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Dirac-type theorems for long Berge cycles in hypergraphs

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The famous Dirac's Theorem gives an exact bound on the minimum degree of an n-vertex graph guaranteeing the existence of a hamiltonian cycle. In the same paper, Dirac also observed that a graph with minimum degree at least $k \geq 2$ contains a cycle of length at least $k+1$. The purpose of this paper is twofold: we prove exact bounds of similar type for hamiltonian Berge cycles as well as for Berge cycles of length at least k in r-uniform, n-vertex hypergraphs for all combinations of k, r and n with $3 \leq r, k \leq n$. The bounds differ for different ranges of r compared to n and k.

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1. Introduction and results

1.1. Terminology and known results

A hypergraph H is a family of subsets of a ground set. We refer to these subsets as the *edges* of H and the elements of the ground set as the *vertices* of H. We use $E(H)$ and $V(H)$ to denote the set of edges and the set of vertices of H, respectively. We say H is *r-uniform* (*an r-graph*, for short) if every edge of H contains exactly r vertices. A *graph* is a 2-graph.

The *degree* $d_H(v)$ of a vertex v in a hypergraph H is the number of edges containing v. The *minimum degree*, $\delta(H)$, is the minimum over degrees of all vertices of H. The *circumference*, $c(G)$, of a graph G, is the length of a longest cycle in G.

A *hamiltonian cycle* in a graph is a cycle that visits every vertex. Sufficient conditions for existence of hamiltonian cycles in graphs have been well studied. A famous result of this type was due to Dirac in the fifties.

Theorem 1.1 *(Dirac [4,5]). Let $n \geq 3$. If G is an n-vertex graph with $\delta(G) \geq n/2$, then G has a hamiltonian cycle.*

Dirac also proved that $c(G) \geq \delta(G) + 1$ for every graph G. We consider similar conditions for *Berge cycles* in hypergraphs.

Definition 1.2. A **Berge cycle of** length s in a hypergraph is a list of s distinct vertices and s distinct edges $v_1, e_1, v_2, \ldots, e_{s-1}, v_s, e_s, v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq s$ (we always take indices of cycles of length s modulo s). We call vertices $v_1, \ldots, v_s$ the **defining vertices** of C and write $V(C) = \{v_1, \ldots, v_s\}$, $E(C) = \{e_1, \ldots, e_s\}$ Similarly, a **Berge path** of length ℓ is a list of $\ell + 1$ distinct vertices and ℓ distinct edges $v_1, e_1, v_2, \ldots, e_\ell, v_{\ell+1}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$, with **defining vertices** $V(P) = \{v_1, \ldots, v_{\ell+1}\}$ and $E(P) = \{e_1, \ldots, e_\ell\}$.

An analogue of Dirac's Theorem for non-uniform hypergraphs was given in [7]. For r-graphs, a well-known approximation of Dirac's bound on circumference and of Theorem 1.1 was proved by Bermond, Germa, Heydemann and Sotteau [1] more than 40 years ago:

Theorem 1.3 *(Bermond, et al. [1]). Let $r \geq 3$ and $k \geq r + 1$. If H is an n-vertex r-graph with $\delta(H) \geq \binom{k-2}{r-1} + r - 1$, then H contains a Berge cycle of length k or longer. In particular, if $\delta(H) \geq \binom{n-2}{r-1} + r - 1$, then H contains a hamiltonian Berge cycle.*

Recently, there was a series of improvements of the hamiltonian part of Theorem 1.3. First, Clemens, Ehrenmüller and Person [2] have proved an asymptotics for $n > 2r - 2$:

Theorem 1.4 *(Clemens et al. [2]). Let H be an r-graph on n vertices. If $n > 2r-2$ and $\delta(H) \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + n - 1$, then H has a hamiltonian Berge cycle.*

Then Coulson and Perarnau [3] proved the exact bound for n much larger than r:

Theorem 1.5 *(Coulson and Perarnau [3]). Let H be an r-graph on n vertices such that $r = o(\sqrt{n})$. If $\delta(H) \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + 1$, then H contains a hamiltonian Berge cycle.*

Then Ma, Hou and Gao [9] improved the bound of Theorem 1.4 for $n \geq 2r+4$.

Theorem 1.6 *(Ma, Hou and Gao [9]). Let $r \geq 4$ and $n \geq 2r+4$, and let H be an r-graph on n vertices. If $\delta(H) \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + \lceil (n-1)/2 \rceil$, then H contains a hamiltonian Berge cycle.*

Very recently, Salia [10] proved sharp results of Pósa type for Berge hamiltonian cycles. It will be easier to describe his results after we state ours in the next section.

1.2. Our results

In this paper we derive exact bounds for all possible $3 \leq r < n$, improving the aforementioned theorems.

Theorem 1.7. *Let $t = t(n) = \lfloor (n-1)/2 \rfloor$, and suppose $3 \leq r < n$. Let H be an n-vertex r-graph. If*

(a) *$r \leq t$ and $\delta(H) \geq \binom{t}{r-1} + 1$ or*
(b) *$r \geq n/2$ and $\delta(H) \geq r$,*

then H contains a hamiltonian Berge cycle.

These bounds are best possible due to the following constructions. We use the notation K_n^r to denote the n-vertex r-graph with all $\binom{n}{r}$ possible edges.

Construction 1. Suppose $r \leq t$. If n is odd, let H_1 consist of two copies of $K^r_{(n+1)/2}$ that share exactly one vertex. If n is even, let H_1 consist of two disjoint $K^r_{n/2}$ and a single edge intersecting both cliques.

Construction 2. Suppose $r \leq t$. Let H_2 have vertex set $X \cup Y$ such that $|X| = t$ and $|Y| = n - t$. The edge set of H_2 consists of every edge with at most one vertex in Y.

Construction 3. Suppose $r \geq n/2$. Let H_3 have vertex set $V(H_3) = \{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$ where $e_i = \{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ for $1 \leq i \leq n-1$ with indices taken modulo n.

It is easy to check that both H_1 and H_2 have minimum degree $\binom{t}{r-1}$. Observe that neither H_1 nor H_2 has a hamiltonian Berge cycle: H_1 has either a cut vertex or a cut edge, and in H_2 a hamiltonian Berge cycle must visit two vertices in Y consecutively, but no edge of H_2 contains any pair of vertices from Y.

Since $r \geq n/2$, $\delta(H_3) = r - 1$. Also, H_3 does not have a hamiltonian Berge cycle because $|E(H_3)| = n - 1$. In fact, removing a single edge from any n-vertex, r-regular, r-graph would also yield an extremal example.

Note that the length of the longest cycle in Construction 1 is $\lceil n/2 \rceil$. Thus Theorem 1.7 yields exact bounds on the minimum degree guaranteeing the existence of any cycle of length at least k in n-vertex r-graphs for all $r \leq t$ and all $k \geq 1 + n/2$.

We also improve the circumference part of Theorem 1.3. Since the bounds for $r \leq t$ and for $r > t$ are different, we state our results as two theorems.

Theorem 1.8. *Let n, k, and r be positive integers such that $n \geq k$ and $t \geq r \geq 3$. Let H be an n-vertex, r-graph. If*

(a) *$k \leq r + 2$ and $\delta(H) \geq k - 1$, or*
(b) *$r + 3 \leq k < t + 2$ and $\delta(H) \geq \binom{k-2}{r-1} + 1$, or*
(c) *$k \geq t + 2$ and $\delta(H) \geq \binom{t}{r-1} + 1$,*

then H contains a Berge cycle of length k or longer.

Theorem 1.9. *Let n, k, and r be positive integers such that $n \geq k \geq r \geq 3$, and $r > t$. If H is an n-vertex r-graph with*

$$\delta(H) \geq \left\lfloor \frac{r(k-1)}{n} \right\rfloor + 1,$$

then H contains a Berge cycle of length k or longer.

Constructions 1 and 2 give sharpness examples for Theorem 1.8(c). The constructions below show that for each $k \geq 3$ the bounds of Theorem 1.8(a,b) are sharp for infinitely many n.

Construction 4. Let $r + 3 \leq k < t + 2$. For $n - 1$ divisible by $k - 2$, let H_4 consist of $(n-1)/(k-2)$ copies of K_{k-1}^r such that all the cliques share exactly one vertex.

Construction 5. Let $k \leq r + 2 \leq t + 2$. For $n - 1$ divisible by r, view $V(H_5)$ as the union of $(n-1)/r$ sets $S_1, \ldots, S_{(n-1)/r}$ of $(r+1)$ vertices, all sharing exactly one vertex. The set $E(H_5)$ has $k - 1$ edges contained in each S_i.

We have $\delta(H_4) = \binom{k-2}{r-1}$ and $\delta(H_5) = k - 2$. A longest Berge cycle in H_4 must be contained in a single clique, and hence has length $k - 1$. Similarly, a longest Berge cycle in H_5 is contained in some S_i, and hence has at most $k - 1$ edges.

For Theorem 1.9, it is easy to construct an analog of Construction 3: an n-vertex r-graph with $k - 1$ edges whose minimum degree is exactly $\lfloor r(k-1)/n \rfloor$.

As mentioned in Section 1.1, after the first version of this paper appeared on arXiv, Salia [10] described the sequences $(d_1, \ldots, d_n)$ with $d_1 \leq d_2 \leq \ldots \leq d_n$ of two types: (a) for $r < n/2$ every n-vertex r-graph with degree sequence $(d'_1, \dots, d'_n)$ such that $d'_i > d_i$ for all i has a hamiltonian Berge cycle and also (b) every n-vertex hypergraph with degree sequence $(d'_1, \dots, d'_n)$ such that $d'_i > d_i$ for all i has a hamiltonian Berge cycle. The first of these nice results implies Part (a) of Theorem 1.7 for odd n.

1.3. Outline of the proofs

As always, $t = t(n) = \lfloor (n-1)/2 \rfloor$. Together, the circumference results, Theorem 1.8 and Theorem 1.9, imply the hamiltonian result Theorem 1.7 by setting $k = n$.

First we will prove Parts (a) and (b) of Theorem 1.8. Then we handle Part (c): for large k, our minimum degree condition guarantees the existence not only of a "long" Berge cycle, but rather of a hamiltonian Berge cycle.

Since $t + 1 \geq n/2$, if $r > t$ then the inequality $\delta(H) \geq \lfloor r(k-1)/n \rfloor + 1$ yields $\delta(H) > \frac{k-1}{2}$ and also $\sum_{v \in V(H)} d(v) > n\frac{r(k-1)}{n} = r(k-1)$; thus $|E(H)| = \frac{1}{r}\sum_{v \in V(H)} d(v) > k-1$. Hence the following theorem implies Theorem 1.9.

Theorem 1.10. *Let n, k, and r be positive integers such that $n \geq k \geq r > t$ and $r \geq 3$. If H is an n-vertex r-graph with at least k edges such that $\delta(H) \geq \lceil k/2 \rceil$, then $c(H) \geq k$.*

So, we will prove Theorem 1.10.

In Section 2, we prove Theorem 1.8(a,b). In Section 3 we describe the setup of the proofs of Theorems 1.8(c) and 1.10. The proofs somewhat differ for $r < t$, $r = t$ and $r > t$. But in all cases we will use the same structure of proofs, namely, a modification of Dirac's original proof of his theorem.

Also, since we always consider only Berge paths and cycles, from now on we drop the word "Berge" and use cycles and paths to exclusively refer to Berge cycles and Berge paths.

2. Proof of Theorem 1.8(a,b)

We will use the following results.

Theorem 2.1 *(Kostochka and Luo [8]). Let $4 \leq k \leq r+1$, and let H be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(H) \leq (k-1)(n-1)/r$.*

Theorem 2.2 *(Ergemlidze, Győri, Methuku, Salia, Thompkins, and Zamora [6]). Let $n \geq r \geq 3$, $k \in \{r+1, r+2\}$, and let H be an n-vertex r-graph with no Berge cycles of length k or longer. Then $e(H) \leq (k-1)(n-1)/r$.*

Proof of Theorem 1.8(a). Recall that $3 \leq k \leq \min\{r+2, n\}$ and $\delta(H) \geq k-1$.

By Theorems 2.1 and 2.2, if $4 \leq k \leq r-1$ or $r \geq 3$ and $k \in \{r+1, r+2\}$, then $e(H) \leq (k-1)(n-1)/r$. It follows that the average degree of H is at most

$$\frac{r}{n} \cdot \frac{(k-1)(n-1)}{r} = \frac{(k-1)(n-1)}{n} < k-1.$$

This gives that H has a vertex of degree at most $k-2$, a contradiction.

Thus to prove the theorem, we need to settle the remaining cases, namely, $k = 3 \leq r$ and $k = r \geq 4$. In both cases, consider a counter-example H with the most edges. Then H contains a path of length at least $k-1$. Among all such paths, let $P = v_1, e_1, v_2, \dots, e_{\ell-1}, v_\ell$ be a longest one.

If there exists a $j \geq k$ such that $v_1 \in e_j$, then $v_1, e_1, v_2, \dots, e_{j-1}, v_j, e_j, v_1$ is a cycle of length at least k. Furthermore, if there exists an edge $e \in E(H) \setminus E(P)$ and a vertex $u \in V(H) \setminus \{v_1, \dots, v_{k-1}\}$ such that $\{v_1, u\} \subset e$, then either $u \notin V(P)$ and we can extend P to a longer path by adding the vertex u and the edge e, or $u \in V(P)$ and we can construct a cycle of length at least k by combining the segment of P from v_1 to u with the edge e. Therefore each edge of H containing v_1 either is in $\{e_1, e_2, \dots, e_{k-1}\}$ or is contained in $\{v_1, \dots, v_{k-1}\}$. Since $k-1 < r$, the latter is impossible. Thus adding the fact that $d(v_1) \geq k-1$, we have that

$$\textit{all edges } e_1, \dots, e_{k-1} \textit{ contain } v_1. \tag{1}$$

Since H has no multiple edges, there is a vertex $v' \in e_1 \setminus e_{k-1}$. If $v' \notin \{v_1, \dots, v_\ell\}$, then we consider path P' obtained from P by replacing v_1 with v' and keeping all the edges. It has the same length as P, but $v' \notin e_{k-1}$, contradicting (1).

So, suppose $v' = v_j$. Since $v' \notin e_{k-1}$ and $v_1 \in e_{k-1}$, $j \notin \{1, k-1, k\}$. If $j \geq k+1$, then we have a cycle $C_2 = v_2, e_2, v_3, \dots, e_{j-1}, v_j, e_1, v_2$ of length $j-1 \geq k$, a contradiction. Thus $2 \leq j \leq k-2$. Consider path

$$P'' = v_j, e_{j-1}, v_{j-1}, \dots, e_1, v_1, e_j, v_{j+1}, e_{j+1}, v_{j+2}, \dots, e_{\ell-1}, v_\ell.$$

Similarly to P', it has the same length as P, but $v' \notin e_{k-1}$, contradicting (1). □

Proof of Theorem 1.8(b). Recall that $k \geq r+3$ and $\delta(H) \geq \binom{k-2}{r-1} + 1$. Suppose the theorem fails, and let H be an edge-maximal counterexample. Then H contains a path of length $k-1$ or greater. Among all such paths, let $P = v_1, e_1, v_2, \dots, e_{\ell-1}, v_\ell$ be a longest one. As in the proof of Theorem 1.8(a), each edge of H containing v_1 either is in $\{e_1, e_2, \dots, e_{k-1}\}$ or is a subset of $\{v_1, \dots, v_{k-1}\}$.

Set $X = \{v_1, \dots, v_{k-1}\}$ and $X' = X \setminus v_1$. Let $E_X = \{e \notin E(P) : e \subseteq X\}$. The previous paragraph implies that every edge containing v_1 belongs to $E_X \cup \{e_1, \dots, e_{k-1}\}$.

Case 1: There exists some $1 \leq i \leq k-2$ such that $v_1 \in e_i$ and $e_i \not\subseteq X$.

Let $u \in e_i \setminus X$. If there exists an edge $f \in E_X$ such that $\{v_1, v_{i+1}\} \subset f$, then

$$u, e_i, v_i, e_{i-1}, v_{i-1}, \dots, e_1, v_1, f, v_{i+1}, e_{i+1}, v_{i+2}, \dots, e_{\ell-1}, v_\ell$$

is longer than P, a contradiction to the maximality of P. So, there are no such edges.

If $r = 3$, then $i = 1$, since otherwise $\{v_1, v_i, v_{i+1}\} \subset e_i$, and there is no room for other vertices in e_i, contradicting our assumption. Therefore

$$d_H(v_1) \le \binom{|X' \setminus \{v_2\}|}{r-1} + |\{e_1, e_2, e_{k-1}\}| = \binom{k-3}{r-1} + 3 \le \binom{k-2}{r-1}, \tag{2}$$

when $k \ge 6$, a contradiction to the minimum degree.

Suppose now that $r \ge 4$. The number of edges in E_X containing v_1 is at most $\binom{|X' \setminus \{v_{i+1}\}|}{r-1} = \binom{k-3}{r-1}$. Since $k \ge r+3$, $k \ge 7$ and $\binom{k-3}{r-2} \ge \binom{k-3}{2} \ge k-1$. So,

$$d_H(v_1) \le \binom{k-3}{r-1} + k - 1 = \binom{k-2}{r-1} - \binom{k-3}{r-2} + k - 1 \le \binom{k-2}{r-1},$$

a contradiction to the minimum degree.

Case 2: For all $1 \le i \le k-2$ with $v_1 \in e_i$, $e_i \subset X$. Then the only possible edge containing v_1 that is not a subset of X is e_{k-1}, and $d_H(v_1) \le \binom{|X'|}{r-1} + 1 = \binom{k-2}{r-1} + 1$ with equality if and only if $v_1 \in e_{k-1}$ and every r-subset of $X' \cup \{v_1\}$ containing v_1 is an edge of H. Hence we may suppose this is the case.

For each $2 \le i \le k-1$, let g_i be the $(r-1)$-subset of X' containing v_i and the $r-2$ previous vertices of X' (with wrap around). I.e., if $i \ge r$, then $g_i = \{v_i, v_{i-1}, \ldots, v_{i-(r-2)}\}$ and if $i \le r-1$, then $g_i = \{v_i, v_{i-1}, \ldots, v_2\} \cup \{v_{k-1}, \ldots, v_{k-1-(r-1-i)}\}$. Then set $f_i = g_i \cup \{v_1\}$. Since $k \ge r+3$ and $\{v_1, v_{k-1}, v_k\} \subset e_{k-1}$, there exists some $2 \le i \le k-2$ such that $v_i \notin e_{k-1}$. Then since $f_j \in E(H)$ for all $2 \le j \le k-1$, the path

$$P_2 = v_i, f_i, v_{i-1}, \ldots, f_2, v_1, f_{i+1}, v_{i+1}, f_{i+2}, v_{i+2}, \ldots, f_{k-1}, v_{k-1}, e_{k-1}, v_k, \ldots, e_{\ell-1}, v_\ell$$

is also a longest path. Note that $f_j \subseteq X$ for each j. Applying the same argument to P_2's first vertex v_i as we did to v_1 in Case 1 and the beginning of Case 2, we have that either $d_H(v_i) \le \binom{k-2}{r-1}$ or $v_i \in e_{k-1}$. In both cases we obtain a contradiction. □

3. Setup of proofs for Theorems 1.8(c) and 1.10 and general lemmas

The original proof by Dirac of Theorem 1.1 involved two steps. In the first step, by looking at a longest path, he greedily found a cycle of length at least $1 + n/2$. In the second step, he considered a *lollipop*, i.e. a pair (C, P) such that C is a cycle, P is a path, $E(C) \cap E(P) = \emptyset$, $|V(C) \cap V(P)| = 1$, and the shared vertex of $v \in V(C) \cap V(P)$ is one of the endpoints of P. Dirac proved that when $\delta(G) \ge n/2$, the lollipop with the largest $|C|$ and modulo this with the largest $|P|$ can be only a hamiltonian cycle.

Our strategy is in the same spirit, only instead of lollipops we will consider pairs of a cycle C and a path P with $V(C) \cap V(P) = E(C) \cap E(P) = \emptyset$. We call such a pair (C, P) a *cycle-path pair*. We will in addition maximize a couple of more parameters.

A cycle-path pair (C, P) *is better* than a cycle-path pair (C', P') if

(i) $|E(C)| > |E(C')|$, or
(ii) $|E(C)| = |E(C')|$ and $|E(P)| > |E(P')|$, or
(iii) $|E(C)| = |E(C')|$, $|E(P)| = |E(P')|$ and the total number of vertices in $V(P)$ in the edges in C (counted with multiplicities) is greater than the total number of vertices in $V(P')$ in the edges in C', or
(iv) all parameters above coincide and the total number of vertices in $V(P)$ in the edges in P (counted with multiplicities) is greater than the total number of vertices in $V(P')$ in the edges in P'.

Similarly to Dirac's proof, we will show that in all cases, a best cycle-path pair is a hamiltonian cycle (or contains a cycle of length at least k when we are looking for such cycles).

In all cases there will be 3 steps: first we find a cycle of length at least $1 + n/2$, then prove that if C is not long enough, then in the best cycle-path pair (C, P), P cannot have only one vertex, and finally show that P also cannot have more than one vertex.

3.1. General lemmas

Suppose (C, P) is a best cycle-path pair with $C = v_1, e_1, \ldots, v_s, e_s, v_1$ and $P = u_1, f_1, \ldots, f_{\ell-1}, u_\ell$.

We consider three subhypergraphs, H_C, H_P and H' of H with the same vertex set $V(H)$: $E(H_C) = \{e_1, \ldots, e_s\}$, $E(H_P) = \{f_1, \ldots, f_{\ell-1}\}$ and $E(H') = E(H) \setminus (E(H_C) \cup E(H_P))$. Observe that the edges of these three subhypergraphs form a partition of the edges of H. We also consider $H - H_C$ with vertex set $V(H)$ and edge set $E(H) \setminus E(H_C)$. For a hypergraph F and a vertex u, we denote by $N_F(u) = \{v \in V(F) : \{u, v\} \subseteq e \text{ for some } e \in F\}$. For $i \in \{1, \ell\}$, set $B_i = \{e_j \in E(C) : u_i \in e_j\}$.

The following claim applies to all best cycle-path pairs (C, P), regardless of the sizes of r and k. It will be used in the sections below.

Claim 3.1. *In a best cycle-path pair (C, P), $N_{H'}(u_1)$ cannot contain a pair of vertices that are consecutive in C.*

Proof. Suppose toward a contradiction that v_i, v_{i+1} are contained in edges of H' with u_1. Let e, e' be edges of H' such that $u_1, v_i \in e$ and $u_1, v_{i+1} \in e'$. If $e \neq e'$, then replacing e_i with e, u_1, e' gives a longer cycle than C, a contradiction. Thus we may assume $e = e'$.

If there is $1 \leq j \leq \ell$ such that $u_j \in e_i$, then by replacing the path v_i, e_i, v_{i+1} in C with the longer path $v_i, e, u_1, f_1, u_2, \ldots, f_{j-1}, u_j, e_i, v_{i+1}$, we obtain a longer cycle than C. Thus $e_i \cap V(P) = \emptyset$. Then replacing e_i with e in C gives a cycle C' with (C', P) better than (C, P) by Rule (iii). $\square$

Symmetrically, the claim holds for u_ℓ as well.

The following claims hold for C as well as for any other longest cycle in H.

Claim 3.2. *Let $C' = v'_1, e'_1, \dots, e'_s, v'_1$ be a longest cycle in H. For any $u \notin V(C')$, if $u \in e'_i$, then $v'_i, v'_{i+1} \notin N_{H-H_{C'}}(u)$.*

Proof. Suppose $v'_i \in N_{H-H_{C'}}(u)$, and let $e \in E(H) \setminus E(H_{C'})$ be such that $\{u, v'_i\} \subseteq e$. Then we can find a longer cycle by replacing e'_i with e, u, e'_i, a contradiction to our choice of C. A similar argument holds for v'_{i+1}. □

Claim 3.3. *Let $C' = v'_1, e'_1, \dots, e'_s, v'_1$ be a longest cycle in H. Suppose there exist vertices $v'_i, v'_j \in V(C')$ and an edge $e \in E(H - H_{C'})$ such that $\{v'_i, v'_j\} \subset e$. Then for any $u \in V(H) \setminus V(C)$, u cannot be contained in both e'_i and e'_j or in both e'_{i-1} and e'_{j-1}.*

Proof. Suppose there exists a vertex $u \notin V(C)$ such that $u \in e'_i$ and $u \in e'_j$ where without loss of generality $i < j$. Then $v'_1, e'_1, \dots, v'_i, e, v'_j, e'_{j-1}, \dots, v'_{i+1}, e'_i, u, e'_j, v'_{j+1}, \dots, e'_s, v_1$ is a cycle longer than C'. The proof for e'_{i-1}, e'_{j-1} is symmetric. □

Claim 3.4. *Suppose (C', P') is a cycle-path pair with $C' = v'_1, e'_1, \dots, e'_s, v'_1$, $P' = u'_1, f'_1, \dots, u'_\ell$, $|V(C')| = |V(C)|$, and $|V(P')| = |V(P)|$. For every e'_i containing u'_1 and e'_j containing u'_ℓ, either $i = j$ or $|i - j| \geq \ell$.*

Proof. Suppose there exist e'_i, e'_j containing u'_1 and u'_ℓ respectively such that without loss of generality $j > i$ and $j - i \leq \ell - 1$. Then that cycle obtained by replacing the segment $v'_i, e'_i, \dots, e'_j, v'_{j+1}$ in C' with $v'_i, e'_i, u'_1, f'_1, \dots, f'_{\ell-1}, u'_\ell, e'_j, v'_{j+1}$ has size $|V(C')| - (i-j) + \ell > |V(C')| = |V(C)|$, contradicting the fact that C is a longest cycle. □

Claim 3.5. *If $C = v_1, e_1, \dots, v_s, e_s, v_1$ is a graph cycle, and A is any set of c edges of C and I is an independent subset of $\{v_1, \dots, v_s\}$ disjoint from all edges in A, then $|I| \leq \lceil (s-1-c)/2 \rceil$.*

Proof. We show the claim by induction on s. If $s = 3$, then either $c = 1$, in which case any independent set disjoint from the edges of A has at most one vertex, or $c \geq 2$, and no vertices are disjoint from A. Hence we get $|I| \leq \lceil (2-c)/2 \rceil$.

Now let $s > 3$ and suppose the lemma holds for $s - 1$. If $A = \emptyset$, then $|I| \leq \lfloor s/2 \rfloor = \lceil (s-1)/2 \rceil$, as desired. So suppose A has at least one edge, say e_i. Let C' be the cycle obtained by contracting e_i. Since $e_i \in A$, $v_i, v_{i+1} \notin I$. Therefore I is still an independent set in C' and is disjoint from the edges in $A \setminus \{e_i\}$. By the induction hypothesis applied to $C', A \setminus \{e_i\}$, and I, $|I| \leq \lceil ((s-1) - 1 - (c-1))/2 \rceil = \lceil (s-1-c)/2 \rceil$. □

Claims 3.1, 3.2 and 3.5 imply the following corollary.

Corollary 3.6. *Let $A = \{e_i \in E(C) : u_1 \in e_i\}$. Then $|N_{H'}(u_1) \cap V(C)| \leq \lceil (s-1-|A|)/2 \rceil$.*

The following general lemmas will be used in conjunction with Claim 3.4 later in our proof.

Lemma 3.7. *Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a graph cycle. Let A and B be nonempty subsets of $E(C)$ such that for any $e_i \in A$ and $e_j \in B$ either $i = j$ or $|i - j| \geq q \geq 2$. Suppose $|B| \geq |A| = a$. Then either*

(a) $a \leq s/2 - q + 1$, or (b) $B = A$ and $a \leq s/q$.

Proof. Suppose first $B = A$. Then between any two edges of A on C there are at least $q - 1$ other edges. This proves (b).

Suppose now $B \neq A$. Let $A = \{e_{i_1}, \ldots, e_{i_a}\}$ with vertices in clockwise order on C. We can view C as the union of a paths $P_1, \ldots, P_a$ where P_j is the part of C from e_{i_j} to $e_{i_{j+1}}$ (modulo a). Since $|B| \geq a$, there is some $f \in B \setminus A$, say $f \in P_a$. Then P_a has at least $2(q-1)$ edges not in $A \cup B$ (and some vertices in B). Also, if $e_{i_j} \in A \cap B$, then $e_{i_j - 1}, e_{i_j + 1} \notin A \cup B$. This means $|E(C) \setminus (A \cup B)| \geq 2(q-1) + (|A \cap B| - 1)$ with equality only if every edge in $E(C) \setminus (A \cup B)$ is one of exactly $2(q-1)$ non-B edges in P_a or appears directly after some edge in $A \cap B$ (which contains e_{i_a}). In this case, we must have that $A \subset B$ as otherwise since $A \neq E(C)$, there will exist some edge $e_k \in A \setminus B$ such that $e_{k+1} \in V(C) \setminus (A \cup B)$.

Thus if $A \not\subset B$, then since $|B| \geq a$,

$$s \geq |A| + |B \setminus A| + 2(q-1) + |A \cap B| \geq 2a + 2(q-1), \tag{3}$$

as claimed. Otherwise, in view of f, $|B| \geq a + 1$, and instead of (3), we get

$$s \geq |A| + |B \setminus A| + 2(q-1) + |A \cap B| - 1 \geq (2a+1) + 2(q-1) - 1 = 2a + 2(q-1),$$

again. $\square$

Lemma 3.8. *Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a graph cycle. Let A and B be nonempty independent subsets in $V(C)$ such that for any $v_i \in A$ and $v_j \in B \setminus A$, $|i - j| \geq q \geq 2$. If $B \setminus A \neq \emptyset$, then $|A| \leq s/2 - q + 1$.*

Proof. Let $A = \{v_{i_1}, \ldots, v_{i_a}\}$ with vertices in clockwise order on C. We view C as the union of a paths $P_1, \ldots, P_a$ where P_j is the part of C from v_{i_j} to $v_{i_{j+1}}$ (modulo a).

Since $B \setminus A \neq \emptyset$, we may assume there is $y \in (B \setminus A) \cap V(P_a)$. Then P_a has at least $2(q-1)$ vertices not in $A \cup B$ and at least one in B. Since A is independent, we also have at least $a - 1$ vertices in $V(C - P_a) \setminus A$. Hence $|V(C)| \geq a + a + 2(q-1)$, as claimed. $\square$

4. Existence of a cycle of length at least $n/2 + 1$

Similarly to Dirac's proof, we show that under the conditions of Theorems 1.8(c) and 1.10 there exists a cycle of length at least $t + 2 \geq n/2 + 1$. We do this in two cases: $r \leq t$ and $r \geq t + 1$.

Lemma 4.1. *If $r \leq t$, and H is an n-vertex r-graph with minimum degree $\delta(H) \geq \binom{t}{r-1} + 1$, then H contains a cycle of length at least $t + 2 = \lfloor (n+3)/2 \rfloor$.*

Proof. Suppose H has no cycles of length at least $t+2$. Let Q be a longest path in H, say $Q = v_1, e_1, v_2, \dots, e_{s-1}, v_s$. Let $q = \min\{t+1, s\}$, $V(q) = \{v_1, \dots, v_q\}$ and let $Q(q)$ denote the subpath of Q with vertex set $V(q)$ and edge set $E(q) = \{e_1, \dots, e_{q-1}\}$. Among such paths Q, choose one in which

$$\begin{aligned}&\textit{(a) the most edges in } E(q) \textit{ are contained in } V(q)\textit{, and}\\ &\textit{(b) modulo (a), the fewest edges in } E(q)\cup\{e_q\} \textit{ contain } v_1.\end{aligned} \tag{4}$$

Let $H_1 = H - E(Q)$. Since H has no cycles of length at least $t+2$ and Q is a longest path,

$$\text{all neighbors of } v_1 \text{ in } H_1 \text{ are in } V(q). \tag{5}$$

Thus $d_{H_1}(v_1) \le \binom{q-1}{r-1}$. By the same reason, the edges e_i for $q+1 \le i \le s-1$ must not contain v_1. So

$$d_H(v_1) \le d_{H_1}(v_1) + \min\{q, s-1\} \le \binom{q-1}{r-1} + \min\{q, s-1\}. \tag{6}$$

If $q = s \le t$, then since $3 \le r \le t$, this is at most $\binom{t-1}{r-1} + t - 1 \le \binom{t}{r-1}$, contradicting the minimum degree condition. Hence $s \ge t+1$ and $q = t+1$. Let $E'(q) = E(q) \cup \{e_q\}$ if e_q exists, and $E'(q) = E(q)$ otherwise.

Let E_0 be the set of edges in $E'(q)$ not containing v_1, E_1 be the set of edges in $E'(q)$ containing v_1 and contained in $V(q)$, and $E_2 = E'(q) \setminus (E_0 \cup E_1)$. In particular, $e_q \in E_0 \cup E_2$ because $v_{q+1} \in e_q \setminus V(q)$.

Let us show that

$$|E_1 \cup E_2| \le \max\{t-1, r\}. \tag{7}$$

Indeed, suppose $|E_1 \cup E_2| = m$. For every $2 \le i \le t+1$ such that $v_1 \in e_i$, we can consider the path Q_i from v_i to v_s obtained from Q by replacing the subpath $v_1, e_1, v_2, \dots, e_i, v_{i+1}$ with the subpath $v_i, e_{i-1}, v_{i-1}, \dots, e_1, v_1, e_i, v_{i+1}$. This path uses the same edges as Q, so by Rule (a) in (4) it is also a valid choice for a best path, and if v_i is in fewer than m edges in $E'(q)$, then Q_i is better by Rule (b). Hence each of the m vertices v_i such that $e_i \in E_1 \cup E_2$ is in at least m edges in $E'(q)$. Since there are at most $q = t+1$ edges in $E'(q)$ each containing r vertices, this gives $m^2 \le r(t+1)$. If $r \le t-1$, then $m^2 \le t^2 - 1$, so $m \le t-1$. Otherwise if $r = t$, we get $m \le t$. This proves (7).

Let $R = R(v_1)$ be the set of r-tuples contained in $V(q)$ that contain v_1 and are not edges of H. By (5), the only edges containing v_1 and not contained in $V(q)$ are those in E_2. Therefore

$$d_H(v_1) = \binom{t}{r-1} + |E_2| - |R|. \tag{8}$$

So, if $E_2 = \emptyset$, then $d_H(v_1) \le \binom{t}{r-1}$, a contradiction to the minimum degree condition. Hence for some $j \in [t+1]$, $e_j \in E_2$, i.e., $v_1 \in e_j$ but $e_j \nsubseteq V(q)$. Choose the smallest such j.

Case 1: $j = 1$. If there is an edge $g \subset V(q)$ in $E(H) \setminus E'(q)$ containing $\{v_1, v_2\}$ (recall that $g \notin \{e_{q+1}, \dots, e_{s-1}\}$), then by replacing e_1 with g we get a contradiction to (4)(a). Thus each of the $\binom{t-1}{r-2}$ r-tuples $g \subset V(q)$ containing $\{v_1, v_2\}$ is in $R \cup E_1$.

Case 1.1: $r = 3$. For any edge e_i containing v_1, $\{v_i, v_{i+1}, v_1\} \subseteq e_i$. Then only e_2 may contain $\{v_1, v_2\}$ and be contained in $V(q)$. Moreover for $2 \le i \le t$, if $v_1 \in e_i$, then $e_i = \{v_1, v_i, v_{i+1}\} \subseteq V(q)$, so the only possible edge in E_2 is e_q. Hence

$$d_H(v_1) \le \binom{t}{2} - |R| + |\{e_2, e_q\}| \le \binom{t}{2} - \binom{t-1}{1} + 2 \le \binom{t}{r-1},$$

a contradiction to the minimum degree condition.

Case 1.2: $r \ge 4$. Set $E_1' = \{e_i \in E_1 : v_2 \in e_i\}$. It follows from (6) that

$$d_H(v_1) \le |E_2| + \binom{t}{r-1} - \left(\binom{t-1}{r-2} - |E_1'|\right) = |E_1' \cup E_2| + \binom{t}{r-1} - \binom{t-1}{r-2}.$$

In order to have $d_H(v_1) \ge 1 + \binom{t}{r-1}$, we need $\binom{t-1}{r-2} \le |E_1' \cup E_2| - 1$.

If either $r \le t-1$ (so $|E_1 \cup E_2| \le t-1$ by (7)) or $r = t$ and $|E_1' \cup E_2| \le t-1$, then since $r - 2 \ge 2$, we have $\binom{t-1}{r-2} \ge t-1 \ge |E_1' \cup E_2|$.

Therefore we may assume that $r = t$ and by (7), $|E_1' \cup E_2| = |E_1 \cup E_2| = t$, implying $E_1 = E_1'$. Then every $e_i \in E_1$ contains v_2. Suppose first that $|E_1| \ge 1$, and let $e_i \in E_1$. If $f := V(q) \setminus \{v_2\}$ is an edge of H, then because $E_1 = E_1'$, $f \notin E(Q)$. We may replace e_i in Q with f and e_1 with e_i because $e_1 \notin V(q)$ to obtain a path that is better than Q by Rule (a). It follows that $f \in R$ and

$$d_H(v_1) \le \binom{t}{r-1} - \left(\binom{t-1}{r-2} + |\{f\}|\right) + |E_1 \cup E_2| = \binom{t}{r-1} - (t-1+1) + t = \binom{t}{r-1},$$

a contradiction to the miminum degree. So we may assume that $|E_1| = 0$, i.e., all edges containing v_1 in $E'(q)$ contain a vertex outside of $V(q)$. If there exists any edge $e \subseteq V(q)$ in H such that $v_1 \in e$, then some $\{v_i, v_{i+1}\} \subseteq e$ since $|e| = t$. Then we may replace the edge e_i with the edge e in Q to obtain a better path by Rule (a). Therefore $|R| = \binom{t}{r-1}$. By (8), $d_H(v) \le |E_2| = t$, contradicting the minimum degree condition.

Case 2: $2 \le j \le t$. In order for e_j to contain v_1, v_j, v_{j+1} and a vertex outside of $V(q)$, we need $r \ge 4$. Similarly to Case 1, if there is an edge $g \subset V(q)$ in $E(H) \setminus E'(q)$ containing $\{v_1, v_{j+1}\}$, then the path

$$v_j, e_{j-1}, v_{j-1}, \dots, e_1, v_1, g, v_{j+1}, e_{j+1}, v_{j+2}, \dots, e_{s-1}, v_s$$

contradicts (4)(a). Hence each of the $\binom{t-1}{r-2}$ r-tuples $g \subset V(q)$ containing $\{v_1, v_{j+1}\}$ is in $R \cup E_1$.

So, now we repeat the argument of Case 1.2 word by word with v_j in place of v_2.

Case 3: $j = t+1$. This means all edges containing v_1 apart from e_{t+1} are contained in $V(q)$. Then $d_H(x) \le \binom{t}{r-1} - |R| + 1$, so we may assume $|R| = 0$. In other words,

$$\textit{all } r\textit{-tuples contained in } V(q) \textit{ and containing } v_1 \textit{ are edges of } H. \tag{9}$$

Since $r \le t$, there is $2 \le i \le t$ such that $v_i \notin e_{t+1}$. By (9), we can construct a path on the vertices $v_i, v_{i-1}, \ldots, v_1, v_{i+1}, v_{i+2}, \ldots, v_{t+1}$ all edges of which are contained in $V(q)$. So, we will have no edges containing v_i and not contained in $V(q)$, a contradiction to (b). □

Next, we prove the result for $r \ge t+1$.

Lemma 4.2. *Let H be an n-vertex r-graph containing at least k edges. If $k \ge r \ge t+1$ and $\delta(H) \ge \lceil k/2 \rceil$, then H contains a cycle of length at least $\min\{k, t+2\}$.*

Proof. Suppose that the lemma does not hold for an n-vertex r-graph H, and

$$\textit{the maximum length of a cycle in } H \textit{ is } s, \textit{ where } s \le \min\{k-1, t+1\}. \tag{10}$$

We start from a series of new notions and auxiliary claims.

For a path $P = v_1, e_1, v_2, \ldots, e_{\ell-1}, v_\ell$ and $i \in \{1, \ell\}$, let $V_i = V_i(P) = \{v_j \in V(P) : v_i \in e_j\}$, and set $V_\ell^+ = V_\ell^+(P) = \{v_{j+1} : v_\ell \in e_j\}$.

For each $v_i \in V_1$, set $P_i^1 = v_i, e_{i-1}, \ldots, e_1, v_1, e_i, v_{i+1}, \ldots, e_{\ell-1}, v_\ell$, and for each $v_j \in V_\ell^+$, set

$P_j^\ell = v_j, e_j, \ldots, e_{\ell-1}, v_\ell, e_j, v_{j-1}, \ldots, e_1, v_1$.

Claim 4.3. *Let $P = v_1, e_1, v_2, \ldots, e_{\ell-1}, v_\ell$ be a longest path in H. Then no edge $e \notin E(P)$ intersects $V_1 \cup V_\ell^+$.*

Proof. Suppose that there exists an edge $e \notin E(P)$ such that $v_1 \in e$. Then by the maximality of P, $e \subseteq \{v_1, \ldots, v_\ell\}$. It follows that there exists some $v_q \in e$ with $q \ge r$, and hence $v_1, e_1, \ldots, e_{q-1}, v_q, e$ is a cycle of length q. Since $r \ge t+1$, (10) implies $q = r = t+1$. This means $e = \{v_1, \ldots, v_r\}$. Swapping e_1 with e in P and repeating the same reasoning we obtain $e_1 = \{v_1, \ldots, v_r\} = e$, a contradiction.

For $v_i \in V_1$ or $v_j \in V_\ell^+$, we apply the same argument for the longest paths P_i^1 or P_j^ℓ (note $E(P_i^1) = E(P_j^\ell) = E(P)$) and obtain our result again. □

Claim 4.4. *The longest path of H contains at least $s+2$ vertices.*

Proof. Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a longest cycle in H. Since $r \ge t+1$, (10) implies $s \le r$, and hence at most one edge of H is contained in $V(C)$ (actually, equals $V(C)$). We may assume that if this happens, then such an edge is one of the e_i.

Case 1: For some $v_1 \in V(C)$, some edge $e \in E(H) \setminus E(C)$ contains v_1. By our assumption, there is a vertex $u \in e \setminus V(C)$. Also at most one of e_1 and e_s is contained in $V(C)$, so we may assume there is $u' \in e_1 \setminus V(C)$. If $u' = u$ then we have cycle $C' = v_2, e_2, \dots, e_s, v_1, e, u, e_1, v_2$ of length $s+1$, otherwise we have path $P = u', e_1, v_2, e_2, \dots, e_s, v_1, e, u$, as claimed.

Case 2: All edges of H incident to $V(C)$ are in $E(C)$. Since H has at least $k > s$ edges, there is an edge f fully disjoint from $V(C)$. Since $|\bigcup_{i=1}^{s} e_i| \geq r+1$ and $n < (r+1)+r$, there is some e_i, say $i = 1$, that contains a vertex $u_1 \in f$. Let u_2 be another vertex of f. Then path $P' = v_2, e_2, \dots, e_s, v_1, e_1, u_1, f, u_2$ is as claimed. □

Claim 4.5. *The longest path of H contains at least $s+3$ vertices.*

Proof. Suppose a longest path $P = v_1, e_1, \dots, e_{\ell-1}, v_\ell$ has at most $s+2$ vertices. By Claim 4.4, $\ell = s+2$.

If there exists some $v_j \in V_1 \cap V_\ell^+$ (i.e., $v_1 \in e_j$ and $v_\ell \in e_{j-1}$), then the cycle

$$C = v_1, e_1 \dots, e_{j-2}, v_{j-1}, e_{j-1}, v_\ell, e_{\ell-1}, \dots, v_{j+1}, e_j, v_1$$

contains all vertices of P except for v_j. Therefore $|V(C)| \geq \ell - 1 \geq s+1$, a contradiction. It follows that

$$V_1 \cap V_\ell^+ = \emptyset. \tag{11}$$

By Claim 4.3, $|V_1| \geq d_H(v_1) \geq \lceil k/2 \rceil$ and $|V_\ell^+| \geq d_H(v_\ell) \geq \lceil k/2 \rceil$. So, (11) yields $|V_1 \cup V_\ell^+| \geq k \geq s+1 = \ell - 1$, which means at most one vertex in $V(P)$ is not contained in $V_1 \cup V_\ell^+$. We now prove that

$$|E(H)| = s+1. \tag{12}$$

Indeed, suppose that H has an edge $e \notin E(P)$. By Claim 4.3, $e \cap (V_1 \cup V_{s+2}^+) = \emptyset$, hence $k + r \leq n$. Since $k \geq r \geq t+1 \geq n/2$, this is only possible when $k = r = s+1 = n/2$ and e is the unique edge with $e = V(H) \setminus (V_1 \cup V_{s+2}^+)$. Moreover, this implies that $E(H) = E(P) \cup \{e\}$. So we have $V(H) \setminus V(P) \subseteq e$, and e contains at least $r-1$ vertices outside of P.

If there exists a vertex $v \in e_{s+1} \setminus V(P)$, then $v \in e$, and there exists another $v' \in e \setminus (V(P) \cup \{v\})$. We get a longer path by replacing the vertex v_{s+2} with the path v, e, v' in P. So $e_{s+1} \subseteq V(P)$. Moreover, if there exists a vertex $v_i \in V_1$ such that $v_i \in e_{s+1}$, then we obtain the cycle $C' = v_1, e_1, \dots, v_i, e_{s+1}, v_{s+1}, e_s, \dots, v_{i+1}, e_i, v_1$ of length $s+1$. Hence $V_1 \cap e_{s+1} = \emptyset$. Therefore $s+2 = |V(P)| \geq |e_{s+1}| + |V_1| \geq r + \lceil k/2 \rceil$, but we assumed $r = k \geq 3$, a contradiction. This proves (12).

By (12) for every cycle C of length s in H, there is exactly one edge e such that $e \notin E(C)$. Among all such pairs (C, e) suppose we chose one to maximize $|e \cap V(C)|$. Let $C = v_1, e_1, \dots, v_s, e_s, v_1$.

Since $s \le r$, if $e \subseteq V(C)$, then $r = s = n/2, k = n/2 + 1$. Let $v \in V(H) \setminus V(C)$. Since $v \notin e$, it is in at least $k/2$ edges of C. So there is a pair of consecutive edges, say e_1, e_2 containing v. Then the cycle

$$C' = v_1, e_1, v, e_2, v_2, e, v_3, e_3, \dots, v_s, e_s, v_1$$

has length $s + 1$, a contradiction.

Therefore $X := e \setminus V(C)$ is nonempty. Define $E_X = \{e_i \in E(C) : e_i \cap X \neq \emptyset\}$. We now show that

$$E_X \text{ cannot contain two consecutive edges in } C. \tag{13}$$

Indeed, suppose $e_1, e_2 \in E_X$. Then there exist $v, v' \in X$ such that $v \in e_1, v' \in e_2$. If $v = v'$, then let C' be the cycle obtained from C by replacing vertex v_2 with v. Since $v \in V(C') \cap e$ and we chose (C, e) to maximize $|V(C) \cap e|$, we need $v_2 \in e$. Then the cycle

$$v_1, e_1, v, e, v_2, e_2, v_3, \dots, v_s, e_s, v_1$$

has length $s + 1$, a contradiction. Therefore we may assume $v \neq v'$. Then by replacing in C the segment v_1, e_1, v_2, e_2, v_3 with the path $v_1, e_1, v, e, v', e_2, v_3$ we again obtain a cycle of length $s + 1$. This contradiction proves (13).

Since $|E_X| \ge \delta(H) - 1 \ge \lfloor (k-1)/2 \rfloor \ge \lfloor s/2 \rfloor$ by (10), we may assume by (13) that if s is odd then $E_X = \{e_1, e_3, e_5, \dots, e_{s-2}\}$ and if s is even, $E_X = \{e_1, e_3, e_5, \dots, e_{s-1}\}$. Moreover, again by (13), $|E_X| = \delta(H) - 1$, and therefore for every $v \in X$, the edges containing v are exactly $E_X \cup \{e\}$. Thus, for every $e_i \in E_X$, $X \cup \{v_i, v_{i+1}\} \subseteq e_i$.

Let $e_i \in E_X$, and suppose $v_i \in e$. Then we may replace in C the segment v_i, e_i, v_{i+1} with v_i, e, v, e_i, v_{i+1} for any $v \in X$ to obtain a cycle of length $s + 1$, a contradiction. Similarly, we have $v_{i+1} \notin e$. If s is even, since the edges of C alternate membership and non-membership in E_X, we have $e \cap V(C) = \emptyset$, i.e., $e = X$. Otherwise, if s is odd, then $e \subseteq X \cup \{v_s\}$.

Recall that if $e_i \in E_X$, then $X \cup \{v_i, v_{i+1}\} \subseteq e_i$. When s is even, we have $|X| = |e| = r$, so $|e_i| \ge r + 2$. When s is odd, we have $|X| \ge r - 1$, and hence $|e_i| \ge r - 1 + 2 \ge r + 1$. In both cases, a contradiction to the uniformity of H proves the claim. □

Among longest paths in H choose $P = v_1, e_1, \dots, v_{\ell-1}, e_{\ell-1}, v_\ell$ so that e_1 has as few vertices outside of $V(P)$ as possible. By Claim 4.5, $\ell \ge s + 3$.

Let $J(1)$ be the maximum j such that $v_j \in e_1$ and $J(\ell - 1)$ be the minimum j such that $v_j \in e_{\ell-1}$

Then

$$J(1) \le \min\{r + 1, s + 1\} \text{ and } J(\ell - 1) \ge 3. \tag{14}$$

Let $\alpha(1)$ (respectively, $\beta(1)$) be the second smallest (respectively, the largest) index i such that $v_1 \in e_i$. By Claim 4.3, $\alpha(1)$ and $\beta(1)$ are well defined. Similarly, let $\alpha(\ell)$ (respectively, $\beta(\ell)$) be the smallest (respectively, the second largest) index i such that $v_\ell \in e_i$.

Since $\ell \geq s+3$ and H has no cycles with length $s+1$ or greater,

$$\beta(1) \leq s, \textit{ and } \alpha(\ell) \geq 3. \tag{15}$$

Claim 4.6. *$J(1) \leq \beta(\ell)$.*

Proof. Suppose $J(1) > \beta(\ell)$. For each i and j such that $v_\ell \in e_i$, $v_j \in e_1$ and $j > i$, the cycle $C_{i,j} = v_j, e_j, v_{j+1}, \ldots, v_\ell, e_i, v_i, e_{i-1}, v_{i-1}, \ldots, v_2, e_1, v_j$ yields that $j \geq i+3$.

In particular, by (14), $\beta(\ell) \leq s-2$. The edge $e_{\beta(\ell)}$ forbids $v_{\beta(\ell)+1}$ and $v_{\beta(\ell)+2}$ from belonging to e_1. By Claim 4.3, v_ℓ belongs only to edges of $E(P)$, and each of the remaining $d(v_\ell)-2$ edges e_i containing v_ℓ other than $e_{\beta(\ell)}$ and $e_{\ell-1}$ also forbids at least one additional v_{i+1} from belonging to e_1. So, by (10) and (14), $|e_1 \cap V(P)| \leq s+1-k/2 \leq (s+1)/2$. Hence $|e_1 \setminus V(P)| \geq r-(s+1)/2$. By the choice of e_1, also $|e_{\ell-1} \setminus V(P)| \geq r-(s+1)/2$. Since $(e_1 \setminus V(P)) \cap (e_{\ell-1} \setminus V(P)) = \emptyset$, we conclude that

$$n \geq |V(P)|+|e_1 \setminus V(P)|+|e_{\ell-1} \setminus V(P)| \geq \ell+2\left(r-\frac{s+1}{2}\right) \geq (s+3)+2r-(s+1) = 2r+2,$$

a contradiction to $r \geq t+1$. $\square$

Define $\beta'(\ell) = \min\{\ell-2, \beta(\ell)+1\}$. If $\beta(\ell) \leq \ell-3$, then let

$$P'(\ell) = v_1, e_1, \ldots, v_{\beta(\ell)}, e_{\beta(\ell)}, v_\ell, e_{\ell-1}, v_{\ell-1}, \ldots, e_{\beta(\ell)+1}, v_{\beta(\ell)+1}.$$

If $\beta(\ell) = \ell-2$ and $v_{\ell-2} \in e_{\ell-1}$, then let $P'(\ell) = v_1, e_1, \ldots, v_{\ell-2}, e_{\ell-1}, v_{\ell-1}, e_{\ell-2}, v_\ell$. In both cases,

$$\textit{$P'(\ell)$ coincides with P up to $v_{\beta(\ell)}$, has the same vertex set as P, and the last edge is $e_{\beta'(\ell)}$.} \tag{16}$$

Let $e_1^- = \{v_j : v_{j+1} \in e_1\}$. If $v_j \in e_{\ell-1} \cap e_1^-$, then the cycle $v_2, e_2, v_3, \ldots, v_j, e_{\ell-1}, v_{\ell-1}, e_{\ell-2}, \ldots, v_{j+1}, e_1, v_2$ has $s+1$ vertices. Thus, $e_{\ell-1} \cap e_1^- = \emptyset$. As we mentioned above, $(e_1 \setminus V(P)) \cap (e_{\ell-1} \setminus V(P)) = \emptyset$. By (15), v_1 and v_2 also cannot belong to $e_{\ell-1}$. So,

$$e_{\ell-1} \cap M(e_1) = \emptyset \quad \textit{where} \quad M(e_1) = M_P(e_1) = e_1^- \cup (e_1 \setminus V(P)) \cup \{v_2\}. \tag{17}$$

Now we consider some cases.

Case 1: $v_3 \notin e_1$. Then $v_2 \notin e_1^-$, and hence $|M(e_1)| = r$. Since $r \geq n/2$, by (17) this is possible only if $r = n/2$ and $e_{\ell-1} = V(H) \setminus M(e_1)$. In particular, $v_{\ell-2} \in e_{\ell-1}$. By

Claim 4.6, $v_s, v_{s+1}, \dots, v_\ell \in e_{\ell-1}$ and in particular, $v_{\ell-2} \in e_{\ell-1}$. Thus by (16), we can apply the same argument to $P'(\ell)$ and get that $e_{\beta'(\ell)} = V(H) \setminus M(e_1)$. But these edges are distinct, a contradiction.

Case 2: $v_3 \in e_1$ and there is $v \in (e_2 - V(P)) \setminus e_1$. Then $v \notin e_{\ell-1}$ as otherwise $v, e_2, v_2, e_1, v_3, e_3, \dots, v_{\ell-1}, e_{\ell-1}, v$ is a cycle with $s+3-2+1 > s$ vertices, contradicting that C is a longest cycle. Hence $e_{\ell-1} = V(H) \setminus (M(e_1) \cup \{v\})$. Now as in Case 1, the same holds for $e_{\beta'(\ell)}$ in place of $e_{\ell-1}$, contradicting the fact that they are distinct.

Case 3: $v_3 \in e_1$, $e_2 - V(P) \subset e_1$ and $v_1 \in e_2$. Let $P_1 = v_1, e_2, v_2, e_1, v_3, e_3, \dots, v_\ell$. Note that $V(P_1) = V(P)$. By the choice of e_1, $|e_2 \setminus V(P)| = |e_1 \setminus V(P)|$, and hence Claim 4.6 holds for e_2 in place of e_1 and P_1 in place of P. Define $M(e_2) = M_{P_1}(e_2)$ similarly to $M(e_1)$. In this case, $e_{\ell-1} \cap (M(e_1) \cup M(e_2)) = \emptyset$, so since $|M(e_1) \cup M(e_2)| \geq r$ (because $|M(e_1)|, |M(e_2)| \geq r-1$ and $e_1 \neq e_2$), we get $e_{\ell-1} = V(H) \setminus (M(e_1) \cup M(e_2))$, and the same holds for $e_{\beta'(\ell)}$, a contradiction again.

Case 4: $v_3 \in e_1$, $e_2 - V(P) \subset e_1$ and $v_1 \notin e_2$. If there is $v \in e_2 \setminus V(P)$, then path $P_2 = v, e_2, v_2, e_1, v_3, e_3, \dots, v_\ell$ differs from P_1 only in the first vertex. So we can repeat the argument of Case 3 word by word.

If e_2 contains a vertex v_i for some $i \geq r+2$, then the cycle $v_3, e_3, v_4, \dots, v_i, e_2, v_2, e_1, v_3$ has $i-1 \geq r+1$ vertices.

The remaining case is $e_2 = \{v_2, v_3, \dots, v_{r+1}\}$. If for some $3 \leq i \leq r$, $v_i \in e_{\ell-1}$, then the cycle $C_i = v_{i+1}, e_{i+1}, \dots, v_{\ell-1}, e_{\ell-1}, v_i, e_{i-1}, v_{i-1}, \dots, v_3, e_1, v_2, e_2, v_{i+1}$ has $\ell-2 \geq s+1$ vertices. Thus $\{v_1, \dots, v_r\} \cap e_{\ell-1} = \emptyset$. It follows that $e_{\ell-1} = V(H) \setminus \{v_1, \dots, v_r\}$, so $P'(\ell)$ exists. If $3 \leq \beta(\ell) \leq r$, then the cycle

$$v_{\beta(\ell)+1}, e_{\beta(\ell)+1}, \dots, v_\ell, e_{\beta(\ell)}, v_{\beta(\ell)}, e_{\beta(\ell)-1}, v_{\beta(\ell)-1}, \dots, v_3, e_1, v_2, e_2, v_{\beta(\ell)+1}$$

has $\ell-1 \geq s+2$ vertices. Thus $\beta(\ell) \geq r+1$, and hence the defining vertices of the last edge $e_{\beta'(\ell)}$ of $P'(\ell)$ are not in $\{v_1, \dots, v_r\}$. This is a contradiction. $\square$

5. The path P in a best cycle-path pair (C, P) is nontrivial

Consider a best cycle-path pair (C, P) with $C = v_1, e_1, v_2, \dots, e_{s-1}, v_s, e_s, v_1$ and $P = u_1, f_1, u_2, \dots, f_{\ell-1}, u_\ell$. In this section, we rule out the case that P contains only one vertex, i.e., $\ell = 1$.

Observe that if $\ell = 1$ and (C, P) is a best cycle-path pair, then every edge of H' contains at most one vertex outside of $V(C)$, otherwise we find a longer path.

5.1. *The case of $\ell = 1$ and $r > t$*

In this subsection we prove the following lemma.

Lemma 5.1. *Let n, k, and r be positive integers such that $n \geq k$ and $r > t$. If H is an n-vertex r-graph with at least k edges such that $\delta(H) \geq \lceil k/2 \rceil$ and $c(H) < k$, then $\ell = |V(P)| \geq 2$.*

Proof. Suppose $\ell = 1$. Since $c(H) < k$, Lemma 4.2 implies that $t+2 \leq k-1$. We consider two cases.

Case 1: Some $e \in E(H')$ contains u_1.

By Claim 3.1, no two vertices of e can be consecutive on C. Since $e \subseteq V(C) \cup V(P)$, e contains $r-1$ vertices of C. Thus $r-1 \leq \lfloor s/2 \rfloor$. We know that $r \geq n/2$, so this implies that either $s = n-2$ and n is even, or $s = n-1$. In either case, there are at most two vertices in $V(C) \setminus e$ that are consecutive along C. Thus any edge $f \in E(H')$ with $f \neq e$ containing u_1 must have the property that $v_i \in e$ and $v_{i+1} \in f$ for some i. However, replacing e_i in C with e, u_1, f extends C, so such an edge f cannot exist. If $u_1 \in e_j$, then $v_j, v_{j+1} \notin e$ by Claim 3.2. Thus u_1 is contained in at most one edge in $E(H')$ and at most one edge in $E(C)$. So $\lceil k/2 \rceil \leq \delta(H) \leq d_H(u_1) \leq 2$, which can only be true if $k \in \{3, 4\}$. Since $3 \leq t+2 \leq s \leq k-1 \leq 3$, $s = 3$, and therefore e must contain at least 2 consecutive vertices in C, contradicting Claim 3.1.

Case 2: Only edges of C contain u_1.

Since $\ell = 1$, we divide the proof into the following two cases.

Case 2.1: There is some edge $e \in E(H')$ with $e \subseteq V(C)$.

By Claim 3.3, u_1 is contained in at most one edge of $\{e_i : v_i \in e\}$ and at most one edge of $\{e_{i-1} : v_i \in e\}$.

If the vertices of e are not all consecutive along C, then there are at least $r+2$ edges in $\{e_i : v_i \in e\} \cup \{e_{i-1} : v_i \in e\}$. Since u_1 is contained in at most two such edges, e prohibits at least r edges of C from containing u_1. Since u_1 is contained in at least $k/2$ edges of C, we have

$$r + k/2 \leq s \leq k-1,$$

which implies $r \leq k/2 - 1$, contradicting that $r \geq n/2 \geq k/2$.

If the vertices of e are consecutive along C, by symmetry say $e = \{v_1, \ldots, v_r\}$, then e prohibits at least $r-1$ edges of C from containing u_1, so

$$r - 1 + k/2 \leq s \leq k-1.$$

This implies $r \leq k/2$, which gives a contradiction unless $k = n$ is even, $r = n/2$, $s = k-1 = n-1$, and u_1 is contained in exactly two edges of $\{e_i : v_i \in e\} \cup \{e_{i-1} : v_i \in e\}$. The only two such edges that u_1 can be contained in are e_r and e_s because every other such edge e_i satisfies $v_i, v_{i+1} \in e$. Thus u_1 must be contained in e_r and e_s. Now consider the cycle C' formed by replacing e_{r-1} with e in C. Since $u_1 \notin e_{r-1}$, (C', u_1) is also a best cycle-path pair. Since $s = n-1$ and $u_1 \notin V(C) = V(C')$, we have that $e_{r-1} \subseteq V(C')$. Let $v_i \in e_{r-1} \setminus e$ (so $i \in \{r+1, \ldots, s\}$). Since $u_1 \in e_r$ and $v_r \in e_{r-1}$, the same argument applied to C' and e_{r-1} implies that $u_1 \notin e_i$. Thus e_{r-1} prohibits u_1 from belonging to an additional edge of C. It follows that at least $r = k/2$ edges of C cannot contain u_1 and $k/2$ edges of C must contain u_1, contradicting that $s = k-1$.

Case 2.2: Each $e \in E(H')$ contains exactly one vertex $v \notin V(C)$. Since C has at most $k-1$ edges, and $|E(H)| \geq k$, $E(H') \neq \emptyset$. Fix an edge $e \in E(H')$ and corresponding vertex $v \notin V(C)$. We must have $v \neq u_1$ because u_1 is contained only in edges of C. As before, u_1 is contained in at most one edge from each set $\{e_i : v_i \in e\}$ and $\{e_{i-1} : v_i \in e\}$. If the vertices of $e \cap V(C)$ are not all consecutive along C, then e prohibits at least $r-1$ edges of C from containing u_1. Since u_1 must be contained in at least $k/2$ edges of C, we have

$$r - 1 + k/2 \leq s \leq k - 1, \tag{18}$$

which implies $r \leq k/2$. This gives a contradiction unless $k = n$ is even, $r = n/2$, and $s = k-1$. However, u_1 and v are both outside of C, so $s \leq n-2 = k-2$, a contradiction.

If the vertices of $e \cap V(C)$ are consecutive along C, then e prohibits at least $r-2$ edges of C from containing u_1, so

$$r - 2 + k/2 \leq s \leq \min\{k-1, n-2\}.$$

This implies $r \leq k/2 + 1$, which gives a contradiction when $k \leq n-3$.

If $k \geq n-2$, then we get a contradiction unless $s = \min\{k-1, n-2\}$ and $r = \lceil n/2 \rceil$. If there exists some $f \in E(H')$ with $v \in f$ and $f \neq e$, then f prohibits at least one additional edge of C from containing u_1, using the same arguments as for e. In this case, we have $r - 1 + k/2 \leq s$, which gives a contradiction similar to (18). Otherwise, v must be contained in at least $k/2 - 1$ edges of C. If $v_i \in e$ then $v \notin e_i, e_{i-1}$ by Claim 3.2. Thus e prohibits at least r edges of C from containing v, so $r + k/2 - 1 \leq s$, giving the same contradiction as (18). □

5.2. The case of $\ell = 1$ and $r = t$

We first prove a claim that will be used in this subsection and the following.

Claim 5.2. *Let n, k, and r be positive integers such that $n \geq k$ and $r \leq t$. If H is an n-vertex r-graph with at least k edges such that $\delta(H) \geq \binom{t}{r-1} + 1$, $c(H) < k$, and $\ell = 1$, then u_1 is contained in at least 2 edges of C.*

Proof. Suppose that u_1 is contained in at most one edge of C. By Claim 3.1 no two vertices of $N_{H'}(u_1)$ are consecutive. Since $s \leq n-1 \leq 2t+1$, this implies that $|N_{H'}(u_1) \cap V(C)| \leq t$. But since $\ell = 1$, $N_{H'}(u_1) \subset V(C)$. So, since $|N_H(u_1)| \geq \binom{t}{r-1} + 1$, u_1 must be contained in an edge of C, say $u_1 \in e_{s-1}$. Then by Claims 3.2 and 3.1, the $\binom{t}{r-1}$ edges of H' containing u_1 must be disjoint from $\{v_{s-1}, v_s\}$ and nonconsecutive along C. This is possible only if $s = 2t+1$ and $|N_{H'}(u_1) \cap V(C)| = t$.

We may assume that $X := N_{H'}(u_1) = \{v_1, v_3, \ldots, v_{2t-1}\}$. Then u_1 must be contained in the $\binom{t}{r-1}$ edges of H' consisting of u_1 and $r-1$ vertices of X.

We now will find an edge $g \neq e_{2t}$ such that $|g \setminus X| \geq 2$ and $|g \cap \{v_2, v_4, \ldots, v_{2t-2}\}| \geq 1$. To do so, choose $v_{2j} \notin e_{2t}$. Since $d_H(v_{2j}) > \binom{t}{r-1}$, there is an edge g containing v_{2j} and at least one additional vertex not in X. Notice that this vertex cannot be u_1, so since X

contains all vertices of odd index other than v_{2t+1}, it must be either v_{2t+1} or be $v_{2j'}$ for some $1 \le j' \le t$, $j' \ne j$.

We use g to find a hamiltonian cycle. Let f_{2j-1} be an edge in $E(H')$ containing both u_1 and v_{2j-1}, which must exist because $v_{2j-1} \in X$. First suppose that $g \in E(H')$. If $v_{2t+1} \in g \setminus X$, then we obtain the hamiltonian cycle

$$C_1 = v_{2j}, g, v_{2t+1}, e_{2t+1}, v_1, e_1, \ldots, v_{2j-1}, f_{2j-1}, u_1, e_{2t}, v_{2t}, e_{2t-1}, \ldots, v_{2j}.$$

Otherwise, we have $v_{2j'} \in g \setminus X$ for some $1 \le j' \le t$, $j' \ne j$. Let $f_{2j'-1} \ne f_{2j-1}$ be an edge of H' containing both u_1 and $v_{2j'-1}$. Then the cycle

$$v_{2j}, g, v_{2j'}, e_{2j'}, v_{2j'+1}, e_{2j'+1}, \ldots, v_{2j-1}, f_{2j-1}, u_1, f_{2j'-1}, v_{2j'-1}, e_{2j'-2}, \ldots, v_{2j}$$

is hamiltonian.

Now we may assume that $g = e_i$ for some $i \ne 2t$. If i is even, we may orient C backwards starting from v_{2t} causing e_i to become an odd-indexed edge. Thus we may assume i is odd. Let $f_i \ne f_{2j-1}$ be an edge of H' containing both u_1 and v_i. If $2j \ne i+1$, then we have the hamiltonian cycle

$$C_2 = v_{2j}, g, v_{i+1}, e_{i+1}, v_{i+2}, e_{i+2}, \ldots, v_{2j-1}, f_{2j-1}, u_1, f_i, v_i, e_{i-1}, \ldots, v_{2j}.$$

If $2j = i+1$ and $v_{2t+1} \in g \setminus X$, then $g = e_{2j-1}$ and we obtain the cycle C_1. Otherwise, $2j = i+1$ and there is some $v_{2j'} \in g \setminus X$ with $j \ne j'$. Swapping the roles of j' with j in the cycle C_2 gives a hamiltonian cycle. □

Lemma 5.3. *Let n, k, and r be positive integers such that $n \ge k$ and $r = t$. If H is an n-vertex r-graph with at least k edges such that $\delta(H) \ge r+1$ and $c(H) < k$, then $\ell = |V(P)| \ge 2$.*

Proof. Suppose $\ell = 1$. We consider cases based on the edges containing u_1 and the edges outside of C. Note that since $\delta(H) \ge r+1$, H must have at least $n(r+1)/r \ge n+3$ edges.

Case 1: Some $e \in E(H')$ contains u_1. Note that no two vertices of $e \cap V(C)$ can be consecutive on C by Claim 3.1. Thus $r - 1 \le \lfloor s/2 \rfloor$, so $s \ge n-3$. Thus we have $n - 3 \le s \le n-1$, and there are at most three edges e_i in C with $v_i, v_{i+1} \notin e$. Observe also that by Claim 3.2, if $v_i \in e$, then $u_1 \notin e_i, e_{i-1}$.

Case 1.1: There are at most two e_i in C with $v_i, v_{i+1} \notin e$. Then there are at least $r + 1 - 2 \ge 2$ edges of $E(H')$ containing u_1, so consider $f \in E(H')$ with $u_1 \in f \ne e$. If for some i, $v_i \in e$ and $v_{i+1} \in f$ (or vice versa), we replace e_i with e, u_1, f to obtain a longer cycle. If no such i exists, then for all $v_j \in f$ we have that $v_{j-1}, v_{j+1} \notin e$. Since $f \ne e$, we can fix a j such that $v_j \in f \setminus e$. Then by Claim 3.2 f prohibits e_{j-1} and e_j from containing u_1, which were not prohibited by e. Therefore no edges of C contain u_1, so there are at least $r+1$ edges in $E(H')$ containing u_1. Then there must exist some such $f' \in E(H')$ and some i (by $r+1 > \binom{r}{r-1}$) such that $v_i \in f'$ and v_{i+1} is in e or f, which allows us to replace e_i and obtain a longer cycle.

Case 1.2: There are three edges e_i in C with $v_i, v_{i+1} \notin e$. This case can only occur when $s = n-1$ and n is even, so we have $s = 2t+1$. We first suppose that $r \geq 4$ and deal with the case $r = 3$ separately. Thus we have at least $r+1-3 \geq 2$ edges of $E(H')$ containing u_1. As in Case 1.1, we consider $f \in E(H')$ with $u_1 \in f \neq e$, and we may assume that for all $v_j \in f$ we have $v_{j-1}, v_{j+1} \notin e$. We also have some j such that $v_j \in f \setminus e$, which gives that $u_1 \notin e_{j-1}, e_j$. Thus at most one edge of C contains u_1.

If there is more than one vertex in $f' \setminus e$ for any $f' \in E(H')$ containing u_1, then no edges of C contain u_1 and we can repeat the arguments of Case 1.1 to obtain a longer cycle. By symmetry, the same holds for the edge f, so $N_{H'}(u_1) = e \cup f$. Notice that $|e \cup f| = r$, so there are at most r edges of $E(H')$ containing u_1. Since $d(u_1) \geq r+1$, this gives that u_1 is contained in exactly those r edges along with one edge of C, contradicting Claim 5.2.

We now handle the case $r = 3$. Notice that in this case, $n = 8$ and $s = 7$. If u_1 is contained in at least two edges of H', then we can in fact follow the above arguments. Thus we may assume that u_1 is contained in exactly one edge of H' and three edges of C. Up to symmetry, we have two cases.

First, consider the case $u_1 \in e = \{u_1, v_2, v_5\}$ and $u_1 \in e_3, e_6, e_7$. The cycle $C_1 = v_1, e_1, v_2, \ldots, v_6, e_6, u_1, e_7, v_1$ has the same edge set as C and misses only the vertex v_7. If v_7 is not contained in an H' edge, then (C_1, v_7) is a better cycle-path pair than (C, u_1), a contradiction. Then $v_7 \in f \in E(H')$, and observe that f cannot contain any vertex in $\{u_1, v_1, v_6\}$ by Claim 3.2 since $v_7 \in e_6, e_7$.

We now consider the possibilities for the edge f. If $v_3 \in f$, then we obtain the hamiltonian cycle $v_7, f, v_3, e_3, \ldots, v_6, e_6, u_1, e, v_2, e_1, v_1, e_7, v_7$. A symmetric argument gives a hamiltonian cycle when $v_4 \in f$. Thus $f = \{v_7, v_2, v_5\}$, and f must be the only H' edge containing v_7. Then $v_7 \in f, e_6, e_7$, and some $e' \in E(C)$. By Claim 3.2, $e' \neq e_1, e_2, e_4, e_5$. Thus $e' = e_3$, but we already have $e_3 = \{u_1, v_3, v_4\}$.

The second case for $r = 3$, up to symmetry, has $u_1 \in e = \{u_1, v_2, v_4\}$ and $u_1 \in e_5, e_6, e_7$. We consider the same cycle C_1 as above, and again we have that the edge $f \in E(H')$ containing v_7 cannot contain any vertices in $\{u_1, v_1, v_6\}$.

If $v_3 \in f$, we obtain the hamiltonian cycle $v_7, f, v_3, e_2, \ldots, v_1, e_7, u_1, e, v_4, e_4, \ldots, e_6, v_7$. If $v_5 \in f$, we have the hamiltonian cycle $v_7, f, v_5, e_5, v_6, e_6, u_1, e, v_4, e_3, \ldots, v_1, e_7, v_7$. Thus $f = \{v_7, v_2, v_4\}$, and f must be the only H' edge containing v_7. By Claim 3.2, $e' \neq e_1, e_2, e_3, e_4$, so $e' = e_5$. But we already have $e_5 = \{u_1, e_5, e_6\}$.

Case 2: Only edges of C contain u_1.

Case 2.1: There is some edge $e \in E(H')$ with $e \subseteq V(C)$.

By Claim 3.3, u_1 is contained in at most one edge of $\{e_i : v_i \in e\}$ and at most one edge of $\{e_{i-1} : v_i \in e\}$.

If the vertices of e are not all consecutive along C, then e prohibits at least r edges from containing u_1. Since u_1 must be contained in at least $r+1$ edges of C, we have

$$r + r + 1 \leq s.$$

Thus we know $r \leq (s-1)/2$, so we reach a contradiction unless $k = n$ and $s = n-1$. Notice that if the vertices of e are in more than two consecutive strings in C, then e prohibits at least $r+1$ edges and we reach a contradiction. Assume without loss of generality that $e = \{v_1, \ldots, v_{i_1}, v_{i_2}, \ldots, v_{i_3}\}$ with $i_2 \geq i_1 + 2$ and $i_3 \leq s-1$. We must also have that u_1 is contained in each edge e_i of C such that $v_i, v_{i+1} \notin e$, and u_1 is contained in exactly one of e_{i_1}, e_{i_3} and exactly one of e_{i_2-1}, e_{n-1}.

Suppose first that u_1 is contained in e_{i_1} and e_{i_2-1}. Let $f \in E(H')$ with $f \neq e$. Since u_1 is the only vertex outside of C, $f \subseteq V(C)$. If there is some $v_i \in f$ such that $u_1 \notin e_{i-1}, e_i$, then f prohibits at least one additional edge from containing u_1, giving a contradiction. Thus $f \subseteq e \cup \{v_{i_3+1}, v_{n-1}\}$. Since $f \neq e$, f must contain at least one of v_{i_3+1}, v_{n-1}. However, if $v_{i_3+1} \in f$, then $v_{i_1} \notin f$ by Claim 3.3 and the fact that $u_1 \in e_{i_3+1}, e_{i_1}$, and similarly if $v_{n-1} \in f$, then $v_{i_2} \notin f$. Therefore we have three distinct possibilities for f ($f = e - v_{i_1} + v_{i_3+1}$, $f = e - v_{i_2} + v_{n-1}$, and $f = e - v_{i_1} - v_{i_2} + v_{i_3+1} + v_{n-1}$), and there are at least $n + 3 - (n-1) - 1 = 3$ edges in $E(H')$ distinct from e. Hence each of the three possibilities are edges in H'. Notice also that for any e_i such that $v_i, v_{i+1} \in e$, we can swap e and e_i to get another maximum cycle (this cycle may not be in a best cycle-path pair). Since $e_i \neq e$ and $e_i \neq f$, $f \in E(H')$, we must have that e_i forbids at least one additional edge from containing u_1, a contradiction.

Now suppose instead that u_1 is contained in e_{i_1} and e_{n-1}. Let $f \in E(H')$ with $f \neq e$. As in the paragraph above, we have $f \subseteq e \cup \{v_{i_2-1}, v_{i_3+1}\}$, unless $i_1 = 1$, which we will handle separately. If $i_1 \neq 1$, then by a similar argument to above we reach a contradiction. If $i_1 = 1$, notice that u_1 must be contained in $r+1$ consecutive edges of C: $e_{i_3+1}, e_{i_3+2}, \ldots, e_{n-1}, e_1, e_2, \ldots, e_{i_2-2}$. In this case, either $f = (e - v_1) \cup \{v_i\}$ for some $v_i \notin e$. Similarly, for any e_j such that $v_j, v_{j+1} \in e$, we must have $e_j = (e - v_1) \cup \{v_i\}$, $v_i \notin e$, because otherwise we may swap e for e_i to see that an additional edge of C is prohibited from containing u_1. This gives that no $f \in E(H')$, $f \neq e$ and no e_j, $i_2 \leq j \leq i_3 - 1$ contains v_1.

Consider the cycle C' formed by swapping u_1 with v_1 and e with the central edge amongst $e_{i_2}, e_{i_2+1}, \ldots, e_{i_3-1}$, call it e_k. That is,

$$C' = u_1, e_1, v_2, e_2, v_3, \ldots, e_{k-1}, v_k, e, v_{k+1}, e_{k+1}, v_{k+2}, \ldots, e_{n-1}, u_1.$$

Then v_1 is contained only in edges of C', so (C', v_1) also is a best cycle-path pair under the same conditions as (C, u_1). If the edges of C' containing v_1 are not all consecutive in along C', then we must be done by a previous argument applied to C' instead of C. If $r \geq 5$, then we immediately see that $v_1 \in e$ but $v_1 \notin e_{k-1}, e_{k+1}$, so we are done. If $r = 3, 4$, then we may assume $k = i_2$ and say $v_1 \in e_{i_2-1}, e_{i_2-2}, e_{i_2-3}$ in order for the edges of C' containing v_1 to be consecutive. Then any $f \in E(H')$ with $v_{i_2} \in f \neq e$ must have $v_{i_2-1}, v_{i_2+1} \notin f$, since if $v_i, v_j \in f$, then v_1 cannot be in both e_i, e_j and cannot be in both e_{i-1}, e_{j-1} by Claim 3.3. However, there is no such edge $f \in E(H')$, so no such f contains v_{i_2}. There is exactly one possibility for f not containing v_{i_2}: $f = (e \setminus \{v_1, v_{i_2}\}) \cup \{v_{i_2-1}, v_{i_3+1}\}$. This contradicts the fact that we have at least 3 edges in $E(H')$ distinct from e.

The case $u_1 \in e_{i_3}, e_{n-1}$ is symmetric to the case $u_1 \in e_{i_1}, e_{i_2-1}$, and the case $u_1 \in e_{i_2-1}, e_{i_3}$ is symmetric to the case $u_1 \in e_{n-1}, e_{i_1}$, so we omit them.

We may now assume that all edges of $E(H')$ contained entirely in $V(C)$ are each consecutive in C, and that $e = \{v_1, v_2, \dots, v_r\}$. Then e prohibits at least $r-1$ edges of C from containing u_1, so

$$r - 1 + r + 1 \leq s$$

and thus $r \leq s/2$. If $s \leq n-3$, we immediately get a contradiction. If $s = n-2$, there exists a unique $v \notin V(C)$ with $v \neq u_1$. We must have $u_1 \in e_r, e_{r+1}, \dots, e_{n-1}$ because otherwise e prohibits r edges of C from containing u_1 and we reach a contradiction. Furthermore, we must have that each edge in $E(H')$ contains v, since any additional consecutive edge of H' contained entirely in $V(C)$ would prohibit at least one additional edge from containing u_1. Thus v is contained in at least $(n+3)-(n-2)-1 = 4$ edges of $E(H')$.

For $e_v \in E(H')$ containing v, we have that if $v_i, v_j \in e_v \cap V(C)$, then by Claim 3.3 u_1 cannot be contained in both e_i and e_j and cannot be contained in both e_{i-1} and e_{j-1}. Thus, any such e_v can contain at most one vertex outside $e \cup \{v\}$, and further that if e_v contains some vertex outside of $e \cup \{v\}$, then $v_1, v_r \notin e_v$. Therefore there exist e_v, e'_v containing v and $v_i, v_{i+1} \in V(C)$ such that say $v_i \in e_v$ and $v_{i+1} \in e'_v$. We are able to extend the cycle C by replacing e_i with e_v, v, e'_v, contradicting the maximality of C.

Therefore $s = n-1$. Then u_1 is the only vertex outside of C, so there are at least 4 edges of $E(H')$, including e, each with their vertices consecutive along C. This prohibits at least $r-1+3$ edges of C from containing u_1, giving a contradiction.

Case 2.2: Each $e \in E(H')$ contains some $v \notin V(C)$.

Let e be such an edge and $v \neq u_1$ the unique vertex in $e \setminus V(C)$ (by $\ell = 1$). Note that as in the previous case, u_1 is contained in at most one edge of $\{e_i : v_i \in e \cap V(C)\}$ and at most one edge of $\{e_{i-1} : v_i \in e \cap V(C)\}$.

If the vertices of $e \cap V(C)$ are not all consecutive along C, then e prohibits at least $r-1$ edges of C from containing u_1. Thus

$$r - 1 + r + 1 \leq s,$$

so $r \leq s/2$. If $s \leq n-3$, we immediately get a contradiction. Since $u_1, v \notin V(C)$, we must have $s = n-2$ and thus every edge of H' contains v. Hence v is contained in at least $(n+3)-(n-2) = 5$ edges of $E(H')$. For $e, f \in E(H')$, if $v_i \in e$, $v_{i+1} \in f$ for some i, then we can replace e_i with e, v, f to extend C. Since e is not all consecutive, it prohibits at least $r+2$ vertices of C from being contained in f. However, C has at most $2r$ vertices and f must contain at least $r-1$ of them, a contradiction as $r+2+r-1 > 2r$.

Thus we may assume the vertices of $e \cap V(C)$ are all consecutive along C. Then we have

$$r - 2 + r + 1 \leq s,$$

and $r \leq (s+1)/2$. If $s \leq n-4$, we get an immediate contradiction. If $s = n-2$, then similarly to above, e prohibits $r+1$ vertices of C from being contained in any $f \in E(H')$. Thus there are only $r-1$ vertices remaining in $V(C)$ that can be contained in any edge of H', but there are at least four edges of H' distinct from e, a contradiction.

Finally, we have $s = n-3$, and there is some $v' \notin V(C)$ distinct from u_1 and v. In this case, there are at least $(n+3)-(n-3) = 6$ edges of H', so we may assume without loss of generality that $v \in f \in E(H')$ for some $f \neq e$. However, e prohibits $r+1$ of the at most $2r-1$ vertices of C from being contained in f, a contradiction. □

5.3. The case of $\ell = 1$ and $r < t$

Lemma 5.4. *Let n, k, and r be positive integers such that $n \geq k$ and $r < t$. If H is an n-vertex r-graph with at least k edges such that $\delta(H) \geq \binom{t}{r-1} + 1$ and $c(H) < k$, then $\ell = |V(P)| \geq 2$.*

Proof. Suppose $\ell = 1$. Since every edge in H' contains at most one vertex outside of C, $N_{H'}(u_1) \subseteq V(C)$.

By Claim 3.1, $|N_{H'}(u_1)| \leq \lfloor s/2 \rfloor \leq t$. Let b_1 be the number of edges in $E(C)$ containing u_1. By Claim 5.2, we must have $b_1 \geq 2$.

Corollary 3.6 additionally gives that if $2 \leq b_1 \leq s-1$, then $|N_{H'}(u_1)| \leq \lceil (s-1-b_1)/2 \rceil$, and if $b_1 = s$, then $|N_{H'}(u_1)| = 0$.

Notice that

$$\binom{t}{r-1} - \binom{t-1}{r-1} \geq \binom{t}{2} - \binom{t-1}{2} = t-1$$

for $t \geq r+2$. Similarly, if $t = r+1$, then $\binom{t}{r-1} - \binom{t-1}{r-1} = \binom{t}{2} - (t-1) \geq t-1$. Thus if $b_1 \leq t-1$, we have

$$d(u_1) \leq b_1 + \binom{|N_{H'}(u_1)|}{r-1} \leq t-1 + \binom{t-1}{r-1} \leq \binom{t}{r-1},$$

a contradiction to the minimum degree. Therefore we may assume $b_1 \geq t$. This gives that $|N_{H'}(u_1)| \leq \lceil (s-t-1)/2 \rceil \leq \lceil t/2 \rceil$.

We have that

$$\binom{t}{r-1} - \binom{\lceil \frac{t}{2} \rceil}{r-1} \geq \binom{t}{2} - \binom{\lceil \frac{t}{2} \rceil}{2} \geq n-1 \geq b_1$$

whenever $\lceil t/2\rceil \geq r+1$ and $t \geq 7$. If $\lceil t/2\rceil = r$, then we instead have $\binom{t}{r-1} - \binom{\lceil t/2\rceil}{r-1} \geq \binom{t}{2} - \lceil t/2\rceil \geq n-1$ when $t \geq 7$. If $\lceil t/2\rceil \leq r-1$, then we have $d(u_1) \leq b_1+1 \leq n \leq \binom{t}{r-1}$ whenever $t \geq 6$. Hence for $t \geq 7$, we have $d(u_1) \leq b_1 + \binom{\lceil t/2\rceil}{r-1} < \delta(H)$, a contradiction.

For the remaining values of t, we consider whether or not $|N_{H'}(u_1)| = 0$. First suppose we have $|N_{H'}(u_1)| \geq r-1$ and hence $\lceil (s-b_1-1)/2\rceil \geq r-1$. When $t=4$, we need $s \in \{8,9\}$, $b_1 \in \{4,5\}$ (since $b_1 \geq t$) to have $\lceil (s-b_1-1)/2\rceil \geq r-1$. In every case we have $|N_{H'}(u_1)| = r-1$, but then $d(u_1) \leq 5+1 < 7 \leq \delta(H)$. When $t=5$, we have $s \leq 11$ and so we need $b_1 \leq 7$ to have $\lceil (s-b_1-1)/2\rceil \geq r-1 \geq 2$. Hence $d(u_1) \leq 7+1 < 11 \leq \delta(H)$. When $t=6$, we have $6 \leq b_1 \leq s \leq 13$, so $\lceil (s-b_1-1)/2\rceil \leq 3$ and hence we are done if $r \geq 5$. If $\lceil (s-b_1-1)/2\rceil = r-1$, then $d(u_1) \leq b_1+1 < 16 \leq \delta(H)$. If $\lceil (s-b_1-1)/2\rceil = r = 3$, then we must have $b_1 \leq 6$, so $d(u_1) \leq 6+3 < 16 \leq \delta(H)$, a contradiction (the case $t=6, r=4$ is done in the preceding paragraph).

For the final case of $|N_{H'}(u_1)| = 0$, we prove a brief claim.

Claim 5.5. *If $|N_{H'}(u_1)| = 0$, then $b_1 \leq s-r+2$.*

Proof. Suppose that $b_1 \geq s-r+3$. Notice that we must have $E(H') \neq \emptyset$ because there are at least $k > s$ edges. Let $e \in E(H')$, and notice that $|e \cap V(C)| \geq r-1 \geq 2$. Thus there must exist $v_i, v_j \in e$ such that $u_1 \in e_i, e_j$ or $u_1 \in e_{i-1}, e_{j-1}$ because u_1 is in all but at most $r-3$ edges of C. However, we can then consider the cycle

$$v_1, e_1, v_2, \dots, e_{i-1}, v_i, e, v_j, e_{j-1}, v_{j-1}, \dots, e_{i+1}, v_{i+1}, e_i, u_1, e_j, v_{j+1}, e_{j+1}, \dots,$$
$$e_{s-1}, v_s, e_s, v_1,$$

which is longer than C, a contradiction. $\square$

If we do have $|N_{H'}(u_1)| = 0$, then Claim 5.5 gives that $b_1 \leq s-r+2 \leq n-r+1$. Then $d(u_1) \leq n-r+1 < \delta(H)$ except in the case $t=4, r=3, b_1 \geq 7$, which we handle separately.

Case 1: $s = n-1 \in \{8,9\}$. Therefore $s - b_1 \leq 2$. Let $e \in E(H')$, and notice that $e \subseteq V(C)$ because $|N_{H'}(u_1)| = 0$. As in the case of $\ell = 1$, $r \geq t$, e prohibits some edges of C from containing u_1. That is, if $v_i, v_j \in e$, then u_1 cannot be contained in both e_i and e_j and cannot be contained in both e_{i-1} and e_{j-1}. If e is not all consecutive, then e prohibits at least 3 edges of C from containing u_1. This contradicts the fact that $s - b_1 \leq 2$. If e is all consecutive, say $e = \{v_i, v_{i+1}, v_{i+2}\}$, notice that if $u_1 \in e_i$, then by Claim 3.3 we must have $u_1 \notin e_{i-1}, e_{i+1}, e_{i+2}$, reaching the same contradiction. Thus we have $u_1 \notin e_i$ and similarly $u_1 \notin e_{i+1}$. Consider the cycle formed by swapping the roles of e and e_i. Then e_i must prohibit at least one additional edge of C from containing u_1, reaching the same contradiction again.

Case 2: $s = 8$, $n = 10$, $b_1 = 7$. If any edge of $E(H')$ is contained fully in $V(C)$, then we follow the same arguments as Case 1 to reach a contradiction. Thus we may assume every edge of $E(H')$ contains the unique vertex $x \neq u_1$ outside C. Let e_i be the unique edge of C which does not contain u_1. For any edge $e \in E(H')$, we must have $v_i, v_{i+1} \in e$, as otherwise by Claim 3.3 e will prohibit at least two edges of C from containing u_1. However, there are at least two such edges $e, e' \in E(H')$, and this gives $e = e'$, a contradiction. $\square$

6. Proof of Theorem 1.8(c)

Proof. Let n, k, and r be positive integers such that $n \geq k$ and $k-2 \geq t \geq r \geq 3$. Recall that $t = \lfloor (n-1)/2 \rfloor$. Let H be an n-vertex, r-graph with $\delta(H) \geq \binom{t}{r-1}+1$. As in previous sections, consider a best cycle-path pair (C, P) with $C = v_1, e_1, v_2, \ldots, e_{s-1}, v_s, e_s, v_1$ and $P = u_1, f_1, u_2, \ldots, f_{\ell-1}, u_\ell$. We use the same notation of H_C, H_P, H', and additionally define the following. For a vertex v of a hypergraph F, $F\{v\}$ will denote the set of the edges of F containing v.

By Lemmas 5.3 and 5.4, $\ell \geq 2$. By Lemma 4.1, $s \geq t+2$. Therefore $\ell \leq n - s \leq 2t + 2 - (t+2) = t$.

Recall for $j \in \{1, \ell\}$, $B_j = H_C\{u_j\}$, and set $b_j = |B_j|$. By symmetry, we may assume $b_\ell \geq b_1$. By Claim 3.4 and Lemma 3.7 applied to the graph cycle with edges $v_1v_2, v_2v_3, \ldots, v_sv_1$, we get that either

$$b_1 \leq (s+2)/2 - \ell, \tag{19}$$

or

$$B_1 = B_\ell \text{ and } \; b_1 \leq s/\ell. \tag{20}$$

Recall that by the maximality of $V(P)$ all edges in H' containing u_1 or u_ℓ are contained in $V(C) \cup V(P)$.

For $j \in \{1, \ell\}$, let $A_j = N_{H'}(u_j) \cap V(C)$ and $a_j = |A_j|$. By Claim 3.1, A_j contains no consecutive vertices of C for $j \in \{1, \ell\}$.

Case 1: $A_1 = \emptyset$. Then all edges in H' containing u_1 are contained in $V(P)$.

Case 1.1: $r = t$. Since $\ell \leq t$, the only possibility of an edge $g \in E(H')$ containing u_1 is that $\ell = t$ and $g = V(P)$. But then we can switch g with f_1, contradicting Part (iv) of choosing (C, P). Thus $N_{H'}(u_1) = \emptyset$. Then

$$b_1 \geq \delta(H) - |E(P)| \geq (t+1) - (\ell - 1) = t - \ell + 2. \tag{21}$$

So, if (19) holds, then since $s \leq n - \ell \leq 2t + 2 - \ell$, $b_1 \leq (2t+2)/2 - \ell$, contradicting (21).

If (20) holds, then comparing with (21) we get $t - \ell + 2 \leq (2t + 2 - \ell)/\ell$, which is equivalent to $\ell(t - \ell + 3) \geq 2t + 2$. This can hold only when $\ell = 2$ and $s = 2t$. In this case $b_1 = t$ and $B_\ell = B_1$. Since an edge in B_ℓ cannot be next to an edge in B_1 on C by Claim 3.4, we may assume that $B_1 = B_\ell = \{e_1, e_3, \ldots, e_{2t-1}\}$. Since $n = s + \ell = s + 2$, f_1 contains a vertex of C, say v_1. But then we get a longer cycle by replacing path v_1, e_1, v_2 in C with path v_1, f_1, u_1, e_1, v_2, a contradiction.

Case 1.2: $3 \le r \le t-1$. The number of edges in H' containing u_1 and contained in $V(P)$ is at most $\binom{\ell-1}{r-1}$. So,

$$\begin{aligned} b_1 &\ge 1 + \binom{t}{r-1} - \binom{\ell-1}{r-1} - (\ell-1) \ge 1 + \binom{t}{2} - \binom{\ell-1}{2} - \ell + 1 \\ &= \frac{(t+\ell-2)(t-\ell+1)}{2} - \ell + 2. \end{aligned} \tag{22}$$

If (19) holds, then since $s \le 2t+2-\ell$, we get

$$\frac{(t+\ell-2)(t-\ell+1)}{2} - \ell + 2 \le \frac{2t+4-\ell}{2} - \ell,$$

which is not true for $2 \le \ell \le t$ as $t \ge 4$.

If (20) holds, then we get

$$\frac{(t+\ell-2)(t-\ell+1)}{2} - \ell + 2 \le \frac{2t+2-\ell}{\ell}. \tag{23}$$

This does not hold in the range $2 \le \ell \le t-1$ as $t \ge 4$. Suppose now $\ell = t \ge 4$. If all $\binom{\ell-1}{r-1}$ r-subsets of $V(P)$ containing u_1 are in H', then we can replace f_1 with $\{u_1, \dots, u_r\}$ contradicting Part (iv) of choosing (C, P). Thus, in this case instead of (22), we have $b_1 \ge (t+\ell-2)(t-\ell+1)/2 - \ell + 3$ and so instead of (23), we have

$$\frac{(t+\ell-2)(t-\ell+1)}{2} - \ell + 3 \le \frac{2t+2-\ell}{\ell},$$

which is not true for $\ell = t \ge 4$. This finishes Case 1.

Case 2: $A_1 \ne \emptyset$ and $B_\ell \ne \emptyset$. If there are $v_i \in A_1$ and $e_j \in B_\ell$ such that $j \ge i$ and $j - i \le \ell - 2$, say $v_i \in g \in E(H'\{u_1\})$, then by replacing in C the path $v_i, e_i, v_{i+1}, \dots, e_j, v_{j+1}$ with the path $v_i, g, u_1, f_1, \dots, f_{\ell-1}, u_\ell, e_j, v_{j+1}$ creates a cycle longer than C, a contradiction. Thus such v_i and e_j do not exist. So each interval of $C \setminus A_1$ contains a vertex not covered by B_ℓ, and each such interval containing an edge in B_ℓ has at least $2(\ell-1)$ such vertices. Since the edges in B_ℓ cover at least $b_\ell + 1$ vertices, we get

$$a_1 + (a_1 - 1 + 2(\ell-1)) + (b_\ell + 1) \le s \le 2t + 2 - \ell. \tag{24}$$

Since $\ell \ge 2$ and by the case $b_\ell \ge 1$, (24) yields $2a_1 + 2\ell - 1 \le 2t$, so by integrality

$$t \ge a_1 + \ell. \tag{25}$$

If $r = t$, (25) yields that $H'\{u_1\}$ contains only one edge, namely, $g = A_1 \cup V(P)$, and $r = t = a_1 + \ell$. But then we can switch g with f_1 and still have the best cycle-path pair (C, P') where P' is obtained from P by deleting f_1 and adding g instead. So, there is a vertex $v_i \in (f_1 \cap V(C)) \setminus A_1$. This is one more vertex that is not next to any $v_j \in A_1$ and is at distance in C at least ℓ from B_ℓ. Thus in this case instead of (24) we get $a_1 + (a_1 + 1 + 2(\ell-1)) + (b_\ell + 1) \leq s$ and hence $t \geq a_1 + \ell + 1$, a contradiction to $r = t = a_1 + \ell$.

Suppose now $3 \leq r \leq t-1$. Then, since $b_\ell \geq b_1$,

$$1 + \binom{t}{r-1} \leq d(u_1) = d_{H'}(u_1) + b_1 + d_{H_P}(u_1) \leq \binom{a_1 + (\ell-1)}{r-1} + b_\ell + (\ell - 1).$$

So,

$$\begin{aligned}\frac{t(t-1) - (a_1+\ell-1)(a_1+\ell-2)}{2} &= \binom{t}{2} - \binom{a_1+\ell-1}{2} \leq \binom{t}{r-1} - \binom{a_1+\ell-1}{r-1} \\ &\leq b_\ell + \ell - 2.\end{aligned}$$

Plugging in the upper bound on $b_\ell + \ell - 2$ from (24) and rewriting $(t(t-1) - (a_1 + \ell - 1)(a_1 + \ell - 2))/2$ as $((t + a_1 + \ell - 2)(t - a_1 - \ell + 1))/2$, we obtain

$$\frac{(t + a_1 + \ell - 2)(t - a_1 - \ell + 1)}{2} \leq 2(t - a_1 - \ell + 1). \tag{26}$$

Since by (25), $t - a_1 - \ell + 1 > 0$, (26) simplifies to $t + a_1 + \ell - 2 \leq 4$. Since $t \geq r + 1 \geq 4$, $a_1 \geq 1$ and $\ell \geq 2$, this is impossible.

Case 3: $A_1 \neq \emptyset$, $B_\ell = B_1 = \emptyset$, and $A_\ell \neq A_1$. By Case 1, $a_1 > 0$ and $a_\ell > 0$.

If $i < i' \leq i + \ell$, and there are distinct $g_1, g_\ell \in E(H')$ such that $\{v_i, u_1\} \subset g_1$ and $\{v_{i'}, u_\ell\} \subset g_\ell$, then replacing path $v_i, e_i, \ldots, v_{i'}$ in C with the path $v_i, g_1, u_1, f_1, \ldots, f_{\ell-1}, u_\ell, g_\ell, v_{i'}$ creates a cycle longer than C, a contradiction.

By Claim 3.1, $A_1 \cup A_\ell$ does not contain consecutive vertices of C. We may assume that $a_1 \leq a_\ell$. Then since $A_\ell \neq A_1$, $A_\ell - A_1 \neq \emptyset$. So, applying Lemma 3.8 with $A = A_1, B = A_\ell$, and $q = \ell + 1$,

$$a_1 \leq (s+2)/2 - \ell - 1 \leq (2t + 2 - \ell)/2 - \ell \leq t - \ell. \tag{27}$$

Also using that $B_1 = \emptyset$,

$$d_{H'}(u_1) \geq d_H(u_1) - b_1 - (\ell - 1) \geq 1 + \binom{t}{r-1} - 0 - \ell + 1 = 2 + \binom{t}{r-1} - \ell. \tag{28}$$

Case 3.1: $r = t$. Then each edge $g \in H'\{u_1\}$ has at least $t - \ell$ vertices in $V(C)$ with equality only when $V(P) \subset g$. By (28) and $\ell \leq t$, $d_{H'}(u_1) \geq 2$. Hence there are at least two edges of H' containing u_1, implying $a_1 \geq (t+1) - \ell$. This contradicts (27).

Case 3.2: $3 \leq r \leq t-1$. Then $d_{H'}(u_1) \leq \binom{a_1+\ell-1}{r-1}$. So, by (27), $d_{H'}(u_1) \leq \binom{t-1}{r-1}$, and together with (28), we get

$$2 + \binom{t}{r-1} - \ell \le \binom{t-1}{r-1},$$

which is not true as $2 \le r-1 \le t-2$ and $\ell \le t$.

Case 4: $A_1 \neq \emptyset$, $B_\ell = B_1 = \emptyset$, and $A_\ell = A_1$. Let $A_1 = \{x_1, \dots, x_{a_1}\}$ with vertices in clockwise order on C.

Case 4.1: Between any x_j and x_{j+1} there are at least ℓ vertices. Then $(\ell+1)a_1 \le s$. If $a_1 \ge 2$, then (27) holds by $2 \le \ell \le t$ and some calculations, and we repeat the argument of Case 3. Suppose (27) does not hold, so $a_1 = 1$ and $A_\ell = A_1 = \{v_1\}$. Since

$$d_{H'}(u_1) \ge 1 + \binom{t}{r-1} - (\ell - 1) \ge 1 + \binom{t-1}{r-1} \tag{29}$$

and each edge in $H'\{u_1\}$ is contained in $V(P) + v_1$, $\ell = t$ and some edge $g \in H'\{u_1\}$ contains u_ℓ. Also, by degree condition, some edge $f \in H\{u_1\}$ is not contained in $V(P) + v_1$. By the case, this is some f_j. By the symmetry between u_1 and u_ℓ, we may assume $j \le \ell/2$. Since H contains path $P_j = u_{j+1}, f_{j+1}, \dots, u_\ell, g, u_1, f_1, \dots, u_j$, the edge f_j is contained in $V(C) \cup C(P)$, and hence f_j contains some v_i for $i \neq 1$. By symmetry, we may assume $i \le s/2 + 1 = t/2 + 2$.

We will show that there is an edge $g_1 \in H'\{u_1\} \setminus g$ not contained in $V(P)$ and hence containing v_1. Indeed, if all other edges of $H'\{u_1\}$ are subsets of $V(P)$, then $d_{H'}(u_1) \le 1 + \binom{t-1}{r-1}$. In particular by (29), all $\binom{t-1}{r-1}$ r-element subsets of $V(P)$ containing u_1 are edges in H' (and not edges of P). This violates Rule (iv) of the choice of (C, P) as we could replace some $f_i \in E(P)$ with an edge of H' to obtain a better cycle-path pair. So suppose such an edge g_1 exists.

When we replace path $v_1, e_1, v_2, \dots, v_i$ in C with path

$$v_1, g_1, u_1, g, u_\ell, f_{\ell-1}, u_{\ell-1}, \dots, u_{j+1}, f_j, v_i,$$

we first delete the $i-2$ internal vertices of the former path and then add $t-j+1$ vertices of the latter. So, the length of the cycle will be at least

$$s - (i-2) + (t-j+1) \ge s - t/2 + t/2 + 1 > s,$$

a contradiction.

Case 4.2: There are indices j such that between x_j and x_{j+1} there are at most $\ell - 1$ vertices. Since $A_\ell = A_1$, for each such j there is an edge $g \in E(H'\{u_1\}) \cap E(H'\{u_\ell\})$ containing x_j and x_{j+1} and no other edge in $E(H'\{u_1\}) \cup E(H'\{u_\ell\})$ contains any of x_j and x_{j+1}, as otherwise there is a longer cycle. In this case, we call g a *private edge of* u_1 and x_j and x_{j+1} – g*-private neighbors* of u_1, or simply *private neighbors*. Suppose that the set of private edges of u_1 is $\{g_1, \dots, g_m\}$ and that A is the set of non-private neighbors of u_1 in C. Let $a = |A|$. By these definitions and remembering that A_1 is independent, $a_1 \ge a + 2m$ and

$$d_{H'}(u_1) \le m + \binom{|V(P) \cup A| - 1}{r-1} \le m + \binom{\ell + a - 1}{r-1} \le \binom{\ell + a + m - 1}{r-1}. \tag{30}$$

Recall that our case is that $m \ge 1$. If $a = 0$ and $m = 1$, then only one edge, say g_1 in $H'\{u_1\}$ intersects $V(C)$. By (28), $H'\{u_1\}$ contains another edge, say g that must be contained in $V(P)$. This yields $\ell = t$ and $g = V(P)$. But then we can switch g with f_1 contradicting Rule (iv) of the choice of (C, P). Thus $a + m \ge 2$.

We may rename the vertices of C in such a way that $x_1 = v_1$ and the vertices x_1 and x_{a_1} are g_1-private neighbors of u_1. Then each interval $I_j = [x_j, x_{j+1}]$ on C with $x_{j+1} \in A$ has length at least $\ell + 1$, and for each private edge g and the minimum j with $x_{j+1} \in g$, the interval I_j also has length at least $\ell + 1$. Thus at least $a + m$ intervals I_j have length at least $\ell + 1$, and since $a_1 \ge a + 2m$,

$$2t \ge n - \ell \ge s \ge (\ell + 1)(a + m) + m. \tag{31}$$

Since $m \ge 1$ and $a + m \ge 2$, (31) yields $2t \ge 2(a + m) + (\ell - 1)(a + m) + m \ge 2(a + m) + (\ell - 1)2 + 1$, and so $t > a + m + \ell - 1$. This means

$$t \ge a + m + \ell. \tag{32}$$

Plugging (32) into (30) and comparing with (28), we get

$$\binom{t}{r-1} - \binom{t-1}{r-1} \le \ell - 2,$$

which does not hold for $\ell \le t$ when $r \le t$. $\square$

7. Proof of Theorem 1.10

Proof. Let $k \ge r > t$ be the smallest integer at least $n/2$ for which the theorem does not hold. Let H be an n-vertex r-graph with at least k edges and $\delta(H) \ge \lceil k/2 \rceil$ such that H has no cycle of length k or longer.

Choose a best cycle-path pair (C, P) with notation as in the previous two sections. By Lemma 4.2, $k > t + 2$. Moreover, by Lemma 5.1, $\ell \ge 2$.

Since the theorem holds for $k' < k$, $s = k - 1$. Also by the maximality of ℓ, each edge in H' containing u_1 or u_ℓ is contained in $V(C) \cup V(P)$ and cannot have two consecutive vertices of C by Claim 3.1.

Case 1: $\ell \ge (1 + k)/2$.

Case 1.1: There are distinct v_i and v_j in $V(C)$ such that $v_i \in f_1$ and $v_j \in f_{\ell-1}$. By symmetry, we may assume that $i = 1$ and $j \le (s + 1)/2$. By the maximality of s, the path $v_1, f_1, u_2, f_2, \dots, u_{\ell-1}, f_{\ell-1}, v_j$ is not longer than the path $v_1, e_1, \dots, e_{j-1}, v_j$. This means $\ell - 2 \le j - 2$. Plugging in the inequalities for ℓ and j, we get

$$(1 + k)/2 \le (s + 1)/2 \le k/2,$$

a contradiction.

Case 1.2: Case 1.1 does not hold.

Then either f_1 or $f_{\ell-1}$ contains at most one vertex in C. Since they overlap in at most one vertex, and $|f_1 \cup V(C)|, |f_{\ell-1} \cup V(C)| \leq n$, this gives $s + r \leq n + 1$. By Lemma 4.2, this is only possible when $r = n/2$ and $s = 1 + n/2$. Since $r + s > n$, each of f_1 and $f_{\ell-1}$ has exactly one vertex in C. Since Case 1.1 does not hold, this is the same vertex, say v_1. Moreover, each of f_1 and $f_{\ell-1}$ must contain $V(G) \setminus V(C)$. But then $f_1 = f_{\ell-1}$ and so $\ell = 2$. By the case, $2 \geq (k+1)/2$, i.e., $k \leq 3$, so $3 > (n+1)/2$, thus $n \leq 4$, and $r \leq n/2 \leq 2$, a contradiction to $r \geq 3$.

Case 2: $2 \leq \ell \leq k/2$. Since $s \geq (n+1)/2$, $\ell \leq n - s < n/2 \leq r$. So, $r - \ell \geq 1$.

Case 2.1: There is an edge $g \in E(H')$ containing u_1. By the maximality of $|V(P)|$, $g \subset V(C) \cup V(P)$. So $|g \cap V(C)| \geq r - \ell$. Since no vertices of g are consecutive on C, the number of vertices in the largest interval of C between vertices of g is at most

$$s - 2(r - \ell) + 1 \leq (n - \ell) - 2r + 2\ell + 1 \leq \ell + 1. \tag{33}$$

This means, the distance on C from any of its vertices to g is at most $1 + \ell/2$.

Case 2.1.1: Some e_i contains u_ℓ, say $i = 1$. If some $v_j \in g$ and $j \leq \ell + 1$, then we can replace the path $v_1, e_1, v_2, \ldots, v_j$ in C with the path $v_1, e_1, u_\ell, f_{\ell-1}, u_{\ell-1}, \ldots, u_1, g, v_j$, and get a longer cycle. Thus the interval of C between two vertices of g that contains e_1 has at least $2 + 2(\ell - 1) = 2\ell$ vertices, contradicting (33).

Case 2.1.2: None of e_i contains u_ℓ. Since $d(u_\ell) \geq k/2 \geq \ell$ and P has only $\ell - 1$ edges, there is an edge $g' \in E(H')$ containing u_ℓ. So, by symmetry we may assume that none of e_i contains u_1.

Suppose first $g' \neq g$. Since the distance on C between any vertex of $g \cap V(C)$ and any vertex of $g' \cap V(C)$ is either 0 or at least $1 + \ell$, all vertices of $g' \cap V(C)$ must belong to g by (33), and the distance on C between any two vertices of g' is at least $1 + \ell$. By symmetry, we get $g \cap V(C) = g' \cap V(C)$. Since $g \neq g'$, the edges must differ in $V(P)$. In particular, $|g \cap V(P)| \leq \ell - 1$, and hence $|g \cap V(C)| \geq r - \ell + 1$. But then

$$n \geq s + \ell \geq (1 + \ell)(r - \ell + 1) + \ell. \tag{34}$$

The minimum of the polynomial $F(\ell) = -\ell^2 + (r+1)\ell + r + 1$ in the right hand side of (34) is attained when ℓ is extremal. We have $F(2) = F(r-1) = -1 + 3r$, which is greater than n when $r \geq \max\{3, n/2\}$.

Suppose now only g is an edge in H' containing u_ℓ. Since $r - \ell \geq 1$, we have $H\{u_1\} = H\{u_\ell\} = E(P) \cup \{g\} =: L$. Moreover, for any $u_i \in V(P)$, the path $P_i^1 = u_i, f_{i-1}, \ldots, f_1, u_1, f_i, u_{i+1}, \ldots, f_{\ell-1}, u_\ell$ has the same length, vertices, and edges as P. We conclude that (C, P_i^1) is also best cycle-path pair, and so we may assume that $H\{u_i\} = L$ for all $1 \leq i \leq \ell$. Therefore $V(P) \subset g$ and $\ell = r - 1$.

Moreover, for every $1 \leq j \leq \ell - 1$, the path $P(j) = u_{j+1}, f_{j+1}, \ldots, u_\ell, g, u_1, f_1, \ldots, u_j$ has the same vertex set as P, and its ends, u_j and u_{j+1} belong to edge f_j not used in $P(j)$. The cycle-path pair $(C, P(j))$ is also a best pair since $V(P) \subset g$. As above we conclude that $f_j \supset V(P)$. In particular, $\ell = |L| = \lceil k/2 \rceil$. Also, each edge in L has exactly one vertex on C and these vertices are distinct. Since $\ell \geq k/2 > s/2$, some vertices of edges in L are consecutive on C. By symmetry, we may assume $v_s \in g$ and $v_1 \in f_1$. Then replacing edge e_s in C by path v_s, g, u_1, f_1, v_1, we get a cycle longer than C.

A symmetric argument applies when there is an edge of H' containing u_ℓ.

Case 2.2: No edge in $E(H')$ contains u_1 or u_ℓ. Recall B_1 (respectively, B_ℓ) is the set of edges e_i that contain u_1 (respectively, u_ℓ). Then for $j \in \{1, \ell\}$, $|B_j| \geq \delta(H) - |E(P)| \geq \lceil k/2 \rceil - \ell + 1$. If B_1 or B_ℓ has size greater than $\lceil k/2 \rceil - \ell + 1$, then we can delete some edges to make both have exactly $\lceil k/2 \rceil - \ell + 1$ edges and be different from each other.

By Claim 3.4, for any distinct $e_i \in B_1$ and $e_j \in B_\ell$, $|i - j| \geq \ell$. So, if $B_1 \neq B_\ell$, then we apply Lemma 3.7 to B_1, B_ℓ and $q = \ell$ to obtain $s \geq 2(\lceil k/2 \rceil - \ell + 1) + 2(\ell - 1) \geq k$, a contradiction. Thus $B_1 = B_\ell$ and $|B_1| = \lceil k/2 \rceil - \ell + 1$. For this, we need $\{u_1, u_\ell\} \subset f_i$ for all $1 \leq i \leq \ell - 1$ and hence for $u \in \{u_1, u_\ell\}$,

$$\textit{the set of edges containing } u \textit{ is } B_1 \cup \{f_1, \dots, f_{\ell-1}\}. \tag{35}$$

If f_1 contains a vertex $u \in V(G) \setminus (V(C) \cup V(P))$, then u can play the role of u_1, and hence (35) holds, as well. Also, for each $1 \leq j < \ell$, since $u_1 \in f_j$, the path $P_j^1 = u_j, f_{j-1}, u_{j-1}, \dots, u_1, f_j, u_{j+1}, f_{j+1}, \dots, u_\ell$ can play role of P. It follows that (35) holds for $u = u_j$ and hence for all $u \in f_{j-1}$.

By symmetry, let $e_1 \in B_1$. By the above, e_1 contains $\{u_1, \dots, u_\ell\}$, all vertices in $f_1 \setminus (V(C) \cup V(P))$, and v_1, v_2. Since $|e_1| = r = |f_1|$, the edge f_1 has at least two vertices in C. These vertices must be at distance in C at least $\ell - 1$ from any edge in B_ℓ. Recalling that B_ℓ is a set of $\lceil k/2 \rceil - \ell + 1$ edges that are distance at least ℓ apart from one another, it follows that

$$s \geq \ell(k/2 - \ell + 1) + (\ell - 2) + 2 = \ell(k/2 - \ell) + 2\ell.$$

For $2 \leq \ell \leq k/2$, the right hand side of the above inequality is at least k, a contradiction. $\square$

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