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The nonabelian Hodge correspondence for balanced Hermitian metrics of Hodge-Riemann type

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This paper extends the nonabelian Hodge correspondence for Kähler manifolds to a larger class of hermitian metrics on complex manifolds called balanced of Hodge-Riemann type. Important examples include balanced metrics arising from multipolarizations of Kähler manifolds. The generalizations rely on a few key observations showing that the known results, especially the Donaldson-Uhlenbeck-Yau theorem and Corlette's theorem, remain valid in this setting. Though not necessarily Kähler, it is shown that the Sampson-Siu Theorem proving that harmonic maps are pluriharmonic holds for a slightly more restrictive class of metrics.

1. Introduction

Let X be a compact, complex manifold of dimension n. Recall that a hermitian metric on X is called *balanced* if $d\omega^{n-1} = 0$, where ω is the fundamental (Kähler) $(1,1)$-form of the metric. The balanced metrics are a more restrictive class than the *Gauduchon* metrics, which satisfy $\partial\bar{\partial}\omega^{n-1} = 0$. Nevertheless, there are many examples of balanced, non-Kähler, metrics (cf. [22, p. 292]).

In this paper we consider a further condition. We say that a balanced metric is of *Hodge-Riemann type*, if it admits an expression:

$$\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \Omega_0 \tag{1.1}$$

where ω_0 (resp. Ω_0) is a real $(1,1)$ (resp. $(n-2, n-2)$) form, and Ω_0 satisfies the Hodge-Riemann bilinear relations (see Definition 2.1 for the precise definition).

The condition of being balanced of Hodge-Riemann type seems very restrictive. However, many examples of non-Kähler metrics satisfying this property come from *multipolarizations*. Namely, let $\omega_0, \omega_1, \ldots, \omega_{n-2}$ be positive $(1,1)$-forms on X, and suppose

$$\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \cdots \wedge \omega_{n-2} \tag{1.2}$$

such that

- $d\omega^{n-1} = 0$;
- $d(\omega_1 \wedge \cdots \wedge \omega_{n-2}) = 0$

(e.g. both conditions are automatic if the ω_i are Kähler). Then by a result of Timorin [27], ω is of Hodge-Riemann type. Even when the ω_i are Kähler metrics, ω is not so in general.

In this note we show that certain properties of Hermitian-Einstein metrics and equivariant harmonic maps familiar for Kähler manifolds continue to hold for balanced metrics of Hodge-Riemann type. Namely, we prove

1) a generalized Bogomolov-Gieseker inequality for ω-polystable holomorphic (and Higgs) bundles (Corollary 3.4);
2) a version of the Sampson-Siu pluriharmonicity theorem for harmonic maps to targets with nonpositive complexified sectional curvature (Theorem 4.1 and Corollary 4.3);
3) the nonabelian Hodge correspondence relating ω-stable Higgs bundles with vanishing Chern classes to irreducible representations of the fundamental group (Theorem 5.1).

Let us remark that for the class of Gauduchon metrics, items (2) and (3) do *not* hold in general (see [2])[1], and the statement of item (1) cannot even be formulated.

The simple idea behind these generalizations is well explained in [25, Lemma 1.1]. Let us focus on item (3) above. Suppose D be a complex connection on a vector bundle $E \to X$. A hermitian metric h on E gives a decomposition $D = D'' + D'$ (see (5.2)). Conversely, a Higgs bundle defines an operator D'', and a metric allows one to complete it to a complex connection D by setting $D' = (D'')^*$. Let $F_D = D^2$ be the curvature of D, and

[1]As pointed out by one of the referees, a version of the nonabelian Hodge correspondence in the non-Kähler surface case was, however, proved by Lübke [19], where the Chern class assumption is slightly different from the one below. The result in this paper is vacuous in the surface case, since the balanced assumption is then equivalent to Kähler.

$G_D = (D'')^2$ the *pseudo-curvature*. Flatness of D is the equation $F_D = 0$, whereas D arises from a Higgs bundle iff $G_D = 0$.

Now suppose ω is a balanced metric on X, so that the degree and slope stability of holomorphic bundles can be defined. Given an ω-slope stable Higgs bundle with $\mathrm{ch}_1(E) = 0$, one can find a metric h so that the associated connection D satisfies

$$F_D \wedge \omega_0 \wedge \Omega_0 = 0 \tag{1.3}$$

Similarly, given a flat connection D one can find a *harmonic metric*, meaning that

$$G_D \wedge \omega_0 \wedge \Omega_0 = 0 \tag{1.4}$$

Thus, under the assumptions, the forms F_D and G_D are "primitive", in the sense of (2.1) below.

The nonabelian Hodge correspondence follows by showing that if in addition $\mathrm{ch}_2(E) = 0$, then (1.3) implies $F_D = 0$, and on the other hand, (1.4) always implies $G_D = 0$ (as pointed out in [25, p. 17], the "pseudo-Chern class" defined by G_D automatically vanishes by the flatness of D). Now, if we assume ω is of Hodge-Riemann type, then these conclusions hold by integrating $\mathrm{tr}(F_D \wedge F_D)$ or $\mathrm{tr}(G_D \wedge G_D)$ against Ω_0, and using the vanishing of the Chern classes and the Hodge-Riemann bilinear relations. Thus, we see that the Kähler condition may be relaxed.

2. Hodge-Riemann forms

In this section, we recall the notion of a Hodge-Riemann form on a polarized complex vector space, i.e. a complex space V with a constant Kähler form ω_0. Denote $\Lambda^{p,q}$ to be the space of constant (p,q) forms over V. We fix Ω_0 to be any real $(n-p-q, n-p-q)$ form. On $\Lambda^{p,q}$, we can define a Hermitian form as

$$Q(\alpha, \beta) := (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} * (\alpha \wedge \overline{\beta} \wedge \Omega_0)$$

The space of *primitive* forms of degree (p,q) associated to (Ω_0, ω_0) is defined as

$$P^{p,q} = \{\alpha \in \Lambda^{p,q} : \alpha \wedge \omega_0 \wedge \Omega_0 = 0\}. \tag{2.1}$$

Definition 2.1. We call Ω_0 a Hodge-Riemann form for degree (p,q) with respect to ω_0 if

1) there exists a Q-orthogonal decomposition

$$\Lambda^{p,q} = P^{p,q} \oplus \omega_0 \wedge \Lambda^{p-1,q-1} \ ;$$

2) Q is positive definite on $P^{p,q}$.

Remark 2.2.
- Later in this paper, without mentioning it, we will apply the Hodge-Riemann property of Ω_0 to a (not necessarily square) matrix of forms. More precisely, suppose (α_{ij}) is a matrix with entries being primitive, i.e. $(\alpha_{ij}) \wedge \omega_0 \wedge \Omega_0 = 0$, then

$$(\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} * ((\mathrm{Tr}((\alpha_{ij}) \wedge (\alpha_{ij})^*) \wedge \Omega_0) = \sum_{ij} Q(\alpha_{ij}, \alpha_{ij}) \geq 0$$

  where $(\alpha_{ij})^* = (\overline{\alpha_{ji}})$ denotes the adjoint of (α_{ij}). This applies, in particular, to forms with values in a hermitian bundle, where the trace is replaced by the hermitian inner product.
- It follows from the classical Hodge-Riemann relation that $\Omega_0 = \omega_0^{n-p-q}$ is a Hodge-Riemann form (cf. [29, Thm. 6.32]).

In general, the Hodge-Riemann property of a form is difficult to verify. However, we have the following result, which has been used to get the mixed Hodge-Riemann relation (see [8]).

Proposition 2.3 ([27, Main Theorem], see also [13]). *For any constant positive $(1,1)$ forms $\omega_1, \cdots, \omega_k$ on (V, ω_0), $\omega_1 \wedge \cdots \wedge \omega_{n-k}$ is a Hodge-Riemann form with respect to ω_0 for any degrees (p,q) satisfying $p+q=k$.*

As a special case, this gives

Corollary 2.4. *For any constant positive $(1,1)$ forms $\omega_1, \ldots, \omega_{n-2}$ on (V, ω_0)*

1) $\omega_0 \wedge \cdots \wedge \omega_{n-2}$ is a strictly positive $(n-1, n-1)$ form. In particular, there exists a positive $(1,1)$ form ω so that

$$\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \cdots \wedge \omega_{n-2}$$

(cf. [22, p. 279], and also [28]).

2) $\omega_1 \wedge \cdots \wedge \omega_{n-2}$ is a Hodge-Riemann form for degrees (p,q) satisfying $p+q=2$ with respect to ω_0.

This combined with [24, Cor. 8.5] implies the following

Proposition 2.5. *For any constant Kähler forms ω_1, ω_2 on (V, ω_0), the form*

$$\omega_1^{n-2} + \omega_1 \wedge \omega_2^{n-3} + \cdots \omega_2^{n-2}$$

is a Hodge-Riemann form for degree (p, q) with $p + q = 2$ with respect to ω_0.

Remark 2.6. It is known that the Hodge-Riemann property is not invariant under convex linear combinations. For example, fix any two Kähler forms ω_1 and ω_2 on $\mathbb{C}^4$, $\omega_1^2 + a\omega_2^2$ is not a Hodge-Riemann form for degree $(1, 1)$ for certain positive values of a (see [24, Rem. 9.3] and also [27, Rem. 3] for other examples).

Timorin's result motivates the following:

Definition 2.7. A hermitian metric ω on a complex manifold X is said to be *balanced of Hodge-Riemann type* if the following hold:

1) we have an expression

$$\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \Omega_0$$

where ω_0 is a positive real $(1, 1)$ form on X and Ω_0 is a real $(n-2, n-2)$;

2) at every point Ω_0 is a Hodge-Riemann form for (p, q), $p + q = 2$;

3) Ω_0 and $\omega_0 \wedge \Omega_0$ are closed.

Note that (3) is equivalent to ω being balanced and Ω_0 being closed.

3. Bogomolov-Gieseker inequality

Below we show how the Donaldson-Uhlenbeck-Yau (resp. Hitchin-Simpson) theorem relating stability of holomorphic (resp. Higgs) bundles to the existence of Hermitian Einstein type metrics results in a Chern class inequality. The main result is Corollary 3.4. In this section, assume (X, ω) is a compact complex Hermitian manifold that satisfies items (1) and (3) of Definition 2.7, as well as the Hodge-Riemann condition (2) for the case $(p, q) = (1, 1)$.

Recall that associated to every coherent analytic sheaf $\mathcal{E} \to X$ is a holomorphic line bundle $\det \mathcal{E}$. The first Chern class of $\mathcal{E}$ is by definition $c_1(\mathcal{E}) := c_1(\det \mathcal{E}) \in H^2(X, \mathbb{Z}) \cap H^{1,1}_{\bar{\partial}}(X)$. We define the ω-degree of $\mathcal{E}$ by:

$$\deg \mathcal{E} := \int_X c_1(\mathcal{E}) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_X c_1(\mathcal{E}) \wedge \omega_0 \wedge \Omega_0$$

Because of the balanced condition, this is well-defined on the class of $c_1(\mathcal{E})$. The *slope* of a (nonzero) torsion-free sheaf is

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$$

Then we say a holomorphic bundle $\mathcal{E} \to X$ is *ω-stable* if $\mu(\mathcal{S}) < \mu(\mathcal{E})$ for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$.

A *Higgs bundle* on X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E} \to X$ is a holomorphic bundle, θ is a holomorphic 1-form with values in $\operatorname{End} \mathcal{E}$, and $\theta \wedge \theta = 0$ (see [14] and [25]). We say that a Higgs bundle is ω-stable if $\mu(\mathcal{S}) < \mu(\mathcal{E})$ for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$ *and* $\theta(\mathcal{S}) \subset \mathcal{S} \otimes \Omega^1_X$, where Ω^1_X is the holomorphic cotangent sheaf of X. Thus, stable vector bundles are a special case of stable Higgs bundles, where $\theta \equiv 0$. Finally, we say that $(\mathcal{E}, \theta)$ is *ω-polystable* if $(\mathcal{E}, \theta)$ splits as a direct sum of stable Higgs subbundles, all with the same slope.

Given a hermitian metric h on $\mathcal{E}$, let $\bar{\partial}_E + \partial_E$ denote the Chern connection of $(\mathcal{E}, h)$, θ^* the hermitian adjoint of θ with respect to h. Thus, θ^* is a $(0,1)$-form with values in $\operatorname{End} E$, satisfying $\partial_E \theta^* = 0$. We will consider the *complex* connection

$$D = \bar{\partial}_E + \partial_E + \theta + \theta^*$$

and its curvature F_D. To be explicit, we will write: $D = (\mathcal{E}, \theta, h)$.

Definition 3.1. Fix a Higgs bundle $(\mathcal{E}, \theta)$ on X. A hermitian metric h on $\mathcal{E}$ is called Hermitian-Einstein (HE) if

$$\sqrt{-1} F_{(\mathcal{E},\theta,h)} \wedge \omega_0 \wedge \Omega_0 = \lambda \cdot \mathrm{Id} \cdot \omega_0^2 \wedge \Omega_0 \tag{3.1}$$

for some constant λ.

Now we have the following generalized Donaldson-Uhlenbeck-Yau, Hitchin-Simpson theorem.

Theorem 3.2 (Donaldson-Uhlenbeck-Yau theorem). *$(\mathcal{E},\theta)$ is ω-polystable if and only if it admits a HE metric. Moreover, if $(\mathcal{E},\theta)$ is ω-stable, such a metric is unique up to scaling.*

The notion of HE metric we use here differs from that in the literature by a conformal factor. For completeness, we provide details showing that this is not an issue. We use the key fact that for balanced metrics the Kähler identities hold for $(1,0)$ and $(0,1)$ forms ([11, Prop. 1]; see also [20, Lemma 7.1.1]).

Lemma 3.3. *Given an n-dimensional hermitian manifold (X,ω) with $d\omega^{n-1}=0$, the following hold:*

$$\bar\partial^*\alpha^{0,1}=-\sqrt{-1}\Lambda\partial\alpha^{0,1}\quad,\quad\partial^*\alpha^{1,0}=\sqrt{-1}\Lambda\bar\partial\alpha^{1,0}$$

for any $(0,1)$-form $\alpha^{0,1}$, and any $(1,0)$-form $\alpha^{1,0}$.

Proof of Theorem 3.2. For the existence of HE metrics, it suffices to assume $(\mathcal{E},\theta)$ is ω-stable. Since ω is balanced, it is in particular Gauduchon, and so by the result of Li-Yau [18], generalized to Higgs bundles by Lübke-Teleman [21], there is a metric $\widetilde{h}$ such that

$$\sqrt{-1}F_{(\mathcal{E},\theta,\widetilde{h})}\wedge\frac{\omega^{n-1}}{(n-1)!}=\widetilde{\lambda}\cdot\mathrm{Id}\cdot\frac{\omega^n}{n!}$$

where $\widetilde{\lambda}=2\pi\mu(\mathcal{E})/\operatorname{vol}(X,\omega)$. Now there is a positive function f such that

$$\omega_0\wedge\frac{\omega^{n-1}}{(n-1)!}=f\cdot\frac{\omega^n}{n!}$$

Choose λ such that

$$\lambda\int_X f\cdot\frac{\omega^n}{n!}=\widetilde{\lambda}\operatorname{vol}(X,\omega)=2\pi\mu(\mathcal{E})\tag{3.2}$$

Then we can find a function φ satisfying: $\Delta_\omega\varphi=2(\lambda f-\widetilde{\lambda})$. Let $h=e^\varphi\widetilde{h}$. Then

$$F_{(\mathcal{E},\theta,h)}=F_{(\mathcal{E},\theta,\widetilde{h})}-\partial\bar\partial\varphi\cdot\mathrm{Id}$$

By Lemma 3.3, the Hodge and Dolbeault laplacians on functions are related: $\Delta_\omega=2\Delta_{\bar\partial}$. Hence,

$$-i\partial\bar\partial\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}=\frac{1}{2}\Delta\varphi\frac{\omega^n}{n!}=\lambda\omega_0\wedge\frac{\omega^{n-1}}{(n-1)!}-\widetilde{\lambda}\frac{\omega^n}{n!}$$

The existence follows. Similarly for the converse, by doing a conformal change we get the existence of HE metrics used in the literature, which implies the polystability. $\square$

As a direct corollary of this, we have the following generalized Bogomolov-Gieseker inequality

Corollary 3.4. *For any ω-polystable rank r Higgs bundle $(\mathcal{E},\theta)$, the following inequality holds:*

$$\int_X (2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2) \wedge \Omega_0 \geq 0$$

where the equality holds if and only if $\mathcal{E}$ is projectively flat.

Proof. Eq. (3.1) implies

$$F^{\perp} = F_{(\mathcal{E},\theta,h)} - \frac{1}{r}\operatorname{tr}(F_{(\mathcal{E},\theta,h)}) \cdot \mathrm{Id} \cdot \omega_0$$

is primitive. Now the conclusion follows as in the usual case by applying the Hodge-Riemann property of Ω_0 to $\sqrt{-1}F^{\perp}$ (see Remark 2.2, and note that $\sqrt{-1}F^{\perp}$ is hermitian). $\square$

For emphasis, we state the following version of the Donaldson-Uhlenbeck-Yau theorem for the slope stability condition defined by multipolarizations (cf. [12]).

Theorem 3.5. *Suppose X is a compact Kähler manifold with $(n-1)$ Kähler forms $\omega_0, \cdots, \omega_{n-2}$. Given a holomorphic vector bundle $\mathcal{E}$ that is slope stable with respect to $[\omega_0] \cup \cdots \cup [\omega_{n-2}]$, there exists a Hermitian-Einstein metric h on $\mathcal{E}$, i.e.*

$$\sqrt{-1}F_{(\mathcal{E},h)} \wedge \omega_0 \wedge \cdots \wedge \omega_{n-2} = \lambda \cdot \mathrm{Id} \cdot \omega_0^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}$$

for some constant λ. Moreover, such a metric is unique up to constant rescalings.

Remark 3.6. We emphasize here that the Chern connection of $(\mathcal{E},h)$ is *not* a Yang-Mills connection in general.

By Corollary 3.4 and Proposition 2.3, we have the following generalization of the Bogomolov-Gieseker inequality for multipolarizations, proven in the projective case by Miyaoka [23, Cor. 4.7].

Corollary 3.7. *Suppose X is a compact Kähler manifold with $(n-1)$ Kähler forms $\omega_0, \cdots, \omega_{n-2}$, and $\mathcal{E}$ is a slope stable holomorphic vector bundle over X with respect to $[\omega_0] \cup \cdots \cup [\omega_{n-2}]$. Then the following holds:*

$$\int_X (2rc_2(\mathcal{E}) - (r-1)c_1^2(\mathcal{E})) \wedge \Omega_j \geq 0$$

for any $j = 0, \ldots, n-2$. Here, $\Omega_j = \omega_0 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j+1} \cdots \wedge \omega_{n-2}$. Moreover, the equality holds for some j if and only if $\mathcal{E}$ is projectively flat.

Remark 3.8. The gauge theoretic side of the HE connections defined via multipolarizations is studied in [4].

4. The Sampson-Siu theorem

In this section, we prove

Theorem 4.1. *Let X be a compact complex manifold with a balanced metric of Hodge-Riemann type, and assume Ω_0 is strongly positive. If N is a Riemannian manifold with nonpositive complexified sectional curvature, then every harmonic map $u : X \to N$ is pluriharmonic.*

In the statement of the theorem, ω satisfies the condition of Definition 2.7, but we make the *additional assumption* that Ω_0 is a *strongly positive* $(n-2, n-2)$-form in the sense of [7, Ch. III, Def. 1.1].

Proof of Theorem 4.1. Let ∇ denote the Levi-Civita connection on N. This induces a connection on u^*TN. The harmonic map equation is: $d_\nabla^* du = 0$. Since ω is balanced, Lemma 3.3 implies that u is harmonic if and only if

$$d_\nabla d^c u \wedge \omega^{n-1} = 0 \tag{4.1}$$

Next, we follow the argument in [1, pp. 73-75]. Since Ω_0 is closed,

$$d\langle d_\nabla d^c u \wedge d^c u\rangle \wedge \Omega_0 = \langle R_N(d^c u) \wedge d^c u\rangle \wedge \Omega_0 + \langle d_\nabla d^c u \wedge d_\nabla d^c u\rangle \wedge \Omega_0$$

and so,

$$0 = \int_X \{\langle R_N(d^c u) \wedge d^c u\rangle \wedge \Omega_0 + \langle d_\nabla d^c u \wedge d_\nabla d^c u\rangle \wedge \Omega_0\}$$

By (4.1) and the Hodge-Riemann property, the second term is nonpositive. We claim that also

$$\langle R_N(d^c u) \wedge d^c u\rangle \wedge \Omega_0 \leq 0$$

Given this, it follows that $d_\nabla d^c u = 0$; hence, the pluriharmonicity. The claim follows from the assumption on Ω_0 and the nonpositivity of the complexified sectional curvature of N. We work at a point x. By definition, we know that

$$\Omega_0 = \sum_i \mu_i \sqrt{-1}\alpha_1^i \wedge \overline{\alpha_1^i} \wedge \cdots \wedge \sqrt{-1}\alpha_{n-2}^i \wedge \overline{\alpha_{n-2}^i}$$

where $\mu_i \geq 0$, and $\{\alpha_1^i, \ldots, \alpha_{n-2}^i\}$ are linearly independent $(1,0)$ forms. Denote by P_i the complex two dimensional subspace of TM where $\alpha_j^i|_{P_i} = 0$ for $j = 1, \cdots, n-2$. Fix X_i, Y_i so that $\{X_i, Y_i, JX_i, JY_i\}$ form an orthogonal basis for P_i. Then

$$\langle R_N(d^c u) \wedge d^c u\rangle \wedge \Omega_0 = \sum_i \mu_i' \langle R_N(d^c u) \wedge d^c u\rangle (X_i, Y_i, JX_i, JY_i)\, \mathrm{dVol}$$

for some $\mu_i' \geq 0$. Now as in [1, p. 75], we know

$$\langle R_N(d^c u) \wedge d^c u\rangle (X_i, Y_i, JX_i, JY_i) = R_N(Z_i, W_i, \overline{W_i}, \overline{Z_i}) \leq 0$$

where $Z_i = du(X_i - JX_i)$ and $W_i = du(W_i - JW_i)$. The claim follows. □

Remark 4.2. If $\widetilde{X}$ is the universal cover of X, then Theorem 4.1 remains valid for harmonic maps $u : \widetilde{X} \to N$ that are equivariant with respect to a representation $\rho : \pi_1(X) \to \mathrm{Iso}(N)$. These play a role in the next section. The existence of equivariant harmonic maps to nonpositively curved targets N is guaranteed if ρ is *reductive* (or *semisimple*) (see [5, 9, 16, 17]).

Theorem 4.1, combined with [7, Prop. III.1.11] implies

Corollary 4.3. *Suppose X is a compact complex manifold with a balanced metric ω of the form* (1.2)*, where ω_i are positive* $(1,1)$*-forms and*

$$d(\omega_1 \wedge \cdots \wedge \omega_{n-2}) = 0$$

If N is a Riemannian manifold with nonpositive complexified sectional curvature, then every harmonic map $u : X \to N$ is pluriharmonic.

5. The nonabelian Hodge correspondence

The goal of this section is to prove the following generalization of [25, Cor. 1.3].

Theorem 5.1. *Suppose X is a compact complex manifold and ω is a balanced metric of Hodge-Riemann type on X. Then we have a categorical 1-1 correspondence between*

1) semisimple flat bundles on X, and

2) isomorphism classes of ω-polystable Higgs bundles $(\mathcal{E}, \theta)$ with $\mathrm{ch}_1(\mathcal{E}) \cup [\omega^{n-1}] = 0$ and $\mathrm{ch}_2(\mathcal{E}) \cup [\Omega_0] = 0$.

Remark 5.2.
- When $\omega_0 = \omega$, $\Omega_0 = \omega^{n-2}/(n-1)!$, then by our assumptions (X, ω) is Kähler. Then Theorem 5.1 reduces to the well known nonabelian Hodge correspondence for compact Kähler manifolds.
- There exist many examples where ω is not a Kähler metric, even when the underlying manifold X is Kähler, or even projective algebraic. For example, take X to be projective with $[\omega_i]$, $i = 0, \ldots, n-2$, all ample classes, and take ω as in (1.2). Then the class ω^{n-1} represents a point in the interior of the cone of *movable curves* (see [3]).
- Notice that if $\Omega_0 = \omega_1^{n-2}$, then $d\Omega_0 = 0$ implies $d\omega_1 = 0$, and the manifold is Kähler. However, closedness of $\Omega_0 = \omega_1 \wedge \cdots \wedge \omega_{n-2}$, for different ω_i, can occur in the non-Kähler setting. One might expect that this will provide new insights for the study the non-Kähler complex manifolds, since the results obtained here already put restrictions on complex manifolds admitting such structures.

Proof of Theorem 5.1. The proof, of course, closely follows the lines of the classical theorem, taking care to avoid the Kähler condition.

First, assume $(\mathcal{E}, \theta)$ is an ω-polystable Higgs bundle. If $\mathrm{ch}_1(\mathcal{E}) \cup [\omega^{n-1}] = 0$, then by Theorem 3.2 there is a hermitian metric h on $\mathcal{E}$ such that (1.3) is satisfied for

$$D = \bar{\partial}_E + \partial_E + \theta + \theta^* \tag{5.1}$$

(note that $\lambda = 0$ by (3.2)). Hence, F_D is primitive. Moreover, $\sqrt{-1}F_D$ is of type $(1,1)$ and hermitian. Since $\mathrm{ch}_2(\mathcal{E}) \cup [\Omega_0] = 0$, we have

$$0 = \int_X \mathrm{tr}(F_D \wedge F_D) \wedge \Omega_0$$

Since Ω_0 is a Hodge-Riemann form, we conclude that D is a flat connection, which is necessarily semisimple.

Now let D be a semisimple flat connection on E. By Corlette's theorem (see Remark 4.2) there exists a *harmonic metric*, which has the following consequence. Decomposing into type we can express D as in (5.1), where $\theta \in \Omega^{1,0}(X, \operatorname{End} E)$, and $\theta + \theta^*$ is essentially du for an equivariant harmonic map $u : \widetilde{X} \to \mathsf{GL}(n, \mathbb{C})/\mathsf{U}(n)$. Let

$$D'' = \bar{\partial}_E + \theta \quad , \quad D' = \partial_E + \theta^* \tag{5.2}$$

We wish to prove that $G_D = (D'')^2 = 0$, for then $\bar{\partial}_E$ is integrable, $\bar{\partial}_E \theta = 0$, and $\theta \wedge \theta = 0$; i.e. $(\bar{\partial}_E, \theta)$ is a Higgs bundle.

Flatness of D implies,

1) $\partial_E \theta = 0$;
2) $(\bar{\partial}_E \theta)^* = -\bar{\partial}_E \theta$;
3) $\partial_E^2 + \frac{1}{2}[\theta, \theta] = 0$;

By (4.1), we have $(\bar{\partial}_E \theta - (\bar{\partial}_E \theta)^*) \wedge \omega^{n-1} = 0$, and combining this with (2) we have

$$\bar{\partial}_E \theta \wedge \omega^{n-1} = \bar{\partial}_E \theta \wedge \omega_0 \wedge \Omega_0 = 0 \tag{5.3}$$

i.e. G_D is primitive (note that we have only used the balanced condition for this part).

To prove that $G_D = 0$, we argue as in the proof of Theorem 4.1 (see also [5, proof of Thm. 5.1]). We have:

$$\begin{aligned} d \operatorname{tr}(\bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0 &= \partial \operatorname{tr}(\bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0 \\ &= \operatorname{tr}(\partial_E \bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0 + \operatorname{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0 \\ &= -\frac{1}{2} \operatorname{tr}([[\theta, \theta^*], \theta] \wedge \theta^*) \wedge \Omega_0 + \operatorname{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0 \\ &= -\frac{1}{4} \operatorname{tr}([\theta, \theta] \wedge [\theta, \theta]^*) \wedge \Omega_0 + \operatorname{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0 \end{aligned}$$

so integrating,

$$0 = -\frac{1}{4} \int_X \operatorname{tr}([\theta, \theta] \wedge [\theta, \theta]^*) \wedge \Omega_0 + \int_X \operatorname{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0$$

By the Hodge-Riemann property of Ω_0, both terms on the right hand side are nonpositive, and hence vanish. We conclude that $\bar{\partial}_E \theta = 0$, and $[\theta, \theta] = 0$. By (3) above, $\bar{\partial}_E$ is integrable, and this completes the proof. $\square$

As in the Kähler case, Theorem 5.1 translates into a statement about moduli spaces. Assume (X,ω) is as above, and let $\mathbf{M}^s_{\mathsf{Dol}}(X,n)$ be the set of isomorphism classes of stable rank n Higgs bundles on X with vanishing Chern classes. Let $\mathbf{M}^{irr}_{\mathsf{dR}}(X,n)$ be the set of isomorphism classes of rank n holomorphic vector bundles on X with integrable connection. Then $\mathbf{M}^s_{\mathsf{Dol}}(X,n)$ and $\mathbf{M}^{irr}_{\mathsf{dR}}(X,n)$ each has the structure of a separated complex analytic space (see [20, Ch. 4] and [15], respectively). We have the following weak version of the nonabelian Hodge correspondence (cf. [26, Thm. 7.18] or [20, Cor. 4.4.4]), whose proof follows exactly as in [26].

Corollary 5.3. *Let (X,ω) be a compact hermitian manifold that is balanced of Hodge-Riemann type. Then the set theoretic identification in Theorem 5.1 gives a homeomorphism:*

$$\mathbf{M}^s_{\mathsf{Dol}}(X,n) \cong \mathbf{M}^{irr}_{\mathsf{dR}}(X,n)\ .$$

As is in the Kähler case, the homeomorphism in Corollary 5.3 is not holomorphic. Indeed, as was pointed out by one of the referees it would be interesting to show that the distinct complex structures fit into a hyperkähler structure on the smooth locus of $\mathbf{M}^{irr}_{\mathsf{dR}}(X,n)$. Such a result would generalize the classical work of Hitchin [14] for the case of Riemann surfaces and of Fujiki [10] for higher dimensional Kähler manifolds. This will be pursued in a future work.

6. Rigidity of representations of fundamental groups

For the sake of completeness, in this last section we point out that two important results of Corlette and Simpson generalize to our setting. Let $G_{\mathbb{R}}$ be a simple real algebraic group acting by isometries on the irreducible bounded symmetric domain $G_{\mathbb{R}}/K$. We assume (X,ω) is a compact complex manifold with a balanced metric of Hodge-Riemann type. Let P be the principle $G_{\mathbb{R}}$ bundle with structure group reduced to K. As in [6], one can associate a volume $\mathrm{vol}(P)$ to P by defining it as a power of the first Chern class of P up to a conformal factor. Now the following generalizes [6, Thm. 0.1].

Theorem 6.1. *Suppose P is flat with $\mathrm{vol}(P)\neq 0$ and $G_{\mathbb{R}}/K$ is not of the form $\mathsf{U}(n,1)/\mathsf{U}(n)\times\mathsf{U}(1)$ or $\mathsf{SO}(2n+1,2)/\mathsf{S}(\mathsf{O}(2n+1)\oplus\mathsf{O}(2))$. Then the monodromy homomorphism of the fundamental group of X into $G_{\mathbb{R}}$ is locally rigid as a homomorphism of the fundamental group of X into the complexification of $G_{\mathbb{R}}$.*

The argument follows by replacing [6, Prop. 2.4] with argument in the proof of Theorem 5.1 to get the holomorphic property of the harmonic sections. This is the only place where the Kähler assumption is needed in [6]. Simpson's argument in the Kähler case also gives (see [25])

Theorem 6.2. *Suppose $\rho : \pi_1(X) \to \mathsf{GL}(n, \mathbb{C})$ is a locally rigid representation of the fundamental group of X. Then the associated flat vector bundle is the underlying vector bundle of a complex variation of Hodge structure.*

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