It follows that $\{\frac{ad}{\log d} \tilde{s}_{0,1}^F\}_{d \ge 1}$ is uniformly integrable, and so is $\{\frac{2ad}{\log d} \tilde{s}_{0,1}^F\}_{d \ge 1}$. □

The proofs of our two main theorems are now straight-forward.

Proof of Theorem 1.2. It follows by combining Propositions 5.1 with the fact that $\mu_d^F(e_1) \le \mathbb{E}\tilde{s}_{0,1}^F$, see [6, pp.246] or [9, Lemma 5.2]. □

Proof of Theorem 1.4. The proof follows by combining Proposition 4.1 and the uniform integrability (5.1). □

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