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Online Learning of Stabilizing Controllers Using Noisy Input-Output Data and Prior Knowledge

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(Intersection of Machine Learning with Control)

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ABSTRACT This paper presents online prior-knowledge-based data-driven approaches for verifying stability and learning a stabilizing dynamic controller for linear stochastic input-output systems. The system is modeled in an autoregressive exogenous (ARX) framework to accommodate cases where states are not fully observable. A key challenge addressed in this article is online stabilizing open-loop unstable systems, where collecting sufficient data for controller learning is impractical due to the risk of failure. To mitigate this, the proposed method integrates uncertain prior knowledge, derived from system physics, with limited available data. Inspired by set-membership system identification, the prior knowledge set is dynamically updated as new data becomes available, reducing conservatism over time. Unlike traditional approaches, this method bypasses explicit system identification, directly designing controllers based on current knowledge and data. A connection between ARX models and behavior theory is established, providing necessary and sufficient stability conditions using strict lossy S-Lemma. Quadratic difference forms serve as a framework for Lyapunov functions, and robust dynamic controllers are synthesized via linear matrix inequalities. The methodology is validated through simulations, including an unstable scalar system visualizing the integration of prior knowledge and data, and a rotary inverted pendulum demonstrating controller effectiveness in a nonlinear, unstable setting.

INDEX TERMS Behavior theory, data-driven control, linear matrix inequalities, robust control, uncertain systems.

I. INTRODUCTION

Data-driven control stability verification and stabilizing controller design have gained significant attention for their ability to autonomously learn controllers and verify their properties in real-time without requiring precise knowledge of system dynamics [1], [3], [4], [5], [6], [7], [8], [28]. This paradigm shift offers new opportunities for managing dynamic and uncertain systems where traditional model-based methods often fall short [9]. However, these methods face challenges. A key issue is the need for rich data to learn controllers and verify system properties, with the data richness requirements varying depending on the control task or property being verified [1]. Moreover, under system disturbance, it is generally impossible to learn an accurate point-based model; instead, a set of models is derived to explain the data. Robust controller design based on such sets can become overly conservative, if not infeasible, if the model set is large. Another critical limitation is the requirement for full-state measurements, which are often unavailable in real-world applications. In such cases, learning a controller from observed input-output data alone is desired but challenging. Thus, it is essential to address these data-related challenges in three key areas: reducing the number of samples required for learning, shrinking the size of the systems conforming to data and prior knowledge online, and enabling control design based solely on observable outputs rather than full-state information.

A. BACKGROUND

Prior knowledge-based data-driven control verification and design has recently demonstrated significant potential in reducing the number of samples required and minimizing controller conservatism. This approach achieves these objectives by integrating prior knowledge about the system's attributes with data collected from an uncertain system [10] during the controller learning process. By utilizing prior knowledge from a well-understood nominal system, controllers for the actual system can be designed with minimal data [11]. This approach is particularly beneficial when the actual system is challenging, hazardous, or costly to interact with frequently for data collection [12]. Additionally, prior-knowledge-based data-driven methods can adapt to parameter variations and maintain stable control strategies despite uncertainties and dynamic changes in system behavior [13]. However, existing methods often assume full access to the system's states, which may not be feasible in many real-world applications. The solutions found in these frameworks are offline, and they do not incorporate online data from the system, which limits the behavior of the controller. Also, the prior knowledge set is static and is not updated as time evolves.

Data-driven control methods have recently been extended to systems with input-output measurements, as highlighted in several studies [14], [15], [16]. These methods often involve transforming the input-output system into an artificial state-space representation, where the state vector consists of shifted inputs and outputs. While this transformation enables the application of state-space data-driven control techniques, it comes with notable limitations. The resulting state-space models are frequently non-minimal and high-dimensional, which significantly increases the data requirements for effective control design. This poses a particular challenge for complex physical systems, where data collection can be costly, time-consuming, or constrained by practical limitations.

To overcome the limitations of state-space approaches, alternative methods rooted in behavioral theory [17], [18] have been proposed [19]. This framework has redefined key control problems, such as data-enabled model predictive control (MPC) [3], [20], [21], by focusing on designing feedback controllers directly from input-output data. These methods bypass state construction by employing higher-order difference equations or autoregressive exogenous (ARX) systems. However, these data-driven approaches typically do not leverage available prior knowledge of the system. Incorporating initial physical insights reduces the number of unknown parameters, enabling parameter estimation with fewer data points and improving efficiency. On top of that, one may shrink the size of the available information using the newly available data.

B. CONTRIBUTIONS

This article addresses the challenges of stability analysis and dynamic stabilizing controller design for systems modeled by ARX representations. It does so by relying solely on noisy input-output data—which alone might not be sufficient for full system identification—and by incorporating prior knowledge that is continuously refined with online data. The approach utilizes the quadratic difference form (QDF) as a natural framework for Lyapunov functions in autonomous AR systems [22], [23]. QDF is particularly effective for describing stability in systems governed by higher-order difference equations, especially in the discrete-time domain [24], [25]. Yakubovich's *S*-Lemma [2], [26], [27], [29] is another critical tool employed in this work. Specifically, the lossy *S*-Lemma [30] is used to integrate multiple information sources, such as prior knowledge and noisy data, for stability analysis and learning stable closed-loop controllers. This methodology enables our algorithm to obtain a solution through a one-shot optimization for the set of systems consistent with both the collected data and the initial prior knowledge, and improve the prior knowledge set on the go. This is in contrast to the iterative nature of some data-driven approaches, such as reinforcement learning techniques [31], [32]. Value iteration methods iteratively learn a sequence of improved stabilizing solutions. However, they typically do not account for prior knowledge and are data-hungry. Besides, they typically require input-state data, in contrast to our presented approach, which relies on input-output data. The size of the overlap between data and the prior knowledge set can vary, potentially impacting controller performance [30]. To this end, this paper introduces an online, adaptive method inspired by set-membership system identification [33] to address this. This approach dynamically updates the prior knowledge set at each step, bypassing explicit system identification and incorporating new data in real-time to prevent performance degradation. The proposed prior-knowledge-based data-driven methods for stability analysis and feedback control synthesis are computationally efficient, leveraging linear matrix inequalities (LMIs) in the presence of measurement disturbance [30], [34] to find solutions. Initially, the method identifies a compact set of systems that includes the actual system using nominal system information and then refines this set with available data. It assumes a close relationship between the nominal and actual systems, ensuring a non-empty overlap of noisy data and prior knowledge.

If initial prior knowledge is available, the collected data do not need to meet the persistent excitation (PE) conditions. Although the set of models consistent with the data may be non-compact, its intersection with the initial prior knowledge can be over-approximated by a compact set. Typically, the persistent excitation (PE) condition is imposed to ensure the compactness of the model set derived from data [35], [36]. In contrast to policy or value iteration methods, which often yield a low proportion of stable controllers under limited data, our approach enables stability analysis and the synthesis of a stabilizing controller. Two simulation examples—a scalar system and a rotary inverted pendulum—are used throughout the paper to illustrate and clarify the methodology.

II. SYSTEM SETTING AND PROBLEM FORMULATION

Notations: $\mathbb{R}^{m\times n}$ represents the real linear space for all real matrices with dimensions $m \times n$. $\mathbb{R}^n_s$ denotes the set of all real symmetric matrices of size $n \times n$. The set of non-negative and positive integers is represented by $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. I_n is used to denote the identity matrix with dimensions $n \times n$, and a zero matrix of size $n \times m$ is represented as $0_{n\times m}$. The notation $Q(\preceq, \succeq, \prec, \succ)0$ indicates that Q is either negative or positive semi-definite and negative or positive definite. When the context is clear, the subscripts indicating the matrix dimensions may be omitted for brevity. The More-Penrose pseudoinverse of a matrix M is shown by $M^\dagger$. We use $col(a, b)$ to represent the column-wise stacking of vectors a and b, i.e., $[a^\top, b^\top]^\top$. A discrete interval from 0 to $N > 0$ with a step size of k is denoted by $[0, N]_k$. A shorthand writing as $\mathrm{M} = \begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ for matrix $\mathrm{M} = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ is used throughout this paper. Random variables uniformly distributed in $[a, b]$ are denoted by $U[a, b]$.

A. SYSTEM REPRESENTATION USING ARX MODEL

In this paper, the actual system that follows linear input-output dynamics with disturbance is described using an input-output system ARX model with disturbance [24]

$$y(t+L) + [-Q_{L-1}\ P_{L-1}] \begin{bmatrix} u(t+L-1) \\ y(t+L-1) \end{bmatrix} + \cdots + [-Q_0\ P_0] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = v(t). \quad (1)$$

where we have $u(t) \in \mathbb{R}^m$ as input, $y(t) \in \mathbb{R}^p$ as the output of the system and $L > 0$ is the order of the system. The process disturbance of the system is shown by the vector $v \in \mathbb{R}^p$. In this representation, $P \in \mathbb{R}^{p\times p}$ and $Q \in \mathbb{R}^{p\times m}$ are the *physical* coefficients of the system. In this paper, the model of the system is the same as the actual system. Choosing an ARX model instead of the input-state-output framework provides a more practical basis for data-driven control. We define the combined physical coefficient matrix $\tilde{R} \in \mathbb{R}^{p\times(qL)}$ as follows

$$\tilde{R} := \begin{bmatrix} -Q_0 & P_0 & -Q_1 & P_1 & \ldots & -Q_{L-1} & P_{L-1} \end{bmatrix}, \quad (2)$$

where $q := p + m$. We differentiate between the nominal parameters, $\tilde{R}^\eta$, and the unknown actual parameters, $\tilde{R}^\star$.

For a system without input, $\tilde{R}$ can be set to

$$\tilde{P} := \begin{bmatrix} 0 & P_0 & 0 & P_1 & \ldots & 0 & P_{L-1} \end{bmatrix}.$$

Under this condition, (1) simplifies to describe autonomous systems as follows

$$y(t+L) + P_{L-1}y(t+L-1) + \cdots + P_0 y(t) = v(t). \quad (3)$$

B. BEHAVIOR THEORY

This subsection explores the behavioral framework as a modeling tool in characterizing dynamical systems. A dynamical system can be characterized by a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where the time axis is defined by $\mathbb{T}$, and $\mathbb{W}$ represents the signal space in which the system variables take their values. The *behavior*, denoted by $\mathfrak{B} \subset \mathbb{W}^\top$, is the set of all possible trajectories of the system variable. To describe a noisy linear time-invariant discrete-time dynamical system, one can set $\mathbb{T} = \mathbb{Z}_+$ and $\mathbb{W} = \mathbb{R}^q$, and represent Σ using the following linear difference algebraic equation [17]

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_L w(t+L) = v(t), \quad (4)$$

where the manifest variable is denoted by $w \in \mathbb{R}^q$. The constant coefficient matrices, represented by $R_i \in \mathbb{R}^{p\times q}$, $i = [1, L]_1$, capture the *physical* parameters of the system. As will be discussed later, these parameters can be learned, grouped into a set of uncertain parameters, and/or estimated. If $v(t) = 0$, $\forall t \in \mathbb{Z}_+$, then, (4) is called a *kernel representation* of the system in the space of solutions $w : \mathbb{Z}_+ \to \mathbb{R}^q$. By employing a shift operator $\sigma x(t) = x(t+1)$, one can represent (4) in the shorthand format [25] as

$$R(\sigma)w = v,$$
$$R(\xi) := R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{p\times q}[\xi], \quad (5)$$

The notation $\sigma^i w(t)$, which represents the repeated application of the shift operator σ, results in $w(t+i)$, where $i = 0, 1, \ldots, L$. This shorthand allows for a concise representation of future values of the manifest variable w at different time steps. Thus, one can describe the set of behaviors of the linear system as [17], [24], [25]

$$\mathfrak{B} = \{w \in \mathbb{R}^q \mid R(\sigma)w = v\}. \quad (6)$$

The *relationship* between the ARX model and behavioral theory becomes clear when we consider the manifest variable as

$$w(t) := \begin{bmatrix} u^\top(t) & y^\top(t) \end{bmatrix}^\top.$$

To facilitate this discussion, we define

$$P(\xi) = I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0, \quad (7a)$$
$$Q(\xi) = Q_{L-1}\xi^{L-1} + \cdots + Q_1\xi + Q_0. \quad (7b)$$

These definitions serve as the basis for linking the ARX framework to behavior theory in the subsequent sections.

Remark 1: The leading coefficient of $P(\xi)$ is taken to be $I_{p\times p}$, while the leading coefficient of $Q(\xi)$ is $0_{p\times m}$. This has several consequences. Firstly, it ensures that $P(\xi)$ is non-singular, which in turn implies that $P(\xi)^{-1}Q(\xi)$ is strictly proper. As a result, (1) effectively represents a causal input-output system, where u is the control input, y is the output, and v is the input disturbance. The strictly proper nature of

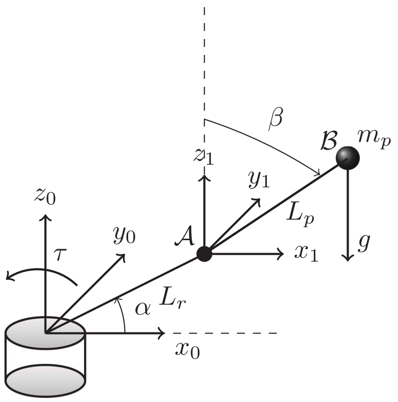

FIGURE 1. The schematic of a rotary inverted pendulum system.

$P(\xi)^{-1}Q(\xi)$ is a crucial property for establishing feedback interconnection.

Definition 1: Let $\bar{\mathfrak{B}}(P)$ be the behavior of the deterministic autonomous system $P(\sigma)y = 0$ as in (3). For the deterministic autonomous system $P(\sigma)y = 0$, it is defined to be stable if all solutions $y \in \bar{\mathfrak{B}}(P)$ over the non-negative integers $\mathbb{Z}_+$ converge to zero as time $t \to \infty$.

C. PROBLEM FORMULATION

This paper presents a novel approach for stability analysis and dynamic controller design under conditions of limited data and substantial uncertainty in prior knowledge. By relying solely on input-output data, our method exhibits robustness against uncertainties in physical coefficients, noisy measurements, and sparse observations. The approach achieves this by constructing a compact set of admissible systems that captures the possible range of system behaviors, effectively integrating available data with domain-specific insights. Specifically, the paper focuses on (i) assessing the stability of systems described by (3), and (ii) synthesizing stabilizing feedback for systems defined by (1). In both cases, the physical coefficients $\tilde{R}$ are estimated from online data and refined with prior knowledge, which may originate from an initial estimate and be iteratively updated with new data, as explained in subsequent sections.

We consider the systems that conform to the intersection of two information sets (i) the set of system coefficients $\tilde{R}$ that is consistent with the prior-knowledge-based information $\mathcal{E}^t_{pk}$, and (ii) the set of system coefficients that are consistent with the online noisy input-output data $\mathcal{E}^t_{data}$. To proceed, we define the intersection of these two information sets as a prior-knowledge-based data-driven overlap set (PDOS) as the following

$$\mathcal{E}^t_{PDOS} := \{\tilde{R} : \tilde{R} \in \mathcal{E}^t_{pk} \cap \mathcal{E}^t_{data}\}. \tag{8}$$

$\mathcal{E}^t_{PDOS}$ is a dynamic set and is updated as new data becomes available. A graphical representation of this set for a scalar system is provided in Fig. 2.

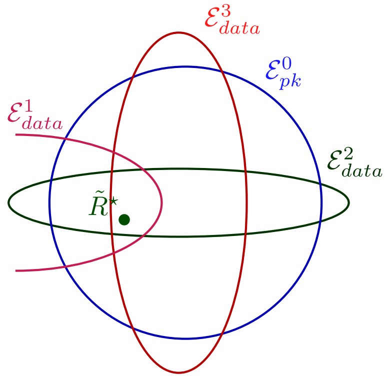

FIGURE 2. A geometric representation of a dynamic system with a single input and output, showing the data-conformity sets $\mathcal{E}^1_{data}$, $\mathcal{E}^2_{data}$, $\mathcal{E}^3_{data}$, the initial prior-knowledge set $\mathcal{E}^0_{pk}$, and the actual system $\tilde{R}^\star$. The overlap between data sets and prior knowledge varies in size and shape with different data sets, potentially degrading controller performance when designed based on the overlap if no refinement on the prior knowledge set is performed.

The formal problem formulations are given below to clearly define the scope and objectives, with the necessary assumptions and structural components introduced later to ensure a logical and comprehensive progression.

Problem 1: **Online Stability Analysis:** Consider the stochastic uncertain autonomous system (3). Assume that the actual system and nominal model are close to each other. At time step t, develop a stability test for the actual system such that $\tilde{P}^\star \in \mathcal{E}^t_{PDOS}$.

Problem 2: **Online Stabilizing Controller Synthesis:** Consider the stochastic uncertain system (1). Assume that the actual system and nominal model are close to each other. At time step t, learn a stabilizing controller for the actual system such that $\tilde{R}^\star \in \mathcal{E}^t_{PDOS}$.

Remark 2: Problem 1 in the deterministic case follows the stability definition as stated in Definition 1.

Remark 3: In both Problems 1 and 2, it is stated that the actual system and the nominal model are closely related. Assumption 1 formally quantifies this condition. Furthermore, since the actual system is unknown, a solution will be provided for the set that encompasses the actual system, ensuring that the obtained solution remains valid for the actual system as well.

III. MOTIVATING EXAMPLE AND PRELIMINARIES

In this section, we provide a motivating example that we will use later in our simulation results. We also present a few preliminary results regarding Matrix Quadratic Functions that will be used later in stability analysis.

A. MOTIVATING EXAMPLE

Stabilizing a rotary inverted pendulum as shown in Fig. 1 is a classic problem in control systems, often used to demonstrate and evaluate control strategies due to its inherently unstable nature. This system consists of a rigid rod (pendulum) attached to a rotating arm, which is mounted on a motor-driven

TABLE 1. Parameters and variables in the model of the rotary inverted pendulum system.

Parameters	Description
L_r	Motor arm length (m)
L_p	Pendulum rod length (m)
m_p	Pendulum ball mass (kg)
J_r	Motor arm inertia ($\mathrm{kg \cdot m^2}$)
C_r	Motor arm friction coefficient (N.m.s/rad)
g	Gravitational acceleration ($\mathrm{m/s^2}$)
Variables	**Description**
τ	DC motor torque input ($\mathrm{N \cdot m}$)
α	Motor arm angle (rad)
β	Pendulum rod angle (rad)

pivot. The motor enables the arm to rotate in the horizontal plane (x, y), while the pendulum oscillates in the vertical plane (x, z). The control task focuses on stabilizing the pendulum in its upright position by adjusting the rotary arm's position. However, collecting data to develop a stabilizing controller poses a potential safety risk, as the pendulum may enter regions where the linearization assumptions no longer hold, leading to inaccuracies in the physical model. To this end, one can develop some physical models for the system, such as the following, and use the uncertain information as a baseline for developing a stabilizing controller. On the other hand, prior knowledge of the system might be uncertain and may not stabilize the actual system when a controller is designed based on it. To this end, a method that considers both information sets and refines the model online is needed. The physical parameter descriptions are brought in Table 1.

Consider the variables α and β, which are both functions of time. For notational simplicity, we have omitted the explicit dependence on t in the expressions that follow. According to the Euler-Lagrange method, the nonlinear dynamics of the rotary inverted pendulum (known as Furuta pendulum [37]) is By linearizing (9), shown at the bottom of this page, around the vertical inverted unstable equilibrium state ($\beta = 0$, $\dot{\beta} = 0$) the following is resulted

$$[J_r + m_p L_r^2]\ddot{\alpha} - [m_p L_P L_r]\ddot{\beta} + [C_r]\dot{\alpha} = \tau, \tag{10a}$$

$$-[m_p L_p L_r]\ddot{\alpha} + [m_p L_p^2]\ddot{\beta} - [m_p L_p g]\beta = 0. \tag{10b}$$

Next, to discretize (10), for a continuous function $f(t)$ we first bring the following standard approximation rule [38]

$$\ddot{f}(t) \approx \frac{f(t + 2\Delta t) - 2f(t + \Delta t) + f(t)}{\Delta t^2}, \tag{11a}$$

$$\dot{f}(t) \approx \frac{f(t + \Delta t) - f(t)}{\Delta t}, \tag{11b}$$

where Δt is the discretization step size. By incorporating (11) in (10) and after some algebraic manipulation one has the linear discretized model of the rotary inverted pendulum system (12), shown at the bottom of this page, with $v(t)$ as the disturbance term which contains the linearization error and also the disturbance on the system's output. By taking $y(t) = \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}$, one can rewrite (12) in the format of ARX systems (1).

Although physical laws define the system model structure, parameters m_p, J_r, C_r may exhibit temporal variations or uncertainties [39]. By considering a range of values for these parameters, we can construct a set of potential model parameters, each representing a permissible configuration. This set forms the foundation of our initial prior knowledge set. However, using only this physical knowledge and discarding the collected data can lead to infeasibility and/or conservativeness. The system data can also be leveraged to identify another set of admissible systems. Integrating uncertain prior physical knowledge with (possibly non-rich) collected data is crucial. Also, refining the prior knowledge set is necessary for the controller to accurately meet the system's parameters. All this motivates us to develop a control methodology that takes advantage of the initial prior knowledge and updates it online as new data becomes available.

$$[J_r + m_p L_r^2]\ddot{\alpha} - \left[m_p L_P L_r \cos(\beta)\right]\ddot{\beta} + \left[C_r + \frac{1}{2} m_p L_p^2 \sin(2\beta)\dot{\beta}\right]\dot{\alpha} + \left[m_p L_p L_r \sin(\beta) + \frac{1}{2} m_p L_p^2 \sin(2\beta)\dot{\alpha}\right]\dot{\beta} = \tau, \tag{9a}$$

$$-\left[m_p L_p L_r \cos(\beta)\right]\ddot{\alpha} + \left[m_p L_p^2\right]\ddot{\beta} - \left[\frac{1}{2} m_p L_p^2 \sin(2\beta)\dot{\alpha}\right]\dot{\alpha} - [m_p L_p g]\sin(\beta) = 0. \tag{9b}$$

$$\underbrace{\begin{bmatrix} \alpha(t+2) \\ \beta(t+2) \end{bmatrix}}_{y(t+2)} = -\underbrace{\begin{bmatrix} -\frac{2J_r - C_r \Delta t}{J_r} & 0 \\ \frac{C_r L_r \Delta t}{J_r L_p} & -2 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} \alpha(t+1) \\ \beta(t+1) \end{bmatrix}}_{y(t+1)} - \underbrace{\begin{bmatrix} \frac{J_r - C_r \Delta t^2}{J_r} & -\frac{L_r \Delta t^2 g m_p}{J_r} \\ -\frac{C_r L_r \Delta t}{J_r L_p} & -\frac{g m_p L_r{}^2 \Delta t^2 + J_r g \Delta t^2 - J_r L_p}{J_r L_p} \end{bmatrix}}_{P_0} \underbrace{\begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}}_{y(t)} + \underbrace{\begin{bmatrix} \frac{\Delta t^2}{J_r} \\ \frac{L_r \Delta t^2}{J_r L_p} \end{bmatrix}}_{Q_0} \tau(t) + v(t) \tag{12}$$

B. MATRIX QUADRATIC FUNCTIONS

As the foundation of the presented material in this paper, matrix quadratic functions (MQF) play an important role in defining sets and S-Lemma in incorporating these sets in stability analysis and stable control design.

Definition 2: A function $\mathcal{F}:\mathbb{R}^{q\times r}\to\mathbb{R}^{q\times q}$ is called a matrix quadratic function if it can be expressed as

$$\mathcal{F}(\theta) := \theta C\theta^\top + \theta B^\top + B\theta^\top + A \tag{13}$$

where $A\in\mathbb{R}^q_s$ and $C\in\mathbb{R}^r_s$ are symmetric matrices and B is a matrix defined in $\mathbb{R}^{q\times r}$. Also, (13) is equivalently defined as the following

$$\mathcal{F}(\theta) := \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}. \tag{14}$$

Define

$$O := \begin{bmatrix} A & B \\ \star & C\end{bmatrix} \in \mathbb{R}^{r+q}_s \tag{15}$$

and the following generalized Schur complement

$$O\,|\,C := A - BC^\dagger B^\top. \tag{16}$$

If $A\prec 0$ and $O\,|\,C\succeq 0$, then

$$\mathcal{Q} := \{\theta\in\mathbb{R}^{q\times r} \;:\; \mathcal{F}(\theta)\succeq 0\} \tag{17}$$

is compact and nonempty and if $O\,|\,C=0$, $\mathcal{Q}$ reduces to a singleton [40].

As an extension of the classic lossy S-lemma [41], one has the following strict lossy S-Lemma. This lemma is useful for combining multiple information sets in this paper.

Lemma 1: (Strict lossy S-Lemma) Let $O_0, O_1,\ldots,O_k\in\mathbb{R}^{r+q}_s$ be symmetric matrices. Assume that

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succeq 0 \quad \text{for all } i=1,2,\ldots,k.$$

Then

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_0 \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succ 0,\ \forall\,\theta\in\mathbb{R}^{q\times r} \tag{18}$$

if there exist positive definite scalars $\tau_i\geq 0,\ i=1,2,\ldots,k$ such that

$$O_0 - \sum_{i=1}^{k}\tau_i O_i \succ 0.$$

Proof: Let θ satisfy the condition

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succeq 0, \quad \forall i=1,2,\ldots,k.$$

Using the given inequality $O_0-\sum_{i=1}^{k}\tau_i O_i\succ 0$ for $\tau_i\geq 0$, we transform it by pre- and post-multiplying with $\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}$, yielding

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top \left(O_0 - \sum_{i=1}^{k}\tau_i O_i\right) \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succ 0.$$

Expanding this expression, we have

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_0 \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} - \sum_{i=1}^{k}\tau_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succ 0.$$

Since $\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}\succeq 0$ for each i, the term

$$\sum_{i=1}^{k}\tau_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_i \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}$$

is non-negative. Thus, we conclude that

$$\begin{bmatrix} I_q \\ \theta^\top\end{bmatrix}^\top O_0 \begin{bmatrix} I_q \\ \theta^\top\end{bmatrix} \succ 0,$$

resulting (18). ■

IV. ASSUMPTIONS ON THE INFORMATION SETS

When only partial information about the system's parameters is available, the actual parameters are assumed to belong to predefined sets. Defining these sets requires establishing baseline assumptions. The following sections outline the necessary assumptions for constructing information sets, which may originate from an initial prior knowledge set—based on the system's physical properties—or from online data gathered during operation. These sets form the foundation for stability analysis and the design of a dynamic feedback controller applicable to all systems with similar characteristics.

A. INITIAL PRIOR KNOWLEDGE SET

Assume that the physical dynamics governing the system's differential equation are known, implying that the system order, denoted by L, is also known. Furthermore, consider that the nominal values of the polynomial physical coefficients, represented by $\tilde{R}^\eta$ as in (2), are known. However, these coefficients are inherently uncertain and belong to a defined uncertainty set. The actual system, characterized by the coefficients $\tilde{R}^\star$, also follows the structure of (2) and is assumed to be in close proximity to the nominal system. Given a reasonable level of prior knowledge regarding the system's physical properties, it follows that the nominal and actual system behaviors must exhibit a degree of similarity. Based on this premise, we establish the following assumption.

Assumption 1 (Actual and nominal systems): The actual and nominal systems are considered to be $\epsilon - close$, $\epsilon>0$ to each other

$$\left\|\tilde{R}^\star - \tilde{R}^\eta\right\| \leq \epsilon. \tag{19}$$

It can be shown that an ϵ satisfying (19) almost always exists. Consequently, Assumption 1 only necessitates an upper bound on the norm (19). This is where the knowledge of the system is embedded. The designer possesses knowledge of the nominal system and the upper bound of the uncertainty ϵ. An initial prior knowledge set $\mathcal{E}^0_{pk}$ that is the set of all system coefficients that are ϵ-close to the nominal system $\tilde{R}^\eta$, which based on Assumption 1 includes the actual system $\tilde{R}^\star$, is denoted by

$$\mathcal{E}^0_{pk} := \left\{\tilde{R} : \left\|\tilde{R} - \tilde{R}^\eta\right\| \leq \epsilon\right\}. \tag{20}$$

Lemma 2: The initial prior knowledge set (20) is equivalently described as the following MQF set

$$\mathcal{E}^0_{pk} := \left\{\tilde{R} : \theta = \tilde{R},\ \begin{bmatrix} I \\ \theta^\top \end{bmatrix}^\top O^0_{pk} \begin{bmatrix} I \\ \theta^\top \end{bmatrix} := \mathcal{F}^0_{pk}(\theta) \succeq 0\right\} \tag{21}$$

where

$$O^0_{pk} := \begin{bmatrix} \epsilon^2 I - \tilde{R}^\eta \tilde{R}^{\eta\top} & \tilde{R}^\eta \\ \star & -I \end{bmatrix}.$$

Proof: By squaring both sides of (20), we obtain the following inequality

$$\begin{aligned} &(\tilde{R} - \tilde{R}^\eta)(\tilde{R} - \tilde{R}^\eta)^\top \preceq \epsilon^2 I \\ &\to -[\tilde{R}\tilde{R}^\top - \tilde{R}^\eta \tilde{R}^\top - \tilde{R}^\eta \tilde{R}^\top + \tilde{R}^\eta \tilde{R}^{\eta\top} - \epsilon^2 I] \succeq 0. \end{aligned} \tag{22}$$

By setting $\theta = \tilde{R}$ and comparing (22) with (13), one concludes the proof of the lemma. ■

The superscript 0 in (20)-(21) denotes that $\mathcal{E}^0_{pk}$ is the initial prior knowledge set. In Lemma 8, we will show how to update the prior knowledge set for $t > L$ and get $\mathcal{E}^t_{pk}$ in Section VI-C.

B. ONLINE NOISY INPUT-OUTPUT MEASUREMENTS

Let $w(t) = \begin{bmatrix} u^\top(t) & y^\top(t) \end{bmatrix}^\top$. Define $h_1(w(t))$ as the concatenation of $w(t)$ over L steps, given by $h_1(w(t)) = [w(t-L)^\top, \ldots, w(t-1)^\top]^\top$. Based on (1), the actual model of the system at the current time t reads

$$\begin{aligned} &y(t) + [-Q_{L-1}\ P_{L-1}] \begin{bmatrix} u(t-1) \\ y(t-1) \end{bmatrix} \\ &\quad + \cdots + [-Q_0\ P_0] \begin{bmatrix} u(t-L) \\ y(t-L) \end{bmatrix} = v(t-L), \end{aligned} \tag{23}$$

which can be equivalently written as

$$\begin{bmatrix} \tilde{R}^\star & I \end{bmatrix} \begin{bmatrix} h_1(w(t)) \\ y(t) \end{bmatrix} = v(t-L). \tag{24}$$

We introduce batch notations for the input-output measurements and disturbance of the system over $[t-N+L, t]$. Let the batch notation of the disturbance samples be

$$V^t := [v(t-N)\ v(t-N+1)\ \ldots\ v(t-L)]. \tag{25}$$

Then, the system can be represented batch-wise via

$$\begin{bmatrix} \tilde{R}^\star & I \end{bmatrix} \begin{bmatrix} H^t_1(w) \\ H^t_2(w) \end{bmatrix} = V^t, \tag{26}$$

where $H^t_1(w)$ contains the first qL rows, with $q := p + m$, and $H^t_2(w)$ the last p rows of a slightly modified following Hankel matrix $H^t(w) \in \mathbb{R}^{(qL+p)\times(N-L+1)}$

$$H^t(w) := \begin{bmatrix} w(t-N) & w(t-N+1) & \ldots & w(t-L) \\ w(t-N+1) & w(t-N+2) & \ldots & w(t-L+1) \\ \vdots & \vdots & & \vdots \\ w(t-N+L-1) & w(t-N+L) & \ldots & w(t-1) \\ y(t-N+L) & y(t-N+L+1) & \ldots & y(t) \end{bmatrix} \tag{27}$$

Although the disturbance sequence V^t from (25) at time step t is unknown, we make the following assumptions about its properties.

Assumption 2 (Disturbance bound): The disturbance samples as shown in (25) are assumed to be from the following set

$$\mathcal{V} := \left\{V^t : \begin{bmatrix} I \\ V^{t\top} \end{bmatrix}^\top O_n \begin{bmatrix} I \\ V^{t\top} \end{bmatrix} \succeq 0\right\} \tag{28}$$

where $O_n \in \mathbb{R}^{p+N-L+1}_s$ is a known partitioned matrix as

$$O_n := \begin{bmatrix} A_n & B_n \\ \star & C_n \end{bmatrix}. \tag{29}$$

In matrix O_n, one has $A_n \in \mathbb{R}^p_s$, $B_n \in \mathbb{R}^{p\times(N-L+1)}$, and $C_n \in \mathbb{R}^{N-L+1}_s$. In order for the set to be nonempty, we take $A_n \prec 0$ and $O_n \mid C_n \succeq 0$.

Remark 4: Assumption 2 provides a unified framework for capturing various types of disturbance encountered in practical scenarios, as discussed in [40]. In our formulation, the disturbance—whether due to measurement errors, linearization inaccuracies, or modeling discrepancies—is assumed to be bounded. Depending on the specific characteristics of the disturbance, the following cases can be represented:

- *Exact Measurements:* By setting $A_n = 0$, $B_n = 0$, and $C_n = -I$, we obtain $V^t = 0$, which corresponds to the ideal case of disturbance-free measurements.
- *Instantaneous Bounds:* If each disturbance sample satisfies $\|v(t)\|^2 \leq \varepsilon_v$, this case can be modeled by choosing $C_n = -I$, $B_n = 0$, and $A_n = \varepsilon_v(N-L+1)I$, thereby enforcing a per-sample disturbance bound.
- *Energy-Bounded Disturbance:* When the cumulative disturbance energy over the measurement interval, expressed as $\sum_i v(i)v(i)^\top = V^t V^{t\top}$, is bounded by ε_v, the disturbance can be captured by setting $C_n = -I$, $B_n = 0$, and $A_n = \varepsilon_v$.

Assumption 2 is sufficiently general to capture disturbance arising from nonlinearities or modeling errors across all these scenarios. However, if the disturbance characteristics are unknown and no bound is available, it becomes impossible to construct a compact set for the collected data.

Under Assumption 2, we define the set of systems, denoted by their physical coefficients $\tilde{R}$, that is conformed with the noisy input-output data. This is because, when defining a set for the disturbance samples, the systems conforming to the data are no longer a singleton but rather a set.

Take (26), one can write the following for the unknown physical coefficients $\tilde{R}$

$$\begin{bmatrix} I \\ V^{t\top} \end{bmatrix} = \begin{bmatrix} I & H_2^t(w) \\ 0 & H_1^t(w) \end{bmatrix}^\top \begin{bmatrix} I \\ \tilde{R}^\top \end{bmatrix}, \tag{30}$$

then, one can define the set of systems conformed with data at time step t with a window of $N \geq L$ using (28) as the following

$$\mathcal{E}_{data}^t := \left\{ \tilde{R} : \theta = \tilde{R},\ \begin{bmatrix} I \\ \theta^\top \end{bmatrix}^\top O_{data}^t \begin{bmatrix} I \\ \theta^\top \end{bmatrix} := \mathcal{F}_{data}^t(\theta) \succeq 0 \right\} \tag{31}$$

where

$$O_{data}^t := \begin{bmatrix} I & H_2^t(w) \\ 0 & H_1^t(w) \end{bmatrix} O_n \begin{bmatrix} I & H_2^t(w) \\ 0 & H_1^t(w) \end{bmatrix}^\top.$$

Remark 5: Since the data is collected from the actual system via (34), the quadratic set of data defined in (31) is nonempty and contains the actual system coefficient matrices $\tilde{R}^\star$.

V. STABILITY CONDITIONS AND TOOLS

To address the challenge of establishing a stability condition, we lay the groundwork for both stability analysis and stabilizing controller design by employing quadratic difference forms (QDFs) as Lyapunov functions within an input-output framework. We then examine the connection between QDFs and stability from a behavioral perspective. Finally, by exploiting the inherent compatibility of ARX models with the behavioral framework, we derive a specific stability condition for ARX models.

A. QDFS AS LYAPUNOV FUNCTIONS

We introduce the necessary definitions and preliminaries of QDFs in this subsection for completeness. However, for a more comprehensive treatment, we refer to [22], [23], [24], [25].

Definition 3: The quadratic difference form (QDF) is a quadratic form of a variable $\ell \in \mathbb{R}^q$ and its shifts, namely

$$Q_\Phi(\ell)(t) := \sum_{i=0}^{\rho} \sum_{j=0}^{\rho} \ell^\top(t+i)\Phi_{i,j}\ell(t+j), \tag{32}$$

where $q, \rho \in \mathbb{N}$, $\Phi_{i,j} \in \mathbb{R}_s^q$. Take $\Phi_{i,j}$ as a symmetric polynomial matrix, meaning $\Phi_{i,j} = \Phi_{i,j}^\top\ \forall\, i, j = 0, \ldots, \rho$. Also, denote the collection of the coefficient matrices $\Phi_{i,j}$ as the following

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \ldots & \Phi_{0,\rho} \\ \Phi_{1,0} & \Phi_{1,1} & \ldots & \Phi_{1,\rho} \\ \vdots & \vdots & & \vdots \\ \Phi_{\rho,0} & \Phi_{\rho,1} & \ldots & \Phi_{\rho,\rho} \end{bmatrix} \in \mathbb{R}_s^{(\rho+1)q}. \tag{33}$$

One can establish a one-to-one correspondence between a QDF $Q_\Phi(\ell)(t)$ and a symmetric two-variable polynomial matrix, where ζ and λ correspond to the shifts on $\ell^\top(t)$ and $\ell(t)$, respectively, as follows

$$\Phi(\zeta, \lambda) := \sum_{i=0}^{\rho} \sum_{j=0}^{\rho} \Phi_{i,j}\zeta^i\lambda^j \in \mathbb{R}_s^q[\zeta, \lambda]. \tag{34}$$

Remark 6: The degree $\mathcal{D}$ of a symmetric polynomial matrix $\Phi_{i,j}$ is defined as the maximum i for which there exists a positive j such that the coefficient matrix $\Phi_{i,j} \neq 0$ and $\Phi_{i,j} = \Phi_{i,j}^\top$. Thus, one can define a QDF in the form of (32) with any $\rho \leq \mathcal{D}$, and consequently, the coefficient matrices $\Phi_{i,j}$ might not be unique. Conversely, one can collect the found coefficient matrices in matrix $\tilde{\Phi} \in \mathbb{R}_s^{(\mathcal{D}+1)q}$. This is useful in defining the dimension of $\tilde{\Phi}$ in stability analysis and control design in the subsequent sections.

Remark 7: Some properties of the QDFs as in (32) are as follows

- if $Q_\Phi(\ell) \geq 0$ for all $\ell \in \mathbb{R}^q$, then, $Q_\Phi(\ell)$ is called nonnegative,
- if $Q_\Phi(\ell) > 0$ for all $\ell \in \mathbb{R}^q$ except for $\ell = 0$, then, $Q_\Phi(\ell)$ is called positive,
- the negative and nonpositive QDFs $Q_\Phi(\ell)$ are defined in the same manner.

Since $\ell \in \mathbb{R}^q$, we have $\tilde{\Phi} \succeq 0$ in all cases.

Definition 4: Let us define $\nabla Q_\Phi(\ell)(t) := Q_\Phi(\ell)(t+1) - Q_\Phi(\ell)(t)$, $\forall t \in \mathbb{Z}+$, as the incremental change of $Q\Phi(\ell)(t)$. Based on (34), we define $\nabla\Phi(\zeta, \lambda) := \zeta\lambda\Phi(\zeta, \lambda) - \Phi(\zeta, \lambda)$. Then, $\nabla Q_\Phi(\ell)(t)$ can be equivalently described by a QDF $Q_{\nabla\Phi}(\ell)(t)$, $\forall t \in \mathbb{Z}_+$.

B. CONNECTION TO BEHAVIOR THEORY AND ARX MODEL

Take the manifest variable $w(t) = \begin{bmatrix} u(t)^\top & y(t)^\top \end{bmatrix}^\top$ as variable $\ell(t)$ in (32). Then, equivalent to (32), one has

$$Q_\Phi(w)(t) := \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+D) \end{bmatrix}^\top \tilde{\Phi} \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+D) \end{bmatrix} \tag{35}$$

Let $\bar{\mathfrak{B}}(R)$ be the behavior of the deterministic system (kernel representation) $R(\sigma)w = 0$. All the definitions of QDFs and their properties carry over for the behavior of the system $\bar{\mathfrak{B}}(R)$ for all the manifest variables $w \in \bar{\mathfrak{B}}(R)$ and one can use (35) as the candidate Lyapunov function to study stability. Based on Remark 6, one can have two $\bar{\mathfrak{B}}(R)$-equivalent QDFs with different collections of coefficient matrices (33), such as $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$, if these QDFs coincide on solutions of $R(\sigma)w = 0$, meaning that for all $w \in \bar{\mathfrak{B}}(R)$, we have $Q_{\tilde{\Phi}_1}(w)(t) = Q_{\tilde{\Phi}_2}(w)(t)$. This property is useful when it turns out that any QDF of the system is equivalent to a QDF whose degree is at most the order of the system minus one, i.e., $(L-1)$. This statement holds for both autonomous and controlled systems. To prove some properties of the developed theorems, we take advantage of the following lemmas.

Lemma 3: [Lemma 5, [19]] For any QDF $Q_{\Phi'}(y)$, there exists an equivalent QDF $Q_{\Phi}(y)$ with respect to the behavior $\mathfrak{B}(P)$, such that the degree of $Q_{\Phi}(y)$ is at most $L-1$, where L is the order of the system. Moreover, if $Q_{\Phi'}(y) \geq 0$ for all $y \in \mathfrak{B}(P)$, then, $Q_{\Phi}(y) \geq 0$ for all $y \in \mathfrak{B}(P)$ as well, which is equivalent to $\tilde{\Phi} \succeq 0$.

Let us consider the QDF Q_{Π} as a Lyapunov function. Then, the following lemma establishes a connection between the stability of the autonomous system as in (3) and the existence of a QDF Q_{Π} with a degree at most $L-1$ as shown in Lemma 3.

Lemma 4: [Lemma 8, [19]] Let $P(\xi)$ be a polynomial matrix for the system (3). The corresponding deterministic autonomous system $P(\sigma)y = 0$ of order L is stable if and only if there exists a QDF $Q_{\Pi}(y)$ with a degree at most $L-1$ satisfying the following conditions

- $Q_{\Pi}(y) \geq 0$ for all $y \in \bar{\mathfrak{B}}(P)$,
- $Q_{\nabla\Pi}(y) < 0$ for all $y \in \bar{\mathfrak{B}}(P)$, where $Q_{\nabla\Pi}(y)(t) = Q_{\Pi}(y)(t+1) - Q_{\Pi}(y)(t)$ from Definition 4.

For stability analysis of a deterministic autonomous system $P(\sigma)y = 0$, we first establish the existence of a QDF Q_{Π} that serves as a Lyapunov function. The following theorem provides the foundational result for the deterministic scenario, and later, we leverage the S-procedure to generalize this concept to a set of systems conformed with both data and prior knowledge. For the sake of completeness, we also bring our version of the proof of this theorem.

Theorem 1: [25] For the deterministic autonomous system $P(\sigma)y = 0$ and physical coefficients defined as $\tilde{P} := [P_0\ P_1\ \ldots\ P_{L-1}]$, this system is stable if and only if there exists a $\Pi \in \mathbb{R}_s^{pL}$ such that $\Pi \succeq 0$ and the following matrix inequality is satisfied

$$\begin{bmatrix} I \\ -\tilde{P} \end{bmatrix}^\top \left(\begin{bmatrix} 0_p & 0 \\ 0 & \Pi \end{bmatrix} - \begin{bmatrix} \Pi & 0 \\ 0 & 0_p \end{bmatrix} \right) \begin{bmatrix} I \\ -\tilde{P} \end{bmatrix} \prec 0. \tag{36}$$

Any such Π defines a QDF Q_{Π} that serves as a Lyapunov function, certifying the stability of the autonomous system.

Proof: Since Π is nonnegative, then, Q_{π} is nonnegative due to the properties of QDF in Remark 7. Thus, based on Lemma 4, if one shows that $Q_{\nabla\Pi}(y)(t) < 0\ \forall\ y \in \bar{\mathfrak{B}}(P)$, then, Q_{Π} is a candidate Lyapunov function. Based on Definition 4, one has $\nabla\Pi(\zeta, \lambda) = \zeta\lambda\Pi(\zeta, \lambda) - \Pi(\zeta, \lambda)$. Based on (32), (33), and $P(\sigma)y = 0$, one has $\sum_{i,j=0}^{L-1} \zeta^i \lambda^j \Pi(\zeta, \rho) = \tilde{P}^\top \Pi \tilde{P}$.

To prove stability, one needs to show that $Q_{\nabla\Pi}(y)(t) < 0\ \forall\ y \in \bar{\mathfrak{B}}(P)$,

$$Q_{\nabla\Pi}(y)(t) = \begin{bmatrix} y(t) \\ \vdots \\ y(t+L-1) \end{bmatrix}^\top \tilde{P}^\top \Pi \tilde{P} \begin{bmatrix} y(t) \\ \vdots \\ y(t+L-1) \end{bmatrix} - \begin{bmatrix} y(t) \\ \vdots \\ y(t+L-1) \end{bmatrix}^\top \Pi \begin{bmatrix} y(t) \\ \vdots \\ y(t+L-1) \end{bmatrix} \prec 0. \tag{37}$$

Note that one can write

$$\tilde{P}^\top \Pi \tilde{P} - \Pi = \begin{bmatrix} I \\ -\tilde{P} \end{bmatrix}^\top \left(\begin{bmatrix} 0_p & 0 \\ 0 & \Pi \end{bmatrix} - \begin{bmatrix} \Pi & 0 \\ 0 & 0_p \end{bmatrix} \right) \begin{bmatrix} I \\ -\tilde{P} \end{bmatrix}.$$

Based on the aforementioned equation, $Q_{\nabla\Pi}(y)(t) \leq 0,\ \forall\, t = 1, 2, \ldots$ and $Q_{\nabla\Pi}(y)(t) = 0$ if and only if $y(t) = 0$ which results in $Q_{\nabla\Pi}(y)(t) < 0\ \forall\, y \in \bar{\mathfrak{B}}$. Thus, the proof is complete, and Q_{Π} is a candidate Lyapunov function. ■

VI. MAIN RESULTS

The main results focus on stability analysis of (3) and designing a stabilizing controller for (1) by combining prior knowledge and noisy input-output data in an online manner. The actual system lies within $\mathcal{E}^t_{PDCS}$, ensuring the validity of the stability condition and controller. $\mathcal{E}^t_{pk}$ is compact under Assumption 1, but $\mathcal{E}^t_{data}$ may not compact be due to limited data. The goal is to determine the stability of an autonomous system and then, stabilize an unstable system using limited data and update prior knowledge based on the data.

A. ONLINE PRIOR-KNOWLEDGE-BASED DATA-DRIVEN STABILITY ANALYSIS FOR AUTOREGRESSIVE AUTONOMOUS SYSTEM MODELS

This subsection provides details for determining the stability of an autonomous system (3) by combining prior knowledge of the system matrices and collected data. In autonomous systems, the modified Hankel matrix (27) only contains the output y as the manifest variable w at time step t as

$$H^t(y) := \begin{bmatrix} y(t-N) & y(t-N+1) & \ldots & y(t-L) \\ y(t-N+1) & y(t-N+2) & \ldots & y(t-L+1) \\ \vdots & \vdots & & \vdots \\ y(t-N+L-1) & y(t-N+L) & \ldots & y(t-1) \\ y(t-N+L) & y(t-N+L+1) & \ldots & y(t) \end{bmatrix} \tag{38}$$

where $H^t_1(y)$ consists of the initial pL rows, while $H^t_2(y)$ comprises the final p rows. The data-conformity set (31) is also for this special case modified by modifying O_{data} to the following

$$O^t_{y,data} := \begin{bmatrix} I & H^t_2(y) \\ 0 & H^t_1(y) \end{bmatrix} O_n \begin{bmatrix} I & H^t_2(y) \\ 0 & H^t_1(y) \end{bmatrix}^\top$$

where O_n is given in (29).

In the derivation of information sets (21) and (31), $\tilde{R}^\top$ or $\tilde{P}^\top$ are used for autonomous systems. However, the stability condition in Theorem 1 is derived for $\tilde{P}$. We restate Theorem 1 using $\tilde{P}^\top$.

Lemma 5: Define a matrix $\tilde{I} := \begin{bmatrix} 0_{p(L-1)\times p} & I_{p(L-1)} \end{bmatrix} \in \mathbb{R}^{p(L-1)\times pL}$ and matrix $\Psi = \Pi^{-1}$. Then, if Ψ satisfies the following inequality

$$\Psi - \begin{bmatrix} \tilde{I} \\ -\tilde{P} \end{bmatrix} \Psi \begin{bmatrix} \tilde{I} \\ -\tilde{P} \end{bmatrix}^\top \succ 0,\ \Psi^{-1} \succ 0 \tag{39}$$

then, Π satisfies

$$\begin{bmatrix} \tilde{I} \\ -\tilde{P} \end{bmatrix}^\top \Pi \begin{bmatrix} \tilde{I} \\ -\tilde{P} \end{bmatrix} - \Pi \prec 0. \quad (40)$$

Proof: One can derive (40) from (39) by applying the Schur complement twice. ∎

Remark 8: The standard Lyapunov stability inequality in (40) can be derived from (36) by inspection. Furthermore, if $\Pi \succeq 0$ satisfies (36), then, both (40) and (39) hold.

Lemma 6: The MQF of the Lyapunov inequality (39) can be written as

$$\mathcal{F}_L(\vartheta) := \vartheta C_L \vartheta^\top + \vartheta B_L^\top + B_L \vartheta^\top + A_L \succ 0 \quad (41)$$

with

$$\vartheta := \tilde{P} \begin{bmatrix} 0 \\ -I_p \end{bmatrix} \quad (42)$$

and

$$A_L := \left[\Psi - \begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} \Psi \begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix}^\top \right], \; B_L := -\begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} \Psi, \; C_L := -\Psi. \quad (43)$$

Proof: If one takes

$$\begin{bmatrix} \tilde{I} \\ -\tilde{P} \end{bmatrix} = \begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\tilde{P} \end{bmatrix} \quad (44)$$

and rewrites (39), then, the following is resulted

$$\underbrace{\left[\Psi - \begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} \Psi \begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix}^\top \right]}_{A_L} + \underbrace{\begin{bmatrix} 0 \\ -\tilde{P} \end{bmatrix}}_{\vartheta} \underbrace{\left(-\begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} \Psi \right)^\top}_{B_L^\top}$$
$$+ \underbrace{\left(-\begin{bmatrix} \tilde{I} \\ 0 \end{bmatrix} \Psi \right)}_{B_L} \underbrace{\begin{bmatrix} 0 \\ -\tilde{P} \end{bmatrix}^\top}_{\vartheta^\top} + \underbrace{\begin{bmatrix} 0 \\ -\tilde{P} \end{bmatrix}}_{\vartheta} \underbrace{(-\Psi)}_{C_L} \underbrace{\begin{bmatrix} 0 \\ -\tilde{P} \end{bmatrix}^\top}_{\vartheta^\top} \succ 0. \quad (45)$$

∎

To apply the strict lossy S-procedure, the same variable must be used in all MQFs (41), (21), and (31). If ϑ is chosen as the variable, where $\vartheta = \theta \begin{bmatrix} 0 & -I_p \end{bmatrix}$, then, the information sets (21) and (31) need to be projected accordingly. The new set of parameters is as follows

$$\bar{O}_{pk}^t := \begin{bmatrix} \begin{bmatrix} 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{pL} \end{bmatrix}^\top O_{pk}^t \begin{bmatrix} \begin{bmatrix} 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{pL} \end{bmatrix}. \quad (46)$$

The projected data conformity set of parameters $\bar{O}_{data}^t$ is projected similarly. With the new variable, one can redefine the MQFs for prior knowledge $\mathcal{F}_{pk}^t(\vartheta)$ and data $\mathcal{F}_d^t(\vartheta)$ as well. Then, the overlap of the two sets is defined as

$$\bar{\Delta}^t := \{\vartheta : \mathcal{F}_d^t(\vartheta) \succeq 0, \mathcal{F}_{pk}^t(\vartheta) \succeq 0\}. \quad (47)$$

The following theorem addresses Problem 1.

Theorem 2: Consider the stochastic uncertain autonomous system (3). Let Assumptions 1 and 2 hold. Then, all systems (3) that conform to the combined information set $\mathcal{E}_{PDCS}^t$ (47) at time step t are stable if there exists $\Psi \in \mathbb{R}_s^{pL}$ such that $\Psi \succ 0$ and the following inequality holds for a scalar $0 \leq \tau \leq 1$

$$\begin{bmatrix} A_L & B_L^\top \\ \star & C_L \end{bmatrix} - \tau \bar{O}_{data}^t - (1-\tau)\bar{O}_{pk}^t \succ 0, \quad (48)$$

where A_L, B_L and C_L are defined in (43).

Proof: Define scalars such that $\tau_d, \tau_{pk} \geq 0$. Using the strict lossy S-procedure from Lemma 1, the preceding inequality (48) indicates the matrix quadratic function $\mathcal{F}_L(\vartheta) \succ 0$ in (41) for all $\vartheta \in \bar{\Delta}^t$ as defined in (47) as the following for $\tau_d, \tau_{pk} \geq 0$

$$\begin{bmatrix} A_L & B_L \\ \star & C_L \end{bmatrix} - \tau_d \bar{O}_{data}^t - \tau_{pk} \bar{O}_{pk}^t \succ 0. \quad (49)$$

Inequality (48) is derived by scaling (49) with $\tau = \frac{\tau_d}{\tau_d + \tau_{pk}}$. Given that $\tau_d, \tau_{pk} \geq 0$, it follows that $0 \leq \tau \leq 1$, ensuring that this transformation preserves the inequality. Since the actual system $\tilde{P}^\star \in \mathcal{E}_{PDCS}^t$, then, the QDF Q_Π with the found solution $\Psi := \Pi^{-1}$ is also a Lyapunov function for the actual system. Thus, the proof is concluded. ∎

B. ONLINE PRIOR-KNOWLEDGE-BASED DATA-DRIVEN DYNAMIC STABILIZING CONTROL SYNTHESIS FOR AUTOREGRESSIVE EXOGENOUS SYSTEM MODELS

This subsection aims to synthesize a stabilizing dynamic feedback controller for the system (1) that stabilizes all systems conformed with $\mathcal{E}_{PDOS}^t$. The controller is defined to have the same dynamical structure as the plant, with the leading coefficient of $D(\xi)$ set to $I_{m\times m}$ and $E(\xi)$ to $0_{m\times p}$ for strict properness. The closed-loop system is defined by interconnecting the system and controller dynamics with the shift operator σ as

$$\begin{bmatrix} D(\sigma) & -E(\sigma) \\ -Q(\sigma) & P(\sigma) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ I_p \end{bmatrix} v, \quad (50)$$

where

$$F(\sigma) := \begin{bmatrix} D(\sigma) & -E(\sigma) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0, \quad (51)$$

is the dynamics of the controller. Its physical coefficients are defined as

$$D(\xi) = I\xi^L + D_{L-1}\xi^{L-1} + \cdots + D_1\xi + D_0,$$
$$E(\xi) = E_{L-1}\xi^{L-1} + \cdots + E_1\xi + E_0.$$

One has the collected physical coefficients of the controller dynamics as

$$\tilde{F} := \begin{bmatrix} D_0 & -E_0 & D_1 & -E_1 & \ldots & D_{L-1} & -E_{L-1} \end{bmatrix}, \quad (52)$$

where $E_k \in \mathbb{R}^{m\times p}$ and $D_k \in \mathbb{R}_s^m$ for $k = 1, 2, \ldots, L-1$. Stabilizing the closed-loop system can be considered analogous to determining the stability of an autonomous system. Take the interconnected system coefficient as

$$\tilde{P}_{cl} := \begin{bmatrix} \tilde{F} \\ \tilde{R} \end{bmatrix}, \quad (53)$$

then, the results presented previously hold for the closed-loop system (50).

Theorem 3: For the deterministic closed-loop system ((50) with $v = 0$) with physical coefficients defined as (53), this system is stable if and only if there exists a $\Pi \in \mathbb{R}_s^{qL}$ such that $\Pi \succeq 0$ and the following matrix inequality is satisfied

$$\begin{bmatrix} I \\ -\tilde{P}_{cl} \end{bmatrix}^\top \left(\begin{bmatrix} 0_q & 0 \\ 0 & \Pi \end{bmatrix} - \begin{bmatrix} \Pi & 0 \\ 0 & 0_q \end{bmatrix} \right) \begin{bmatrix} I \\ -\tilde{P}_{cl} \end{bmatrix} \prec 0. \quad (54)$$

Any such Π defines a QDF Q_Π that serves as a Lyapunov function, certifying the stability of the autonomous system.

Proof: The proof is similar to the one of Theorem 1. ■

From (54), it follows that the closed-loop system coefficients $\tilde{P}_{cl}$ satisfy a Lyapunov stability inequality similar to (40). To this end, one can define the following

$$\tilde{I}_{cl} := \begin{bmatrix} 0_{q(L-1)\times q} & I_{q(L-1)} \end{bmatrix}.$$

If $\Pi \succeq 0 \in \mathbb{R}_s^{qL}$ satisfies (54) then, one can see by inspection that it satisfies the following standard Lyapunov inequality for $\Pi \succ 0$

$$\begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{P}_{cl} \end{bmatrix}^\top \Pi \begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{P}_{cl} \end{bmatrix} - \Pi \prec 0. \quad (55)$$

and also using Lemma 5, for $\Psi = \Pi^{-1}$ one has the following from

$$\Psi - \begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{P}_{cl} \end{bmatrix} \Psi \begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{P}_{cl} \end{bmatrix}^\top \succ 0. \quad (56)$$

The above inequity (56) resolves the problem of having the information sets in terms of $\tilde{R}^\top$ and (55) in terms of $\tilde{R}$.

Lemma 7: The Lyapunov inequality (56) is equivalent to the following MQF

$$\mathcal{F}_{cl}(\vartheta_{cl}) := \vartheta_{cl} C_{cl} \vartheta_{cl}^\top + \vartheta_{cl} B_{cl}^\top + B_{cl} \vartheta_{cl}^\top + A_{cl} \succ 0 \quad (57)$$

with

$$\vartheta_{cl} := \tilde{R} \begin{bmatrix} 0 \\ 0 \\ -I_p \end{bmatrix}, \quad G := -\tilde{F}\Psi \quad (58)$$

and

$$A_{cl} := \Psi - \begin{bmatrix} \tilde{I}_{cl}\Psi \\ G \\ 0 \end{bmatrix} \Psi^{-1} \begin{bmatrix} \tilde{I}_{cl}\Psi \\ G \\ 0 \end{bmatrix}^\top, \quad (59)$$

$$B_{cl} := - \begin{bmatrix} \tilde{I}_{cl}\Psi \\ G \\ 0 \end{bmatrix}, \quad C_{cl} := -\Psi. \quad (60)$$

Proof: One has

$$\begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{P}_{cl} \end{bmatrix} = \begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{F} \\ -\tilde{R} \end{bmatrix} = \begin{bmatrix} \tilde{I}_{cl} \\ -\tilde{F} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\tilde{R} \end{bmatrix}. \quad (61)$$

To derive the results from (56), we introduce the change of variable $G := -\tilde{F}\Psi$ and apply the Schur complement. This approach is analogous to the proof of Lemma 6. ■

To use the strict lossy S-procedure, all MQFs (57), (21), and (31) must have the same variable. If we choose ϑ_{cl} as the variable, since $\vartheta_{cl} = \theta \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix}$, we need to project the prior-knowledge set (21) and the data-conformity set (31) to match the new variable ϑ_{cl}. The new set of parameters is given by

$$\hat{O}_{pk}^t := \begin{bmatrix} \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{qL} \end{bmatrix}^\top O_{pk}^t \begin{bmatrix} \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{qL} \end{bmatrix}. \quad (62)$$

The projected data-conformity set $\hat{O}_{data}^t$ is computed similarly. With the new variable, the MQFs for prior knowledge $\mathcal{F}_{pk}^t(\vartheta_{cl})$ and data $\mathcal{F}_d^t(\vartheta_{cl})$ can be redefined. The overlap of the two sets is

$$\hat{\Delta}^t := \{\vartheta_{cl} : \mathcal{F}_d^t(\vartheta_{cl}) \succeq 0, \mathcal{F}_{pk}^t(\vartheta_{cl}) \succeq 0\}. \quad (63)$$

Theorem 4: Consider the stochastic uncertain system (1). Let Assumptions 1 and 2 hold. If there exist $\Psi \in \mathbb{R}_s^{qL}$ such that $\Psi \succ 0$ and $\tilde{F} = -G\Psi^{-1} \in \mathbb{R}^{m\times qL}$ and also, the LMI (64) shown at the bottom of this page, holds for a scalar $0 \le \tau \le 1$, then, the dynamic controller with the coefficient matrix $\tilde{F}$ stabilizes all systems (1) that conform to the combined information set $\mathcal{E}_{PDOS}^t$ (63) at time step t.

Proof: Define scalars such that $\tau_d, \tau_{pk} \ge 0$. Applying the strict lossy S-procedure from Lemma 1, the LMI (64) implies that the matrix quadratic function $\mathcal{F}_{cl}(\vartheta_{cl}) \succ 0$ for all $\vartheta_{cl} \in \hat{\Delta}^t$, as defined in (63) as the following for $\tau_d, \tau_{pk} \ge 0$

$$\begin{bmatrix} A_{cl} & B_{cl} \\ \star & C_{cl} \end{bmatrix} - \tau_d \bar{O}_{data}^t - \tau_{pk} \bar{O}_{pk}^t \succ 0. \quad (65)$$

Inequality (64) is derived by scaling (65) by $\tau = \frac{\tau_d}{\tau_d + \tau_{pk}}$. Given that $\tau_d, \tau_{pk} \ge 0$, it follows that $0 \le \tau \le 1$, ensuring that this transformation preserves the inequality. Since the actual system $\tilde{R}^\star \in \mathcal{E}_{PDOS}^t$, the QDF Q_Π with the found solution $\Psi := \Pi^{-1}$ is also a Lyapunov function for the actual system. Consequently, the controller with coefficient matrix $\tilde{F}$ stabilizes the actual system as well. This concludes the proof. ■

Remark 9: When an initial prior knowledge set $\mathcal{E}_{pk}^0$ is available, the collected system data (see (27) or (38)) need not be full-rank for (64) and (48) to admit a solution; that is, the online data is not required to satisfy excitation conditions.

$$\begin{bmatrix} \Psi & -\begin{bmatrix} \tilde{I}_{cl}\Psi \\ G \\ 0 \end{bmatrix} & \begin{bmatrix} \tilde{I}_{cl}\Psi \\ G \\ 0 \end{bmatrix} \\ \star & -\Psi & 0 \\ \star & \star & \Psi \end{bmatrix} - \tau \begin{bmatrix} \hat{O}_{data}^t & 0 \\ \star & 0_{qL} \end{bmatrix} - (1-\tau) \begin{bmatrix} \hat{O}_{pk}^t & 0 \\ \star & 0_{qL} \end{bmatrix} \succ 0 \quad (64)$$

This is because even if the set conformed with the data itself is noncompact (as defined in Definition 2), its intersection with the initial prior knowledge set (21) can be over-approximated by a compact set leveraging the lossy S-procedure, as shown in Fig. 2. In contrast, without an initial prior knowledge set, the data matrices (27) or (38) must be full-rank in the first computation round. The PE condition is typically required in the literature to ensure the compactness of the set of models conformed with collected data [35]. Once the prior knowledge set is formed using Lemma 8, however, the compactness of the collected data is no longer necessary—a consequence of the properties of the lossy S-procedure [29].

Remark 10: Theorem 2 and Theorem 4 provide sufficient conditions for stability analysis and control synthesis, respectively, by integrating data with prior knowledge. For specific values of $\mu_d, \mu_{pk} \in \mathbb{R}$, the LMIs (48) and (64) become necessary and sufficient if [2]

$$\mu_d O^t_{data} + \mu_{pk} O^t_{pk} \succ 0, \tag{66a}$$

$$\mathcal{F}_{pk}(\bar{\theta}) \succ 0, \quad \mathcal{F}_d(\bar{\theta}) \succ 0. \tag{66b}$$

These constraints guarantee the existence of a matrix $\bar{\theta} \in \mathcal{E}^t_{data} \cap \mathcal{E}^t_{pk}$ for which both $\mathcal{F}^t_{pk}(\bar{\theta})$ and $\mathcal{F}^t_d(\bar{\theta})$ are nonsingular if (66a) holds. In particular, constraint (66b), which serves as a Slater-type condition, is satisfied by virtue of Assumption 1 and Remark 5.

C. REFINING THE PRIOR KNOWLEDGE SET

Let $\mathcal{E}^t_{PDOS}$ denote the set of systems for which controllers are designed at time t. A significant challenge lies in preventing this set from expanding or fluctuating unpredictably over time, as such changes can adversely affect controller behavior. A visual depiction is provided in Fig. 2.

The following lemma proposes a dynamic update strategy for the prior-knowledge set $\mathcal{E}^t_{pk}$ to address this.

Lemma 8: Define the projected prior knowledge set's parameter update rule at time t as

$$\bar{O}^{t+1}_{pk} := \tau \bar{O}^t_{data} + (1-\tau)\bar{O}^t_{pk} \tag{67}$$

for the stability test case and

$$\hat{O}^{t+1}_{pk} := \tau \hat{O}^t_{data} + (1-\tau)\hat{O}^t_{pk} \tag{68}$$

for the learning stabilizing controller case. Note that $0 \leq \tau \leq 1$ controls the rate at which prior knowledge and data conformity sets are combined. τ is computed from (48) or (64) at each step for the stability check or stabilizing controller design cases, respectively. Then, using the refined prior knowledge at each time step in (48) or (64) leads to behavior alteration prevention.

Proof: The prior-knowledge set $\mathcal{E}^0_{pk}$ is initially defined based on the designer's knowledge of the system, as described by the uncertain set $\mathcal{E}^0_{pk} := \{\tilde{R} : \mathcal{F}^0_{pk}(\theta) \preceq 0\}$, where $\mathcal{F}^0_{pk}(\theta)$ is a quadratic representation of the prior knowledge set derived from the system's physical properties. In the first step, the LMIs (48) or (64) are solved without using additional data, addressing either the stability test or stabilizing controller design problems. After that, at each time $t > L$, as the system evolves and new data becomes available, the data-conformity set $\mathcal{E}^t_{data}$ is defined based on collected data. To dynamically update the prior-knowledge set, we compute the overlap between the projected $\mathcal{E}^t_{pk}$ and $\mathcal{E}^t_{data}$ based on the strict lossy $\mathcal{S}$-procedure's properties [2] as an outer-approximated ellipsoidal set. The resulting set, which refines $\mathcal{E}^t_{pk}$, is given by $\mathcal{E}^{t+1}_{pk} := \mathcal{E}^t_{pk} \cap \mathcal{E}^t_{data}$. By substituting the quadratic forms of $\mathcal{E}^t_{pk}$ and $\mathcal{E}^t_{data}$ and projecting them based on the problem requirements, the parameters of the updated projected prior-knowledge set are obtained as (67) or (68). This concludes the proof. ■

Remark 11: In Lemma 8, for each time step $t > L$, the prior knowledge set is updated by recalculating the overlap between $\mathcal{E}^t_{pk}$ and $\mathcal{E}^t_{data}$. This process incorporates the informative data (present in the overlap set) into the prior knowledge for subsequent steps while discarding non-informativeness, as illustrated in Fig. 2, whether in the prior knowledge set or data. This dynamic adjustment controls both the size and composition of $\mathcal{E}^t_{PDOS}$, mitigating behavioral deviations caused by fluctuations in the overlap through an effective filtering mechanism.

D. ONLINE PRIOR-DATA DRIVEN INPUT-OUTPUT STABILIZING CONTROL ALGORITHM

Algorithm 1 outlines the procedure of designing an online Prior Data-Driven (P-DD) input-output controller based on Theorem 4 and Lemma 8. In this algorithm, the system input is assigned a random value from the uniform distribution $U[\underline{U}, \bar{U}]$ until the LMI (64) has a solution. This ensures persistent excitation of the input signal, facilitating the collection of informative data.

Additionally, at each step, if the updated prior knowledge set is larger than the previous one, we discard it and continue using the previous set to maintain consistent performance.

VII. SIMULATION RESULTS

This section presents case studies to validate the robust stabilizing controller. First, an unstable scalar system is stabilized using the dynamic feedback controller in an offline setting. Next, a rotary inverted pendulum with uncertain parameters is regulated by incorporating prior knowledge and data-driven information sets to design a robust controller using the proposed online method. In both cases, the matrices $\Psi \succ 0$, G, and the scalar $0 \leq \tau \leq 1$ are determined to satisfy the LMI (64) using YALMIP and Mosek as the solver [42].[1]

A. UNSTABLE SCALAR SYSTEM:

This subsection serves two main purposes. First, it visually illustrates the initial prior knowledge set, the data-conformity set, and their overlap. Second, it constructs a dynamic feedback controller to stabilize an unstable scalar system. To illustrate these concepts, consider the following scalar system

[1] https://github.com/NarimanNiknejad/Ph-Learning-ARX

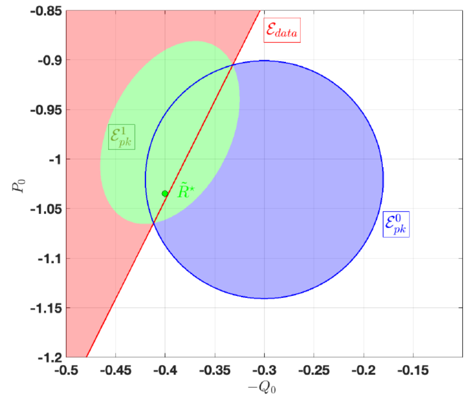

FIGURE 3. **Overlap illustration for the scalar system (69). The initial prior knowledge set $\mathcal{E}^0_{pk}$ (blue), based on the designer's estimate, and the data-conformity set $\mathcal{E}_{data}$ (red), corresponding to the actual physical coefficients $\tilde{R}^\star$, are shown. The updated prior knowledge set $\mathcal{E}^1_{pk}$ (green) is smaller than each of the individual sets, demonstrating the refinement achieved by their intersection.**

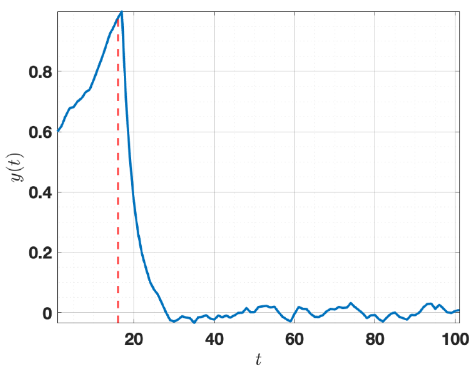

FIGURE 4. **The closed-loop performance of the robust controller resulting from (64) on the unstable scalar system (69).**

with an order of $L = 1$

$$y(t+1) + \underbrace{\begin{bmatrix} -Q_0 & P_0 \end{bmatrix}}_{R_0} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = v(t). \tag{69}$$

The actual system's physical coefficients are taken as $R_0^\star = \begin{bmatrix} -0.4 & -1.035 \end{bmatrix}$ leading to an unstable open-loop and the nominal one is $R_0^\eta = \begin{bmatrix} -0.3 & -1.021 \end{bmatrix}$. Consequently, the closeness value from (19) is set to $\epsilon = 0.12$. The disturbance energy is assumed to be bounded, corresponding to the third instance of the disturbance description in Remark 4 with $\varepsilon_v = 0.01$. The number of collected data points is $N = 1$ with a random input $u \sim U[-0.1, 0.1]$ in an open-loop way. Fig. 3 illustrates that stabilizing the intersection of the information sets $\mathcal{E}^t_{PDOS}$ is significantly less conservative than stabilizing either $\mathcal{E}_{pk}$ or $\mathcal{E}_{data}$ individually. Solving (64) for these information sets yields a control dynamics with $\tilde{F} = \begin{bmatrix} 0.4183 & 1.1340 \end{bmatrix}$. Starting the simulation from $y(0) = 0.6$, the controller is activated at $t = 16$ to demonstrate its effectiveness in stabilizing the actual system. As shown in Fig. 4, the system is stabilized with a single data point.

Algorithm 1: `Online P-DD IO Control`

```
Inputs: N, p, m, R̃^η, L, ε, O_n
check = 0;
t = 0;
Compute projected prior-knowledge set Ô^0_pk using
  (21) and (62);
if LMI (64) holds for Ô^0_pk and Ô^0_data = 0 then
    compute G and Ψ, compute F̃ = −GΨ^{−1}, and
      form controller (52);
    check = 1;
else
    set u ~ U[U̲, Ū];
while Simulation do
    if t > L then
        Refine Ô^t_pk using Lemma 8;
    if Size(Ô^t_pk) > Size(Ô^{t−1}_pk) then
        Ô^t_pk = Ô^{t−1}_pk;
    Give system u;
    Let the system evolve;
    Collect input u and corresponding y;
    Compute Ô^t_data using (31) and (62);
    if LMI (64) holds for Ô^t_pk and Ô^t_data then
        compute G and Ψ, compute the feedback
          control F̃ = −GΨ^{−1}, and form the
          controller (52);
        check =1;
    else
        if check==1 then
            Use previous feedback control F̃;
        else
            set u ~ U[U̲, Ū]
    t = t + 1;
```

In this example, we demonstrate that incorporating data reduces the size of the prior knowledge set, as illustrated in Fig. 3.

B. UNSTABLE HIGHER ORDER SYSTEM: (ROTARY INVERTED PENDULUM)

In the online case study, the objective is to design a robust controller for stabilizing the rotary inverted pendulum system (12) using two distinct approaches. The first approach relies solely on online data for controller design, following the methodology of [19] but applied in an online setting. The second approach integrates both online data and initial prior system knowledge to design the controller and updates the prior knowledge set. Both simulations start at $y(0) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^\top$. The nominal and actual parameter values for the physical coefficients $\tilde{R}^\eta$ of the ARX system in (12) are provided in Table 2. The discretization step size is set $\Delta t = 0.01$. The

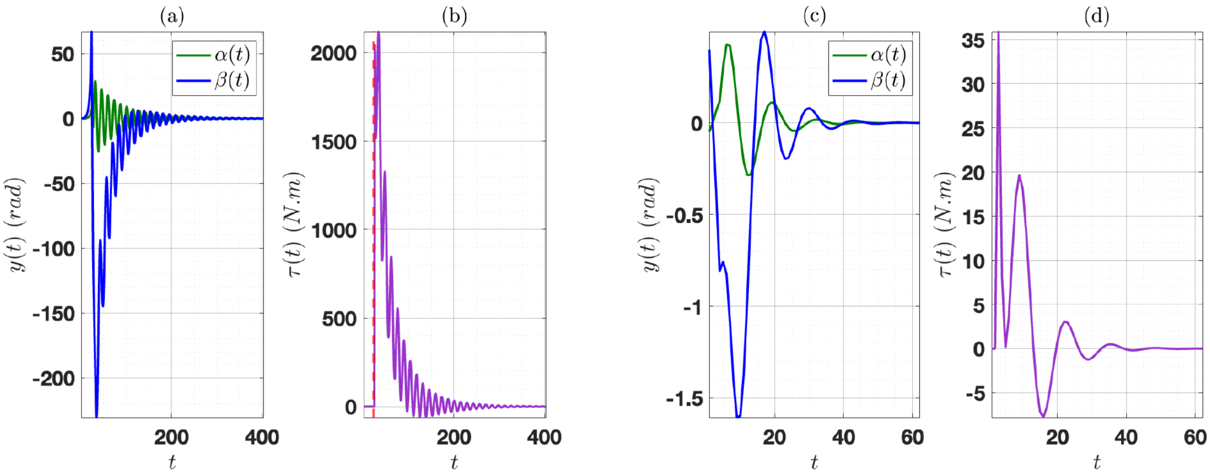

FIGURE 5. (a) and (b) show the response and control input of the rotary inverted pendulum under a dynamic controller designed solely based on data collected throughout the experiment. The controller is found at step $t = 23$ of the simulation, using a window size encompassing the entire experiment until then. The red vertical line indicates the point at which a stabilizing controller is achieved. In contrast, (c) and (d) present the online results with a sliding window of $N = 2$ data points, incorporating refinement of the prior knowledge set.

TABLE 2. The nominal and actual parameter values in the rotary inverted pendulum.

Parameters	Nominal	Actual
L_r	0.752	0.75
L_p	0.241	0.239
m_p	1.17	1.2
J_r	2.0e-2	1.98e-2
C_r	0.47	0.46
g	10	9.81

uncertainty range is set to $\epsilon = 0.007$, and (19) verifies Assumption 1 by confirming that the computed ϵ exceeds the distance between the nominal and actual systems.

For a fair comparison between the two approaches, the window size in the first case is all the data that is collected until the point of the simulation when a controller is feasible to be designed, and the window size for the presented algorithm in this paper is set to be $N = 2$. In both cases, the simulation begins by inputting a random input $u \sim U[-1, 1]$ and then trying to design a controller at each step with only all the data until that point in the first case, and with a window of $N = 2$ data and initial prior knowledge in the second case. The assumption on energy-bounded disturbance is $VV^\top \leq \epsilon_v I_2$, where $\epsilon_v = 1e-2 \times \Delta t^4$. Fig. 5(a) and (b) show the behavior of the system with online data and without refinement and (c) and (d) show the behavior with refinement of the prior knowledge set, leading to safe operation, less conservative control, and faster convergence. In both cases, the system is stabilized.

VIII. CONCLUSION

This paper presents necessary and sufficient conditions for stability analysis and the synthesis of a stabilizing controller in an online framework using an input-output ARX system model with data and uncertain prior knowledge. The parameters in this model are assumed to be uncertain. Stability analysis shows that, with data collected from an autonomous AR system and initial physical knowledge, the stability of the system can be determined, and its certainty can be improved by updating the prior knowledge set. For the closed-loop system, a robust stabilizing controller is designed by combining data from the actual system with uncertain physical knowledge. The method demonstrates that stabilization is achievable even with insufficient data for system identification or stabilization, as waiting for additional data might lead to irreversible states, as shown in the simulation study. The strict lossy S-lemma proves that the designed feedback controller stabilizes a set of parameters within the overlap set $\mathcal{E}^t_{PDOS}$, which dynamically updates as more data becomes available. This framework departs from traditional input-state data-driven control, enabling stabilization with limited data and improving the resulting controller by managing the size of the information set. Simulations illustrate reduced conservatism in controller design and demonstrate its applicability to nonlinear real-world cases, achieving stabilization with limited data.

Future research will investigate how window size, convergence speed, and system performance interrelate and extend the algorithm's applicability to parameter-varying systems using adaptive strategies. Moreover, exploring the potential for transfer learning across systems with similar models presents an exciting opportunity for further study. It will focus on integrating performance criteria into our framework to bridge this gap, as well as using other set representations, such as zonotopes, for lossless prior knowledge set improvement design.

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