

Original Article

Practice-based Teacher Development in Its Mathematical Context: Lessons and Their Representation in Approximations of Practice

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Abstract

Addressing the need for practice-based teacher education to attend to context, we describe anticipations of a lesson as a case of approximations of practice that may anchor practice to its disciplinary context. After providing a general framework for how to think about anchoring practice in context we consider the *StoryCircles* process as an approximation of practice and illustrate the variability in anticipations of a lesson that can be generated by different *StoryCircles*. We use this variability to propose a conceptualization of lesson which is of particular value to practice-based teacher education.

Keywords: Lesson, Disciplinary context of teaching practice, Approximations of practice, Anticipation of a lesson

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I. INTRODUCTION

Practice-based teacher education is teacher education which draws its content from practice, relies on practicing as a pedagogy, and aims at developing practitioners—or as Lampert (2010) puts it, it means learning teaching in, from, and for practice. Practice-based teacher education (PBTE) has been criticized for avowedly emphasizing transposable skills without equipping prospective teachers (PT) with the intellectual dispositions to attend to context (Philip et al., 2019; Zeichner, 2012). Though such criticism may not apply with the same force to all instances of PBTE (e.g., see Kavanagh, 2022), we agree that contexts are crucial anchors that shape the practices that can be enacted and believe that manifold considerations of context need to be made in PBTE. This is particularly needed as mathematics teacher educators create or select approximations of practice—that is, activities in which teachers can practice teaching in settings of reduced complexity (Grossman et al., 2009). In this paper, we argue that lessons are units of practice that permit a kind of approximation of practice—the anticipation of a lesson—in which the disciplinary context of mathematics teaching practice can be attended. We show three cases of using the *StoryCircles* process for teachers to anticipate the same lesson, which are based on an informal notion of what a lesson is. We use that analysis to eventually define lesson in such a way that it can support the creation of opportunities for PT to learn teaching practice in its disciplinary context.

II. ATTENTION TO CONTEXT IN PRACTICE

In their model of research on classroom teaching, Dunkin and Biddle (1974, p. 38) had depicted relationships among four classes of variables: Presage variables that alluded to teacher characteristics, Process variables that referred to classroom behaviors, Product variables that identified outcomes of interest (including subject matter learning), and Context variables. The latter included pupil characteristics, school characteristics, community and class characteristics (including ethnic composition of community and class size), and the curriculum (Dunkin & Biddle, 1974, pp. 41-43). We build on this early account of the context of instruction a theoretical reconstruction of context that aims to serve PBTE (which for us includes the initial training of new teachers and the professional development of practicing teachers) to build approximations of practice that attend to context.

We propose this theorization from the theory of practical rationality, which conceives the process of instruction as interactions among teacher, students, and content and which organizes the context of instruction in terms of four stakeholders: The client, knowledge, society, and organization (Chazan et al., 2016). Our conceptualization of these stakeholders attempts to generalize the context variables in Dunkin and Biddle (1974) as well as to posit sources of regulation of instruction that connect to the products that instruction aims at. Thus, by the *client* we allude to the child and their parents or guardians as stakeholders of a

process that seeks student growth and identity formation. By *society* we refer to the communities and societies where children come from and into which they will incorporate later as members and workers; communities and societies are stakeholders of a process that seeks children socialization into shared values and acquisition of socially valued skills. By *organization* we refer to the school and its embedded and embedding organizations as stakeholders of a process of which such organizations expect effectiveness, efficiency, and legality in its production function. And *knowledge* names in our case the discipline of mathematics as a stakeholder of a process that seeks the communication of warrantable knowledge, skills, and dispositions that support its production and use. Knowledge as a stakeholder is not a monolith and it includes several mathematical disciplines (e.g., mathematics as produced by mathematicians, mathematics as used by STEM professionals, etc.). Within this account of the context of instruction, we interpret the notion that approximations of practice put practice in context as a call for PBTE, and, in particular, for approximations of practice, to be designed in such a way that they activate consideration of ranges of variability with which practice may respond to the demands of particular stakeholders, and the contexts of teaching they create.

The design of manifold considerations of context can be scaffolded by the professional obligations of mathematics teaching which correspond to how teachers experience the stakeholders identified above. As Herbst and Chazan (2012) proposed, any mathematics teacher is obligated to the discipline of mathematics, to their individual students, to community and society, and to the institutions of schooling. Individual teachers teach in specific locales that instantiate those obligations, though the specific ways in which such locales will do that can be anticipated and approximated by the conceptualization of the obligations. For example, the students whom a teacher needs to serve are unique individuals, but their needs can be anticipated, for example, with the assistance of resources from psychological and medical frameworks that serve contemporary conceptualizations of children and youth. Similarly, the expectations of community and society a teacher needs to attend to may have some characteristics which are decidedly local and others which may apply more generally to the larger society that receives the students when they graduate from school. Social and economic research (e.g., on local customs, on the demands of the world of work) and research on area studies (e.g., African American culture) can provide resources to create approximations of practice to support PBTE that also approximate such social context. The same could be said about the institutions in which teachers teach—where research on educational organizations, curriculum, and assessment can support the development of approximations of practice that attend to variability in institutional characteristics (e.g., the length of class periods, the nature of departmental oversight, etc.). Finally, the discipline of mathematics provides context in the form of the intellectual sources of the content of instruction. An interpretation of the disciplinary distinctions that make a difference in how an approximation of practice attends to the mathematical content of instruction (e.g., is the goal of instruction the formulation of a new conjecture or the use of a learned theorem?) can approximate such context.

Approximations of practice have been described as environments of reduced complexity in which PT

can practice teaching (Grossman et al., 2009). The specific reductions of complexity used in an approximation of mathematics teaching practice may naturally vary depending on the specific aspects of practice that are to be practiced. Mathematics teaching practice will involve a person occupying the position of teacher and engaging in instruction, where they will transact mathematical content with a class of students. Because instruction happens in context, those transactions will be modulated by attention to the various professional obligations to which the stakeholders in this context call on the teacher to attend. The design of approximations of practice which are used to learn to do the work of teaching will rely on a core representation of instruction, will simplify some of the complexities of mathematics instruction in context while representing more richly other complexities, and will provide a position, a timescale, and a scope of work for the teacher-learner to get involved in practicing. In those circumstances, it may be possible for various professional obligations to be instantiated as opportunities for teacher-learners to consider the variability of potential action in context, as well as limits to such variability. As we strive to approximate practice in such a way that the disciplinary context of instruction is kept in focus, we consider lessons as the units of time for such approximations.

III. ANTICIPATIONS OF LESSONS IN *StoryCircles*

Our paper is dedicated to conceptualizing anticipations of lessons as approximations of practice that, albeit reductive of much complexity, afford opportunities for practicing mathematics teaching in the context of the complexities of mathematical work, knowledge, and dispositions. Specifically, while approximations of mathematics instruction are always likely to involve some representation of the content of instruction, some approximations may be particularly good in enabling the practitioner to attend to the disciplinary obligation (thus, to the disciplinary context of teaching practice). We argue that anticipations of lessons, as done through the *StoryCircles* process (which we describe below) illustrate one such approximation.

Our present contribution considers approximations of practice in units that are primarily organized according to the disciplinary considerations a teacher needs to make—that is, where the context specificity of practice consists of the mathematical work the teacher needs to manage. In this paper, we describe multimodal anticipation of lessons as approximations of practice that instantiate such disciplinary context. In the section below on Approximations of Practice, we review in more detail what we mean by multimodal anticipation of lessons.

We use lessons as the unit for our approximations of practice. We first dive into illustrating how lessons can be units for approximations of practice that require practitioners to respond to the disciplinary obligation. To do that we account for variations in the anticipations of a lesson observed in the context of *StoryCircles* (Herbst & Milewski, 2018). Then we use observations of such anticipations of lessons to propose a conceptualization of lesson which is serviceable for enabling attention to the disciplinary obligation.

1. Anticipating a lesson as an approximation of practice

Anticipating a lesson (i.e., anticipating how a lesson may unfold) is one type of approximation of practice that, we argue, is particularly helpful for PT to experience the demands the discipline may make on the work of the teacher (Brown et al., 2022). The pedagogy of practice called *approximation of practice*, described by Grossman et al. (2009), has been a part of teacher education for decades and is often exemplified in rehearsals, live simulations, and digital simulations of various sorts. Though the timescale of approximations of practice has not been an explicit consideration in their theorization and though the specific genres of approximations of practice do not necessarily imply a timescale or a particular unit of approximation, we observe that distinctions of timescale and unitization of practice are present in the way different approximations are used in teacher education and suggest that they might make a difference in understanding the exchanges that can be accomplished in each type of approximation.¹⁾ Rehearsals, for example, have focused on instructional activities (such as number talks; Lee et al., 2021) that can be realized at the timescale of a few utterances and that provide an opportunity to practice questioning, listening, and responding while attending to student contributions to the whole class (Arbaugh et al., 2019; Campbell et al., 2020; Lampert et al., 2013; McDonald et al., 2013). Some simulations (both live and digital) have inherited from medical education's simulated patient the focus on a single student and attention to a single student's processes in solving problems, providing context for the teacher to notice student thinking, interpret it, and decide on the next questions to ask (Shaughnessy & Boerst, 2018; Stockero et al., 2020; Webel & Conner, 2017). Other simulations have put an emphasis on issues of class engagement and attention to students' personalities though involved with some content (Grant & Ferguson, 2021; Ledet et al., 2015). Also, animation and comic genres have been leveraged by scholars to enable PT to approximate practice through the representation of instructional moments or scenes of interest to the mathematics teacher educator or the PT (e.g., Estapa et al., 2018). The search for approximations that enable teachers to practice teaching in the context of instructional exchanges where they could respond to the demands of the discipline has taken us to a practice of anticipating lessons among groups of teachers using a process called *StoryCircles* (e.g., Milewski et al., 2018). Chen (2012) introduced lesson anticipation in preservice teacher education as a dynamic narrative of a lesson plan. Thus, to explain how lesson anticipation is an approximation of practice, we contrast it with lesson planning.

Lesson planning has had a consistent presence as a teacher education activity and could be considered an approximation of pre-active aspects of teaching practice (Westerman, 1991). But we differentiate lesson anticipation from lesson planning in that we consider lesson anticipation to approximate some of the active, rather than the pre-active, aspects of teaching practice by going beyond planning lessons into visualizing what concretely will happen in lessons. For almost 15 years, our group has been invested in storyboarding lessons multimodally for a variety of purposes. One of these purposes is the anticipation of a lesson, that

1) That is, what instructional goal of teacher education is at stake (and possibly attained) by the completion of an approximation of practice (see Chazan & Herbst, 2023; Herbst & Chazan, 2023).

is, the creation of a prototype (using one or more communication modalities) of how the lesson may flow and the experiencing of that lesson by perusing the prototype. Though such anticipation of a lesson occurs separately from, and usually before, a lesson enactment, it is not to be assimilated with lesson planning. Chen (2012) studied anticipations of lessons that had been previously planned on paper by comparing pairs of PT in two conditions: Narrating orally how the lesson would flow and depicting how the lesson would flow by using a storyboarding tool. In both conditions, the narration of the lesson evinced elements ignored in the plan. But those PT who visually depicted lessons attended to more instructional details and better specified teacher actions (e.g., crafting exactly what the teacher would say and the representations they would show) compared to those PT who simply talked through a lesson plan (and alluded to what they would say and show without being explicit). Depicting lessons led to more specific considerations of teacher actions, student individuality, and instructional nuances, suggesting its potential to enhance teacher preparation beyond traditional lesson planning methods (see also Herbst et al., 2014). In her study, Chen (2012) documented how the storyboard medium not only enabled expressing what PT planned to do but also examining the flow of the scripted events and engaging in discussions among the pairs of PT that led to revisions of the storyboard. Depictions and storyboards are used interchangeably here to mean the combined use of visual and print modalities in sequential frames (as in the comics genre; see McCloud, 1993) to represent moments in instruction.

In his teaching of a methods class for secondary mathematics, the first author had used since 2004 scripting assignments for PT to demonstrate how they would engage in particular activities in the classroom (i.e., made up classroom dialogues demonstrating, e.g., how they might explain a concept). At about the same time, Rina Zazkis and her colleagues had been using the notion of *lesson plays* (scripting classroom dialogue in print) for compatible purposes (see Zazkis et al., 2013). Beginning in 2011, the first author started to use storyboards as a scripting multimodality with his methods class, on account that storyboards afforded resources to better represent the co-occurrence of verbal and nonverbal actions (e.g., speaking about a diagram on the board). Herbst et al. (2014) argued that engaging PT in storyboarding is a form of approximation of practice, because, like print-based scripting, storyboarding leaves a trace that, when read, creates a milieu that provides feedback to the authors. Because print scripts and storyboards continue to exist after they have been composed, authors and others can peruse them (e.g., in reenactments or read alouds) to visualize the lesson and can rely on their experience in such perusals to decide whether details are missing or whether something needs to be corrected. Rougée and Herbst (2018) compared the work PT did in print scripting and storyboarding assignments and they found that while print scripts often included vague descriptions of teacher and student actions without specifying the details of enactment, storyboard representations allowed for these actions to be realized through speech bubbles and text, leading to a more detailed portrayal of teaching practices. Additionally, the storyboard medium facilitated the inclusion of affect and movement across the room in representations, providing a richer depiction of classroom interactions compared to print-based representations.

Clearly, while scripting and storyboarding enable practicing of speech and diagramming coordinated sequentially over time as well as the anticipation of student responses, they do not enable participants' engagement in a fully embodied and timely practice. They don't support engagement with real students or with proxies (e.g., simulated students) that could surprise the teacher learners. The feedback from experience comes when participants read aloud scripts or storyboards to visualize the lessons, something they usually need to be prompted to do. As they compose and read these scripts, one might wonder in what sense they are practicing teaching. We argue that the storyboard multimodality allows PT to practice composing what they will look for, say, show, and do, both strategically (e.g., as non-contingent moves to teach the lesson) and tactically (e.g., as actions contingent on the various things they anticipate having to deal with). Reading or visualizing them by perusing a series of frames as a continuous story organizes all those actions and speech as a sequence of events taking place in time and participants can contrast this experience (e.g., hearing themselves say what they would say to students) against prior experiences in real classrooms (e.g., things they have heard teachers say to students). As they do this, they get a sense of what might need to be changed (e.g., if something they say is a mouthful or poorly phrased, they might realize that and change it; see Brown et al., 2022). Both in the sense of anticipating possible reactions from the students and practicing what they would look for, say, show, and do, we contend storyboarding a lesson or anticipating a lesson through scripting it multimodally is an approximation of practice.

Furthermore, we see the storyboarding of lessons using cartoon characters as only a developmental step in the creation of more fully embodied, technology-assisted approximations of practice in which PT could script or prototype practice and then visualize it (see Kosko et al., 2021). Tools for collecting multimodal and embodied data (e.g., 3D cameras, sensors), for producing student responses (e.g., large multimodal language models powered by AI), and for analyzing multimodal data (e.g., machine learning) are making progress toward turning the notion of anticipating a lesson into one with face validity much closer to practicing in front of actual students. With those technological advances, it is possible to envision simulated teaching environments in which PT are fully instrumented with sensor technology that records their body expression and movement adding to 3D video and audio technology that captures their speech and inscriptions and their interactions with holographic virtual students animated by intelligent agents. These instrumented actions would be a form of embodied scripting that, similar to storyboarding, would leave a trace. This trace could then be projected holographically for PT to examine their practice and tinker with it (i.e., edit and play again). That is, just as the scripting phase of anticipation could move from the written modality to the storyboard multimodality, it could over time evolve into an embodied multimodal scripting and the anticipation of student responses could optionally be done independently. We see the storyboarding of lessons not only as a precursor of those future embodied ways of anticipating lessons but also, particularly in this paper, as helping us conceptualize the space of lessons that could be anticipated with those tools.

The notion that anticipating lessons using storyboards could be an approximation of practice has been reinforced by the parallel development of a professional learning process we call *StoryCircles* (Herbst &

Milewski, 2018). Herbst and Milewski (2018) characterized *StoryCircles* as the work of a group of teachers working with the support of a facilitator to anticipate a lesson through scripting, visualizing, and arguing about alternatives. Scripting is done multimodally by creating lines of classroom talk, classroom artifacts (e.g., diagrams), and laying those out in storyboard frames with cartoon characters. *StoryCircles* uses a socio-technical infrastructure for scripting: a storyboarding tool (*LessonDepict*) and an assistant called the storyboarder (usually an undergraduate student, expert in the use of the software but without any teaching or mathematical credentials). Visualization involves perusing and annotating partially completed versions of that storyboard to experience the sequence of events as they unfold over the timeline set by the sequence of storyboard frames, noticing how the lesson flows against the background of participants' own teaching experience. Usually, such visualization enables participants to notice aspects of the lesson, raises questions about what happened, and brings up participants' need to see intermediate actions that resolve discontinuities or alternative actions that may make more sense than what had been laid out before. Arguments about those needed actions can develop asynchronously within the lesson commentary as participants read and respond to the annotations from other participants (Milewski et al., 2020). These arguments continue over subsequent synchronous videoconference meetings where the additional or alternative scenes are scoped out. *StoryCircles* is a facilitated process, but the role of the facilitator is to orient participants to each other and offer resources (Brown et al., 2021)—not to guide them to what is good to do in the target lesson.

Through the continuous design of *StoryCircles*, the goal is to delineate the roles that multimodal scripting, visualization, and argumentation about a lesson play in the elicitation and improvement of professional discourse and to identify the technologies and resources that best support such work. As in earlier uses with PT (e.g., Chen, 2012; Herbst et al., 2014; Rougée & Herbst, 2018), the key hypothesis is that the visualization of a lesson through a storyboard perusal serves as a feedback mechanism on the actions considered while scripting (Brown & Herbst, 2023). As in Chen's (2012) design, the collaborative nature of the visualization seeks to support the sharing of claims of what the lesson needs and the argumentation for or against those claims. *StoryCircles* is focused on entire lessons rather than on component activities. Although the basic format of *StoryCircles* could be used to anticipate any lesson, our usages have focused on problem-based lessons that aim at an instructional goal and seek to involve students in whole class discussions. By problem-based lessons we mean lessons that are constructed around a problem that can be understood and begun to be approached using students' prior knowledge, and where the work on the problem can get students to a point in which it is relevant to state a new item of knowledge (the instructional goal). In this basic sense, problem-based lessons are characterized by an epistemology of knowledge development through mathematical work (i.e., that the essence of mathematical ideas is founded in their existence within the problems they help solve) and not by any particular pedagogy (i.e., it is analytically possible to conceive a problem-based lesson in which students only listen to the teacher while the teacher works on the problem until they arrive at the goal). However, we have been interested in eliciting practitioners' anticipations of lessons that require the teacher to also engage the class in discussion, considering that the emergence

of new knowledge from problematic situations involves navigating a diversity of perspectives that may not in general be consistent with each other,²⁾ and those would very naturally come from students who are empowered to work on their own and share their work: We refer to this as the multivocality of classroom knowledge (Herbst et al., 2011). For practitioners to develop a disposition to accommodate such disciplinary features into their lessons, it is important that they can anticipate and manage the challenges that multivocality may bring about. In particular, problem-based lessons that use a pedagogy of launch, explore, discuss, and summarize (Stein et al., 2008) are likely to introduce desirable difficulties for the practitioner, in the form of student work and contributions that vary in regard to their correctness, responsiveness to the problem, normative use of prior knowledge, and serviceability for developing the knowledge at stake (Herbst et al., 2023). Anticipating a lesson in which the teacher must be prepared for such variability to show up and in which the teacher still has to manage the class's advancement toward the instructional goal is a challenging approximation of practice (see Brown et al., 2021). Indeed, one where some of those challenges are sourced in the discipline as different ideas students may bring up will need to be scrutinized with regard to their mathematical significance and orchestrated to support the development of the instructional goal. Not only there are different ways in which the teacher may have to attend to the disciplinary obligation, but also the other obligations are likely to come up as reasons to avoid short-cuts (e.g., finding the one student who did the best work and having the class discuss this work would likely be seen as playing favorites, which would be hard to defend on account of the interpersonal obligation; ignoring other students' difficulties would be hard to defend on account of the individual obligation; designing a lesson that takes several days would be hard to defend on account of the institutional obligation).

StoryCircles provides both resources and constraints to handle the anticipation of such a lesson. Resources come in the form of the diverse experiences thinking and learning mathematically of each participant as well as what they have seen students say and do in classrooms. Participants may therefore contribute what they think students might say and do in the context of their work on the problem as they script a lesson. Constraints come from participants' classroom experience as well: Anything that is scripted by someone needs to appear sensible for the group, including what ideas and verbiage are attributed to students. If it is not sensible (e.g., if an idea is seen as too advanced for high school students to have come up with it), it may be struck out of the script, but if it is sensible it may pose interesting challenges to figure out how to respond to it (e.g., an idea that is incorrect but has the potential to be used to develop a better idea may require careful judgment in deciding how to make it available to the whole class and how to promote students' further building on it). One important resource provided to begin each StoryCircle is an initial or starter lesson representation or multimodal script which participants are asked to annotate

2) As Brousseau (1997) noted, "The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory ...; not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc." (p. 81)

asynchronously (the activity has been called "Leave Tracks") and for which they are asked to leave comments based on their observations or reactions to the storyline in order to help the group to eventually envision alternative trajectories for the lesson storyline. The *StoryCircles* process has been used with both PT (Brown et al., 2022) and practicing teachers (Milewski et al., 2018) in the understanding that both groups have had experiences in classrooms and in doing mathematics that can serve as learning resources. Most of the research to date, however, has concerned practicing teachers. As we consider the role that *StoryCircles* with its focus on lessons can play in the practice-based preparation of PT, we surmise that observations about its use with practicing teachers might be instructive.

One important question for our research and development of *StoryCircles* is what sort of starter lesson representation groups of teachers need to receive in order for them to engage in visualization and argumentation that contribute to subsequent scripting that improves the lesson, particularly by making it contain more student participation and classroom discussion. We address this question by illustrating how three different starter scripts of a lesson called The Tangent Circle resulted in different visualizations by *StoryCircles* participants. This lesson takes place in a US high school geometry course right after students have studied the definition of tangent to a circle and the theorem that describes the tangent as perpendicular to the radius of the circle at the point of tangency and where the goal of the lesson is to install the theorem that establishes the congruence of segments formed between points of tangency and the intersection of the intersecting tangent lines (hereafter, the Tangent Segments Theorem). In order to focus participants' attention on anticipating and managing multivocality in a problem-based lesson aimed at that goal, every *StoryCircle* is asked to first pose the problem "Suppose that you have two intersecting lines and two points, one on each of those lines. How can you find a circle tangent to those lines at those points?" The question that we attempt to answer here is what starter script seems most productive to move participants into arguing about how to improve the lesson. This question is crucial for our design of *StoryCircles* but not solely relevant for the development of this PD process. Rather, the question delves into a fundamental issue in PBTE: How should we scaffold teachers' engagement in anticipations of problem-based lessons and what role could the starting representations of the lesson play in this process (e.g., Sherin et al., 2009). The question makes sense because previous experiences with *StoryCircles* revealed that merely starting with a problem may not actually get teachers to dedicate their efforts to other segments of the lesson beyond the launch of the work. Similarly, Chen (2012) started her participants by having them plan a lesson whose only given was the instructional goal (understanding slope) but participants did not often make those lessons problem-based. In our case, we also wanted to start from a particular problem and have participants include a discussion of that problem in the lesson but have that discussion lead to the instructional goal. Hence, some representation of the lesson might need to be shared for participants to visualize an initial lesson before starting to argue how to improve it. The need to understand what representation of a lesson is best to use to start a *StoryCircle* is particularly salient when the *StoryCircle* will involve PT, as they will have to anticipate the lesson but do not have a large stock of experiential knowledge

of practice to script a lesson from scratch. The problem may be posed as follows: If in order to promote engagement in arguing about what a lesson should look like and script such a lesson, PT need to visualize an initial representation of a lesson, what qualities does such an initial lesson need to have to support that engagement?

IV. WHAT QUALITIES DOES AN INITIAL REPRESENTATION OF A LESSON NEED TO HAVE TO SUPPORT ANTICIPATION?

A possible way to start a *StoryCircle* could include having participants view a version of the lesson recorded in video and annotate it with alternative moves that could be made. But framing the work of suggesting alternatives can be tricky. Prior literature on PBTE that has relied on video representations of practice has run into this problem: The decision to show something to teachers in a professional learning environment always implies having made a choice that the artifact being shown has some value, but sometimes it is not productive to disclose what this value is before showing the artifact and letting teachers work with it. Thus, participants' work with the artifact needs to be framed in reference to something else than that value. This is the case with the starters of *StoryCircles*, and in that sense, the way prior uses of representations of practice have been framed provide a useful antecedent.

Though video representations of practice have not always been used as starting points for the improvement of a lesson, the framing of a video representation to get teachers' initial reactions has taken two forms. Firstly, sometimes video representations have been framed as demonstrating accomplished practice to be consumed as models for imitation (Reed et al., 2018). Secondly, sometimes video representations have been framed as expressions of individuals' developing practice, in search of formative feedback (Horn et al., 2023). Both ways of framing video representations in PBTE encounters may support some teacher education functions. For example, exemplary videos may make the case that something is possible (Yung et al., 2007) and may be useful to engage teachers in the decomposition of practice into its component parts (Grossman et al., 2009). And peer feedback on teaching may be good too. Teacher educators have had difficulties using video to promote inquiry stances into teaching that might lead to improving a lesson—not only evaluations of what was done may focus too much on the specific actions of a person (Sherin et al., 2009) and be perceived as evaluating the person but also any refocusing on actions and what could have been done instead may be perceived as a waste of time as the real actions have already occurred. Interestingly, these problems with videos have led some mathematics teacher educators to restrain their clients from evaluative comments on videos and include more deference when relating to the actions represented (e.g., "I notice" and "I wonder" language; see Anderson & Dobie, 2022; McGugan et al., 2023). It is unclear, however, whether restricting the impulse to evaluate, despite its relational virtue and its usefulness to develop an inquiry stance, is productive to develop alternative ideas for what to do in practice, as the proposal of

an alternative action does connote an evaluation (e.g., Achinstein, 2002; Chieu et al., 2015; Dobie & Anderson, 2015; Lord, 1994). The use of storyboards instead of videos as starters for *StoryCircles* is meant to support the development of alternatives, as there is no need to prevent evaluations when the teacher in the representation of practice is a nondescript cartoon character (Bieda et al., 2015; Herbst et al., 2011).

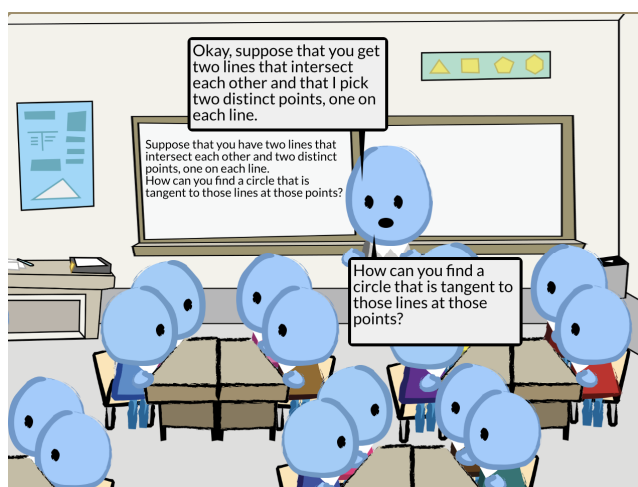
In our case, the use of storyboards with cartoon characters has allowed two ways of avoiding the problems of video noted above. The characteristics that Herbst et al. (2011) called *individuality* and *temporality* make these storyboards different from videos. First, because the cartoons used are not only hollow but also nondescript, evaluative comments on the storyboards can hardly be taken as comments on a person and can more easily be used to consider what their actions meant and what they could have done instead. Second, because temporality is modeled as a sequence of frames, the storyboard multimodality allows for several ways of reflecting the passing of time in lessons, hence the possibility to evoke rather than inscribe possibilities for action (e.g., moments of a lesson show up without a clear trace of how they came to be) hence making the question of how something could have been done a more sensible one to inquire about (Herbst et al., 2023). Asking for what happened before, between, or after events does not call for a historical or psychic inquiry; it rather encourages the creative proposal of many alternatives, only subject to the constraint that these alternatives be sensible within what has already been laid out. Arguing about what actions appear more sensible is something that we posit can happen within a group discussion about a starter storyboard.

1. Three starter scripts

We discuss three different representations of the Tangent Circle Lesson that were used in the context of a "Leave tracks" activity to get a *StoryCircle* started. In all cases, the participants knew that the representation showed one way a lesson based on the tangent circle problem could lead to the Tangent Segments Theorem, and they were asked to annotate the lesson with ideas for how to include more discussion. We describe each lesson representation first and the expectations we had for how visualizing each lesson would support each *StoryCircle*. In later sections we introduce the groups that started their *StoryCircle* with each of them and describe what reactions they had and finally what version of the lesson they accomplished at the end of the *StoryCircle*. In all starters of the lesson, the lesson starts by reminding students that the day prior they had defined tangents and introduced the theorem that states that the tangent is perpendicular to the radius at the point of tangency. Then, in all starters, the teacher poses the problem (see Figure 1).

Finally, all three starters feature a final frame in which the teacher is standing at the board with the statement of the Tangent Segments Theorem spelled out (i.e., Two intersecting lines are tangent to a circle if and only if the points of tangency are equidistant from the point of intersection of the lines) and says "Here's the statement of the theorem we just discovered. For homework, I'll want you to write up the proof we just sketched."

Figure 1 *Frame 2 of all three versions of the lesson featuring the teacher posing the problem*

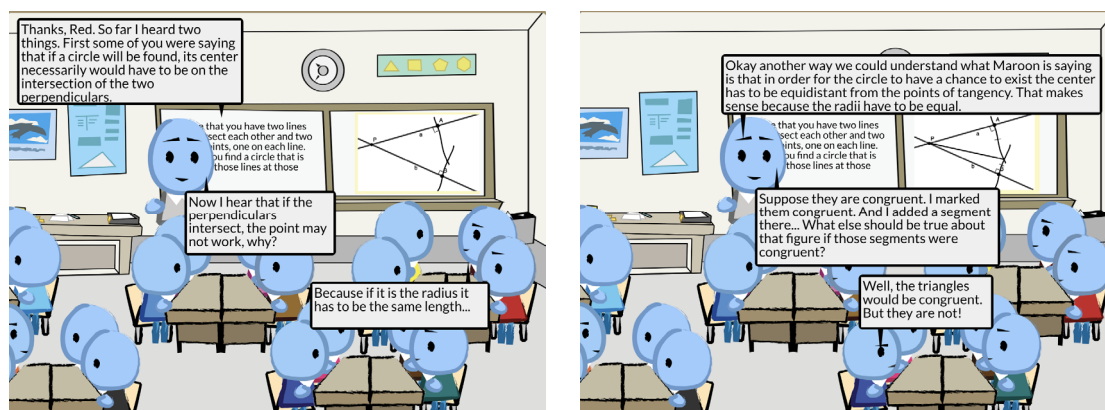


The first lesson representation, 21d0100³) (Starter 1, hereafter), can be described as a relatively dense lesson representation, comprising 30 frames. This lesson representation illustrates the fact that the problem may not always be solvable, as it depends on where students place the points on the line. And the representation demonstrates that if a teacher does not resolve this fact earlier (e.g., asking students to consider carefully where to place the points or placing equidistant points for them), then the exploratory work students do can guide them to deduce conditions necessary for its hypothetical resolution, thus enabling them to formulate the theorem. In particular, the teacher treats an inaccurate diagram as appropriate for a proof and pushes students to reason deductively about the figure it represents (see Figure 2).

The starter shows a lesson that has some student participation, both in thinking about the problem on their own and, later on, in giving some ideas to the teacher who leads the discussion of the problem at the board. In giving this starter representation, and particularly because it relies on the teacher treating the problem as if it has been solved and proceeding with the sketching of a proof that relies on an inaccurate diagram, we expected teachers might predict students' difficulties and hoped that prediction would motivate them to increase the student questions and comments in the discussion parts, involving more classroom discussion.

3) The unique IDs of each storyboard are important mainly for authors to respond to questions from readers that require going back to the data sources. For readers' convenience, we are also providing Starters 1–3 as IDs that support the reading of this article so that readers don't need to attend to the unique IDs.

Figure 2 *Two pivotal frames in Starter 1, where the teacher transitions away from construction and into proof by asking students to reason on an inaccurate diagram*



The second lesson representation, 21DMS0T523 (Starter 2, hereafter), can be described as a sparser representation of the lesson (with frames like the clothespins along a clothesline from which a large sheet would hang), including little more than the beginning and end frames—only some intermediate keyframes are shown that describe transitions to different activity types (e.g., setting students to work in groups, calling students' attention to share ideas, etc.). This starter had only 8 frames and left much for the reader to imagine. We expected participating teachers would have much freedom to choose what to do. We thought this representation would provide an interesting contrast with the prior one in that nothing in the storyboard would have been so well anticipated that participants might feel stifled from using their own ideas. At the same time, this starter had little to take issue with, as even the included frames contained very generic cues (see Figure 3) and meant to simply indicate the passing of time and the changes in activity type. Therefore, we anticipated little that could be countered and only fresh ideas to add about what might happen next or what might have come before.

Like Starter 1, the third starter representation, 21DMSBT523_G3_T (Starter 3, hereafter), is dense with a total of 20 frames. It can be described as a problem-based lesson (in that the lesson uses a novel problem as a vehicle to introduce a new idea) yet the lesson does not involve any student participation. The storyboard shows a monologue, with the teacher answering their own questions (see Figure 4). Clearly, this version of the lesson is inconsistent with what participants are asked to develop. With this starter we hoped people would be able to use a less desirable representation to think with and against—and from that more critical position they could construct alternatives that informed their own eventual depiction (see Brown et al, 2024).

Figure 3 An example frame from Starter 2, showing the teacher redirecting the classes work with a question after reviewing their initial attempts on the problem

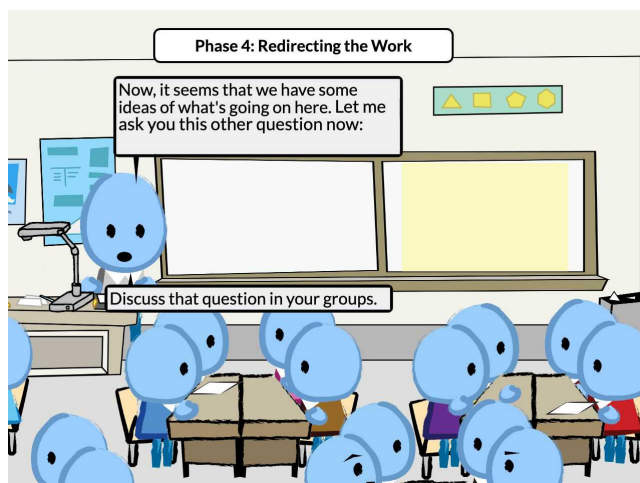
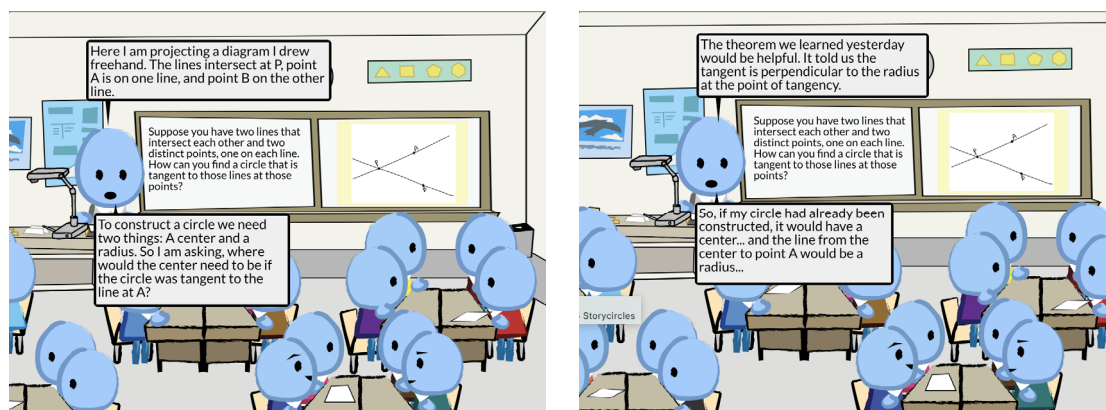


Figure 4 Two example frames from Starter 3, illustrating the teacher-dominated dialogue



V. THE StoryCircles AND THEIR PARTICIPANTS

The three distinct starter scripts⁴⁾ described above were employed by different groups of secondary geometry teachers participating in three different StoryCircles. Starter 1 was utilized by a cohort of 14 secondary geometry teachers from the American Midwest (hereafter, Midwest) within a StoryCircle convened in the spring of 2021 and led by a facilitator named Quincy, who is an experienced geometry teacher herself. Over a two-week span in February and March of that year, participants annotated this specific lesson representation and then moved on to discussing how to revise the storyboard. A total of 78 comments

4) We call these scripts as a shorthand for multimodal scripts, to refer to the phase of scripting which is a component of StoryCircles.

were contributed by 11 of the 14 active participants in the group to the 30-frame starter representation. The engagement levels varied considerably, with some participants leaving only one to three comments (five participants), while others contributed four to eight comments (four participants). Remarkably, one participant left 18 comments, while another participant provided 27. Additionally, certain frames garnered more attention than others, with some receiving no comments (two frames), others receiving only one or two comments (12 frames), and about half of the frames receiving between three and six comments (16 frames). Following the annotation phase, the group expanded the original 30-frame script into a 40-frame storyboard by incorporating 10 new frames prior to the pivotal transition frames depicted in Figure 2. These additional frames delineated various approaches through which a teacher could repair the inaccuracies in the diagram.

Starter 2 was annotated by a *StoryCircles* cohort of five secondary geometry teachers from the Appalachian region of the US (Appalachian, hereafter) assembled in the fall of 2023 and facilitated by Zane, a mathematics teacher educator. This starter script underwent annotation over a two-week period in October 2023. Surprisingly, only one comment was generated during this period, contributed by one of the five active participants, across the eight frames presented. Subsequently, the group revised the eight-frame script, expanding it into a 37-frame storyboard. This revision entailed modifying three existing frames, retaining five frames in their original form, and introducing 29 new frames. Among the newly constructed frames, three provided additional details on how the problem was presented to the class, while 15 elaborated on students' thought processes during the exploration phase. One frame offered more insights into a whole-class review of student work, four frames delved into strategies for redirecting student efforts, and six frames expanded on potential directions for whole-class discussions.

Starter 3 was used by a *StoryCircles* group which included three teachers from the East Coast of the US (East Coast, hereafter) that gathered in Spring of 2024 under the facilitation of Lucy, a mathematics teacher educator and experienced former teacher. This starter script was annotated by teachers over a three-week period in February and March of 2024. Over that span of time, the participants generated a total of 27 comments across the 20-frame storyboard. The levels of engagement differed substantially: one participant left three comments, another provided ten comments, and the last one contributed 14 comments. Additionally, there was a slight variation in the number of comments per frame; five frames received no comments, 12 frames had between one and two comments, one frame elicited four comments, and one frame elicited six comments. Following the annotation phase, the group discarded two of the original 20 frames, revised two of the original frames, and then constructed two new frames. The focus of their revisions was on the first part of the lesson, where the problem is posed. In that section, they revised the way the problem was stated on the board, they also revised or constructed six of the teachers' dialogue bubbles and added six new student dialogue bubbles.

We looked across the contributions participants made to each of these three starters, in terms of the annotations made on the frames. Specifically, using the coding scheme from Chieu et al. (2015), we parsed

the annotations into clauses and then classified clauses into three categories: Evaluative (capturing comments that appraise either the students, the teacher, or other aspects of the frame), Alternative (capturing comments that suggest alternative possibilities for the students, the teacher, or other aspects of the frame), and Reflective/ Interpretative (capturing comments that express inquiry or uncertainty on the actions or otherwise interpret the actions of students, the teacher, or other aspects of the frame).⁵⁾ We also document the revisions made to the storylines. We use both our coding and the documented revisions to illustrate how different starters were associated with different kinds of interactions and opportunities to learn for teachers.

VI. HOW THE *StoryCircles* ANTICIPATED THE LESSON

1. *Midwestern StoryCircle*: Cultivating teachers' dialogue through the consideration of a breach of instruction

Starter 1 prompted the highest number of annotations ($n=78$), averaging about 7 contributions per active participant. The bulk of these annotations (66%, $n=52$) comprised single clauses, such as "I appreciate the diverse student responses." The remaining annotations contained two or three clauses, like "Blue's question greatly aided the class, with additional guidance from the teacher." In total, these annotations could be parsed into 105 distinct clauses.

Table 1 illustrates that a significant portion of clauses focused on students, either offering evaluative remarks (e.g., "Green is smart.", "The students' observations are very insightful so far.") or interpretations (e.g., "The question by Blue might indicate that he is having trouble visually picturing what he needs to construct."). Few clauses suggested alternative actions for students (e.g., "I think this student (and probably others) would need to see that even if we draw a circle that connects the two points; the radii of that circle might not be perpendicular to our original lines."). Clauses referring to the teacher were evenly distributed among evaluation (e.g., "I like how the teacher is encouraging Bluegreen to use their words to explain their thinking."), reflection (e.g., "Teacher helps but it is still a student lead conversation"), and suggestions for alternative teacher actions (e.g., "At this point; the teacher should have drawn 2 intersecting lines and the points referenced").

5) Chieu et al.'s (2015) coding scheme is categorical, hence it does not posit any hierarchy among those codes, in spite of the use of similar words as in other coding schemes that consider those codes as part of a hierarchy.

Table 1 *Distribution of comments from the annotations of Starter 1*

	Evaluative	Alternative	Reflective/ Interpretive	Total
Student	23%, $n=18$	4%, $n=3$	32%, $n=25$	44%, $n=46$
Teacher	16%, $n=13$	19%, $n=15$	17%, $n=13$	39%, $n=41$
Other aspects of frame	8%, $n=6$	5%, $n=4$	9%, $n=7$	16%, $n=17$
Total	36%, $n=37$	21%, $n=22$	43%, $n=45$	$n=104$ comments

The evaluative clauses were predominantly positive (80%, $n=30$), both towards students (e.g., "Love that the students explored the theorem and reasoning instead of just reading the theorem and applying it!") and the teacher (e.g., "I like the setup here. The teacher has proposed a question and has not provided a diagram; allowing for more free thinking."). However, a few negative evaluations surfaced across the participants' annotations. The bulk of these kinds of evaluations were elicited by frames depicting students engaged in a debate about whether or not it is possible to find a circle tangent to the two lines from the points provided by the teacher (which appeared not to be equidistant from the point of intersection). Across the frames representing this discussion, participants made evaluative comments about the teacher such as "Missed opportunity for students to explain and show thinking. The practice of doing and verbalizing is important to the learning process." They also made critical comments about the students such as "I would find what '[B]rown' said to be slightly inappropriate." as well as about the tenor of the overall interaction, "Is it just me or did this conversation escalate quickly." As the represented dialogue unfolded, the represented teacher is shown bringing a student named Red up to the board to share his approach (see left column of Figure 5) which includes an imprecise sketch of the construction, participants' critiques of both the teacher and the students continued along with alternate suggestions for the ways things could have gone (see right column of Figure 5).

Given the abrupt transition in participants' annotations from predominantly positive to negative evaluations, accompanied by corresponding suggestions spanning frames 16 to 23, it's hardly surprising that this StoryCircle opted to create ten additional storyboard frames to bridge the gap between frames 21 and 22. These frames depict six alternative scenarios illustrating how the situation presented in frame 21 could have unfolded differently to address the issue of imprecision in Red's diagram (see Figure 6). These variations include envisioning Red having a different, more precise contribution to start with (Alternative 1); having the class or the teacher assist Red in moving towards a more precisely constructed diagram (Alternatives 2, 3, 5); the teacher posing the problem differently (Alternative 4); and the teacher framing Red's solution as incorrect but valuable for consideration (Alternative 6).

Figure 5 Frames 21, 22, and 23 from Starter 1 and some of the teachers' annotations provided by teachers

The figure displays three sequential frames (21, 22, and 23) from a video commentary tool. Each frame shows a classroom scene with blue cartoon characters and a whiteboard. The interface includes a 'Comments' sidebar on the right with teacher annotations.

Frame 21: The whiteboard shows a diagram of two intersecting lines, a and b , with a point P on line a and a point Q on line b . A line segment connects P and Q . The text on the whiteboard says: "See what happens when you actually draw the perpendiculars". The characters are discussing the point where the lines intersect.

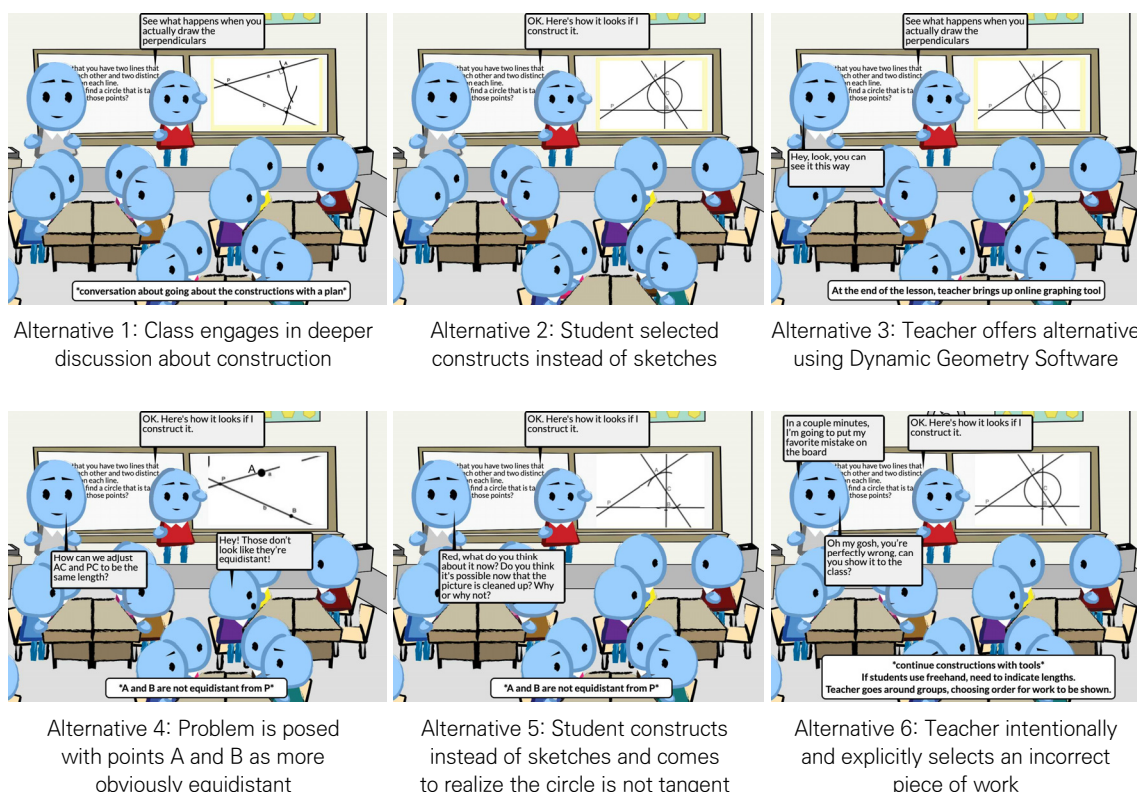
Frame 22: The whiteboard shows a diagram of two intersecting lines, a and b , with a point P on line a and a point Q on line b . A line segment connects P and Q . The text on the whiteboard says: "Thanks, Red. So far I heard two things. First some of you were saying that if a circle will be found, its center necessarily would have to be on the intersection of the two perpendiculars." The characters are discussing the intersection point.

Frame 23: The whiteboard shows a diagram of two intersecting lines, a and b , with a point P on line a and a point Q on line b . A line segment connects P and Q . The text on the whiteboard says: "Okay another way we could understand what Maroon is saying is that in order for the circle to have a chance to exist the center has to be equidistant from the points of tangency. That makes sense because the radii have to be equal." The characters are discussing the equidistant property.

Comments:

- Slide 21:**
 - Pink Hummingbird** (March 6, 2021): Love the student being able to share his work. Love the class discussion.
 - Maroon Jackal** (March 7, 2021): Why did the teacher shift from "you tell me what to do" to having a student come up and draw?
 - Maroon Cheetah** (March 7, 2021): I would encourage them to not use freehand to draw as it is not precise.
- Slide 22:**
 - Beige Panda** (March 1, 2021): At this point, I am wondering if we need to stop a minute and have some students restate what the teacher/maroon said. The teacher used a LOT of mathematical vocabulary and it may be a good time to have a few students restate what we know and what we are trying to figure out so that more students are engaged and understanding.
- Slide 23:**
 - Beige Panda** (March 1, 2021): At this point, I am wondering if we need to stop a minute and have some students restate what the teacher/maroon said. The teacher used a LOT of mathematical vocabulary and it may be a good time to have a few students restate what we know and what we are trying to figure out so that more students are engaged and understanding.
 - Pink Hummingbird** (March 6, 2021): Love the vocab. Great questions. The students are starting to piece things together.
 - Maroon Cheetah** (March 7, 2021): By the way, why are all the kids blue? Would it be nice to have some different colors there? But maybe it would be too distracting.
 - Orange Ape** (March 6, 2021): The teacher really sums it up with her statement. I like the fact that the kids are starting to work there way back to congruent triangles.
 - Orange Ape** (March 6, 2021): This is a good place to have a reminder about "drawings may not be to scale" and what that means.

Figure 6 Six alternative ways represented by participants for handling a teachers' puzzling decision to ask students to work on a proof using an inaccurate diagram



2. Appalachian StoryCircle: Nurturing teacher collective lesson visualization through a sparser representation

As previously observed, Starter 2 garnered the fewest annotations, with only one participant leaving a comment ("What is the other question?") upon reaching the sixth frame of the representation (shown in Figure 3). The scarcity of annotations may owe to the fact that the starter had so little mathematical interactions to react to. When coupled with this solitary remark, there may also be reason to believe there was a degree of confusion among participants regarding what to make of this sparser representation, unclear for example, why the frames were so generic and missing so much information about what might have happened from one frame to the next. Nonetheless, such a representation, characterized by significant time gaps between frames, invites participants to embark on a sort of choose your own adventure journey (Goodbrey, 2013; see also McCloud, 1993). This aptly describes the activity facilitated by this particular starter, as evidenced by the group's abundant construction of new storyboard frames: modifying three existing frames and crafting 29 new ones. The resulting revised storyboard bears little resemblance to the original, with only five frames retained in their initial form, three of which remained unaltered as per request

of the PD designers.

Throughout this reimagined lesson, the group of participants introduced crucial scaffolding elements to ease students into the geometric challenge at hand. A handout displaying the initial intersecting lines offered a visual anchor, aiding students as they weighed the mathematical implications of which points to select along those lines. Additionally, they incorporated 15 examples of student work showcasing diverse approaches to the problem, thereby illustrating how a teacher could address such variations within the context of small group instruction. These examples ranged from very rough sketches to well-articulated constructions, work well aligned with the lesson's objective (i.e., identifying the necessity of equidistant points A and B for tangent circle formation), and work in which students complied with the request to produce a tangent circle.

As the storyboard unfolds, the mathematical ideas bubble up from ongoing conversations between the teacher and students. Participants used the frames to articulate students' challenge of choosing the center of the circle, showing students searching along the perpendicular from point A , highlighting that such a selection ensures tangency to only one line. The issue remains unresolved for quite some time, with the participants electing to leave the spatial relation to the other line unresolved. As the storyboard unfolds, so does the mathematical complexity; it captures the introduction of complex concepts like the fact that the intersection of two constructed perpendiculars represents the precise center of the desired circle—illustrating the crucial moments of insights as ones that emerge from students' independent work on the problem which is an important realization for teachers' own professional development.

The storyboard deftly captures the delicate balance of teacher guidance and student exploration. During the whole-class check-in and redirect sections of the lesson, one scene begins with students' frustration that they could not find the circle and then eventually illustrated the students' realization that the radii must lie on the perpendicular lines constructed from each of the given points. This vignette is emblematic of the learning landscape that teachers navigate—a terrain where perseverance, frustration, and eventual revelation intertwine to shape the profound learning experiences at the heart of problem-based instruction (Smith, 1996).

Finally, the storyboard concluded by bringing to life scenes from the whole class discussion featuring students, one after another, approaching the board to share their work and emerging conjectures. Early frames in the discussion depict students sharing their early thinking such as pondering how to get a circle that is tangent to both lines rather than just one line or realizing that the two points determined by the arc used to construct an angle bisector as useful for finding two points where a tangent circle can be sketched. These less developed ideas eventually lead to discussions regarding what had to be true about the configuration. The whole class discussion culminates with an exploration and eventual proof of the conjecture that the two points must be equidistant from the point of intersection to make a tangent circle through those points to those lines possible. These later frames feature one student proving a pair of congruent triangles by Hypotenuse-Leg (HL) congruency and then another student leveraging the fact that Corresponding

Parts of Congruent Triangles are Congruent (CPCTC) to cement the fact the points of tangency must be equidistant for the circle to be tangent to both lines.

Across the storyboard, the represented teacher scaffolded responses serving to coalesce students' contributions into a collective understanding of the Tangent Segments Theorem. This storyboard not only showcased the range of moves a teacher might employ to guide students towards key conjectures but also encapsulated the instructional journey of transitioning from individual student exploration to collaboratively proving a new mathematical theorem. This is the crux of the lesson—that through hands-on practice with these mathematical ideas, teachers themselves undergo a parallel process of exploration and insight, developing their own expertise by melding their content knowledge and pedagogical know-how.

3. East Coast Story *Circle*: Challenging instructional norms and amplifying teachers' perspectives

While participants only made 27 annotations when visualizing Starter 3, these included clauses about instructional practice that are notably different in nature from those crafted in response to Starter 1 (see Table 2). Specifically, Starter 3 compelled participants to share a disproportionately high number of alternative ideas (70%, $n=19$), for both how the teacher and students might contribute within such a lesson. Furthermore, while there were only a handful of evaluative clauses offered, these evaluations were uniformly critical of the teacher, suggesting dissatisfaction with how the teacher taught the lesson (e.g., "After 'listening' to the teacher do all the talking; I disagree that the teacher could characterize it as an 'exploration' and claim that 'we just discovered' a theorem.") The participants expressed what they believed was flawed about the instructional approach throughout their annotations; with comments focused on the problematic aspects of the teacher-centric delivery, noting the ways that such an approach suppressed student engagement and exploration in mathematical reasoning (e.g., "Students might be thinking 'Why is the teacher talking about the wrong way to do this; it's confusing'", "Students might be thinking 'Duh!'").

Table 2 *Distribution of comments from the annotations of Starter 3*

	Evaluative	Alternative	Reflective/ Interpretive	Total
Student	0%, $n=0$	44%, $n=12$	7%, $n=2$	52%, $n=14$
Teacher	15%, $n=4$	26%, $n=7$	7%, $n=2$	48%, $n=13$
Total	15%, $n=4$	70%, $n=19$	15%, $n=4$	$n=27$ comments

While at first glance, Starter 3 may have seemed inferior to Starter 2 in enabling the development of a problem-based lesson that included discussion, closer examination suggests more complexity to the outcome. Participants working with Starter 3 managed to revise only the first six frames of the storyboard; however, their ambition (as evidenced by the annotations) was not limited to merely tweaking the launch of the lesson. Their goal was to pivot the lesson's trajectory entirely, an endeavor that naturally has far-reaching

implications on the storyboard's subsequent frames. This ripple effect potentially rendered the task more challenging, as it might have been simpler to begin anew—as was the case with Starter 2—rather than amend an existing structure.

The breadth and scope of the revisions envisioned by participants working with Starter 3 differed notably from the single-issue focus that emerged within the revisions participants made in reaction to Starter 1. Instead of generating numerous alternatives for a single issue the group found objectionable, those working with Starter 3 were navigating a plethora of possible changes prompted by a critical examination of many problem points throughout the lesson. This broader perspective could have contributed to the group feeling overwhelmed, which may explain why their progress stalled after modifying only six frames.

The participants' approach to the first four frames of their revised storyboard (visible in Figure 7) showcases a significant investment in constructing a better lesson, as evidenced by their crafting of many new dialogue bubbles and modifications to the existing ones, alongside multiple adjustments to the board content. They were hesitant to discard the entire storyboard, choosing instead to meticulously revise and adapt, weaving their modifications into a coherent narrative. This intricate decision-making process might have slowed their progress as they deliberated on what to retain, revise, or create anew.

The numerous annotations peppered across the storyboard show several ideas for teaching the lesson differently. This suggests that, while the progress in terms of frames altered may appear modest, the breadth and depth of the participants' engagement and critical thinking might ultimately yield a more substantial and nuanced transformation of the lesson narrative.

Figure 7 *The first four frames of the newly revised storyline (on the right) which emerged from review of the first six frames in Starter 3*

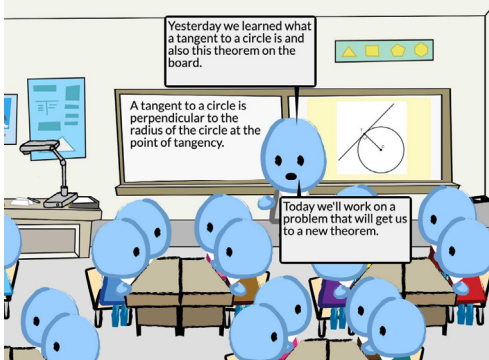
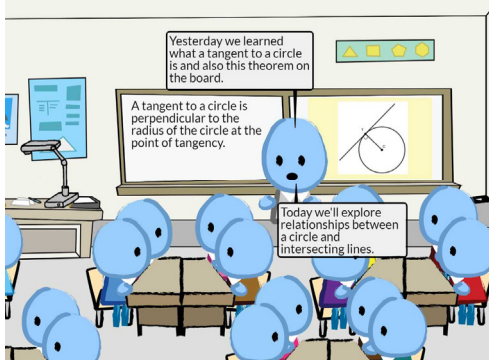
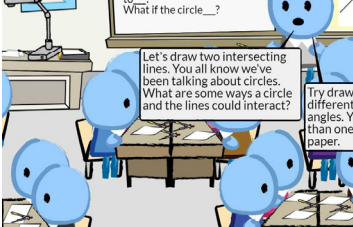
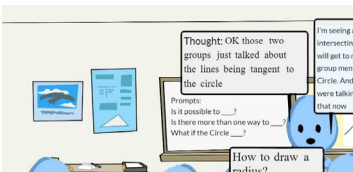
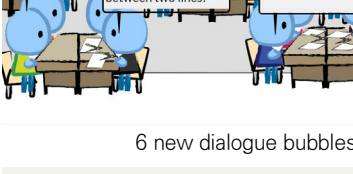
Original Frame #	Starter 3	Revisions or New Frames
1		 1 dialogue bubble revised

Figure 7 Continued

Original Frame #	Starter 3	Revisions or New Frames
NEW		 2 new dialogue bubbles Left-side board newly constructed
NEW		 6 new dialogue bubbles
6		 3 new dialogue bubbles, 1 dialogue bubble revised, new board content

4. A graph of the three anticipations of the lesson as a representation of the learning opportunity offered by the Tangent Circle Lesson

In Figure 8, the branching tree diagram serves as a visual metaphor, tracing the unique paths taken by teacher-participants as they grappled with anticipating teaching the Tangent Segments Theorem through the lens of their assigned starter scripts. Anchored in a shared engagement with this open problem set against the backdrop of high school geometry, the trunk of our tree splits into three branches, each diverging according to the script from which it sprang—hereafter marked as Anticipations 1, 2, and 3 corresponding to their sourcing starters.

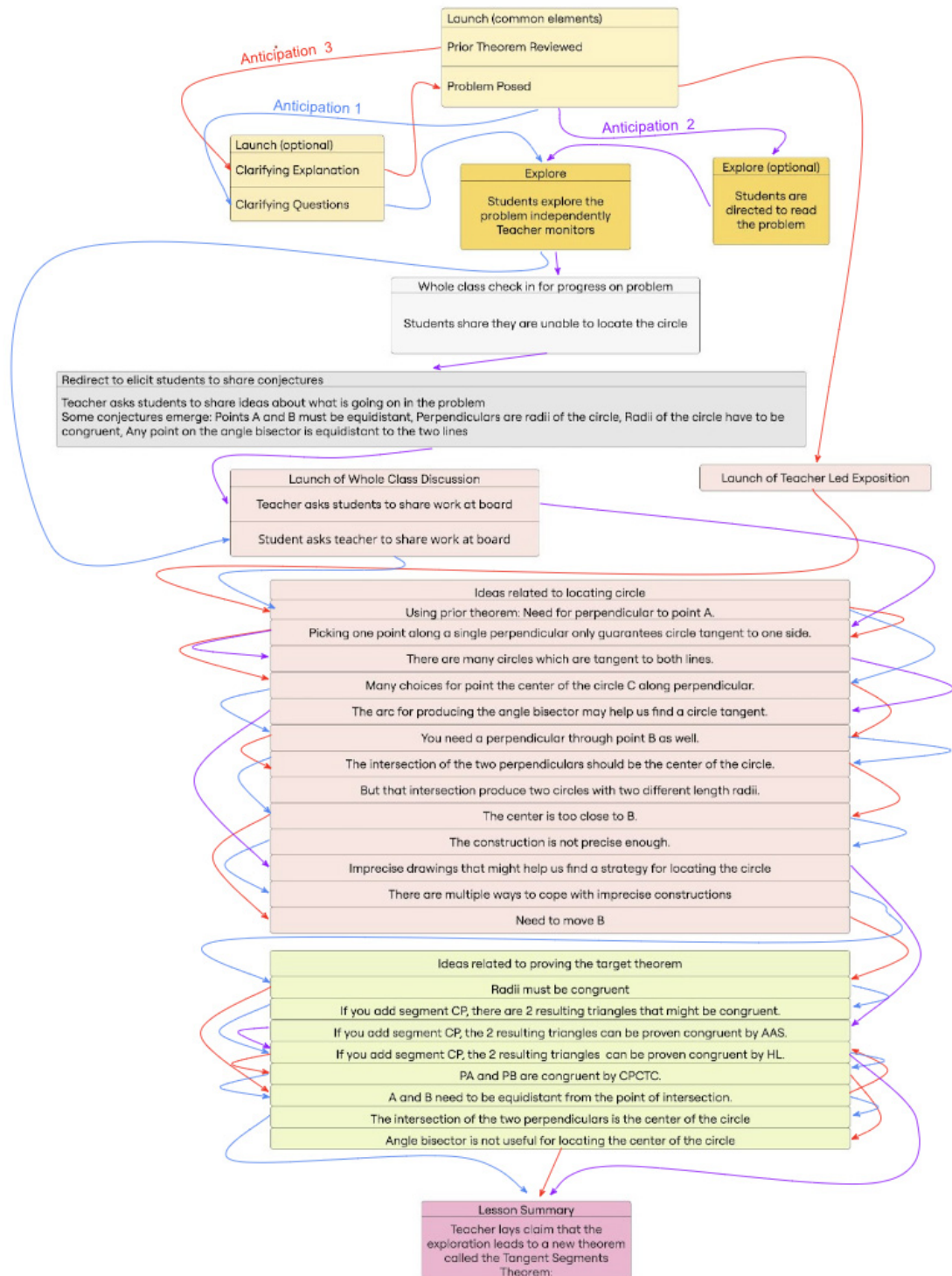
Anticipation 1, traversed by the blue line, displays a robust interaction with the originally presented frames and emerged from a plethora of annotations and insights for nuanced lesson enhancement. The annotations, largely concentrated around the crucial shifts between student exploration and theorem proving, echo a collective commitment to delving into the critical instructional moments of the lesson. Significantly, this pathway focused on resolving inaccuracies in students' diagrams through the construction of a diverse set of strategies for facilitating whole class discourse. In this way, the group's interactions provided participants with ample opportunities to discuss the critical role that diagrams play in developing and solidifying geometric reasoning.

Emerging from a solitary annotation, Anticipation 2, traversed by the purple line, underwent a notable growth from its starter, adding substantial new elements to the storyboard. Despite its sparse beginning, this representation encompasses significant portions of the pedagogical journey both akin to and beyond what the other groups were able to accomplish. The sparse nature of Starter 2 provided fertile ground for ideation, unfolding into a rich storyline of envisioned classroom interactions and pedagogical decisions.

In Anticipation 3, traversed by the red line, we observe a narrative marked by sincere, albeit restrained, engagement in whole class discussion. The moderate volume of annotations suggests a deliberate focus on reshaping the teacher-driven content. However, the revisions in that direction were only evident at the beginning of the lesson, where the teacher-participants envisioned more exploratory experiences for students. Notably, while the annotations represent a desire for a substantial shift away from a teacher-centric approach across the entire storyline, the resultant branch conveys this ambition as a project in progress.

In considering the nuances of each anticipation of the lesson—shown in Anticipations 1, 2, and 3—it becomes evident that this graph shown in Figure 8 not only chronicles varied instructional trajectories but also illuminates a spectrum of opportunities for learning through practice for the teacher-participants that constructed them. The resulting alternative storylines are not only different ways to teach the lesson but represent the distinct possibilities of practice that each group considered while engaging with and refining their storyboard. Each branch unfolds a different set of experiences wherein participants grappled, adjusted, and anticipated instructional moves, allowing them to practice aspects of teaching within a supportive yet challenging context.

Figure 8 Pathways through the Tangent Circle Lesson emerging from the three starter scripts



As we delve into the points of convergence across these storyboards, we are compelled to consider what constitutes the essence of these narratives. While each Anticipation took a distinct form, influenced by its originating script, there is resonance in how each group's exertions interpreted the task of teaching the theorem through a problem. Each storyboard illustrates elements of students' understanding of the problem and the use of relevant previously learned ideas in addressing the problem as the class moves toward the new theorem. They do so within practices that might be called generic, such as launch, explore, discuss, and summarize, but draw substantively on elements of the disciplinary context of this lesson, incorporating mathematical nuances that build toward the theorem. They all in their own ways illustrate what the quote from Brousseau (1997, p. 81, cited first in footnote 3 means ("The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory..., not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc." (p. 81)

This suggests the question—what is the source of these diverse ideas? Or more specifically, from where in the discipline of mathematics come the elements that put a problem-based lesson in its disciplinary context? Though each of the experiences that teacher-learners had in developing each storyboard contributes to the learning of the group that produced it, Figure 8 provides a glimpse of an aggregate. Each new StoryCircle focused on the Tangent Circle Lesson provides material for a conceivable extension of the graph shown in Figure 8, which presents potentially a larger narrative—a narrative of shared content, a lesson as an element of a multiverse of sorts yet anchored in the disciplinary neighborhood of the Tangent Segments Theorem. Though the actual storyboard a group creates represents the work that group of teachers did to anticipate the Tangent Circle Lesson, Figure 8 represents the aggregate opportunity to learn offered to the participants of the three StoryCircles focused on the Tangent Circle problem. Moreover, Figure 8 suggests a larger, maximal graph, where all the things that could possibly feature in any storyboard of the same lesson could be found. Conceptualizing what such maximal graph represents is important because it accounts for the opportunity to learn available for anticipations of this lesson. Thus, there is the actual lesson anticipated and the potential aggregate, maximal lesson as an instructional goal for teachers (Chazan & Herbst, 2023). This suggests that while the actual learning pathways diverged, the destination remained stable: Anticipating how to support students in engaging deeply with the problem towards reaching an understanding of the theorem at stake. As we conceive StoryCircles as a concrete pedagogy of practice in which teacher learners practice teaching attending to its disciplinary context, conceptualizing this opportunity to learn seems essential in order to decide which lessons are particularly good for StoryCircles and for the larger enterprise of teacher development.

In sum, what these storyboards offer goes beyond exercises in lesson planning; they are instruments for shaping the praxis of mathematics education. They are manifestations of the varied experiences that, when synthesized by an adept educator, have the potential to transform into a cohesive understanding

of what teaching this lesson entails, which can contribute to understanding what teaching mathematics through problems might mean. Such an understanding is vital to the proliferation of the knowledge base for teachers and exemplifies the profound potential for growth that lies at the heart of the kind of practice-based teacher development envisioned by Hiebert and Morris (2009)—a system for transforming teaching (rather than teachers) through the collective development and sharing of lesson artifacts.

VII. DISCUSSION: *StoryCircles* AND LESSONS

The foregoing description of how the three starters were commented on and led to three eventual anticipations of the Tangent Circle Lesson illustrates the great variability that can be found within a lesson which has been anticipated using the *StoryCircles* process. Different storyboards were chosen as starters to explore affordances and liabilities of each for getting a group of teachers going into anticipating a lesson, under the premise that this would be facilitated by engaging participants in an initial visualization of the lesson. This, itself, had been a design principle we discovered after earlier iterations that only started with a problem and a goal and where participants spent most of their efforts in the launching of the problem, dedicating little thought to activity types that would happen later in the lesson (e.g., discussion). The three starters chosen all meant to prime the participants' initial discussions about what the full lesson should include. All three of these starters assumed that the class shared the same prior knowledge (i.e., prior definition of tangent and knowledge of the theorem that tangent is perpendicular to radius), started with the same problem, and pursued the same instructional goal—the Tangent Segments Theorem. But the starters also suggested different features in the lesson that participants might react to. Along those lines, Starter 1 represented a lesson with qualities similar to what was asked of the group (e.g., it involved classroom discussion) but had the teacher doing something unexpected (e.g., providing a diagram that was not accurate and expecting students to reason deductively from it). Starter 2 included very little substance and only suggested some changes in activity types that would take place along the lesson. And Starter 3 was a fully developed problem-based lesson that did not include any classroom participation; in that sense, it was structurally unresponsive to the goal of the *StoryCircles* though it did present a few of the mathematical ideas constitutive of the meaning of the theorem. In this discussion, we take both the similarities and the differences among the storyboards involved to consider the fundamental question that underlies this paper: How should *lesson* be conceptualized to serve the *StoryCircles* process and more generally to serve the anticipation of a lesson as an approximation of practice in a context that makes room for the demands of the discipline?

Our analysis showed that the lessons anticipated through *StoryCircles* are sensitive to features of the starters. Starter 2 was surely the one that assumed the least about the lesson and participants did eventually produce an anticipation of the lesson that contained several details originally absent in the starter. But

its initial annotation provided little cues as to how participants arrived there; it seems that Starter 2 was not much of a source of arguments about what to do and the subsequent scripting instead relied on a shared sense that the starter needed filling in. In contrast, Starters 1 and 3 provided a lot of detail. Starter 3 seems to have been efficacious in eliciting participants' appraisals indicating preferred alternatives—even if the work to flesh out those alternatives seemed to demand more time than what participants had available, to the point that they only redeveloped one section of the lesson. Starter 1 was not so efficacious in eliciting alternatives and participants' edits on the script mostly added to it rather than replaced events in it. All three *StoryCircles* ended up producing different storyboard representations to anticipate the Tangent Circle Lesson.

A frame-by-frame comparison could show where the ending storyboards were similar and different; though such work is premature at this stage, Figure 8 summarizes what the ending storyboards shared (nodes common to different paths) and how they differed (paths of different color joining different nodes). Underlying such observations is the assumption that these final storyboards are comparable and our goal here is to conceptualize this comparability against the background metaphor that a maximal graph containing all possible anticipations of the lesson constitutes the learning opportunity of this approximation of practice. Thus, the comparability among storyboards goes beyond counting the number of frames or the number of comments that one might use to argue for the efficacy of a starter to enable a group to work. This comparability alludes instead to the various events that each of the completed storyboards include and how those refer to possibly expected events in the lesson. The fact that in all of the final storyboards the same problem is posed and the same goal is aimed for, permits to compare the storyboards in regard to how different groups filled the span of time between the posing of the problem and the claiming of the instructional goal. The storyboards achieved by the three *StoryCircles* permit to inductively build the categories of events one might expectedly see in any anticipation of the lesson.

The storyboard produced by the East Coast group, for example (see Figure 7, right hand side) shows the teacher doing more than stating the prior knowledge for students. The teacher engages students with the prior knowledge and uses it to frame the problem. One might notice that, as originally posed (see Figure 1), the problem did not include a diagram and it asked students to "find" a circle; hence, it might not seem so clear to students what they might be expected to do. These observations suggest that analysis of the storyboards eventually constructed could include noticing whether and how participants script the teacher connecting the prior knowledge to the problem of the day or using any instructional situation (e.g., construction, proof) to frame the problem as something that the class knows how to do (see Herbst et al., 2023).

The storyboard produced by the Appalachian group illustrates another important possibility from the lessons anticipated via *StoryCircles*. Starter 2 suggested some time in which students would work on their own and when the teacher might visit with individuals or groups, providing tailored feedback. As these *StoryCircles* included some resources for scripting, such as samples of what students might have done

in response to the original problem, these could be picked up by groups of participants scripting the lesson and those resources could anchor local comparisons across anticipations of the lesson. Once an instance of student work (e.g., a diagram showing the putative tangent circle) was included, analysis could focus on the events that surround such inclusion, such as how the student work was introduced in the lesson (e.g., as an individual idea responded privately by the teacher, as a public contribution) and what students were made to say if and when presenting the work. Said student work could also be analyzed in terms of how it was attended to, that is, whether the teacher introduced it publicly in some form, invited comments about it from the class, or, otherwise, how the teacher responded to it.

The storyboard produced by the Midwestern group illustrates how the process of iteratively representing a lesson using storyboards allows for problematizing events that at one time might have been seen as unproblematic. Thus, while the transition between frames 21 and 22 in the starter might be understood by assuming the teacher had seen and heard things not present in the storyboard, participants might find different meanings to that event by proposing alternative ways in which the event might have come to be. Thus, uncovering discontinuities and resolving them with different alternatives is yet another way in which different anticipations of a lesson might differ.

1. What we should say a lesson is

Over those local differences, we want to conceptualize the commonality of pursuing the Tangent Segments Theorem through the posing of a problem that draws on the same prior knowledge. To provide a first working definition of *lesson*, we start with instruction and its fundamental connection to the discipline of mathematics. Instruction refers to the transactions of content between students and their teacher in the environment of a course of studies (Cohen et al., 2003). As Herbst and Chazan (2020) have noted, these transactions involve, for the teacher, the need to consider the possible symbolic exchange between the work students do in the context of activities the teacher organizes for them and the instructional goal(s) on whose behalf those activities are organized. The discipline of mathematics plays a key role as stakeholder of that instructional exchange, providing warrants for the elements of content that become instructional goals and providing resources for the teacher to connect observations of students' work to the instructional goals—for example, in the form of mathematical properties that justify the teacher's noticing students' emergent behaviors as serviceable for the development of mathematical ideas at stake (Herbst et al., 2023).

Lesson plans often include goals and activities to accomplish the goals. Among the goals are to develop understanding or skill with ideas or processes; among the activities, teachers sometimes rely on generic activity sequences such as "I do, you do, we do" or "Launch, explore, discuss, summarize." Prior scholarship on lessons (e.g., Leinhardt, 1989) brought in cognitive science constructs, such as *agenda* to name teachers' mental representations of goals and activities. We take the categories of *goal* and *activity* initially at face

value and use this to provide a provisional definition of a mathematics lesson that tracks the notion of instructional exchange recalled above. We will say that a mathematics lesson is a set of activities whose pursuit over time the teacher may take to mean that a specific, pre-established instructional goal has been met. Note that in this definition a lesson may unfold over variable time: Though one could see one such lesson taking a whole class period, some lessons might take more time and some lessons might take less. *Lesson* as a category of instruction events that could be used to organize approximations of practice can be compared and contrasted with categories of events of larger grain size such as courses of study (e.g., high school geometry) which can take a semester or a year, or instructional units (e.g., circles), which can take a couple of weeks, and with categories of events of smaller grain size such as the question-answer exchange, which can take around a minute, or distinguishable lesson segments or instances of activity types (e.g., the launch of a task), which can take several minutes. In order to be able to compare the three anticipations of the lesson produced by our three *StoryCircles*, and each of them with the starters given to each group, we propose that a lesson is a set of activities that put an instructional goal at stake and assumes the same initial conditions (particularly the prior knowledge students are expected to have, given the placement of the lesson in a unit of instruction of a course of studies). As noted, a lesson can take an amount of time of the order of a class period but not be defined by that duration. Structurally, lessons are sequences of activities or segments composed in turn of several discursive exchanges; lessons themselves are components of instructional units and these units are components of courses of study. Within that hierarchical structure for parsing teaching, a lesson is distinguished by being associated with a specific instructional goal (that is for generating interactions that can be exchanged for a claim on an instructional goal). This definition allows us to argue why lesson, so defined, is a useful category to build approximations of practice that put practice in a context in which PT can attend to the disciplinary obligation. This possibility lessons offer for teaching practice to attend to the demands of the discipline relies just as much on the instructional goal as on the activities to be done to lay claim on it.

Indeed, while from the perspective of the lesson planning documents used in schools or in teacher education classes, instructional goals and instructional activities might strike the reader as instruments with which the institution of schooling exercises an influence in teaching, we also see them as potential⁶⁾ instruments for the influence of the discipline. More often than not, for something to become an instructional goal in mathematics, a disciplinary warrant is expected to be available, even if buried in and transformed by the history of the curriculum (Chevallard, 1991). Given an instructional goal for a mathematics course, it is likely that the discipline of mathematics provides the warrants for it as something that can be known, including selecting the predicates that befit each item of content (be this item a definition, a proposition, a proof, etc.) as knowledge: predicates such as consistency, truth, validity, etc.⁷⁾

6) We say potential because instructional goals could still exist that are unaccountable to the discipline of mathematics for warrants—the introduction of new technologies, for example, can require substantial instructional activities to the point of becoming instructional goals themselves before being used to learn any mathematical idea.

For example, in the course of studies in geometry, which US students take in their first or second year in high school (ages 14 or 15), one instructional goal is to understand and be able to use the Tangent Segments Theorem which asserts that two lines tangent to a circle that pass through a common exterior point, have their points of tangency equidistant to the common exterior point. The field of Euclidean geometry provides the disciplinary warrant for this instructional goal, including precise concepts of tangency and equidistance, and an array of theorems (e.g., the tangent is perpendicular to the radius at the point of tangency) that permit proving the Tangent Segments Theorem as true for any circle and any point exterior to the circle. Regardless of whether the class will see the proof of the theorem, the fact that such a proof exists warrants the Tangent Segments Theorem as something that can be known to be true and hence can be chosen as instructional content. The Tangent Segments Theorem is thus a warrantable instructional goal in geometry.⁸⁾

The activities by which an instructional goal can be claimed as accomplished are similar to the instructional goal in being, primarily, institutional resources for the regulation of instruction. That is, for the teacher to claim that an instructional goal has been accomplished, the class needs to have done something about it, however bureaucratic that can be (e.g., memorizing pages of a textbook). However, the institutional requirement of these activities allows room for the discipline to exercise its influence. In traditional university instruction, the students' following of the instructor's presentation of the statement and proof of a theorem in class have often been the activities that warrant the instructor's claim that the theorem has been taught. These activities can vary in how they involve the students in doing mathematics. Memorizing pages of a textbook, following a presentation by the teacher (as shown in Starter 3), or working independently on a problem are among possible genres of activities that might differ in the extent to which the students' actions may be described as doing mathematical work. Though any and all of those genres of activities might be involved in a lesson that aims at the same instructional goal, because the specific design of these activities can involve students in work that is more or less mathematical, the activities themselves are another locus in which the obligation to the discipline may be anchored. Because those activities are not equally compelling as mathematical work, it is useful to be able to define lesson in such a way that multiple different activities can be enacted to arrive at the instructional goal: If those different activities belong to different anticipations of the same lesson, these anticipations can be compared in terms of their mathematical qualities.

Brousseau's (1997) theory of didactical situations (TDS) contributes an important element to this

7) In this sense instructional goals in mathematics are different than instructional goals in other school subjects which might rely on other disciplines for stewardship, as those other disciplines may favor completely different ontologies, epistemologies, and axiologies.

8) Incidentally, this mathematical warrantability of knowledge might point to a different way in which teacher education can be practice-based. Though the institutional obligation calls teachers to accept what is included in the curriculum as content to be taught, it is not immediate that all content to be taught is warrantable as something that can be known mathematically. Mathematics teacher preparation (particularly in mathematics courses for teachers) might benefit from engaging PT in warranting mathematically the content they have to teach.

consideration. The foundational postulate of TDS is that for each item of knowledge at stake (i.e., each item of knowledge to be taught and learned) there is at least one fundamental situation that gives that knowledge its meaning. This fundamental situation is a problematic context in which the item of knowledge makes a difference or is worth asserting, against the background of many other possible statements that could have been made instead. This postulate is what warrants the engineering of didactical and a-didactical situations where students can participate in mathematical work that eventuates with the introduction of new knowledge. For our consideration here, this suggests that every instructional goal can be warranted in some mathematical work that constructs the meaning for the knowledge at stake. Note that this work may or may not be how items of knowledge are warranted in textbooks. For example, textbooks don't always provide intellectual motivation before they define concepts or state theorems; the so-called definition-theorem-proof sequence (Weber, 2004) tends to hide precisely the fundamental situations that make such concepts and theorems worth defining or stating. But the history of the ideas organized by a field of study (e.g., the history of geometry), the textual development of those ideas (e.g., the definitions and theorems immediately before in a particular geometry text), or some field of application of those ideas can help identify an intellectual context that gives meaning to that specific item of knowledge which is now an instructional goal. In other words, the discipline of mathematics as a practice and the knowledge environment in the neighborhood of an instructional goal (e.g., the prior knowledge that students can be assumed to have) also suggest elements for the activities that might lead to claiming that instructional goal as accomplished. As we look into lessons and the role that their anticipation can play as approximations of practice that connect teaching practice to its disciplinary context, we will want to highlight the possibility that these lessons include activities in which students' work creates mathematical meaning for the items of knowledge which are the instructional goal of lessons.

Our definition of lesson thus contains the instructional goal, the prior knowledge that supports providing motivation and warrant for that instructional goal, and a variable, extensive set of potential activities in which the latter can be articulated to warrant claiming the former. This variable and extensive set of potential activities is illustrated in the set of nodes shown in Figure 8, while the lesson itself is illustrated as the set of all possible anticipations (i.e., the paths) connecting the problem and the goal in Figure 8. Lessons, so defined, are good units of teaching practice for the creation of approximations that put practice in its disciplinary context. Lessons, thus defined, allow for practicing not only the installing of items of knowledge at stake but also the development of their meaning, one of whose options is through engagement in activities that foreshadow the knowledge at stake. More specifically, these activities permit students to produce evidence that they are developing a sense of the intellectual need and warrant for the instructional goal. As illustrated, patently, in teachers' comments on Starter 3, the evidence offered in that storyboard was nowhere close to warranting that students had discovered the theorem. Our definition of lesson affords us the possibility to see alternative activities (e.g., having students answer the questions the teacher posed instead of having the teacher answering them) starting from the same problem and

ending with the same instructional goal.

More importantly, this definition of lesson, as a design concept that complements the approximation of practice we call anticipation of a lesson as produced through *StoryCircles*, permits a group of teachers to engage in arguments about the different activities that feature in a lesson. In particular, they can engage in arguments about competing activities, distinguishing them in terms of their merits providing evidence of mathematical work which serves to warrant the item of knowledge at stake. As Chazan and Herbst (2023; see also Herbst & Chazan, 2023) showed, the concept of instructional exchange, developed to describe mathematics teaching, also has an interpretation in teacher education and especially in PBTE. In PBTE, educators also engage their charges (teacher learners) in work on behalf of instructional goals and seek evidence that those goals can be claimed as accomplished in and through their charges' engagement in such work. The aforementioned definition of lesson helps make more concrete this instructional exchange for the case of anticipations of lessons using *StoryCircles*. The interactive work being done involves the facilitator's (e.g., the teacher educator's) engagement of the participants in various activities that involve scripting, visualizing, and arguing about alternatives in a lesson, producing a consensual storyboard of an anticipation of the lesson. Though the product of these activities represents an anticipation of the lesson, it emerges against the background of a larger set of possibilities that is the whole lesson as learning opportunity. Hence, beyond pointed learning opportunities through the scripting of a storyboard (e.g., realizing that a given verbiage may be too stilted or too informal for a classroom), the whole lesson as a container of disciplinary meanings is a resource for the facilitator to appraise what the group learned. The anticipation of a lesson through *StoryCircles* is an approximation of practice that serves to learn, in particular, the practice of teaching in its disciplinary mathematical context and illustrates, as Hiebert and colleagues (2019) have argued, that the study of lessons in teacher preparation supports the building of mathematical knowledge for teaching.

VIII. CONCLUSION

The paper inspects the *StoryCircles* process, in which teachers as a group anticipate a lesson over a few weeks, as an approximation of practice. An examination of records from *StoryCircles* leads to the observation of commonalities and differences among different variants of a lesson, some of which variants respond to pedagogical design decisions (e.g., which starter to use) and others to what each group of participants eventually finds compelling to do (e.g., the ending storyboards). To explicate how these anticipations of a lesson serve to create an opportunity to practice teaching in its disciplinary context, we propose a definition of lesson as a maximal graph containing all possible paths connecting activities with a problem and an instructional goal. This definition serves in turn to show how the disciplinary context of instruction can be the knowledge at stake in the process of lesson anticipation for teachers.

The paper substantiates the contention that lessons are an appropriate unit to use in designing approximations of practice that, while amenable to bringing in some of the complexities indexed by the obligations to individuals, society, and institution, it especially permits practicing teaching that attends to the obligation to the discipline, particularly to attend to and practice alternative ways of transacting specific mathematical content with students. As the content of instruction is one of the salient context variables traditionally considered in teacher education, the notion of lesson proposed supports the claim that anticipating a lesson using the StoryCircles process is a type of approximation of practice that allows practicing in a context in which attention to disciplinary demands is possible.

CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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