IEEE CSS

Received 1 February 2024; revised 4 May 2024; accepted 11 June 2024. Date of publication 1 July 2024; date of current version 17 July 2024. Recommended by Senior Editor C. Ocampo-Martinez

Digital Object Identifier 10.1109/OJCSYS.2024.3421372

Risk-Aware Stochastic MPC for Chance-Constrained Linear Systems

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This work was supported in part by the Office of Naval Research under Grant N00014- 22-1-2159, and in part by the National Science Foundation under Grant ECCS-2227311.

ABSTRACT This paper presents a fully risk-aware model predictive control (MPC) framework for chance-constrained discrete-time linear control systems with process noise. Conditional value-at-risk (CVaR) as a popular coherent risk measure is incorporated in both the constraints and the cost function of the MPC framework. This allows the system to navigate the entire spectrum of risk assessments, from worst-case to risk-neutral scenarios, ensuring both constraint satisfaction and performance optimization in stochastic environments. The recursive feasibility and risk-aware exponential stability of the resulting risk-aware MPC are demonstrated through rigorous theoretical analysis by considering the disturbance feedback policy parameterization. In the end, two numerical examples are given to elucidate the efficacy of the proposed method.

INDEX TERMS Chance constraints, conditional value at risk, distributionally robust optimization, risk - aware MPC.

I. INTRODUCTION

Model Predictive Control (MPC) is a highly effective control strategy extensively applied in various industries, including automotive, energy, chemical, robotics, and aerospace [1], [2]. This approach excels in managing complex, multivariable control challenges and adhering to system constraints. However, effective implementation of MPC requires accounting for uncertainties and disturbances in system dynamics. Strategies that are developed to tackle these issues include robust MPC, which optimizes for the worst disturbances [3]; tube-based MPC, which accounts for uncertainty through invariant sets [4], [5]; and stochastic MPC, which treats uncertainty as noise and minimizes the expected value of the cost function [6], [7].

Despite the advantages that these methods add to the MPC, they still suffer from some shortcomings. Robust MPC can be overly conservative, focusing on rare, extreme scenarios, which may hinder overall performance [8]. Stochastic MPC typically considers only the expected cost value, overlooking the informative potential of the full cost distribution. This can lead to fluctuations in performance, particularly in low-probability, high-impact scenarios [9]. To address these issues, risk-aware optimal control strategies have been developed, aiming to minimize both the expected value and the variance of the cost function for more predictable outcomes. This approach balances the conservatism of robust MPC with the risk neutrality of stochastic MPC [10].

In stochastic optimal control, the use of risk measures such as Conditional Value-at-Risk (CVaR) in the objective function offers a flexible approach that bridges the gap between worst-case and expectation-based (risk-neutral) formulations [11]. By incorporating CVaR, our framework not only aims to minimize the expected or mean cost but also reduces the variance, thereby leading to more predictable and stable outcomes. This risk-aware approach provides a balanced solution that mitigates the overly conservative nature of min-max strategies and purely expectation-based methods [9].

Even though the integration of various risk-aware optimization criteria into the MPC framework has been considered in the literature [11], [12], [13], these adaptations have taken into account either the cost function's risk or the safety constraints' risk, but not both simultaneously. This paper introduces a multi-stage risk approach in the MPC framework, embedding risk considerations in both the constraints and the cost function. This holistic approach enables comprehensive risk assessments, from worst-case to risk-neutral scenarios, ensuring optimal performance and constraint satisfaction.

However, challenges persist, especially when the noise distribution is unknown, making standard risk-aware methods ineffective. Distributionally robust MPC offers a solution, accommodating a range of uncertainty distributions [14]. This concept is intrinsically linked to risk measures, providing a comprehensive approach to dealing with uncertainty in control design [15].

The main contributions of this paper are twofold. First, a fully risk-aware MPC is developed in which risk considerations are embedded in both constraints and cost function. This is in sharp contrast to existing risk-aware MPC [14], [16], [17] that accounts for the risk of either the cost or the constraints. By applying the concept of CVaR, a popular coherent risk measure, in both the performance function and constraints, the closed-loop system can explore the full range of risk assessments. These assessments range from worst-case to risk-neutral scenarios, affecting both constraint satisfaction and performance. Second, rigorous theoretical properties such as risk-aware exponential stability [18], which is absent in [16], and recursive feasibility are provided for the presented MPC by considering the disturbance feedback policy parameterization [19]. In contrast to the approach in [17], our proposed method introduces critical enhancements that significantly broaden the applicability and robustness of distributionally robust MPC. Notably, we incorporate risk considerations directly into the cost function, which is crucial for addressing rare but severe outcomes in the loss function's distribution tail. Additionally, our model adopts more flexible ellipsoidal state constraints, unlike the affine, polyhedral state constraints used in [17], allowing for the handling of more complex scenarios. Setting aside the consideration of soft input constraints in [17], our framework can be seen as an extension of [17].

The presented risk-aware MPC offers three key advantages: 1) It is computationally tractable, as it can be reduced to a semi-definite programming (SDP) optimization problem. 2) The risk measure can be manually adjusted in terms of both performance and constraint satisfaction to adapt to various control applications, allowing for customized control strategies. 3) Beyond the need for second-moment information, this method does not require specific noise information. 4) By satisfying certain conditions, risk-aware exponential stability and recursive feasibility are guaranteed, which are crucial for ensuring reliable performance in real-world applications.

A. ORGANIZATION AND NOTATIONS

The following notations will be used throughout this paper. $\|x\|_2$ denotes the Euclidean (spectral) l_2-norm of vector (matrix) x. Consider the matrices (or vectors) X, Y, and the cone $\mathcal{K}$. The notation $X \leq_{\mathcal{K}} Y$ ($X \geq_{\mathcal{K}} Y$) implies that $Y - X \in \mathcal{K}$ ($X - Y \in \mathcal{K}$). In cases where $\mathcal{K}$ is positive semi-definite matrices, we utilize the designated symbol $\leq$ ($\geq$). The dual cone $\mathcal{K}^*$ is defined as $\mathcal{K}^* := \{\lambda \in \mathbb{R}^n | \lambda^*. \lambda \geq 0, \forall \lambda^* \in \mathcal{K}\}$. (X, Y) is defined as $(X, Y) = \begin{bmatrix} Vec(X) \\ Vec(Y) \end{bmatrix}$, where $Vec(A)$ means the vectorization of matrix A. $(.)^+$ means $(.)^+ = \max(0, .)$. $\lambda_{\min}(A)$ denotes the minimum eigenvalue of A. We use the notation $\mathbb{N}$ to represent the set of natural numbers and $\mathbb{R}$ ($\mathbb{R}_+$) to represent the set of real (nonnegative) numbers. $\mathbb{S}_d$ refers to the set of $d \times d$ symmetric matrices, while $\mathbb{S}_d^{++}$ (or $\mathbb{S}_d^+$) represents the set of positive definite (or positive semidefinite) matrices. $\boldsymbol{w}_{a:b}$ denotes the sequence of $\{\omega_i\}_{i \in \mathbb{N}_{a:b}}$, where $\mathbb{N}_{a:b} = \{a, \ldots, b\}$. $\mathbb{N}_0$ denotes as $\mathbb{N}_0 := \{0, 1, \ldots\}$. $Tr(A)$ represents the trace of matrix A. $\mathbb{I}_N$ denotes $N \times N$ identity matrix. $\bar{\mathbb{R}} : \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ denotes the set of extended real numbers. Let $\mathrm{diag}(A, B)$ be a block diagonal matrix where A and B are matrices with compatible dimensions.

Define the probability space $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \mathcal{P}_x^*)$, where the sample space is defined as $\mathbb{R}^n$ with its associated Borel σ-algebra $\mathfrak{B}$ for the random vector $\boldsymbol{x} \in \mathbb{R}^n$. $\mathcal{P}_x^* \in \mathcal{P}_0$ denotes the *true* probability measure, where $\mathcal{P}_0$ denotes the space of all probability measures defined on the measurable space $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$.

II. PRELIMINARIES

The requirement to constrain the random vector $\boldsymbol{x}$ within the set $\mathbb{X}_x$ with high probability can be expressed using the concept of chance constraint [20], as follows:

$$\mathcal{P}_x^*(\boldsymbol{x} \in \mathbb{X}_x) \geq 1 - \epsilon_x \tag{1}$$

where $0 < \epsilon_x \leq 1$ is a confidence level to control the acceptable level of constraint violation.

Chance constraints are an efficient tool for softening constraints on uncertain variables. However, verifying the feasibility of these constraints usually leads to a non-convex problem, which can make computations intractable. To address this issue, CVaR as an effective tool is introduced as follows.

Definition 1 (Conditional Value-at-Risk [21]): For a given measurable loss function $Z : \mathbb{Z} \to \mathbb{R}$ as a function of random vector $\boldsymbol{x} \in \mathbb{R}^n$ distributed with the probability measure $\mathcal{P}_x^*$, and tolerance $\epsilon_x \in (0, 1]$, the CVaR of loss function Z at level ϵ_x with respect to the probability distribution $\mathcal{P}_x^*$ is defined as

$$CVaR_{\epsilon_x}^{\mathcal{P}_x^*}(Z(\boldsymbol{x})) = \inf_{\beta' \in \mathbb{R}} \left\{ \beta' + \frac{1}{\epsilon_x} \mathbb{E}_{\mathcal{P}_x^*} \left[\left(Z(\boldsymbol{x}) - \beta' \right)^+ \right] \right\} \tag{2}$$

Fig. 1 depicts the comparison among the mean, VaR_{ϵ_x} and $CVaR_{\epsilon_x}$ for a given confidence level $\epsilon_x \in (0, 1]$, where VaR_{ϵ_x} denotes the ϵ_x-quantile value of the loss function Z and is defined as $VaR_{\epsilon_x}(Z) := \inf\{z | \mathcal{P}_x^*(Z \leq z) \geq 1 - \epsilon_x\}$ [21].

CVaR is a coherent risk measure defined as follows [22].

Definition 2 (Coherent Risk Measures [22]): The risk measure $\rho : \mathbb{Z} \to \bar{\mathbb{R}}$ is coherent if it satisfies the following axioms:

1) *Convexity:* $\rho(aZ_1 + (1-a)Z_2) \leq a\rho(Z_1) + (1-a)\rho(Z_2)$, $\forall Z_1, Z_2 \in \mathbb{Z}$, and $a \in [0, 1]$.

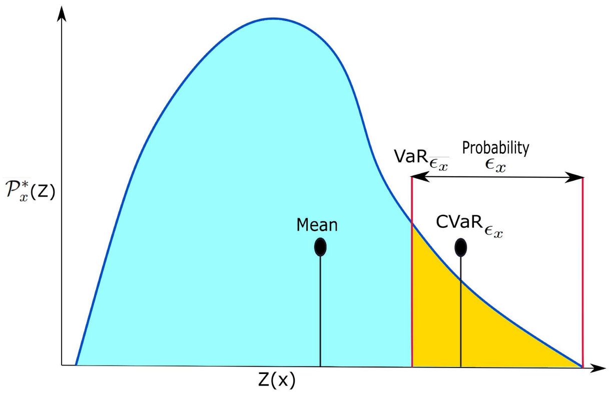

FIGURE 1. Comparison of the mean, VaR_{ϵ_x} and $CVaR_{\epsilon_x}$ for a given confidence level $\epsilon_x \in (0, 1]$. The yellow shaded area denotes the %ϵ_x of the area under $\mathcal{P}(Z)$. Setting $\epsilon_x = 0$ represents the worst-case scenario, while $\epsilon_x = 1$ corresponds to the expectation value.

2) *Monotonicity:* If $Z_1, Z_2 \in \mathbb{Z}$ and $Z_1 \geq Z_2$, then $\rho(Z_1) \geq \rho(Z_2)$.
3) *Translation equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathbb{Z}$, then $\rho(Z + a) = \rho(Z) + a$.
4) *Positive homogeneity:* If $t > 0$ and $Z \in \mathbb{Z}$, then $\rho(tZ) = t\rho(Z)$.

The following lemma states that every coherent risk measure can be represented as an optimization problem in the dual form.

Lemma 1 ([23]): The risk measure $\rho : \mathbb{Z} \to \bar{\mathbb{R}}$ is coherent if and only if $\rho(Z)$ can be represented in the following form:

$$\rho(Z) := \sup_{\mathcal{P} \in \mathcal{A}} \mathbb{E}_{\mathcal{P}}[Z] \tag{3}$$

where $\mathcal{A} \subseteq \mathfrak{R}$ is called an ambiguity set, which is a convex, non-empty, and closed set.

Define $\mathbb{X}_x$ as $\mathbb{X}_x = \{\boldsymbol{x} \in \mathbb{R}^n | Z(\boldsymbol{x}) \leq 0\}$. According to [24], CVaR provides a convex (upper) approximation of the chance constraint (1).

$$CVaR_{\epsilon_x}^{\mathcal{P}_x^*}(Z(\boldsymbol{x})) \leq 0 \Longrightarrow \mathcal{P}_x^*(\boldsymbol{x} \in \mathbb{X}_x) \geq 1 - \epsilon_x \tag{4}$$

As can be seen from (4), the true probability measure $\mathcal{P}_x^*$ is required. However, obtaining $\mathcal{P}_x^*$ may not be feasible in real-world scenarios. Typically, only a limited understanding of $\mathcal{P}_x^*$ is accessible. This constrained knowledge is encapsulated by an ambiguity set $\mathcal{A}_x$, which includes a range of probability measures within which the chance constraint (4) holds true. Consequently, to ensure the resilience of the chance constraint against all probability measures within the ambiguity set $\mathcal{A}_x$, the following distributionally robust chance constraint is introduced.

$$\begin{aligned} \inf_{\mathcal{P} \in \mathcal{A}_x} \mathcal{P}(\boldsymbol{x} \in \mathbb{X}_x) \geq 1 - \epsilon_x \Longleftrightarrow \mathcal{P}(\boldsymbol{x} \in \mathbb{X}_x) \\ \geq 1 - \epsilon_x, \forall \mathcal{P} \in \mathcal{A}_x \end{aligned} \tag{5}$$

In [25], an upper convex approximation is given for the non-convex distributionally robust chance constraint (5) using CVaR as follows

$$\sup_{\mathcal{P} \in \mathcal{A}_x} CVaR_{\epsilon_x}^{\mathcal{P}}(Z(\boldsymbol{x})) \leq 0 \Longrightarrow \inf_{\mathcal{P} \in \mathcal{A}_x} \mathcal{P}(\boldsymbol{x} \in \mathbb{X}_x) \geq 1 - \epsilon_x \tag{6}$$

Assume only the first- and second-order moments of the random vector $\boldsymbol{x}$ are available. As a result, the following ambiguity set is defined as

$$\mathcal{A}_x = \left\{ \mathcal{P} \middle| \mathbb{E}_{\mathcal{P}} \left[(1, \boldsymbol{x}^T)^T (1, \boldsymbol{x}^T) \right] = M_x \right\} \tag{7}$$

where $M_x = \begin{bmatrix} 1 & \mu_x^T \\ \mu_x & \Sigma_x + \mu_x \mu_x^T \end{bmatrix}$. μ_x and $\Sigma_x > 0$ are the mean and covariance of the random variable $\boldsymbol{x}$, respectively.

In the following theorem, a tractable semidefinite program (SDP) solution is given to the distributionally robust CVaR (6) using the defined ambiguity set (7).

Theorem 1 ([26]): Assume the ambiguity set is defined as (7), and $Z(\boldsymbol{x}) = \boldsymbol{x}^T E \boldsymbol{x} + 2F^T \boldsymbol{x} + G'$, where $E \geq 0$. Then, the distributionally robust CVaR (6) is equivalent to the following SDP

$$\sup_{\mathcal{P} \in \mathcal{A}_x} CVaR_{\epsilon_x}^{\mathcal{P}}(Z(\boldsymbol{x})) = \inf_{\beta' \in \mathbb{R},\, X \geq 0} \beta' + \frac{1}{\epsilon_x} \mathrm{Tr}(M_x X) \tag{8a}$$

$$X - \begin{bmatrix} G' - \beta' & F^T \\ F & E \end{bmatrix} \geq 0 \tag{8b}$$

The following Lemmas are extensively used throughout the remainder of the paper.

Lemma 2 (S-Lemma [27]): Let P_0 and P_1 be symmetric matrices of equal size. Then, the following statements are equivalent: i) For any vector x, if $x^T P_1 x \leq 0$, then $x^T P_0 x \leq 0$. ii) There exists $\lambda_p \geq 0$ such that $P_0 \leq \lambda_p P_1$.

Lemma 3 ([18]): For any $M \in \mathbb{S}_n$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}^n$, the following relation is true for every $\epsilon \in (0, \infty)$

$$(y + z)^T M (y + z) \leq (1 + \epsilon) y^T M y + \left(1 + \frac{1}{\epsilon}\right) z^T M z \tag{9}$$

Lemma 4 (Schur Complement Lemma [28]): Let S be a symmetric matrix defined as

$$S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \tag{10}$$

where A is a symmetric and square matrix, and C is a symmetric positive definite matrix. Then, the following statements are equivalent:

- S is positive semi-definite.
- The matrix $A - BC^{-1}B^T$ is positive semi-definite.

III. PROBLEM FORMULATION

A. SYSTEM DESCRIPTION AND PROBLEM DEFINITION

Consider the following constrained discrete-time linear stochastic control system as

$$x_{t+1} = Ax_t + Bu_t + D\omega_t, \tag{11}$$

where $x_t \in \mathbb{X} \subset \mathbb{R}^{n_x}$ and $u_t \in \mathbb{U}_u \subseteq \mathbb{R}^{n_u}$ are the state of the system and control input, respectively. The input and state constraints $\mathbb{U}_u$ and $\mathbb{X}$ are defined as

$$\mathbb{U}_u := \{u \in \mathbb{R}^{n_u} |\ \|C_u u\|_2 \leq u_{\max}\},$$
$$\mathbb{X} := \{x \in \mathbb{R}^{n_x} \big| \phi(x) \leq 0\},$$

where $\phi(x) = x^T Gx + 2\, g^T x + \gamma$, $u_{\max}$ is a positive design parameter, and $G \geq 0$. Moreover, $\omega_t \in \mathbb{R}^{n_w}$ represents the system noise with a true but unknown probability measure $\mathcal{P}^*$ satisfying the following assumption.

Assumption 1: The system noise ω_t is assumed to be a wide sense stationary (W.S.S.) white noise process with the covariance matrix $\Sigma_{\omega_t} := \Sigma_w$ for all $t \in \mathbb{N}_0$.

Given the W.S.S. Assumption 1, only available information about the system noise ω_t is its auto-correlation $R_{ww}(t) = \mathbb{E}_{\mathcal{P}^*}\{\omega_i \omega_{i-t}^T\}$, where $R_{ww}(0) = \Sigma_w$ and $R_{ww}(t) = \mathbf{0}_{n_\omega \times n_\omega}$, otherwise. Therefore, the true probability measure $\mathcal{P}^*$ belongs to the following ambiguity set using W.S.S. given in Assumption 1

$$\mathcal{A} = \Bigg\{\mathcal{P} \in \mathcal{P}_0 \Bigg| \mathbb{E}_{\mathcal{P}} \left\{(\omega_i^T, 1)^T(\omega_j^T, 1)\right\} = \begin{bmatrix} \Sigma_w \delta_{ij} & \mathbf{0}_{n_\omega \times 1} \\ \mathbf{0}_{n_\omega \times 1}^T & 1 \end{bmatrix}, i, j \in \mathbb{N}_0 \Bigg\} \tag{12}$$

We adopt a risk-aware approach to optimizing a cost function while satisfying the system's constraints. Therefore, rather than the satisfaction of state constraints in expectation, a distributionally robust chance constraint is formalized to penalize the anticipated violation at the ϵ-quantile. This modification allows for a more nuanced handling of the state constraint and performance in a stochastic setting.

The following risk-aware MPC is now formalized for the linear system described by (11) as

$$V^*(x_t) = \min_{\boldsymbol{u}_t \in \mathbb{U}_u} \sup_{\mathcal{P} \in \mathcal{A}} CVaR_{\epsilon_1}^{\mathcal{P}} \left[\sum_{k=0}^{N-1} r(x_{k|t}, u_{k|t}) + r_N(x_{N|t})\right] \tag{13a}$$

$$\text{s.t.}\ \ x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + D\omega_{k|t},\ \ k \in \mathbb{N}_{0:N-1} \tag{13b}$$

$$\inf_{\mathcal{P} \in \mathcal{A}} \mathcal{P}\left[x_{k|t} \in \mathbb{X}\right] \geq 1 - \epsilon_2,\ \ k \in \mathbb{N}_{1:N-1} \tag{13c}$$

$$x_{N|t} \in \mathbb{O}_\infty \tag{13d}$$

where $x_{0|t} = x_t$, $\boldsymbol{u}_t = \left[u_{0|t}^T, \ldots, u_{N-1|t}^T\right]^T$, $r(x_{k|t}, u_{k|t}) := x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t}$, $r_N(P) := x_{N|t}^T P x_{N|t}$, where $Q > 0$, $R > 0$, and $P > 0$. The terminal set $\mathbb{O}_\infty(P, \gamma_p^w)$ is given by

$$\mathbb{O}_\infty\left(P, \gamma_p^w\right) = \left\{x \in \mathbb{R}^{n_x} |\ x^T P x \leq \gamma_p^w\right\}, \tag{14}$$

where $\gamma_p^w = \gamma_p - b_w$ and $b_w = \frac{1}{\epsilon_\infty}(1 + \frac{1}{\epsilon})Tr[D^T D\Sigma]$ with $\gamma_p > 0$, $\epsilon > 0$, and $0 < \epsilon_\infty \leq 1$. Hence, the MPC controller can be obtained as $u^{MPC}(x_t) = u_{0|t}^*$. As will be detailed in Section V, the terminal set (14) is proposed to bring the recursive feasibility and stability to the closed-loop system.

Remark 1: Incorporating the CVaR into the MPC cost function (13a) enables us to capture extreme outcomes in the right tail of the cost distribution, as illustrated in Fig. 1. This approach contrasts with the methods given in [14], [17], which focus solely on the mean (expected value) of the cost function. Our presented risk-aware method offers a comprehensive spectrum of risk assessments. It ranges from a risk-neutral scenario—achieved by setting $\epsilon_1 = 1$, thereby converting the CVaR to an expectation—to a worst-case scenario.

It is worth mentioning that although the system dynamics considered here are linear, the closed-loop system under the presented MPC controller is nonlinear. Even so, a sub-optimal linear feedback controller with a closed-loop analysis is obtained for the CVR cost function in [29]; their controller suffers from two limitations. First, as mentioned in [29], the solution is sub-optimal, i.e., the performance under the proposed linear controller is not optimal. Second, using the linear controller for a constrained system results in a small feasible region, i.e., a region where, for every initial state, the system satisfies the state and input constraints at all time steps. According to [30], nonlinear controllers can indeed offer a larger feasible set for systems with nonlinear constraints compared to linear controllers.

IV. SOLUTION ALGORITHMS

A. CONTROLLER PARAMETRIZATION

Inspired by the affine disturbance feedback policy [14], a control policy in the following form will be leveraged in the subsequent sections to find a solution to the optimization problem (13).

$$\boldsymbol{u}_t = U_t \bar{\boldsymbol{\omega}}_N^t \tag{15}$$

where $\bar{\boldsymbol{\omega}}_N^t = [1, \omega_{0|t}^T, \ldots, \omega_{N-1|t}^T]^T$ and $U_t \in \mathbb{U}$, where $\mathbb{U}$ is defined as

$$\mathbb{U} := \left\{U \in \mathbb{R}^{Nn_u \times Nn_w + 1} \middle| U = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & * & 0 \end{bmatrix}\right\} \tag{16}$$

By using (15), the system trajectory can be rewritten in terms of the past disturbances as

$$\boldsymbol{x}_{N|t} = (\boldsymbol{B}U_t + \boldsymbol{D})\bar{\boldsymbol{\omega}}_N^t \tag{17}$$

where $\boldsymbol{x}_{N|t} = [x_{0|t}^T, \ldots, x_{N|t}^T]^T$,

$$\boldsymbol{B} = \begin{bmatrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \mathcal{B}_2 \\ \vdots \\ \vdots \\ \mathcal{B}_N \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ B & 0 & & & \\ AB & B & 0 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & B & 0 \\ A^{N-1}B & A^{N-2}B & \ldots & AB & B \end{bmatrix},$$

and

$$\boldsymbol{D} = \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_N \end{bmatrix} = \begin{bmatrix} x_{0|t} & & & & \\ Ax_{0|t} & D & & & \\ A^2 x_{0|t} & AD & D & & \\ \vdots & & \ddots & \ddots & \\ A^N x_{0|t} & A^{N-1}D & \dots & AD & D \end{bmatrix}.$$

In this section, we aim at solving the following problem.

Problem 1: Finding an equivalent SDP optimization for the proposed MPC algorithm (13) by considering the disturbance feedback policy (15).

The following subsections are given to find equivalent SDPs for the CVaR cost function (13a) and chance constraints (13c)-(13d).

B. DISTRIBUTIONALLY ROBUST CVAR COST FUNCTION

By using (17), the cost function (13a) can be rewritten as

$$\min_{\boldsymbol{u}_t \in \mathbb{U}_u} \sup_{\mathcal{P}\in\mathcal{A}} CVaR_{\epsilon_1}^{\mathcal{P}} \left[\sum_{k=0}^{N-1} r(x_{k|t}, u_{k|t}) + r_N(x_{N|t}) \right]$$
$$= \min_{U_t \in \mathbb{F}_u} \sup_{\mathcal{P}\in\mathcal{A}} CVaR_{\epsilon_1}^{\mathcal{P}} \left[(\bar{\boldsymbol{\omega}}_N^t)^T F_w \bar{\boldsymbol{\omega}}_N^t \right] \tag{18}$$

where $F_w = U_t^T(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{B})U_t + 2\boldsymbol{D}^T\boldsymbol{Q}\boldsymbol{B}U_t + \boldsymbol{D}^T\boldsymbol{Q}\boldsymbol{D}$, $\boldsymbol{Q} = \mathrm{diag}(Q, \dots, Q, P)$, $\boldsymbol{R} = \mathrm{diag}(R, \dots, R)$, and $\mathbb{F}_u$ is obtained as follows based on the input constraint $\mathbb{U}_u$, controller parameterization (15), and using Schur complement Lemma 4 as

$$\mathbb{F}_u = \left\{ U \in \mathbb{U} \middle| \begin{bmatrix} \boldsymbol{S}_u & U^T C_u^T \\ C_u U & I \end{bmatrix} \geq 0 \right\} \tag{19}$$

where $\boldsymbol{S}_u = \mathrm{diag}(u_{\max}^2, \boldsymbol{0}_{Nn_w \times Nn_w})$.

Remark 2: The proposed risk-aware objective function (18) does not depend explicitly on noise realizations. This independence arises because, according to Theorem 1 and Assumption 1, the expression $\sup_{\mathcal{P}\in\mathcal{A}} \mathrm{CVaR}_{\epsilon_1}^{\mathcal{P}}[(\bar{\boldsymbol{\omega}}_N^t)^T F_w \bar{\boldsymbol{\omega}}_N^t]$ depends solely on the initial condition x_0, U_t, and the constant parameters Σ_ω, ϵ_1, Q, R, N, A, B, and D. Additionally, by considering causal policies (16), the feedback policy $\boldsymbol{u}_t$ presented in (15) is $\mathcal{F}_t^\omega$-measurable, according to [14] and [31, Sec. 14.4.2].

Lemma 5: Suppose Assumption 1 is true and the control input u_t is parameterized as (15). Then, the distributionally robust CVaR MPC cost function (18) is equivalent to the following SDP optimization problem:

$$\min_{U_t \in \mathbb{F}_u,\ \beta \in \mathbb{R},\ M \in \mathbb{S}_{Nn_\omega+1}^+} \beta + \frac{1}{\epsilon_1}\mathrm{Tr}(\Omega M) \tag{20a}$$

$$\begin{bmatrix} M' - 2\boldsymbol{D}^T\boldsymbol{Q}\boldsymbol{B}U_t - \boldsymbol{D}^T\boldsymbol{Q}\boldsymbol{D} & U_t^T(\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{B})^{1/2} \\ (\boldsymbol{R} + \boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{B})^{1/2}U_t & \mathbb{I}_{Nn_u} \end{bmatrix} \geq 0 \tag{20b}$$

where $M' = M + \mathrm{diag}(\beta, \boldsymbol{0})$ and $\Omega = \mathrm{diag}(1, \mathbb{I}_N \otimes \Sigma_\omega)$.

Proof: By holding Assumption 1 in which the noise process ω_t is assumed to be W.S.S., one has $\mathbb{E}_{\mathcal{P}^*}(\bar{\boldsymbol{\omega}}_N^t(\bar{\boldsymbol{\omega}}_N^t)^T) = \Omega$. As a result, by applying Theorem 1, one obtains the cost function (20a). By following [26, Theorem 2.1 and Lemma A.1], with a slight modification, one can rewrite (8b) as

$$(\bar{\boldsymbol{\omega}}_N^t)^T \left(M' - F_w\right) \bar{\boldsymbol{\omega}}_N^t \geq 0 \Longleftrightarrow M' - F_w \geq 0 \tag{21}$$

By applying Schur complement (Lemma 4) into (21) and using the definition of F_w, (20b) is obtained. ■

C. DISTRIBUTIONALLY ROBUST CHANCE CONSTRAINT

Lemma 6: Suppose Assumption 1 is true and the control input u_t is parameterized as (15). Then, the distributionally robust chance constraint (13c) is true if there exist $\beta_k \in \mathbb{R}$, $M_k \geq 0$, and $Z_k \geq 0$ such that the following linear matrix inequalities (LMIs) hold:

$$\beta_k + \frac{1}{\epsilon_2}\mathrm{Tr}(\Omega' M_k) \leq 0 \tag{22a}$$

$$M_k - \begin{bmatrix} Z_k & (\mathcal{B}_k U_t + \mathcal{D}_k)^T g \\ g^T(\mathcal{B}_k U_t + \mathcal{D}_k) & \gamma - \beta_k \end{bmatrix} \geq 0 \tag{22b}$$

$$\begin{bmatrix} Z_k & (\mathcal{B}_k U_t + \mathcal{D}_k)^T G^{\frac{1}{2}} \\ G^{\frac{1}{2}}(\mathcal{B}_k U_t + \mathcal{D}_k) & \mathbb{I}_{n_x} \end{bmatrix} \geq 0 \tag{22c}$$

where $\Omega' = \mathrm{diag}(1, \mathbb{I}_N \otimes \Sigma_\omega, 1)$ and $k = 1, \dots, N-1$.

Proof: According to (6), one has the following statement for the distributionally robust chance constraint (13c) as

$$\sup_{\mathcal{P}\in\mathcal{A}} CVaR_{\epsilon_2}^{\mathcal{P}}(\phi(x_{k|t})) \leq 0 \Longrightarrow \inf_{\mathcal{P}\in\mathcal{A}} \mathcal{P}(x_{k|t} \in \mathbb{X}) \geq 1 - \epsilon_2,$$

with $k = 1, \dots, N-1$. As a result, if $\sup_{\mathcal{P}\in\mathcal{A}} CVaR_{\epsilon_2}^{\mathcal{P}}(\phi(x_{k|t})) \leq 0$ is true, then the distributionally robust chance constraint (13c) is satisfied as well. By following controller parametrization (15) and considering (17), one has

$$\phi(x_{k|t}) = \begin{bmatrix} \bar{\boldsymbol{\omega}}_N^t \\ 1 \end{bmatrix}^T \begin{bmatrix} (\mathcal{B}_k U_t + \mathcal{D}_k)^T G (\mathcal{B}_k U_t + \mathcal{D}_k) & (\mathcal{B}_k U_t + \mathcal{D}_k)^T g \\ g^T(\mathcal{B}_k U_t + \mathcal{D}_k) & \gamma \end{bmatrix} \times \begin{bmatrix} \bar{\boldsymbol{\omega}}_N^t \\ 1 \end{bmatrix} \tag{23}$$

By considering $\Omega' = \mathbb{E}_{\mathcal{P}^*}\left[\begin{bmatrix} \bar{\boldsymbol{\omega}}_N^t \\ 1 \end{bmatrix}\begin{bmatrix} (\bar{\boldsymbol{\omega}}_N^t)^T & 1 \end{bmatrix}\right] = \mathrm{diag}(1, \mathbb{I}_N \otimes \Sigma_\omega, 1)$, using Assumption 1 in which the noise process ω_t is assumed to be W.S.S., following [26, Lemma A.1 and Theorem 2.1], using (2), and applying Schur complement Lemma 4, one can obtain (22). ■

Regarding the terminal constraint (13d), the following lemma is given to find an equivalent LMI solution.

Lemma 7: If Assumption 1 is true, and the control input u_t is parametrized as (15), then the terminal constraint (13d) is equivalent to solving the following LMI:

$$\begin{bmatrix} \mathrm{diag}(\gamma_p^w, \mathbf{0}) & (\mathcal{B}_N U_t + \mathcal{D}_N)^T P^{\frac{1}{2}} \\ P^{\frac{1}{2}} (\mathcal{B}_N U_t + \mathcal{D}_N) & \mathbb{I}_{n_x} \end{bmatrix} \geq 0 \tag{24}$$

Proof: The terminal constraint (13d) is translated into the following inequality

$$(\bar{\boldsymbol{\omega}}_N^t)^T \left((\mathcal{B}_N U_t + \mathcal{D}_N)^T P (\mathcal{B}_N U_t + \mathcal{D}_N) - \mathrm{diag}\left(\gamma_p^w, \mathbf{0}\right)\right) \times \bar{\boldsymbol{\omega}}_N^t \leq 0 \tag{25}$$

By applying Schur complement (Lemma 4), one has (24). ∎

V. CLOSED-LOOP ANALYSIS

In this section, conditions for recursive feasibility and risk-informed exponential stability are provided, followed by their definitions.

Definition 3 (Recursive Feasibility): Define $\mathbb{X}_N$ to be the set of initial states for which the robust CVaR MPC (13) is feasible. Assume $x_t \in \mathbb{X}_N$, and the control input is defined as $u^{MPC}(x_t) = u^*_{0|t}$. The MPC optimization (13) is recursively feasible if $x_{t+1} \in \mathbb{X}_N$ almost surely.

Definition 4 (Risk-aware exponential stability [18]): For a given risk functional $\rho : \mathbb{Z} \to \bar{\mathbb{R}}$, a subset $\mathbb{S} \subseteq \mathbb{R}^{n_x}$, and a state energy function $\psi(x)$, which is Borel-measurable, non-negative, and $\psi(x) = 0$ if and only if $x = 0_{n_x}$, the system (11) is risk-aware exponentially stable with an offset with respect to ρ in the region $\mathbb{S}$ if there exist parameters $\lambda_s \in [0, 1)$, $a \in [0, \infty)$, and $b \in \mathbb{R}$ such that for every time $t \in \mathbb{N}$, and $\psi(x_t) \in \mathbb{Z}$, one has

$$\rho(\psi(x_t)) \leq a\lambda_s^t \psi(x_0) + b, \quad \forall x_0 \in \mathbb{S}. \tag{26}$$

Problem 2: Consider the risk-aware MPC (13), design the terminal cost matrix $P^* > 0$ and terminal set $\mathbb{O}_\infty$ such that:

1) The MPC problem (13) is recursive feasible according to Definition 3.
2) By applying $u^{MPC}(x_t)$, and considering $\rho := \sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_1}(.)$ as a risk assessment in (26), the closed-loop system (11) is risk-aware exponential stable according to Definition 4.

The following subsections are given to solve Problem 2.

A. RECURSIVE FEASIBILITY

To guarantee the recursive feasibility of the proposed MPC controller (13), the terminal set $\mathbb{O}_\infty$ is required to be a distributionally robust CVaR positively invariant set (as will be shown later in Theorem 3), which is defined as follows.

Definition 5 (Distributionally robust CVaR positively invariant set): The ellipsoidal set $\mathbb{O}_\infty$ is a distributional robust CVaR positively invariant set with certainty level $0 < \epsilon_\infty \leq 1$ if for any state $x_t \in \mathbb{O}_\infty(P, \gamma_p^w)$, then $\inf_{\mathcal{P}\in\mathcal{A}} \mathcal{P}(x_{t+1} \in \mathbb{O}_\infty(P, \gamma_p)|x_t) \geq 1 - \epsilon_\infty$.

Theorem 2: Given the closed-loop system (11) consisting the state-feedback controller $u_t = Fx_t$. The ellipsoidal set $\mathbb{O}_\infty$, defined as (14), is a distributional robust CVaR positively invariant set with certainty level $0 < \epsilon_\infty \leq 1$ if

$$(1 + \epsilon)A_c^T PA_c - P + (Q + F^T RF) \leq 0 \tag{27}$$

where $\epsilon > 0$ and $A_c = A + BF$.

Proof: See Appendix A. ∎

To find P and F satisfying (27) in Theorem 2, the following lemma is given.

Lemma 8 ([32], [33]): The inequality (27) is equivalent to solving the following LMI

$$\min_{X>0, Z\geq 0, L} tr(Z) \tag{28a}$$

$$\begin{bmatrix} X & XA^T + L^T B^T & XQ^{1/2} & L^T \\ AX + BL & \frac{1}{1+\epsilon}X & 0 & 0 \\ Q^{1/2}X & 0 & I & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \geq 0 \tag{28b}$$

$$\begin{bmatrix} Z & I \\ I & X \end{bmatrix} \geq 0 \tag{28c}$$

Given the X^* and L^* as the LMI optimal solutions, P^* and F^* can be obtained as $P^* = (X^*)^{-1}$ and $F^* = L^* P^*$.

The following Lemma is given to find the largest γ_p such that the input and state constraints $\mathbb{U}_u$ and $\mathbb{X}$ are satisfied.

Lemma 9: Given the optimal terminal cost P^* and controller gain F^*, if the optimization problem (29) yields a feasible solution, then for all $x \in \mathbb{O}_\infty(P^*, \gamma_p^w)$, both the state constraint $\mathbb{X}$ and the input constraint $\mathbb{U}_u$ are satisfied.

$$\max_{\lambda\geq 0, \gamma_p > 0} \gamma_p \tag{29a}$$

$$P^* - \gamma_p \left(F^{*T} \frac{C_u^T C_u}{u_{\max}^2} F^* \right) \geq 0, \tag{29b}$$

$$\begin{bmatrix} \lambda P^* - G & -g \\ -g^T & -\lambda\gamma_p - \gamma \end{bmatrix} \geq 0. \tag{29c}$$

Proof: See Appendix B. ∎

The optimization problem (29) is bilinear in terms of decision variables λ and γ_p in (29c), which can be solved efficiently by using iterative methods [34] or by applying McCormick envelopes, as proposed in [35], to transform the nonconvex bilinear constraint in (29c) into a tractable convex constraint.

To solve (29) efficiently, one can assume either $G = 0$ or $g = 0$ to simplify (29c) as follows

- If $g = 0$, then (29c) is equivalent to $P^* + \gamma_p(\frac{G}{\gamma}) \leq 0$.
- If $G = 0$, then (29c) is equivalent to $\|2P^{*1/2} g\|_2^2 \gamma_p \leq \gamma^2$ [36].

As a result, by assuming either $g = 0$ or $G = 0$, (29) simplifies to a linear programming problem.

Until now, we could find a CVaR terminal invariant set $\mathbb{O}_\infty(P^*, \gamma_p^w)$ satisfying both input and state constraints, where P^* and $\gamma_p^{*w} = \gamma_p^* - b_w$ are obtained from LMI (28) and (29), respectively.

Theorem 3 (Recursive Feasibility): Let $\boldsymbol{\Pi}_t^N(x_t, \mathcal{A})$ represent the feasible set of (13) for the given ambiguity set $\mathcal{A}$ and the initial condition x_t. If the following Assumptions are satisfied:

1) There exists a feasible optimal control policy $\boldsymbol{u}_t^* = \left[{u_{0|t}^*}^T, \ldots, {u_{N-1|t}^*}^T\right]^T$, i.e., $\boldsymbol{u}_t^* \in \boldsymbol{\Pi}_t^N(x_t, \mathcal{A})$.
2) P, F and γ_p are the feasible solutions of (28) and (29).

then, the presented MPC (13) with the parameterized control policy (15) is recursive feasible according to Definition 3.

Proof: See Appendix D. ■

B. RISK-AWARE EXPONENTIAL STABILITY

The following Lemma provides a sufficient condition for system (11) to be risk-aware local exponential stable (RLES).

Lemma 10 (Sufficient condition for being RLES): Consider the closed-loop system (11), and the given state energy function $\psi(x)$. The closed-loop system (11) is RLES with an offset $b' \geq 0$ with respect to a coherent risk-function ρ if there exist a function $V(x) : \mathbb{R}^{n_x} \to \mathbb{R}$ and scalars $c_3 \geq 0$, $c_1, c_2, c_4 > 0$, such that $\forall x_t \in \mathbb{R}^{n_x}$, the following conditions are satisfied

$$c_1\psi(x_t) \leq V(x_t) \leq c_2\psi(x_t) + c_3 \tag{30}$$

$$\rho(V(x_{t+1})) - V(x_t) \leq -c_4\psi(x_t) + b' \tag{31}$$

for every time $t \in \mathbb{N}$.

Proof: See Appendix C. ■

Theorem 4: Let $\mathbb{X}_N$ be the set of initial states for which MPC (13) is feasible. The control policy $u_t^* = u_{0|t}^*$ obtained from the MPC problem (13) makes the closed-loop system (11) locally exponentially stable with domain $\mathbb{X}_N$ if (27) is satisfied.

Proof: See Appendix E. ■

C. CVAR MPC ALGORITHM

By using the previous subsections, the proposed CVaR MPC (13) is cast into the following SDP optimization

$$\begin{aligned} &\min_{\substack{U_t \in \mathbb{U},\ \beta \in \mathbb{R},\ M \geq 0 \\ \beta_k \in \mathbb{R},\ M_k \geq 0,\ Z_k \geq 0 \\ k=1,\ldots,N-1}} \beta + \frac{1}{\epsilon_1}\mathrm{Tr}(\Omega M) \\ &\text{s.t.} \quad (19),\ (20b),\ (22),\ (24),\ x_{0|t} = x_t \end{aligned} \tag{32}$$

Algorithm 1 is provided to show the implementation of the proposed risk-aware MPC (13).

VI. SIMULATION RESULTS

In this section, we first evaluate and compare the performance of the proposed risk-aware MPC Algorithm 1, with several state-of-the-art MPC methods. These methods include the Chebyshev–Cantelli stochastic MPC (SMPC) approach [37], [38], Robust MPC (RMPC) [4], and the recent Distributionally Robust MPC (DRMPC) [17]. Our evaluation focuses on a practical application involving a DC-DC converter and the lateral control of autonomous vehicles to demonstrate the efficacy of the proposed method.

Algorithm 1: Risk-Aware MPC with Distributionally Robust Chance Constraints.

Input: system matrices A, B and D, initial condition x_0, performance indices $Q > 0$ and $R > 0$, state constraint matrices $G \geq 0$, g, and γ, input constraint parameters $u_{\max} > 0$ and C_u, CVaR cost and constraint tolerances $0 < \epsilon_1 \leq 1$ and $0 < \epsilon_2 \leq 1$, second noise moment Σ_w.
Output: $u^{MPC}(x_t) = u_{0|t}^*$
Obtain the CVaR Invariant Set $\mathbb{O}_\infty(P^*, \gamma_p^*)$: Solve LMI (28) for P^* and F^*, then solve (29) for γ_p^*.
for $t = t_0$ **to** T_f **do**
  Set $x_{0|t} = x_t$ and solve the LMI (32) for U_t. Set $u_t^{MPC} = f_{11}^t$.
  Apply u_t^{MPC} to system (11) and obtain x_{t+1}.
end for

A. EXAMPLE 1: VOLTAGE CONTROL FOR A DC-DC CONVERTER

To evaluate the performance and feasibility of the proposed method, we consider a real-world example, voltage control for a DC-DC converter, which was originally introduced in [39] to test the effectiveness of the nonlinear MPC method. This example is adapted from [17], [40]. Accordingly, A, B, and D in (11) are considered as

$$A = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix}, B = \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The main goal is to regulate the DC-DC converter system [17] using the proposed risk-aware MPC (13) in the presence of unknown non-Gaussian disturbances. The state and input constraints $\mathbb{X}$ and $\mathbb{U}_u$ are considered as

$$\mathbb{U}_u := \{u \in \mathbb{R} |\ \|u\|_2 \leq 1\},$$

$$\mathbb{X} := \{x \in \mathbb{R}^2 | x_1 - 3 \leq 0\}.$$

The chance constraint parameter ϵ_2 in (13c) is set as $\epsilon_2 = 0.2$ similar to the settings given in [17]. The risk-aware parameter ϵ_1 is also set as $\epsilon_1 = 0.1$. The performance parameters Q and R are defined as $Q = \mathrm{diag}(1, 10)$ and $R = 1$. Similar to [17], the non-Gaussian distribution considered here is approximated with the Gaussian mixture model [41], where the PDF of ω_t is considered as [17]

$$f(\omega_i) = \sum_{l=1}^{2} \frac{\lambda_l}{\pi\sigma_l^2} \exp\left\{-\frac{\omega_t^2}{\sigma_l^2}\right\},\ i = 1, 2 \tag{33}$$

with $\sigma_1^2 = 0.1$, $\lambda_1 = 0.4$, $\sigma_2^2 = 0.15$, and $\lambda_2 = 0.6$. The prediction horizon is set as $N = 10$, and the initial condition is considered as $x_0 = [2.5 \quad 2.8]^T$. By setting $\epsilon = 1$, one can find $P = \begin{bmatrix} 1.30 & -1.54 \\ -1.54 & 18.96 \end{bmatrix}$ satisfying (28).The terminal constraint is considered as $\mathbb{O}_\infty = \{x \in \mathbb{R}^2 | x^T P x \leq \gamma_p - b_\omega\}$,

TABLE 1 The average optimal value functions.

	Risk-aware MPC	RMPC	SMPC	DRMPC [17]
$V(x_0)$	281.2	282.4	275.4	276.3

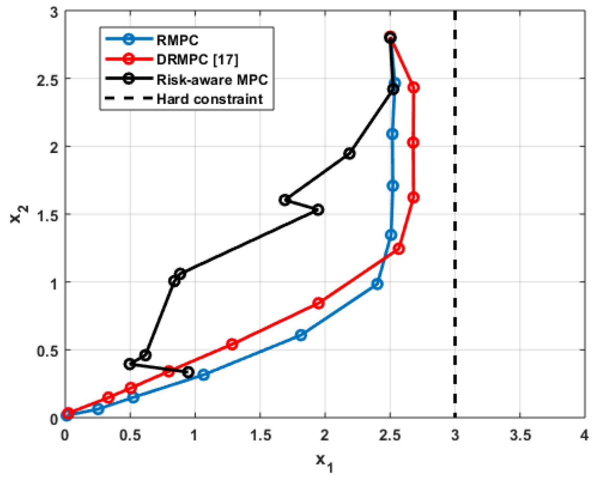

FIGURE 2. The mean trajectories.

TABLE 2 The number of constraint violations.

	Risk-aware MPC	SMPC	DRMPC [17]
$t = 1$	0	0	0
$t = 2$	2	4	2
$t = 3$	0	4	1
$t = 4$	0	4	2
$t = 5$	0	4	2
$t \geq 6$	0	0	0

where $\epsilon_\infty = 0.99$ and $\gamma_p = 6.9$ satisfying (29). The system is simulated for $T_f = 20$ time-steps.

In Table 1, we present a comparison of the performance of our proposed risk-aware MPC by evaluating 100 independent experiments against that of DRMPC, SMPC, and RMPC. The performance metrics for DRMPC, SMPC, and RMPC are based on data adopted from [17]. Our method, which incorporates the CVaR operator into the cost function to account for extreme outcomes in the distribution's tail, manifests a more conservative behavior as a trade-off for considering risk. Notably, RMPC yields the worst performance, which can be attributed to its conservative stance on worst-case disturbance scenarios.

In Fig. 2, we present a comparison between RMPC, DRMPC, and our risk-aware MPC method, focusing on the mean trajectories. This illustration clearly demonstrates that DRMPC, represented in red, exhibits the most unsafe behavior in terms of constraint handling. In contrast, our risk-aware MPC method demonstrates considerably safer constraint handling. Table 2 lists the number of constraint violations for each algorithm, including SMPC and DRMPC in different time steps. According to this table, the risk-aware MPC method outperforms the others, notably outperforming SMPC, which exhibits the highest number of constraint violations. It is important to note that the data for SMPC, RMPC, and DMPC illustrated in both Fig. 2 and Table 2 are sourced from [17].

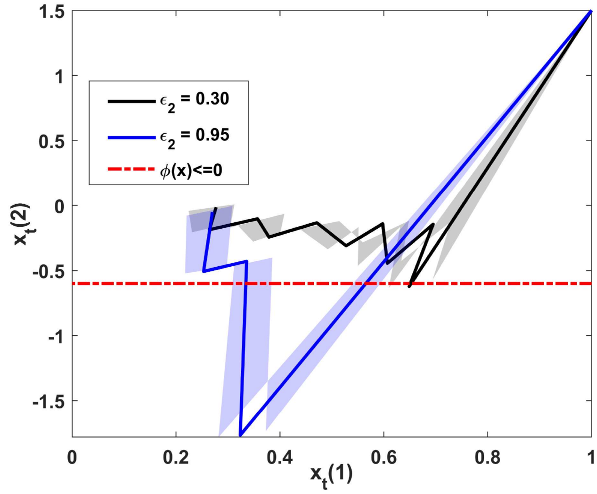

FIGURE 3. State trajectories for different CVaR constraint levels ϵ_2.

B. EXAMPLE 2: LATERAL CONTROL FOR AUTONOMOUS CARS

In this section, the proposed MPC controller (13) is applied to the discretized linear model of the steering system in autonomous cars [29], [42], which is described as:

$$x_{t+1} = \begin{bmatrix} 1 & 0.20 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.06 \\ 0.20 \end{bmatrix} u_t + \begin{bmatrix} 0.10 & 0 \\ 0 & 0.10 \end{bmatrix} \omega_t, \tag{34}$$

where $x_t = \begin{bmatrix} x_t(1) \\ x_t(2) \end{bmatrix}$ with $x_t(1)$ being the lateral position and $x_t(2)$ being the heading angle. The unknown system noise is considered as a Gaussian noise with zero mean and covariance $\Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix}$. The initial condition is considered as $x_0 = [1\ 1.5]^T$. The performance matrices are defined as $Q = \mathbb{I}_2$ and $R = 2$. By setting $\epsilon = 0.1$, one can find $P = \begin{bmatrix} 6.44 & 3.59 \\ 3.59 & 9.05 \end{bmatrix}$ satisfying (28). Regarding the state and input constraints, we impose a constraint on the heading angle as $\phi(x_t) = -x_t(2) - 0.6 \leq 0, \forall t \in \{1, \ldots, t_f\}$, and the input constraint is considered as $\|u_t\|_2 \leq 15$. The terminal constraint is considered as $\mathbb{O}_\infty = \{x \in \mathbb{R}^2 | x^T P x \leq \gamma_p - b_\omega\}$, where $\epsilon_\infty = 0.99$ and $\gamma_p = 0.01$ satisfying (29). The system is simulated for $T_f = 20$ time-steps.

We investigate two experiments to assess the effect of the CVaR parameters ϵ_1 and ϵ_2 on constraint satisfaction and performance.

As illustrated in Fig. 3, evaluated by 20 independent experiments with fixed $\epsilon_1 = 0.1$, two different constraint satisfaction parameters ϵ_2 are considered. The solid blue line

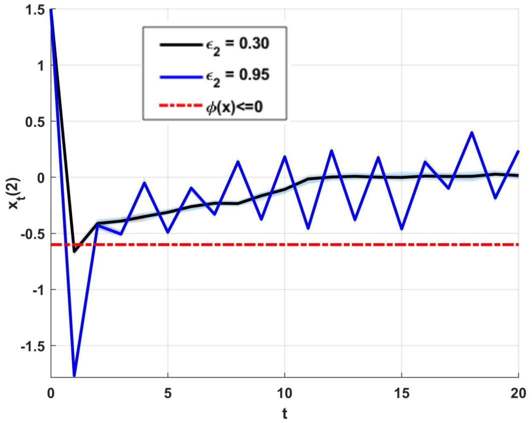

FIGURE 4. Heading angle $x_t(2)$ versus t for different CVaR constraint levels ϵ_2.

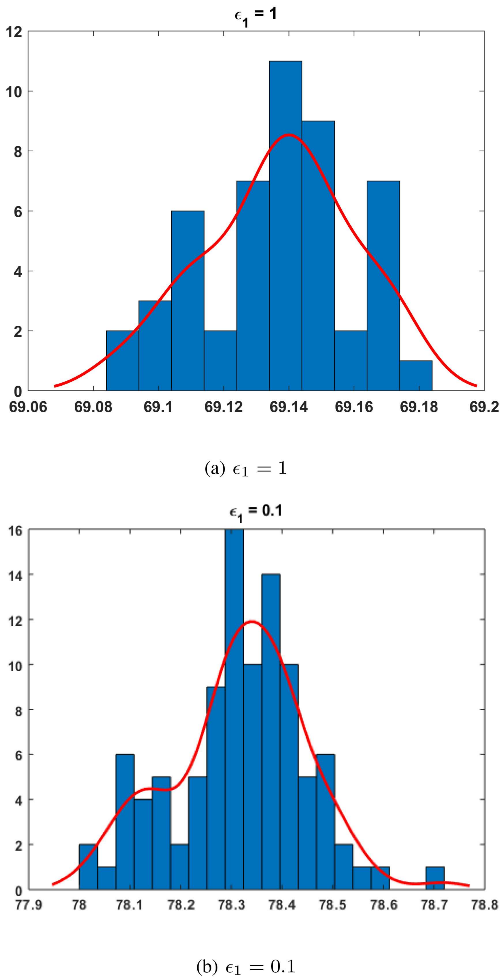

FIGURE 5. Histograms of loss function for different risk levels ϵ_1

represents the system trajectory when $\epsilon_2 = 0.95$, corresponding to the imposition of an almost risk-neutral constraint. In contrast, the solid black line illustrates the constraint satisfaction under a risk-aware approach with $\epsilon_2 = 0.3$. As can be seen from Fig. 3, by decreasing the probability level ϵ_2, the number of trajectories violating the safe constraints is decreased. Shaded areas represent the 75% confidence bounds. Additionally, Fig. 4 is plotted to show the dynamic evolution of the second state for different CVaR constraint levels, capturing the fact that reducing ϵ_2 decreases safety violations.

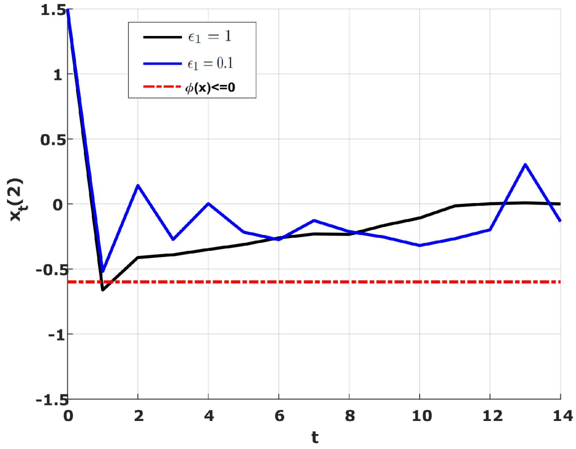

FIGURE 6. Heading angle $x_t(2)$ versus t for different CVaR confidence level ϵ_1.

To demonstrate the effect of ϵ_1 on performance, two different situations are considered: the risk-neutral case, where $\epsilon_1 = 1$, and a case with $\epsilon_1 = 0.1$ to capture the tail of the loss function. Fig. 5 depicts the histogram of the loss function evaluated by 100 independent experiments with a fixed safety probability $\epsilon_2 = 0.3$. In comparing the two histograms, it is evident that the risk-aware scenario (Fig. 5(b)) results in a system that behaves more conservatively, as reflected by the lower variance and higher mean cost. This increased average cost represents the price of considering risks, showing that the system is potentially foregoing lower costs in some instances to avoid higher costs in others, which could be associated with riskier outcomes.

Fig. 6 plots the evolution of the heading angle over time for different CVaR confidence levels ϵ_1. As observed in the risk-aware scenario depicted with the black color, the system inherently accounts for fluctuations and uncertainties within its operating environment, leading to a more resilient performance at the cost of incurring a higher average cost.

VII. CONCLUSION

In this paper, a distributionally robust MPC has been presented, incorporating the concept of multi-stage CVaR into both the cost function and constraints. By employing CVaR in both the cost function and constraints, the ability to calibrate risk assessment from worst-case scenarios to risk-neutral positions has been achieved, based on specific control applications. Under certain conditions, it has been demonstrated that the proposed MPC exhibits risk-aware exponential stability [18], and recursive feasibility, assuming the disturbance feedback policy parameterization. Finally, a numerical example was provided to underscore the efficacy of the proposed method.

APPENDIX

A. PROOF OF THEOREM 2

According to Definition 5, we need to show that under condition (27), if $x_t \in \mathbb{O}_\infty(P, \gamma_p^w)$, then $\inf_{\mathcal{P}\in\mathcal{A}} \mathcal{P}(x_{t+1} \in \mathbb{O}_\infty(P, \gamma_p)|x_t) \geq 1-\epsilon_\infty$. As can be seen in (6), it was shown $\sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_\infty}(x_{t+1}^T P x_{t+1}|x_t) \leq \gamma_p \Rightarrow \inf_{\mathcal{P}\in\mathcal{A}} \mathcal{P}(x_{t+1} \in \mathbb{O}_\infty(P, \gamma_p)|x_t) \geq 1-\epsilon_\infty$. As a result, one needs to show that $\sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_\infty}(x_{t+1}^T P x_{t+1}|x_t) \leq \gamma_p$ given $x_t \in \mathbb{O}_\infty(P, \gamma_p^w)$. Hence, one has

$$x^T P x + b_\omega \leq \gamma_p, \ \forall x \in \mathbb{O}_\infty\left(P, \gamma_p^w\right) \tag{35}$$

By applying (27), and using this fact $x^T(Q + F^T RF)x \geq 0$, one can obtain

$$(1+\epsilon)x^T A_c^T P A_c x + b_\omega \leq \gamma_p \tag{36}$$

By using Lemma 3, considering [14, Corollary I.3], and leveraging the monotonicity and translation equivariance property of CVaR given in Definition 2, one has

$$\sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_\infty}(x_{t+1}^T P x_{t+1}|x_t) \leq (1+\epsilon)x_t^T A_c^T P A_c x_t + b_w \tag{37}$$

As a result, by considering (36) and (37), we have

$$\sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_\infty}(x_{t+1}^T P x_{t+1}|x_t) \leq \gamma_p \tag{38}$$

which completes the proof.

B. PROOF OF LEMMA 9

According to Theorem 2, under the condition (27), if $u_t = F x_t, \forall x_t \in \mathbb{O}_\infty(P^*, \gamma_p^w) \subset \mathbb{O}_\infty(P^*, \gamma_p)$, is applied to the system (11), then $x_{t+1} \in \mathbb{O}_\infty(P^*, \gamma_p)$. As a result, by designing a proper set level γ_p, one can ensure that the trajectories starting from $\mathbb{O}_\infty(P^*, \gamma_p^w)$ never leaves $\mathbb{O}_\infty(P^*, \gamma_p)$ while satisfying the input and state constraints. Satisfying control input constraint can be translated into if $\bar{x}^T \begin{bmatrix} P^* & 0 \\ 0 & -\gamma_p \end{bmatrix} \bar{x} \leq 0$, then $\bar{x}^T \begin{bmatrix} F^{*T} C_u^T C_u F^* & 0 \\ 0 & -u_{\max}^2 \end{bmatrix} \bar{x} \leq 0$, where $\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$. As a result, by applying S-Lemma, one has $\begin{bmatrix} F^{*T} C_u^T C_u F^* - \lambda P^* & 0 \\ 0 & -u_{\max}^2 + \lambda\gamma_p \end{bmatrix} \leq 0$, where $\lambda_p \geq 0$. By setting $\lambda\gamma_p = u_{\max}^2$, one obtains the largest feasible set for $\lambda\gamma_p$. By substituting $\lambda\gamma_p$, one has (29b). Equation (29c) also can be shown in an identical fashion by using S-Lemma 2. It indicates that if $x \in \mathbb{O}_\infty(P^*, \gamma_p^w) \subset \mathbb{O}_\infty(P^*, \gamma_p)$, then $x \in \mathbb{X}$, i.e., $\bar{x}^T \begin{bmatrix} P^* & 0 \\ 0 & -\gamma_p \end{bmatrix} \bar{x} \leq 0 \Rightarrow \bar{x}^T \begin{bmatrix} G & g \\ g^T & \gamma \end{bmatrix} \bar{x} \leq 0$.

C. PROOF OF LEMMA 10

By using the monotonicity property of the risk measures, and considering (30) and (31), one has

$$\begin{aligned} \rho(c_1\psi(x_{t+1})) &\leq \rho(V(x_{t+1})) - V(x_t) + V(x_t) \\ &\overset{(31)}{\leq} -c_4\psi(x_t) + V(x_t) + b' \\ &\overset{(30)}{\leq} (c_2 - c_4)\psi(x_t) + (b' + c_3) \end{aligned} \tag{39}$$

We claim that $c_2 - c_4 \geq 0$. To show this, we use a proof by contradiction in which we assume $c_4 > c_2$, and as a result, there exists a time step t_1 in which $(c_2 - c_4)\psi(x_{t_1}) + (b' + c_3) < 0$ since $\psi(\cdot)$ is a strictly positive monotonic function and $b' + c_3 \geq 0$. Thus, $\rho(c_1\psi(x_{t+1}))$ takes a negative value, which is a contradiction since $\rho(c_1\psi(x_{t+1}))$ has a positive value due to the monotonicity and positive homogeneity properties given in Definition 2. As a result, c_2 should be greater than or equal to c_4, which implies $0 \leq (1 - \frac{c_4}{c_2}) < 1$.

By using (39), one has

$$\begin{aligned} \rho(V(x_{t+1})) &\leq -c_4\psi(x_t) + V(x_t) + b' \\ &\overset{(30)}{\leq} \left(1 - \frac{c_4}{c_2}\right) V(x_t) + \left(b' + \frac{c_3 c_4}{c_2}\right) \end{aligned} \tag{40}$$

Then, by applying the risk measure ρ on both sides of the above inequality, one has

$$\rho(V(x_{t+1})) \leq \left(1 - \frac{c_4}{c_2}\right) \rho(V(x_t)) + \left(b' + \frac{c_3 c_4}{c_2}\right) \tag{41}$$

By recursion, the following results can be obtained:

$$\begin{aligned} \rho(V(x_{t+1})) \leq & \left(1 - \frac{c_4}{c_2}\right)^t V(x_0) \\ & + \left(b' + \frac{c_3 c_4}{c_2}\right) \sum_{i=0}^{t-1} \left(1 - \frac{c_4}{c_2}\right)^i \end{aligned} \tag{42}$$

By using the geometric series formula, one has $(b' + \frac{c_3 c_4}{c_2}) \sum_{i=0}^{t-1} (1 - \frac{c_4}{c_2})^i \leq (b' + \frac{c_3 c_4}{c_2}) \frac{c_2}{c_4}$. As a result, we have

$$\begin{aligned} \rho(\psi(x_{t+1})) \leq & \frac{c_2}{c_1}\left(1 - \frac{c_4}{c_2}\right)^t \psi(x_0) \\ & + \frac{c_3}{c_1}\left(1 - \frac{c_4}{c_2}\right)^t + (b' + \frac{c_4 c_3}{c_2})\frac{c_2}{c_1 c_4} \\ \leq & \frac{c_2}{c_1}\left(1 - \frac{c_4}{c_2}\right)^t \psi(x_0) \\ & + \left(\frac{c_3}{c_1} + \left(b' + \frac{c_4 c_3}{c_2}\right)\frac{c_2}{c_1 c_4}\right) \end{aligned} \tag{43}$$

By setting $a = \frac{c_2}{c_1}$, $\lambda = (1 - \frac{c_4}{c_2})$, and $b = (\frac{c_3}{c_1} + (b' + \frac{c_4 c_3}{c_2})\frac{c_2}{c_1 c_4})$, the proof is completed.

D. PROOF OF THEOREM 3

In this proof, we will show that if (13) is feasible for given x_t and $\mathcal{A}$ defined as (12), then it will also be feasible for the next time step x_{t+1} by applying $u^*_{MPC} = u^*_{0|t}$ to the system (11) for any fixed $\omega_t \in \mathbb{W}$.

Consider the optimal control policy $\boldsymbol{u}_t^*$, to which we append $u^*_{N|t} = F x_{N|t}$, where F satisfies Lemma 8. Then, define the shifted control policy $\boldsymbol{u}^{*+}_t$ as

$$\boldsymbol{u}^{*+}_t = \left[u^{*}_{1|t}{}^T, \ldots, u^{*}_{N-1|t}{}^T, u^{*}_{N|t}{}^T\right]^T \tag{44}$$

Also, the shifted sequence of states $x^*_{0:N|t}$ is defined as follows by applying $\boldsymbol{u}^*_t$ and considering the parameterized control policy (15).

$$x^{*}_{0:N|t}{}^{+} = \left[x^{*}_{1|t}{}^{T}(\omega_{0|t}, \boldsymbol{\omega}_{1:N|t}), \ldots, x^{*}_{N+1|t}{}^{T}(\omega_{0|t}, \boldsymbol{\omega}_{1:N|t}) \right]^{T} \tag{45}$$

where $\omega_{0|t} = \omega_t$ is fixed, and $\boldsymbol{\omega}_{1:N|t}$ is considered as a variable. We assume the existence of $\boldsymbol{u}^*_t$ and $x^*_{0:N|t}$ by satisfying condition (1.

We claim that the shifted control policy $\boldsymbol{u}^{*+}_t$ and the shifted states $x^*_{0:N|t}{}^{+}$ are feasible solutions at the time step $t+1$. To prove this claim, first, from Lemma 9, it was shown that $u^*_{N|t} = F x_{N|t} \in \mathbb{U}_u$, where $x_{N|t} \in \mathbb{O}_\infty$ by holding 2. As a result, the shifted control policy (44) satisfies the control constraint, i.e., $\boldsymbol{u}^{*+}_t \in \mathbb{U}_u$. Also, by holding (2, if $x^*_{N|t} \in \mathbb{O}_\infty$, then $x^*_{N+1|t} \in \mathbb{O}_\infty$ by applying $u^*_{N|t} = F x_{N|t}$ according to Theorem 2. Consequently, the shifted states, as given by (45), satisfy the terminal constraint (13d).

To show that the shifted sequence of states $x^*_{0:N|t}{}^{+}$ satisfies the CVaR constraint (13c) given that $x^*_{0:N|t}$ is feasible at t, the following steps are taken. First, by using (6), an upper-approximate to (13c) is obtained. Then, by leveraging the CVaR definition in (2), interchanging the sup and inf using the stochastic saddle point theorem given in [43], and considering (3), one has

$$\begin{aligned}
&\sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_2}\left[\phi(x^*_{k|t})\right] \\
&\quad = \inf_{\beta\in\mathbb{R}} \left\{ \beta + \frac{1}{\epsilon_2} \rho_{\mathcal{A}}\left(\phi_{k|t}\right) \right\}
\end{aligned} \tag{46}$$

where $\phi_{k|t} := [\phi(x^*_{k|t}) - \beta]^+$. $\rho_{\mathcal{A}}(\phi_{k|t})$ can be obtained using Lemma 1 as

$$\begin{aligned}
\rho_{\mathcal{A}}\left(\phi_{k|t}\right) &= \sup_{\mathcal{P}\in\mathcal{A}} \mathbb{E}_{\mathcal{P}}[\phi_{k|t}] \\
&= \sup_{\mathcal{P}\in\mathcal{A}} \left\{ \int_{\mathbb{W}^k} \phi_{k|t}(\boldsymbol{w}_{0:k-1|t}) d\mathcal{P}(\boldsymbol{w}_{0:k-1|t}) \right\}
\end{aligned} \tag{47}$$

where (47) can be split up as

$$\begin{aligned}
&\sup_{\mathcal{P}\in\mathcal{A}} \left\{ \int_{\mathbb{W}^k} \phi_{k|t}(\boldsymbol{w}_{0:k-1|t}) d\mathcal{P}(\boldsymbol{w}_{0:k-1|t}) \right\} \\
&\quad = \sup_{\mathcal{P}\in\mathcal{A}} \sup_{\boldsymbol{\omega}_{0:k-2|t}\in\mathbb{W}^{k-1}} H(\boldsymbol{\omega}_{0:k-2|t})
\end{aligned} \tag{48}$$

where $H(\boldsymbol{\omega}_{0:k-2|t}) = \int_{\mathbb{W}} \phi_{k|t}(\boldsymbol{w}_{0:k-1|t}) d\mathcal{P}(\omega_{k-1})$ is a continuous random variable $\mathbb{W}^{k-1} \to \mathbb{R}$.

By considering $\omega_{0|t}$ as a fix (deterministic) value, one has

$$\begin{aligned}
\rho_{\mathcal{A}}\left(\phi_{k|t}\right) &\geq \sup_{\mathcal{P}\in\mathcal{A}} \sup_{\boldsymbol{\omega}_{1:k-2|t}\in\mathbb{W}^{k-2}} H(\omega_0, \boldsymbol{\omega}_{1:k-2|t}) \\
&= \sup_{\mathcal{P}\in\mathcal{A}} \left\{ \int_{\mathbb{W}^{k-1}} \phi_{k|t}(\omega_0, \boldsymbol{\omega}_{1:k-2|t}) d\mathcal{P}(\boldsymbol{w}_{1:k-2|t}) \right\}
\end{aligned} \tag{49}$$

By reverting (47)-(48), the right-hand side of the above inequality can be rewritten as follows

$$\rho_{\mathcal{A}}\left(\phi_{k|t}\right) \geq \rho_{\mathcal{A}}\left(\phi_{k-1|t}\right) \tag{50}$$

where $\phi_{k-1|t} = [\phi(x^*_{k-1|t}) - \beta]^+$. By applying (46), and using the CVaR property given in Definition 2, one has

$$0 \geq \sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_2}\left[\phi(x^*_{k|t})\right] \geq \sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_2}\left[\phi\left(x^*_{k-1|t}{}^{+}\right)\right] \tag{51}$$

As a result, for $k = 1 : N-1$, the aforementioned inequality remains valid, leading to the conclusion that the shifted states $x^*_{0:N-2|t}{}^{+}$ satisfy the CVaR constraint (13c).

E. PROOF OF THEOREM 4

By defining $\mathcal{R}(.) := \sup_{\mathcal{P}\in\mathcal{A}} CVaR^{\mathcal{P}}_{\epsilon_1}(.)$, the optimal value function $V^*(x_t)$ in (13) can be rewritten as:

$$V^*(x_t) = r\left(x_t, u^*_{0|t}\right) + \mathcal{R}\left(\sum_{k=1}^{N-1} r\left(x^*_{k|t}, u^*_{k|t}\right) + x^{*T}_{N|t} P x^*_{N|t} \right) \tag{52}$$

Define $\mathcal{Z}_N := x^{*T}_{N|t}(-P + Q + F^T R F) x^*_{N|t}$ and $\mathcal{Z}_{N+1} := ((A+BF)x^*_{N|t} + D\omega_{N|t})^T P((A+BF)x^*_{N|t} + D\omega_{N|t})$.

By applying Lemma 3, and using the properties of coherent risk measures defined in Definition 2, one has

$$\begin{aligned}
&\mathcal{R}(\mathcal{Z}_{N+1}) \\
&\quad \leq (1+\epsilon)\mathcal{R}\left(x^{*T}_{N|t}(A+BF)^T P (A+BF) x^*_{N|t} \right) + b_\omega
\end{aligned} \tag{53}$$

where $b_\omega = (1 + \frac{1}{\epsilon})\lambda_{\max}(D^T P D)\mathcal{R}(\omega^T_{N|t}\omega_{N|t})$.

By using the definition of $\mathcal{Z}_N$, (52) can be rewritten as

$$\begin{aligned}
V^*(x_t) &= r(x_t, u^*_{0|t}) \\
&\quad + \mathcal{R}\left(\sum_{k=1}^{N-1} r\left(x^*_{k|t}, u^*_{k|t}\right) + \|x^*_{N|t}\|^2_Q + \|x^*_{N|t}\|^2_{F^T R F} - \mathcal{Z}_N \right)
\end{aligned} \tag{54}$$

By considering (53), adding and subtracting $\mathcal{R}(\mathcal{Z}_{n+1})$ from (52), the following inequality can be obtained

$$\begin{aligned}
V^*(x_t) &\geq r(x_t, u^*_{0|t}) + \mathcal{R}\Bigg(\sum_{k=1}^{N-1} r(x^*_{k|t}, u^*_{k|t}) + \|x^*_{N|t}\|^2_Q \\
&\quad + \|x^*_{N|t}\|^2_{F^T R F} + \mathcal{R}(\mathcal{Z}_{N+1}) - \mathcal{Z}_N \\
&\quad - (1+\epsilon)\mathcal{R}\left(x^{*T}_{N|t}(A+BF)^T P (A+BF) x^*_{N|t} \right) - b_\omega \Bigg)
\end{aligned} \tag{55}$$

Let's claim that $\mathcal{Z}_N + (1+\epsilon_p)\mathcal{R}(x^{*T}_{N|t}(A+BF)^T P(A+BF)x^*_{N|t}) \leq 0$, which will be proved later. Thus, (55) becomes

$$\begin{aligned}
V^*(x_t) &\geq r(x_t, u^*_{0|t}) + \mathcal{R}\Bigg(\sum_{k=1}^{N-1} r(x^*_{k|t}, u^*_{k|t}) + \|x^*_{N|t}\|^2_Q \\
&\quad + \|x^*_{N|t}\|^2_{F^T R F} + \mathcal{R}(\mathcal{Z}_{N+1}) - b_\omega \Bigg) \\
&\geq r(x_t, u^*_{0|t}) + \mathcal{R}\left(\bar{V}(x^*_{1|t})\right) - b_\omega \\
&\geq r(x_t, u^*_{0|t}) + \mathcal{R}\left(V^*(x_{t+1})\right) - b_\omega
\end{aligned}$$

$$\geq \lambda_{\min}(Q)\|x_t\|^2 + \mathcal{R}\left(V^*(x_{t+1})\right) - b_\omega \tag{56}$$

Finally, we can get

$$\mathcal{R}\left(V^*(x_{t+1})\right) - V^*(x_t) \leq -\lambda_{\min}(Q)\|x_t\|^2 + b_\omega \tag{57}$$

By setting $c = \lambda_{\min}(Q)$, $\psi(x_t) = \|x_t\|^2$, and $b = b_\omega$, (57) resembles the sufficient condition (31).

To show that the first condition (30) is satisfied as well, the following procedures are taken.

The lower bound of $V^*(x_t)$ is easy to show according to:

$$V^*(x_t) \geq x_t^T Q x_t \geq \lambda_{\min}(Q)\|x_t\|^2 \tag{58}$$

where $\lambda_{\min}(Q) > 0$ since $Q > 0$.

To analyze the upper bound of $V^*(x_t)$, consider the control input $u(x_{k|t}) = F x_{k|t}$, $k = 0, \ldots, N-1$ is applied to the system (11), where F satisfies (27). Since this control input is feasible for all $x_{k|t} \in \mathbb{O}_\infty$ according to Theorem 3, as a result, by defining $\theta = \|Q + F^T R F\|_2$, one has

$$V^*(x_t) \leq \theta\|x_t\|^2 + \mathcal{R}\left(\theta \sum_{k=1}^{N-1} \|x_{k|t}\|^2 + \|P\|_2\|x_{N|t}\|^2\right) \tag{59}$$

By substituting $x_{k|t} = (A+BF)^k x_t + \sum_{j=0}^{k-1}(A+BF)^{k-j-1} D\omega_{j|t}$ in the above equation, and applying Lemma 3, one can get

$$V^*(x_t) \leq a_1\|x_t\|^2 + b_1,\ \forall x_t \in \mathbb{O}_\infty \tag{60}$$

where $a_1 = (\theta(1 + 2\sum_{k=1}^{N-1}\alpha_k) + \|P\|_2\alpha_N)$, $b_1 = 2\theta\mathcal{R}(\sum_{k=1}^{N-1}\sum_{j=0}^{k-1}\|(A+BF)^{k-j-1}\|^2\|D\omega_{j|t}\|^2)$, and $\alpha_k = \|(A+BF)^k\|^2$.

However, the inequality (60) is true for all $x_t \in \mathbb{O}_\infty \subseteq \mathbb{X}_N$. Now, we want to obtain an upper bound for $V^*(x_t)$ with a similar structure defined in (60) for all $x_t \in \mathbb{X}_N$.

The feasible set $\mathbb{X}_N$ is a closed set because the noise ω_t belongs to the compact set $\mathbb{W}$, and according to the set closure preservation property for the inverse of continuous functions, one can easily show that the feasible set $\mathbb{X}_N$ is closed. Also, $\mathbb{X}_N$ is bounded since it is necessarily a subset of the bounded set $\mathbb{X}$. Thus, $\mathbb{X}_N$ is a compact set. Consequently, there exists a positive constant J_N such that $V^*(x_t) \leq J_N$ for all $x_t \in \mathbb{X}_N$. By considering that the $\mathbb{O}_\infty$ is a compact and non-empty set, there exists $d > 0$ such that $\mathbb{O}_d := \{x \in \mathbb{R}^{n_x} | \|x\|_2 \leq d\} \subset \mathbb{O}_\infty$. Let $\beta_x := \max\{a_1\|x\|^2 + b_1 |\ \|x\|_2 \leq d\}$. As a result, one has $a_1\|x\|^2 + b_1 > \beta_x$ for all $x \in \mathbb{X}_N \setminus \mathbb{O}_d$, which results in $J_N \leq \frac{a_1 J_N}{\beta_x}\|x_t\|^2 + \frac{b_1 J_N}{\beta_x}$ for all $x_t \in \mathbb{X}_N$. Finally, one has

$$V^*(x_t) \leq \frac{a_1 J_N}{\beta_x}\|x_t\|^2 + \frac{b_1 J_N}{\beta_x},\ x_t \in \mathbb{X}_N \tag{61}$$

By setting $c_2 = \frac{a_1 J_N}{\beta_x} > 0$, and $c_3 = \frac{b_1 J_N}{\beta_x} \geq 0$, (61) resembles the upper bound condition on $V^*(x_t)$ given in (30).

By utilizing (27), one can demonstrate that $\mathcal{Z}_N + (1+\epsilon_p)\mathcal{R}({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}) \leq 0$. To this end, we rewrite (27) as follows:

$$0 \leq (1+\epsilon_p)\left({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}\right) \leq -\mathcal{Z}_N \tag{62}$$

By using [44, Eq. 3.7] and [45, Eq.4], we obtain the following inequality:

$$(1+\epsilon_p)\mathbb{E}_{\mathcal{P}^N}\left({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}\right) \leq -\mathcal{Z}_N \tag{63}$$

Taking the supremum on the left side of the above inequality, one has

$$(1+\epsilon_p)\sup_{\mathcal{P}^N \in \mathcal{A}^N}\mathbb{E}_{\mathcal{P}^N}\left({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}\right) \leq -\mathcal{Z}_N \tag{64}$$

By applying Lemma 1, we can substitute the coherent risk measure ρ as follows:

$$(1+\epsilon_p)\rho_{\mathcal{A}^N}\left({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}\right) \leq -\mathcal{Z}_N \tag{65}$$

Using the fact that $\sup_{\mathcal{P}^N \in \mathcal{A}^N} CVaR_{\epsilon_1}^{\mathcal{P}^N}(.)$ is indeed a coherent risk measure, we can conclude that $\mathcal{Z}_N + (1+\epsilon_p)\mathcal{R}({x^*}_{N|t}^T(A+BF)^T P(A+BF)x^*_{N|t}) \leq 0$, thereby completing the proof.

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