When does $\aleph_1$-categoricity imply ω-stability? *

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April 7, 2024

Abstract

For an $\aleph_1$-categorical atomic class, we clarify the space of types over the unique model of size $\aleph_1$. Using these results, we prove that if such a class has a model of size $\beth_1^+$ then it is ω-stable.

1 Introduction

Our principal result is

Theorem 1.1. *If an atomic class* At *is* $\aleph_1$*-categorical and has a model of size* $(2^{\aleph_0})^+$*, then* At *is* ω*-stable.*

This result springs from several related problems in the study of $L_{\omega_1,\omega}$: the role of $\beth_{\omega_1}$, the possible necessity of the weak continuum hypothesis, the absoluteness of $\aleph_1$-categoricity.

For first order logic, Morley [Mor65] proved, enroute to his categoricity theorem, that an $\aleph_1$-categorical first order theory is ω-stable (né totally transcendental). The existence of a saturated Ehrenfeucht-Mostowski model of cardinality $\aleph_1$ that is generated by a well-ordered set of indiscernibles is crucial to the proof. The construction of such indiscernibles via the Erdős-Rado theorem and Ehrenfeucht-Mostowski models is tied closely to the existence of 'large' (i.e. of size $\beth_{\omega_1}$) models of the theory.

*All authors are grateful to an individual who prefers to remain anonymous' for providing typing services that were used during the work on the paper. **The second author is grateful to Danielle Ulrich for helpful discussions.** This is paper 1244 in the Shelah Archive.

†Research partially supported by Simons travel grant G3535

‡Partially supported by NSF grant DMS-2154101.

§The third author would like to thank the NSF and BSF for partially supporting this research through grant NSF 2051825/ BSF 3013005232 with M. Malliaris. The reader should note that the version in my website is usually more up-to-date than the one in arXiv.

The compactness of first order logic yields the full upward Löwenheim-Skolem-Tarski (LST) theory for $L_{\omega,\omega}$: if ψ has an infinite model it has arbitrarily large models. But for $L_{\omega_1,\omega}$, the LST-theorem replaces 'an infinite model' by 'a model of size $\beth_{\omega_1}$.' The proof proceeds by using iterations of the Erdős-Rado theorem to find infinite sets of indiscernibles and to transfer size via Ehrenfeucht-Mostowski models.

By an atomic class we mean the *atomic* models (i.e., each finite sequence in each model realizes a principal type over the empty set) of a complete theory in a countable first order language. Every complete sentence in $L_{\omega_1,\omega}$ defines such a class because Chang's theorem translates the sentence to a first order theory omitting types and the language can be expanded to make all realized types atomic [Bal09, Chapter 6].

Shelah calls an atomic class excellent if it satisfies an n-amalgamation property for all n and structures of arbitrary cardinality. He proved [She83a, She83b] in ZFC: If an atomic class K is excellent and has an uncountable model then 1) it has models of arbitrarily large cardinality; 2) if it is categorical in one uncountable power it is categorical in all uncountable powers. He also obtained a partial converse; under the very weak generalized continuum hypothesis ($2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$): an atomic class $\boldsymbol{K}$ that has at least one uncountable model and is categorical in $\aleph_n$ for each $n < \omega$ is excellent. Thus under VWGCH the 'Hanf number' for existence and for categorical atomic classes is reduced from $\beth_{\omega_1}$ to $\aleph_\omega$.

This raises the question. Does an $\aleph_1$-categorical atomic class have arbitrarily large models? Shelah [She75] showed it has a model in $\aleph_2$.

For the authors, work on this problem began by searching for sentences of $L_{\omega_1,\omega}$ for which $\aleph_1$-categoricity can be altered by forcing.[1] The third author proposed an example, but the first author objected to the proof and the second author proved in ZFC that the putative example was not $\aleph_1$-categorical.

In a series of papers the authors show that $\aleph_1$categorical atomic classes (or even simply $< 2^{\aleph_1}$ atomic models in $\aleph_1$) exhibit some 'superstable-like' behavior. In [BLS16] we introduced the appropriate notion of an algebraic type for atomic classes, *pseudo-algebraic* (Definition 3.2.2) and proved there that for an atomic class with $< 2^{\aleph_1}$ models in $\aleph_1$ the pseudo-algebraic types were dense. This is analogous to every non-algebraic formula being extendible to a weakly minimal formula in a superstable theory. In [LS19] it is shown that an atomic class with few models in $\aleph_1$ is 'pcl-small', i.e., there are few types over the pseudo-closure of any finite set (which is a weakening of ω-stability) and here we show that $\aleph_1$ categoricity and the existence of an atomic model of size $\beth_1^+$ implies ω-stability.

The search for weakened conditions for ω-stability is partially motivated by asking whether the absoluteness of $\aleph_1$-categoricity for first order logic (given by the equivalence to ω-stable and no two-cardinal model) extends to atomic classes. [Bal12] proves that either arbitrarily large models ($\beth_{\omega_1}$) or ω-stability

[1]For sentences of $L_{\omega_1,\omega}(Q)$, such sentences exist, see [She87, §6], expounded as [Bal09, §17]. A non-ω-stable sentence with no models above the continuum is given, where $\aleph_1$-categoricity fails under CH but holds under Martin's Axiom.

sufficed for such an absolute characterization. Our main theorem reduces the $\beth_{\omega_1}$ to $\beth_1^+$.

In Section 2 we investigate *constrained* types over models and investigate their relation to $\aleph_1$-categoricity and ω-stability. The notion of a constrained type is just a renaming; a type $p \in S(M)$ is constrained just if it does not split over a finite subset. Such a type is definable in the standard use in model theory – the existence of a schema such that for all $\mathbf{m} \in M$, $\phi(\mathbf{x}, \mathbf{m}) \in p \leftrightarrow d_\phi(\mathbf{x}, \mathbf{m})$. In Sections 2.2 and 2.3.1 we introduce 'constrained' and limit types (over models) and investigate them under the assumption of $\aleph_1$-categoricity. From this, we prove the main theorem. However, our results in Section 2.2 depend on a major hypothesis, the existence of an uncountable model in which every limit type is constrained. In Section 3 we pay back our debt. By proving Theorem 2.3.2, we show the existence of a model of size $\aleph_1$ in which every limit type is constrained, using only the existence of an uncountable model. Although the proof there uses forcing, by appealing to the absoluteness given by Keisler's model existence theorem for sentences of $L_{\omega_1,\omega}(Q)$, the result is really a theorem of ZFC.

2 Constrained types, $\aleph_1$-categoricity and ω-stability

Throughout this article, T denotes a complete theory in a countable language for which there is an uncountable atomic model. At denotes the class of atomic models of T. In everything that follows, we only consider atomic sets, i.e., sets for which every finite tuple is isolated by a complete formula. Throughout, M, N denote atomic models and A, B atomic sets. We write $\boldsymbol{a}, \mathbf{b}$ for finite atomic tuples, and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote finite tuples of variables.

We repeatedly use the fact that the countable atomic model M is unique up to isomorphism. Vaught [Vau61] showed the existence of an uncountable atomic model is equivalent to the countable atomic model having a proper elementary extension. The only types we consider are either over an atomic model or are over a finite subset of a model. In either case, we only consider types realized in atomic sets.

For general background see [Bal09] and more specifically [BLS16].

2.1 Constrained types and filtrations

Definition 2.1.1. Fix a countable complete theory T with monster model $\mathcal{M}$. $At = At_T$ denotes the collection of atomic models of T.

1. For $M \in At$, $S_{at}(M)$ is the collection of $p(\mathbf{x}) \in S(M)$ such that if $\boldsymbol{a} \in \mathcal{M}$ realizes p, $M\boldsymbol{a}$ is an atomic set.

2. At is *ω-stable* if for every/some countable $M \in$ At, $S_{at}(M)$ is countable.

The reader is cautioned that the definition of ω-stability is not equivalent to the classical notion (i.e., $S(M)$ countable) but within the context of atomic sets, this revised notion of ω-stability plays an analogous role. The spaces $S_{at}(M)$

are typically not compact. However, if M is countable, then $S_{at}(M)$ is a G_δ subset of the full Stone space $S(M)$, and thus is a Polish space. In particular, if At is not ω-stable, then $S_{at}(M)$ contains a perfect set.

Definition 2.1.2. *1.* A type $p \in S_{at}(M)$ *splits over* $F \subseteq M$ if there are tuples $\mathbf{b}, \mathbf{b}' \subseteq M$ and a formula $\phi(\mathbf{x}, \mathbf{y})$ such that $\mathrm{tp}(\mathbf{b}/F) = \mathrm{tp}(\mathbf{b}'/F)$, but $\phi(\mathbf{x}, \mathbf{b}) \wedge \neg\phi(\mathbf{x}, \mathbf{b}') \in p$.

2. We call $p \in S_{at}(M)$ *constrained* if p does not split over some finite $F \subseteq M$ and *unconstrained* if p splits over every finite subset of M.

3. For any atomic model M, let $C_M := \{p \in S_{at}(M) : p \text{ is constrained}\}$. We say At has *only constrained types* if $S_{at}(N) = C_N$ for every atomic model N.

We use the term constrained in place of 'does not split over a finite subset' for its brevity, which is useful in subsequent definitions.

jb 2/7/24: I found the next paragraph awkward to read: I hope I improved matters.

Remark 2.1.3. The concepts in clauses (2) and (3) above give a method of proving that an atomic class is ω-stable. We show At is ω-stable holds if and only both a) C_M is countable for some/every countable atomic M and b) At has only constrained types. Right to left is well-known: ω-stability immediately implies a) and the deduction of b) is standard [Bal09, Lemma 20.8]. Under the assumption of $\aleph_1$-categoricity, Theorem 2.2.1 gives a) and Theorem 2.4.4 gives three equivalents of b). However, the short proof of Theorem 2.4.4 makes crucial use of Theorem 2.3.2, whose lengthy proof is relegated to Section 3.

The constrained types $p \in C_M$ are those that have a defining scheme over a uniform finite set of parameters, i.e., if $p \in S_{at}(M)$ does not split over $\boldsymbol{a}$, then for every parameter-free $\phi(\mathbf{x}, \mathbf{y})$, there is an $\boldsymbol{a}$-definable formula $d_p\mathbf{x}\phi(\mathbf{x}, \mathbf{y})$ such that for any $\mathbf{b} \in M^{|\mathbf{y}|}$, $\phi(\mathbf{x}, \mathbf{b}) \in p$ if and only if $M \models d_p\mathbf{x}\phi(\mathbf{x}, \mathbf{b})$. We record three easy facts about extensions and restrictions of types.

Lemma 2.1.4. *1. If M is a countable atomic model and $p \in S_{at}(M)$ then p is realized in an atomic extension of M.*

2. For any atomic models $M \preceq N$ and $A \subseteq M$ is finite, then for any $q \in S_{at}(N)$ that does not split over A, the restriction $q\restriction_M$ does not split over A; and any $p \in S_{at}(M)$ that does not split over A has a unique non-splitting extension $q \in S_{at}(N)$.

3. If some atomic N has an unconstrained $p \in S_{at}(N)$, then for every countable $A \subseteq N$, there is a countable $M \preceq N$ with $A \subseteq M$ for which the restriction $p\restriction_M$ is unconstrained.

4. At *has only constrained types if and only if* $S_{at}(M) = C_M$ *for every/some countable atomic model* M.

Proof. (1) Suppose $\boldsymbol{a}$ realizes p in the monster model $\mathcal{M} \succeq M$. $\mathcal{M}$ need not be atomic, but $M \cup \{\boldsymbol{a}\}$ is a countable atomic subset. Since every atomic model N is ω-homogeneous, a 'forth construction' shows that for every countable atomic $S \subseteq \mathcal{M}$, there is an $(\mathcal{M}, N)$-elementary map $f : S \to N$. Thus there is an atomic $M' \succeq M$ containing $\boldsymbol{a}$.

(2) The first statement is immediate. For the second, given $p(\mathbf{x}) \in S_{at}(M)$ non-splitting over A, put

$$q(\mathbf{x}) := \{\phi(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in N^{|\mathbf{y}|}, \phi(\mathbf{x}, \mathbf{b}') \in p \text{ for some } \mathbf{b}' \in M \text{ with } \mathrm{tp}(\mathbf{b}'/A) = \mathrm{tp}(\mathbf{b}/A)\}$$

(3) We construct $M \preceq N$ as the union of an increasing elementary ω-chain $M_n \preceq N$ of countable, elementary substructures of N with $A \subseteq M_0$ and, for each $n \in \omega$, $p\restriction_{M_{n+1}}$ splits over every finite $F \subseteq M_n$. It follows that $M^* := \bigcup\{M_n : n \in \omega\}$ is as required.

(4) Left to right is immediate. For the converse, assume there is some atomic N with an unconstrained type $p \in S_{at}(N)$. By (2) there is a countable $M \preceq N$ with $p\restriction_M$ unconstrained. ■

Much of the paper concerns analyzing atomic models N of size $\aleph_1$. It is useful to consider any such N as a direct limit of a family of countable, atomic submodels.

Definition 2.1.5. For a model N of size $\aleph_1$, a *filtration of* N is a continuous, increasing sequence $(M_\alpha : \alpha \in \omega_1)$ of countable, elementary substructures with $N = \bigcup_{\alpha \in \omega_1} M_\alpha$.

When N is atomic, then in any filtration $(M_\alpha : \alpha \in \omega_1)$ of N, each of the countable models are isomorphic. As well, any two filtrations $(M_\alpha : \alpha \in \omega_1)$ and $(M'_\alpha : \alpha \in \omega_1)$ agree on a club. Thus, for any given countable $M \preceq N$, $\{\alpha \in \omega_1 : M \preceq M_\alpha \text{ and } M_\alpha = M'_\alpha\}$ is club as well.

2.2 $\aleph_1$-categoricity implies C_M is countable

Throughout this subsection, At is an atomic class that admits an uncountable model and M *denotes a fixed copy of the countable atomic model*. We aim to count the set $C_M = \{p \in S_{at}(M) : p \text{ is constrained}\}$. Theorem 2.2.5 yields the main result of the subsection:

Theorem 2.2.1. *If* At *is* $\aleph_1$*-categorical, then* C_M *is countable for every/some countable atomic model* M.

As M is countable, the natural action of $Aut(M)$ on the set M induces an action of $Aut(M)$ on $S_{at}(M)$. When M is atomic, a useful characterization of $p \in C_M$ is: C_M consists of those elements of $S_{at}(M)$ whose orbits are countable. However, for the results in this section we only require the easy half of this statement.

Lemma 2.2.2. *Suppose $p \in C_M$ and M' is any countable, atomic model. Then:*

1. *$\{\pi(p) : \pi : M \to M'$ an isomorphism$\}$ is a countable set of constrained types in $S_{at}(M')$.*

2. *There is a countable atomic $M^* \succ M'$ realizing $\pi(p)$ for every isomorphism $\pi : M \to M'$.*

Proof. (1) Choose a finite $A \subseteq M$ over which p does not split. As M' is countable, A has only countably many images under isomorphisms $\pi : M \to M'$, and it follows immediately from non-splitting that if $\pi_1, \pi_2 : M \to M'$ are isomorphisms satisfying $\pi_1(a) = \pi_2(a)$ for each $a \in A$, then $\pi_1(p) = \pi_2(p)$.

(2) Using (1), let $\{q_i : i < \gamma \leq \omega\} \subseteq S_{at}(M')$ be the set of all images of p under isomorphisms $\pi : M \to M'$. We recursively construct an increasing sequence of countable models $\{M_i : i < \gamma\}$ with $M_0 = M'$ and, for each $i < \gamma$, M_i contains a realization of q_j for every $j < i$. Supposing $i < \gamma$ and M_i has been defined, let $q_i^* \in S_{at}(M_i)$ be the unique ([Bal09, Theorem 19.9]) non-splitting extension of $q_i \in S_{at}(M')$. Then letting d_i realize q_i^*, let $M_{i+1} \in At$ be an elementary extension of M_i containing $M_i \cup \{d_i\}$. Then $\bigcup_{i<\omega} M_i$ works. ∎

Definition 2.2.3. Suppose $(M_\beta : \beta < \omega_1)$ is a filtration of some $N \in \mathrm{At}$ of size $\aleph_1$. For each $\beta < \omega_1$, let

$$R_N^\beta := \{p \in C_M : \pi(p) \text{ is realized in } N \text{ for every isomorphism } \pi : M \to M_\beta\}$$

and let $R_N := \{p \in C_M : p \in R_N^\beta \text{ for a stationary set of } \beta \in \omega_1\}$.

As any two filtrations of N agree on a club, it follows that R_N is independent of the choice of filtration of N. Similarly, R_N is an isomorphism invariant, i.e., if $N \cong N'$ are each atomic models of size $\aleph_1$, then $R_N = R_{N'}$. We record two facts about R_N.

Lemma 2.2.4.

1. *For any $N \in \mathrm{At}$ of size $\aleph_1$, $|R_N| \leq \aleph_1$.*

2. *For any $p \in C_M$ there is some $N \in \mathrm{At}$ of size $\aleph_1$ such that $p \in R_N$.*

Proof. (1) Choose any sequence $\langle p_i : i \in \omega_2\rangle$ from R_N and we will show that $p_i = p_j$ for some distinct i, j. Fix a filtration (M_α) of N. We shrink the sequence in two stages. First, for each $i < \omega_2$, let $\alpha(i) \in \omega_1$ be least such that $p_i \in R_N^{\alpha(i)}$. By pigeonhole and reindexing we may assume $\alpha(i) = \alpha^*$ for all i, i.e., each $p_i \in R_N^{\alpha^*}$. Now fix any isomorphism $\pi : M \to M_{\alpha^*}$. By definition of $R_N^{\alpha^*}$, $\pi(p_i)$ is realized in N for every p_i. But, as $|N| = \aleph_1$, there is $c^* \in N$ realizing both $\pi(p_i)$ and $\pi(p_j)$ for some distinct i, j. Thus, $\pi(p_i) = \pi(p_j)$, hence $p_i = p_j$.

(2) Fix $p \in C_M$. Using Lemma 2.2.2(2) at each level, construct a continuous, increasing elementary sequence M_α of countable atomic models such that, for every $\alpha < \omega_1$, $\pi(p)$ is realized in $M_{\alpha+1}$ for every isomorphism $\pi : M \to M_\alpha$. Put $N := \bigcup_{\alpha<\omega_1} M_\alpha$. Then (M_α) is a filtration of N and $p \in R_N^\alpha$ for every $\alpha < \omega_1$. Thus, $p \in R_N$. ∎

We are now able to prove the theorem below, which clearly implies Theorem 2.2.1.

Theorem 2.2.5. *If C_M is uncountable, then $I(\mathrm{At}, \aleph_1) = 2^{\aleph_1}$.*

Proof. It is easily verified that C_M is an F_σ subset of the Polish space $S_{at}(M)$, so on general grounds, C_M is either countable or else it contains a perfect set.

Our proof is non-uniform, depending on the relative sizes of $2^{\aleph_0}$ and $2^{\aleph_1}$. First, under weak CH, i.e., $2^{\aleph_0} < 2^{\aleph_1}$ then combining arguments of Keisler [Kei70] and Shelah [Bal09, Theorem 18.16] shows if $I(\mathrm{At}, \aleph_1) \neq 2^{\aleph_1}$, then At is ω-stable, so $S_{at}(M)$ is countable. As $C_M \subseteq S_{at}(M)$, C_M is countable as well.

On the other hand, assume $2^{\aleph_0} = 2^{\aleph_1}$, so in particular WCH fails. Under this assumption, we will prove that if C_M is uncountable, then $I(\mathrm{At}, \aleph_1) = 2^{\aleph_0}$, which equals $2^{\aleph_1}$ under our cardinal hypotheses for this case. Indeed, choose representatives $\{N_i : i \in \kappa\}$ for the isomorphism classes of atomic models of size $\aleph_1$. If C_M is uncountable, then as noted the first sentence of the proof, C_M contains a perfect set and so $|C_M| = 2^{\aleph_0}$. But by Lemma 2.2.4, $C_M \subseteq \bigcup\{R_{N_i} : i \in \kappa\}$ and $|R_{N_i}| \leq \aleph_1$ for each $i \in \kappa$. As we are assuming $2^{\aleph_0} > \aleph_1$, we conclude $\kappa \geq 2^{\aleph_0}$, as required. ∎

2.3 Limit types and $\aleph_1$-categoricity

Definition 2.3.1. A type $p \in S_{at}(N)$ is a *limit type* if the restriction $p{\restriction}_M$ is realized in N for every countable $M \preceq N$.

Trivially, for every N, every type in $S_{at}(N)$ realized in N is a limit type. Since we allow $M = N$ in the definition of a limit type, if M is countable, then the only limit types in $S_{at}(M)$ are those realized in M.

Also, if $(M_\alpha : \alpha \in \omega_1)$ is a filtration of N, then a type $p \in S_{at}(N)$ is a limit type if and only if N realizes $p{\restriction}_{M_\alpha}$ for cofinally many α.

The long proof of the following crucial theorem is relegated to Section 3. Note that there are no additional assumptions on At, other than the existence of an uncountable, atomic model.

Theorem 2.3.2. *If* At *admits an uncountable, atomic model, then there is some $N \in \mathrm{At}$ with $|N| = \aleph_1$ for which every limit type in $S_{at}(N)$ is constrained.*

Here, we sharpen this result under the additional assumption of $\aleph_1$-categoricity.

Corollary 2.3.3. *If* At *is $\aleph_1$-categorical and $N \in \mathrm{At}$ has size $\aleph_1$, then for every $p \in S_{at}(N)$, $\in C_N \leftrightarrow p \in \{$limit types in $S_{at}(N)\}$.*

Proof. The hard direction of the equality is Theorem 2.3.2. For the converse, by the assumption of $\aleph_1$-categoricity it suffices to construct **some** $N \in \mathrm{At}$ of size $\aleph_1$ for which every $p \in C_N$ is a limit type. For this, first note that for every countable atomic M, since C_M is countable by Theorem 2.2.1, iterating Lemma 2.1.4(1) ω times yields a countable atomic $M' \succeq M$ that realizes every $p \in C_M$. Using this, construct a strictly increasing, continuous elementary chain $(M_\alpha : \alpha \in \omega_1)$ of countable, atomic models such that for each $\alpha \in \omega_1$, $M_{\alpha+1}$

realizes every $p \in C_{M_\alpha}$. Put $N := \bigcup_{\alpha \in \omega_1} M_\alpha$. We claim that every $p \in C_N$ is a limit type. So fix $p \in C_N$ and choose any countable $M \preceq N$. Choose a finite $A \subseteq N$ for which p does not split over A and choose $\alpha \in \omega_1$ so that $M \cup A \subseteq M_\alpha$. By Lemma 2.1.4(2), $p\restriction_{M_\alpha}$ is constrained, hence it is realized in $M_{\alpha+1} \subseteq N$. As any such realization in N realizes $p\restriction_M$, p is a limit type. ■

2.4 Characterizing ω-stability

In this Subsection, we first derive Lemma 2.4.3 that gives three consequences of ω-stability in terms of the behavior of constrained types. Then, taking Theorem 2.3.2 as a black box (proved in Section 3), Lemma 2.4.4 shows that each of these conditions is equivalent to ω-stability under the assumption of $\aleph_1$-catgoricity. Finally, Theorem 2.4.5 asserts that the existence of a model in $\beth_1^+$ and $\aleph_1$-categoricity implies condition 1) of Lemma 2.4.4 and thus ω-stability.

Definition 2.4.1.

- A *proper constrained pair* is a pair $N \precneqq N'$ of atomic models such that $\mathrm{tp}(\mathbf{c}/N)$ is constrained for every tuple $\mathbf{c} \in N'$.
- A *proper relatively $\aleph_1$-saturated pair* is a proper pair $N \precneqq N'$ such that, for every countable $M \preceq N$, every type $p \in S(M)$ realized in N' is realized in N.

Note that in (2), both models must be uncountable, whereas (1) makes sense for countable models as well. Of course, in (2) it would be equivalent to say that 'every type over every countable set $A \subseteq N$ that is realized in N' is realized in N,' but we choose the definition above to conform with our convention about only looking at types over models.

Lemma 2.4.2. *Let* At *be any atomic class.*

1. *If both (M, M') and (M', M'') are constrained pairs, then (M, M'') is a constrained pair as well.*
2. *If (M, M') is a constrained pair of countable atomic models, then there is an uncountable N with a filtration $(M_\alpha : \alpha \in \omega_1)$ such that (M_α, N) is a constrained pair for every $\alpha \in \omega_1$.*

Proof. (1) Choose any $\mathbf{c} \in M''$. As (M', M'') is a constrained pair, choose $\mathbf{b} \in M'$ such that $\mathrm{tp}(\mathbf{c}/M')$ does not split over $\mathbf{b}$. As (M, M') is a constrained pair, choose $\boldsymbol{a} \in M$ such that $\mathrm{tp}(\mathbf{b}/M)$ does not split over $\boldsymbol{a}$. We claim that $\mathrm{tp}(\mathbf{cb}/M)$ does not split over $\boldsymbol{a}$, which clearly suffices. To see this, choose any $\mathbf{m}_1, \mathbf{m}_2$ from M such that $\mathrm{tp}(\mathbf{m}_1\boldsymbol{a}) = \mathrm{tp}(\mathbf{m}_2\boldsymbol{a})$. By non-splitting, this implies $\mathrm{tp}(\mathbf{m}_1\boldsymbol{a}\mathbf{b}) = \mathrm{tp}(\mathbf{m}_2\boldsymbol{a}\mathbf{b})$. Now both $\mathbf{m}_1\boldsymbol{a}$ and $\mathbf{m}_2\boldsymbol{a}$ are from M', hence $\mathrm{tp}(\mathbf{m}_1\boldsymbol{a}\mathbf{bc}) = \mathrm{tp}(\mathbf{m}_2\boldsymbol{a}\mathbf{bc})$ as $\mathrm{tp}(\mathbf{c}/M')$ does not split over $\mathbf{b}$.

(2) As M is a countable atomic model that is the lower part of a constrained pair, so is any other countable, atomic model. Thus, we can form a continuous, increasing chain $(M_\alpha : \alpha \in \omega_1)$ of countable atomic models with $(M_\alpha, M_{\alpha+1})$ a constrained pair for each α. This chain is a filtration of the atomic $N := \bigcup\{M_\alpha : \alpha \in \omega_1\}$. That each (M_α, N) is a constrained pair follows from (1). ■

We record the following consequences of ω-stability in atomic classes. It is noteworthy that $\aleph_1$-categoricity plays no role in Lemma 2.4.3, and without additional assumptions, none of these imply ω-stability. However, following this, with Theorem 2.4.4 we see that when coupled with $\aleph_1$-categoricity, each of these conditions implies ω-stability.

Lemma 2.4.3. *Suppose* At *is an ω-stable atomic class that admits an uncountable atomic model. Then:*

1. At *has only constrained types;*
2. At *has a proper constrained pair; and*
3. At *has a proper, relatively $\aleph_1$-saturated pair.*

Proof. (1) For an ω-stable atomic class, one can define ([Bal09, Definition 19.1]) a splitting rank on types $p \in S_{at}(N)$ for any model N such that ([Bal09, Theorem 19.8]): for any atomic model N and any $p \in S_{at}(N)$, then choosing $\phi(x, \boldsymbol{a}) \in p$ to be a complete formula of smallest rank, p does not split over $\boldsymbol{a}$. That is, p is constrained.

(2) Choose any countable, atomic model M. Since At admits an uncountable atomic model, there is a countable, proper, atomic elementary extension $M' \succ M$. By (1), $\text{tp}(c/M)$ is constrained for every $c \in M'$, hence (M, M') is a proper constrained pair.

(3) We first argue that there is an *atomically saturated* model N of size $\aleph_1$. That is, for every countable $M \preceq N$, N realizes every $p \in S_{at}(M)$. The existence of an uncountable, atomically saturated N is easy. Using Lemma 2.4.3(1) all types for At are constrained. Then, using 2.1.4(1) and (2) as in the proof of Corollary 2.3.3, build a union of a continuous elementary chain $(M_\alpha : \alpha \in \omega_1)$ of countable atomic models with the property that for each $\alpha < \omega_1$, $M_{\alpha+1}$ realizes every $p \in S_{at}(M_\alpha)$. The existence of such an $M_{\alpha+1}$ is immediate since $S_{at}(M_\alpha)$ is countable and every $p \in S_{at}(M_\alpha)$ can be realized in some countable, atomic elementary extension.

Now, given an atomically saturated model N of size $\aleph_1$, recall that if At is ω-stable, then every model of size $\aleph_1$ has a proper atomic extension N', see e.g., the proof of 19.26 of [Bal09]. But then (N, N') is a proper, relatively $\aleph_1$-saturated pair. ∎

Given Theorem 2.2.1 and Corollary 2.3.3 (the latter depending on the promised Theorem 2.3.2), we give short proofs of our main results.

Theorem 2.4.4. *The following are equivalent for an $\aleph_1$-categorical atomic class* At.

1. At *has a proper, relatively $\aleph_1$-saturated pair;*
2. At *has a proper constrained pair;*
3. At *has only constrained types; and*

4. At *is ω-stable.*

Proof. We will show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), which in light of Lemma 2.4.3 suffices.

(1) $\Rightarrow$ (2) : Suppose (M^*, M^{**}) is a proper, relatively $\aleph_1$-saturated pair of atomic models, and by way of contradiction suppose that (M^*, M^{**}) is not a proper constrained pair. Choose $c \in M^{**}$ such that $p := \mathrm{tp}(c/M^*)$ is unconstrained. Then, by iterating Lemma 2.1.4(3), we construct a continuous, elementary chain $(M_\alpha : \alpha \in \omega_1)$ of countable, elementary substructures of M^* such that, for every $\alpha \in \omega_1$, $p\restriction_{M_\alpha}$ is unconstrained, but is realized in $M_{\alpha+1}$. To accomplish this, by Lemma 2.1.4(3), choose a countable $M_0 \preceq M^*$ such that $p\restriction_{M_0}$ is unconstrained. At countable limits, take unions. Finally, given a countable $M_\alpha \preceq M^*$, by relative $\aleph_1$-saturation choose $c_\alpha \in M^*$ realizing $p\restriction_{M_\alpha}$ and then apply Lemma 2.1.4(3) to the set $M_\alpha \cup \{c_\alpha\}$ to get $M_{\alpha+1} \preceq M^*$ with $p\restriction_{M_{\alpha+1}}$ unconstrained. Let $N := \bigcup\{M_\alpha : \alpha \in \omega_1\}$. Then N has size $\aleph_1$ and the type $p\restriction_N$ is an unconstrained limit type, contradicting $\aleph_1$-categoricity by Corollary 2.3.3.

(2) $\Rightarrow$ (3) : Assume that (N^*, N^{**}) is a proper constrained pair (of any cardinality). By an easy Löwenheim-Skolem argument (in the pair language) there is a proper constrained pair (M, M') of countable atomic models. By Lemma 2.4.2(2), there is an atomic model N of size $\aleph_1$ with a filtration $(M_\alpha : \alpha \in \omega_1)$ such that (M_α, N) is a constrained pair for every $\alpha \in \omega_1$.

Now, by way of contradiction, assume (3) fails. By Lemma 2.1.4(4), $S_{at}(M)$ contains an unconstrained type for every countable atomic model M. Thus, for any such M, there is a countable atomic $M' \succ M$ containing a realization of an unconstrained type. By iterating this ω_1 times, we construct a continuous, elementary chain $(M'_\alpha : \alpha \in \omega_1)$ for which $M'_{\alpha+1}$ contains a realization of an unconstrained type in $S_{at}(M'_\alpha)$. Let $N' := \bigcup\{M'_\alpha : \alpha \in \omega_1\}$. Note that (M'_α, N') is never a constrained pair. But this contradicts $\aleph_1$-categoricity: If $f : N \to N'$ were an isomorphism, then there would be (club many) $\alpha \in \omega_1$ such that $f\restriction_{M_\alpha}$ maps M_α onto M'_α, hence maps the pair (M_α, N) onto (M'_α, N'). As the former is a constrained pair, while the latter is not, we obtain a contradiction.

(3) $\Rightarrow$ (4) : Assume At has only constrained types and let M be any countable, atomic model. This means that $S_{at}(M) = C_M$. However, as At is $\aleph_1$-categorical, C_M is countable by Theorem 2.2.1. Thus, $S_{at}(M)$ is countable, which is the definition of At being ω-stable. ∎

With this result in hand, it is easy to deduce the main theorem. This is the *only* use of the existence of a model in $\beth_1^+$. We imitate the classical proof that every $\kappa \geq |L|$, every L-theory with an infinite model has a κ^+-saturated model of size 2^κ, to prove clause 1) of Lemma 2.4.4 and thus deduce ω-stability.

Theorem 2.4.5. *If an atomic class* At *is $\aleph_1$-categorical and has a model of size $(2^{\aleph_0})^+$, then* At *is ω-stable.*

Proof. Let M^{**} be an atomic model of size $(2^{\aleph_0})^+$. We construct a relatively $\aleph_1$-saturated elementary substructure $M^* \preceq M^{**}$ of size $2^{\aleph_0}$ as the union of a continuous chain $(N_\alpha : \alpha \in \omega_1)$ of elementary substructures of M^{**}, each of size $2^{\aleph_0}$, where, for each $\alpha < \omega_1$ and each of the $2^{\aleph_0}$ countable $M \preceq N_\alpha$, $N_{\alpha+1}$ realizes each of the at most $2^{\aleph_0}$ $p \in S(M)$ that is realized in M^{**}. ω-stability is immediate from (1) $\Rightarrow$ (4) in Lemma 2.4.4. ∎

3 Paying our debt

The whole of this section is aimed at proving Theorem 2.3.2: If a countable theory T has an uncountable atomic model, then it has one in which every limit type is constrained.[2] The proof relies heavily on Keisler's completeness theorem that implies 'model existence' of sentences of $L_{\omega_1,\omega}(Q)$ is absolute between forcing extensions. In the first subsection, we explicitly give an $L_{\omega_1,\omega}(Q)$ sentence Ψ^* in a countable language extending the language of T such that in any set-theoretic universe, Ψ^* has a model of size $\aleph_1$ if and only if there is an atomic model of size $\aleph_1$ with every limit type constrained.

The second subsection describes family of striated formulas ([BLS16]). Such formulas are used to describe a c.c.c. forcing notion $(\mathbb{P}, \leq)$ in the third subsection. There, we prove that $(\mathbb{P}, \leq)$ forces the existence of an atomic model of size $\aleph_1$ with every limit type constrained. Thus, we conclude that Ψ^* has a model of size $\aleph_1$ in a c.c.c. forcing extension, so by the absoluteness described above, $\mathbb{V}$ has a model of Ψ^* of size $\aleph_1$, yielding our requested model.

3.1 Finding a requisite sentence Ψ^* of $L_{\omega_1,\omega}(Q)$

This subsection is devoted to proving the following Proposition.

Proposition 3.1.1. *Let T be a first order L-theory for a countable language with an uncountable model in* At*, the class of atomic models. There is a sentence $\Psi^* \in (L^*)_{\omega_1,\omega}(Q)$ in an expanded (but still countable) language $L^* \supseteq L$ for which the following are equivalent:*

1. *There is a model $N^* \models \Psi^*$; and*
2. *There is an atomic model $N \models T$ of size $\aleph_1$ such that every limit type of N is constrained.*

Whereas the L-reduct of any $N^* \models \Psi^*$ will satisfy (2), it is noteworthy that in proving (2) $\Rightarrow$ (1), the model $N^* \models \Psi^*$ we produce is not necessarily an expansion of a given N witnessing (2).

The relevant Ψ^* is defined in Definition 3.1.6. As we will be interested in arbitrary models of a sentence and because "is a well ordering" is not expressible in $L_{\omega,\omega}(Q)$, we need to generalize the notion of a filtration.

[2]By the correspondence described in the introduction, it follows immediately that any complete $L_{\omega_1,\omega}$-sentence with an uncountable model has an uncountable model with every limit type constrained.

Definition 3.1.2. A linear order $(I, \leq)$ is ω_1-*like* if it has cardinality $\aleph_1$, but, letting pred(i) denote $\{j \in I : j < i\}$, for every $i \in I$, $|\operatorname{pred}(i)| \leqslant \aleph_0$.

If N is any set and $(I, \leq)$ is ω_1-like, then an $(I, \leq)$-*scale* is a surjective function $f : N \to I$ such that $f^{-1}(i)$ is countable for every $i \in I$.

If $f : N \to I$ is a scale, put $A_i := f^{-1}(\operatorname{pred}(i))$ for every $i \in I$, and note that each A_i is countable.

Note that 'being an ω_1-like linear order' is expressible by a sentence of $L_{\omega_1,\omega}(Q)$ – the point is that any uncountable linear order $(I, \leq)$ for which pred(i) is countable for every $i \in I$ has both size and cofinality $\aleph_1$. Similarly, if an uncountable set N has an $(I, \leq)$-scale, then N must have size $\aleph_1$.

We consider the sets $(A_i : i \in I)$ to be a surrogate for a filtration of N; A_i replaces M_α. We now define a tree order on types over certain countable subsets of a model with cardinality $\aleph_1$ of T.

Definition 3.1.3. Fix T, N as in Proposition 3.1.1. Suppose $(I, \leq)$ is an ω_1-like linear order and $f : N \to I$ is a scale.

1. Define an equivalence relation E_f on $(N \times I)$ as $(a, i)E_f(b, j)$ if and only if $i = j$ and $\operatorname{tp}(a/A_i) = \operatorname{tp}(b/A_i)$. Thus each equivalence class corresponds to a type.

2. Define a strict partial order $\prec_f$ on $(N \times I)/E_f$ as: $[(a, i)] \prec_f [(b, j)]$ if and only if $i <_I j$; $\operatorname{tp}(a/A_i) = \operatorname{tp}(b/A_i)$; and $\operatorname{tp}(b/A_j)$ splits over every finite $F \subseteq A_i$.

3. A $\prec_f$-*chain* is a sequence of types linearly ordered by $\prec_f$ (hence splitting).

It is evident that $((N \times I)/E_f, \prec_f)$ is *tree-like* in that the $\prec_f$-predecessors of every E_f-class are linearly ordered by $\prec_f$. Moreover, since $(I, \leq)$ is ω_1-like, every E_f-class has only countably many $\prec_f$-predecessors.

Lemma 3.1.4. *Let N be any atomic model of size $\aleph_1$, $(I, \leq)$ be ω_1-like, $f : N \to I$ be any scale and I, E_f, A_i, and $\prec_f$ be as in Definition 3.1.3. The following are equivalent:*

1. *there exists an f such that $\mathcal{T}_f = ((N \times I)/E_f, \prec_f)$ has an uncountable $\prec_f$-chain;*

2. *Some limit type in $S_{at}(N)$ is unconstrained;*

3. *for every f, $\mathcal{T}_f = ((N \times I)/E_f, \prec_f)$ has an uncountable $\prec_f$-chain.*

Proof. (3) $\Rightarrow$ (1) is immediate. For (1) $\Rightarrow$ (2), suppose for some f, $C \subseteq \mathcal{T}_f$ is an uncountable $\prec_f$-chain. As $[(a, i)] \prec_f [(b, j)]$ implies $i < j$ and since $(I, \leq)$ is ω_1-like, $\pi_2(C) := \{i \in I : \exists a \in N[(a, i)] \in C\}$ is cofinal in I, hence $\bigcup\{A_i : i \in \pi_2(C)\} = N$. Also, as $[(a, i)] \prec_f [(b, j)]$ implies $\operatorname{tp}(a/A_i) = \operatorname{tp}(b/A_i)$, there is a unique $p \in S_{at}(N)$ defined as $p := \bigcup\{\operatorname{tp}(a/A_i) : (a, i) \in C\}$. Furthermore, as $\operatorname{tp}(b/A_j)$ splits over every finite $F \subseteq A_i$, it follows that p is unconstrained. Recalling Definition 2.3.1(2), it remains to show that p is a limit type. Choose a filtration $\overline{M} = (M_\alpha)$ of N and argue that $p\restriction_{M_\alpha}$ is realized in N for every $\alpha \in \omega_1$. Given $\alpha \in \omega_1$, choose $i \in \pi_2(C)$ such that $M_\alpha \subseteq A_i$. Then each $a \in N$ for which $(a, i) \in C$ realizes $p\restriction_{A_i}$ and hence realizes $p\restriction_{M_\alpha}$. So p is a limit type.

(2) $\Rightarrow$ (3). Suppose N has an unconstrained limit type $p \in S_{at}(N)$ and fix a scale f. Also choose a filtration $(M_\alpha : \alpha \in \omega_1)$ of N. To construct an uncountable chain $\mathcal{T}_f$ we repeatedly use the following claim.

Claim 3.1.5. *For every countable $B \subseteq N$ there is $i \in I$ such that*

- $B \subseteq A_i$;
- $p\restriction_{A_i}$ *is realized; and*
- $p\restriction_{A_i}$ *splits over every finite* $F \subseteq B$.

Proof. Given a countable $B \subseteq N$, since $p \in S_{at}(N)$ splits over every finite $F \subseteq N$, there is a countable $B^* \supseteq B$ such that $p\restriction_{B^*}$ splits over every finite $F \subseteq B$. Now choose $i \in I$ such that $B^* \subseteq A_i$ and then choose $\alpha \in \omega_1$ such that $A_i \subseteq M_\alpha$. Since p is a limit type, choose $c \in N$ realizing $p\restriction_{M_\alpha}$ and hence $p\restriction_{A_i}$. ■

Iterating Claim 3.1.5 ω_1 times yields a strictly increasing sequence $(i_\alpha : \alpha \in \omega_1)$ from $(I, \leq)$ and $(c_\alpha : \alpha \in \omega_1)$ from N, where at each stage α, we take $B = \bigcup\{A_{i_\beta} : \beta < \alpha\}$. It follows directly from the definition of $\prec_f$ that $(c_\beta, i_\beta) \prec_f (c_\alpha, i_\alpha)$ whenever $\beta < \alpha$, so $((N \times I)/E, \prec_f)$ has an uncountable chain. ■

With Lemma 3.1.4 in hand, we now define the sentence Ψ^* described in Proposition 3.1.1.

> jb 2/8/24: There is a confusion between N^* and its reduct N in the statement of Proposition 3.1.1, Definition 3.1.6, and the 'proof Proposition 3.1.1. I think this is what prompted the referee's objection to 'model' vs 'atomic model' that I addressed poorly in the first paragraph of 3.2 – you properly deleted by non-fix. In the statement of 3.1.1, N^* is an L^*-structure, while in Definition 3.1.6, N is an L^*-structure with an unnamed reduct. The second sentence 'proof of Proposition 3.1.1' contained a proof of the tautology that a reduct has the same cardinality as the structure; I deleted that. But I have been thinking in circles for 1/2 hour about the fewest number of changes to avoid this confusion. I propose but did not do:
> 2nd line of Definition 3.1.6: $N \models \psi^*$ becomes $N^* \models \Psi^*$ with universe N.
> I did correct several typos toward the end of 'proof of Proposition 3.1.1'.

Definition 3.1.6. Let $L^* := L \cup \{I, \leq_I, f, E, \prec_f\} \cup \{\mathbb{Q}, \leq_{\mathbb{Q}}, H\}$ and let Ψ^* be a set of $L_{\omega_1,\omega}(Q)$-axioms ensuring that for any $N \models \Psi^*$:

1. The L-reduct of N is an atomic model of T;
2. N is uncountable;

3. $I \subseteq N$ and $(I, \leq_I)$ is an ω_1-like linear order;

4. $f : N \to I$ is a scale; (recall: $A_i := f^{-1}(\text{pred}(i))$);

5. $E \subseteq N \times I$ satisfies $(a, i)E(b, j)$ iff $i = j$ and $\text{tp}_L(a/A_i) = \text{tp}_L(b/A_i)$;

6. For all $[(a, i)], [(b, j)] \in (N \times I)/E$, $[(a_i)] \prec_f [(b, j)]$ iff $i < j$, $\text{tp}_L(a/A_i) = \text{tp}_L(b/A_i)$, and $\text{tp}_L(b/A_j)$ splits over every finite $F \subseteq A_i$;

7. $\mathbb{Q} \subseteq N$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is a countable model of DLO;

8. $H : N \times I \to \mathbb{Q}$ satisfies: For all $(a, i), (b, j)$,
   (a) If $(a, i)E(b, j)$ then $H(a, i) = H(b, j)$; and
   (b) If $[(a, i)] \prec_f [(b, j)]$, then $H(a, i) <_{\mathbb{Q}} H(b, j)$.

We verify that this sentence Ψ^* works for Proposition 3.1.1.

Proof of Proposition 3.1.1

For (1) $\Rightarrow$ (2) assume $N^* \models \Psi^*$ and let N be the L-reduct of N^*. Then$|N| = \aleph_1$, $(I, \leq)$ is an ω_1-ordering and $f : N \to I$ is a scale. Moreover, as the ordering on $(\mathbb{Q}, \leq)$ forbids a strictly increasing ω_1 sequence, the existence of the function H forbids $T = ((N \times I)/E, \prec_f)$ having an uncountable $\prec_f$-chain. Thus, by Lemma 3.1.4, every limit type in $S_{at}(N)$ is constrained.

The converse is more involved. Assume we are given $N \in \text{At}$ of size $\aleph_1$ with every limit type in $S_{at}(N)$ constrained. Under this assumption, with the help of Lemma 3.1.7 we will show that a model $N^* \models \Psi^*$ can be found in some generic extension $\mathbb{V}[G]$ of $\mathbb{V}$ by a c.c.c. forcing extension. Once we have that, it follows by the absoluteness gleaned from Keisler's model existence theorem for sentences of $L_{\omega_1,\omega}(Q)$ that a model of Ψ^* exists in $\mathbb{V}$, giving 3.1.1.(1).

So, given N as in Proposition 3.1.1.(2), choose arbitrary subsets $I, \mathbb{Q} \subseteq N$ of cardinality $\aleph_1, \aleph_0$, respectively and choose orderings $\leq_I$ and $\leq_{\mathbb{Q}}$ as required by Ψ^*. Fix an arbitrary scale $f : N \to I$ and interpret E and $\prec_f$ as required. Since N has every limit type constrained, it follows from Lemma 3.1.4 that $\prec_f$ has no uncountable chains.

It only remains to find a function $H : N \times I \to \mathbb{Q}$ as requested by Ψ^*. For this, we turn to forcing, and invoke the following general Lemma[3], taking X to be $(N \times I/E)$ and $\prec$ to be $\prec_f$.

Lemma 3.1.7. *Suppose $(X, \prec)$ is any strict partial order satisfying*

1. $|X| = \aleph_1$;

2. for every $a \in X$, the induced suborder $(\text{pred}(a), \prec)$ is a countable linear order; and

3. $(X, \prec)$ has no uncountable chain.

[3]The statement of Lemma 3.1.7 is reminiscent of how one specializes Aronszajn trees by forcing, and the ideas of the proof can be found in Section 2 of [Bau70].

Then there is a c.c.c. forcing $(\mathbb{P}, \leq)$ such that in any generic $\mathbb{V}[G]$ there is a function $H : X \to \mathbb{Q}$ such that if $a \prec b$, then $H(a) <_{\mathbb{Q}} H(b)$.

Proof. The partial order $(\mathbb{P}, \leq)$ is simply the set of all finite approximations of such an H. That is, $\mathbb{P}$ is the set of all functions $h : X_0 \to \mathbb{Q}$ with $X_0 \subseteq X$ finite such that for all $a, b \in X_0$, if $a \prec b$, then $h(a) <_{\mathbb{Q}} h(b)$, ordered by inclusion, i.e., ($\sharp$) $h \leq h'$ if and only if $h \subseteq h'$. It is easily checked that this forcing will produce (in $\mathbb{V}[G]$) a total function $H : X \to \mathbb{Q}$ as desired. The non-trivial part is showing that $(\mathbb{P}, \leq)$ has the c.c.c. For this, choose any uncountable set $Y = \{h_\alpha : \alpha \in \omega_1\} \subseteq \mathbb{P}$ and assume, by way of contradiction, that $h_\alpha \cup h_\beta \notin \mathbb{P}$ for distinct $\alpha, \beta \in \omega_1$. By passing to a subset of Y, we may assume $|\mathrm{dom}(h_\alpha)| = n$ for some fixed $n \in \omega$ and we argue by contradiction. If $n = 1$, i.e., $\mathrm{dom}(h_\alpha) = \{a^\alpha\}$, then by passing to a further subset, there is a single $m^* \in \mathbb{Q}$ such that $h_\alpha(a^\alpha) = m^*$ for every α. The only way we could have $h_\alpha \cup h_\beta \notin \mathbb{P}$ would be if a^α, a^β were distinct, but $\prec$-comparable. But then $C = \{a^\alpha : \alpha \in \omega_1\}$ would be an uncountable chain in $(X, \prec)$, contradicting our assumption.

So, assume $n > 1$ and we have proved (c.c.c.) for all $n' < n$. To ease notation, enumerate the universe X with order type ω_1. For each α, write $\mathrm{dom}(h_\alpha) = (a_1^\alpha, \ldots, a_n^\alpha)$ in increasing order, subject to this enumeration. By the Δ-system lemma, there is an uncountable subset and a root r such that $\mathrm{dom}(h_\alpha) \cap \mathrm{dom}(h_\beta) = r$ for all distinct pairs α, β. If $r \neq \emptyset$, we can apply our inductive hypothesis to the family of sets $\{\mathrm{dom}(h_\alpha) \setminus r : \alpha \in \omega_1\}$, so we may assume $r = \emptyset$, i.e., the domains $\{\mathrm{dom}(h_\alpha) : \alpha \in \omega_1\}$ are pairwise disjoint. Again, passing to a subsequence, we may assume that with respect to the global enumeration of X $a_n^\alpha < a_1^\beta$ for all $\alpha < \beta$. Additionally, we may assume there are $\{m_1, \ldots, m_n\} \subseteq \mathbb{Q}$ such that $h_\alpha(a_i^\alpha) = m_i$ for all $\alpha \in \omega_1$ and $i \in \{1, \ldots, n\}$.

Now fix $\alpha < \beta$. In order for $h_\alpha \cup h_\beta$ to not be in $\mathbb{P}$, there must be some $p(\alpha, \beta), q(\alpha, \beta) \in \{1, \ldots, n\}$ such that $a_{p(\alpha,\beta)}^\alpha$ and $a_{q(\alpha,\beta)}^\beta$ are $\prec$-comparable. As a bookkeeping device, fix a uniform[4] ultrafilter $\mathcal{U}$ on ω_1.

Thus, for any $\alpha \in \omega_1$, there is some $S_\alpha \in \mathcal{U}$, some $p(\alpha), q(\alpha) \in \{1, \ldots, n\}$ such that, by ($\sharp$), for every $\beta \in S_\alpha$, $a_{p(\alpha)}^\alpha$ and $a_{q(\alpha)}^\beta$ are $\prec$-comparable. However, since $pred(a_{p(\alpha)}^\alpha)$ was assumed to be countable, there is $S_\alpha^* \subseteq S_\alpha$, $S_\alpha^* \in \mathcal{U}$ such that $a_{p(\alpha)}^\alpha \prec a_{q(\alpha)}^\beta$ for all $\beta \in S_\alpha^*$.

Similarly, there is some $S \in \mathcal{U}$ and some $p^*, q^* \in \{1, \ldots, n\}$ such that for all $\alpha \in S$ and for all $\beta \in S_\alpha^*$ we have $a_{p^*}^\alpha \prec a_{q^*}^\beta$. We obtain our contradiction by showing that

$$C = \{a_{p^*}^\alpha : \alpha \in S\}$$

is an uncountable chain in $(X, \prec)$. Since $\mathcal{U}$ is uniform, C is uncountable. To get comparability, choose any $\alpha, \gamma \in S$. As $S_\alpha^*, S_\gamma^* \in \mathcal{U}$, there is $\beta \in S_\alpha^* \cap S_\gamma^*$. It follows that $a_{p^*}^\alpha \prec a_{q^*}^\beta$ and $a_{p^*}^\gamma \prec a_{q^*}^\beta$. From our assumptions on $(X, \prec)$, $(\mathrm{pred}(a_{q^*}^\beta), \prec)$ is a linear order, so $a_{p^*}^\alpha$ and $a_{p^*}^\gamma$ are $\prec$-comparable. ∎

[4] That is, every $Y \in \mathcal{U}$ has cardinality $\aleph_1$. Equivalently, $\mathcal{U}$ contains all of the co-countable subsets of ω_1.

3.2 Extendible and striated formulas

Throughout this section, we work with the atomic models of a complete, first order theory T in a countable language that has an uncountable atomic model. We expound model theoretic properties needed in the forcing construction of Section 3.3.

Remark 3.2.1. In this section we work with complete formulas $\theta(\mathbf{w})$, usually with a prescribed partition of the free variables. Regardless of the partition, for any subsequence $\mathbf{v} \subseteq \mathbf{w}$, we use the notation $\theta\restriction_{\mathbf{v}}$ to denote the complete formula in the variables $\mathbf{v}$ that is equivalent to $\exists \mathbf{u}\theta(\mathbf{v}, \mathbf{u})$ where $\mathbf{u} = (\mathbf{w} \setminus \mathbf{v})$.

Definition 3.2.2.

1. A complete formula $\phi(x, \boldsymbol{a})$ is *pseudo-algebraic*[5] if for some/any countable M with $\boldsymbol{a} \in M$ and any $N \succneqq M$, $\phi(N, \boldsymbol{a}) = \phi(M, \boldsymbol{a})$.
2. $b \in \mathrm{pcl}(\boldsymbol{a}, M)$, written $b \in \mathrm{pcl}(\boldsymbol{a})$ if and only if every $N \preceq M$ with $\boldsymbol{a} \in N$, $b \in N$.
3. A complete formula $\theta(\mathbf{z}; \mathbf{x})$ is *extendible* if there is a pair $M \preceq N$ of countable, atomic models and $\mathbf{b} \subseteq M$, $\boldsymbol{a} \subseteq N \setminus M$ such that $N \models \theta(\mathbf{b}, \boldsymbol{a})$.

Note that an atomic class has an uncountable model if and only if it has a non-pseudo-algebraic type.

The definition of an extendible formula depends on the partition of its free variables. As we require extendible formulas to be complete, they are not preserved under adjunction of dummy variables. If $\lg(\mathbf{x}) = 1$, then $\theta(\mathbf{z}, x)$ being extendible is equivalent to it being complete, with $\theta(\mathbf{z}, x)$ not pseudo-algebraic. Much of the utility of the notion is given by the following fact.

Fact 3.2.3.

1. *If $\theta(\mathbf{z}; \mathbf{x})$ is extendible, then for any countable, atomic M and any $\mathbf{b} \in M^{\lg(\mathbf{z})}$ and $\boldsymbol{a} \in M^{\lg(\mathbf{x})}$ such that $M \models \theta(\mathbf{b}, \boldsymbol{a})$, there is $M_0 \preceq M$ such that $\mathbf{b} \subseteq M_0$ and $\boldsymbol{a} \subseteq M \setminus M_0$.*
2. *If $\theta(\mathbf{z}; \mathbf{x})$ is extendible and $\mathbf{z}' \subseteq \mathbf{z}$ and $\mathbf{x}' \subseteq \mathbf{x}$, then the restriction $\theta\restriction_{\mathbf{z}';\mathbf{x}'}$ is extendible as well.*
3. *Any complete formula $\theta(\mathbf{z}; \mathbf{x})$ is extendible if and only if $\theta\restriction_{\mathbf{z},x_i}$ is not pseudo-algebraic for every $x_i \in \mathbf{x}$.*

Proof. (1) As $\theta(\mathbf{z}; \mathbf{x})$ is extendible, choose countable atomic models $M' \preceq N'$, $\mathbf{b}' \subseteq M$ and $\boldsymbol{a}' \subseteq N' \setminus M'$ such that $N' \models \theta(\mathbf{b}', \boldsymbol{a}')$. As $\theta(\mathbf{z}; \mathbf{x})$ is complete, there is an isomorphism $f : N' \to M$ with $f(\mathbf{b}') = \mathbf{b}$ and $f(\boldsymbol{a}') = \boldsymbol{a}$. Then $M_0 := f(M')$ is as desired.

(2) This follows easily from the proof of (1).

(3) Left to right follows easily from (2). We prove the converse by induction on $\lg(\mathbf{x})$. For $\lg(\mathbf{x}) = 1$ this is immediate, so assume this holds when $\lg(\mathbf{x}) = n$. Choose a complete $\theta(\mathbf{z}; \mathbf{x}, x_n)$ such that $\lg(\mathbf{x}) = n$ and $\theta\restriction_{\mathbf{z},x_i}$ is non-pseudoalgebraic for each $i \leq n$. Choose any countable, atomic N and $\mathbf{b}, \boldsymbol{a}, a_n$

[5]The careful distinctions of pseudoalgebraicity 'in a model' of [BLS16] are avoided because we have assumed there is an uncountable atomic model.

from N so that $N \models \theta(\mathbf{b}, \boldsymbol{a}, a_n)$. By (1), it suffices to find some $M_0 \preceq N$ with $\mathbf{b} \subseteq M_0$ and $\boldsymbol{a}a_n \subseteq N \setminus M_0$. To obtain this, since $\exists x_n \theta(\mathbf{z}; \mathbf{x}, x_n)$ is extendible by (2), (1) implies there is $M \preceq N$ with $\mathbf{b} \subseteq M$ and $\boldsymbol{a} \subseteq N \setminus M$. Thus, if $a_n \in N \setminus M$, we can take $M_0 := M$ and we are done. If not, then as $\mathbf{b}a_n \subseteq M$ we can apply (1) to M and the extendible $\exists \mathbf{x}\theta(\mathbf{z}; \mathbf{x}, x_n)$ to get $M_0 \preceq M$ with $\mathbf{b} \subseteq M_0$ and $a_n \in M \setminus M_0$. ∎

Next, we consider the 'transitive closure' of extendibility.

Definition 3.2.4. An *n-striated formula* is a complete formula $\theta(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$ whose free variables are partitioned into n pieces such that, for every $i < n$, letting $\mathbf{z} = (\mathbf{y}_0, \ldots, \mathbf{y}_i)$ and $\mathbf{x} = (\mathbf{y}_i, \ldots, \mathbf{y}_{n-1})$, we have $\theta(\mathbf{z}, \mathbf{x})$ extendible.

A *striated formula* is an n-striated formula for some n.

A *realization* of an n-striated formula $\theta(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$ is an n-chain $M_0 \preceq M_1 \preceq M_{n-1}$ of countable, atomic models, together with tuples $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}$ with $\boldsymbol{a}_0 \subseteq M_0$ and $\boldsymbol{a}_i \subseteq M_i \setminus M_{i-1}$ for every $0 < i < n$ such that $M_{n-1} \models \theta(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1})$.

Iterating Fact 3.2.3, we see that a partitioned complete formula $\theta(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$ is n-striated if and only if for some countable atomic M and some $(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1})$ from M with $M \models \theta(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1})$, there are $M_0 \preceq M_1 \preceq \cdots \preceq M_{n-2} \preceq M$ with $\boldsymbol{a}_0 \subseteq M_0$, $\boldsymbol{a}_i \subseteq M_i \setminus M_{i-1}$ for $0 < i < n-2$ and $\boldsymbol{a}_{n-1} \cap M_{n-2} = \emptyset$.

Using this characterization, if $\theta(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$ is n-striated and we modify the partition of θ by fusing together two adjacent tuples, then the resulting partition yields an $(n-1)$-striated formula. Going forward, we have the following amalgamation property for striated formulas.

Lemma 3.2.5. *Suppose $\alpha(\mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\beta(\mathbf{z}, \mathbf{y}_1, \ldots, \mathbf{y}_m)$ are striated and $\alpha\restriction_{\mathbf{z}}$ is equivalent to $\beta\restriction_{\mathbf{z}}$. Then there is a striated $\psi(\mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_m)$ extending $\alpha(\mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_n) \wedge \beta(\mathbf{z}, \mathbf{y}_1, \ldots, \mathbf{y}_m)$.*

Proof. Choose an $(n+1)$-chain $M_0 \preceq M_1 \preceq \cdots \preceq M_n$ and $\mathbf{b}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ realizing α (so $\mathbf{b} \subseteq M_0$ and $\boldsymbol{a}_i \subseteq M_i \setminus M_{i-1}$ for each i) and choose similarly an $(m+1)$-chain $N_0 \preceq N_1 \preceq \cdots \preceq N_m$ and $\mathbf{c}, \mathbf{d}_1, \ldots, \mathbf{d}_m$ realizing β. As $\alpha\restriction_{\mathbf{z}}$ is equivalent to $\beta\restriction_{\mathbf{z}}$, there is an isomorphism $f : N_0 \to M_n$ with $f(\mathbf{c}) = \mathbf{b}$. Choose $M_{n+m} \succeq M_n$ for which there is an isomorphism $f^* : N_m \to M_{n+m}$ extending f. Now, for $i \leq m$ put $M_{n+i} := f^*(N_i)$. [Note this is compatible with our previous placements.] Also, for each $1 \leq i \leq m$, put $\boldsymbol{a}_{n+i} := f^*(\mathbf{d}_i)$. Finally, put $\psi(\mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_m) := \mathrm{tp}(\mathbf{b}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n+m})$. Then the $(n+m+1)$-chain $M_0 \preceq \cdots \preceq M_{n+m}$, together with $\mathbf{b}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n+m}$ witness that ψ is striated. ∎

3.3 The forcing

We continue our assumption that we have a fixed complete theory T in a countable language with an uncountable atomic model. We fix an ω_1-like dense linear order $(I, \leq)$ with least element 0 and fix a continuous, increasing (necessarily cofinal) sequence $\langle J_\alpha : \alpha \in \omega_1 \rangle$ of initial segments of I. Also, fix a set $X = \{x_{t,m} : t \in I, m \in \omega\}$ of distinct variable symbols and, for each $\alpha \in \omega_1$, let $X_\alpha = \{x_{t,m} : t \in J_\alpha, m \in \omega\}$. Our forcing below will describe a complete diagram in the variables X corresponding to an atomic model N of size $\aleph_1$ and the countable substructures N_α corresponding to the variables X_α will be a filtration of N.

Definition 3.3.1. The forcing $(\mathbb{P}, \leq)$ consists of all conditions

$$p = (u_p, \ell(p), \{k_{p,i} : i < \ell(p)\}, \theta_p(\mathbf{y}_0, \ldots, \mathbf{y}_{\ell(p)-1}))$$

satisfying the following properties:

1. u_p is a finite subset $\{s_0, \ldots, s_{\ell(p)-1}\} \subseteq I$. We always write the elements of u_p in ascending order.
2. $\ell(p) = |u_p|$;
3. If $u_p \neq \emptyset$, then $0 \in u_p$;
4. Each $k_{p,i} \in \omega$ and denotes $\lg(\mathbf{y}_i)$ in θ_p;
5. $\theta_p(\mathbf{y}_0, \ldots, \mathbf{y}_{\ell(p)-1}))$ is an $\ell(p)$-striated formula, where each $\mathbf{y}_i = (\mathbf{x}_{s_i,j} : j < k_{p,i})$ is the initial segment of the s_i'th row of X of length $k_{p,i}$.

The ordering on $\mathbb{P}$ is natural, i.e., $p \leq_{\mathbb{P}} q$ if and only if $u_p \subseteq u_q$, the free variables of θ_p are contained in the free variables of θ_q and $\theta_q \vdash \theta_p$.

We remark that the effect of requiring $0 \in u_p$ whenever u_p is non-empty is to ensure that if θ_p entails '$x_{\alpha_i,j} \in \mathrm{pcl}(\emptyset)$', then $\alpha_i = 0$. That is, in the generic model we construct, all pseudo-algebraic complete types of singletons will be contained in M_0.

It is easily verified that $(\mathbb{P}, \leq)$ is c.c.c. (See [BLS15, Claim 4.3.7] for a verification of this in an extremely similar setting.) As well, $(\mathbb{P}, \leq)$ is highly homogeneous. In particular, we record the following facts, with (1) following from $(I, \leq)$ being dense and ω_1-like.

Fact 3.3.2.

1. *For all $\alpha < \omega_1$ and for all finite $u_1, u_2 \subseteq I \setminus J_\alpha$ with $|u_1| = |u_2|$ and $min(I \setminus J_\alpha)$ (if it exists) $\notin u_1 \cup u_2$, then there is an order isomorphism $\sigma \in Aut(I, \leq)$ with $\sigma(u_1) = u_2$ and $\sigma \restriction_{J_\alpha} = id$.*
2. *Any order isomorphism $\sigma \in Aut(I, \leq)$ induces both a permutation $\sigma' \in Sym(X)$ given by $\sigma'(x_{t,m}) = x_{\sigma(t),m}$ and an automorphism $\sigma^* \in Aut(\mathbb{P}, \leq)$ given by*

$$\sigma^*(p) = (\sigma(u_p), \ell_p, \{k_{p,i} : i < \ell(p)\}, \theta_p(\sigma'(\mathbf{y}_0), \ldots, \sigma'(\mathbf{y}_{\ell(p)-1}))$$

We record three additional density conditions about $(\mathbb{P}, \leq)$ whose verifications depend on the following fact.

Lemma 3.3.3. *Suppose $\delta(x)$ is a non-pseudoalgebraic 1-type. Then for every countable atomic N and every $\mathbf{e} \subseteq N$, there are $M \preceq N$ and $c \in N \setminus M$ such that $\mathbf{e} \subseteq M$ and $N \models \delta(c)$.*

Proof. From the definition of (non)-pseudoalgebraicity, fix countable atomic $M^* \preceq N^*$ and $c^* \in N^* \setminus M^*$ with $N^* \models \delta(c^*)$. Choose any isomorphism $f : N \to M^*$ and put $\mathbf{e}^* := f(\mathbf{e})$. Now, choose an isomorphism $g : N^* \to N$ with $g(\mathbf{e}^*) = \mathbf{e}$. Put $M := g(M^*)$ and $c := g(c^*)$. Then $\mathbf{e} \subseteq M$, $c \in N \setminus M$, and $N \models \delta(c)$. ∎

The forcing is surjective in the sense that for every condition p and every variable there is an extension of p that includes the variable.

Lemma 3.3.4 (**Surjective**). *For every $p \in \mathbb{P}$ and $x_{t,m} \in X$, there is $q \in \mathbb{P}$, $q \geq p$ with $\mathbf{x}_q = \mathbf{x}_p \cup \{x_{t,m}\}$.*

Proof. We may assume that $p \neq 0$ and that $x_{t,m} \notin \mathbf{x}_p$. Choose $M_0 \preceq M_1 \preceq \cdots \preceq M_{n-1}$ and $\mathbf{e}_0 \ldots \mathbf{e}_{n-1}$ realizing θ_p (so $\mathbf{e}_0 \subseteq M_0$, $\mathbf{e}_i \subseteq M_i \setminus M_{i-1}$ for $0 < i < n$ and $M_{n-1} \models \theta(\mathbf{e}_0, \ldots, \mathbf{e}_{n-1})$.

We first handle the case where $m = 0$. In this case, it must be that $t \notin u_p$. Choose j maximal such that $s_j < t$. Apply Fact 3.3.3 to M_j and $\mathbf{e}_0 \ldots \mathbf{e}_j$ to get $M_j^* \preceq M_j$ and $c \in M_j \setminus M_j^*$ with $M_j \models \delta(c)$ and $\mathbf{e}_0 \ldots, \mathbf{e}_j \subseteq M_j^*$. Now let $f : M_j \to M_j^*$ be an isomorphism fixing $\mathbf{e}_0 \ldots, \mathbf{e}_j$ pointwise. Then the type $\mathrm{tp}(\mathbf{e}_0, \ldots, \mathbf{e}_j, c, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_{n-1})$ and the $(n+1)$-chain $f(M_0) \preceq \ldots f(M_j) \preceq M_j \preceq M_{j+1} \preceq \ldots M_{n-1}$ describes an $(n+1)$-striated formula θ. Let $q \in \mathbb{P}$ be the element with $\mathbf{x}_q = \mathbf{x}_p \cup \{x_{t,0}\}$ with $\theta_q(\mathbf{x}_q)$ being the complete formula generating this type.

If $m > 0$, then we apply the previous case to ensure that $x_{t,0} \in \mathbf{x}_p$. Say $t = s_j$, the j'th element of u_p. But then, given any $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ and $M_0 \preceq \cdots \preceq M_{n-1}$ realizing θ_p, extend $\mathbf{x}_{p,t}$ to include $x_{t,m}$ by making each 'new' element of $\mathbf{e}_j$ equal to the element $e_{j,0} \in M_j$. ∎

The notational issue in what follows is the placement of free variables, For $p \in \mathbb{P}$, there is an explicit ordering to the variables $\mathbf{x}_p$ occurring in $\theta(\mathbf{x}_p)$, but when we consider extensions $\phi(\mathbf{v}, \mathbf{x}_p)$, we do not want to specify where the v_i's fit in the sequence. Recall Definition 3.3.1 (5).

Lemma 3.3.5 (**Henkin**). *Suppose $p \in \mathbb{P}$ and $\theta_p(\mathbf{x}_p) \vdash \exists \mathbf{v} \phi(\mathbf{v}, \mathbf{x}_p)$. Then there is $q \in \mathbb{P}$, $q \geq p$ for which the variables in $(\mathbf{x}_q \setminus \mathbf{x}_p)$ consist of a realization of $\phi(\mathbf{v}, \mathbf{x}_p)$ (in some order). Moreover, if $p \neq 0$, then can be chosen with $u_q = u_p$.*

Proof. Arguing by induction, we may assume $\mathbf{v} = \{v\}$ is a singleton, and we may further assume that $\phi(v, \mathbf{x}_p)$ describes a complete type. Let $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ and $M_0 \preceq \cdots \preceq M_{n-1}$ witness the truth and striation of θ_p and choose any $b \in M_{n-1}$ such that $M_{n-1} \models \phi(b, \mathbf{e}_p)$. Let $j \leq n-1$ be least such that $b \in M_j$. (Note that if $\phi(v, \mathbf{x}_p) \vdash$ '$v \in \mathrm{pcl}(\emptyset)'$, then we must have $j = 0$.) Let $\mathbf{x}_q = \mathbf{x}_p \cup \{x_{s_j, k_{p,j}}\}$. Then, letting $\mathbf{e}_j^* = \mathbf{e}_j b$, we have a striation $\mathbf{e}_0, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_j^*, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_{n-1}$ using the same n-chain of models $M_0 \preceq \ldots M_{n-1}$. Put

$$\theta_q(\mathbf{x}_q) := \mathrm{tp}(\mathbf{e}_0, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_j^*, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_{n-1}).$$

Then $q \in \mathbb{P}$ and $q \geq p$. ∎

Lemma 3.3.6. *Suppose $p, q, r \in \mathbb{P}$ with $p \le q$, $p \le r$, $\mathbf{x}_q \cap \mathbf{x}_r = \mathbf{x}_p$, and for some $t \in I$, $u_q \subseteq I_{<t}$ and $(u_r \setminus u_p) \subseteq I_{>t}$. Suppose further that there are $M \preceq N$ and $\boldsymbol{a}, \mathbf{b}, \mathbf{c}$ with $\mathbf{b} \cap \mathbf{c} = \boldsymbol{a}$, $\mathbf{b} \subseteq M$, and $(\mathbf{c} \setminus \boldsymbol{a}) \subseteq N \setminus M$ with $N \models \theta_p(\boldsymbol{a}) \wedge \theta_q(\mathbf{b}) \wedge \theta_r(\mathbf{c})$. Then there is $r^* \in \mathbb{P}$, $r^* \ge q$, $r^* \ge r$ with $\mathbf{x}_{r^*} = \mathbf{x}_q \cup \mathbf{x}_r$ and $\theta_{r^*} = \mathrm{tp}(\mathbf{b}, (\mathbf{c} \setminus \boldsymbol{a}))$.*

Proof. Arguing by induction, we may additionally assume that $u_r = u_p \cup \{s^*\}$ for some single $s^* > t$. That is, $\mathbf{x}_q \setminus \mathbf{x}_p$ lies on a single level of X. Since $q \in \mathbb{P}$, there is a striation of $\mathbf{b} = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1}$ induced by the rows of $\mathbf{x}_q$. As $\mathbf{b} \subseteq M$, we can find an n-chain $M_0 \preceq M_{n-1}$ of models with $M_{n-1} = M$, $\mathbf{b}_0 \subseteq M_0$ and $\mathbf{b}_i \subseteq M_i \subseteq M_{i-1}$ for all $0 < i < n$. As $(\mathbf{c} \setminus \boldsymbol{a}) \subseteq N \setminus M$ and as $(\mathbf{x}_q \setminus \mathbf{x}_p)$ consists of a single row (and since $s^* > t$) it follows that the $(n+1)$-tuple $\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}, (\mathbf{c} \setminus \boldsymbol{a})$ is realized in the $(n+1)$-chain $M_0 \preceq \cdots \preceq M \preceq N$. Choose $\mathbf{x}_{r^*} = \mathbf{x}_q \cup \mathbf{x}_r$ and put $\theta_{r^*} = \mathrm{tp}(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}, (\mathbf{c} \setminus \boldsymbol{a}))$. Then $r^* \in \mathbb{P}$ and both $r^* \ge q$, $r^* \ge r$ hold. ∎

Armed with these lemmas, we can now prove the main fact about the forcing $(\mathbb{P}, \le)$ and the generic model $\underset{\sim}{N}$ of T. For general forcing notation see [Kun80]. However, note that contrary to Kunen, we use the convention that $p \leqslant q$, means q is a stronger condition, carrying more information.

Notation 3.3.7. In what follows, when dealing with L-formulas, we will use the letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$, possibly with decorations to denote free variables. By contrast, tuples denoted by $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote finite tuples from X. Thus, for example, $\eta(\mathbf{v}, \mathbf{z})$ has free variables $\mathbf{v}$, and $\mathbf{z}$ is a fixed tuple from X.

We first establish that $(\mathbb{P}, \le)$ forces an uncountable atomic model of T. This initial Lemma only uses the Surjective and Henkin density conditions (Lemmas 3.3.4 and 3.3.5). More details of this initial construction can be found in [BLS16, §4.4].

Lemma 3.3.8. *There are $\mathbb{P}$-names $\underset{\sim}{N}$ and $\underset{\sim}{N}_\alpha$ for each $\alpha \in \omega_1$ such that*

$$(\mathbb{P}, \le) \Vdash \text{ `}\underset{\sim}{N} \in \mathrm{At},\ \underset{\sim}{N} = \aleph_1, \textit{ and } (\underset{\sim}{N}_\alpha : \alpha \in \omega_1) \textit{ is a filtration of } \underset{\sim}{N}\text{'}$$

> jb 2/7/24: I may just be very confused about p names but I have several questions about this proof -which I have not touched.
>
> In statement above should $\underset{\sim}{N} = \aleph_1$ be $|\underset{\sim}{N}| = \aleph_1$? In choosing that $\mathbb{P}$-name $\underset{\sim}{N}$ below are we thinking of a structure as a being a set and an interpretaton each formula in the Morleyization?
>
> In the displayed formula a) is the 'u' in the restriction to 'u', the same as the variable of ϕ; b) How does 'u' related to $\mathbf{x}$?

Proof. For every n-ary atomic L-formula $\phi(\mathbf{u})$, choose a $\mathbb{P}$-name $\underset{\sim}{\phi}$ such that, for every generic subset $G \subseteq \mathbb{P}$,

$$\underset{\sim}{\phi}[G] = \{\mathbf{x} \in X^n : T + \theta_p\restriction_{\mathbf{u}} \vdash \phi(\mathbf{u}) \text{ for some } p \in G\}$$

In particular, for the atomic formula of equality, we have a $\mathbb{P}$-name $\underset{\sim}{E}$ representing 'equality' on X^2. As each θ_p is consistent with T, it follows that $(\mathbb{P}, \leq)$ forces that $\underset{\sim}{E}$ is an L-congruence. Choose a $\mathbb{P}$-name $\underset{\sim}{N}$ representing L-structure whose universe is the quotient $X/\underset{\sim}{E}$ and whose atomic formulas are interpreted as $\underset{\sim}{\phi}$, and choose $\mathbb{P}$-names $\underset{\sim}{N}_\alpha$ for the substructure with universe $X_\alpha/\underset{\sim}{E}$.

Continuing, for every L-formula $\psi(\mathbf{u})$ (with quantifiers) choose a $\mathbb{P}$-name $\underset{\sim}{\psi}$ analogously to $\underset{\sim}{\phi}$. Using the Henkin density conditions, a straightforward induction on the quantifier complexity of ψ shows that for every $\mathbf{x} \in X^n$ and generic $G \subseteq \mathbb{P}$,

$$\mathbb{V}[G] \models \text{`}\underset{\sim}{N} \models \psi(\mathbf{x}/\underset{\sim}{E})\text{'} \quad \iff \quad \mathbf{x} \in \underset{\sim}{\psi}[G]$$

and similarly for each $\underset{\sim}{N}_\alpha$. From this, it follows that $(\mathbb{P}, \leq)$ forces that each $\underset{\sim}{N}_\alpha \preceq \underset{\sim}{N}$. As each θ_p is a (consistent) complete formula with respect to T, $(\mathbb{P}, \leq)$ also forces that $\underset{\sim}{N}$ is an atomic model of T. Finally, since each θ_p is a striated formula, we see that $(\mathbb{P}, \leq)$ also forces $\underset{\sim}{N}_{\alpha+1}$ properly extends $\underset{\sim}{N}_\alpha$, hence forces $|\underset{\sim}{N}| = \aleph_1$. ∎

It remains to show that $(\mathbb{P}, \leq)$ forces that $\underset{\sim}{N}$ has every limit type constrained. For this, we note a consequence of splitting inside an atomic model.

Remark 3.3.9. Suppose $M \preceq N$ are atomic, $\boldsymbol{a} \in M$, $\mathbf{b} \in N$, but $\text{tp}(\mathbf{b}/M)$ splits over $\boldsymbol{a}$. Then, letting $\theta(\mathbf{u})$ isolate the complete type of $\boldsymbol{a}$ and $\theta'(\mathbf{w}, \mathbf{u})$ isolate the complete type of $\mathbf{b}\boldsymbol{a}$, there must be a complete formula $\eta(\mathbf{v}, \mathbf{u}) \vdash \theta(\mathbf{u})$ and two contradictory complete formulas $\delta_1(\mathbf{w}, \mathbf{v}, \mathbf{u})$ and $\delta_2(\mathbf{w}, \mathbf{v}, \mathbf{u})$, each extending the (incomplete) formula $\eta(\mathbf{v}, \mathbf{u}) \wedge \theta'(\mathbf{w}, \mathbf{u})$.

Proposition 3.3.10. $(\mathbb{P}, \leq)$ *forces every limit type in* $S_{at}(\underset{\sim}{N})$ *is constrained.*

Proof. To ease notation, in what follows write ψ in place of the more cumbersome $\underset{\sim}{\psi}$ throughout the argument. Call a function $b : \omega_1 \to \underset{\sim}{N}$ a *limit sequence* if, for all $\alpha \leq \beta$, $\text{tp}(b(\alpha)/\underset{\sim}{N}_\alpha) = \text{tp}(b(\beta)/\underset{\sim}{N}_\alpha)$. Now, if $(\mathbb{P}, \leq)$ does not force that every limit type is constrained, then there is some $p^* \in \mathbb{P}$ and some $\mathbb{P}$-name $\underset{\sim}{\mathbf{b}}$ and some club $C \subseteq \omega_1$ such that

$$p^* \Vdash \underset{\sim}{\mathbf{b}} \text{ is a limit sequence with } \text{tp}(\underset{\sim}{\mathbf{b}}(\alpha)/\underset{\sim}{N}_\alpha) \text{ unconstrained for every } \alpha \in C.$$

(Since $(\mathbb{P}, \leq)$ is c.c.c. we can find such a club in $\mathbb{V}$.)

For each $\alpha \in C$, choose $p_\alpha \in \mathbb{P}$, $p_\alpha \geq p^*$ and $x^*_\alpha \in X$ such that

$$p_\alpha \Vdash \underset{\sim}{\mathbf{b}}(\alpha) = x^*_\alpha$$

We will eventually reach a contradiction by finding some $q^* \geq p^*$ and some $\alpha < \beta$ from C such that

$$q^* \Vdash \text{tp}(x^*_\alpha/N_\alpha) \neq \text{tp}(x^*_\beta/N_\alpha)$$

contradicting that $p^* \Vdash \underset{\sim}{\mathbf{b}}$ is a limit sequence. By a routine Δ-system argument, find a 'root' $p_0 \in \mathbb{P}$, some $\gamma^* \in \omega_1$, and a stationary set $S \subseteq C$ satisfying:

- $p_0 \le p_\alpha$ for all $\alpha \in S$;
- $u_{p_0} \subseteq J_{\gamma^*}$ (first paragraph of Section 3.3); and
- for all $\alpha < \beta$ in S,
  - $\mathbf{x}_{p_\alpha} \cap X_{\gamma^*} = \mathbf{x}_{p_0}$;
  - $\max(u_{p_\alpha}) < \min(u_{p_\beta} \setminus u_{p_0})$;
  - $\lg(p_\alpha) = \lg(p_\beta)$ and $k_{p_\alpha} = k_{p_\beta}$; and
  - The formulas θ_{p_α} and θ_{p_β} have the same syntactic shape [one formula can be obtained from the other by substituting the free variables].

Note that we do not require $p_0 \ge p^*$. As notation, we write $\mathbf{z}$ for $\mathbf{x}_{p_0}$ and note that $\mathbf{z} \subseteq X_{\gamma^*}$. Now fix, for the remainder of the argument, some $\alpha < \beta$ from S. To obtain our desired contradiction, we first concentrate on p_α. Write $\theta_{p_\alpha}(\mathbf{y}, \mathbf{z})$ and note that $\mathbf{y}$ is disjoint from X_{γ^*}. We apply Remark 3.3.9, noting that $p_\alpha \Vdash \mathrm{tp}(x^*_\alpha/\underset{\sim}{N}_\alpha)$ splits over $\mathbf{z}$. Choose a complete formula $\eta(\mathbf{v}, \mathbf{z})$ implying $\theta_{p_0}(\mathbf{z})$ and contradictory complete formulas $\delta_1(x^*_\alpha, \mathbf{v}, \mathbf{z})$ and $\delta_2(x^*_\alpha, \mathbf{v}, \mathbf{z})$, each extending $\eta(\mathbf{v}, \mathbf{z}) \wedge \theta^*_{p_\alpha}(x^*_\alpha, \mathbf{z})$, where $\theta^*_{p_\alpha}$ is the restriction of the compete formula $\theta_{p_\alpha}(\mathbf{y}, \mathbf{z})$.

By Henkin, choose $q_0 \in \mathbb{P}$, $q_0 \ge p_0$ with $u_{q_0} \subseteq J_\alpha$ and $\theta_{q_0}(\mathbf{z}', \mathbf{z}) := \eta(\mathbf{z}', \mathbf{z})$. Next, we use Lemma 3.3.6 twice. In both cases we start with $p_0 \le q_0$ and $p_0 \le p_\alpha$. Our first application gives $r^1_\alpha \in \mathbb{P}$ extending both q_0 and p_α with $\theta_{r^1_\alpha}(\mathbf{y}, \mathbf{z}', \mathbf{z}) \vdash \delta_1(x^*_\alpha, \mathbf{z}', \mathbf{z})$. The second application gives $r^2_\alpha \in \mathbb{P}$, also extending both q_0 and p_α with $\theta_{r^2_\alpha}(\mathbf{y}, \mathbf{z}', \mathbf{z}) \vdash \delta_2(x^*_\alpha, \mathbf{z}', \mathbf{z})$.

Next, we use the fact that the forcing $(\mathbb{P}, \le)$ is highly homogeneous. Due to the similarity of p_α and p_β found by the Δ-system argument and described in the third bullet point just above, Fact 3.3.2 gives an automorphism σ of $(\mathbb{P}, \le)$ sending p_α to p_β, fixing q_0. Put $r^2_\beta := \sigma(r^2_\alpha)$. We now apply Lemma 3.2.5 to $q_0 \le r^1_\alpha$ and $q_0 \le r^2_\beta$ to get $q^* \in \mathbb{P}$ with $q^* \ge r^1_\alpha$ and $q^* \ge r^2_\beta$. However, this is impossible, as

$$q^* \Vdash \delta_1(x^*_\alpha, \mathbf{z}', \mathbf{z}) \wedge \delta_2(x^*_\beta, \mathbf{z}', \mathbf{z})$$

contradicting $p^* \Vdash \mathrm{tp}(x^*_\alpha/\underset{\sim}{N}_\alpha) = \mathrm{tp}(x^*_\beta/\underset{\sim}{N}_\alpha)$ since δ_1 and δ_2 were chosen to be contradictory. ■

3.4 Proof of Theorem 2.3.2

Theorem 2.3.2 follows easily from Propositions 3.1.1 and 3.3.10 and Keisler's model existence result for $L_{\omega_1,\omega}(Q)$. In particular, in some c.c.c. forcing extension $\mathbb{V}[G]$, by Proposition 3.3.10, there is an uncountable atomic model of T with every limit type constrained. Hence, by (2) $\Rightarrow$ (1) of Proposition 3.1.1, there is a model of Ψ^* in $\mathbb{V}[G]$. By the absoluteness of model existence from Keisler's theorem, there is also a model of Ψ^* in $\mathbb{V}$. Whence, by (1) $\Rightarrow$ (2) of Proposition 3.1.1, we obtain the existence of an atomic model of T in $\mathbb{V}$ such that all limit types are constrained.

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