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Asymptotic behavior of endemic equilibria for a SIS epidemic model in convective environments

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Abstract

In this paper, we study a reaction-convection-diffusion SIS epidemic model with standard incidence function in a heterogeneous environment. The convection term is allowed to vary from positive to negative and a sign-changing function is used to specify convective direction. In particular, such a sign-changing function is allowed to be high-order degenerate at its critical points. We first establish the existence of endemic equilibria through the basic reproduction number $\mathcal{R}_0$ and investigate the asymptotic profile of $\mathcal{R}_0$ for large convection rate and small diffusion rate of the infectives, respectively. We further study the asymptotic behavior of endemic equilibria as convection approaches to infinity and the diffusion rate of infectives tends to zero, respectively. Our findings show that for large convection rate, both susceptible and infectious populations concentrate only at the critical points of the convection function, behaving exactly like a delta function; and for small diffusion rate of infectives, the density of susceptible population is positive while the total biomass of infectious population vanishes.

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Keywords: SIS epidemic model; Inhomogeneous convective environment; Endemic equilibrium; Basic reproduction number; Asymptotic behavior

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Contents

1. Introduction

Understanding host movements is of crucial importance in studying the transmission dynamics of infectious diseases. Hosts' social and movement behaviors can impact the transmission dynamics of infectious diseases, especially for pathogens transmitted through close contact between hosts or through contact with infectious stages in the environment (Han et al. [23]). The spatial structure of a host population determines the spatial probability distribution of interaction between individuals, and therefore influences the spatio-temporal dynamics of disease transmission within the host population (Gudelj and White [22]). Environmental heterogeneity is likely to be a key factor in determining the spatial distribution of host individuals (Cronin and Reeve [9]), thus affects disease dynamics via its influence on the spatial distribution of host individuals. Beardmore and Beardmore [4] investigated the spread of infectious diseases in a single social group of hosts and took into account spatial inhomogeneities, in which the members were divided into two classes: susceptible individuals $S(x,t)$ and infectious individuals $I(x,t)$ at location x and time t. Assume that the host movement comprises two components: (convective) movement towards the den to feed the offspring and (diffusive) dispersal in order to food and protect the group. Based on these assumptions, Beardmore and Beardmore [4] proposed a parabolic system, including inhomogeneous diffusive and convective terms to represent the presence of a focal point, described by reaction-convection-diffusion equations on a bounded domain of the following form

$$\begin{cases} \bar{S}_t = (D_1(x)\bar{S}_x + C_1(x)\bar{S})_x + (a-b)\bar{S} + a\bar{I} - \lambda\bar{S}\bar{I}, & (x,t) \in (0,1)\times(0,T), \\ \bar{I}_t = (D_2(x)\bar{I}_x + C_2(x)\bar{I})_x - c\bar{I} + \lambda\bar{S}\bar{I}, & (x,t) \in (0,1)\times(0,T), \\ D_1(x)\bar{S}_x + C_1(x)\bar{S} = D_2(x)\bar{I}_x + C_2(x)\bar{I} = 0, & (x,t) \in \{0,1\}\times(0,T), \\ \bar{S}(x,0) = S_0(x),\ \bar{I}(x,0) = I_0(x), & x \in (0,1). \end{cases} \tag{1.1}$$

In such a model, movement towards the den (or focal point) is modeled by convection (also known as advection) towards a single point in the domain $x_D \in (0,1)$. Mathematically, the convection coefficients $C_i(x)(i=1,2)$ are assumed to be continuously differentiable functions satisfying

$$C_i(0) < 0 \quad \text{and} \quad C_i(1) > 0, \qquad i = 1, 2.$$

The diffusion coefficients $D_i(x)(i = 1, 2)$ are continuously differentiable and strictly positive on $[0, 1]$. Parameters a, b, c and λ are positive constants representing the per-capita birth rate of both susceptible and infectious individuals, the per-capita natural death rate of the susceptible individuals, the per-capita disease-induced death rate, and the contact rate between susceptible and infectious individuals, respectively. They showed that a vertical bifurcation of steady-state solutions occurs in the model when the birth rate is taken as the bifurcation parameter, and used singular perturbation technique to present a description of the limiting spatial structure.

To investigate how the movement scenarios of susceptible and infectious individuals together with the between-group contact parameter affect the survival rate of susceptible individuals in each group, Gudelj et al. [21,22] further developed and analyzed a spatially explicit model of two interacting social groups of animals. It should be noted that there is no vertical transmission, and once infected, individuals cannot recover, in the models considered in [4,21,22]. Such a model is sometimes said of SI-type.

Allen et al. [2] proposed a SIS epidemic model given by the following coupled parabolic equations:

$$\begin{cases} \bar{S}_t = d_S \Delta \bar{S} - \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S} + \bar{I}} + \gamma(x)\bar{I}, & (x, t) \in \Omega \times \mathbb{R}_+, \\ \bar{I}_t = d_I \Delta \bar{I} + \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S} + \bar{I}} - \gamma(x)\bar{I}, & (x, t) \in \Omega \times \mathbb{R}_+, \\ \partial_{\mathbf{n}} \bar{S} = \partial_{\mathbf{n}} \bar{I} = 0, & (x, t) \in \partial\Omega \times \mathbb{R}_+, \\ \bar{S}(x, 0) = \bar{S}_0(x), \ \bar{I}(x, 0) = \bar{I}_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The main results in [2] were on the existence, uniqueness and asymptotic behavior of the endemic equilibrium as the diffusion rate of susceptible individuals converges to zero. Peng et al. [32,34] considered the effect of large and small diffusion rates of susceptible and infectious populations on the existence and extinction of infectious diseases. Peng and Liu [33] studied asymptotic stability of the endemic equilibrium with some additional conditions (i.e. (i) $d_S = d_I$ or (ii) $\beta(x) = r\gamma(x), r \in (0, \infty)$). If the environment featured by a predominantly unidirectional flow, such as river/stream or water column, one can describe these biased or passive movement in certain direction by adding an advection term to model (1.2). Cui et al. [13] investigated the following spatial epidemic model with advection in heterogeneous environment:

$$\begin{cases} \bar{S}_t = d_S \bar{S}_{xx} - q\bar{S}_x - \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S} + \bar{I}} + \gamma(x)\bar{I}, & 0 < x < L, t > 0, \\ \bar{I}_t = d_I \bar{I}_{xx} - q\bar{I}_x + \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S} + \bar{I}} - \gamma(x)\bar{I}, & 0 < x < L, t > 0, \\ d_S \bar{S}_x - q\bar{S} = d_I \bar{I}_x - q\bar{I} = 0, & x = 0, L, t > 0, \\ \bar{S}(x, 0) = \bar{S}_0(x), \ \bar{I}(x, 0) = \bar{I}_0(x), & 0 < x < L. \end{cases} \tag{1.3}$$

It is of interest in their paper to understand how diffusion and advection affect the persistence and extinction of infectious diseases. They focused on the relationship between the parameters

d_I, q and the basic reproduction number. The asymptotic behaviors of the endemic equilibrium (if it exists) with respect to d_S, d_I, q were investigated in [11] (see also in [25] where a different technical route was taken). In the follow-up studies, researchers examined the concentration phenomenon of the endemic equilibrium (with respect to d_S, d_I, q, respectively) with saturated incidence mechanism [6,7,10,19,37,40], mass action incidence mechanism [12], linear source [5], etc. Regarding other related studies and backgrounds on (1.2) and (1.3), one can refer to [1,3,14,18,20,24,26–28,35,39] and the references therein.

In this paper, we consider the following susceptible-infectious-susceptible (SIS) epidemic model

$$\begin{cases} \bar{S}_t = (d_S\bar{S}_x - qh(x)\bar{S})_x - \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} + \gamma(x)\bar{I}, & 0 < x < 1, t > 0, \\ \bar{I}_t = (d_I\bar{I}_x - qh(x)\bar{I})_x + \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} - \gamma(x)\bar{I}, & 0 < x < 1, t > 0, \\ d_S\bar{S}_x - qh(x)\bar{S} = d_I\bar{I}_x - qh(x)\bar{I} = 0, & x = 0, 1, t > 0, \\ \bar{S}(x,0) = \bar{S}_0(x),\ \bar{I}(x,0) = \bar{I}_0(x), & 0 < x < 1, \end{cases} \tag{1.4}$$

where $\bar{S}(x,t)$ and $\bar{I}(x,t)$ denote the densities of susceptible and infectious individuals at time t and position x, respectively; the positive constants d_S and d_I are diffusion rates for susceptible and infectious populations; $qh(x)$ forms the convection at position x, where the constant $q > 0$ is convection rate and the nonconstant $h(x) \in C^\infty([0,1])$ is used to specify the convective direction. (In fact, it suffices to assume that $h(x) \in C^{2k-1}([0,1])$ for some positive integer k throughout of this paper.) The functions $\beta(x)$ and $\gamma(x)$ are supposed to be positive and Hölder continuous on [0, 1] and represent the rates of disease transmission and recovery at x, respectively. We suppose that there is a positive number of infectious individuals initially; that is,

$$\int_0^1 \bar{I}_0(x)\mathrm{d}x > 0 \text{ with } \bar{S}_0(x), \bar{I}_0(x) \geq 0 \text{ for } x \in (0,1).$$

By using the argument of standard parabolic regularity and embedding theorem, we know that system (1.4) admits a unique classical solution $(\bar{S}(x,t), \bar{I}(x,t))$ which exists globally in time. Denote the initial total population on [0, 1] by $N := \int_0^1 \left[\bar{S}_0(x) + \bar{I}_0(x)\right]\mathrm{d}x > 0$. As argued in [2], it turns out that the total population is constant in time; that is,

$$\int_0^1 \left[\bar{S}(x,t) + \bar{I}(x,t)\right]\mathrm{d}x = N, \ \forall\, t \geq 0. \tag{1.5}$$

In addition to epidemic models, we can also find many applications of convection (advection) in other biological models. For instance, Zhou and Xiao [43] considered a Lotka-Volterra type competition-diffusion-advection model on a bounded domain $\bar{\Omega} \in \mathbb{R}^n$, in which they used a nonconstant function $P(x) \in C^2(\bar{\Omega})$ to specify the convective direction. Such examples can also be found in prey-predator models [38], phytoplankton models [15,16,30], etc. Various forms for these kind of active movement of animals have been discussed in [31].

We assume that the convection function can change signs. More specifically, we make the following assumption on h.

(H) Suppose that k is an integer in $\mathbb{N}^+$ and $h(x) \in C^{2k-1}([0,1])$. There exists a $\kappa \in (0,1)$ such that $h(x) > 0$ if $x \in [0, \kappa)$; $h(x) < 0$ if $x \in (\kappa, 1]$; and at $x = \kappa$, it holds

$$h^{(i)}(\kappa) = 0 \text{ if } i = 0, 1, \cdots, 2k-2, \quad \text{and } -\infty < h^{(2k-1)}(\kappa) < 0.$$

Here and after, $h^{(i)}(x)$ $(i = 0, 1, 2\cdots)$ represents the ith-order derivative of $h(x)$ at $x \in [0,1]$. Hypothesis (H) requires that $0 < \kappa < 1$. In fact, κ could be 0 or 1. We assume that

(H0) Suppose that k is an integer in $\mathbb{N}^+$ and $h(x) \in C^{2k-1}([0,1])$. $h(x) < 0$ if $x \in (0,1]$; and at $x = 0$, it holds

$$h^{(i)}(0) = 0 \text{ if } i = 0, 1, \cdots, 2k-2, \quad \text{and } -\infty < h^{(2k-1)}(0) < 0.$$

(H1) Suppose that k is an integer in $\mathbb{N}^+$ and $h(x) \in C^{2k-1}([0,1])$. $h(x) > 0$ if $x \in [0,1)$; and at $x = 1$, it holds

$$h^{(i)}(1) = 0 \text{ if } i = 0, 1, \cdots, 2k-2, \quad \text{and } -\infty < h^{(2k-1)}(1) < 0.$$

In this paper, we are mainly concerned about the asymptotic behaviors of the nonnegative steady-state solutions of system (1.4); that is, the nonnegative solutions of

$$\begin{cases} (d_S S_x - qh(x)S)_x - \beta(x)\dfrac{SI}{S+I} + \gamma(x)I = 0, & 0 < x < 1, \\ (d_I I_x - qh(x)I)_x + \beta(x)\dfrac{SI}{S+I} - \gamma(x)I = 0, & 0 < x < 1, \\ d_S S_x - qh(x)S = d_I I_x - qh(x)I = 0, & x = 0, 1. \end{cases} \tag{1.6}$$

Observing the fact that (1.5) holds, we impose an additional hypothesis

$$\int_0^1 [S(x) + I(x)]\mathrm{d}x = N. \tag{1.7}$$

Since in practical considerations, only nonnegative solutions $(S(x), I(x))$ of (1.6)-(1.7) are of interest. We say such $(S(x), I(x))$ is a *disease-free equilibrium* (DFE) of (1.4)-(1.5) if $I(x) \equiv 0$ on $[0,1]$, and an *endemic equilibrium* (EE) of (1.4)-(1.5) if $I(x) \not\equiv 0$ on $[0,1]$.

The rest of paper is organized as follows. In section 2 we aim to investigate the existence of the EE, in which we additionally examine the asymptotic behavior of $\mathcal{R}_0$ for large convection rate q and small infectious diffusion rate d_I. In section 3 we derive some priori estimates related to the EE $(S(x), I(x))$ which are useful in studying the asymptotic profiles of the EE. Section 4 is devoted to studying the asymptotic behavior of the EE as $q \to \infty$. In section 5, we analyze the asymptotic behavior of the EE as $d_I \to 0$. Finally, we make some comparisons of our results to the model in homogeneously convective environment and prospects for future research.

2. The disease-free and endemic equilibria

In this section, we aim to obtain conditions on the stability/instability of the DFE, thus the existence of the EE.

2.1. *Stability of the* DFE

Let $(\hat{S}, 0)$ be any DFE of (1.4). Hence, $\hat{S}(x)$ is a solution of

$$\begin{cases} (d_S S_x - qSh(x))_x = 0, & x \in (0,1), \\ d_S S_x - qSh(x) = 0, & x = 0, 1, \\ \int\limits_0^1 S(x)\mathrm{d}x = N. \end{cases} \tag{2.1}$$

Solving (2.1), one obtains that its unique solution can be represented as

$$\hat{S}(x) = \frac{N e^{\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}}{\int_0^1 e^{\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}\mathrm{d}x}. \tag{2.2}$$

Linearizing (1.4) around the DFE, we obtain

$$\begin{cases} \bar{\xi}_t = (d_S\bar{\xi}_x - q\bar{\xi}h(x))_x - (\beta(x) - \gamma(x))\bar{\eta}, & (x,t) \in (0,1) \times \mathbb{R}_+, \\ \bar{\eta}_t = (d_I\bar{\eta}_x - q\bar{\eta}h(x))_x + (\beta(x) - \gamma(x))\bar{\eta}, & (x,t) \in (0,1) \times \mathbb{R}_+. \end{cases} \tag{2.3}$$

Let $\bar{\xi}(x,t) = e^{-\lambda t}\xi(x)$, $\bar{\eta}(x,t) = e^{-\lambda t}\eta(x)$ be the solution of (2.3). We then derive the following eigenvalue problem

$$\begin{cases} (d_S\xi_x - q\xi h(x))_x - (\beta(x) - \gamma(x))\eta + \lambda\xi = 0, & x \in (0,1), \\ (d_I\eta_x - q\eta h(x))_x + (\beta(x) - \gamma(x))\eta + \lambda\eta = 0, & x \in (0,1), \\ d_S\xi_x - q\xi h(x) = d_I\eta_x - q\eta h(x) = 0, & x = 0, 1. \end{cases} \tag{2.4}$$

In view of the fact $\int_0^1 \left(\hat{S}_0(x) + \hat{I}_0(x)\right)\mathrm{d}x = N$ and (2.2), we additionally suppose that

$$\int\limits_0^1 (\xi(x) + \eta(x))\mathrm{d}x = 0. \tag{2.5}$$

Since the η-equation is decoupled from (2.4), in light of (2.5), we only need to consider the following eigenvalue problem

$$\begin{cases} (d_I\eta_x - q\eta h(x))_x + (\beta(x) - \gamma(x))\eta + \lambda\eta = 0, & x \in (0,1), \\ d_I\eta_x - q\eta h(x) = 0, & x = 0, 1. \end{cases} \tag{2.6}$$

Set $\zeta(x) = e^{-\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\eta(x)$. Then one has

$$\begin{cases} -d_I\zeta_{xx} - qh(x)\zeta_x - (\beta(x)-\gamma(x))\zeta = \lambda\zeta, & x \in (0,1), \\ \zeta_x = 0, & x = 0, 1. \end{cases} \tag{2.7}$$

By Krein-Rutman theorem, we know that (2.7) admits a unique simple real principal eigenvalue and its corresponding positive eigenfunction is also unique up to a scalar multiple. Denote the principal eigen-pair by (λ_1, ϕ_1).

Following the definition of the basic reproduction number in [2,3,11,13], we define the basic reproduction number for system (1.4) by

$$\mathcal{R}_0 = \sup_{\varphi \in H^1((0,1)), \varphi \neq 0} \frac{\int_0^1 \beta(x) e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\varphi^2 \mathrm{d}x}{d_I \int_0^1 e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}|\varphi_x|^2\mathrm{d}x + \int_0^1 \gamma(x) e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\varphi^2\mathrm{d}x}. \tag{2.8}$$

Obviously, $\mathcal{R}_0 > 0$ for all $q > 0,\ d_I > 0$. It is a well known fact (see, e.g., [2,13]) that there exists a positive function $\Phi(x) \in C^2((0,1)) \cap C^1([0,1])$ such that

$$\begin{cases} -(d_I\Phi_x - qh(x)\Phi)_x + \gamma(x)\Phi = \dfrac{1}{\mathcal{R}_0}\beta(x)\Phi, & x \in (0,1), \\ d_I\Phi_x - qh(x)\Phi = 0, & x = 0, 1. \end{cases} \tag{2.9}$$

Lemma 2.1. $1 - \mathcal{R}_0$ *has the same sign as* λ_1.

Proof. Recall that (λ_1, ϕ_1) satisfies

$$\begin{cases} -d_I\phi_{1,xx} - qh(x)\phi_{1,x} - (\beta(x)-\gamma(x))\phi_1 = \lambda_1\phi_1, & x \in (0,1), \\ \phi_{1,x} = 0, & x = 0, 1. \end{cases} \tag{2.10}$$

Multiplying (2.9) by ϕ_1 and (2.10) by Φ, integrating by parts in $(0,1)$, and substracting the resulting equations, one gets

$$\lambda_1 \int_0^1 \Phi\phi_1 \mathrm{d}x = \left(\frac{1}{\mathcal{R}_0} - 1\right)\int_0^1 \beta(x)\Phi\phi_1\mathrm{d}x.$$

Due to the positivity of Φ and ϕ_1, we then have that $\mathcal{R}_0 > 1$ if $\lambda_1 < 0$; $\mathcal{R}_0 = 1$ if $\lambda_1 = 0$; and $\mathcal{R}_0 < 1$ if $\lambda_1 > 0$. □

In fact, we further have the following result on the stability of the DFE.

Lemma 2.2. *If* $\mathcal{R}_0 < 1$, *then the* DFE *is linearly stable; if* $\mathcal{R}_0 > 1$, *then the* DFE *is unstable.*

2.2. Uniform persistence and existence of the EE

Now we are concerned with the uniform persistence property of solutions to system (1.4) provided that $\mathcal{R}_0 > 1$, which will in turn imply the existence of the EE. Let $(\bar{S}, \bar{I})$ be any positive solution (1.4). We make the transformations

$$\bar{u}(x,t) = e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}\bar{S}(x,t) \text{ and } \bar{v}(x,t) = e^{-\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\bar{I}(x,t).$$

Then $(\bar{u}, \bar{v})$ satisfies the following system:

$$\begin{cases} \bar{u}_t = d_S\bar{u}_{xx} + qh(x)\bar{u}_x \\ \qquad - \beta(x)\dfrac{\bar{u}\bar{v}e^{\left(\frac{q}{d_S}+\frac{q}{d_I}\right)\int_0^x h(s)\mathrm{d}s}}{\bar{u}e^{\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s} + \bar{v}e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}} + \gamma(x)\bar{v}e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}, & (x,t) \in (0,1)\times\mathbb{R}_+, \\ \bar{v}_t = d_I\bar{v}_{xx} + qh(x)\bar{v}_x \\ \qquad + \beta(x)\dfrac{\bar{u}\bar{v}e^{\left(\frac{q}{d_S}+\frac{q}{d_I}\right)\int_0^x h(s)\mathrm{d}s}}{\bar{u}e^{\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s} + \bar{v}e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}} - \gamma(x)\bar{v}e^{\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}, & (x,t) \in (0,1)\times\mathbb{R}_+, \\ \bar{u}_x = \bar{v}_x = 0, & (x,t) \in \{0,1\}\times\mathbb{R}_+, \\ \bar{u}(x,0) = e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}\bar{S}(x,0),\ \bar{v}(x,0) = e^{-\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\bar{I}(x,0), & x \in (0,1) \end{cases} \tag{2.11}$$

with

$$\int_0^1 \left[\bar{u}(x,t)e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s} + \bar{v}(x,t)e^{-\frac{q}{d_I}\int_0^x h(s)\mathrm{d}s}\right]\mathrm{d}x = N. \tag{2.12}$$

Condition (2.12) implies that $\|\bar{u}(\cdot,t)\|_{L^1((0,1))} + \|\bar{v}(\cdot,t)\|_{L^1((0,1))}$ is bounded. Use [17, Lemma 2.1] (with $p_0 = 1, q = 2$ and $\varrho = 0$ in [17, Lemma 2.1]) to obtain the uniform bounds (w.r.t. $t > 0$) of $\|\bar{u}(\cdot,t)\|_{L^\infty((0,1))}$ and $\|\bar{v}(\cdot,t)\|_{L^\infty((0,1))}$. As a consequence, these yield the uniform bounds of $\|\bar{S}(\cdot,t)\|_{L^\infty((0,1))}$ and $\|\bar{I}(\cdot,t)\|_{L^\infty((0,1))}$ for all $t > 0$.

Lemma 2.3. *There exists a positive constant C independent of $(\bar{S}_0, \bar{I}_0)$ with $\int_0^1(\bar{S}_0 + \bar{I}_0)\mathrm{d}x = N$ such that*

$$\|\bar{S}(\cdot,t)\|_{L^\infty((0,1))} + \|\bar{I}(\cdot,t)\|_{L^\infty((0,1))} < C$$

for all $t > 0$, where $(\bar{S}, \bar{I})$ is the solution of (1.4).

We use the theory of abstract dynamical systems (see, e.g., [29,41,42]) to investigate the uniform persistence of (1.4) when $\mathcal{R}_0 > 1$.

Let (X, d) be a complete metric space and $\Theta(t) : X \to X$ be a continuous semiflow. Define the distance of a point $z \in X$ to a subset A of X by $d(z, A) := \inf_{x\in A} d(z, x)$, where d is a norm-induced distance. Suppose that X_0 is an open set of X with $X = \bar{X}_0$, then the boundary $\partial X_0 := X - X_0$

is closed in X. We say that the semiflow $\Theta(t)$ is *uniformly persistent* with respect to $(X_0, \partial X_0)$ if there is an $\epsilon > 0$ such that

$$\liminf_{t\to\infty} d(\Theta(t)x, \partial X_0) \geq \epsilon \quad \text{for all } x \in X_0.$$

In the following we set $X = C([0,1]; \mathbb{R}^2_+)$ with the metric induced by the supremum norm $\|\cdot\|_\infty$. Denote

$$Y := \left\{ (\upsilon_1, \upsilon_2) \in X : \int_0^1 (\upsilon_1 + \upsilon_2)\mathrm{d}x = N \right\}.$$

For each $(\bar{S}_0, \bar{I}_0) \in Y$, (1.4) has a unique solution $(\bar{S}(x,t), \bar{I}(x,t)) \in Y$ for any $t \geq 0$. Define

$$X_0 := \{(\upsilon_1, \upsilon_2) \in Y : \upsilon_2 \not\equiv 0\} \ \text{ and } \ \partial X_0 := \{(\upsilon_1, \upsilon_2) \in Y : \upsilon_2 \equiv 0\}.$$

Then $Y = X_0 \cup \partial X_0$, X_0 and ∂X_0 are respectively open and closed subsets of Y. Moreover, X_0 is convex. We now define a semiflow $\Theta(t) : Y \to Y$ by

$$\Theta(t)(\bar{S}_0, \bar{I}_0) = (\bar{S}(x,t), \bar{I}(x,t)), \ (\bar{S}_0, \bar{I}_0) \in Y, \ t \geq 0.$$

Clearly, $\Theta(t)X_0 \subset X_0$ and $\Theta(t)\partial X_0 \subset \partial X_0$ for $t \geq 0$. By Lemma 2.3, using the standard parabolic theory and embedding theorem one obtains that the semiflow $\Theta(t)$ is continuous and compact on Y for all $t > 0$. Lemma 2.3 also implies that $\Theta(t)$ is point dissipative in X. Represent the omega-limit set of $(\bar{S}_0, \bar{I}_0) \in Y$ by $\omega((\bar{S}_0, \bar{I}_0))$, $(\hat{S}(x), 0)$ (where $\hat{S}(x)$ has obtained in (2.2)) by M_0.

Theorem 2.4. *Assume that* $\mathcal{R}_0 > 1$. *Then there exists a constant* $\tau > 0$*, independent of the initial data* $(\bar{S}_0, \bar{I}_0)$*, such that any solution* $(\bar{S}, \bar{I})$ *of* (1.4) *satisfies*

$$\liminf_{t\to\infty} \bar{S}(x,t) \geq \tau \ \textit{and} \ \liminf_{t\to\infty} \bar{I}(x,t) \geq \tau \quad \textit{uniformly on } [0,1].$$

Moreover, (1.4) *admits at least one* EE.

Proof. We proceed the proof in several steps.

Step 1. $\omega((\bar{S}_0, \bar{I}_0)) = \{M_0\}$ for all $(\bar{S}_0, \bar{I}_0) \in \partial X_0$. Clearly, $\bar{I}(x,t) \equiv 0$ if $\bar{I}(x,0) \equiv 0$. Hence, $\bar{S}(x,t)$ solves

$$\begin{cases} \bar{S}_t = (d_S \bar{S}_x - qh(x)\bar{S})_x, & 0 < x < 1, t > 0, \\ d_S \bar{S}_x - qh(x)\bar{S} = 0, & x = 0, 1, t > 0, \\ \bar{S}(x,0) = \bar{S}_0(x) \geq, \not\equiv 0, & 0 < x < 1, \\ \int_0^1 \bar{S}(x,t)\mathrm{d}x = N, & t \geq 0. \end{cases}$$

One can check that $\bar{S}(x,t)$ converges uniformly to $\hat{S}(x)$ on [0, 1] (i.e. $\lim_{t\to\infty} \bar{S}(x,t) = \hat{S}(x)$ for all $x \in [0,1]$). This follows $\bigcup_{(\bar{S}_0,\bar{I}_0)\in\partial X_0} \omega(\bar{S}_0,\bar{I}_0) = \{M_0\}$.

Step 2. There exists a constant $\delta > 0$ such that $\limsup_{t\to\infty} \|\Theta(t)(\bar{S}_0,\bar{I}_0) - M_0\| \ge \delta, \forall\, (\bar{S}_0,\bar{I}_0) \in X_0$. Since $\mathcal{R}_0 > 1$, we have $\lambda_1 < 0$. We then can find a small constant $\epsilon_0 > 0$ such that $\lambda_1(\epsilon_0) + \epsilon_0 < 0$, where $\lambda_1(\epsilon_0)$ is the principal eigenvalue of

$$\begin{cases} -(d_I\phi_x - qh(x)\phi)_x - \left(\beta(x)\dfrac{\hat{S}(x)-\epsilon_0}{\hat{S}(x)+2\epsilon_0} - \gamma(x)\right)\phi = \lambda\phi, & 0<x<1, \\ d_I\phi_x - qh(x)\phi = 0, & x = 0, 1. \end{cases} \tag{2.13}$$

Now we suppose to the contrary that there exists some $(\bar{S}_0^*, \bar{I}_0^*) \in X_0$ such that the unique solution $\Theta(t)(\bar{S}_0^*, \bar{I}_0^*) = (\bar{S}^*(x,t), \bar{I}^*(x,t))$ satisfies

$$\limsup_{t\to\infty} \|\Theta(t)(\bar{S}_0^*, \bar{I}_0^*) - M_0\|_\infty \le \delta_0,$$

where $\delta_0 < \epsilon_0$. This implies that there exists $T_0 > 0$ such that for all $x \in [0,1]$,

$$\hat{S}(x) - \epsilon_0 < \hat{S}(x) - \delta_0 \le \bar{S}^*(x,t) \le \hat{S}(x) + \delta_0,\ 0 \le \bar{S}_0^* \le \delta_0 < \epsilon_0,\ \forall\, t \ge T_0. \tag{2.14}$$

Let ϕ_{ϵ_0} be the positive eigenfunction corresponding to $\lambda_1(\epsilon_0)$ in (2.13). By the strong maximum principle for parabolic equations, one sees that $(\bar{S}^*(\cdot,t), \bar{I}^*(\cdot,t)) \in \mathrm{int}(Y)$ for any given $t \ge 0$ if $(\bar{S}_0^*, \bar{I}_0^*) \in X_0$. Furthermore, one can find a small positive constant c^* such that $\bar{I}_0^* \ge c^*\phi_{\epsilon_0}$ on [0, 1]. In view of (2.14), we see that $\bar{I}^*(x,t)$ is a supersolution of

$$\begin{cases} w_t = (d_I w_x - qh(x)w)_x + \left(\beta(x)\dfrac{\hat{S}-\epsilon_0}{\hat{S}+2\epsilon_0} - \gamma(x)\right)w, & 0<x<1, t>T_0, \\ d_I w_x - qh(x)w = 0, & x = 0, 1, t > T_0, \\ w(x, T_0) = c^*\phi_{\epsilon_0}, & 0<x<1. \end{cases} \tag{2.15}$$

On the other hand, one can check that $c^* e^{-\lambda_1 t}\phi_{\epsilon_0}(x)$ is the unique solution of (2.15). We then deduce a contradiction by parabolic comparison principle that

$$\bar{I}^*(x,t) \ge c^* e^{-\lambda_1 t}\phi_{\epsilon_0}(x) \to \infty \text{ uniformly on } [0,1] \text{ as } t \to \infty.$$

Step 3. $\Theta(t)$ is uniformly persistent and (1.4) admits at least one EE. By Step 2, we see that $\{M_0\}$ is an isolated invariant set for $\Theta(t)$ in Y and $W^x(M_0) \cap X_0 = \emptyset$, where $W^x(M_0)$ is the stable set of M_0 for $\Theta(t)$. We then can use [41, Theorem 2.2] (or [42, Theorem 1.3.1 and Remarks 1.3.1]) to conclude that $\Theta(t)$ is uniformly persistent with respect to $(X_0, \partial X_0)$. We can also use [41, Theorem 2.3] to obtain that $\Theta(t) : X_0 \to X_0$ has a compact global attractor A_0. This immediately follows that (1.4) admits at least one EE. For any EE (S_e, I_e), we see $(S_e, I_e) \in A_0$. Let $B_0 := \bigcup_{t\ge0} \Theta(t)A_0$, then it follows that $B_0 \subset X_0$ and $\lim_{t\to\infty} d(\Theta(t)(S_e, I_e), B_0) = 0$. Clearly,

$A_0 \subset \text{int}X_0$ and then $B_0 \subset \text{int}X_0$. Obviously, $\Theta(t)(S_e, I_e) \in B_0$. Using the compactness and global attractiveness of B_0 for $\Theta(t)$ in X_0, we can find a $\tau > 0$ such that $\liminf_{t\to\infty} \Theta(t)(\upsilon_1, \upsilon_2) \geq (\tau, \tau)$ for any $(\upsilon_1, \upsilon_2) \in X_0$. This completes the proof. □

It is necessary to find some explicit conditions guaranteeing the existence of the EE for large q before we discuss the asymptotic profiles of the EE later. Thus, we are going to derive the limiting behavior of the basic reproduction number $\mathcal{R}_0$.

2.3. Asymptotic behavior of $\mathcal{R}_0$

Recall (2.8) and (2.9). Set $w = w(x; q) := \Phi(x; q)e^{-\frac{q}{2d_I}\int_0^x h(s)\mathrm{d}s}$, then one can check that $(\mathcal{R}_0, w)$ satisfies equations

$$\begin{cases} w_{xx} = \left(\dfrac{q}{2d_I}h'(x) + \dfrac{q^2}{4d_I^2}h^2(x) + \dfrac{\gamma(x) - \beta(x)/\mathcal{R}_0}{d_I}\right)w, & x \in (0,1), \\ w_x = \dfrac{q}{2d_I}wh(x), & x = 0, 1 \end{cases} \tag{2.16}$$

and the integral identity

$$d_I \int_0^1 \left|w_x - \frac{q}{2d_I}wh(x)\right|^2 \mathrm{d}x + \int_0^1 \gamma(x)w^2\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_0^1 \beta(x)w^2\mathrm{d}x. \tag{2.17}$$

Later, we say "w is a positive solution of (2.16)" means that $w = \Phi e^{-\frac{q}{2d_I}\int_0^x h(s)\mathrm{d}s}$ with Φ being the principal eigenfunction of (2.9). Set $\phi(x) = e^{\frac{q}{2d_I}\int_0^x h(s)\mathrm{d}s}\varphi(x)$ in (2.8), then

$$\mathcal{R}_0 = \sup_{\phi \in H^1((0,1)), \phi \neq 0} \frac{\int_0^1 \beta(x)\phi^2\mathrm{d}x}{d_I \int_0^1 \left|\phi_x - \frac{q}{2d_I}\phi h(x)\right|^2 \mathrm{d}x + \int_0^1 \gamma(x)\phi^2\mathrm{d}x}. \tag{2.18}$$

We see that $\mathcal{R}_0$ is bounded from (2.18) for all $q > 0, d_I > 0$. Indeed, we have

$$\mathcal{R}_0 \leq \sup_{\varphi \in H^1((0,1)), \varphi \neq 0} \frac{\int_0^1 \beta\varphi^2\mathrm{d}x}{\int_0^1 \gamma\varphi^2\mathrm{d}x} \leq \frac{\beta^*}{\gamma_*},$$

where $\beta^* = \max_{x\in[0,1]} \beta(x), \gamma_* = \min_{x\in[0,1]} \gamma(x)$. One of the main results in this section is stated as below.

Theorem 2.5. *Assume that one of* (H), (H0) *and* (H1) *holds. Then* $\lim_{q\to\infty} \mathcal{R}_0(q) = \frac{\beta(\kappa)}{\gamma(\kappa)}$.

Proof. We consider only the case (H), since the case (H0) or (H1) can be proved similarly. Indeed, we only need to show that

$$\liminf_{q\to\infty}\mathcal{R}_0(q)\geq\frac{\beta(\kappa)}{\gamma(\kappa)}\text{ and }\limsup_{q\to\infty}\mathcal{R}_0(q)\leq\frac{\beta(\kappa)}{\gamma(\kappa)}.$$

Lower bound. $\liminf_{q\to\infty}\mathcal{R}_0(q)\geq\frac{\beta(\kappa)}{\gamma(\kappa)}$. Let $H(x)$ be a primitive function of $h(x)$; namely, $H(x)=\int_0^x h(s)\mathrm{d}s$. From assumption (H), one can see that

$$H(x)-H(\kappa)=\frac{h^{(2k-1)}(\kappa)}{(2k)!}(x-\kappa)^{2k}+o((x-\kappa)^{2k})\quad\text{for all } x \text{ near } \kappa.$$

Hence, there exist constants $C_1, C_2>0$ such that

$$C_1(x-\kappa)^{2k}\leq H(\kappa)-H(x)\leq C_2(x-\kappa)^{2k},\quad\forall\, x\in(\kappa-2R,\kappa+2R)\tag{2.19}$$

for some small $R>0$ with $(\kappa-2R,\kappa+2R)\subset(0,1)$. Thanks to (2.18), for any nonzero $w\in H^1((0,1))$, we have

$$\mathcal{R}_0\geq\frac{\int_0^1\beta(x)w^2\mathrm{d}x}{d_I\int_0^1\left|w_x-\frac{q}{2d_I}wh(x)\right|^2\mathrm{d}x+\int_0^1\gamma(x)w^2\mathrm{d}x}.$$

To obtain the lower bound, we construct the test function as $w(x)=\frac{1}{\sqrt{c}}\eta(x)e^{\frac{q}{2d_I}(H(x)-H(\kappa))}$, in which the function $\eta(x)$ is defined as

$$\eta(x)=\begin{cases}1, & x\in(\kappa-R,\kappa+R),\\ 2-|x-\kappa|/R, & x\in[\kappa-2R,\kappa-R]\cup[\kappa+R,\kappa+2R],\\ 0, & x\in\mathbb{R}\setminus[\kappa-2R,\kappa+2R],\end{cases}$$

and $c>0$ is a constant such that $c=\int_0^1\eta^2(x)e^{\frac{q}{d_I}(H(x)-H(\kappa))}\mathrm{d}x$. Then one can check that

$$d_I\left|w_x-\frac{q}{2d_I}h(x)w\right|^2=\frac{d_I}{c}e^{\frac{q}{d_I}(H(x)-H(\kappa))}|\eta_x(x)|^2\leq\frac{1}{c}e^{\frac{q}{2d_I}(H(x)-H(\kappa))}e^{-\frac{C_1q}{2d_I}R^{2k}+\ln\frac{d_I}{R^2}}.$$

In light of inequality (2.19), there exists a constant $C>0$, independent of $q>1$, such that

$$\frac{1}{c}\int\limits_{(0,1)\cap(\kappa-2R,\kappa+2R)}e^{\frac{q}{2d_I}(H(x)-H(\kappa))}\mathrm{d}x<\frac{\int_{(0,1)\cap(\kappa-2R,\kappa+2R)}e^{-\frac{q}{2d_I}C_1(x-\kappa)^{2k}}\mathrm{d}x}{\int_{(0,1)\cap(\kappa-R,\kappa+R)}e^{-\frac{q}{d_I}C_2(x-\kappa)^{2k}}\mathrm{d}x}<C.$$

Here and after, the constant $C>0$ may vary from line to line, but independent of $q>1$. Then one has

$$\mathcal{R}_0\geq\frac{\int_0^1\beta(x)w^2\mathrm{d}x}{Ce^{-\frac{C_1q}{2d_I}R^{2k}+\ln\frac{d_I}{R^2}}+\int_0^1\gamma(x)w^2\mathrm{d}x}\qquad\forall\, q>1.$$

Therefore, we get

$$\liminf_{q\to\infty}\mathcal{R}_0(q) \geq \frac{\min\limits_{x\in[0,1]\cap[\kappa-2R,\kappa+2R]}\beta(x)}{\max\limits_{x\in[0,1]\cap[\kappa-2R,\kappa+2R]}\gamma(x)} \geq \frac{\beta(\kappa)}{\gamma(\kappa)}.$$

The last inequality holds by the arbitrariness of $R > 0$.

Upper bound. $\limsup\limits_{q\to\infty}\mathcal{R}_0(q) \leq \frac{\beta(\kappa)}{\gamma(\kappa)}$. We first select a sequence $\{q_n\}_{n\in\mathbb{N}^+}$ of $q \to \infty$ such that $\lim\limits_{n\to\infty}\mathcal{R}_0(q_n) = \limsup\limits_{q\to\infty}\mathcal{R}_0(q)$. Let $\{(w, \mathcal{R}_0; q_n)\}_{n\in\mathbb{N}^+}$ be a family positive solutions of (2.16) with $\int_0^1 w^2\mathrm{d}x = 1$. Obviously, the set $\{w^2(\cdot; q_n)\}_{n\in\mathbb{N}^+}$ is weakly compact. Hence, there exists a subsequence, still denoted by $\{q_n\}$, such that $\{w^2(\cdot; q_n)\}_{n=1}^{\infty}$ converges weakly to a certain probability measure μ in the sense of

$$\lim_{n\to\infty}\int_0^1 w^2(x; q_n)c(x)\mathrm{d}x = \int_0^1 c(x)\mu(\mathrm{d}x), \qquad \forall\, c(x) \in C([0, 1]). \tag{2.20}$$

Note that

$$\mathcal{R}_0(q_n) = \frac{\int_0^1 \beta(x)w^2\mathrm{d}x}{d_I\int_0^1 \left|w_x - \frac{q_n}{2d_I}wh(x)\right|^2\mathrm{d}x + \int_0^1 \gamma(x)w^2\mathrm{d}x} \leq \frac{\int_0^1 \beta(x)w^2\mathrm{d}x}{\int_0^1 \gamma(x)w^2\mathrm{d}x}.$$

From (2.20), then we only need to show that μ supports only at κ. Denote q_n by q for simplicity of notation.

From (2.17) one sees that

$$d_I\int_0^1 \left|w_x - \frac{q}{2d_I}wh(x)\right|^2\mathrm{d}x \quad \text{uniformly bounded for all } q > 1.$$

Recall the cut-off function $\rho(\cdot)$, which is a smooth function satisfies

$$\rho = 1 \text{ in } (-1, 1), \quad \rho = 0 \text{ in } \mathbb{R}\setminus(-2, 2), \quad 0 \leqslant \rho \leqslant 1, |\rho_x| \leqslant 2 \text{ in } (-2, 2).$$

Fix an arbitrary $z \in (0, 1) \setminus \{\kappa\}$. There exist positive constants δ and R such that

$$|h(x)| > \delta \quad \text{in } (z - 2R, z + 2R) \subset (0, 1).$$

Set $\zeta(x) = \rho(|x - z|/R)$. Then

$$C \geqslant \int_0^1 \zeta^2(x)\left|w_x(x; q) - \frac{q}{2d_I}wh(x)\right|^2\mathrm{d}x$$

$$=\int_0^1 \zeta^2\left\{|w_x|^2+\frac{q^2}{4d_I^2}w^2|h(x)|^2\right\}\mathrm{d}x-\frac{q}{2d_I}\int_0^1 \zeta^2(w^2)_x h(x)\mathrm{d}x$$

$$=\int_0^1 \zeta^2\left\{|w_x|^2+\left[\frac{q^2}{4d_I^2}|h(x)|^2+\frac{q}{2d_I}h'(x)\right]w^2\right\}\mathrm{d}x+\frac{q}{d_I}\int_0^1 w^2\zeta\zeta_x h(x)\mathrm{d}x$$

$$\geqslant\int_0^1 \left\{\frac{q^2}{8d_I^2}|h(x)|^2+\frac{q}{2d_I}h'(x)\right\}w^2\zeta^2\mathrm{d}x-2\int_0^1 |\zeta_x|^2w^2\mathrm{d}x$$

$$\geqslant\int_0^1 \left\{\frac{q^2}{8d_I^2}|h(x)|^2+\frac{q}{2d_I}h'(x)\right\}w^2\zeta^2\mathrm{d}x-\frac{8}{R^2},$$

where we have used integration by parts in the second equation for the second integral, the Cauchy inequality $\frac{q}{d_I}|h(x)\zeta\zeta_x|\leqslant\frac{q^2}{8d_I^2}\zeta^2|h(x)|^2+2|\zeta_x|^2$ in the second inequality, and finally the estimate $|\zeta_x|\leqslant 2/R$ and the identity $\int_0^1 w^2\mathrm{d}x=1$ in the last inequality.

Thus, when $q\geqslant 8d_I\|h'\|_\infty/\delta^2$, we have $\frac{q^2}{8d_I^2}|h(x)|^2+\frac{q}{2d_I}h'(x)\geqslant\frac{q^2\delta^2}{16d_I^2}$ in the support of ζ so that

$$\int_0^1 \zeta^2(x)w^2(x;q)\mathrm{d}x\leqslant\frac{16d_I^2}{q^2\delta^2}\int_0^1\left\{\frac{q^2}{8d_I^2}|h(x)|^2+\frac{q}{2d_I}h'(x)\right\}w^2\zeta^2\mathrm{d}x\leqslant\frac{16d_I^2}{q^2\delta^2}\left(C+\frac{8}{R^2}\right).$$

Letting $q\to\infty$, we then obtain from (2.20) that $\mu((z-R,z+R))\leqslant\lim_{q\to\infty}\int_0^1 w^2(x;q)\zeta^2(x)\mathrm{d}x$ $=0$.

If $z=0$ or 1, which is not a maximum point of $H(x)$, we only need to note that at the boundary, $h(0)>0$ and $h(1)<0$, and then use a similar argument to show that $\mu(\{z\})=0$. As a consequence, we have shown that μ supports only at κ, which implies the upper bound of $\mathcal{R}_0$. This completes the proof. □

Theorem 2.6. *Assume that one of* (H), (H0) *and* (H1) *holds. Then* $\lim_{d_I\to 0}\mathcal{R}_0(d_I)=\frac{\beta(\kappa)}{\gamma(\kappa)}$.

Proof. Without loss of generality, we mainly verify the results under assumption (H). Similarly, we need only to show that

$$\liminf_{d_I\to 0}\mathcal{R}_0(d_I)\geq\frac{\beta(\kappa)}{\gamma(\kappa)}\text{ and }\limsup_{d_I\to 0}\mathcal{R}_0(d_I)\leq\frac{\beta(\kappa)}{\gamma(\kappa)}.$$

Lower bound. $\liminf_{d_I\to 0}\mathcal{R}_0(d_I)\geq\frac{\beta(\kappa)}{\gamma(\kappa)}$. We construct a test function w as

$$w(x)=\frac{1}{\sqrt{c}}\zeta(x),\qquad\text{where }\zeta(x)=\frac{1}{\sqrt{\varepsilon}}\exp\left(\frac{1}{\varepsilon^{2k}}\frac{h^{(2k-1)}(\kappa)}{(2k)!}(x-\kappa)^{2k}\right).$$

Here, the positive integer k is given in (H), $\varepsilon=(2d_I/q)^{\frac{1}{2k}}$ and $c=c(\varepsilon)=\int_0^1\zeta^2\mathrm{d}x$. Moreover, one can check that $c>0$ holds uniformly for all $\varepsilon>0$. By using the Taylor's extension of $h(x)$, one gets

$$(\ln w)_x-\frac{q}{2d_I}h(x)=\frac{1}{\varepsilon^{2k}}O\left(|x-\kappa|^{2k}\right)\qquad \text{as } x\to\kappa.$$

Hence, noting $\mathcal{R}_0(d_I)=\mathcal{R}_0\left(q\varepsilon^{2k}/2\right)$ and setting $y=(x-\kappa)/\varepsilon$, $x\in[0,1]$, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to0}\mathcal{R}_0\left(q\varepsilon^{2k}/2\right)\\
&\quad\ge\liminf_{\varepsilon\to0}\frac{\int_0^1\beta(x)w^2(x)\mathrm{d}x}{d_I\int_0^1|w_x(x)-\frac{q}{2d_I}w(x)h(x)|^2\mathrm{d}x+\int_0^1\gamma(x)w^2(x)\mathrm{d}x}\\
&\quad\ge\liminf_{\varepsilon\to0}\frac{\int_0^1\beta(x)w^2(x)\mathrm{d}x}{d_I\int_0^1|(\ln w(x))_x-\frac{q}{2d_I}h(x)|^2w^2(x)\mathrm{d}x+\int_0^1\gamma(x)w^2(x)\mathrm{d}x}\\
&\quad=\liminf_{\varepsilon\to0}\frac{\int_{-\kappa/\varepsilon}^{(1-\kappa)/\varepsilon}\beta(\kappa+\varepsilon y)w^2(\kappa+\varepsilon y)\mathrm{d}y}{\frac{q}{2}O(\varepsilon)\int_{-\kappa/\varepsilon}^{(1-\kappa)/\varepsilon}y^{4k-2}w^2(\kappa+\varepsilon y)\mathrm{d}y+\int_{-\kappa/\varepsilon}^{(1-\kappa)/\varepsilon}\gamma(\kappa+\varepsilon y)w^2(\kappa+\varepsilon y)\mathrm{d}y}\\
&\quad=\frac{\int_{\mathbb{R}}\beta(\kappa)w^2(\kappa+\varepsilon y)\mathrm{d}y}{\int_{\mathbb{R}}\gamma(\kappa)w^2(\kappa+\varepsilon y)\mathrm{d}y}\\
&\quad=\frac{\beta(\kappa)}{\gamma(\kappa)}.
\end{aligned}$$

We make a remark here to the proof under assumption (H0) or (H1). We only need to note that $[-\kappa/\varepsilon,(1-\kappa)/\varepsilon]$ tends to $[0,\infty)$ if $\kappa=0$ and $(-\infty,0]$ if $\kappa=1$ as $\varepsilon\to0$, respectively. The rest of the discussion is exactly similar to the proof of Theorem 2.5.

Upper bound. $\limsup\limits_{d_I\to0}\mathcal{R}_0(d_I)\le\frac{\beta(\kappa)}{\gamma(\kappa)}$. As in the previous theorem, we can find a sequence $\{d_{I,n}\}_{n\in\mathbb{N}^+}$ of $d_I\to0$ such that $\lim\limits_{n\to0}\mathcal{R}_0(d_{I,n})=\limsup\limits_{d_I\to0}\mathcal{R}_0(d_I)$. Normalize the corresponding functions $w(\cdot;d_{I,n})$ by $\int_0^1w^2(x;d_{I.n})\mathrm{d}x=1$, $n\in\mathbb{N}^+$. Then, by weak compactness of $\{w^2(\cdot;d_{I,n})\}_{n\in\mathbb{N}^+}$, we know that $\{w^2(\cdot;d_{I,n})\}_{n\in\mathbb{N}^+}(\lim\limits_{n\to\infty}d_{I,n}=0)$ converges weakly to a certain probability measure, still represented by μ. Then we can show that $\mu(\cdot)$ supports only at $x=\kappa$. Indeed, reasoning as those in the proof of Theorem 2.5, one can easily verify that $\mu((0,1)\setminus\{\kappa\})=0$. The arguments for cases (H0) and (H(1)) are analogous. This ends the proof. □

Remark 2.7. In Theorems 2.5 and 2.6, we discussed the asymptotic behavior of the basic reproduction number $\mathcal{R}_0$ as $q\to\infty$ and $d_I\to0$. We can also study the asymptotic behavior of λ_1, the principal eigenvalue of (2.6) (also (2.7)), to guarantee the existence of the EE for large q and small d_I. For instance, by [8, Theorem 1.1] or [36, Theorem 1.2], one has $\lim\limits_{q\to\infty}\lambda_1(q)=\gamma(\kappa)-\beta(\kappa)$. Due to Lemma 2.1, one sees that $\mathcal{R}_0(\infty)>1$ if $\lambda_1(\infty)<0$.

3. Priori estimates

In this section and next section, we first consider the case $\kappa \in (0, 1)$, namely, hypothesis (H) holds. Our main purpose in this section is to deduce some useful priori estimates. The process to establish Lemmas 3.1, 3.3 and 3.5 is mainly based on assumption (H). However, Lemma 3.1 is also valid for cases (H0) and (H1), Lemmas 3.4 and 3.6 are the counterparts to Lemmas 3.3 and 3.5 for cases (H0) and (H1). If one of (H0) and (H1) happens, then one of ranges $[0, \kappa]$ and $[\kappa, 1]$ degenerates to a point respectively. For simplicity of notation, denote

$$\beta^* := \max_{x\in[0,1]} \beta(x),\ \gamma^* := \max_{x\in[0,1]} \gamma(x) \text{ and } \gamma_* := \min_{x\in[0,1]} \gamma(x).$$

Lemma 3.1. *Assume that* (H) *holds. Let* $(S(x), I(x))$ *be any* EE *of* (1.6) *and* $C_3 = (\beta^* + \gamma^*)N$. *For any* $q, d_S, d_I > 0$*,*

(i) *we have*

$$S(x) \le \begin{cases} S(\kappa)e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s} + \dfrac{C_3}{d_S}\displaystyle\int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y, & \forall\, x \in [0,\kappa], \\ S(\kappa)e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s} + \dfrac{C_3}{d_S}\displaystyle\int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y, & \forall\, x \in [\kappa, 1] \end{cases} \tag{3.1}$$

with

$$\begin{cases} S(\kappa) \le \dfrac{\frac{C_3}{d_S}\int_0^\kappa\int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_0^\kappa S(x)\mathrm{d}x}{\int_0^\kappa e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x}, \\ S(\kappa) \le \dfrac{\frac{C_3}{d_S}\int_\kappa^1\int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_\kappa^1 S(x)\mathrm{d}x}{\int_\kappa^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x}; \end{cases} \tag{3.2}$$

(ii) *we have*

$$I(x) \le \begin{cases} I(\kappa)e^{-\frac{q}{d_I}\int_x^\kappa h(s)\mathrm{d}s} + \dfrac{C_3}{d_I}\displaystyle\int_x^\kappa e^{-\frac{q}{d_I}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y, & \forall\, x \in [0,\kappa], \\ I(\kappa)e^{\frac{q}{d_I}\int_\kappa^x h(s)\mathrm{d}s} + \dfrac{C_3}{d_I}\displaystyle\int_\kappa^x e^{\frac{q}{d_I}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y, & \forall\, x \in [\kappa, 1] \end{cases} \tag{3.3}$$

with

$$\begin{cases} I(\kappa) \le \dfrac{\frac{C_3}{d_I}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_I}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_0^\kappa I(x)\mathrm{d}x}{\int_0^\kappa e^{-\frac{q}{d_I}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x}, \\ I(\kappa) \le \dfrac{\frac{C_3}{d_I}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_I}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_\kappa^1 I(x)\mathrm{d}x}{\int_\kappa^1 e^{\frac{q}{d_I}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x}. \end{cases} \tag{3.4}$$

Proof. We proceed our proof in several steps for clarity.

Step 1. $\|d_S S_x - qh(x)S\|_\infty \le C_3, \quad \|d_I I_x - qh(x)I\|_\infty \le C_3$. Integrating the S-equation and the I-equation from 0 to x respectively, where $x \in [0, 1]$, we have

$$d_S S_x - qh(x)S = \int_0^x \left(\beta(y)\frac{SI}{S+I} - \gamma(y)I\right)\mathrm{d}y,$$

$$d_I I_x - qh(x)I = -\int_0^x \left(\beta(y)\frac{SI}{S+I} - \gamma(y)I\right)\mathrm{d}y.$$

Since

$$\left|\int_0^x \left(\beta(y)\frac{SI}{S+I} - \gamma(y)I\right)\mathrm{d}y\right| = \left|\int_0^x \left(\beta(y)\frac{S}{S+I} - \gamma(y)\right)I\mathrm{d}y\right| \le (\beta^* + \gamma^*)N,$$

we obtain

$$|d_S S_x - qh(x)S| \le (\beta^* + \gamma^*)N \text{ and } |d_I I_x - qh(x)I| \le (\beta^* + \gamma^*)N, \quad \forall\, x \in [0, 1].$$

The desired result follows.

Step 2. We prove (3.2). Since $-C_3 \le d_S S_x - qh(x)S \le C_3$, we have

$$d_S\left(Se^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}\right)_x \le C_3 e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}, \quad x \in [0, \kappa]$$

and

$$d_S\left(Se^{\frac{q}{d_S}\int_x^1 h(s)\mathrm{d}s}\right)_x \ge -C_3 e^{\frac{q}{d_S}\int_x^1 h(s)\mathrm{d}s}, \quad x \in [\kappa, 1].$$

Integrating the above inequalities from x to κ and from κ to x respectively, one has

$$S(\kappa)e^{-\frac{q}{d_S}\int_0^\kappa h(s)\mathrm{d}s} - S(x)e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s} \le \frac{C_3}{d_S}\int_x^\kappa e^{-\frac{q}{d_S}\int_0^y h(s)\mathrm{d}s}\mathrm{d}y, \quad x \in [0, \kappa]$$

and

$$S(x)e^{\frac{q}{d_S}\int_x^1 h(s)\mathrm{d}s} - S(\kappa)e^{\frac{q}{d_S}\int_\kappa^1 h(s)\mathrm{d}s} \geq -\frac{C_3}{d_S}\int_\kappa^x e^{\frac{q}{d_S}\int_y^1 h(s)\mathrm{d}s}\mathrm{d}y, \quad x \in [\kappa, 1].$$

Or equivalently,

$$S(\kappa)e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s} \leq \frac{C_3}{d_S}\int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y + S(x), \quad x \in [0, \kappa]$$

and

$$S(\kappa)e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s} \leq \frac{C_3}{d_S}\int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y + S(x), \quad x \in [\kappa, 1].$$

Integrating above inequalities on $[0, \kappa]$ and $[\kappa, 1]$ respectively, we obtain

$$S(\kappa) \leq \frac{\frac{C_3}{d_S}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_0^\kappa S(x)\mathrm{d}x}{\int_0^\kappa e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x}$$

and

$$S(\kappa) \leq \frac{\frac{C_3}{d_S}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_\kappa^1 S(x)\mathrm{d}x}{\int_\kappa^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x}.$$

This proves the results.

Step 3. We prove (3.1). As in Step 2, multiplying $-C_3 \leq d_S S_x - qh(x)S \leq C_3$ by $e^{-\frac{q}{d_S}\int_0^x h(s)\mathrm{d}s}$ if $x \leq \kappa$ and by $e^{\frac{q}{d_S}\int_x^1 h(s)\mathrm{d}s}$ if $x \geq \kappa$, and then integrating over (x, κ) and (κ, x) respectively, we are led to

$$S(x) \leq S(\kappa)e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s} + \frac{C_3}{d_S}\int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y, \quad x \in [0, \kappa]$$

and

$$S(x) \leq S(\kappa)e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s} + \frac{C_3}{d_S}\int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y, \quad x \in [\kappa, 1].$$

The inequalities in (3.1) are proved.

Step 4. By using similar arguments in Steps 2 and 3, it is easy to obtain estimates (3.3) and (3.4). The proof is complete. □

Lemma 3.2. *Let* (S, I) *be any* EE *of* (1.6). *Then*

$$\int_0^1 S(x)\mathrm{d}x \geq \frac{\gamma_* N}{\beta^* + \gamma^*} > 0. \tag{3.5}$$

Proof. Integrating (1.6) over (0, 1), we see

$$\int_0^1 \gamma(x)I(x)\mathrm{d}x = \int_0^1 \beta(x)\frac{S(x)I(x)}{S(x)+I(x)}\mathrm{d}x.$$

Adding $\int_0^1 \gamma(x)S(x)\mathrm{d}x$ to both sides we get

$$\int_0^1 \gamma(x)\left(I(x)+S(x)\right)\mathrm{d}x = \int_0^1 \left(\beta(x)\frac{I(x)}{S(x)+I(x)} + \gamma(x)\right)S(x)\mathrm{d}x.$$

Owing to

$$\int_0^1 \gamma(x)\left(I(x)+S(x)\right)\mathrm{d}x \geq \gamma_* \int_0^1 \left(I(x)+S(x)\right)\mathrm{d}x = \gamma_* N$$

and

$$\int_0^1 \left(\beta(x)\frac{I(x)}{S(x)+I(x)} + \gamma(x)\right)S(x)\mathrm{d}x \leq (\beta^* + \gamma^*)\int_0^1 S(x)\mathrm{d}x,$$

we obtain

$$\gamma_* N \leq (\beta^* + \gamma^*)\int_0^1 S(x)\mathrm{d}x,$$

which is equivalent to (3.5). This completes the proof. □

Since $h(x)$ vanishes at κ, we need to carefully understand the property of $h(x)$ near κ. Let $K > 0$ be any given constant. In light of assumption (H) on h, one sees that

$$h\left(\kappa \pm K\Lambda^{-\frac{1}{2k}}\right) \to 0 \quad \text{as } \Lambda \to \infty.$$

Additionally, one further has $h^{(2k-1)}(x) < 0$ in the neighborhood of κ because of $h^{(i)}(\kappa) = 0,\ i = 0, 1, \cdots, 2k-2$, and $h^{(2k-1)}(\kappa) < 0$. Therefore, without loss of generality, we suppose that

$$h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\begin{cases} > h\left(\kappa-K\Lambda^{-\frac{1}{2k}}\right) & \text{if } t\in\left(-\Lambda^{\frac{1}{2k}}\kappa,-K\right),\\ < h\left(\kappa+K\Lambda^{-\frac{1}{2k}}\right) & \text{if } t\in\left(K,\Lambda^{\frac{1}{2k}}(1-\kappa)\right)\end{cases} \tag{3.6}$$

and

$$h\left(\kappa+K\Lambda^{-\frac{1}{2k}}\right)<h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)<h\left(\kappa-K\Lambda^{-\frac{1}{2k}}\right) \quad \text{if } t\in(-K,K) \tag{3.7}$$

hold for all $\Lambda>\max\left\{\frac{K^2}{\kappa^2},\frac{K^2}{(1-\kappa)^2}\right\}$. Note that as $\Lambda\to\infty$, $K\Lambda^{-\frac{1}{2k}}\to 0$. Applying the Taylor's extension to $h(x)$ near κ, one has

$$h\left(\kappa\pm K\Lambda^{-\frac{1}{2k}}\right)=\pm\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}\Lambda^{\frac{1}{2k}-1}+o\left(\Lambda^{\frac{1}{2k}-1}K^{2k-1}\right) \quad \text{as } \Lambda\to\infty.$$

Thereby, we can find a constant $\Lambda_1>0$ such that if $\Lambda>\Lambda_1$, then

$$\begin{cases} \Lambda^{1-\frac{1}{2k}}h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)>\Lambda^{1-\frac{1}{2k}}h\left(\kappa-K\Lambda^{-\frac{1}{2k}}\right)\\ \qquad > -\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}, \qquad \forall\, t\in\left(-\Lambda^{\frac{1}{2k}}\kappa,-K\right),\\ \Lambda^{1-\frac{1}{2k}}h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)<\Lambda^{1-\frac{1}{2k}}h\left(\kappa+K\Lambda^{-\frac{1}{2k}}\right)\\ \qquad < \frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}, \qquad \forall\, t\in\left(K,\Lambda^{\frac{1}{2k}}(1-\kappa)\right)\end{cases} \tag{3.8}$$

and

$$\begin{cases} 0<\Lambda^{1-\frac{1}{2k}}h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)<\Lambda^{1-\frac{1}{2k}}h\left(\kappa-K\Lambda^{-\frac{1}{2k}}\right)<-\frac{3}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}, & \forall\, t\in(-K,0),\\ 0>\Lambda^{1-\frac{1}{2k}}h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)>\Lambda^{1-\frac{1}{2k}}h\left(\kappa+K\Lambda^{-\frac{1}{2k}}\right)>\frac{3}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}, & \forall\, t\in(0,K).\end{cases} \tag{3.9}$$

Lemma 3.3. *Assume that* (H) *holds. Then we have*

$$\lim_{\Lambda\to\infty}\int_0^{\kappa}\Lambda^{\frac{1}{2k}}e^{-\Lambda\int_x^{\kappa}h(s)\mathrm{d}s}\mathrm{d}x=\lim_{\Lambda\to\infty}\int_{\kappa}^{1}\Lambda^{\frac{1}{2k}}e^{\Lambda\int_{\kappa}^{x}h(s)\mathrm{d}s}\mathrm{d}x=\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right), \tag{3.10}$$

where $\Gamma(\cdot)$ *is a Gamma's function, namely,*

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}\mathrm{d}t.$$

Proof. We only need to show that

$$\lim_{\Lambda\to\infty}\int_0^{\kappa}\Lambda^{\frac{1}{2k}}e^{-\Lambda\int_x^{\kappa}h(s)\mathrm{d}s}\mathrm{d}x=\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right),\tag{3.11}$$

since the second one in (3.10) can be proved by using similar arguments. By change of variables $y=\Lambda^{\frac{1}{2k}}(x-\kappa)$, we can rewrite (3.11) as

$$\lim_{\Lambda\to\infty}\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{0}e^{-\Lambda^{1-\frac{1}{2k}}\int_y^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y=\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right).\tag{3.12}$$

Let $\varepsilon>0$ be any small constant. Then there exists a $K_1=K_1(\varepsilon)>1$, such that

$$-\frac{2(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}<\frac{\varepsilon}{3},\qquad \forall\, K>K_1,\tag{3.13}$$

and

$$\left|\int_{-K}^{0}e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y-\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right)\right|<\frac{\varepsilon}{3},\qquad \forall\, K>K_1,\tag{3.14}$$

where (3.14) follows from the fact

$$\lim_{K\to\infty}\int_{-K}^{0}e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y=\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right),\tag{3.15}$$

We first fix such a K satisfying (3.13) and (3.14). For all $\Lambda>(K\kappa^{-1})^{2k}$, we have

$$\begin{aligned}&\left|\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{0}e^{-\Lambda^{1-\frac{1}{2k}}\int_y^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y-\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right)\right|\\ &\le\left|\int_{-K}^{0}e^{-\int_y^0\Lambda^{1-\frac{1}{2k}}h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y-\int_{-K}^{0}e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\right|\\ &\quad+\left|\int_{-K}^{0}e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y-\frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right)\right|\end{aligned}\tag{3.16}$$

$$+\left|\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{-\Lambda^{1-\frac{1}{2k}}\int_y^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y\right|$$
$$=: I_1(\Lambda, K) + I_2(K) + I_3(\Lambda, K).$$

Clearly, $I_2(K) < \varepsilon$ follows from (3.14). We now consider $I_1(K, \Lambda)$. By using Taylor's extension to $h(x)$ near κ, we have

$$h\left(\kappa + y\Lambda^{-\frac{1}{2k}}\right) = \frac{h^{(2k-1)}(\kappa)}{(2k-1)!}y^{2k-1}\Lambda^{\frac{1}{2k}-1} + o\left(\Lambda^{\frac{1}{2k}-1}y^{2k-1}\right), \quad \forall\, y \in [-K, 0], \text{ as } \Lambda \to \infty.$$

This implies that

$$\Lambda^{1-\frac{1}{2k}} h\left(\kappa + y\Lambda^{-\frac{1}{2k}}\right) \to \frac{h^{(2k-1)}(\kappa)}{(2k-1)!}y^{2k-1} \quad \text{uniformly in } [-K, 0] \text{ as } \Lambda \to \infty. \tag{3.17}$$

Hence, in light of (3.9) and (3.17), we can apply the Lebesgue dominant convergent theorem to obtain that, as $\Lambda \to \infty$,

$$\int_{-K}^{0} e^{-\int_y^0 \Lambda^{1-\frac{1}{2k}} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y \to \int_{-K}^{0} e^{-\int_y^0 \frac{h^{(2k-1)}(\kappa)}{(2k-1)!}t^{2k-1}\mathrm{d}t}\mathrm{d}y = \int_{-K}^{0} e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y.$$

Therefore, one can find a $\Lambda_2 = \Lambda_2(\varepsilon, K) > \max\left\{\Lambda_1, (K\kappa^{-1})^{2k}\right\}$ (Λ_1 is a constant defined before (3.8)) such that

$$I_1(\Lambda, K) = \left|\int_{-K}^{0} e^{-\int_y^0 \Lambda^{1-\frac{1}{2k}} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y - \int_{-K}^{0} e^{\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\right| < \frac{\varepsilon}{3}, \quad \forall\, \Lambda > \Lambda_2. \tag{3.18}$$

We then consider $I_3(\Lambda, K)$. Indeed, from (3.8) and (3.13), one sees that for all $\Lambda > \Lambda_2$, it holds

$$\begin{aligned}
I_3(\Lambda, K) &= \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{-\Lambda^{1-\frac{1}{2k}}\int_y^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y \\
&= e^{-\Lambda^{1-\frac{1}{2k}}\int_{-K}^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{-\Lambda^{1-\frac{1}{2k}}\int_y^{-K} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y \\
&\le \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{\int_y^{-K}\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}\mathrm{d}t}\mathrm{d}y \\
&\le -\frac{2(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}} < \frac{\varepsilon}{3}.
\end{aligned} \tag{3.19}$$

Together with (3.16), (3.18), (3.14) and (3.19), we see that

$$\left| \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{0} e^{-\Lambda^{1-\frac{1}{2k}} \int_y^0 h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y - \frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}} \Gamma\left(\frac{1}{2k}\right) \right| < \varepsilon,\ \forall\, K > K_1, \forall\, \Lambda > \Lambda_2.$$

By the arbitrariness of ε, we obtain (3.12), and then (3.11). This completes the proof. $\square$

Next we consider the cases (H0) and (H1). Review the process of analysis for case (H), where we have carefully discussed and analyzed the properties of $h(x)$ on both sides of κ. However, we need only to investigate the properties on the right side neighborhood of $h(x)$ at 0 for case (H0) while on the left side neighborhood of $h(x)$ at 1 for case (H1). Let $K > 0$ be any given constant, then as $\Lambda \to \infty$,

$$h\left(K\Lambda^{-\frac{1}{2k}}\right) \to 0 \text{ for case (H0)}, \quad \text{and } h\left(1 - K\Lambda^{-\frac{1}{2k}}\right) \to 0 \text{ for case (H1)}.$$

As done in case (H),

(i) for case (H0), since $h^{(2k-1)}(0) < 0$ we can assume that for sufficiently large Λ,

$$h\left(t\Lambda^{-\frac{1}{2k}}\right) > h\left(K\Lambda^{-\frac{1}{2k}}\right) \ \ \forall t \in [0, K) \ \text{ and } \ h\left(t\Lambda^{-\frac{1}{2k}}\right) < h\left(K\Lambda^{-\frac{1}{2k}}\right) \ \forall\, t \in \left(K, \Lambda^{\frac{1}{2k}}\right];$$

(ii) for case (H1), since $h^{(2k-1)}(1) < 0$ we can assume that for sufficiently large Λ,

$$h\left(1 - t\Lambda^{-\frac{1}{2k}}\right) < h\left(1 - K\Lambda^{-\frac{1}{2k}}\right) \qquad \forall\, t \in [0, K)$$

and

$$h\left(1 - t\Lambda^{-\frac{1}{2k}}\right) > h\left(1 - K\Lambda^{-\frac{1}{2k}}\right) \qquad \forall\, t \in \left(K, \Lambda^{\frac{1}{2k}}\right].$$

Furthermore, one can check that as $\Lambda \to \infty$,

$$\Lambda^{1-\frac{1}{2k}} h\left(K\Lambda^{-\frac{1}{2k}}\right) \to \frac{h^{(2k-1)}(0)}{(2k-1)!} K^{2k-1} \quad \text{for case (H0)},$$

and

$$\Lambda^{1-\frac{1}{2k}} h\left(1 - K\Lambda^{-\frac{1}{2k}}\right) \to -\frac{h^{(2k-1)}(1)}{(2k-1)!} K^{2k-1} \quad \text{for case (H1)}.$$

Therefore, one can find a constant, still denoted by $\Lambda_1 > 0$, such that if $\Lambda > \Lambda_1$, then

(i) for case (H0),

$$\begin{cases} h\left(t\Lambda^{-\frac{1}{2k}}\right) > h\left(K\Lambda^{-\frac{1}{2k}}\right) > \dfrac{3}{2}\dfrac{h^{(2k-1)}(0)}{(2k-1)!}K^{2k-1}\Lambda^{\frac{1}{2k}-1} & \forall\, t \in [0, K), \\ h\left(t\Lambda^{-\frac{1}{2k}}\right) < h\left(K\Lambda^{-\frac{1}{2k}}\right) < \dfrac{1}{2}\dfrac{h^{(2k-1)}(0)}{(2k-1)!}K^{2k-1}\Lambda^{\frac{1}{2k}-1} & \forall\, t \in \left(K, \Lambda^{\frac{1}{2k}}\right]; \end{cases}$$

(ii) for case (H1),

$$\begin{cases} h\left(1-t\Lambda^{-\frac{1}{2k}}\right) < h\left(1-K\Lambda^{-\frac{1}{2k}}\right) < -\dfrac{3}{2}\dfrac{h^{(2k-1)}(1)}{(2k-1)!}K^{2k-1}\Lambda^{\frac{1}{2k}-1} & \text{if } t \in [0, K), \\ h\left(1-t\Lambda^{-\frac{1}{2k}}\right) > h\left(1-K\Lambda^{-\frac{1}{2k}}\right) > -\dfrac{1}{2}\dfrac{h^{(2k-1)}(1)}{(2k-1)!}K^{2k-1}\Lambda^{\frac{1}{2k}-1} & \text{if } t \in \left(K, \Lambda^{\frac{1}{2k}}\right]. \end{cases}$$

We now can state the results for cases (H0) and (H1) as follows.

Lemma 3.4. *The following assertions hold:*

(i) *Suppose that* (H0) *holds, then*

$$\lim_{\Lambda\to\infty}\int_0^1 \Lambda^{\frac{1}{2k}} e^{\Lambda\int_0^x h(s)\mathrm{d}s}\mathrm{d}x = \frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(0)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right).$$

(ii) *Suppose that* (H1) *holds, then*

$$\lim_{\Lambda\to\infty}\int_0^1 \Lambda^{\frac{1}{2k}} e^{-\Lambda\int_x^1 h(s)\mathrm{d}s}\mathrm{d}x = \frac{1}{2k}\left(-\frac{(2k)!}{h^{(2k-1)}(1)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right).$$

The proof of Lemma 3.4 is similar to Lemma 3.3 and even easier. We now continue to consider the case (H), and deduce some other estimates.

Lemma 3.5. *Assume* (H) *holds. Then one has*

$$\lim_{\Lambda\to\infty}\int_x^\kappa \Lambda^{\frac{1}{2k}} e^{-\Lambda\int_x^z h(s)\mathrm{d}s}\mathrm{d}z = 0 \quad \textit{uniformly on } [0,\kappa] \tag{3.20}$$

and

$$\lim_{\Lambda\to\infty}\int_\kappa^x \Lambda^{\frac{1}{2k}} e^{\Lambda\int_z^x h(s)\mathrm{d}s}\mathrm{d}z = 0 \quad \textit{uniformly on } [\kappa, 1]. \tag{3.21}$$

Proof. We only prove (3.20) since (3.21) can be verified in a similar way with modifications. In fact, since the integral in (3.20) is nonnegative and decreasing in $x \in [0, \kappa]$, we only need to show that

$$\limsup_{\Lambda\to\infty} \int_0^\kappa \Lambda^{\frac{1}{2k}} e^{\Lambda \int_z^x h(s)\mathrm{d}s} \mathrm{d}z = 0.$$

Let $\varepsilon > 0$ be any given small constant. Let $K_2 > 1$ be a constant such that

$$-\frac{2(2k-1)!}{h^{(2k-1)}(\kappa)} \frac{1}{K^{2k-1}} < \frac{\varepsilon}{2}, \qquad \forall\, K > K_2. \tag{3.22}$$

Now we fix such a $K > K_2$ first. Then one can find a $\Lambda_3 = \Lambda_3(K) > (K\kappa^{-1})^{2k}$ such that

$$K \exp\left(\frac{1}{2} \frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1} \left(\Lambda^{\frac{1}{2k}}\kappa - K\right)\right) < \frac{\varepsilon}{2}, \qquad \forall\, \Lambda > \Lambda_3. \tag{3.23}$$

So for all $\Lambda > \Lambda_3$, combining with (3.22) and (3.23), we have

$$\begin{aligned}
&\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{-\Lambda^{1-\frac{1}{2k}} \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{y} h\left(\kappa + t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y \\
&\le \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{y} \frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1}\mathrm{d}t} \mathrm{d}y \\
&= \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1}\left[y + \Lambda^{\frac{1}{2k}}\kappa\right]} \mathrm{d}y \\
&= \frac{2(2k-1)!}{h^{(2k-1)}(\kappa)} \frac{1}{K^{2k-1}} e^{\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1}\Lambda^{\frac{1}{2k}}\kappa} \cdot e^{-\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1} y}\Big|_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} \\
&= -\frac{2(2k-1)!}{h^{(2k-1)}(\kappa)} \frac{1}{K^{2k-1}} \left(1 - e^{\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1}\left(\Lambda^{\frac{1}{2k}}\kappa - K\right)}\right) \\
&\le -\frac{2(2k-1)!}{h^{(2k-1)}(\kappa)} \frac{1}{K^{2k-1}} < \frac{\varepsilon}{2}
\end{aligned} \tag{3.24}$$

and

$$\int_{-K}^{0} e^{-\Lambda^{1-\frac{1}{2k}} \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{y} h\left(\kappa + t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y$$

$$
\begin{aligned}
&= \left(\int_{-K}^{0} e^{-\int_{-K}^{y} \Lambda^{1-\frac{1}{2k}} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y \right) e^{-\Lambda^{1-\frac{1}{2k}} \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \\
&\le K \exp\left(\int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} \frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1}\mathrm{d}t \right) \\
&= K \exp\left(\frac{1}{2}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!} K^{2k-1} \left(\Lambda^{\frac{1}{2k}}\kappa - K\right)\right) < \frac{\varepsilon}{2}.
\end{aligned} \tag{3.25}
$$

By change of variables $z = \kappa + y\Lambda^{-\frac{1}{2k}},\ s = \kappa + t\Lambda^{-\frac{1}{2k}}$, together with (3.24) and (3.25), we get

$$
\begin{aligned}
&\int_0^{\kappa} \Lambda^{\frac{1}{2k}} e^{-\Lambda\int_x^z h(s)\mathrm{d}s}\mathrm{d}z \\
&= \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{0} e^{-\Lambda^{1-\frac{1}{2k}} \int_{\Lambda^{\frac{1}{2k}}(x-\kappa)}^{y} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y \\
&= \int_{-K}^{0} e^{-\Lambda^{1-\frac{1}{2k}} \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{y} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y + \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{-K} e^{-\Lambda^{1-\frac{1}{2k}} \int_{-\Lambda^{\frac{1}{2k}}\kappa}^{y} h\left(\kappa+t\Lambda^{-\frac{1}{2k}}\right)\mathrm{d}t} \mathrm{d}y \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{3.26}
$$

By the arbitrariness of $\varepsilon > 0$, we obtain the desired results. This completes the proof. □

With some modifications to the proof of Lemma 3.5, one can verify following lemma.

Lemma 3.6. *The following limits hold:*

(i) *Assume that* (H0) *holds. Then* $\lim_{\Lambda\to\infty} \int_0^x \Lambda^{\frac{1}{2k}} e^{\Lambda\int_z^x h(s)\mathrm{d}s}\mathrm{d}z = 0$ *uniformly on* $[0, 1]$.

(ii) *Assume that* (H1) *holds. Then* $\lim_{\Lambda\to\infty} \int_x^1 \Lambda^{\frac{1}{2k}} e^{-\Lambda\int_x^z h(s)\mathrm{d}s}\mathrm{d}z = 0$ *uniformly on* $[0, 1]$.

4. Asymptotic behavior of the EE as $q \to \infty$

The purpose of this section is to investigate the asymptotic profile of the EE as $q \to \infty$. We first study the case (H).

Lemma 4.1. *Assume that* (H) *holds. Let* $(S(x;q), I(x;q))$ *be any positive solution of* (1.6). *For fixed* $d_S, d_I > 0$*, it holds that* $(S(\cdot;q), I(\cdot;q)) \to (0,0)$ *locally uniformly in* $[0,\kappa) \cup (\kappa, 1]$ *as* $q \to \infty$.

Proof. We firstly claim that for given $d_S, d_I > 0$, there exists a constant $C_4 > 0$ independent of q such that for sufficiently large q,

$$S(\kappa) < C_4 q^{\frac{1}{2k}} \text{ and } I(\kappa) < C_4 q^{\frac{1}{2k}}. \tag{4.1}$$

We can rewrite inequalities (3.2) and (3.4) in Lemma 3.1 as

$$\begin{cases} S(\kappa) \le \dfrac{\frac{C_3}{d_S}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_0^\kappa S(x)\mathrm{d}x}{q^{\frac{1}{2k}}\int_0^\kappa e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x} q^{\frac{1}{2k}}, \\ S(\kappa) \le \dfrac{\frac{C_3}{d_S}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_\kappa^1 S(x)\mathrm{d}x}{q^{\frac{1}{2k}}\int_\kappa^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x} q^{\frac{1}{2k}} \end{cases}$$

and

$$\begin{cases} I(\kappa) \le \dfrac{\frac{C_3}{d_I}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_I}\int_x^y h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_0^\kappa I(x)\mathrm{d}x}{q^{\frac{1}{2k}}\int_0^\kappa e^{-\frac{q}{d_I}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x} q^{\frac{1}{2k}}, \\ I(\kappa) \le \dfrac{\frac{C_3}{d_I}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_I}\int_y^x h(s)\mathrm{d}s}\mathrm{d}y\mathrm{d}x + \int_\kappa^1 I(x)\mathrm{d}x}{q^{\frac{1}{2k}}\int_\kappa^1 e^{\frac{q}{d_I}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x} q^{\frac{1}{2k}}. \end{cases}$$

Then we need only to show the boundedness of the four fractions in above inequalities. Obviously, one easily sees that the following four inequalities

$$\frac{C_3}{d_S}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}x + \int_0^\kappa S(x)\mathrm{d}x < \infty,$$

$$\frac{C_3}{d_S}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}x + \int_\kappa^1 S(x)\mathrm{d}x < \infty,$$

$$\frac{C_3}{d_I}\int_0^\kappa \int_x^\kappa e^{-\frac{q}{d_S}\int_x^y h(s)\mathrm{d}s}\mathrm{d}x + \int_0^\kappa I(x)\mathrm{d}x < \infty,$$

$$\frac{C_3}{d_I}\int_\kappa^1 \int_\kappa^x e^{\frac{q}{d_S}\int_y^x h(s)\mathrm{d}s}\mathrm{d}x + \int_\kappa^1 I(x)\mathrm{d}x < \infty$$

uniformly hold for all large q. Using Lemma 3.3, one easily yields that for fixed $d_S,\ d_I > 0$,

$$\lim_{q\to\infty} q^{\frac{1}{2k}} \int_0^\kappa e^{-\frac{q}{d_S}\int_x^\kappa h(s)\mathrm{d}s}\mathrm{d}x > 0 \text{ and } \lim_{q\to\infty} q^{\frac{1}{2k}} \int_\kappa^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x > 0$$

with $\Lambda = q/d_S$ in (3.10), and

$$\lim_{q\to\infty} q^{\frac{1}{2k}} \int_0^{\kappa} e^{-\frac{q}{d_I}\int_x^{\kappa} h(s)\mathrm{d}s}\mathrm{d}x > 0 \text{ and } \lim_{q\to\infty} q^{\frac{1}{2k}} \int_{\kappa}^{1} e^{\frac{q}{d_I}\int_{\kappa}^{x} h(s)\mathrm{d}s}\mathrm{d}x > 0$$

with $\Lambda = q/d_I$ in (3.10). Then we can find positive constants $Q > 0$ and $C_4 > 0$ (independent of all large q) such that (4.1) holds for all $q > Q$. For convenience, in the following we assume that $Q = 1$.

Because of $\int_x^{\kappa} h(s)\mathrm{d}s > 0$, $\forall\, x \in [0, \kappa)$ and $\int_{\kappa}^{x} h(s)\mathrm{d}s < 0$, $\forall\, x \in (\kappa, 1]$, we immediately obtain from (3.2) and (4.1) that as $q \to \infty$,

$$S(\kappa)e^{-\frac{q}{d_S}\int_x^{\kappa} h(s)\mathrm{d}s} \le C_4 q^{\frac{1}{2k}} e^{-\frac{q}{d_S}\int_x^{\kappa} h(s)\mathrm{d}s} \to 0 \quad \text{locally uniformly in } [0,\kappa) \tag{4.2}$$

and

$$S(\kappa)e^{\frac{q}{d_S}\int_{\kappa}^{x} h(s)\mathrm{d}s} \le C_4 q^{\frac{1}{2k}} e^{\frac{q}{d_S}\int_{\kappa}^{x} h(s)\mathrm{d}s} \to 0 \quad \text{locally uniformly in } (\kappa, 1]. \tag{4.3}$$

Similarly, from (3.4) and (4.1), one can verify that as $q \to \infty$,

$$I(\kappa)e^{-\frac{q}{d_I}\int_x^{\kappa} h(s)\mathrm{d}s} \le C_4 q^{\frac{1}{2k}} e^{-\frac{q}{d_I}\int_x^{\kappa} h(s)\mathrm{d}s} \to 0 \quad \text{locally uniformly in } [0,\kappa) \tag{4.4}$$

and

$$I(\kappa)e^{\frac{q}{d_I}\int_{\kappa}^{x} h(s)\mathrm{d}s} \le C_4 q^{\frac{1}{2k}} e^{\frac{q}{d_I}\int_{\kappa}^{x} h(s)\mathrm{d}s} \to 0 \quad \text{locally uniformly in } (\kappa, 1]. \tag{4.5}$$

Furthermore, using Lemma 3.5, one can show that for given $x \in [0, 1] \setminus \{\kappa\}$, as $q \to \infty$,

$$\frac{C_3}{d_S}\int_x^{\kappa} e^{-\frac{q}{d_S}\int_x^{y} h(s)\mathrm{d}s}\mathrm{d}y \to 0 \quad \text{if } x \in [0,\kappa),\ \frac{C_3}{d_S}\int_{\kappa}^{x} e^{\frac{q}{d_S}\int_y^{x} h(s)\mathrm{d}s}\mathrm{d}y \to 0 \quad \text{if } x \in (\kappa, 1] \tag{4.6}$$

and

$$\frac{C_3}{d_I}\int_x^{\kappa} e^{-\frac{q}{d_I}\int_x^{y} h(s)\mathrm{d}s}\mathrm{d}y \to 0 \quad \text{if } x \in [0,\kappa),\ \frac{C_3}{d_I}\int_{\kappa}^{x} e^{\frac{q}{d_I}\int_y^{x} h(s)\mathrm{d}s}\mathrm{d}y \to 0 \quad \text{if } x \in (\kappa, 1]. \tag{4.7}$$

Using estimates (3.1)-(3.4), (4.2)-(4.7) and letting $q \to \infty$, we get that as $q \to 0$,

$$S(x;q) \to 0 \text{ and } I(x;q) \to 0 \qquad \forall\, x \in [0,1] \setminus \{\kappa\},$$

which further implies that $(S(\cdot;q), I(\cdot;q)) \to (0,0)$ locally uniformly on $[0,\kappa) \cup (\kappa, 1]$, as $q \to \infty$. The proof is complete. □

For any EE $(S(x;q), I(x;q))$ of (1.6), set $S(\xi_S^q;q) = \max_{x\in[0,1]} S(x;q)$ and $I(\xi_I^q;q) = \max_{x\in[0,1]} I(x)$. Then $\xi_S^q, \xi_I^q \neq 0, 1$ for all $q > 0$ due to boundary conditions. Furthermore, we have the following result.

Lemma 4.2. *Assume that* (H) *holds. Let ξ_S^q, ξ_I^q be defined as above. Then we have*

$$\lim_{q\to\infty} q^{\frac{1}{2k}} |\xi_S^q - \kappa| = 0 \text{ and } \lim_{q\to\infty} q^{\frac{1}{2k}} |\xi_I^q - \kappa| = 0, \tag{4.8}$$

which directly imply that

$$\lim_{q\to\infty} q^{\frac{1}{2k}} |\xi_S^q - \xi_I^q| = 0. \tag{4.9}$$

Proof. Since $\xi_S^q, \xi_I^q \in (0, 1)$, one has $S_x(\xi_S^q;q) = 0$ and $I_x(\xi_I^q;q) = 0$. Integrating the S-equation from 0 to ξ_S^q and the I-equation from 0 to ξ_I^q, respectively, we have

$$\left|qh(\xi_S^q)\right| S(\xi_S^q;q) = \left|\int_0^{\xi_S^q} \left(\frac{\beta(x)S(x;q)}{S(x;q)+I(x;q)} - \gamma(x)\right) I(x;q)\mathrm{d}x\right| \leq \left(\beta^* + \gamma^*\right) N \tag{4.10}$$

and

$$\left|qh\left(\xi_I^q\right)\right| I(\xi_I^q;q) = \left|\int_0^{\xi_I^q} \left(\beta(x)\frac{S(x;q)}{S(x;q)+I(x;q)} - \gamma(x)\right) I(x;q)\mathrm{d}x\right|, \tag{4.11}$$

where $\beta^* = \max_{x\in[0,1]} \beta(x)$ and $\gamma^* = \max_{x\in[0,1]} \gamma(x)$. In view of Lemma 3.2, arguing by a way of contradiction, one can get that $\liminf_{q\to\infty} S(\xi_S^q;q) > 0$. Together with (4.10) and (4.11), one then has

$$\left|qh(\xi_S^q)\right| \leq \frac{(\beta^* + \gamma^*) N}{S(\xi_S^q;q)} < \infty$$

and

$$\left|qh(\xi_I^q)\right| \leq \int_0^{\xi_I^q} \left|\beta(x)\frac{S(x;q)}{S(x;q)+I(x;q)} - \gamma(x)\right| \frac{I(x;q)}{I(\xi_I^q;q)}\mathrm{d}x \leq \beta^* + \gamma^* < \infty$$

uniformly for all $q > 0$. As a result, one easily gets

$$\lim_{q\to\infty} \left|q^{1-\frac{1}{2k}} h(\xi_S^q)\right| = 0 \text{ and } \lim_{q\to\infty} \left|q^{1-\frac{1}{2k}} h(\xi_I^q)\right| = 0. \tag{4.12}$$

In view of assumption (H), we obtain that $\lim_{q\to\infty} \xi_S^q = \kappa = \lim_{q\to\infty} \xi_I^q$, which further implies that

$$\lim_{q\to\infty} \frac{h(\xi_S^q) - h(\kappa)}{(\xi_S^q - \kappa)^{(2k-1)}} = \frac{h^{(2k-1)}(\kappa)}{(2k-1)!} = \lim_{q\to\infty} \frac{h(\xi_I^q) - h(\kappa)}{(\xi_I^q - \kappa)^{(2k-1)}}. \tag{4.13}$$

By virtue of (4.12) and (4.13), it is readily to check that as $q \to \infty$,

$$\left(q^{\frac{1}{2k}} \left|\xi_S^q - \kappa\right|\right)^{2k-1} \left|\frac{h(\xi_S^q) - h(\kappa)}{(\xi_S^q - \kappa)^{(2k-1)}}\right| = q^{1-\frac{1}{2k}} \left|h(\xi_S^q) - h(\kappa)\right| \to 0$$

and

$$\left(q^{\frac{1}{2k}} \left|\xi_I^q - \kappa\right|\right)^{2k-1} \left|\frac{h(\xi_I^q) - h(\kappa)}{(\xi_I^q - \kappa)^{(2k-1)}}\right| = q^{1-\frac{1}{2k}} \left|h(\xi_I^q) - h(\kappa)\right| \to 0.$$

One then immediately obtains (4.8) since $h^{(2k-1)}(\kappa) < 0$. This completes the proof. $\square$

For any EE $(S(x;q), I(x;q))$ of (1.6), we introduce the following transformations:

$$u(y;q) = q^{-\frac{1}{2k}} S\left(\xi_S^q + yq^{-\frac{1}{2k}}; q\right) \text{ and } v(y;q) = q^{-\frac{1}{2k}} I\left(\xi_S^q + yq^{-\frac{1}{2k}}; q\right) \tag{4.14}$$

for all $y \in \left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right]$. Corresponding to (1.6)-(1.7), $(u(y;q), v(y;q))$ satisfies the following differential equations over $\left(-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right)$,

$$\begin{cases} d_S u_{yy} - q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) u_y - q^{1-\frac{1}{k}} h'\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) u & \\ \qquad - q^{-\frac{1}{k}} \left(\dfrac{\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) u}{u+v} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right) v = 0, & y \in \left(-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right), \\ d_I v_{yy} - q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) v_y - q^{1-\frac{1}{k}} h'\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) v & \\ \qquad + q^{-\frac{1}{k}} \left(\dfrac{\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) u}{u+v} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right) v = 0, & y \in \left(-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right), \\ d_S u_y - q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) u = 0, & y = -q^{\frac{1}{2k}}\xi_S^q,\ q^{\frac{1}{2k}}(1-\xi_S^q), \\ d_I v_y - q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) v = 0, & y = -q^{\frac{1}{2k}}\xi_S^q,\ q^{\frac{1}{2k}}(1-\xi_S^q) \end{cases} \tag{4.15}$$

together with

$$\int_{-q^{\frac{1}{2k}}\xi_S^q}^{q^{\frac{1}{2k}}(1-\xi_S^q)} [u(y;q)+v(y;q)]\,\mathrm{d}y = N, \quad \forall\, q>0. \tag{4.16}$$

Theorem 2.4 establishes the existence of the EE under condition that $\mathcal{R}_0 > 1$. We next state the limiting behavior of the EE (if it exists) as the convection rate q approaches infinity.

Theorem 4.3. *Assume that* (H) *and* $\beta(\kappa) > \gamma(\kappa)$ *hold. Let* $d_S, d_I > 0$ *be fixed,* $(S(x;q), I(x;q))$ *be any* EE *of* (1.6). *Then the following assertions hold:*

(i) *As* $q \to \infty$, $(S(x;q), I(x;q)) \to (0,0)$ *locally uniformly in* $[0,\kappa)\cup(\kappa,1]$.
(ii) *Let* $\xi_S^q, \xi_I^q \in [0,1]$ *be points such that* $S\left(\xi_S^q;q\right) = \max_{x\in[0,1]} S(x;q)$ *and* $I(\xi_I^q;q) = \max_{x\in[0,1]} I(x;q)$, *respectively. Then we have*

$$\lim_{q\to\infty} q^{\frac{1}{2k}}|\xi_S^q - \kappa| = 0 \text{ and } \lim_{q\to\infty} q^{\frac{1}{2k}}|\xi_I^q - \kappa| = 0.$$

(iii) *Around* $x=\kappa$, *set* $x = \kappa + yq^{-\frac{1}{2k}}$, *then*

$$\lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}} S(\kappa+yq^{-\frac{1}{2k}};q) = C_S e^{\frac{h^{(2k-1)}(\kappa)}{d_S(2k)!}y^{2k}}, \quad \lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}} I(\kappa+yq^{-\frac{1}{2k}};q) = C_I e^{\frac{h^{(2k-1)}(\kappa)}{d_I(2k)!}y^{2k}},$$

holds uniformly in any compact subset of $\mathbb{R}$, *where* (C_S, C_I) *is the unique positive solution of*

$$\begin{cases} C_S d_S^{\frac{1}{2k}} + C_I d_I^{\frac{1}{2k}} = kN\left(-\dfrac{h^{(2k-1)}(\kappa)}{(2k)!}\right)^{\frac{1}{2k}}\Gamma^{-1}\left(\dfrac{1}{2k}\right), \\ \displaystyle\int_0^1 \frac{1}{(\ln z^{-1})^{1-\frac{1}{2k}}\left(1+\frac{C_I}{C_S}z^{1-\frac{d_I}{d_S}}\right)}\mathrm{d}z = \frac{\gamma(\kappa)}{\beta(\kappa)}\Gamma\left(\frac{1}{2k}\right). \end{cases} \tag{4.17}$$

Furthermore, the ratio $\frac{C_I}{C_S}$ *is increasing in* $\frac{\beta(\kappa)}{\gamma(\kappa)}$ *and* $\frac{d_I}{d_S}$ *respectively.*

Proof. The assertions (i) follows from Lemma 4.1 and (ii) follows from Lemma 4.2. We next prove the third assertion by the following several steps.

Step 1. Convergence of $\{(u(y;q), v(y;q))\}$. It follows from Lemma 3.1 and (4.1) that when q is sufficiently large,

$$u(y) \leq \begin{cases} C_4 e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\ \quad + \dfrac{C_3}{qd_S}\displaystyle\int_y^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \quad \forall\, y\in\left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)\right], \\ C_4 e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_{q^{\frac{1}{2k}}(\kappa-\xi_S^q)}^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\ \quad + \dfrac{C_3}{qd_S}\displaystyle\int_{q^{\frac{1}{2k}}(\kappa-\xi_S^q)}^y e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_z^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \quad \forall\, y\in\left[q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right), q^{\frac{1}{2k}}(1-\xi_S^q)\right] \end{cases} \tag{4.18}$$

and

$$v(y) \leq \begin{cases} C_4 e^{-\frac{q^{1-\frac{1}{2k}}}{d_I}\int_y^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\ \quad + \dfrac{C_3}{qd_I}\displaystyle\int_y^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_I}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \quad \forall\, y\in\left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)\right], \\ C_4 e^{\frac{q^{1-\frac{1}{2k}}}{d_I}\int_{q^{\frac{1}{2k}}(\kappa-\xi_S^q)}^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\ \quad + \dfrac{C_3}{qd_I}\displaystyle\int_{q^{\frac{1}{2k}}(\kappa-\xi_S^q)}^y e^{\frac{q^{1-\frac{1}{2k}}}{d_I}\int_z^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \quad \forall\, y\in\left[q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right), q^{\frac{1}{2k}}(1-\xi_S^q)\right]. \end{cases} \tag{4.19}$$

Therefore, for any fixed $K>0$ with $[-K,K]\subset\left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right]$,

$$\{(u(y;q), v(y;q))\} \text{ is uniformly bounded on } [-K,K]$$

with respect to all large $q>1$. Furthermore, one can easily check that the coefficients in (4.15) are all uniformly bounded ($w.r.t.\ q>1$) in $[-K,K]$, especially,

$$q^{1-\frac{1}{2k}}h\left(\xi_S^q+yq^{-\frac{1}{2k}}\right) \text{ and } q^{1-\frac{1}{k}}h'\left(\xi_S^q+yq^{-\frac{1}{2k}}\right) \quad \text{are uniformly bounded on } [-K,K].$$

Consequently, for given constant $K>0$, by standard L^p-estimates for elliptic equations and embedding theorem, we have

$$\|u(\cdot;q)\|_{C^{1,\alpha}([-K,K])}\leq C \text{ and } \|v(\cdot;q)\|_{C^{1,\alpha}([-K,K])}\leq C \tag{4.20}$$

for some $\alpha \in (0, 1)$ and positive constant $C = C(K)$ which is independent of all large q. Now we can apply Ascoli-Arzelà theorem, along with a diagonal argument to find a positive sequence $\{q_n\}$ going to infinity as $n \to \infty$ and a pair of nonnegative functions $(u^*(y), v^*(y))$ satisfying

$$\lim_{n\to\infty} (u(y; q_n), v(y; q_n)) = (u^*(y), v^*(y)) \quad \text{in } C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}). \tag{4.21}$$

Since $(u(y; q_n), v(y; q_n))$ satisfies equation (4.15) with $q = q_n$, sending $n \to \infty$, then (u^*, v^*) satisfies the following limiting system (in the weak sense and thus in the classical sense)

$$\begin{cases} d_S u^*_{yy} - \dfrac{h^{(2k-1)}(\kappa)}{(2k-1)!} y^{2k-1} u^*_y - \dfrac{h^{(2k-1)}(\kappa)}{(2k-2)!} y^{2k-2} u^* = 0, & y \in \mathbb{R}, \\ d_I v^*_{yy} - \dfrac{h^{(2k-1)}(\kappa)}{(2k-1)!} y^{2k-1} v^*_y - \dfrac{h^{(2k-1)}(\kappa)}{(2k-2)!} y^{2k-2} v^* = 0, & y \in \mathbb{R} \end{cases} \tag{4.22}$$

with

$$u^*_y(0) = 0 \text{ and } v^*_y(0) = 0. \tag{4.23}$$

We give a brief explanation to equalities in (4.23), the first equality $u^*_y(0) = 0$ follows from that $u_y(y; q) = q^{-\frac{1}{k}} S_x \left(\xi^q_S + y q^{-\frac{1}{2k}}; q\right) = 0$ when $y = 0$. The second equality means that the maximum point of $v(y; q_n)$, denoted by y_{q_n}, converges to 0 (which is the maximum point of $u(y; q_n)$). By the definition of ξ^q_I, we have $0 = I_x\left(\xi^q_I + y q^{-\frac{1}{2k}}\right) = I_x\left(\xi^q_S + \frac{q^{\frac{1}{2k}}(\xi^q_I - \xi^q_S) + y}{q^{\frac{1}{2k}}}\right)$ at $y = 0$. In view of (4.9), as $n \to \infty$, we have

$$\frac{1}{q_n} I_x\left(\xi^{q_n}_I + \frac{y + q_n^{\frac{1}{2k}}(\xi^{q_n}_S - \xi^{q_n}_I)}{q_n^{\frac{1}{2k}}}; q_n\right) = v_y\left(q_n^{\frac{1}{2k}}\left(\xi^{q_n}_I - \xi^{q_n}_S\right) + y; q_n\right) \to v^*_y(y),$$
$$y \in [-K, K].$$

Hence, combining with above arguments, one gets $v^*_y(0) = 0$.

Solving (4.22)-(4.23), we have

$$u^*(y) = C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}} \text{ and } v^*(y) = C_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}}$$

for some nonnegative constants C_S, C_I. So we further have from (4.21) that

$$\lim_{n\to\infty} (u(y; q_n), v(y; q_n)) = \left(C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}}, C_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}}\right) \quad \text{in } C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R}). \tag{4.24}$$

It should be noted that C_S and C_I are bounded by a constant which is independent of the sequence $\{q_n\}$, due to (4.20).

Step 2. $C_S > 0$. We denote q_n by q for simplicity of notation here and next steps. Firstly, by employing the change of variable $x = \xi^q_S + y q^{-\frac{1}{2k}}$, we obtain from Lemma 3.2 that

$$\int_{-q^{\frac{1}{2k}}\xi_S^q}^{q^{\frac{1}{2k}}(1-\xi_S^q)} u(y;q)\mathrm{d}y \geq \frac{N\gamma_*}{\beta^*+\gamma^*} > 0. \tag{4.25}$$

We will claim that

$$\lim_{n\to\infty} \int_{-q^{\frac{1}{2k}}\xi_S^q}^{q^{\frac{1}{2k}}(1-\xi_S^q)} u(y;q)\mathrm{d}y = \int_{-\infty}^{\infty} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y. \tag{4.26}$$

It should be noted that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y =& 2\int_{0}^{\infty} e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y \\ =& \frac{1}{k}\left(-d_S\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\int_0^\infty z^{\frac{1}{2k}-1}e^{-z}\mathrm{d}z \\ =& \frac{1}{k}\left(-d_S\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right).\end{aligned}$$

Then, once identity (4.26) is true, by virtue of (4.25), it naturally holds that

$$C_S\frac{1}{k}\left(-d_S\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right) \geq \frac{\gamma_* N}{\beta^*+\gamma^*},$$

that is,

$$C_S \geq \frac{\gamma_* N}{\beta^*+\gamma^*}k\left(-d_S\frac{(2k)!}{h^{(2k-1)}(\kappa)}\right)^{-\frac{1}{2k}}\Gamma^{-1}\left(\frac{1}{2k}\right) > 0.$$

We next verify (4.26). Let $\varepsilon > 0$ be any given small constant. Thanks to (3.15), there is a constant $K_3 = K_3(\varepsilon) > 1$ such that for any $K > K_3$, there hold

$$\int_{-\infty}^{-K} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y < \frac{\varepsilon}{5} \text{ and } \int_{K}^{\infty} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y < \frac{\varepsilon}{5} \tag{4.27}$$

and

$$\left(-2d_S C_4\frac{(2k-1)!}{h^{(2k-1)}(\kappa)} + \frac{2C_3}{d_S} - 2C_3\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\right)\frac{1}{K^{2k-1}} < \frac{\varepsilon}{5}, \tag{4.28}$$

where C_3 and C_4 are the constants defined in Lemma 3.1 and (4.1). We have the following fact.

Claim. For each $K > K_3$, there exist constants $Q_1, Q_2 > 1$ such that

$$\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} u(y;q)\mathrm{d}y < \frac{\varepsilon}{5} \text{ and } \int_{K}^{q^{\frac{1}{2k}}(1-\xi_S^q)} u(y;q)\mathrm{d}y < \frac{\varepsilon}{5}, \quad \forall\, q > \max\left\{Q_1, Q_2, K^{\frac{4k^2}{2k-1}}\right\}. \tag{4.29}$$

Indeed, for any y in $\mathbb{R}$, one has

$$q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) = \left[q^{\frac{1}{2k}}(\xi_S^q - \kappa) + y\right]^{2k-1} \frac{h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) - h(\kappa)}{\left(\xi_S^q + yq^{-\frac{1}{2k}} - \kappa\right)^{2k-1}}.$$

This, together with (4.8) $\left(\text{that is, } \lim\limits_{q\to\infty} q^{\frac{1}{2k}}(\xi_S^q - \kappa) = 0\right)$ and the fact

$$\frac{h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) - h(\kappa)}{\left(\xi_S^q + yq^{-\frac{1}{2k}} - \kappa\right)^{2k-1}} \to \frac{1}{(2k-1)!} h^{(2k-1)}(\kappa) \quad \text{locally uniformly in } y \in \mathbb{R},$$

implies that

$$q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) \to \frac{h^{(2k-1)}(\kappa)}{(2k-1)!} y^{2k-1} \quad \text{locally uniformly in } \mathbb{R}, \text{ as } q \to \infty. \tag{4.30}$$

In view of the fact (4.8) and the assumptions on h, reasoning as (3.8), we can find a constant $Q_1 = Q_1(K) > 1$ such that, for all $q > Q_1$, there hold $\left|q^{\frac{1}{2k}}(\xi_S^q - \kappa)\right| < K$ and

$$-q^{1-\frac{1}{2k}} h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right) < \frac{h^{(2k-1)}(\kappa)}{(2k-1)!} \frac{K^{2k-1}}{2}, \quad \forall\, t \in \left(-q^{\frac{1}{2k}}\kappa, -K\right].$$

Thereby, for any $y < -K$ and $q > Q_1$,

$$\begin{aligned}
& e^{-\frac{1}{d_S}\int_y^{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)} q^{1-\frac{1}{2k}} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\
&= e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_{-K}^{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t - \frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{-K} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\
&\le e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{-K} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \\
&= e^{\frac{1}{d_S}\int_y^{-K}\left[-q^{1-\frac{1}{2k}} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\right]\mathrm{d}t} \\
&\le e^{\frac{1}{d_S}\int_y^{-K}\left(\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}\frac{K^{2k-1}}{2}\right)\mathrm{d}t}
\end{aligned} \tag{4.31}$$

$$= e^{-\frac{1}{2d_S}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}(K+y)}.$$

The second inequality follows from the fact $h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right) > 0$ for $t \in \left(-K, q^{\frac{1}{2k}}\left(\kappa - \xi_S^q\right)\right)$. Arguing by a similar way, one can find a constant $Q_2 > 1$, such that

$$e^{\frac{1}{d_S}\int_{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)}^{y} q^{1-\frac{1}{2k}} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \le e^{-\frac{1}{2d_S}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}(K-y)}, \qquad \forall\, y > K \text{ and } q > Q_2. \tag{4.32}$$

On the other hand, one should note that for all $q > \max\{Q_1, Q_2\}$,

$$\begin{aligned} &\int_{-K}^{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z \le 2K, \quad y \in \left(-q^{\frac{1}{2k}}\xi_S^q, -K\right), \qquad \text{and} \\ &\int_{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)}^{K} e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_z^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z \le 2K, \qquad y \in \left(K, q^{\frac{1}{2k}}(1-\xi_S^q)\right). \end{aligned} \tag{4.33}$$

Arguing as those in (3.19) and (3.24), we have

$$\int_{y}^{-K} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z \le -2d_S\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}, \quad \text{if } y < -K \tag{4.34}$$

and

$$\int_{K}^{y} e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_z^y h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z \le -2d_S\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}, \quad \text{if } y > K. \tag{4.35}$$

We now verify (4.29). Indeed, for all $q > \max\{Q_1, Q_2, K^{\frac{4k^2}{2k-1}}\}$, from (4.18) we have

$$\begin{aligned} &\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} u(y;q)\mathrm{d}y \\ \le{}& C_4 \int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&+\frac{C_3}{qd_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K}\int_{y}^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z\mathrm{d}y \\
=&C_4\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{q^{\frac{1}{2k}}\left(\kappa-\xi_S^q\right)} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}y \\
&+\frac{C_3}{qd_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K}\int_{-K}^{q^{\frac{1}{2k}}(\kappa-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z\mathrm{d}y \\
&+\frac{C_3}{qd_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K}\int_{y}^{-K} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z\mathrm{d}y \\
\le&C_4\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} e^{-\frac{1}{2d_S}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}(K+y)}\mathrm{d}y+\frac{C_3}{qd_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} 2K\mathrm{d}y \\
&+\frac{C_3}{qd_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K}\left(-2d_S\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}\right)\mathrm{d}y \\
\le&-2d_SC_4\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}+\frac{C_3\xi_S^q}{d_S}\frac{2K}{q^{1-\frac{1}{2k}}}+\frac{C_3\xi_S^q}{d_Sq^{1-\frac{1}{2k}}}\left(-2d_S\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\frac{1}{K^{2k-1}}\right) \\
\le&\left(-2d_SC_4\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}+\frac{2C_3\xi_S^q}{d_S}\frac{K^{2k}}{q^{1-\frac{1}{2k}}}-2\frac{C_3\xi_S^q}{q^{1-\frac{1}{2k}}}\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\right)\frac{1}{K^{2k-1}}<\frac{\varepsilon}{5}.
\end{aligned}
$$

The second inequality follows from (4.31), (4.33) and (4.34), and the last one follows from (4.28) and $q > \max\{Q_1, Q_2, K^{\frac{4k^2}{2k-1}}\}$. Similarly, one can check the second one in (4.29) with some modifications. Thus we have shown (4.29) as claimed.

We now complete the proof of (4.26). By (4.24), we can find a $Q_3 > 1$ such that

$$
\left|\int_{-K}^{K}\left(u(y;q)-C_Se^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\right)\mathrm{d}y\right|<\frac{\varepsilon}{5},\qquad \forall\, q>Q_3. \tag{4.36}
$$

Then for any $q > \max\{Q_1, Q_2, Q_3, K^{\frac{4k^2}{2k-1}}\}$, by estimates (4.27), (4.29) and (4.36), we have

$$
\begin{aligned}
&\left| \int_{-q^{\frac{1}{2k}}\xi_S^q}^{q^{\frac{1}{2k}}(1-\xi_S^q)} u(y;q)\mathrm{d}y - \int_{-\infty}^{\infty} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y \right| \\
&\le \left| \int_{-K}^{K} \left(u(y;q) - C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}} \right)\mathrm{d}y \right| + \int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} u(y;q)\mathrm{d}y \\
&\quad + \int_{K}^{q^{\frac{1}{2k}}(1-\xi_S^q)} u(y;q)\mathrm{d}y + \int_{-\infty}^{-K} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y + \int_{K}^{\infty} C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y \\
&\le \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.
\end{aligned}
$$

This completes the proof of (4.26).

Step 3. $C_I > 0$. Suppose by contradiction that $C_I = 0$. Then it follows from (4.24) that

$$\lim_{q\to\infty} v(y;q) = 0 \quad \text{in } C^1_{loc}(\mathbb{R}).$$

Review the definition of ξ_I^q, that is, $I(\xi_I^q;q) = \max_{x\in[0,1]} I(x;q)$ for all $q > 0$. By scaling transformation $v(y;q) = q^{-\frac{1}{2k}} I\left(\xi_S^q + yq^{-\frac{1}{2k}};q\right)$, we see that $v(y;q)$ attains its global maximum on $\left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right]$ at $y^q := q^{\frac{1}{2k}}(\xi_I^q - \xi_S^q)$. We have shown in (4.9) that $y^q \to 0$ as $q \to \infty$. Denote

$$\tilde{v}(y;q) = \frac{v(y;q)}{v(y^q;q)}.$$

We immediately see that $\tilde{v}(y;q) \le 1$ on $\left[-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)\right]$ and $\tilde{v}(y^q;q) = 1$ for all $q > 0$. By the similar arguments done in Step 1, there exists a nonnegative function $\tilde{v}^*$ such that

$$\lim_{q\to\infty} \tilde{v}(y;q) = \tilde{v}^*(y) \quad \text{in } C^1_{loc}(\mathbb{R}), \tag{4.37}$$

up to a extra subsequence of $q \to \infty$, and still denoted by q. Dividing (4.15) by $v(y^q;q)$, one gets

$$
\begin{cases}
d_I\tilde{v}_{yy} - q^{1-\frac{1}{2k}}h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\tilde{v}_y - q^{1-\frac{1}{k}}h'\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\tilde{v} & \\
\quad + \dfrac{1}{q}\left(\dfrac{\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)u}{u+v} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right) \right)\tilde{v} = 0, & y \in \left(-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}\left(1-\xi_S^q\right)\right), \\
d_I\tilde{v}_y - q^{1-\frac{1}{2k}}h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\tilde{v} = 0, & y = -q^{\frac{1}{2k}}\xi_S^q,\ q^{\frac{1}{2k}}\left(1-\xi_S^q\right), \\
\tilde{v}(y^q;q) = 1, \quad \tilde{v}_y(y^q;q) = 0.
\end{cases}
\tag{4.38}
$$

By letting $q \to \infty$, reasoning as those in Step 1, we obtain the following limiting problem

$$\begin{cases} d_I \tilde{v}^*_{yy} - \dfrac{h^{(2k-1)}(\kappa)}{(2k-1)!} y^{2k-1} \tilde{v}^*_y - \dfrac{h^{(2k-1)}(\kappa)}{(2k-2)!} y^{2k-2} \tilde{v}^* = 0, & y \in \mathbb{R}, \\ \tilde{v}^*_y(0) = 0, \quad \tilde{v}^*(0) = 1. \end{cases} \tag{4.39}$$

Solving problem (4.39), one obtains $\tilde{v}^*(y) = e^{\frac{1}{d_I} \frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}}$ for all $y \in \mathbb{R}$. In view of (4.37), one further yields

$$\lim_{q \to \infty} \tilde{v}(y;q) = e^{\frac{1}{d_I} \frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}} \qquad \text{in } C^1_{loc}(\mathbb{R}).$$

Integrating (4.38) over $\left(-q^{\frac{1}{2k}} \xi^q_S, q^{\frac{1}{2k}} \left(1 - \xi^q_S\right)\right)$, one has

$$\int_{-q^{\frac{1}{2k}} \xi^q_S}^{q^{\frac{1}{2k}}(1-\xi^q_S)} \left(\frac{\beta\left(\xi^q_S + y q^{-\frac{1}{2k}}\right) u(y;q)}{u(y;q) + v(y;q)} - \gamma\left(\xi^q_S + y q^{-\frac{1}{2k}}\right)\right) \tilde{v}(y;q) \mathrm{d}y = 0. \tag{4.40}$$

On the other hand, we will claim that

$$\begin{aligned} &\lim_{q \to \infty} \int_{-q^{\frac{1}{2k}} \xi^q_S}^{q^{\frac{1}{2k}}(1-\xi^q_S)} \left[\beta\left(\xi^q_S + y q^{-\frac{1}{2k}}\right) \frac{u(y;q)}{u(y;q) + v(y;q)} - \gamma\left(\xi^q_S + y q^{-\frac{1}{2k}}\right)\right] \tilde{v}(y;q) \mathrm{d}y \\ &= \int_{-\infty}^{\infty} (\beta(\kappa) - \gamma(\kappa)) e^{\frac{1}{d_I} \frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}} \mathrm{d}y. \end{aligned} \tag{4.41}$$

Once (4.41) is true, together with (4.40), we see that

$$\int_{-\infty}^{\infty} (\beta(\kappa) - \gamma(\kappa)) e^{\frac{1}{d_I} \frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}} \mathrm{d}y = 0,$$

namely, $\beta(\kappa) - \gamma(\kappa) = 0$. This contradicts our assumption $\beta(\kappa) > \gamma(\kappa)$. Thus, we have $C_I > 0$.

So the remain work is to show (4.41). Let $\varepsilon > 0$ be any given small constant. Similarly, we can find a constant $K_4 = K_4(\varepsilon) > 1$ such that, for all $K > K_4$, there hold

$$(\beta(\kappa) - \gamma(\kappa)) \left[\int_{(-\infty,-K] \cup [K,\infty)} e^{\frac{1}{d_I} \frac{h^{(2k-1)}(\kappa)}{(2k)!} y^{2k}} \mathrm{d}y \right] < \frac{\varepsilon}{3} \tag{4.42}$$

and

$$2(\beta^* + \gamma^*)\left[-\frac{4d_I(2k-1)!}{K^{2k-1}h^{(2k-1)}(\kappa)}\right] < \frac{\varepsilon}{3}. \tag{4.43}$$

Define $x_{+K,q} := \xi_S^q - Kq^{-\frac{1}{2k}}$ and $x_{-K,q} := \xi_S^q + Kq^{-\frac{1}{2k}}$. Reasoning as in (3.8), we may assume that there exists a $Q_4 > 1$ such that, for all $q > Q_4$,

$$\left|q^{1-\frac{1}{2k}} h\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right| > -\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}\frac{K^{2k-1}}{2}, \ \forall\, y \in \left[-q^{\frac{1}{2k}}\xi_S^q, -K\right] \cup \left[K, q^{\frac{1}{2k}}(1-\xi_S^q)\right], \tag{4.44}$$

or equivalently,

$$\begin{aligned} &q^{1-\frac{1}{2k}} h(x) > -\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}\frac{K^{2k-1}}{2} &&\text{if } x \in [0, x_{+K,q}], \quad \text{and} \\ &q^{1-\frac{1}{2k}} h(x) < \frac{h^{(2k-1)}(\kappa)}{(2k-1)!}\frac{K^{2k-1}}{2} &&\text{if } x \in [x_{-K,q}, 1]. \end{aligned} \tag{4.45}$$

Define $\tilde{I}(x;q) := \frac{I(x;q)}{I(\xi_I^q;q)}$. Recall that $\beta^* = \max\limits_{x\in[0,1]} \beta(x)$ and $\gamma_* = \min\limits_{x\in[0,1]} \gamma(x)$. We introduce an auxiliary function

$$w(x;q) = \begin{cases} \tilde{I}(x;q)e^{\frac{q}{d_I}\int_x^\kappa h(s)\left(1-\frac{2d_I\beta^*}{q^2h^2(s)}\right)\mathrm{d}s} & \text{if } x \in [0,\kappa), \\ \tilde{I}(x;q)e^{-\frac{q}{d_I}\int_\kappa^x h(s)\left(1-\frac{2d_I\beta^*}{q^2h^2(s)}\right)\mathrm{d}s} & \text{if } x \in (\kappa, 1]. \end{cases} \tag{4.46}$$

Then, by some simple calculations, one has

$$\begin{cases} 0 = d_I w_{xx} + \left(qh(x) - \dfrac{4d_I\beta^*}{qh(x)}\right)w_x + \\ \left[-2\beta^*\left(1 - \dfrac{2d_I\beta^*}{q^2h^2(x)}\right) + \dfrac{2d_I\beta^* h'(x)}{qh^2(x)} + \left(\dfrac{\beta(x)S(x;q)}{S(x;q)+I(x;q)} - \gamma(x)\right)\right]w, & x \in (0,1)\setminus\{\kappa\}, \\ d_I w_x = \dfrac{2d_I\beta^*}{qh(x)}w, & x = 0, 1. \end{cases} \tag{4.47}$$

We now study the maximum of $w(x;q)$ on $[0, x_{+K,q}]$ and $[x_{-K,q}, 1]$, respectively. By the boundary condition, for all $q > 0$, $w(x;q)$ can not attain its maximum at $x = 0$ on $[0, x_{+K,q}]$ and at $x = 1$ on $[x_{-K,q}, 1]$, respectively. Now we claim that $w(x;q)$ can not attain its maximum in $(0, x_{+K,q})$ for all sufficiently large $q > 0$. Suppose to the contrary that there is a subsequence $\{q_k\}_{k\in\mathbb{N}}$ of $q \to \infty$, (for convenience, still denoted q_k by q in the following), such that $w(x;q)$ attains its maximum at $x^q \in (0, x_{+K,q})$. Then, one has $w_x(x^q;q) = 0$, $w_{xx}(x^q;q) \le 0$. Consequently,

$$\begin{aligned} -d_I w_{xx}(x^q) = \Bigg[-2\beta^*\left(1 - \frac{2d_I\beta^*}{q^2h^2(x^q)}\right) + \frac{2d_I\beta^* h'(x^q)}{qh^2(x^q)} \\ + \left(\frac{\beta(x^q)S(x^q;q)}{S(x^q;q)+I(x^q;q)} - \gamma(x^q)\right)\Bigg]w(x^q). \end{aligned} \tag{4.48}$$

We need to consider two cases: (i) $x^q \to \kappa$ and (ii) $x^q \to \tau \neq \kappa$ along a subsequence of $q \to \infty$ (still denoted by q), where $\tau \in [0, \kappa)$. If $x^q \to \kappa$, then there exists a constant $Q_5 > 1$, such that $\frac{2d_I \beta^* h'(x^q)}{qh^2(x^q)} < 0$ for all $q > Q_5$. If $x^q \to \tau$, then $\lim\limits_{q\to\infty} |h(x^q)| > 0$. Therefore there exists a constant $Q_6 > 1$, such that $\frac{2d_I \beta^* h'(x^q)}{qh^2(x^q)} < \gamma_*$ for all $q > Q_6$. In view of (4.45), we see

$$\frac{2d_I\beta^*}{q^2h^2(x)} = q^{-\frac{1}{k}} \frac{2d_I\beta^*}{\left|q^{1-\frac{1}{2k}}h(x)\right|^2} \le q^{-\frac{1}{k}} \left(\frac{(2k-1)!}{h^{(2k-1)}(\kappa)}\right)^2 \frac{8d_I\beta^*}{K^{4k-2}} \quad \forall\, x \in \left[0, x_{+K,q}\right],\ \forall\, q > Q_4. \tag{4.49}$$

Hence, we can find another constant $Q_7 > 1$, such that $\frac{2d_I\beta^*}{q^2h^2(x^q)} < \frac{1}{4}$ for all $q > Q_7$. As a consequence, one can check that the right hand of (4.48) is negative for all $q > \max\{Q_4, Q_5, Q_6, Q_7\}$. This makes a contradiction to the left hand of (4.48), which is nonnegative. From above discussion, we conclude that $w(x; q)$ can only attain its maximum on $[0, x_{+K,q}]$ at $x = x_{+K,q}$ for all $q > \max\{Q_4, Q_5, Q_6, Q_7\}$. Similarly, one can find a constant $Q_8 > 1$ such that $w(x; q)$ can only attain its maximum on $[x_{-K,q}, 1]$ at $x = x_{-K,q}$ for all $q > Q_8$. We then obtain that for all $q > \max\{Q_4, Q_5, Q_6, Q_7, Q_8\}$,

$$w(x;q) \le w(x_{+K,q};q) \text{ if } x \in [0, x_{+K,q}] \text{ and } w(x;q) \le w(x_{-K,q};q) \quad \text{if } x \in [x_{-K,q}, 1]. \tag{4.50}$$

Reasoning as in (4.49), we see that $\frac{2\beta^*}{q^{1-\frac{1}{2k}}h(x)}$ is uniformly bounded ($w.r.t.$ $q \to \infty$) on $[0, x_{+K,q}]$ and $[x_{-K,q}, 1]$ for all sufficiently large q. So we can find a constant $Q_9 > 1$, such that

$$e^{\int_0^{x_{+K,q}} \frac{2\beta^*}{qh(s)}\mathrm{d}s} < 2 \text{ and } e^{-\int_{x_{-K,q}}^1 \frac{2\beta^*}{qh(s)}\mathrm{d}s} < 2, \quad q > Q_9.$$

Consequently, in view of (4.46), together with the fact $\tilde{I}(x_{-K,q}; q) \le 1$ and $\tilde{I}(x_{+K,q}; q) \le 1$, we obtain that if $x \in [0, x_{+K,q}]$, then

$$\tilde{I}(x;q) \le \frac{\tilde{I}(x;q)}{\tilde{I}(x_{+K,q};q)} \le e^{-\frac{q}{d_I}\int_x^{x_{+K,q}} h(s)\mathrm{d}s} e^{\int_x^{x_{+K,q}} \frac{2\beta^*}{qh(s)}\mathrm{d}s} \le 2e^{-\frac{q}{d_I}\int_x^{x_{+K,q}} h(s)\mathrm{d}s}, \tag{4.51}$$

and if $x \in [x_{-K,q}, 1]$, then

$$\tilde{I}(x;q) \le \frac{\tilde{I}(x;q)}{\tilde{I}(x_{-K,q};q)} \le e^{\frac{q}{d_I}\int_{x_{-K,q}}^x h(s)\mathrm{d}s} e^{-\int_{x_{-K,q}}^x \frac{2\beta^*}{qh(s)}\mathrm{d}s} \le 2e^{\frac{q}{d_I}\int_{x_{-K,q}}^x h(s)\mathrm{d}s}. \tag{4.52}$$

Using the transformation $x = \xi_S^q + yq^{-\frac{1}{2k}}$, one gets

$$\tilde{v}(y;q) = \frac{v(y;q)}{v(y^q;q)} = \frac{I\left(\xi_S^q + yq^{-\frac{1}{2k}};q\right)}{I(\xi_I^q;q)} = \frac{I(x;q)}{I(\xi_I^q;q)} = \tilde{I}(x;q). \tag{4.53}$$

In view of (4.44), (4.51)-(4.53), we deduce that if $y < -K$, then

$$\tilde{v}(y;q) \le 2e^{-\frac{q}{d_I}\int_{\xi_S^q+yq^{-\frac{1}{2k}}}^{\xi_S^q-Kq^{-\frac{1}{2k}}} h(s)\mathrm{d}s} = 2e^{-\frac{q^{1-\frac{1}{2k}}}{d_I}\int_y^{-K} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \le 2e^{-\frac{1}{2d_I}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}(K+y)},$$

and if $y > K$, then

$$\tilde{v}(y;q) \le 2e^{\frac{q}{d_I}\int_{\xi_S^q+Kq^{-\frac{1}{2k}}}^{\xi_S^q+yq^{-\frac{1}{2k}}} h(s)\mathrm{d}s} = 2e^{\frac{q^{1-\frac{1}{2k}}}{d_I}\int_K^{y} h\left(\xi_S^q+tq^{-\frac{1}{2k}}\right)\mathrm{d}t} \le 2e^{-\frac{1}{2d_I}\frac{h^{(2k-1)}(\kappa)}{(2k-1)!}K^{2k-1}(K-y)}.$$

Hence, we obtain that

$$\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} \tilde{v}(y;q)\mathrm{d}y \le \int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} 2e^{-\frac{1}{2d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}K^{2k-1}(K+y)}\mathrm{d}y \le -\frac{4d_I(2k-1)!}{K^{2k-1}h^{(2k-1)}(\kappa)} \tag{4.54}$$

and

$$\int_{K}^{q^{\frac{1}{2k}}(1-\xi_S^q)} \tilde{v}(y;q)\mathrm{d}y \le \int_{K}^{q^{\frac{1}{2k}}(1-\xi_S^q)} 2e^{-\frac{1}{2d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}K^{2k-1}(K-y)}\mathrm{d}y \le -\frac{4d_I(2k-1)!}{K^{2k-1}h^{(2k-1)}(\kappa)}. \tag{4.55}$$

In addition, by using Lebesgue dominant convergent theorem, there is a constant $Q_{10} > 1$, such that

$$\begin{aligned}\Bigg|&\int_{-K}^{K}(\beta(\kappa)-\gamma(\kappa))e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\\ &-\int_{-K}^{K}\left[\beta\left(\xi_S^q+yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)}-\gamma\left(\xi_S^q+yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\Bigg| < \frac{\varepsilon}{3}.\end{aligned} \tag{4.56}$$

Now we complete our proof of (4.41). Together with (4.42), (4.43), (4.54), (4.55) and (4.56), we see that, for any given $K > K_4$ and $q > \max\{Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\}$, it holds

$$\begin{aligned}&\Bigg|\int_{-\infty}^{\infty}(\beta(\kappa)-\gamma(\kappa))e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y-\\ &\int_{-q^{\frac{1}{2k}}\xi_S^q}^{q^{\frac{1}{2k}}(1-\xi_S^q)}\left[\beta\left(\xi_S^q+yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)}-\gamma\left(\xi_S^q+yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\Bigg|\\ &\le\left|\int_{(-\infty,-K]\cup[K,\infty)}(\beta(\kappa)-\gamma(\kappa))e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\right|+\left|\int_{-K}^{K}(\beta(\kappa)-\gamma(\kappa))e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{-K}^{K} \left[\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\Bigg| \\
&\quad + \left|\int_{K}^{q^{\frac{1}{2k}}(1-\xi_S^q)} \left[\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\right| \\
&\quad + \left|\int_{-q^{\frac{1}{2k}}\xi_S^q}^{-K} \left[\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\right| \\
&\leq (\beta(\kappa)-\gamma(\kappa))\left[\int_{(-\infty,-K]\cup[K,\infty)} e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y\right] + \Bigg|\int_{-K}^{K}(\beta(\kappa)-\gamma(\kappa))e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\mathrm{d}y \\
&\quad - \int_{-K}^{K} \left[\beta\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\frac{u(y;q)}{u(y;q)+v(y;q)} - \gamma\left(\xi_S^q + yq^{-\frac{1}{2k}}\right)\right]\tilde{v}(y;q)\mathrm{d}y\Bigg| \\
&\quad + 2(\beta^* + \gamma^*)\left[-\frac{4d_I(2k-1)!}{K^{2k-1}h^{(2k-1)}(\kappa)}\right] \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

This completes the proof (4.41).

Step 4. We now prove the assertion (iii). By letting $q \to \infty$, arguing similarly to those done in Step 3, from (4.24) we can obtain that

$$
\int_{-\infty}^{\infty} \left(C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}} + C_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\right)\mathrm{d}y = N,
$$

that is,

$$
C_S d_S^{\frac{1}{2k}} + C_I d_I^{\frac{1}{2k}} = kN\left(-\frac{h^{(2k-1)}(\kappa)}{(2k)!}\right)^{\frac{1}{2k}}\Gamma^{-1}\left(\frac{1}{2k}\right).
$$

On the other hand, integrating (4.15) over $\left(-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}\left(1-\xi_S^q\right)\right)$, and using the similar argument as done in Step 3 leads to

$$
\int_{-\infty}^{\infty} C_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\left(\beta(\kappa)\frac{C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}}{C_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}} + C_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}} - \gamma(\kappa)\right)\mathrm{d}y = 0,
$$

which is equivalent to

$$\int_0^1 \frac{1}{\left(\ln z^{-1}\right)^{1-\frac{1}{2k}}\left(1+\frac{C_I}{C_S}z^{1-\frac{d_I}{d_S}}\right)}\mathrm{d}z = \frac{\gamma(\kappa)}{\beta(\kappa)}\Gamma\left(\frac{1}{2k}\right). \tag{4.57}$$

To simplify notations, set $\eta = \frac{C_I}{C_S}$, $\theta = \frac{\beta(\kappa)}{\gamma(\kappa)} > 0$ and $\vartheta = \frac{d_I}{d_S} > 0$. Then we can rewrite (4.57) as

$$\int_0^1 \frac{1}{\left(\ln z^{-1}\right)^{1-\frac{1}{2k}}\left(1+\eta z^{1-\vartheta}\right)}\mathrm{d}z = \frac{1}{\theta}\Gamma\left(\frac{1}{2k}\right). \tag{4.58}$$

By differentiating (4.58) with respect to θ, we see

$$\eta'(\theta) = \frac{\theta^{-2}\Gamma\left(\frac{1}{2k}\right)}{\int_0^1 \frac{z^{1-\vartheta}}{\left(\ln z^{-1}\right)^{1-\frac{1}{2k}}(1+\eta z^{1-\vartheta})^2}\mathrm{d}z} > 0.$$

Similarly, we can differentiate (4.58) with respect to ϑ to obtain

$$\eta'(\vartheta) = \frac{\int_0^1 \frac{\eta z^{1-\vartheta}\left(\ln z^{-1}\right)^{\frac{1}{2k}}}{(1+\eta z^{1-\vartheta})^2}\mathrm{d}z}{\int_0^1 \frac{z^{1-\vartheta}}{\left(\ln z^{-1}\right)^{1-\frac{1}{2k}}(1+\eta z^{1-\vartheta})^2}\mathrm{d}z} > 0.$$

As a result, one easily yields that $\frac{C_I}{C_S}$ increases in $\frac{\beta(\kappa)}{\gamma(\kappa)}$ and $\frac{d_I}{d_S}$, respectively.

Finally, we need to show the uniqueness of (C_S, C_I). Indeed we need only to show the ratio C_S/C_I is uniquely determined by above identity. Set

$$F(\eta) = \int_0^1 \frac{1}{\left(\ln z^{-1}\right)^{1-\frac{1}{2k}}\left(1+\eta z^{1-\frac{d_I}{d_S}}\right)}\mathrm{d}z - \frac{1}{\theta}\Gamma\left(\frac{1}{2k}\right).$$

One easily checks that

$$F'(\eta) < 0,\ \lim_{\eta\to 0} F(\eta) = \left(1-\frac{1}{\theta}\right)\Gamma\left(\frac{1}{2k}\right) > 0,\ \lim_{\eta\to\infty} F(\eta) < 0.$$

Since $\gamma(\kappa) < \beta(\kappa)$, there must be a unique $\eta > 0$ solving $F(\eta) = 0$.

By the uniqueness of (C_S, C_I), which is independent of the sequence $\{q\}$, we conclude that assertion (iii) on Theorem 4.3 holds true. This completes the proof of Theorem 4.3. □

From Theorems 2.5-2.6 and assumption $\beta(\kappa) > \gamma(\kappa)$, it follows that $\mathcal{R}_0 > 1$ for all sufficiently large q and fixed $d_S, d_I > 0$, which guarantees the existence of EE by using Theorem 2.4. In addition, it should be noted that $\lim_{q\to\infty} \frac{\|I(\cdot;q)\|_\infty}{\|S(\cdot;q)\|_\infty} = \frac{C_I}{C_S}$, and for fixed $d_S, d_I > 0$, it is actually true that $\lim\limits_{q\to\infty}\frac{\int_0^1 I(x;q)\mathrm{d}x}{\int_0^1 S(x;q)\mathrm{d}x}=\frac{d_I^{\frac{1}{2k}}C_I}{d_S^{\frac{1}{2k}}C_S}$. It should be also noted that $\frac{\beta(\kappa)}{\gamma(\kappa)}$ is known as the local basic reproduction number at κ. Thus, the monotonicity result in Theorem 4.3 (iii) means for fixed $d_S, d_I>0$, larger local basic reproduction number results in larger ratio $\frac{C_I}{C_S}$ and so for the ratio of total biomass for two groups, in large convection environment. On the other hand, for fixed local reproduction number $\frac{\beta(\kappa)}{\gamma(\kappa)}$, the ratio $\frac{C_I}{C_S}$ increases as $\frac{d_I}{d_S}$ increases. So in infectious disease prevention and control strategies, one can also consider controlling the ratio of the free diffusion rates of infectious and susceptible individuals, under suitably large convection rate.

When $\kappa=0$ or $\kappa=1$, that is, under the assumption (H0) or (H1), we have

Corollary 4.4. *Let $d_S, d_I>0$ be fixed, $(S(x;q), I(x;q))$ be any* EE *of* (1.6)*, ξ_S^q and ξ_I^q be the maximum points of $S(x;q)$ and $I(x;q)$ on* $[0,1]$ *respectively. Then the following assertions hold:*

(i) *For the case* (H0)*, assume that $\beta(0)>\gamma(0)$. Then as $q\to\infty$, one has* $(S(x;q), I(x;q))\to(0,0)$ *locally uniformly in* $(0,1]$*, both $q^{\frac{1}{2k}}\xi_S^q$ and $q^{\frac{1}{2k}}\xi_I^q$ converge to zero. Moreover,*

$$\lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}}S\left(yq^{-\frac{1}{2k}};q\right)=2C_Se^{\frac{h^{(2k-1)}(0)}{d_S(2k)!}y^{2k}},\quad \lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}}I\left(yq^{-\frac{1}{2k}};q\right)=2C_Ie^{\frac{h^{(2k-1)}(0)}{d_I(2k)!}y^{2k}}$$

uniformly in any compact subset of $\mathbb{R}$, where (C_S, C_I) is the solution of (4.17) *with $\kappa=0$. Furthermore, the ratio $\frac{C_I}{C_S}$ is increasing in $\frac{\beta(\kappa)}{\gamma(\kappa)}$ and $\frac{d_I}{d_S}$ respectively.*

(ii) *For the case* (H1)*, assume that $\beta(1)>\gamma(1)$. Then as $q\to\infty$, one has* $(S(x;q), I(x;q))\to(0,0)$ *locally uniformly in* $[0,1)$*, both $q^{\frac{1}{2k}}(1-\xi_S^q)$ and $q^{\frac{1}{2k}}(1-\xi_I^q)$ converge to zero. Moreover,*

$$\lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}}S(1-yq^{-\frac{1}{2k}};q)=2C_Se^{\frac{h^{2k-1}(1)}{d_S(2k)!}y^{2k}},\quad \lim_{q\to\infty}\frac{1}{q^{\frac{1}{2k}}}I(1-yq^{-\frac{1}{2k}};q)=2C_Ie^{\frac{h^{(2k-1)}(1)}{d_I(2k)!}y^{2k}}$$

uniformly in any compact subset of $\mathbb{R}$, where (C_S, C_I) is the solution of (4.17) *with $\kappa=1$. Furthermore, the ratio $\frac{C_I}{C_S}$ is increasing in $\frac{\beta(\kappa)}{\gamma(\kappa)}$ and $\frac{d_I}{d_S}$ respectively.*

Sketch of the proof of Corollary 4.4. One can prove Corollary 4.4 as discussed before. Here we only give the key steps to the proof of Corollary 4.4. We still denote q_n by q for convenience.

Step 1. Convergence of EE as convection rate $q\to\infty$.

(i) If (H0) holds, then $(S(\cdot;q), I(\cdot;q))$ converges to $(0,0)$ locally uniformly in $(0,1]$;
(ii) If (H1) holds, then $(S(\cdot;q), I(\cdot;q))$ converges to $(0,0)$ locally uniformly in $[0,1)$.

Step 2. The asymptotic behaviors of maximum points ξ_S^q and ξ_I^q of $S(x;q)$ and $I(x;q)$ respectively.

(i) If (H0) holds, then $\lim\limits_{q\to\infty}q^{\frac{1}{2k}}\xi_S^q=\lim\limits_{q\to\infty}q^{\frac{1}{2k}}\xi_I^q=0$, and hence $\lim\limits_{q\to\infty}q^{\frac{1}{2k}}|\xi_S^q-\xi_I^q|=0$;

(ii) If (H1) holds, then $\lim_{q\to\infty} q^{\frac{1}{2k}}(1-\xi_S^q) = \lim_{q\to\infty} q^{\frac{1}{2k}}(1-\xi_I^q) = 0$, and hence $\lim_{q\to\infty} q^{\frac{1}{2k}}|\xi_S^q - \xi_I^q| = 0$.

Step 3. Convergence of $(u(y;q), v(y;q))$.

(i) If (H0) holds, we have estimates

$$\begin{cases} u(y) \leq C_4 e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{y} h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right)\mathrm{d}t} + \dfrac{C_3}{qd_S}\displaystyle\int_{-q^{\frac{1}{2k}}\xi_S^q}^{y} e^{\frac{q^{1-\frac{1}{2k}}}{d_S}\int_z^y h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \\ \qquad\qquad \forall\, y \in [-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)], \\ v(y) \leq C_4 e^{\frac{q^{1-\frac{1}{2k}}}{d_I}\int_{-q^{\frac{1}{2k}}\xi_S^q}^{y} h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right)\mathrm{d}t} + \dfrac{C_3}{qd_I}\displaystyle\int_{-q^{\frac{1}{2k}}\xi_S^q}^{y} e^{\frac{q^{1-\frac{1}{2k}}}{d_I}\int_z^y h\left(\xi_S^q + tq^{-\frac{1}{2k}}\right)\mathrm{d}t}\mathrm{d}z, \\ \qquad\qquad \forall\, y \in [-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)]. \end{cases}$$

Hence, one can show that for any given $K > 1$,

$$\|u(\cdot;q)\|_{C^{1,\alpha}([0,K])} \leq C \text{ and } \|v(\cdot;q)\|_{C^{1,\alpha}([0,K])} \leq C$$

for some positive constant C. Furthermore, there exist nonnegative constants C_S', C_I' such that (passing to a subsequence of $q \to \infty$ if necessary)

$$\lim_{q\to\infty}(u(y;q), v(y;q)) = \left(C_S' e^{\frac{1}{d_S}\frac{h^{(2k-1)}(0)}{(2k)!}y^{2k}}, C_I' e^{\frac{1}{d_I}\frac{h^{(2k-1)}(0)}{(2k)!}y^{2k}}\right) \text{ in } \left(C^1_{loc}([0,K])\right)^2.$$

(ii) If (H1) holds, we have estimates

$$\begin{cases} u(y) \leq C_4 e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^{q^{\frac{1}{2k}}(1-\xi_S^q)} h\left(\xi_S^q + \frac{t}{q^{\frac{1}{2k}}}\right)\mathrm{d}t} + \dfrac{C_3}{qd_S}\displaystyle\int_y^{q^{\frac{1}{2k}}(1-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_S}\int_y^z h\left(\xi_S^q + \frac{t}{q^{\frac{1}{2k}}}\right)\mathrm{d}t}\mathrm{d}z, \\ \qquad\qquad \forall\, y \in [-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)], \\ v(y) \leq C_4 e^{-\frac{q^{1-\frac{1}{2k}}}{d_I}\int_y^{q^{\frac{1}{2k}}(1-\xi_S^q)} h\left(\xi_S^q + \frac{t}{q^{\frac{1}{2k}}}\right)\mathrm{d}t} + \dfrac{C_3}{qd_I}\displaystyle\int_y^{q^{\frac{1}{2k}}(1-\xi_S^q)} e^{-\frac{q^{1-\frac{1}{2k}}}{d_I}\int_y^z h\left(\xi_S^q + \frac{t}{q^{\frac{1}{2k}}}\right)\mathrm{d}t}\mathrm{d}z, \\ \qquad\qquad \forall\, y \in [-q^{\frac{1}{2k}}\xi_S^q, q^{\frac{1}{2k}}(1-\xi_S^q)]. \end{cases}$$

Therefore, using a similar discussion to Step 1 in proof of Theorem 4.3 one sees that for any given $K > 1$,

$$\|u(\cdot;q)\|_{C^{1,\alpha}([-K,0])} \le C \text{ and } \|v(\cdot;q)\|_{C^{1,\alpha}([-K,0])} \le C$$

for some positive constant C. Furthermore, there exist nonnegative constants C'_S, C'_I such that (passing to a subsequence of $q \to \infty$ if necessary)

$$\lim_{q\to\infty}(u(y;q), v(y;q)) = \left(C'_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(1)}{(2k)!}y^{2k}}, C'_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(1)}{(2k)!}y^{2k}}\right) \text{ in } \left(C^1_{loc}([-K,0])\right)^2$$

Step 4. $C'_S, C'_I > 0$ for both cases (H0) and (H1). The conclusion can be verified as the discussion to Steps 2-3 in the proof of Theorem 4.3 with some modifications.

Step 5. Determine the constants C'_S and C'_I. Observe that for both cases (H0) and (H1),

$$\int_0^\infty \left(C'_S e^{\frac{1}{d_S}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}} + C'_I e^{\frac{1}{d_I}\frac{h^{(2k-1)}(\kappa)}{(2k)!}y^{2k}}\right) \mathrm{d}y = N, \qquad \kappa = 0, 1.$$

Hence, C'_S and C'_I satisfy

$$\begin{cases} C'_S d_S^{\frac{1}{2k}} + C'_I d_I^{\frac{1}{2k}} = 2kN\left(-\dfrac{h^{(2k-1)}(\kappa)}{(2k)!}\right)^{\frac{1}{2k}} \Gamma^{-1}\left(\dfrac{1}{2k}\right), \\ \displaystyle\int_0^1 \frac{1}{(\ln z^{-1})^{1-\frac{1}{2k}}\left(1+\frac{C'_I}{C'_S}z^{1-\frac{d_I}{d_S}}\right)} \mathrm{d}z = \frac{\gamma(\kappa)}{\beta(\kappa)}\Gamma\left(\frac{1}{2k}\right). \end{cases}$$

By differentiating the second identity of the above equations with respect to $\frac{\gamma(\kappa)}{\beta(\kappa)}$ and $\frac{d_I}{d_S}$, one easily gets that $\frac{C'_I}{C'_S}$ increases in $\frac{\beta(\kappa)}{\gamma(\kappa)}$ and $\frac{d_I}{d_S}$, respectively. By the same reason as in Step 4 in the proof of Theorem 4.3, one obtains the uniqueness of constant pair (C'_S, C'_I). In addition, one easily checks that $(C'_S, C'_I) = (2C_S, 2C_I)$, where (C_S, C_I) is the solution of (4.17). The proof is complete. □

Remark 4.5. We would like to make some notes and comparisons between our results and existing works.

(i) Regardless of cases (H), (H0) or (H1), we always assume that $\beta(\kappa) > \gamma(\kappa)$ to guarantee the existence of EE. If $\beta(\kappa) < \gamma(\kappa)$, then there is no EE for large q, which is not of interest of this paper.

(ii) Comparing Theorem 4.4 with the conclusions in [11] or [25] for large convection rate, there are some obvious differences in asymptotic profiles. For constant convective environment, the densities of both susceptible and infectious populations behave like a function $\exp(-cy)$ nearing the boundary for some positive constant c, while the densities behave like a function $\exp(-cy^{2k})$ nearing the boundary for some positive constant c if convection term degenerate to zero at boundary.

5. Asymptotic behavior of the EE as $d_I \to 0$

Theorem 2.6 established the existence of the EE of (1.6) under the assumption $\beta(\kappa) > \gamma(\kappa)$. In this section, we would like to investigate the limiting distribution of the EE as $d_I \to 0$. We always fix $q > 0$ in this section. Let x_{d_I} be the maximum point of $I(x; d_I)$. We will first show that $\lim_{d_I \to 0} x_{d_I} = \kappa$.

Lemma 5.1. *Assume that* (H) *holds. For any given small constant $\epsilon > 0$, there exists a suitably small $\hat{d}_I = \hat{d}_I(\epsilon) > 0$, such that, for all $0 < d_I < \hat{d}_I$, it holds*

$$I_x(x) > 0 \quad \forall\, 0 \le x \le \kappa - \epsilon, \qquad \textit{and} \qquad I_x(x) < 0 \quad \forall\, \kappa + \epsilon \le x \le 1.$$

As a result, we have $\lim_{d_I \to 0} x_{d_I} = \kappa$.

Proof. We need only to show $I_x(x) > 0$, $\forall\, x \in [0, \kappa - \epsilon]$. Assume that $I(x; d_I)$ attains its maximum at $\hat{x}_{d_I}$ on $[0, \kappa - \epsilon]$. Consider the auxiliary function

$$w(x; d_I) = I(x; d_I) e^{-\frac{q}{2d_I}\int_0^x h(s)\mathrm{d}s}, \quad x \in [0, \kappa).$$

Then, by some simple computations, one can check that w satisfies

$$\begin{cases} d_I w_{xx} - \dfrac{q^2 h^2(x)}{4d_I} w = \left(\dfrac{qh'(x)}{2} + \gamma(x) - \beta(x)\dfrac{S}{S+I}\right) w, & x \in (0, \kappa), \\ w_x(0) = \dfrac{qh(0)}{2d_I} w(0). \end{cases}$$

Let $x = \hat{x}_{d_I} + d_I y$, $y \in [-\hat{x}_{d_I}/d_I, (\kappa - \hat{x}_{d_I})/d_I]$, and $\tilde{w}(y; d_I) = w(\hat{x}_{d_I} + d_I y; d_I)$. We then have

$$\begin{cases} \tilde{w}_{yy} - \dfrac{q^2 h^2(\hat{x}_{d_I} + d_I y)}{4} \tilde{w} = \\ d_I \left[\dfrac{qh'(\hat{x}_{d_I} + d_I y)}{2} + \gamma(\hat{x}_{d_I} + d_I y) - \beta(\hat{x}_{d_I} + d_I y)\dfrac{S}{S+I}\right] \tilde{w}, & y \in \left(-\dfrac{\hat{x}_{d_I}}{d_I}, \dfrac{\kappa - \hat{x}_{d_I}}{d_I}\right), \\ \tilde{w}_y(-\hat{x}_{d_I}/d_I) = \dfrac{qh(0)}{2} \tilde{w}(-\hat{x}_{d_I}/d_I). \end{cases}$$

Since $\left[qh'(\hat{x}_{d_I} + d_I y)/2 + \gamma(\hat{x}_{d_I} + d_I y) - \beta(\hat{x}_{d_I} + d_I y)S/(S+I)\right]$ is bounded, we can find a suitably small $\hat{d}_I > 0$ such that

$$\frac{q^2 h^2(x)}{4} + d_I \left|\frac{qh'(x)}{2} + \gamma(x) - \beta(x)\frac{S}{S+I}\right| > 0, \quad \forall\, 0 < d_I < \hat{d}_I,\ \forall\, x \in [0, \kappa - \epsilon].$$

Consequently, if $0 < d_I < \hat{d}_I$, then

$$\tilde{w}_{yy} \ge 0 \quad \text{for all } y \in (-\hat{x}_{d_I}/d_I, (\kappa - \epsilon - \hat{x}_{d_I})/d_I).$$

Observing that $\tilde{w}_y(-\hat{x}_{d_I}/d_I) = \frac{qh(0)}{2}\tilde{w}(-\hat{x}_{d_I}/d_I) > 0$, we have $\tilde{w}_y > 0$ for all $y \in (-\hat{x}_{d_I}/d_I, (\kappa - \epsilon - \hat{x}_{d_I})/d_I)$. This implies that $w_x > 0$ for all $x \in [0, \kappa - \epsilon]$. Since

$$I_x(x) = e^{\frac{q}{2d_I}\int_0^x h(s)\mathrm{d}s}\left(w_x + \frac{qh(x)}{2d_I}w\right) > 0$$

for all $x \in [0, \kappa - \epsilon]$ and $d_I \in (0, \hat{d}_I)$, which follows that $\hat{x}_{d_I} = \kappa - \epsilon$. Similarly, there exists another suitably small positive constant, denoted still by $\hat{d}_I$, such that $I_x(x) < 0$ for all $x \in [\kappa + \epsilon, 1]$ and $d_I \in (0, \hat{d}_I)$, and more, $\hat{x}_{d_I} = \kappa + \epsilon$. By the arbitrariness of $\epsilon > 0$, we easily see that as $d_I \to 0$, the maximum point x_{d_I} of $I(x; d_I)$, converges to κ. This ends the proof of the lemma. □

For cases (H0) and (H1), we have

Lemma 5.2. *Assume that* (H0) *or* (H1) *holds. For any given small constant $\epsilon > 0$, there exists a suitably small $\hat{d}_I = \hat{d}_I(\epsilon) > 0$, such that, for all $0 < d_I < \hat{d}_I$, it holds*

$$I_x(x) < 0 \quad \forall\, \epsilon \le x \le 1 \text{ if (H0) holds,} \qquad \text{and} \qquad I_x(x) > 0 \quad \forall\, 0 \le x \le 1 - \epsilon \text{ if (H1) holds.}$$

As a result, we have $\lim_{d_I \to 0} x_{d_I} = 0$ if (H0) *holds, and $\lim_{d_I \to 0} x_{d_I} = 1$ if* (H1) *holds.*

We now derive the asymptotic behavior of EE as $d_I \to 0$. Recall that $\beta^* = \max_{x\in[0,1]} \beta(x)$, $\gamma_* = \min_{x\in[0,1]} \gamma(x)$ and $\gamma^* = \max_{x\in[0,1]} \gamma(x)$. We only prove Theorem 5.3 under assumption (H), since it is easy to obtain Theorem 5.3 under assumption (H0) or (H1) with slight modifications. We introduce another auxiliary function, denoted still by $w(x; d_I)$, which is defined by

$$w(x; d_I) = \begin{cases} I(x; d_I)e^{-\int_0^x \frac{qh(s)}{d_I}\left(1 - \frac{2\beta^* d_I}{q^2h^2(s)}\right)\mathrm{d}s}, & x \in [0, \kappa), \\ I(x; d_I)e^{\int_x^1 \frac{qh(s)}{d_I}\left(1 - \frac{2\beta^* d_I}{q^2h^2(s)}\right)\mathrm{d}s}, & x \in (\kappa, 1]. \end{cases} \tag{5.1}$$

Then $w(x; d_I)$ satisfies

$$\begin{cases} 0 = d_I w_{xx} + \left(qh(x) - \dfrac{4\beta^* d_I}{qh(x)}\right) w_x + \\ \qquad \left\{-2\beta^*\left[1 - \dfrac{d_I}{qh^2(x)}\left(2\beta^* - h'(x)\right)\right] + \beta(x)\dfrac{S}{S+I} - \gamma(x)\right\} w, & x \in [0, \kappa) \cup (\kappa, 1], \\ w_x = \dfrac{2\beta^*}{qh(x)} w, & x = 0, 1. \end{cases} \tag{5.2}$$

Theorem 5.3. *Let $q, d_S > 0$ be fixed. Assume that* (H) $(resp.$ (H0) *or* (H1)) *holds and $\beta(\kappa) > \gamma(\kappa)$. Then the following assertions hold.*

(i) *We have*

$$S(x; d_I) \to \frac{N}{\int_0^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x} e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s} \quad \textit{uniformly on } [0,1], \textit{ as } d_I \to 0.$$

(ii) $I(x; d_I) \to 0$ *locally uniformly in* $[0,\kappa) \cup (\kappa, 1]$ *as* $d_I \to 0$*, and* $\lim\limits_{d_I \to 0} \int_0^1 I(x; d_I)\mathrm{d}x = 0$.

Proof. Let $0 < \varsigma << \min\{\kappa, 1-\kappa\}$ be a small constant, which can be chosen arbitrarily small. Then there exists a $\hat{d}_I = \hat{d}_I(\varsigma) > 0$ such that

$$\frac{d_I}{qh^2(x)}\left(2\beta^* - h'(x)\right) < \frac{1}{2} \quad \forall\, x \in [0, \kappa - \varsigma] \cup [\kappa + \varsigma, 1],\ \forall\, d_I \in (0, \hat{d}_I]. \tag{5.3}$$

Then we are going to show that w attains its maximum at downstream on $[0, \kappa - \varsigma]$ and at upstream on $[\kappa + \varsigma, 1]$, for all small $d_I > 0$. In view of the boundary condition in (5.2), we see that w cannot attain its maximum at $x = 0$ or 1. If $w(x; d_I)$ attains its maximum at $x = x^* \in (0, \kappa - \varsigma)$ on $[0, \kappa - \varsigma]$, then $w_x(x^*; d_I) = 0$ and $w_{xx}(x^*; d_I) \le 0$. By (5.3), one then can check that

$$-2\beta^*\left[1 - \frac{d_I}{qh^2(x^*)}\left(2\beta^* - h'(x^*)\right)\right] + \beta(x^*)\frac{S}{S+I} - \gamma(x^*) < 0 \quad \forall\, d_I \in (0, \hat{d}_I].$$

Substituting these facts into (5.2), we arrive at a contradiction. Similarly, one can derive the conclusion on $[\kappa + \varsigma, 1]$. Therefore, we have

$$w(x; d_I) \le \begin{cases} w\left(\kappa - \varsigma; d_I\right), & \forall\, x \in \left[0, \kappa - \varsigma\right], \\ w\left(\kappa + \varsigma; d_I\right), & \forall\, x \in \left[\kappa + \varsigma, 1\right]. \end{cases}$$

Or equivalently,

$$\begin{cases} I\left(x; d_I\right) \le I\left(\kappa - \varsigma; d_I\right) e^{-\int_x^{\kappa-\varsigma} \frac{qh(s)}{d_I}\left(1 - \frac{2\beta^* d_I}{q^2h^2(s)}\right)\mathrm{d}s}, & \forall\, x \in \left[0, \kappa - \varsigma\right), \\ I\left(x; d_I\right) \le I\left(\kappa + \varsigma; d_I\right) e^{\int_{\kappa+\varsigma}^{x} \frac{qh(s)}{d_I}\left(1 - \frac{2\beta^* d_I}{q^2h^2(s)}\right)\mathrm{d}s}, & \forall\, x \in \left(\kappa + \varsigma, 1\right]. \end{cases}$$

We now claim that $I\left(\kappa - \varsigma; d_I\right)$ is uniformly bounded for all small $d_I > 0$. Indeed, let $\epsilon = \frac{\varsigma}{2}$ in the previous lemma firstly. Then, without loss of generality, we may assume that $I(x; d_I)$ increases in x on $[0, \kappa - \varsigma/2]$ for all $0 < d_I < \hat{d}_I$. If $I(\varsigma; d_I) \to \infty$ along some sequence $\{d_{I,n}\}$ of $d_I \to 0$, then

$$N \ge \int_0^1 I(x; d_I)\mathrm{d}x \ge \int_{\kappa-\varsigma}^{\kappa-\varsigma/2} I(x; d_I)\mathrm{d}x \to \infty \quad \text{as } d_I = d_{I,n} \to 0,$$

which arrives at a contradiction. Similarly, $I\left(\kappa + \varsigma; d_I\right)$ is also uniformly bounded for all $d_I \in (0, \hat{d}_I]$. Then, by the arbitrariness of $\varsigma > 0$, one yields that $I(x; d_I) \to 0$ locally uniformly in $[0, \kappa) \cup (\kappa, 1]$ as $d_I \to 0$.

We next show that

$$\lim_{d_I \to 0} \int_0^1 I(x; d_I)\mathrm{d}x = 0.$$

Indeed, from Lemma 3.1, we easily see that $S(x; d_I)$ is uniformly bounded for all $d_I > 0$ on $[0, 1]$. Observing that

$$\gamma_* \int_0^1 I(x; d_I)\mathrm{d}x \leq \int_0^1 \gamma(x) I(x; d_I)\mathrm{d}x = \int_0^1 \beta(x) \frac{S(x; d_I) I(x; d_I)}{S(x; d_I) + I(x; d_I)}\mathrm{d}x,$$

we have

$$\begin{aligned}
&\gamma_* \lim_{d_I \to 0} \int_0^1 I(x; d_I)\mathrm{d}x \\
&\leq \lim_{d_I \to 0} \int_0^1 \beta(x) \frac{S(x; d_I) I(x; d_I)}{S(x; d_I) + I(x; d_I)}\mathrm{d}x \\
&\leq \lim_{\varepsilon \to 0} \left[\lim_{d_I \to 0} \int_{[0,1]\setminus[\kappa-\varepsilon,\kappa+\varepsilon]} \beta(x) \frac{S(x; d_I)}{S(x; d_I) + I(x; d_I)} I(x; d_I)\mathrm{d}x \right. \\
&\qquad \left. + \lim_{d_I \to 0} \int_{\kappa-\varepsilon}^{\kappa+\varepsilon} \beta(x) \frac{I(x; d_I)}{S(x; d_I) + I(x; d_I)} S(x; d_I)\mathrm{d}x \right] \\
&= 0
\end{aligned}$$

as claimed. One should further observe that

$$-(\beta^* + \gamma^*) \int_0^x I(s; d_I)\mathrm{d}s \leq d_S S_x - q h(x) S \leq (\beta^* + \gamma^*) \int_0^x I(s; d_I)\mathrm{d}s, \quad x \in (0, 1).$$

Appealing the standard regular arguments for elliptic equations to the S-equation, passing to a sequence of $d_I \to 0$ if necessary, we may assume that $\lim_{d_I \to 0} S(x; d_I) = \check{S}(x)$ in $C^1([0, 1])$. Combining with the boundary condition of S, one easily obtains that $d_S \check{S}_x - q h(x) \check{S} = 0$ for all $x \in [0, 1]$, which has a solution represented by $\check{S}(x) = \check{S}(\kappa) e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}$, $x \in [0, 1]$. Since

$$\int_0^1 S(x; d_I)\mathrm{d}x = N - \int_0^1 I(x; d_I)\mathrm{d}x \to N \text{ as } d_I \to 0,$$

we see that $\int_0^1 \check{S}(x)\mathrm{d}x = N$. Thus, $\check{S}(\kappa) = \frac{N}{\int_0^1 e^{\frac{q}{d_S}\int_\kappa^x h(s)\mathrm{d}s}\mathrm{d}x}$. This completes the proof. $\square$

Remark 5.4. We would like to point out that, if (H0) or (H1) holds, namely, $\kappa = 0$ or 1, one can obtain similar conclusions. The asymptotic behavior of the EE as $d_S \to 0$ is also of our interest. However, it seems more difficult to discuss. We leave this for future consideration.

6. Discussion

In this paper we proposed a reaction-diffusion-convection SIS epidemic model in a variational convective environment. We first established the uniform persistence property of (1.4) if $\mathcal{R}_0 > 1$, and the limiting behavior of basic reproduction number $\mathcal{R}_0$. Then we studied the asymptotic behavior of the EE (if it exists) for large convection rate q. Our main results show that the asymptotic behaviors of susceptible and infectious individuals for large convection rate q are different from those SIS models with always positive or negative convection coefficients. That is, we see that susceptible and infectious individuals are able to concentrate around some interior points rather than only to boundary points (upstream or downstream). We think this has important implications for infectious disease prevention and control strategies. Because in addition to concentrating individuals at upstream or downstream, we can also flexibly guide individuals actively move to appropriate locations (low-risk) to control or eliminate disease by optimizing the distribution of medical resources or epidemic prevention policies.

Besides the case that convection rate approaches infinity, we were also of interest in cases that small diffusion rates of susceptible and infectious individuals respectively. As mentioned above, the convection function $h(x)$ models the direction of individuals movement, yet the assumption (H) is only one special case. It is natural to ask that what will happen to the DFE and EE for other more general cases. For example, define a convection function h as

$$h(x) = \begin{cases} \sqrt{0.5 - x}, & \text{if } x \in [0, 0.5)\,, \\ 0, & \text{if } x = 0.5, \\ -\sqrt{x - 0.5}, & \text{if } x \in (0.5, 1]\,. \end{cases}$$

The first derivative of h is

$$h'(x) = \begin{cases} -\dfrac{1}{2\sqrt{0.5 - x}}, & \text{if } x \in [0, 0.5)\,, \\ -\dfrac{1}{2\sqrt{x - 0.5}}, & \text{if } x \in (0.5, 1]\,. \end{cases}$$

Obviously, such a $h(x)$ is in $C([0, 1])$ but not in $C^1([0, 1])$. However, $h(x) \in C^{1/2}([0, 1])$. So, for such kind of h, what are the asymptotic behaviors of the EE? More generally, through our model is based on a one-dimensional bounded region, we believe that it will be more realistic to consider epidemic models in a high-dimensional convective environment. We plan to investigate these interesting problems in follow-up research.

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Data availability

No data was used for the research described in the article.

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