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Root of unity quantum cluster algebras and discriminants

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Abstract

We describe a connection between the subjects of cluster algebras, polynomial identity algebras, and discriminants. For this, we define the notion of root of unity quantum cluster algebras and prove that they are polynomial identity algebras. Inside each such algebra we construct a (large) canonical central subalgebra, which can be viewed as a far reaching generalization of the central subalgebras of big quantum groups constructed by De Concini, Kac, and Procesi and used in representation theory. Each such central subalgebra is proved to be isomorphic to the underlying classical cluster algebra of geometric type. When the root of unity quantum cluster algebra is free over its central subalgebra, we prove that the discriminant of the pair is a product of powers of the frozen variables times an integer. An extension of this result is also proved for the discriminants of all subalgebras generated by the cluster variables of nerves in the exchange graph. These results can be used for the effective computation of discriminants. As an application we obtain an explicit formula for the discriminant of the integral form over $\mathbb{Z}[\varepsilon]$ of each quantum unipotent cell of De Concini, Kac, and Procesi for arbitrary symmetrizable Kac–Moody algebras, where ε is a root of unity.

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1 INTRODUCTION

1.1 Cluster algebras and discriminants

Cluster algebras were introduced by Fomin and Zelevinsky in [14] and since then, they have played a fundamental role in a number of diverse areas such as representation theory, combinatorics, Poisson and algebraic geometry, mathematical physics, and others [17, 35].

Discriminants of number fields were defined by Dedekind in the 1870s. They have proven an invaluable tool in number theory, algebraic geometry, combinatorics, and orders in central simple algebras [23, 37, 38]. In more recent years, new applications of discriminants have been found in the noncommutative setting. Bell, Ceken, Palmieri, Wang, and Zhang used the discriminant as an invariant in determining the automorphism groups of certain polynomial identity (PI) algebras [7, 8] and to address the Zariski cancellation problem (when $A[t] \simeq B[t]$ implies $A \simeq B$) [1]. Discriminant ideals are also intrinsically related to the representation theory of the corresponding noncommutative algebra [6].

In this paper, we connect the subjects of cluster algebras, polynomial identity algebras, and discriminants (we refer the reader to [5, section I.13– 14 and part III] and [33, chapter 13] for an overview of polynomial identity algebras and their representation theory). We define the notion of root of unity quantum cluster algebra, show that these algebras are polynomial identity algebras, and construct a canonical large central subalgebra in each of them which is shown to be isomorphic to the underlying classical cluster algebra. These special central subalgebras can be viewed as far reaching generalizations of the De Concini–Kac–Procesi central subalgebras of big quantum groups [11, 12]. We prove a theorem giving an explicit formula for the discriminant of a root of unity quantum cluster algebra, and apply it to compute the discriminants of the big quantum unipotent cells for all symmetrizable Kac–Moody algebras at roots of unity.

1.2 Root of unity quantum cluster algebras

Let $\varepsilon^{1/2}$ be a primitive ℓth root of unity for a positive integer ℓ. We define a root of unity quantum cluster algebra by constructing mutations in the skew field of fraction of the based quantum torus over $\mathbb{Z}[\varepsilon^{1/2}]$ with basis $\{X^f \mid f \in \mathbb{Z}^N\}$ and relations

$$X^f X^g = \varepsilon^{\Lambda(f,g)/2} X^{f+g}, \quad \forall f, g \in \mathbb{Z}^N$$

for a skew-symmetric bilinear form $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}/\ell$. Quantum frames M_ε are introduced in this setting as in the quantum setting of Berenstein and Zelevinsky [4], but weaker compatibility assumptions between the bilinear form Λ and the exchange matrix $\widetilde{B}$ are imposed (Definition 3.2). *In particular, $\widetilde{B}$ need no longer have a full rank as the quantum case in* [4]. A subset **inv** of the exchange indices **ex** is allowed to be inverted and the corresponding $\mathbb{Z}[\varepsilon^{1/2}]$-algebra generated by all cluster variables and the inverses of the frozen ones in **inv** is denoted by $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ (Definition 3.7).

In the special case of $\ell = 1$ this construction exactly recovers the definition of a classical cluster algebra of geometric type. Quantum Weyl algebras and quantum unipotent cells at roots of unity for all symmetrizable Kac–Moody algebras are examples of root of unity quantum cluster algebras (Subsections 5.6 and 8.3). In addition to the standard properties of classical and quantum cluster algebras, such as the Laurent phenomenon, we prove the following key results for the algebras $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$:

Theorem A. *Let $\varepsilon^{1/2}$ be a primitive ℓth root of unity for a positive integer ℓ.*

(1) *All root of unity quantum cluster algebras $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ are PI algebras.*
(2) *The subring $\mathsf{C}(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ generated by the ℓth powers of all cluster variables and the inverses of the ℓth powers of the frozen ones in **inv** is in the center of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$. If ℓ is odd and coprime to the entries of the symmetrizing diagonal matrix for the principal part of $\widetilde{B}$, this subring is isomorphic to the corresponding classical cluster algebra $\mathsf{A}(\widetilde{B}, \mathbf{inv})$.*
(3) *Under the assumption in part (2) the exchange graphs of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ and $\mathsf{A}(\widetilde{B}, \mathbf{inv})$ are canonically isomorphic.*

In [13], Fock and Goncharov defined and studied root of unity quantum cluster algebras in the setting of cluster $\mathcal{X}$-varieties. They constructed an isomorphism between the (upper) cluster algebra of a cluster $\mathcal{X}$-variety and a central subalgebra of the corresponding root of unity (upper) quantum cluster algebra under the following assumption:

(*) the order of the root of unity is coprime to the entries of the exchange matrices of all seeds of the algebra.

This isomorphism in [13] is called the quantum Frobenius map. The differences between our setting and the setting of [13] are as follows. First, compared to the assumption (*), the assumption in Theorem A(2) is weaker and explicit in the sense that it requires knowledge of only one seed, while (*) involves the exchange matrices of all seeds which are very rarely known except the case of surface cluster algebras. Second, the cluster $\mathcal{X}$-variety is a regular Poisson manifold and the representations of the corresponding root of unity upper quantum cluster $\mathcal{X}$-algebra have the same dimension, that is, that setting captures only the Azumaya locus of a root of unity quantum algebra. Our setting of the algebras $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is suitable to the study of all irreducible representations of root of unity quantum algebras, for instance the spectrum of the central subring of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ (when the base is extended from $\mathbb{Z}$ to $\mathbb{C}$) is extremely rarely a regular Poisson manifold. The proofs of the quantum Frobenius map in [13] is different from ours. It relies to specializations of quantum dilogarithms defined for generic q, while we work directly with the root of unity algebra without the use of specialization. Finally, we note that in the setting of [13], Mandel [34] proved the quantum Frobenius conjecture of [13] on the specialization of quantum theta functions to roots of unity.

In the setting of Theorem A, denote by

$$\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \tag{1.1}$$

the $\mathbb{Z}[\varepsilon^{1/2}]$-extension of the subring $\mathsf{C}(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$. It is isomorphic to $\mathsf{A}(\widetilde{B}, \mathbf{inv}) \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon^{1/2}]$. In concrete important situations $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is module finite over $\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ (Subsections 6.4 and 8.3). For quantum unipotent cells at roots of unity, the latter is proved to be precisely the special De Concini–Kac–Procesi subalgebra [12]. The punchline of part (2) of the theorem is that it not only constructs a large central subalgebra in vast generality, but it also gives a full control on it via cluster theory. As an upshot, the representation theory of the algebras in [12] can be studied within the framework of root of unity and classical cluster algebras.

The proof of part (3) uses a different strategy from the Berenstein–Zelevinsky [4] result for the isomorphism between classical and quantum exchange graphs. It is based on the special central subalgebras from part (2).

Root of unity quantum cluster algebras do not necessarily arise as specializations of quantum cluster algebras. For instance, in the case $\ell = 1$ we recover all cluster algebras of geometric type. For these reasons, we introduce a subclass of *strict root of unity quantum cluster algebras*, defined as those for which the skew-symmetric bilinear form $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}/\ell$ comes from a skew-symmetric bilinear form $\mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}$ which is compatible with the exchange matrix $\widetilde{B}$ in the sense of [4]. In the case $\ell = 1$, that notion is the same as the notion of a classical cluster algebra with a compatible Poisson structure in the sense of Gekhtman–Shapiro–Vainshtein [24]. If the quantum cluster algebra for $\widetilde{B}$ equals the corresponding upper quantum cluster algebra, then we prove that the root of unity $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ arises as a specialization from a quantum cluster algebra (Section 5). This gives an effective tool for the construction of root of unity quantum cluster algebras (Subsections 5.6 and 8.3).

1.3 | Discriminants

Knowing the explicit form of the discriminant of a noncommutative algebra has a number of important applications, but its calculation is very difficult. Only a few results are known to date and they concern concrete classes of algebras. Skew-polynomial algebras were treated in [7, 8], their Veronese subrings in [10], low dimension Artin–Schelter regular algebras in [1, 40, 41], Ore extensions without skew-derivations and skew group extensions in [19], quantized Weyl algebras in [9, 31], Taft algebra smash products in [20] and others. A Poisson geometric method for computing discriminants via deformation theory was given in [36].

We prove the following general results for the computation of the discriminants of all root of unity quantum cluster algebras over their special central subalgebras (1.1) arising from Theorem A(2):

Theorem B. *Let $\varepsilon^{1/2}$ be a primitive ℓth root of unity and $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ be a root of unity quantum cluster algebra such that ℓ is odd and coprime to the entries of the skew-symmetrizing diagonal matrix for the principal part of $\widetilde{B}$. Let Θ be any collection of seeds that is a nerve (in the sense of [18] and Definition 6.4) and $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$, $\mathsf{A}(\Theta, \mathbf{inv})$ (resp., $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$) be the subalgebras of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$, $\mathsf{A}(\widetilde{B}, \mathbf{inv})$ (resp., $\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$) generated by the cluster variables from the seeds in Θ (resp., their ℓth powers).*

(1) *If $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a free module over $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$, then $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a finite rank $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$-module of rank ℓ^N, where as before N denotes the number of variables in each seed, and its discriminant $d\,(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})/\mathsf{C}_\varepsilon(\Theta, \mathbf{inv}))$ with respect to the regular trace function equals*

$$\ell^{N\ell^N} \prod_{i \in [1,N] \setminus (\mathbf{ex} \sqcup \mathbf{inv})} X_i^{\ell a_i} \quad \text{for some } a_i \in \mathbb{N}$$

up to multiplication by a unit of $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$ *(discriminants are nonuniquely defined up to such unit). Here* X_i *denote the frozen variables of* $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ *and* $\mathbb{N} := \{0, 1, \ldots\}$.

(2) *If* $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ *is a free module over* $\mathsf{A}(\Theta, \mathbf{inv})$*, then* $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ *is a finite rank* $\mathsf{A}(\Theta, \mathbf{inv})$*-module of rank* $\ell^N \varphi(\ell)$ *and its discriminant* $d\left(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})/\mathsf{A}(\Theta, \mathbf{inv})\right)$ *with respect to the regular trace function equals*

$$\left(\frac{\ell^{(N+1)\varphi(\ell)}}{\prod_{p|\ell} p^{\varphi(\ell)/(p-1)}}\right)^{\ell^N} \prod_{i\in[1,N]\setminus(\mathbf{ex}\sqcup\mathbf{inv})} X_i^{\ell c_i} \text{ for some } c_i \in \mathbb{N}$$

up to multiplication by a unit of $\mathsf{A}(\Theta, \mathbf{inv})$*, where* $\varphi(.)$ *denotes Euler's* φ*-function.*

In the theorem, one can choose Θ to be the set of all seeds, which gives a formula for the discriminant of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ over $\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$. The choice of any nerve Θ in the collection of all seeds allows for the extra flexibility in computing discriminants of subalgebras of root of unity quantum cluster algebras that do not have cluster structures on their own. The very specific form of the discriminant in the theorem makes the computation of the integers a_i easy by degree and filtration arguments (see, e.g., Subsection 8.5).

1.4 | The De Concini–Kac–Procesi quantum unipotent cells

Many PI algebras are secretly root of unity quantum cluster algebras or, more generally, algebras of the form $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$. Let $\mathfrak{g}$ be an arbitrary symmetrizable Kac–Moody algebra and w a Weyl group element. In Theorems 8.4 and 8.5, we prove that this is the case for the integral forms over $\mathbb{Z}[\varepsilon]$ of all big quantum unipotent cells $A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ of [12] (when ℓ is odd and coprime to the symmetrizing integers of the Cartan matrix of $\mathfrak{g}$), namely that

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]} \cong \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \varnothing) \tag{1.2}$$

for a certain exchange matrix $\widetilde{B}$, and that the corresponding De Concini–Kac–Procesi central subalgebra $C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ is precisely the underlying classical cluster algebra

$$C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]} \cong \mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \varnothing) \cong \mathsf{A}(\widetilde{B}, \varnothing) \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]. \tag{1.3}$$

The DKP central subalgebras $C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ play a fundamental role [11, 12] in the study of the representation theory of big quantum unipotent cells $A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$. The power of the isomorphisms (1.2)–(1.3) is that we get a full control on the pair $(A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}, C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]})$ as a pair of a root of unity quantum cluster algebra and the underlying classical cluster algebra. Furthermore, using Theorem B, we prove:

Theorem C. *For all symmetrizable Kac–Moody algebras* $\mathfrak{g}$*, Weyl group elements* w *and primitive* ℓ*th roots of unity* ε *such that* ℓ *is odd and coprime to the symmetrizing integers of the Cartan matrix of* $\mathfrak{g}$*, the discriminant* $d\big(A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}/C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}\big)$ *of the integral form of the corresponding quantum unipotent cell* $A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ *over its De Concini–Kac–Procesi central subalgebra* $C_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$

with respect to the regular trace equals

$$\ell^{(N\ell^N)} \prod_{i \in S(w)} \overline{D}_{\varpi_i, w\varpi_i}^{\ell^N(\ell-1)},$$

up to multiplication by a unit of $\mathbb{Z}[\varepsilon]$*, where* $S(w)$ *is the support of* w *and* $\overline{D}_{\varpi_i, w\varpi_i}$ *are the standard unipotent quantum minors in* $A_\varepsilon(\mathfrak{n}_+(w))_{\mathbb{Z}[\varepsilon]}$ *associated to the fundamental weights* ϖ_i*.*

A special and weaker case of this theorem was proved in [36]. It only dealt with the case of finite-dimensional simple Lie algebras $\mathfrak{g}$, due to the use of Poisson geometric results from [11, 12]. Furthermore, [36] only applied to the case of discriminants of algebras over $\mathbb{C}(\varepsilon)$ and not over $\mathbb{Z}[\varepsilon]$, because of the use of Poisson geometric techniques.

Remark D. We expect that other important pairs of the form

(PI algebra, previously constructed central subalgebra)

will be shown to be special cases of pairs of the form

$$(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv}), \mathsf{C}_\varepsilon(\Theta, \mathbf{inv}) \cong \mathsf{A}(\Theta, \mathbf{inv}) \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon^{1/2}])$$

and that cluster algebras can provide a strong new tool for the study of the representation theory of such PI algebras.

1.5 | Notation

We will use the following notation throughout the paper. For a pair of integers $j \leqslant k$, denote $[j,k] := \{j, j+1, \dots, k\}$. For a pair of positive integers m, n, denote $0_{m \times n}$ the zero matrix of size $m \times n$.

2 | PRELIMINARIES ON CLASSICAL AND QUANTUM CLUSTER ALGEBRAS

In this section, we gather background material on cluster algebras of geometric type and quantum cluster algebras which will be used in the rest of the paper.

2.1 | Cluster algebras of geometric type

Cluster algebras were defined by Fomin and Zelevinsky in [14]. Let N be a positive integer, $\mathbf{ex}$ be a subset of $[1, N]$, and $\mathcal{F}$ be a purely transcendental extension of $\mathbb{Q}$ of transcendence degree N. A pair $(\widetilde{\mathbf{x}}, \widetilde{B})$ is called a *seed* if

(1) $\widetilde{\mathbf{x}} = \{x_1, \dots, x_N\}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{Q}$ which generates $\mathcal{F}$;
(2) $\widetilde{B} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$ and its $\mathbf{ex} \times \mathbf{ex}$ submatrix B (called the principal part of $\widetilde{B}$) is skew-symmetrizable; that is DB is skew-symmetric for a matrix $D = \mathrm{diag}(d_j, j \in \mathbf{ex})$ with $d_j \in \mathbb{Z}_+$.

We call $\widetilde{B}$ the *exchange matrix* of the seed, $\widetilde{\mathbf{x}}$ the *cluster* of the seed, x_i the *cluster variables*. The subset $\mathbf{ex} \subseteq [1, N]$ is called set of *exchangeable indices*. The columns of $\widetilde{B}$ are indexed by this set. The mutation of $\widetilde{B}$ in direction $k \in \mathbf{ex}$ is given by

$$\mu_k(\widetilde{B}) = (b'_{ij}) := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Equivalently, $\mu_k(\widetilde{B}) = E_s \widetilde{B} F_s$ where $s = \pm$ is a sign and the matrices $E_s \in M_N(\mathbb{Z})$, $F_s \in M_{\mathbf{ex}}(\mathbb{Z})$ are defined by

$$E_s := (e_{ij}) = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(0, -sb_{ik}) & \text{if } i \neq j = k, \end{cases} \quad F_s := (f_{ij}) = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \\ \max(0, sb_{kj}) & \text{if } j \neq i = k. \end{cases}$$

The principal part of $\mu_k(\widetilde{B})$ is the mutation $\mu_k(B)$ of the principal part B of $\widetilde{B}$ and the matrix $\mu_k(B)$ is skew-symmetrizable with respect to the same diagonal matrix D that skew-symmetrizes B, [14]. Mutation μ_k of the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ in the direction of $k \in \mathbf{ex}$ is given by $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) := (\widetilde{\mathbf{x}}', \mu_k(\widetilde{B}))$ where the mutation of $\widetilde{\mathbf{x}}$ is given by

$$\widetilde{\mathbf{x}}' = \{x'_k\} \cup \widetilde{\mathbf{x}} \backslash \{x_k\} \quad \text{and} \quad x_k x'_k := \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}. \tag{2.1}$$

Mutation is an involution, $\mu_k^2 = \text{id}$, [14]. We say that two seeds $(\widetilde{\mathbf{x}}', \widetilde{B}')$, $(\widetilde{\mathbf{x}}'', \widetilde{B}'')$ are *mutation-equivalent* if $(\widetilde{\mathbf{x}}'', \widetilde{B}'')$ can be obtained from $(\widetilde{\mathbf{x}}', \widetilde{B}')$ via a finite sequence of mutations. Denote this by $(\widetilde{\mathbf{x}}', \widetilde{B}') \sim (\widetilde{\mathbf{x}}'', \widetilde{B}'')$. All seeds that are mutation-equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$ contain the cluster variables $\mathbf{c} := \{x_i \mid i \in [1, N] \backslash \mathbf{ex}\}$, called the *frozen variables*.

The cluster algebra $\mathsf{A}(\widetilde{B})$ is defined as the $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables in the seeds $(\widetilde{\mathbf{x}}', \widetilde{B}') \sim (\widetilde{\mathbf{x}}, \widetilde{B})$. For the purposes of applications to coordinate rings, instead of inverting all frozen variables, we often need to pick a subset $\mathbf{inv} \subseteq [1, N] \backslash \mathbf{ex}$ to invert. Then $\mathsf{A}(\widetilde{B}, \mathbf{inv})$, denotes the $\mathbb{Z}[\mathbf{c}, x_k^{-1}, k \in \mathbf{inv}]$-subalgebra generated by all cluster variables in the seeds $(\widetilde{\mathbf{x}}', \widetilde{B}') \sim (\widetilde{\mathbf{x}}, \widetilde{B})$. In particular, $\mathsf{A}(\widetilde{B}) = \mathsf{A}(\widetilde{B}, [1, N] \backslash \mathbf{ex})$.

The upper cluster algebra $\mathsf{U}(\widetilde{B}, \mathbf{inv})$ is the intersection of all mixed polynomial/Laurent polynomial subrings

$$\mathbb{Z}[x'_1, \dots, x'_N][(x'_i)^{-1}, i \in \mathbf{ex} \sqcup \mathbf{inv}]$$

of $\mathcal{F}$ for the seeds $((x'_1, \dots, x'_N), \widetilde{B}') \sim (\widetilde{\mathbf{x}}, \widetilde{B})$. The *Laurent phenomenon* of Fomin–Zelevinsky [15] established that $\mathsf{A}(\widetilde{B}, \mathbf{inv}) \subseteq \mathsf{U}(\widetilde{B}, \mathbf{inv})$.

2.2 | Quantum cluster algebras

Quantum cluster algebras were defined by Berenstein and Zelevinsky in [4]. Let $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}$ be a skew-symmetric bilinear form. By abuse of notation, we will denote its matrix in the standard basis $e_1, \dots, e_N$ of $\mathbb{Z}^N$ by the same symbol $\Lambda = (\Lambda(e_i, e_j))$, and we will use interchangeably both notions. The bilinear form is uniquely reconstructed from this matrix. Using a formal variable $q^{1/2}$, we work with the Laurent polynomial ring

$$\mathcal{A}_q^{1/2} := \mathbb{Z}[q^{\pm 1/2}]. \tag{2.2}$$

Definition 2.1. The *based quantum torus* $\mathcal{T}_q(\Lambda)$ associated with Λ is defined as the $\mathcal{A}_q^{1/2}$-algebra with a $\mathcal{A}_q^{1/2}$-basis $\{X^f \mid f \in \mathbb{Z}^N\}$ and multiplication given by

$$X^f X^g = q^{\Lambda(f,g)/2} X^{f+g}, \quad \text{where} \quad f, g \in \mathbb{Z}^N.$$

The bilinear form Λ can be recovered from the commutation relations of the generators X^{e_1}, … , X^{e_N} of $\mathcal{T}_q(\Lambda)$, because $X^f X^g = q^{\Lambda(f,g)} X^g X^f$. We denote by $\mathcal{F}$ the skew-field of fractions of $\mathcal{T}_q(\Lambda)$, which is a $\mathbb{Q}(q^{1/2})$-algebra. Each $\sigma \in GL_N(\mathbb{Z})$ gives rise to the based quantum torus $\mathcal{T}_q(\Lambda')$ associated to the form $\Lambda'(f, g) = \Lambda(\sigma f, \sigma g)$. Note that if we consider Λ' as a matrix, then $\Lambda' = \sigma^\top \Lambda \sigma$. Also, we have an $\mathcal{A}_q^{1/2}$-algebra isomorphism $\Psi_\sigma : \mathcal{T}_q(\Lambda) \to \mathcal{T}_q(\Lambda')$ given by $X^f \mapsto X^{\sigma^{-1} f}$.

Definition 2.2. Let $\mathcal{F}_q$ be a division algebra over $\mathbb{Q}(q^{1/2})$. A *toric frame* M_q for $\mathcal{F}_q$ is defined as a map $M_q : \mathbb{Z}^N \to \mathcal{F}_q$ for which there exists a skew-symmetric matrix $\Lambda \in M_N(\mathbb{Z})$ satisfying the following.

(1) There is an $\mathcal{A}_q^{1/2}$-algebra embedding $\phi : \mathcal{T}_q(\Lambda) \hookrightarrow \mathcal{F}_q$ with $\phi(X^f) = M_q(f)$ for all $f \in \mathbb{Z}^N$.
(2) $\mathcal{F}_q = \mathrm{Fract}(\phi(\mathcal{T}_q(\Lambda)))$.

The skew-symmetric matrix associated to a toric frame M_q will be denoted by Λ_{M_q}. For any $\sigma \in GL_N(\mathbb{Z})$, $\rho \in \mathrm{Aut}(\mathcal{F}_q)$, and toric frame M_q, the map $\rho M_q \sigma$ is a toric frame with $\Lambda_{\rho M_q \sigma} = \sigma^\top \Lambda \sigma$. The embedding ϕ for M_q gives rise to an embedding $\phi' : \mathcal{T}_q(\Lambda_{\rho M_q \sigma}) \hookrightarrow \mathcal{F}_q$ by $\phi' = \rho \circ \phi \circ \Psi_{\sigma^{-1}}$, which satisfies the two properties above for $\rho M_q \sigma$.

For a toric frame M_q, we indicate the based quantum torus that lies in $\mathcal{F}_q$ with basis $\{M_q(f) \mid f \in \mathbb{Z}^N\}$ by $\mathcal{T}_q(M_q)$. We have the canonical isomorphism $\mathcal{T}_q(M_q) \simeq \mathcal{T}_q(\Lambda_{M_q})$.

As in the previous subsection fix $\mathbf{ex} \subseteq [1, N]$. View $\Lambda = (\lambda_{ij})$ as a skew-symmetric matrix and let $\widetilde{B}$ be an $N \times \mathbf{ex}$ matrix. We call the pair $(\Lambda, \widetilde{B})$ *compatible* if

$$\sum_{k=1}^{N} b_{kj} \lambda_{ki} = \delta_{ij} d_j \text{ for all } i \in [1, N],\ j \in \mathbf{ex} \tag{2.3}$$

for some $d_j \in \mathbb{Z}_+$. Equivalently $\widetilde{B}^\top \Lambda = \widetilde{D}$ where $d_{jj} = d_j$ for $j \in \mathbf{ex}$ and otherwise $d_{ij} = 0$. Denote by $D := \mathrm{diag}(d_j, j \in \mathbf{ex})$ the principal part of $\widetilde{D}$. If $(\Lambda, \widetilde{B})$ is a compatible pair, then $\widetilde{B}$ has full rank and its principal part B is skew-symmetrized by D, [4]

A pair $(\Lambda, \widetilde{B})$ is mutated in the direction of $k \in \mathbf{ex}$, by setting $\mu_k(\Lambda, \widetilde{B}) := (\Lambda', \widetilde{B}')$ where $\widetilde{B}' = E_s \widetilde{B} F_s$ as in the classical case and $\Lambda' := E_s^\top \Lambda E_s$, which is independent on the choice of sign s, [4]. As in the classical case μ_k is an involution, [4].

We call a pair $(M_q, \widetilde{B})$ (consisting of a toric frame M_q for a division algebra $\mathcal{F}_q$ and a matrix $\widetilde{B} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$) a *quantum seed* if the pair $(\Lambda_{M_q}, \widetilde{B})$ is compatible. We call $\{M_q(e_j) \mid j \in [1, N]\}$ the *cluster variables* of the seed $(M_q, \widetilde{B})$. The subset of cluster variables $\{M_q(e_j) \mid j \notin \mathbf{ex}\}$ are called frozen variables.

Proposition 2.3. *Suppose M_q is a toric frame, $k \in [1, N]$ and $g = (n_1, \dots, n_N) \in \mathbb{Z}^N$ is such that $\Lambda_{M_q}(g, e_j) = 0$ for $j \neq k$ and $n_k = 0$. Then for each $s = \pm$, there is an automorphism $\rho_{g,s} = \rho_{g,s}^{M_q}$ of $\mathcal{F}_q$, such that*

$$\rho_{g,s}(M_q(e_j)) = \begin{cases} M_q(e_k) + M_q(e_k + sg) & \text{if } j = k \\ M_q(e_j) & \text{if } j \neq k. \end{cases}$$

This is a variation of [4, Proposition 4.2], proved in [25, Lemma 2.8], which will be more suitable for our root of unity treatment and its relation to the quantum picture via the homomorphism (5.3).

Mutation $\mu_k(M_q, \widetilde{B})$ of a quantum seed in the direction of $k \in \mathbf{ex}$ is defined as

$$(\mu_k(M_q), \mu_k(\widetilde{B})) := \left(\rho_{b^k,s}^{M_q} M_q E_s, E_s \widetilde{B} F_s\right),$$

which is independent on the choice of sign, and $\Lambda_{\mu_k(M_q)} = \mu_k(\Lambda_{M_q})$, [4]. Explicitly, mutation of toric frames is given by

$$\begin{aligned} \mu_k(M_q)(e_j) &= M_q(e_j) \text{ for } j \neq k, \\ \mu_k(M_q)(e_k) &= M_q(-e_k + [b^k]_+) + M_q(-e_k - [b^k]_-), \end{aligned} \tag{2.4}$$

[4]. Here, for $b = (b_1, \dots, b_N) \in \mathbb{Z}^N$, set $[b]_\pm := (c_1, \dots, c_N) \in \mathbb{Z}^N$ where $c_i := b_i$ if $\pm b_i \geqslant 0$ and $c_i := 0$ otherwise.

We fix a subset $\mathbf{inv} \subseteq [1, N] \backslash \mathbf{ex}$ corresponding to frozen variables that will be inverted.

Definition 2.4. The *quantum cluster algebra* $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$ is the $\mathcal{A}_q^{1/2}$-subalgebra of $\mathcal{F}_q$ generated by all cluster variables $M'_q(e_j)$, $j \in [1, N]$ of quantum seeds $(M'_q, \widetilde{B}')$ mutation equivalent to $(M_q, \widetilde{B})$ and by the inverses $M_q(e_j)^{-1}$ for $j \in \mathbf{inv}$.

The *upper quantum cluster algebra* $\mathsf{U}_q(M_q, \widetilde{B}, \mathbf{inv})$ is defined as the intersection over quantum seeds $(M'_q, \widetilde{B}') \sim (M_q, \widetilde{B})$ of all $\mathcal{A}_q^{1/2}$-subalgebras of $\mathcal{F}_q$ of the form

$$\mathcal{A}_q^{1/2}\langle M'_q(e_i), M'_q(e_j)^{-1} \mid i \in [1, N],\ j \in \mathbf{ex} \sqcup \mathbf{inv} \rangle.$$

These subalgebras of $\mathcal{F}_q$ are called *mixed quantum tori.*

The quantum Laurent phenomenon states that

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \subseteq \mathsf{U}_q(M_q, \widetilde{B}, \mathbf{inv}).$$

Berenstein and Zelevinsky [4] proved this in the case when all frozen variables are inverted, that is, when $\mathbf{inv} = [1, N] \backslash \mathbf{ex}$. The general case was proved in [25, Theorem 2.5], where the result is stated over $\mathbb{C}(q^{\pm 1/2})$ but the proof works over $\mathcal{A}_q^{1/2}$.

The *exchange graphs* of a cluster algebra $\mathsf{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ and a quantum cluster algebra $\mathsf{A}_q(M_q, \widetilde{B})$ are the labeled graphs with vertices corresponding to seeds mutation-equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$, respectively, $(M_q, \widetilde{B})$, and edges given by seed mutation and labeled by the corresponding mutation number.

Those graphs will be denoted by $E(\widetilde{B})$ and $E_q(\Lambda_{M_q}, \widetilde{B})$. A map between two labeled graphs is a graph map that preserves labels of edges. Berenstein and Zelevinsky [4] proved that there is a (unique) isomorphism between the exchange graphs $E_q(\Lambda_{M_q}, \widetilde{B})$ and $E(\widetilde{B})$ obtained by sending the vertex corresponding to seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ to that of $(M_q, \widetilde{B})$. Obviously, the exchange graphs do not depend on the choice of inverted set **inv**.

3 | ROOT OF UNITY QUANTUM CLUSTER ALGEBRAS AND ELEMENTARY PROPERTIES

In this section, we define root of unity quantum cluster algebras and describe their elementary properties that are similar to those for quantum cluster algebras. We furthermore prove that all of them are PI algebras.

3.1 | Construction

Let ℓ be a positive integer. For a matrix $C \in M_{n\times m}(\mathbb{Z})$ denote its image in $M_{n\times m}(\mathbb{Z}/\ell)$ by $\overline{C}$. Let $\varepsilon^{1/2} \in \mathbb{C}$ be a primitive ℓth root of unity and set

$$\mathcal{A}_\varepsilon^{1/2} := \mathbb{Z}[\varepsilon^{1/2}]. \tag{3.1}$$

Note that in the case of ℓ odd, ε is also a primitive ℓth root of unity and $\mathbb{Z}[\varepsilon^{1/2}] = \mathbb{Z}[\varepsilon]$.

By abuse of notation, for a skew-symmetric bilinear form $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}/\ell$ we will denote by the same letter its matrix $(\Lambda(e_i, e_j)) \in M_N(\mathbb{Z}/\ell)$. For such a bilinear form define the *root of unity based quantum torus* $\mathcal{T}_\varepsilon(\Lambda)$ to be the $\mathcal{A}_\varepsilon^{1/2}$-algebra with an $\mathcal{A}_\varepsilon^{1/2}$-basis $\{X^f \mid f \in \mathbb{Z}^N\}$ and multiplication given by

$$X^f X^g = \varepsilon^{\Lambda(f,g)/2} X^{f+g} \quad \text{where} \quad f, g \in \mathbb{Z}^N.$$

Hence, $X^f X^g = \varepsilon^{\Lambda(f,g)} X^g X^f$. The bilinear form Λ can be recovered from the based quantum torus by

$$\varepsilon^{\Lambda(f,g)/2} = X^f X^g X^{-f-g}, \quad \forall f, g \in \mathbb{Z}^N$$

by using the assumption that ε is a primitive ℓth root of unity.

Definition 3.1. A *root of unity toric frame* M_ε of a division algebra $\mathcal{F}_\varepsilon$ over $\mathbb{Q}(\varepsilon^{1/2})$ is a map $M_\varepsilon : \mathbb{Z}^N \to \mathcal{F}_\varepsilon$ such that there is a skew-symmetric matrix $\Lambda \in M_N(\mathbb{Z}/\ell)$ satisfying the following conditions.

(1) There is an $\mathcal{A}_\varepsilon^{1/2}$-algebra embedding $\phi : \mathcal{T}_\varepsilon(\Lambda) \hookrightarrow \mathcal{F}_\varepsilon$ with $\phi(X^f) = M_\varepsilon(f)$ for all $f \in \mathbb{Z}^N$.
(2) $\mathcal{F}_\varepsilon \simeq \mathrm{Fract}(\mathcal{T}_\varepsilon(\Lambda))$.

The matrix $\Lambda \in M_N(\mathbb{Z}/\ell)$ is uniquely reconstructed from the root of unity toric frame M_ε. It will be called *matrix of the frame* M_ε and we will denote $\Lambda_{M_\varepsilon} := \Lambda$.

Fix a subset of $\mathbf{ex} \subseteq [1, N]$.

Definition 3.2. Let $\widetilde{B} \in M_{N\times \mathbf{ex}}(\mathbb{Z})$ and $\Lambda = (\lambda_{ij}) \in M_N(\mathbb{Z}/\ell)$ be skew-symmetric. The pair $(\Lambda, \widetilde{B})$ will be called *ℓ-compatible* if there exists a diagonal matrix $D := \mathrm{diag}(d_j, j \in \mathbf{ex})$ with $d_j \in \mathbb{Z}_+$ such that

(1) the principal part B of $\widetilde{B}$ is skew-symmetrized by D; that is DB is skew-symmetric;
(2) $\sum_{k=1}^{N} \overline{b}_{kj}\lambda_{ki} = \delta_{ij}\overline{d}_j \pmod{\ell}$ for all $i \in [1,N]$, $j \in \mathbf{ex}$; that is $\Lambda^{\top}\overline{\widetilde{B}} = \begin{bmatrix}\overline{D}\\ 0\end{bmatrix}$, where 0 denotes the zero matrix of size $([1,N]\backslash\mathbf{ex}) \times \mathbf{ex}$.

We will not require any conditions on $\overline{d}_j$, so the matrix $\widetilde{B}$ need not have full rank like in the case of quantum cluster algebras.

Similar to the generic case, we define the *mutation* in direction $k \in \mathbf{ex}$ of ℓ-compatible pairs to be

$$\mu_k(\Lambda, \widetilde{B}) := \left(\overline{E}_s^{\top}\Lambda\overline{E}_s, E_s\widetilde{B}F_s\right) \text{ for a choice of sign } s.$$

The proof of the following proposition is analogous to [4, Propositions 3.4 and 3.6].

Proposition 3.3. *The pair $\mu_k(\Lambda, \widetilde{B})$ is independent of the choice of sign s. If the pair $(\Lambda, \widetilde{B})$ is ℓ-compatible with respect to a diagonal matrix D, then the pair $\mu_k(\Lambda, \widetilde{B})$ is also ℓ-compatible with respect to the same diagonal matrix D. Mutation μ_k of ℓ-compatible pairs is an involution.*

Definition 3.4. We will call a pair $(M_\varepsilon, \widetilde{B})$ a *root of unity quantum seed* if

(1) M_ε is a root of unity toric frame of $\mathcal{F}_\varepsilon$,
(2) $\widetilde{B} \in M_{N\times\mathbf{ex}}(\mathbb{Z})$ and $(\Lambda_{M_\varepsilon}, \widetilde{B})$ is an ℓ-compatible pair.

Proposition 3.5. *Suppose M_ε is a root of unity toric frame, $k \in [1,N]$, and $g = (n_1, \dots, n_N) \in \mathbb{Z}^N$ is such that $\Lambda_{M_\varepsilon}(g, e_j) \equiv 0 \pmod{\ell}$ for $j \neq k$ and $n_k = 0$. Then for each $s = \pm$, there is a unique automorphism $\rho_{g,s}^{M_\varepsilon}$ of $\mathcal{F}_\varepsilon$, such that*

$$\rho_{g,s}^{M_\varepsilon}(M_\varepsilon(e_j)) = \begin{cases} M_\varepsilon(e_k) + M_\varepsilon(e_k + sg) & \text{if } j = k \\ M_\varepsilon(e_j) & \text{if } j \neq k. \end{cases} \tag{3.2}$$

Our argument is similar to [25, Lemma 2.8] but we spell out the details because they will be needed later.

Proof. Denote $\mathrm{Fract}(\mathcal{T}_\varepsilon(M_\varepsilon))$ by $\mathcal{F}_\varepsilon$. We have a homomorphism $\rho_{g,s} : \mathcal{T}_\varepsilon(M_\varepsilon) \to \mathcal{F}_\varepsilon$ because

$$(M_\varepsilon(e_k) + M_\varepsilon(e_k + sg))M_\varepsilon(e_j) = \varepsilon^{\Lambda(e_k,e_j)}M_\varepsilon(e_j)(M_\varepsilon(e_k) + M_\varepsilon(e_k + sg))$$

for $j \neq k$. On the $\mathcal{A}_\varepsilon^{1/2}$-basis $\{M_\varepsilon(f)\}$, one calculates that

$$\rho_{g,s}(M_\varepsilon(f)) = \begin{cases} P_{g,s,+}^{M_\varepsilon,m_k} M_\varepsilon(f) & \text{if } m_k \geqslant 0 \\ \left(P_{g,s,-}^{M_\varepsilon,-m_k}\right)^{-1} M_\varepsilon(f) & \text{if } m_k < 0 \end{cases}$$

for $f = (m_1, \dots, m_N) \in \mathbb{Z}^N$, where

$$P^{M_\varepsilon, m_k}_{g,s,\pm} := \prod_{p=1}^{m_k} \left(1 + \varepsilon^{\mp s(2p-1)\Lambda_{M_\varepsilon}(g,e_k)/2} M_\varepsilon(sg)\right) \text{ for } m_k \geqslant 0.$$

Let $G := \mathcal{A}_\varepsilon^{1/2}[M_\varepsilon(sg)]\backslash\{0\} \subset \mathcal{T}_\varepsilon(M_\varepsilon)$. Note that $G \cdot M_\varepsilon(f) = M_\varepsilon(f) \cdot G$ for any $f \in \mathbb{Z}^N$, and hence G is an Ore set. Moreover, as $\mathrm{im}(\rho_{g,s}) \subset \mathcal{T}_\varepsilon(M_\varepsilon)G^{-1}$, we may consider $\rho_{g,s} : \mathcal{T}_\varepsilon(M_\varepsilon) \to \mathcal{T}_\varepsilon(M_\varepsilon)G^{-1}$. As g has $n_k = 0$, the map $\rho_{g,s}$ acts by the identity on G. We can clearly extend the map to an endomorphism $\rho_{g,s} : T_\varepsilon(M_\varepsilon)G^{-1} \to T_\varepsilon(M_\varepsilon)G^{-1}$.

We can similarly construct an algebra endomorphism $\rho'_{g,s} : T_\varepsilon(M_\varepsilon)G^{-1} \to T_\varepsilon(M_\varepsilon)G^{-1}$ defined by

$$\rho'_{g,s}(M_\varepsilon(e_j)) = \begin{cases} \left(P^{M_\varepsilon,1}_{g,s,+}\right)^{-1} M_\varepsilon(f) & \text{if } j = k \\ M_\varepsilon(e_j) & \text{if } j \neq k \end{cases}.$$

Clearly, $\rho_{g,s}$ and $\rho'_{g,s}$ are inverse to each other and are automorphisms of $T_\varepsilon(M_\varepsilon)G^{-1}$. In particular, they are injective and can be extended to automorphisms of $\mathcal{F}_\varepsilon$. Uniqueness follows because $M_\varepsilon(e_j)$ are skew-field generators of $\mathcal{F}_\varepsilon$. □

Similar to the generic case, we define *mutation* of a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$ in the direction of $k \in \mathbf{ex}$ by

$$\mu_k(M_\varepsilon, \widetilde{B}) := \left(\rho^{M_\varepsilon E_s}_{b^k,s} M_\varepsilon E_s, E_s \widetilde{B} F_s\right). \tag{3.3}$$

The proof of the following proposition is analogous to [4, Propositions 4.7 and 4.10].

Proposition 3.6. *Given a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$, the following hold.*

(1) *For $k \in \mathbf{ex}$ and either sign $s = \pm$:*

$$\rho^{M_\varepsilon E_s}_{b^k,s} M_\varepsilon E_s(e_j) = M_\varepsilon(e_j) \textit{ for } j \neq k,$$

$$\rho^{M_\varepsilon E_s}_{b^k,s} M_\varepsilon E_s(e_k) = M_\varepsilon(-e_k + [b^k]_+) + M_\varepsilon(-e_k - [b^k]_-).$$

In particular, mutation does not depend on the sign used.

(2) *$\mu_k(M_\varepsilon, \widetilde{B})$ is also a root of unity quantum seed.*

Moreover, mutation is an involution.

We consider the equivalence classes under finite sequences of mutations of root of unity quantum seeds. Fix a subset $\mathbf{inv} \subseteq [1, N]\backslash\mathbf{ex}$ corresponding to frozen variables that we will set as invertible.

Definition 3.7. Given a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$, we define the *quantum cluster algebra at a root of unity* $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ as the $\mathcal{A}_\varepsilon^{1/2}$-subalgebra of $\mathcal{F}_\varepsilon$ generated by all cluster variables of quantum seeds $(M'_\varepsilon, \widetilde{B}')$ mutation equivalent to $(M_\varepsilon, \widetilde{B})$ and by the inverses of the frozen variables corresponding to **inv**,

$$\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) := \mathcal{A}_\varepsilon^{1/2}\langle M'_\varepsilon(e_i), M_\varepsilon(e_j)^{-1} \mid i \in [1,N],\ j \in \mathbf{inv}, (M'_\varepsilon, \widetilde{B}') \sim (M_\varepsilon, \widetilde{B})\rangle.$$

We have associated to each skew-symmetric bilinear form Λ a based quantum torus. Given subsets **ex** and **inv**, we can also associate an algebra in between the corresponding skew-polynomial algebra and the quantum torus,

$$\mathcal{T}_\varepsilon(\Lambda)_\geqslant := \mathcal{A}_\varepsilon^{1/2}\Big\langle X_i,\ X_j^{-1} \mid i \in [1,N],\ j \in \mathbf{ex} \sqcup \mathbf{inv}\Big\rangle \subset \mathcal{T}_\varepsilon(\Lambda). \tag{3.4}$$

We call this a *mixed based quantum torus*. Equivalently, it is the algebra

$$\mathcal{A}_\varepsilon^{1/2}\text{–Span}\Big\{X^f \mid f \in \mathbb{Z}^N_\geqslant\Big\} \quad \text{with the product} \quad X^f X^g = \varepsilon^{\Lambda(f,g)/2} X^{f+g},\ \forall f, g \in \mathbb{Z}^N_\geqslant,$$

where

$$\mathbb{Z}^N_\geqslant := \{f = (f_1, \dots, f_N) \in \mathbb{Z}^N \mid f_i \geqslant 0,\ \forall i \notin \mathbf{ex} \sqcup \mathbf{inv}\}. \tag{3.5}$$

We similarly define

$$\mathcal{T}_\varepsilon(M_\varepsilon)_\geqslant := \langle\, M_\varepsilon(e_i),\ M_\varepsilon(e_j)^{-1} \mid i \in [1,N],\ j \in \mathbf{ex} \sqcup \mathbf{inv}\rangle \subset \mathcal{T}_\varepsilon(M_\varepsilon).$$

Definition 3.8. Given a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$ and specified subsets **ex** and **inv**, we define the *upper quantum cluster algebra at a root of unity* $\mathsf{U}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ as the intersection of mixed quantum tori corresponding to quantum seeds mutation equivalent to $(M_\varepsilon, \widetilde{B})$,

$$\mathsf{U}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) := \bigcap_{(M_\varepsilon, \widetilde{B}) \sim (M'_\varepsilon, \widetilde{B}')} \mathcal{T}_\varepsilon(M'_\varepsilon)_\geqslant.$$

Remark 3.9. In the case when $\varepsilon^{1/2} = 1$ (i.e., $\ell = 1$), a root of unity quantum cluster algebra can be identified with a classical cluster algebra (of geometric type)

$$\mathsf{A}_1(M_1, \widetilde{B}, \mathbf{inv}) = \mathsf{A}((M_1(e_1), \dots, M_1(e_N)), \widetilde{B}, \mathbf{inv}),$$

and similarly a root of unity upper quantum cluster algebra with an upper cluster algebra

$$\mathsf{U}_1(M_1, \widetilde{B}, \mathbf{inv}) = \mathsf{U}((M_1(e_1), \dots, M_1(e_N)), \widetilde{B}, \mathbf{inv}).$$

3.2 | The quantum Laurent phenomenon at roots of unity

Theorem 3.10. *For any root of unity quantum cluster algebra* $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$

$$\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \subseteq \mathsf{U}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}).$$

Proof. The case when $\widetilde{B}$ has full rank is proved analogously to [25, Theorem 2.15]. We deduce the general case of the theorem from the full rank one as follows.

For simplicity of notation, assume that $\mathbf{ex} = [1, n]$ for some integer $n \leqslant N$. Consider the augmented skew-symmetric bilinear form with matrix

$$\Lambda_{\text{aug}} := \begin{bmatrix} \Lambda & 0_{N\times n} \\ 0_{n\times N} & 0_{n\times n} \end{bmatrix},$$

where $0_{i\times j}$ denotes the zero matrix of size $i \times j$. Denote the augmented exchange matrix

$$\widetilde{B}_{\text{aug}} := \begin{bmatrix} \widetilde{B} \\ I_n \end{bmatrix}$$

whose principal part is the same as $\widetilde{B}$. The pair $(\Lambda_{\text{aug}}, \widetilde{B}_{\text{aug}})$ is ℓ-compatible with respect to the same diagonal matrix D because

$$\overline{\widetilde{B}}_{\text{aug}}^{\top} \Lambda_{\text{aug}} = \begin{bmatrix} \overline{D} & 0 \end{bmatrix}.$$

Denote by $\widehat{\mathcal{F}}_{\varepsilon}$ the skew-field $\mathrm{Fract}(\mathcal{T}_{\varepsilon}(\Lambda_{\text{aug}}))$ and consider the toric frame $(M_{\varepsilon})_{\text{aug}}$ with matrix Λ_{aug} such that $(M_{\varepsilon})_{\text{aug}}(e_k) := X_k$ for all $k \in [N+1, N+n]$. Clearly, $((M_{\varepsilon})_{\text{aug}}, \widetilde{B}_{\text{aug}})$ is a root of unity quantum seed. We have a canonical surjective $\mathcal{A}_{\varepsilon}^{1/2}$-algebra homomorphism

$$\pi : \mathcal{T}_{\varepsilon}((M_{\varepsilon})_{\text{aug}})_{\geqslant} \to \mathcal{T}_{\varepsilon}(M_{\varepsilon})_{\geqslant} \text{ given by } \pi((M_{\varepsilon})_{\text{aug}}(e_k)) := \begin{cases} M_{\varepsilon}(e_k), & 1 \leqslant k \leqslant N \\ 1, & N < k \leqslant N+n \end{cases}$$

because the elements $(M_{\varepsilon})_{\text{aug}}(e_k)$ are in the center of $\mathcal{T}_{\varepsilon}((M_{\varepsilon})_{\text{aug}})_{\geqslant}$ for $N < k \leqslant N+n$.

By induction on $m \geqslant 0$ one easily shows that

$$\pi\Big(\mu_{i_1} \dots \mu_{i_m}((M_{\varepsilon})_{\text{aug}})(e_k)\Big) = \mu_{i_1} \dots \mu_{i_m}(M_{\varepsilon})(e_k)$$

for all $k \in [1, N]$. As the matrix $\widetilde{B}_{\text{aug}}$ has full rank, by the validity of the root of unity quantum Laurent phenomenon in the full rank case we have

$$\mu_{i_1} \dots \mu_{i_m}((M_{\varepsilon})_{\text{aug}})(e_k) \in \mathcal{T}_{\varepsilon}((M_{\varepsilon})_{\text{aug}})_{\geqslant}.$$

Hence, $\mu_{i_1} \dots \mu_{i_m}(M_{\varepsilon})(e_k) \in \mathcal{T}_{\varepsilon}(M_{\varepsilon})_{\geqslant}$ for all $k \in [1, N]$, which completes the proof of the theorem in the general case. □

3.3 | PI properties of root of unity quantum cluster algebras

Theorem 3.11. *All root of unity quantum cluster algebras $\mathsf{A}_{\varepsilon}(M_{\varepsilon}, \widetilde{B}, \mathbf{inv})$ and root of unity upper quantum cluster algebras $\mathsf{U}_{\varepsilon}(M_{\varepsilon}, \widetilde{B}, \mathbf{inv})$ are PI domains, see, for example, [5, section I.13] or [33, chapter 13].*

Proof. By Theorem 3.10, for every toric frame M_ε of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ we have the embeddings

$$\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \subseteq \mathsf{U}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \subseteq \mathcal{T}_\varepsilon(M_\varepsilon) \cong \mathcal{T}_\varepsilon(\Lambda_{M_\varepsilon}).$$

As each root of unity quantum torus $\mathcal{T}_\varepsilon(\Lambda)$ is a PI domain, the same is true for the first two algebras in the chain. □

4 | CANONICAL CENTRAL SUBRINGS OF ROOT OF UNITY QUANTUM CLUSTER ALGEBRAS

The main results of this section are the construction of a canonical central subring of a root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ and a theorem that it is isomorphic to the classical cluster algebra $\mathsf{A}(\widetilde{B}, \mathbf{inv})$.

4.1 | Central embedding of commutative cluster algebras

Lemma 4.1. *If $(M'_\varepsilon, \widetilde{B}')$ is mutation-equivalent to $(M_\varepsilon, \widetilde{B})$, then the element $M'_\varepsilon(e_j)^l \in \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is central for any $j \in [1, N]$.*

Proof. We only need show that $M_\varepsilon(e_j)^l \in Z(\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}))$ for $j \in [1, N]$, as $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) = \mathsf{A}_\varepsilon(M'_\varepsilon, \widetilde{B}', \mathbf{inv})$. Now $M_\varepsilon(e_j)^l$ is central in $\mathcal{T}_\varepsilon(M_\varepsilon)$ as

$$M_\varepsilon(e_j)^l M_\varepsilon(f) = M_\varepsilon(le_j) M_\varepsilon(f) = \varepsilon^{\Lambda(le_j, f)} M_\varepsilon(f) M_\varepsilon(le_j) = M_\varepsilon(f) M_\varepsilon(e_j)^l.$$

Thus, it is central in $\mathrm{Fract}(\mathcal{T}_\varepsilon(M_\varepsilon))$ and in $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$. □

For a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$ and for $j \in \mathbf{ex}$, consider the commutation of elements $M_\varepsilon(-e_j + [b^j]_+)$ and $M_\varepsilon(-e_k - [b^j]_-)$. The relation in the quantum torus is

$$M_\varepsilon(-e_j - [b^j]_-) M_\varepsilon(-e_j + [b^j]_+) = \varepsilon^{\Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+)} M_\varepsilon(-e_j + [b^j]_+) M_\varepsilon(-e_j - [b^j]_-).$$

Set $t_j := \Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+)$ for brevity.

Lemma 4.2. *Let $(M_\varepsilon, \widetilde{B})$ be a root of unity quantum seed, so $(\Lambda_{M_\varepsilon}, \widetilde{B})$ is an ℓ-compatible pair with respect to a diagonal matrix $D = \mathrm{diag}(d_j, j \in \mathbf{ex})$ with $d_j \in \mathbb{Z}_+$. Then for $j \in \mathbf{ex}$, $t_j = \overline{d}_j$.*

Proof. We have that

$$\begin{aligned} t_j &= \Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+) \\ &= \Lambda(-e_j, -e_j) + \Lambda(-e_j, [b^j]_+) + \Lambda(-[b^j]_-, -e_j) + \Lambda(-[b^j]_-, [b^j]_+) \\ &= \Lambda(b^j, e_j) + \Lambda([b^j]_+, [b^j]_-) = \overline{d}_j + \Lambda([b^j]_+, [b^j]_-). \end{aligned}$$

To evaluate $\Lambda([b^j]_+, [b^j]_-)$, we note that $b^j - [b^j]_+ = [b^j]_-$, so

$$\Lambda([b^j]_+, [b^j]_-) = \Lambda([b^j]_+, b^j) - \Lambda([b^j]_+, [b^j]_+) = \Lambda([b^j]_+, b^j).$$

As $b_{jj} = 0$,

$$\Lambda([b^j]_+, [b^j]_-) = \Lambda([b^j]_+, b^j) = \sum_{b_{ij}>0} b_{ij}\Lambda(e_i, b^j) = \sum_{b_{ij}>0} -b_{ij}\delta_{i,j}\overline{d}_j = 0.$$

Thus, $t_j = \overline{d}_j$. □

We will often require the following condition on our root of unity quantum seed $(M_\varepsilon, \widetilde{B})$:

(Coprime) *ℓ is an odd integer coprime to d_k for $k \in$ **ex**, where $D = \mathrm{diag}(d_j, j \in \mathbf{ex})$ is the matrix that skew-symmetrizes the principal part B of $\widetilde{B}$.*

The condition **(Coprime)** only concerns the ℓth root of unity ε and the compatible pair $(\Lambda_{M_\varepsilon}, \widetilde{B})$, and not the root of unity toric frame M_ε.

Remark 4.3. The diagonal matrix D that skew-symmetrizes exchange matrices is invariant under mutation. Therefore, if a root of unity quantum seed satisfies condition **(Coprime)**, then any mutation equivalent seed does so as well. So, **(Coprime)** is a condition on a root of unity quantum cluster algebra and not on individual seeds.

The main use of Lemma 4.2 is the following result. The formula appearing should be compared to the mutation relation of (2.1).

Proposition 4.4. *Let $(M_\varepsilon, \widetilde{B})$ be a root of unity quantum seed satisfying the condition* (**Coprime**). *Then for $k \in$ **ex**,*

$$M_\varepsilon(e_k)^\ell(\mu_k M_\varepsilon(e_k))^\ell = \prod_{b_{ik}>0} (M_\varepsilon(e_i)^\ell)^{b_{ik}} + \prod_{b_{ik}<0} (M_\varepsilon(e_i)^\ell)^{-b_{ik}}.$$

Proof. Denote

$$Y := M_\varepsilon(-e_k + [b^k]_+),\ Z := M_\varepsilon(-e_k - [b^k]_-) \in \mathcal{T}_\varepsilon(M_\varepsilon).$$

As $ZY = \varepsilon^{d_k} YZ$ (by Lemma 4.2) and ε^{d_k} is an ℓth primitive root of unity,

$$(Y+Z)^\ell = Y^\ell + Z^\ell.$$

Thus,

$$\begin{aligned}(\mu_k M_\varepsilon(e_k))^\ell &= \big(M_\varepsilon(-e_k + [b^k]_+) + M_\varepsilon(-e_k - [b^k]_-)\big)^\ell = (Y+Z)^\ell \\ &= M_\varepsilon(-e_k + [b^k]_+)^\ell + M_\varepsilon(-e_k - [b^k]_-)^\ell\end{aligned}$$

$$= M_\varepsilon(-\ell e_k + \ell[b^k]_+) + M_\varepsilon(-\ell e_k - \ell[b^k]_-)$$
$$= M_\varepsilon(-\ell e_k) \prod_{b_{ik}>0} M_\varepsilon(\ell b_{ik} e_i) + M_\varepsilon(-\ell e_k) \prod_{b_{ik}<0} M_\varepsilon(-\ell b_{ik} e_i). \qquad \square$$

Example 4.5. The previous proposition does not hold if the condition that ℓ is coprime to the integers d_k is dropped. Consider the following example when $\ell = 9$. Let

$$\varepsilon^{1/2} = e^{2\pi i/9}, \quad \Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}.$$

Let $\mathcal{F}_\varepsilon := \mathrm{Fract}\,(\mathcal{T}_\varepsilon(\Lambda))$ and $M_\varepsilon : \mathbb{Z}^2 \to \mathcal{F}_\varepsilon$ be the toric frame related to Λ such that $M_\varepsilon(f) = X^f$ and $\Lambda_{M_\varepsilon} := \Lambda$. Clearly, $(M_\varepsilon, \widetilde{B})$ is a root of unity quantum seed. Here we have

$$\widetilde{B}^\top \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

In particular, $d_1 = 3$ is not coprime to $\ell = 9$. For $Y := M_\varepsilon(-e_1 + [b^1]_+) = M_\varepsilon(-e_1)$ and $Z := M_\varepsilon(-e_1 - [b^1]_-) = M_\varepsilon(-e_1 + 3e_2)$, by a direct computation one obtains

$$(Y+Z)^9 = Y^9 + 3Y^6Z^3 + 3Y^3Z^6 + Z^9 \neq Y^9 + Z^9,$$

so the conclusion of Proposition 4.4 fails.

In a similar way, dropping the odd root of unity condition will result in a failure of the statement of Proposition 4.4. Consider the same choice for Λ and $\widetilde{B}$, but with $\varepsilon^{1/2} = i$, a primitive fourth root of unity. Then $\varepsilon = -1$ and

$$(Y+Z)^4 = Y^4 + (1 + \varepsilon + 2\varepsilon^2 + \varepsilon^3 + \varepsilon^4)Y^2Z^2 + Z^4$$
$$= Y^4 + 2Y^2Z^2 + Z^4 \neq Y^4 + Z^4$$

leading once again to a failure of the conclusion of Proposition 4.4. The issue in the even case is that ε is a primitive $(\ell/2)$th root of unity, not a primitive ℓth root of unity.

Define the $\mathbb{Z}$-subring

$$\mathsf{C}(M_\varepsilon, \widetilde{B}, \mathbf{inv}) := \mathbb{Z}\langle M'_\varepsilon(e_i)^\ell, M'_\varepsilon(e_j)^{-\ell} \mid (M'_\varepsilon, \widetilde{B}') \sim (M_\varepsilon, \widetilde{B}), i \in [1, N], j \in \mathbf{inv} \rangle$$

of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$.

Theorem 4.6. *Suppose that $(M_\varepsilon, \widetilde{B})$ satisfies condition **(Coprime)**. Then the subring $\mathsf{C}(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is isomorphic to $\mathsf{A}(\widetilde{B}, \mathbf{inv})$.*

Proof. As $\mathsf{A}(\{x_1, \dots, x_N\}, \widetilde{B}, \varnothing)$ is constructed as a subalgebra of $\mathbb{Q}(x_1, \dots, x_N)$, consider the isomorphism $\phi : \mathbb{Q}(x_1, \dots, x_N) \to \mathrm{Fract}\left(\mathbb{Z}[M_\varepsilon(e_1)^\ell, \dots, M_\varepsilon(e_N)^\ell]\right)$ given by $x_j \mapsto M_\varepsilon(e_j)^\ell$. Proposition 4.4 gives us that $\phi(\mu_i(x_j)) = (\mu_i M_\varepsilon(e_j))^\ell$ for all $i \in \mathbf{ex}$, $j \in [1, N]$. By induction on the length of the mutation sequence, $\phi(\mu_{i_k} \dots \mu_{i_1}(x_j)) = (\mu_{i_k} \dots \mu_{i_1} M_\varepsilon(e_j))^\ell$.

As the generators of $\mathbb{Z}\langle M'_\varepsilon(e_i)^\ell \mid (M'_\varepsilon, \widetilde{B}') \sim (M_\varepsilon, \widetilde{B}),\, i \in [1,N]\rangle$ are the images of the generators of $\mathsf{A}(\{x_1, \dots, x_N\}, \widetilde{B}, \varnothing)$ under the isomorphism ϕ, then we have an isomorphism of $\mathbb{Z}$-algebras. The more general case, when $\mathbf{inv} \neq \varnothing$, is obtained by adjoining the appropriate inverses of frozen variables. □

Corollary 4.7. *If $(M_\varepsilon, \widetilde{B})$ satisfies condition **(Coprime)**, then the $\mathcal{A}_\varepsilon^{1/2}$-subalgebra*

$$\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) := \mathcal{A}_\varepsilon^{1/2}\langle M'_\varepsilon(e_i)^\ell, M'_\varepsilon(e_j)^{-\ell} \mid (M'_\varepsilon, \widetilde{B}') \sim (M_\varepsilon, \widetilde{B}),\, i \in [1,N],\, j \in \mathbf{inv}\rangle$$

of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is isomorphic to $\mathcal{A}_\varepsilon^{1/2} \otimes_{\mathbb{Z}} \mathsf{A}(\widetilde{B}, \mathbf{inv})$.

4.2 | Exchange graphs of root of unity quantum cluster algebras

For a root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B})$, define its *exchange graph* $E_\varepsilon(M_\varepsilon, \widetilde{B})$ to be the labeled graph with vertices corresponding to root of unity quantum seeds mutation-equivalent to $(M_\varepsilon, \widetilde{B})$ and with edges given by seed mutation labeled by the corresponding letters.

Theorem 4.8. *Let $(M_\varepsilon, \widetilde{B})$ be a root of unity quantum seed satisfying condition* (**Coprime**). *There is a unique isomorphism of labeled graphs from the exchange graph $E_\varepsilon(M_\varepsilon, \widetilde{B})$ to the exchange graph $E(\widetilde{B})$ which sends the vertex corresponding to the seed $(M_\varepsilon, \widetilde{B})$ to the vertex corresponding to the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, where $\widetilde{\mathbf{x}} = (M_\varepsilon(e_1)^\ell, \dots, M_\varepsilon(e_N)^\ell)$.*

We will need the following two propositions for the proof of the theorem which are of independent interest. Recall that an exchange matrix $\widetilde{B}$ is indecomposable if it cannot be represented in a block diagonal form with blocks of strictly smaller size.

The first proposition establishes a leading term statement for cluster expansions.

Proposition 4.9. *Assume that $(M_\varepsilon, \widetilde{B})$ and $(M'_\varepsilon, \widetilde{B}')$ are two seeds of a root of unity quantum cluster algebra, where $\widetilde{B}$ is indecomposable and $\widetilde{B} \neq 0$. Then for every $k \in [1,N]$ there exists a functional $\theta : \mathbb{Z}^N \to \mathbb{Z}$ such that*

$$M'_\varepsilon(e_k) = M_\varepsilon(f) + \sum_i a_i M_\varepsilon(f_i)$$

for some $a_i \in \mathcal{A}_\varepsilon^{1/2}$ and $f, f_i \in \mathbb{Z}^N$ such that $\theta(f) > \theta(f_i)$ for all i.

The statement fails when $\widetilde{B} = 0$, because in that case $\mu_1(M_\varepsilon)(e_1) = 2M_\varepsilon(-e_1)$.

Proof. We prove the proposition by induction on the distance between the vertices in the exchange graph corresponding to the seeds $(M_\varepsilon, \widetilde{B})$ and $(M'_\varepsilon, \widetilde{B}')$. The case when the distance equals 1 is trivial because the condition that $\widetilde{B}$ is indecomposable and $\widetilde{B} \neq 0$ implies that $\mu_j(M_\varepsilon)(e_j) = M_\varepsilon(f_1) + M_\varepsilon(f_2)$ for some $f_1 \neq f_2 \in \mathbb{Z}^N$.

Assume the validity of the statement when the distance equals m. Consider two seeds $(M_\varepsilon, \widetilde{B})$ and $(M'_\varepsilon, \widetilde{B}')$ whose vertices are at distance $m+1$ in the exchange graph. Then there exists a seed $(M''_\varepsilon, \widetilde{B}'')$ such that $(M''_\varepsilon, \widetilde{B}'') = \mu_j(M_\varepsilon, \widetilde{B})$ for some $j \in [1,N]$ and the distance between the vertices of the exchange graph corresponding to the seeds $(M''_\varepsilon, \widetilde{B}'')$ and $(M'_\varepsilon, \widetilde{B}')$ equals m. The exchange matrices $\widetilde{B}'$ and $\widetilde{B}''$ are necessarily indecomposable. We have

$$M''_\varepsilon(e_l) = M_\varepsilon(e_l) \text{ for } l \neq j,$$

$$M''_\varepsilon(e_j) = M_\varepsilon(-e_j + [b^j]_+) + M_\varepsilon(-e_j - [b^j]_-).$$

By the induction hypothesis, there exists a functional $\theta'' : \mathbb{Z}^N \to \mathbb{Z}$ such that

$$M'_\varepsilon(e_k) = M''_\varepsilon(g) + \sum_i a''_i M''_\varepsilon(g_i) \tag{4.1}$$

for some $a''_i \in \mathcal{A}_\varepsilon^{1/2}$ and $g, g_i \in \mathbb{Z}^N$ such that $\theta''(g) > \theta''(g_i)$ for all i.

Denote by s the sign $\pm$ for which $\theta''([b^j]_+)$ or $-\theta''([b^j]_-)$ is minimal. Define the functional $\theta : \mathbb{Z}^N \to \mathbb{Z}$ by

$$\theta(e_j) = \theta''(-e_j + s[b^j]_s), \quad \theta(e_l) = \theta''(e_l) \text{ for } l \neq j.$$

Let $\widehat{\mathcal{T}}_\varepsilon(M_\varepsilon)$ be the completion of the quantum torus $\mathcal{T}_\varepsilon(M_\varepsilon)$ spanned by formal sums of the form

$$\sum_{m=0}^{\infty} c_m M_\varepsilon(h - msb^j)$$

for $h \in \mathbb{Z}^N$ and $c_m \in \mathcal{A}_\varepsilon^{1/2}$. It is an $\mathcal{A}_\varepsilon^{1/2}$-algebra on its own. We have $-s[b^j]_{-s} = s[b^j]_s - sb^j$. As $M_\varepsilon(-e_j + [b^j]_+)$ and $M_\varepsilon(-e_j - [b^j]_-)$ skew-commute up to a power of ε, for all $n \in \mathbb{Z}$,

$$\begin{aligned} M''_\varepsilon(ne_j) &= \big(M_\varepsilon(-e_j + s[b^j]_s) + M_\varepsilon(-e_j - s[b^j]_{-s})\big)^n \\ &= M_\varepsilon\big(n(-e_j + s[b^j]_s)\big) + \sum_{m=1}^{\infty} c_m M_\varepsilon\big(n(-e_j + s[b^j]_s) - msb^j\big) \end{aligned} \tag{4.2}$$

for some $c_m \in \mathcal{A}_\varepsilon^{1/2}$. Denote $\mathbb{Z}^{N-1} := \bigoplus_{j \neq l} \mathbb{Z}e_j \subset \mathbb{Z}^N$. For all $h \in \mathbb{Z}^{N-1}$ we have

$$\Lambda''(e_j, h) = \Lambda(-e_j + [b^j]_+, h) = \Lambda(-e_j - [b^j]_-, h)$$

and thus, by using (4.2) and the definition of root of unity toric frames,

$$M''_\varepsilon(ne_j + h) = M_\varepsilon\big(n(-e_j + s[b^j]_s) + h\big) + \sum_{m=1}^{\infty} c_m M_\varepsilon\big(n(-e_j + s[b^j]_s) + h - msb^j\big) \tag{4.3}$$

for some $c_m \in \mathcal{A}_\varepsilon^{1/2}$. Write the elements $g, g_i \in \mathbb{Z}^N$ in (4.1) in the form $g = ne_j + h$, $g_i = n_i e_j + h_i$, for $n, n_i \in \mathbb{Z}$, $h, h_i \in \mathbb{Z}^{N-1}$ and apply (4.3) to obtain,

$$\begin{aligned} M'_\varepsilon(e_k) &= M_\varepsilon\big(n(-e_j + s[b^j]_s) + h\big) + \sum_{m=1}^{\infty} c_m M_\varepsilon\big(n(-e_j + s[b^j]_s) + h - msb^j\big) \\ &\quad + \sum_i a''_i M_\varepsilon\big(n_i(-e_j + s[b^j]_s) + h_i\big) + \sum_{m=1}^{\infty} c_{i,m} M_\varepsilon\big(n_i(-e_j + s[b^j]_s) + h_i - msb^j\big) \end{aligned}$$

for some $c_{i,m} \in \mathcal{A}_\varepsilon^{1/2}$. By the root of unity quantum Laurent phenomenon (Theorem 3.10), the sum in the right-hand side belongs to $\mathcal{T}_\varepsilon(M_\varepsilon)$. Furthermore, the definition of the functional θ implies that

$$\theta(n(-e_j + s[b^j]_s) + h) = \theta''(g) > \theta''(g_i) = \theta(n_i(-e_j + s[b^j]_s) + h_i),$$
$$\theta(sb^j) = \theta(s[b^j]_s) + \theta(s[b^j]_{-s}) < 0.$$

Hence, the above expansion of $M'_\varepsilon(e_k)$ in $\mathcal{T}_\varepsilon(M_\varepsilon)$ has the desired properties with respect to the functional θ. □

Remark 4.10. The proof of Proposition 4.9 directly translates to the case of quantum cluster algebras to yield the validity of the obvious analog of it in that situation.

The second auxiliary proposition for the proof of Theorem 4.8 is a recognition statement for toric frames of root of unity quantum cluster algebras in terms of the ℓth powers of the cluster variables in them.

Proposition 4.11. *Assume that $(M_\varepsilon, \widetilde{B})$ and $(M'_\varepsilon, \widetilde{B}')$ are two seeds of a root of unity quantum cluster algebra. Then $\big(M'_\varepsilon(e_1), \dots, M'_\varepsilon(e_N)\big)$ is a permutation of $\big(M_\varepsilon(e_1), \dots, M_\varepsilon(e_N)\big)$ if and only if $\big(M'_\varepsilon(e_1)^\ell, \dots, M'_\varepsilon(e_N)^\ell\big)$ is a permutation of $\big(M_\varepsilon(e_1)^\ell, \dots, M_\varepsilon(e_N)^\ell\big)$.*

Proof. The forward direction is obvious. For the reverse direction it is sufficient to consider the case when $\widetilde{B}$ is indecomposable. If $\widetilde{B} = 0$, the statement is clear. In the remaining part we assume that $\widetilde{B}$ is indecomposable and $\widetilde{B} \neq 0$. Suppose that

$$M'_\varepsilon(e_k)^\ell = M_\varepsilon(e_{\sigma(k)})^\ell \quad \text{for some} \quad \sigma \in S_N. \tag{4.4}$$

Consider a root of unity quantum torus $\mathcal{T}_\varepsilon(\Lambda)$ with generators $X_1^{\pm 1}, \dots, X_N^{\pm 1}$. By using the standard basis of $\mathcal{T}_\varepsilon(\Lambda)$, one easily sees that the only solutions of the equation $y^\ell = X_k^\ell$ for $y \in \mathcal{T}_\varepsilon(\Lambda)$ and $1 \leqslant k \leqslant N$ are $y = (\varepsilon^{1/2})^m X_k$ for $m \in [0, \ell)$. By Theorem 3.10, $M'_\varepsilon(e_k) \in \mathcal{T}_\varepsilon(M_\varepsilon)$, and (4.4) implies that for all $1 \leqslant k \leqslant N$

$$M'_\varepsilon(e_k) = (\varepsilon^{1/2})^{m_k} M_\varepsilon(e_{\sigma(k)}) \quad \text{for some} \quad m_k \in [0, \ell).$$

Proposition 4.9 implies that $m_k = 0$ for all $1 \leqslant k \leqslant N$, so

$$M'_\varepsilon(e_k) = M_\varepsilon(e_{\sigma(k)}), \quad \forall 1 \leqslant k \leqslant N.$$ □

Proof of Theorem 4.8. Any map of labeled graphs from $E_\varepsilon(M_\varepsilon, \widetilde{B})$ to $E(\widetilde{B})$ that sends the vertex corresponding to the seed $(M_\varepsilon, \widetilde{B})$ to the vertex corresponding to the seed $((M_\varepsilon(e_1)^\ell, \dots, M_\varepsilon(e_N)^\ell), \widetilde{B})$ necessarily sends the vertex $\mu_{i_1} \dots \mu_{i_m}(M_\varepsilon, \widetilde{B})$ to the vertex $\mu_{i_1} \dots \mu_{i_m}((M_\varepsilon(e_1)^\ell, \dots, M_\varepsilon(e_N)^\ell), \widetilde{B})$ for all sequences $i_1, \dots, i_m$ in **ex**. Proposition 4.11 implies that this map is well-defined. It is obviously surjective. It is injective by Proposition 4.11. □

4.3 | The full centers of roots of unity quantum cluster algebras

In Corollary 4.7, we constructed a central subalgebra $\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ of each root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ satisfying condition **(Coprime)**. We will call it the *special central subalgebra*. One can provide a characterization of the full center of the algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$, as follows. For a skew-symmetric bilinear form $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}/\ell$ denote

$$\mathrm{Ker}\Lambda := \{f \in \mathbb{Z}^N \mid \Lambda(f, g) = 0, \forall g \in \mathbb{Z}^N\}.$$

Proposition 4.12. *Let $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ be a root of unity quantum cluster algebra. For every seed $(M'_\varepsilon, \widetilde{B}') \sim (M_\varepsilon, \widetilde{B})$, the center of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is given by*

$$Z(\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})) = \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \cap \mathcal{A}_\varepsilon^{1/2}\text{–Span}\{M'_\varepsilon(f) \mid f \in \mathrm{Ker}\Lambda_{M'_\varepsilon}\}.$$

Proof. Using the standard basis of a root of unity quantum torus, one easily shows that

$$Z(\mathcal{T}_\varepsilon(M'_\varepsilon)) = \mathcal{A}_\varepsilon^{1/2}\text{–Span}\{M'_\varepsilon(f) \mid f \in \mathrm{Ker}\Lambda_{M'_\varepsilon}\} \tag{4.5}$$

The root of unity quantum Laurent phenomenon (Theorem 3.10) implies that $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \subseteq \mathcal{T}_\varepsilon(M'_\varepsilon)$. As $\mathrm{Fract}(\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})) = \mathrm{Fract}(\mathcal{T}_\varepsilon(M'_\varepsilon))$,

$$Z(\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})) = \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \cap Z(\mathcal{T}_\varepsilon(M'_\varepsilon))$$

and the proposition follows from (4.5). □

Remark 4.13. Using the full form of the root of unity quantum Laurent phenomenon (Theorem 3.10), one analogously proves the following stronger (but more technical) description of the center of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$:

$$Z(\mathsf{A}_\varepsilon(M_\varepsilon,\widetilde{B}, \mathbf{inv})) = \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}) \cap \mathcal{A}_\varepsilon^{1/2}\text{–Span}\{M_\varepsilon(f) \mid f = (f_1, \dots, f_N) \in \mathrm{Ker}\Lambda_{M_\varepsilon}, f_i \geqslant 0, \forall i \notin \mathbf{ex} \sqcup \mathbf{inv}\}.$$

5 | STRICT ROOT OF UNITY QUANTUM CLUSTER ALGEBRAS AND SPECIALIZATIONS

In this section, we introduce the notion of strict root of unity quantum cluster algebras and show that, under certain general assumptions, they arise as specializations.

5.1 | Construction

Definition 5.1. Consider a root of unity quantum seed $(M_\varepsilon, \widetilde{B})$, so that $(\Lambda_{M_\varepsilon}, \widetilde{B})$ is ℓ-compatible with respect to a diagonal matrix D. We say that this seed is *strict* if there exists a skew-symmetric integer matrix $\Lambda \in M_N(\mathbb{Z})$ such that

(1) $\overline{\Lambda} = \Lambda_{M_\varepsilon}$ and
(2) the pair $(\Lambda, \widetilde{B})$ is compatible with respect to the diagonal matrix D.

Recall that $\overline{C}$ denotes the image of a matrix $C \in M_{n\times m}(\mathbb{Z})$ in $M_{n\times m}(\mathbb{Z}/\ell)$. Clearly, condition (2) is stronger than requiring that $(\Lambda_{M_\varepsilon}, \widetilde{B})$ be ℓ-compatible with respect to D. The choice of matrix Λ is not unique.

Proposition 5.2. *If $(M_\varepsilon, \widetilde{B})$ is a strict root of unity quantum seed with respect to a skew-symmetric integer matrix $\Lambda \in M_N(\mathbb{Z})$, then $\mu_k(M_\varepsilon, \widetilde{B})$ is also a strict root of unity quantum seed with respect to the skew-symmetric integer matrix*

$$\Lambda' = E_s^\top \Lambda E_s$$

Proof. The pair $(E_s^\top \Lambda E_s, E_s\widetilde{B}F_s)$ is the mutation of the compatible pair of matrices $(\Lambda, \widetilde{B})$. By [4, Proposition 3.4] the first pair is compatible with respect to the matrix D. We have

$$\Lambda_{\mu_k(M_\varepsilon)} = \overline{E}_s^\top \Lambda_{M_\varepsilon} \overline{E}_s = \overline{E}_s^\top \overline{\Lambda}\,\overline{E}_s = \overline{\Lambda}'. \qquad \square$$

Definition 5.3. We call a root of unity quantum cluster algebra *strict* if one, and thus every of its seeds, is strict.

Remark 5.4. The class of strict root of unity quantum cluster algebras is a proper subset of the class of root of unity quantum cluster algebras. For example, by Remark 3.9, for $\ell = 1$, a root of unity quantum cluster algebra is the same object as a classical cluster algebra. At the same time, it is easy to see that a strict root of unity quantum cluster algebra for $\ell = 1$ is the same object as a classical cluster algebra with a compatible Poisson structure in the sense of Gekhtman–Shapiro–Vainshtein [24].

5.2 | Specialization of quantum tori

Denote the ℓth cyclotomic polynomial by

$$\Phi_\ell(t) \in \mathbb{Z}[t]. \tag{5.1}$$

We have the isomorphism $\mathcal{A}_q^{1/2}/(\Phi_\ell(q^{1/2})) \simeq \mathcal{A}_\varepsilon^{1/2}$ given by $q^{1/2} \mapsto \varepsilon^{1/2}$. This makes $\mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2}))$ an $\mathcal{A}_\varepsilon^{1/2}$-algebra.

Lemma 5.5. *There is an isomorphism of $\mathcal{A}_\varepsilon^{1/2}$-algebras $\mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2})) \simeq \mathcal{T}_\varepsilon(\Lambda)$.*

Proof of Lemma 5.5. It follows that $\mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2})) \simeq \mathcal{T}_\varepsilon(\Lambda)$ because the free $\mathcal{A}_q^{1/2}$-module $\mathcal{T}_q(\Lambda)$ and the free $\mathcal{A}_\varepsilon^{1/2}$-module $\mathcal{T}_\varepsilon(\Lambda)$ both have the basis $\{X^f \mid f \in \mathbb{Z}^N\}$. $\square$

Denote the specialization map

$$\kappa_\varepsilon : \mathcal{T}_q(\Lambda) \twoheadrightarrow \mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2})) \simeq \mathcal{T}_\varepsilon(\Lambda). \tag{5.2}$$

It is a homomorphism of $\mathcal{A}_q^{1/2}$-algebras, where $\mathcal{A}_q^{1/2}$ acts on $\mathcal{T}_\varepsilon(\Lambda)$ via the map $q \mapsto \varepsilon$.

Construction 5.6. Let $(M_\varepsilon, \widetilde{B})$ be a strict root of unity quantum seed associated to a skew-symmetric integer matrix $\Lambda \in M_N(\mathbb{Z})$. To it we associate the unique quantum seed $(M_q, \widetilde{B})$ of $\mathcal{F}_q := \mathrm{Fract}(\mathcal{T}_q(\Lambda))$ such that $\Lambda_{M_q} = \Lambda$. The compatibility of the pair $(\Lambda, \widetilde{B})$ with respect to the matrix D implies that $(M_q, \widetilde{B})$ is indeed a quantum seed.

The isomorphisms of quantum tori $\mathcal{T}_q(M_q) \simeq \mathcal{T}_q(\Lambda)$ and $\mathcal{T}_\varepsilon(M_\varepsilon) \simeq \mathcal{T}_\varepsilon(\Lambda)$ and the specialization map (5.2) give rise to the specialization map (an $\mathcal{A}_q^{1/2}$-algebra homomorphism)

$$\kappa_\varepsilon : \mathcal{T}_q(M_q) \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon) \tag{5.3}$$

with kernel $(\Phi_\ell(q^{1/2}))$. It is given by $\kappa_\varepsilon(M_q(f)) := M_\varepsilon(f)$ for $f \in \mathbb{Z}^N$.

The next theorem provides a general realization of a root of unity quantum cluster algebra in terms of the specialization maps (5.3) for toric frames.

Theorem 5.7. *Let $(M_\varepsilon, \widetilde{B})$ be a root of unity quantum toric frame associated to a skew-symmetric integer matrix $\Lambda \in M_N(\mathbb{Z})$ and $(M_q, \widetilde{B})$ be the corresponding quantum toric frame from Construction 5.6. We have the isomorphism of $\mathcal{A}_\varepsilon^{1/2}$-algebras*

$$\kappa_\varepsilon(\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})) \simeq \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv}). \tag{5.4}$$

In the special case $\varepsilon^{1/2} = 1$ (i.e., $\ell = 1$), the theorem provides a realization of classical cluster algebras with a compatible Poisson structure (in the sense of [24]) in terms of toric frame specializations of quantum cluster algebras, recall Remark 3.9.

Proof. As the elements $M_q(e_k), 1 \leqslant k \leqslant N$ generate $\mathcal{T}_q(M_q)$ and $\kappa_\varepsilon : \mathcal{T}_q(M_q) \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon)$ is a surjective ring homomorphism,

$$\mathrm{Fract}\big(\kappa_\varepsilon(\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}))\big) \simeq \mathrm{Fract}(\mathcal{T}_\varepsilon(M_\varepsilon)). \tag{5.5}$$

We claim that the following hold for all quantum seeds $(M'_q, \widetilde{B}')$ of $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$:

(i) $(\kappa_\varepsilon M'_q, \widetilde{B}')$ is a root of unity quantum seed of $\kappa_\varepsilon(\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}))$.
(ii) $\mu_k\big(\kappa_\varepsilon M'_q, \widetilde{B}'\big) = (\kappa_\varepsilon(M''_q), \mu_k(\widetilde{B}'))$ where M''_q is the toric frame of the seed $\mu_k(M'_q, \widetilde{B}')$.

Property (i): $\kappa_\varepsilon M'_q$ is a root of unity quantum toric frame of $\mathcal{T}_\varepsilon(M_\varepsilon)$ because of (5.5) and the fact that κ_ε is a homomorphism of $\mathcal{A}_q^{1/2}$-algebras. The compatibility of the pair $(\Lambda_{M_q}, \widetilde{B}')$ implies that the matrix of the frame $\kappa_\varepsilon M'_q$ and the exchange matrix $\widetilde{B}'$ are ℓ-compatible. Property (ii) follows from the mutation formulae in Equation (2.4) and Proposition 3.6(1), and once again the fact that fact that κ_ε is an $\mathcal{A}_q^{1/2}$-algebra homomorphism.

The properties (i)–(ii), the fact that the image $\kappa_\varepsilon(\mathbf{A}_q(M_q, \widetilde{B}, \mathbf{inv}))$ is generated by the elements $\kappa_\varepsilon M'_q(e_j)$ for quantum seeds $(M'_q, \widetilde{B}') \sim (M_q, \widetilde{B})$, $1 \leqslant j \leqslant N$ and by the inverses of these elements for $j \in \mathbf{inv}$ imply the isomorphism in (5.4). ☐

5.3 | Generalized specialization

For a commutative ring $\mathcal{A}$ and an ideal $\mathcal{I}$ of it, denote the factor ring $\mathcal{A}' := \mathcal{A}/\mathcal{I}$.

Lemma 5.8.

(1) *For every $\mathcal{A}$-module V we have the short exact sequence of $\mathcal{A}$-modules*

$$0 \to \mathcal{I}V \to V \to \mathcal{A}' \otimes_{\mathcal{A}} V \to 0,$$

where the third map is $v \mapsto 1 \otimes v$ for $v \in V$ and $\mathcal{A}' \otimes_{\mathcal{A}} V$ is made into an $\mathcal{A}$-module via the surjection $\mathcal{A} \twoheadrightarrow \mathcal{A}'$.

(2) *For an $\mathcal{A}$-submodule $W \subseteq V$, the following are equivalent:*
  (a) *the induced map $\mathcal{A}' \otimes_{\mathcal{A}} - : \mathcal{A}' \otimes_{\mathcal{A}} W \to \mathcal{A}' \otimes_{\mathcal{A}} V$ is injective,*
  (b) $W \cap \mathcal{I}V = \mathcal{I}W$.

Proof. The first part is well-known, see, for example, [22, Lemma 3.1].

(2) Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}W & \xrightarrow{\iota_W} & W & \xrightarrow{\eta_W} & \mathcal{A}' \otimes_{\mathcal{A}} W & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow \theta & & \big\downarrow \theta' & & \\
0 & \longrightarrow & \mathcal{I}V & \xrightarrow{\iota_V} & V & \xrightarrow{\eta_V} & \mathcal{A}' \otimes_{\mathcal{A}} V & \longrightarrow & 0
\end{array}$$

where the horizontal maps are the ones from part (1) and the vertical ones are induced from the embedding $\theta : W \hookrightarrow V$.

(a) $\Rightarrow$ (b) Let $v_0 \in \mathcal{I}V$ and $w \in W$ be such that $\iota_V(v_0) = \theta(w)$. Then $\theta'\eta_W(w) = \eta_V\theta(w) = \eta_V\iota_V(v_0) = 0$. As (a) holds, $\eta_W(w) = 0$, and so $w \in \operatorname{Im}\iota_W$.

(b) $\Rightarrow$ (a) Let $w' \in \mathcal{A}' \otimes_{\mathcal{A}} W$ be such that $\theta'(w') = 0$. Choose $w \in W$ such that $w' = \eta_W(w)$. Because $\eta_V\theta(w) = \theta'\eta_W(w) = 0$, $\theta(w) \in \operatorname{Im}\iota_V$. As (b) holds, $w = \iota_W(w_0)$ for some $w_0 \in \mathcal{I}W$, and thus $w' = \eta_W\iota_W(w_0) = 0$.

A special case of the second part of the lemma for principal ideal domains $\mathcal{A}$, prime ideals $\mathcal{I}$ and free modules V is stated in [22, Lemma 2.1]. ☐

The $\mathcal{A}'$-module $V/\mathcal{I}V \simeq \mathcal{A}' \otimes_{\mathcal{A}} V$ is called the (generalized) specialization of V at $\mathcal{I}$; traditionally, specialization deals with the special case when $\mathcal{I}$ is a principal ideal. The canonical projection map

$$\eta_V : V \twoheadrightarrow V/\mathcal{I}V \simeq \mathcal{A}' \otimes_{\mathcal{A}} V$$

is called the *specialization map*. It is a homomorphism of $\mathcal{A}$-modules. If an $\mathcal{A}$-submodule $W \subseteq V$ satisfies the equivalent conditions in Lemma 5.8(2), then

$$W/\mathcal{I}W \simeq \eta_V(W) \quad \text{and} \quad \eta_W = \eta_V|_W.$$

5.4 | A general specialization result for quantum cluster algebras

Recall from (5.1) that $\Phi_\ell(t)$ denotes the ℓth cyclotomic polynomial and $\mathcal{A}_q^{1/2}/(\Phi_\ell(q^{1/2})) \simeq \mathcal{A}_\varepsilon^{1/2}$. For a quantum cluster algebra $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$ denote the corresponding specialization map

$$\eta_\varepsilon : \mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \twoheadrightarrow \mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})/(\Phi_\ell(q^{1/2})) \simeq \mathcal{A}_\varepsilon^{1/2} \otimes_{\mathcal{A}_q^{1/2}} \mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}),$$

which is a surjective homomorphism of $\mathcal{A}_q^{1/2}$-modules.

Similarly to (3.4), for a quantum seed $(M'_q, \widetilde{B}') \sim (M_q, \widetilde{B})$ denote the subalgebra

$$\mathcal{T}_q(M'_q)_{\geqslant} := \mathcal{A}_q^{1/2}\langle M'_q(e_i), M'_q(e_j)^{-1} \mid i \in [1, N],\ j \in \mathbf{ex} \sqcup \mathbf{inv}\rangle \tag{5.6}$$

of the quantum torus $\mathcal{T}_q(M'_q)$. It is isomorphic to the mixed (based) quantum torus/skew polynomial algebra

$$\mathcal{A}_\varepsilon^{1/2}\text{–Span}\{X^f \mid f \in \mathbb{Z}^N_{\geqslant}\} \quad \text{with the product} \quad X^f X^g = q^{\Lambda'(f,g)/2} X^{f+g},\ \forall f, g \in \mathbb{Z}^N_{\geq}$$

for $\mathbb{Z}^N_{\geq}$ as in (3.5). The specialization map $\kappa_\varepsilon : \mathcal{T}_q(M_q) \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon) \cong \mathcal{T}_q(M_q)/(\Phi_\ell(q^{1/2}))$ form (5.3) restricts to the specialization map

$$\kappa_\varepsilon : \mathcal{T}_q(M_q)_{\geqslant} \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon)_{\geqslant} \cong \mathcal{T}_q(M_q)_{\geqslant}/(\Phi_\ell(q^{1/2})), \tag{5.7}$$

which, by abuse of notation, will be denoted by the same symbol.

The following result gives a general way of constructing root of unity quantum cluster algebras as specializations from quantum cluster algebras.

Theorem 5.9. *Let $(M_\varepsilon, \Lambda, \widetilde{B})$ be a root of unity quantum toric frame and $(M_q, \widetilde{B})$ be the corresponding quantum toric frame from Construction 5.6. If*

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \cap \left(\Phi_\ell(q^{1/2})\mathcal{T}_q(M_q)_{\geqslant}\right) = \Phi_\ell(q^{1/2})\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}), \tag{5.8}$$

then the root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is a specialization of the quantum cluster algebra $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$:

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})/(\Phi_\ell(q^{1/2})) \simeq \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$$

and the specialization map η_ε is a restriction of the specialization map $\kappa_\varepsilon : \mathcal{T}_q(M_q)_{\geqslant} \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon)_{\geqslant}$ to $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$.

Verifying the condition (5.8) in concrete cases is difficult. Theorem 5.11 presents another result of the form that $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is a specialization of $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$ under an assumption that is stronger but more natural and easier to verify. The proof of Theorem 5.11 uses Theorem 5.9.

Proof of Theorem 5.9. In light of Lemma 5.8(2), the assumption (5.8) implies that

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})/(\Phi_\ell(q^{1/2})) \simeq \kappa_\varepsilon(\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})) \quad \text{and} \quad \eta_\varepsilon = \kappa_\varepsilon|_{\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})}.$$

Thus, we have the commutative diagram

$$\begin{array}{ccccc} \mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) & \hookrightarrow & \mathcal{T}_q(M_q)_\geq & \hookrightarrow & \mathrm{Fract}(\mathcal{T}_q(M_q)) \\ & \searrow^{\eta_\varepsilon} & \downarrow\!\!\downarrow^{\kappa_\varepsilon} & & \\ & & \mathcal{T}_\varepsilon(M_\varepsilon)_\geq & \hookrightarrow & \mathrm{Fract}(\mathcal{T}_\varepsilon(M_\varepsilon)) \end{array}$$

The theorem now follows from Theorem 5.7. □

5.5 | Specialization results for quantum cluster algebras

The following is an extension of [22, Proposition 3.5]:

Proposition 5.10. *For each prime element $p \in \mathcal{A}_q^{1/2}$ and $k \in \mathbf{ex}$,*

$$\mathcal{T}_q(M_q)_\geqslant \cap (p\mathcal{T}_q(\mu_k M_q)_\geqslant) = (p\mathcal{T}_q(M_q)_\geqslant) \cap \mathcal{T}_q(\mu_k M_q)_\geqslant.$$

Proof. We follow the line of argument of [22, Proposition 3.5] but include the proof because the original result in [22] is stated over the base ring $k[q^{\pm 1/2}]$, where k is a field, and for a concrete choice of p.

Denote by $\mathcal{T}_q(M_q)^\circ_\geqslant$ the subalgebra $\mathcal{T}_q(M_q)_\geqslant$ with those generators as in (5.6) such that $i, j \neq k$. Let $X_k := M_q(e_k)$ and $X'_k := \mu_k(M_q)(e_k)$. $\mathcal{T}_q(M_q)_\geqslant$ is a free (left and right) $\mathcal{T}_q(M_q)^\circ_\geqslant$-module with basis $\{X_k^j \mid j \in \mathbb{Z}\}$:

$$\mathcal{T}_q(M_q)_\geqslant = \bigoplus_{j\in\mathbb{Z}} X_k^j \mathcal{T}_q(M_q)^\circ_\geqslant. \tag{5.9}$$

For $j \in \mathbb{Z}$ denote

$$Q^j = q^{j\Lambda(e_k,[b^k]_+)/2} M_q([b^k]_+) + q^{-j\Lambda(e_k,[b^k]_-)/2} M_q(-[b^k]_-) \in \mathcal{T}_q(M_q)^\circ_\geqslant.$$

We have

$$Q^1 = X_k X'_k \quad \text{and} \quad Q^j X_k = X_k Q^{j-2}, \ \forall j \in \mathbb{Z}.$$

If $y \in \mathcal{T}_q(M_q)_\geqslant \cap (p\mathcal{T}_q(\mu_k M_q)_\geqslant)$, then

$$y = \sum_{j\in\mathbb{Z}} X_k^j c_j = \sum_{j\in\mathbb{Z}} (X'_k)^j d_j,$$

where both sums are finite and $c_j \in \mathcal{T}_q(M_q)^\circ_\geqslant$, $d_j \in p\mathcal{T}_q(M_q)^\circ_\geqslant$ for all $j \in \mathbb{Z}$. The free module structure (5.9) implies that

$$\begin{aligned} c_0 &= d_0 \\ c_j &= Q^{-2j-1} \dots Q^3 Q^1 d_{-j} && \text{for } j < 0, \\ Q^{-1}Q^{-3} \dots Q^{-2j+1} c_j &= d_{-j} && \text{for } j > 0. \end{aligned}$$

Therefore, $c_j \in p\mathcal{T}_q(M_q)^\circ_\geqslant$ for all $j \leqslant 0$. For the case $j > 0$, first note that $\mathcal{A}_q^{1/2}/(p)$ is an integral domain because $p \in \mathcal{A}_q^{1/2}$ is prime. As a consequence, $\mathcal{T}_q(M_q)^\circ_\geqslant / p\mathcal{T}_q(M_q)^\circ_\geqslant$ is a domain because it is a subalgebra of a quantum torus with coefficients in $\mathcal{A}_q^{1/2}/(p)$. If $\tau : \mathcal{T}_q(M_q)^\circ_\geqslant \twoheadrightarrow \mathcal{T}_q(M_q)^\circ_\geqslant / p\mathcal{T}_q(M_q)^\circ_\geqslant$ denotes the canonical projection, then

$$\tau(c_j)\tau(Q^{2j-1}) \dots \tau(Q^3)\tau(Q^1) = \tau(d_{-j}) = 0.$$

Because $Q^i \notin p\mathcal{T}_q(M_q)^\circ_\geqslant$ for all $i \in \mathbb{Z}$ and $\mathcal{T}_q(M_q)^\circ_\geqslant / p\mathcal{T}_q(M_q)^\circ_\geqslant$ is a domain, $\tau(c_j) = 0$ and thus $c_j \in p\mathcal{T}_q(M_q)^\circ_\geqslant$ for $j > 0$. Hence, $y \in p\mathcal{T}_q(M_q)_\geqslant$. □

Theorem 5.11. *Let $(M_\varepsilon, \Lambda, \widetilde{B})$ be a root of unity quantum toric frame and $(M_q, \widetilde{B})$ be the corresponding quantum toric frame from Construction 5.6. If*

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) = \mathsf{U}_q(M_q, \widetilde{B}, \mathbf{inv}),$$

then the root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ is a specialization of the quantum cluster algebra $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$:

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})/(\Phi_\ell(q^{1/2})) \simeq \mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$$

and the specialization map η_ε is a restriction of the specialization map $\kappa_\varepsilon : \mathcal{T}_q(M_q)_\geqslant \twoheadrightarrow \mathcal{T}_\varepsilon(M_\varepsilon)_\geqslant$ from (5.7) to $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$.

Proof. Applying Proposition 5.10, one proves that for all quantum seeds $(M_q', \widetilde{B}') \sim (M_q, \widetilde{B})$,

$$\begin{aligned} \mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \cap \left(\Phi_\ell(q^{1/2})\mathcal{T}_q(M_q)_\geqslant\right) &= \mathsf{U}_q(M_q, \widetilde{B}, \mathbf{inv}) \cap \left(\Phi_\ell(q^{1/2})\mathcal{T}_q(M_q)_\geqslant\right) \\ &\subseteq \Phi_\ell(q^{1/2})\mathcal{T}_q(M_q')_\geqslant \end{aligned}$$

by induction on the distance from $(M_q, \widetilde{B})$ to $(M_q', \widetilde{B}')$ in the exchange graph. Hence

$$\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \cap \left(\Phi_\ell(q^{1/2})\mathcal{T}_q(M_q)_\geqslant\right) \subseteq \Phi_\ell(q^{1/2})\mathsf{U}_q(M_q, \widetilde{B}, \mathbf{inv}) = \Phi_\ell(q^{1/2})\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$$

and clearly $\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv}) \cap \left(\Phi_\ell(q^{1/2})\mathcal{T}_q(M_q)_\geqslant\right) \supseteq \Phi_\ell(q^{1/2})\mathsf{A}_q(M_q, \widetilde{B}, \mathbf{inv})$. This verifies the condition (5.8) and the theorem now follows from Theorem 5.9. □

5.6 | An example: Quantized Weyl algebras at roots of unity

Let $Q = (a_{ij}) \in M_n(\mathbb{Z})$ be a skew-symmetric integer matrix and $\varepsilon^{1/2} \in \mathbb{C}$ be a primitive ℓth root of unity for $\ell > 1$. Denote by $A^Q_{n,\varepsilon,\mathbb{C}}$ the quantized Weyl algebra at the root of unity ε, which is a $\mathbb{C}$-algebra generated by x_i, y_i for $i \in [1, n]$ with relations

$$y_i y_j = \varepsilon^{a_{ij}} y_j y_i \ \forall i, j, \qquad x_i x_j = \varepsilon^{1+a_{ij}} x_j x_i \text{ for } i < j,$$

$$x_i y_j = \varepsilon^{-a_{ij}} y_j x_i \text{ for } i < j, \quad x_i y_j = \varepsilon^{1-a_{ij}} y_j x_i \text{ for } i > j,$$

$$x_j y_j = 1 + \varepsilon y_j x_j + (\varepsilon - 1) \sum_{r=1}^{j-1} y_r x_r.$$

Note that $\{x_i, (\varepsilon - 1)y_i \mid 1 \leqslant i \leqslant n\}$ is another set of generators for this algebra. Denote by $A^Q_{n,\varepsilon,\mathbb{Z}}$ the $\mathcal{A}^{1/2}_\varepsilon$-subalgebra generated by $x_i, (\varepsilon - 1)y_i$. It is an $\mathcal{A}^{1/2}_\varepsilon$-form of $A^Q_{n,\varepsilon,\mathbb{C}}$. The algebra $A^Q_{n,\varepsilon,\mathbb{Z}}$ is a specialization of the $\mathcal{A}^{1/2}_q$-algebra $A^Q_{n,q,\mathbb{Z}}$ with generators and relations as in [26, eq. (4.9)]:

$$A^Q_{n,\varepsilon,\mathbb{Z}} \cong A^Q_{n,q,\mathbb{Z}} / \Phi_\ell(q^{1/2}).$$

This easily follows by using bases for both algebras.

By [26, Example 4.10] $A^Q_{n,q,\mathbb{Z}}$ has a quantum cluster algebra structure of type $(A_1)^n$ and by [26, Theorem 4.8] this quantum cluster algebra equals the corresponding upper quantum cluster algebra. Proposition 5.10 implies that $A^Q_{n,\varepsilon,\mathbb{Z}}$ has a strict root of unity quantum cluster algebra structure. The root of unity quantum toric frame for its initial seed is given by

$$M_\varepsilon(e_i) := (-1)^i \varepsilon^{1/2} x_i, \quad M_\varepsilon(e_{i+n}) := (-1)^i [x_i, y_i] = (-1)^i + (-1)^i (\varepsilon - 1) \sum_{r=1}^{i} x_r y_r$$

for $1 \leqslant i \leqslant n$, and the corresponding matrix is

$$\Lambda = \begin{bmatrix} Q' & -R \\ R & 0_{n\times n} \end{bmatrix}$$

whose blocks are the $n \times n$ integer matrices

$$(Q')_{ij} = \begin{cases} a_{ij} + 1 & \text{if } i < j \\ -a_{ji} - 1 & \text{if } i > j \\ 0 & \text{if } i = j \end{cases} \qquad (R)_{ij} = \begin{cases} 1 & \text{if } i < j \\ a_{ji} & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}.$$

The set of exchangeable indices is $\mathbf{ex} = [1, n]$ and the set of inverted frozen variables is empty, $\mathbf{inv} = \emptyset$. The exchange matrix of the seed is

$$\widetilde{B} = \begin{bmatrix} 0_{n\times n} \\ S \end{bmatrix}$$

where the entries of $S \in M_n(\mathbb{Z})$ are $(S)_{i,n+1-i} = 1, (S)_{i,n-i} = -1, (S)_{ij} = 0$ otherwise.

6 | DISCRIMINANTS OF ROOT OF UNITY QUANTUM CLUSTER ALGEBRAS

In this section, we prove a general result for the computation of discriminants of root of unity quantum cluster algebras.

6.1 | Background on discriminants

For an algebraic number field K, consider its trace function $\mathrm{tr} = \mathrm{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$ obtained from the composition $K \hookrightarrow M_N(\mathbb{Q}) \xrightarrow{\mathrm{Tr}} \mathbb{Q}$, where the first embedding is obtained from the K-action on $K \simeq \mathbb{Q}^N$ (for some positive integer N) and the second map is the trace map on matrices. The discriminant of K is defined by

$$\Delta_K := \det\left(\mathrm{tr}(y_i y_j)\right)_{i,j=1}^{N},$$

where $\{y_1, y_2, \ldots, y_N\}$ is a $\mathbb{Z}$-basis of the ring of integers O_K of K. The discriminant does not depend on the choice of basis. More generally, we consider *algebras with trace*:

Definition 6.1. An algebra with trace is a ring R with a central subring C and a C-linear map $\mathrm{tr} : R \to C$ such that

$$\mathrm{tr}(xy) = \mathrm{tr}(yx), \quad \forall x, y \in R.$$

Such a ring R is naturally a C-algebra.

Example 6.2. Consider a ring R which is free and of finite rank N over a central subring $C \subseteq Z(R)$. Choosing a C-basis of R gives rise to a C-module isomorphism $R \simeq C^N$, and the left action of R on itself gives rise to an algebra homomorphism $R \to M_N(C)$. The *regular trace* of R is defined as the composition

$$\mathrm{tr}_{\mathrm{reg}} : R \to M_N(C) \xrightarrow{\mathrm{Tr}} C \subseteq R,$$

where the second maps is the trace map on matrices. The trace map $\mathrm{tr}_{\mathrm{reg}}$ is independent of the choice of C-basis used to construct the homomorphism $R \to M_N(C)$.

For a commutative ring C, denote by $C^\times$ its group of *units* (i.e., invertible elements under the product operation). Two elements $c_1, c_2 \in C$ are called *associates* (denoted $c_1 =_{C^\times} c_2$) if $c_1 = uc_2$ for some $u \in C^\times$.

Definition 6.3. Assume that R is an algebra with trace $\mathrm{tr} : R \to C$ such that R is a free and of finite rank N over the central subring $C \subseteq Z(R)$. The *discriminant* of R over C is defined by

$$d(R/C) :=_{C^\times} \det\left(\mathrm{tr}(y_i y_j)\right)_{i,j=1}^{N}, \tag{6.1}$$

where $\{y_1, y_2, \ldots, y_N\}$ is a C-basis of R. For different choices of C-bases of R, the right-hand sides of (6.1) are associates of each other.

6.2 | Nerves and the algebras $\mathbf{A}_\varepsilon(\Theta, \mathbf{inv})$ and $\mathbf{C}_\varepsilon(\Theta, \mathbf{inv})$

Let $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ be a quantum cluster algebra with exchange graph $E_\varepsilon(M_\varepsilon, \widetilde{B})$.

For a collection of seeds Θ in $E_\varepsilon(M_\varepsilon, \widetilde{B})$, let

(1) $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ be $\mathcal{A}_\varepsilon^{1/2}$-subalgebra of $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ generated by $M'_\varepsilon(e_j)$ for $j \in [1, N]$ and $M'_\varepsilon(e_i)^{-1}$ for $i \in \mathbf{inv}$, for all $(M'_\varepsilon, \widetilde{B}') \in \Theta$, and
(2) $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$ be $\mathcal{A}_\varepsilon^{1/2}$-subalgebra of $\mathsf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ generated by $M'_\varepsilon(e_j)^\ell$ for $j \in [1, N]$ and $M'_\varepsilon(e_i)^{-\ell}$ for $i \in \mathbf{inv}$, for all $(M'_\varepsilon, \widetilde{B}') \in \Theta$.

Thus, $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$ is in the center of $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$.

Definition 6.4. A subset of seeds Θ that satisfies the following conditions is called a *nerve*:

(1) The subgraph in $E_\varepsilon(M_\varepsilon, B)$ induced by Θ is connected.
(2) For each mutable direction $k \in \mathbf{ex}$, there are at least two seeds in Θ mutation equivalent by μ_k.

The concept of nerves was introduced in [18] for a practical way of specifying a quasi-homomorphism of a cluster algebra. A basic example of a nerve would be a star neighborhood in $E_\varepsilon(M_\varepsilon, B)$ of any particular seed.

6.3 | The discriminant of $\mathbf{A}_\varepsilon(\Theta, \mathbf{inv})$ over $\mathbf{C}_\varepsilon(\Theta, \mathbf{inv})$

For the proof of the main theorem on discriminants we will need the following lemma. Its proof was communicated to us by Greg Muller.

Lemma 6.5. *If*

$$n \prod_{i=1}^{N} x_i^{a_i} \in \mathsf{A}(\widetilde{\mathbf{x}}, \widetilde{B}, \mathbf{inv}) \tag{6.2}$$

for some $a_i, n \in \mathbb{Z}$, $n \neq 0$, then $a_i \geqslant 0$ for $i \notin \mathbf{inv}$.

Proof. It is sufficient to prove the statement in the case $\mathbf{inv} = \varnothing$ because (6.2) implies that $n \prod_{i=1}^{N} x_i^{a_i} \prod_{i \in \mathbf{inv}} x_i^{a_i} \in \mathsf{A}(\widetilde{\mathbf{x}}, \widetilde{B}, \varnothing)$ for some $c_i \in \mathbb{N}$. For the rest of the proof we assume that $\mathbf{inv} = \varnothing$.

If $i \in \mathbf{ex}$, then $a_i \geqslant 0$ because, if $a_i < 0$, then expressing the Laurent monomial in terms of the cluster variables of the seed $\mu_i(\widetilde{\mathbf{x}}, \widetilde{B})$ would contradict the Laurent phenomenon. If $i \notin \mathbf{ex}$, the statement follows from [16, Proposition 3.6]. ∎

Theorem 6.6. *Consider a root of unity quantum cluster algebra $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ satisfying the condition **(Coprime)**, where $\varepsilon^{1/2}$ is a primitive ℓth root of unity. Let Θ be a collection of seeds which is a nerve.*

(1) *If $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a free $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$-module, then $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a finite rank $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$-module of rank ℓ^N and its discriminant with respect to the regular trace function is given as a product of noninverted frozen variables raised to the ℓth power,*

$$d(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})/\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})) =_{\mathsf{C}_\varepsilon(\Theta,\mathbf{inv})^\times} \ell^{(N\ell^N)} \prod_{i\in[1,N]\backslash(\mathbf{ex}\sqcup\mathbf{inv})} M_\varepsilon(e_i)^{\ell a_i} \quad \textit{for some} \;\; a_i \in \mathbb{N}.$$

(2) *If $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a free $\mathsf{C}(\Theta, \mathbf{inv})$-module, then $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a finite rank $\mathsf{C}(\Theta, \mathbf{inv})$-module of rank $\ell^N \varphi(\ell)$ and its discriminant with respect to the regular trace function is given by*

$$d(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})/\mathsf{C}(\Theta, \mathbf{inv})) =_{\mathsf{C}(\Theta,\mathbf{inv})^\times} \left(\frac{\ell^{(N+1)\varphi(\ell)}}{\prod_{p|\ell} p^{\varphi(\ell)/(p-1)}}\right)^{\ell^N} \prod_{i\in[1,N]\backslash(\mathbf{ex}\sqcup\mathbf{inv})} M_\varepsilon(e_i)^{\ell c_i}$$

for some $c_i \in \mathbb{N}$.

Proof. Throughout the proof all discriminants are computed with respect to the regular traces of the algebras that are involved.

(1) For a root of unity quantum frame M'_ε denote the skew polynomial subalgebra of $\mathcal{T}_\varepsilon(M'_\varepsilon)$

$$\mathcal{S}_\varepsilon(M'_\varepsilon) := \mathcal{A}_\varepsilon^{1/2}\langle M'_\varepsilon(e_i), 1 \leqslant i \leqslant N\rangle \simeq \mathcal{A}_\varepsilon^{1/2}\langle X_1, \dots, X_N\rangle/(X_iX_j - \varepsilon^{\lambda'_{ij}} X_jX_i),$$

where $\lambda'_{ij} := \Lambda_{M'_\varepsilon}(e_i, e_j)$. By [8, Proposition 2.8], the discriminant of $\mathcal{S}_\varepsilon(M'_\varepsilon)$ over the central subalgebra $\mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^\ell]_{i=1}^N$ is given by

$$d\Big(\mathcal{S}_\varepsilon(M'_\varepsilon)/\mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^\ell]_{i=1}^N\Big) =_{\mathcal{A}_\varepsilon^{1/2\times}} \ell^{N\ell^N} \prod_{i\in[1,N]} \Big(M'_\varepsilon(e_i)^{\ell^N(\ell-1)}\Big).$$

Therefore, the discriminant of its localization

$$\mathcal{T}_\varepsilon(M'_\varepsilon) \simeq \mathcal{S}_\varepsilon(M'_\varepsilon)[M'_\varepsilon(e_i)^{-\ell}]_{i=1}^N$$

is given by

$$d\Big(\mathcal{T}_\varepsilon(M'_\varepsilon)/\mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^{\pm\ell}]_{i=1}^N\Big) =_{\left(\mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^{\pm\ell}]_{i=1}^N\right)^\times} \ell^{N\ell^N}.$$

For the rest of the proof assume that $(M'_\varepsilon, \widetilde{B}') \in \Theta$. Applying Theorem 4.6 (using the assumption that $\mathsf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \mathbf{inv})$ satisfies the condition **(Coprime)**) and the Laurent phenomenon, we obtain that

$$\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})[M'_\varepsilon(e_i)^{-\ell}]_{i=1}^N \simeq \mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^{\pm\ell}]_{i=1}^N.$$

The root of unity quantum Laurent phenomenon (Theorem 3.10) implies

$$\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})[M'_\varepsilon(e_i)^{-\ell}]_{i=1}^N \simeq \mathcal{T}_\varepsilon(M'_\varepsilon).$$

Therefore, the rank of $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ as an $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$-module equals the rank of $\mathcal{T}_\varepsilon(M'_\varepsilon)$ as an $\mathcal{A}_\varepsilon^{1/2}[M'_\varepsilon(e_i)^{\pm\ell}]_{i=1}^N$-module. As the latter rank equals ℓ^N, $\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})$ is a finite rank $\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})$-module of rank ℓ^N. Furthermore,

$$d(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv})[M'_\varepsilon(e_i)^{-\ell}]_{i=1}^N / \mathsf{C}_\varepsilon(\Theta, \mathbf{inv})[M'_\varepsilon(e_i)^{-\ell}]_{i=1}^N) =_{\mathcal{T}_\varepsilon(M'_\varepsilon)^\times} \ell^{N\ell^N}. \tag{6.3}$$

Theorem 4.6 implies that

$$\mathsf{C}_\varepsilon(\Theta, \mathbf{inv}) \cap \mathcal{T}_\varepsilon(M'_\varepsilon)^\times \subseteq \{(\mathcal{A}_\varepsilon^{1/2})^\times M'_\varepsilon(e_1)^{\ell a_1} \dots M'_\varepsilon(e_N)^{\ell a_N} \mid a_i \in \mathbb{Z}\}. \tag{6.4}$$

Combining (6.3) and (6.4) gives that for all seeds $(M'_\varepsilon, \widetilde{B}') \in \Theta$,

$$d(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv}) / \mathsf{C}_\varepsilon(\Theta, \mathbf{inv})) =_{\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})^\times} \ell^{N\ell^N} \prod_{i \in [1,N]} \left(M'_\varepsilon(e_i)^\ell\right)^{a_i} \tag{6.5}$$

for some integers a_i (depending on each seed). We will assume that $a_i = 0$ for $i \in \mathbf{inv}$ because $M_\varepsilon(e_i)^\ell \in \mathsf{C}_\varepsilon(\Theta, \mathbf{inv})^\times$ for $i \in \mathbf{inv}$. Theorem 4.6 and Lemma 6.5 imply that $a_i \geqslant 0$ for $i \notin \mathbf{inv}$.

Fix $k \in \mathbf{ex}$. As Θ is a nerve, there exists $(M'_\varepsilon, \widetilde{B}') \in \Theta$ such that $\mu_k(M'_\varepsilon, \widetilde{B}') \in \Theta$. Applying (6.5) to the two seeds gives

$$\begin{aligned} d(\mathsf{A}_\varepsilon(\Theta, \mathbf{inv}) / \mathsf{C}_\varepsilon(\Theta, \mathbf{inv})) &=_{\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})^\times} \ell^{N\ell^N} \left(M'_\varepsilon(e_k)^\ell\right)^{a_k} \prod_{i \in [1,N] \backslash (\mathbf{inv} \sqcup \{k\})} \left(M'_\varepsilon(e_i)^\ell\right)^{a_i} \\ &=_{\mathsf{C}_\varepsilon(\Theta, \mathbf{inv})^\times} \ell^{N\ell^N} \left(\mu_k M'_\varepsilon(e_k)^\ell\right)^{c_k} \prod_{i \in [1,N] \backslash (\mathbf{inv} \sqcup \{k\})} \left(M'_\varepsilon(e_i)^\ell\right)^{c_i} \end{aligned}$$

for some $a_i, c_i \in \mathbb{Z}$, $i \in [1, N] \backslash \mathbf{inv}$. By Proposition 4.4,

$$\mu_k M'_\varepsilon(e_k)^\ell = M'_\varepsilon(-e_k + [b^k]_+)^\ell + M'_\varepsilon(-e_k - [b^k]_-)^\ell,$$

which is not a monomial of the $M'_\varepsilon(e_i)$'s for $i \in [1, N] \backslash (\mathbf{inv} \sqcup \{k\})$. Hence,

$$a_k = 0 = c_k,$$

$$a_i = c_i \text{ for } i \neq k.$$

Because of the connectedness assumption in Definition 6.4(1), for all seeds $(M'_\varepsilon, \widetilde{B}') \in \Theta$ and $k \in \mathbf{ex} \sqcup \mathbf{inv}$, $a_k = 0$ in (6.5).

(2) For every root of unity quantum frame M'_ε, $\mathcal{S}_\varepsilon(M'_\varepsilon)$ is a free $\mathbb{Z}[M'_\varepsilon(e_i)^\ell]_{i=1}^N$-module of rank $\ell^N \varphi(\ell)$. The discriminant of the cyclotomic field extension $\mathbb{Q}(\varepsilon^{1/2})$ of $\mathbb{Q}$ equals

$$\frac{(-1)^{\varphi(\ell)/2} \ell^{\varphi(\ell)}}{\prod_{p|\ell} p^{\varphi(\ell)/(p-1)}}.$$

From this one easily deduces that

$$d\big(\mathcal{S}_\varepsilon(M'_\varepsilon)/\mathbb{Z}[M'_\varepsilon(e_i)^\ell]_{i=1}^N\big) =_{\mathbb{Z}^\times} \left(\frac{\ell^{(N+1)\varphi(\ell)}}{\prod_{p|\ell} p^{\varphi(\ell)/(p-1)}}\right)^{\ell^N} \prod_{i\in[1,N]} \Big(M'_\varepsilon(e_i)^{\ell^N(\ell-1)\varphi(\ell)}\Big).$$

Using this formula, the proof of part (2) is carried out using exactly the same arguments as part (1). □

Remark 6.7. As it is unknown whether $\mathbf{C}_\varepsilon(\Theta, \mathbf{inv})$ is a free $\mathbf{C}(\Theta, \mathbf{inv})$-module, part (2) of the theorem is not a consequence of part (1) and the formula for the discriminants of cyclotomic field extensions.

6.4 | An example: Discriminants of quantized Weyl algebras at roots of unity

By the construction in Subsection 5.6,

$$A^Q_{n,\varepsilon,\mathbb{Z}} \cong \mathbf{A}_\varepsilon(M_\varepsilon, \widetilde{B}, \varnothing)$$

for the toric frame M_ε and exchange matrix $\widetilde{B}$ specified there. The underlying cluster algebra is of finite type $(A_1)^n$. Let $\varepsilon^{1/2}$ be a primitive ℓth root of unity for an odd integer $\ell > 1$, $\mathcal{A}_\varepsilon^{1/2} = \mathcal{A}_\varepsilon$. Denote

$$C^Q_{n,\varepsilon,\mathbb{Z}} := \mathcal{A}_\varepsilon[x_i^\ell, ((\varepsilon-1)y_i)^\ell, 1 \leqslant k \leqslant n].$$

It is well-known and easy to verify that $C^Q_{n,\varepsilon,\mathbb{Z}}$ is in the center of $A^Q_{n,\varepsilon,\mathbb{Z}}$. We apply Theorem 6.6 for Θ equal to the set of all seeds of the root of unity quantum cluster algebra. It is easy to see that it has 2^n seeds with cluster variables

$$(t_1, \dots, t_n, -z_1, \dots, (-1)^i z_i, \dots, (-1)^n z_n) \quad \text{where} \quad t_i = (-1)^i \varepsilon^{1/2} x_i \ \text{ or } \ t_i = (\varepsilon-1)y_i.$$

This implies that $\mathbf{C}_\varepsilon(M_\varepsilon, \widetilde{B}, \varnothing) = C^Q_{n,\varepsilon,\mathbb{Z}}$. The algebra $A^Q_{n,\varepsilon,\mathbb{Z}}$ is a free $C^Q_{n,\varepsilon,\mathbb{Z}}$-module with basis

$$\{x_1^{j_1} \dots x_n^{j_n} y_1^{m_1} \dots y_n^{m_n} \mid j_1, \dots, j_n, m_1, \dots, m_n \in [0, \ell-1]\}.$$

Applying Theorem 6.6 gives that

$$d(A^Q_{n,\varepsilon,\mathbb{Z}}/C^Q_{n,\varepsilon,\mathbb{Z}}) =_{\mathcal{A}_\varepsilon^\times} \ell^{2n\ell^{2n}} z_1^{\ell a_1} \dots z_n^{\ell a_n} \tag{6.6}$$

for some $a_k \in \mathbb{N}$ (here and below discriminants are computed with respect to the regular trace). To determine the integers a_k, consider the filtration of $A^Q_{n,\varepsilon,\mathbb{Z}}$ given by $\deg x_k = \deg y_k = k$ for $k \in [1, n]$. The associated graded is isomorphic to a skew-polynomial algebra with generators given by the images of $x_k, (\varepsilon-1)y_k$ for $k \in [1, n]$, which will be denoted by $\overline{x_k}, \overline{(\varepsilon-1)y_k}$. The discriminants of skew-polynomial algebras are given by [8, Proposition 2.8]:

$$d(\mathrm{gr}A^Q_{n,\varepsilon,\mathbb{Z}}/\mathrm{gr}C^Q_{n,\varepsilon,\mathbb{Z}}) =_{\mathcal{A}_\varepsilon^\times} \ell^{2n\ell^{2n}}(\overline{x_1}\overline{(\varepsilon-1)y_1})^{(\ell-1)\ell^n}\dots(\overline{x_n}\overline{(\varepsilon-1)y_n})^{(\ell-1)\ell^n}.$$

Applying [8, Proposition 4.10] to (6.6) gives that $a_1 = \dots = a_n = (\ell-1)\ell^{n-1}$, which proves the following proposition. It recovers results in [9, 31].

Proposition 6.8. *For each root of unity $\varepsilon^{1/2}$ of odd order ℓ, the discriminant of the quantized Weyl algebra $A^Q_{n,\varepsilon,\mathbb{Z}}$ over its central subalgebra $C^Q_{n,\varepsilon,\mathbb{Z}}$ with respect to the regular trace is given by*

$$d\left(A^Q_{n,\varepsilon,\mathbb{Z}}/C^Q_{n,\varepsilon,\mathbb{Z}}\right) =_{\mathcal{A}_\varepsilon^\times} \ell^{2n\ell^{2n}} z_1^{(\ell-1)\ell^n}\dots z_n^{(\ell-1)\ell^n}.$$

7 | QUANTUM GROUPS

In this section, we gather material about quantized universal enveloping algebras of symmetrizable Kac–Moody algebras, their integral forms and specializations to roots of unity.

7.1 | Quantized universal enveloping algebras

We will follow the notation of Kashiwara for quantized universal enveloping algebras of symmetrizable Kac–Moody algebras, [28]. Let $I := [1, r]$ serve as an index set and $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum composed of the following.

(i) A symmetrizable, generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. In particular, $a_{ii} = 2$ for $i \in I$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$, and there exists a diagonal matrix $D = (d_i)_{i\in I}$ consisting of positive, relatively prime integers d_i such that DA is symmetric.
(ii) A free abelian group P (*weight lattice*).
(iii) A linearly independent subset $\Pi = \{\alpha_i \mid i \in I\} \subset P$ (*set of simple roots*).
(iv) The dual group $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ (*coweight lattice*).
(v) Two linearly independent subsets $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ (*set of simple coroots*), such that $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$, and $\{\varpi_i \mid i \in I\} \subset P$ (*set of fundamental weights*), such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$ for $i, j \in I$.

Let $P_+ := \{\gamma \in P \mid \langle h_i, \gamma\rangle \in \mathbb{Z}_{\geqslant 0}\}$. Denote the root lattice $Q := \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ and set $Q_+ := \bigoplus_{i\in I}\mathbb{Z}_{\geqslant 0}\alpha_i$. Set $\mathfrak{h} := \mathbb{Q}\otimes_{\mathbb{Z}} P^\vee$. There is a $\mathbb{Q}$-valued nondegenerate, symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^* = \mathbb{Q}\otimes_{\mathbb{Z}} P$ that satisfies

$$\langle h_i, \mu\rangle = \frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad (\alpha_i, \alpha_i) = 2d_i \quad \text{for all } \ i \in I, \mu \in \mathfrak{h}^*.$$

Note that the existence of such a bilinear form is equivalent to the symmetrizability of the generalized Cartan matrix A. Denote $\|\mu\| := (\mu, \mu)$ for $\mu \in \mathfrak{h}^*$.

Let $\mathfrak{g}$ be the symmetrizable Kac–Moody algebra over $\mathbb{Q}$ associated to this Cartan datum. It is the Lie algebra generated by $\mathfrak{h}$, e_i, and f_i for $i \in I$ with Serre relations for $h \in \mathfrak{h}$ and $i, j \in I$,

$$\mathfrak{h} \text{ is an abelian Lie subalgebra,}$$

$$[h, e_i] = \langle h, \alpha_i \rangle e_i, \quad [h, f_i] = -\langle h, \alpha_i \rangle f_i, \quad [e_i, f_j] = \delta_{ij} h_i,$$

$$(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0.$$

Let W be the Weyl group of $\mathfrak{g}$, acting on $(\mathfrak{h}^*, (\cdot,\cdot))$ by isometries. Denote its generators by s_i for $i \in I$. The length function on W will be written as $l : W \to \mathbb{Z}_{\geqslant 0}$. The Bruhat order will be denoted by $\geq$. Let $\Delta_+ \subset Q_+$ be the set of positive roots of $\mathfrak{g}$.

Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ denote the Lie subalgebras of $\mathfrak{g}$ generated by $\{e_i \mid i \in I\}$ and $\{f_i \mid i \in I\}$. So,

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^{\pm\alpha},$$

where $\mathfrak{g}^\alpha$ is the root space in $\mathfrak{g}$ corresponding to α. The root spaces are one-dimensional for real roots; that is roots in $W\{\alpha_i \mid i \in I\}$. For $w \in W$, we denote the nilpotent Lie subalgebras

$$\mathfrak{n}_\pm(w) \ := \bigoplus_{\alpha \in \Delta_+ \cap w^{-1}(-\Delta_+)} \mathfrak{g}^{\pm\alpha}.$$

If w has a reduced expression $w = s_{i_1} \cdots s_{i_N}$, then $\mathfrak{n}_+(w)$ is generated by the root vectors corresponding to the real roots $\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N})$.

Let $U_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra defined over $\mathbb{Q}(q)$, which is generated by e_i, f_i, and q^h for $i \in I$, $h \in \mathfrak{h}$ subject to the relations

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i,$$

$$[e_i, f_i] = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q_i - q_i^{-1}},$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_i e_i^{1-a_{ij}-s} e_j e_i^s = 0, \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_i f_i^{1-a_{ij}-s} f_j f_i^s = 0, \quad i \neq j$$

for $h, h' \in \mathfrak{h}$, $i, j \in I$, where

$$q_i = q^{d_i}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = [n]_i \dots [1]_i, \quad \text{and} \quad \begin{bmatrix} n \\ s \end{bmatrix}_i = \frac{[n]_i!}{[n-s]_i!\,[s]_i!}.$$

The standard Hopf algebra structure on $U_q(\mathfrak{g})$ has counit, coproduct, and antipode given by

$$\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0,$$

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + q^{d_i h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-d_i h_i} + 1 \otimes f_i,$$

$$S(q^h) = q^{-h}, \quad S(e_i) = -q^{-d_i h_i} e_i, \quad S(f_i) = -f_i q^{d_i h},$$

where $h \in P^\vee$ and $i \in I$. The unital subalgebras generated by $\{e_i \mid i \in I\}$, $\{q^h \mid h \in P^\vee\}$, and $\{f_i \mid i \in I\}$ will be denoted by $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{h})$, and $U_q(\mathfrak{n}_-)$. The algebras $U_q(\mathfrak{b}_\pm) := U_q(\mathfrak{n}_\pm)U_q(\mathfrak{h})$ are Hopf subalgebras of $U_q(\mathfrak{g})$.

For a $U_q(\mathfrak{g})$-module V and $\mu \in P$, denote the root space $V_\mu := \{v \in V \mid q^h \cdot v = q^{\langle h,\mu\rangle} v, \forall h \in P^\vee\}$.

Let $\{T_i \mid i \in I\}$ be the standard generators of the braid group of W. For a reduced expression $s_{i_1} \dots s_{i_N}$ of $w \in W$, let $T_w := T_{i_1} \dots T_{i_N}$ in the braid group of W (this element is independent on choice of reduced expression). We use the same notation for Lusztig's braid group action [32] on $U_q(\mathfrak{g})$ and on integrable $U_q(\mathfrak{g})$-modules (i.e., modules V on which e_i and f_i act locally nilpotent for $i \in I$ and $V = \bigoplus_{\mu \in P} V_\mu$). For $\mu \in P_+$, let $V(\mu)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight μ, and v_μ be a highest weight vector of it. For $w \in W$, denote $v_{w\mu} = T^{-1}_{w^{-1}} v_\mu$. In $(V(\mu)_{w\mu})^*$, let $\xi_{w\mu}$ be such that $\langle \xi_{w\mu}, v_{w\mu}\rangle = 1$. The quantum minors (viewed as functionals on $U_q(\mathfrak{g})$) are defined as the matrix coefficients $\Delta_{u\mu,w\mu} := c_{\xi_{u\mu}, v_{w\mu}}$ for $u, w \in W$ and $\mu \in P_+$. Note that $\Delta_{u\mu,w\mu}\Delta_{u\nu,w\nu} = \Delta_{u(\mu+\nu),w(\mu+\nu)}$ because $T^{-1}_{w^{-1}}(v_\mu \otimes v_\nu) = T^{-1}_{w^{-1}} v_\mu \otimes T^{-1}_{w^{-1}} v_\nu$.

7.2 | Hopf pairings and integral forms

Recall that a Hopf pairing between Hopf $\mathbb{K}$-algebras A and H is a bilinear form $(\cdot,\cdot) : A \times H \to \mathbb{K}$ such that

(1) $(ab, h) = (a, h_{(1)})(b, h_{(2)})$

(2) $(a, gh) = (a_{(1)}, g)(a_{(2)}, h)$

(3) $(a, 1) = \varepsilon_A(a)$ and $(1, h) = \varepsilon_H(h)$

for all $a, b \in A$ and $g, h \in H$ in terms of Sweedler notation.

Let $d \in \mathbb{Z}_+$ be an integer such that $(P^\vee, P^\vee) \subseteq \frac{1}{d}\mathbb{Z}$. The Rosso–Tanisaki form $(\cdot,\cdot)_{RT} : U_q(\mathfrak{b}_-) \times U_q(\mathfrak{b}_+) \to \mathbb{Q}(q^{1/d})$ is the Hopf pairing defined by

$$(f_i, e_j)_{RT} = \delta_{ij}\frac{1}{q_i^{-1} - q_i}, \ (q^h, q^{h'})_{RT} = q^{-(h,h')}, \ (f_i, q^h)_{RT} = 0 = (q^h, e_i)_{RT}$$

for all $i \in [1, r]$ and $h \in P^\vee$. The Rosso–Tanisaki form has the following useful properties,

$$\begin{gathered}\left(xq^h, yq^{h'}\right)_{RT} = (x, y)_{RT} q^{-(h,h')},\\ \left(U_q(\mathfrak{n}_-), U_q(\mathfrak{n}_+)\right)_{RT} \subset \mathbb{Q}(q),\\ \text{and } \left(U_q(\mathfrak{n}_-)_{-\gamma}, U_q(\mathfrak{n}_+)_\delta\right)_{RT} = 0\end{gathered} \tag{7.1}$$

for $x \in U_q(\mathfrak{n}_-)$, $y \in U_q(\mathfrak{n}_+)$, and $\gamma, \delta \in Q_+$ with $\gamma \neq \delta$, see [27, chapter 6].

Recall (2.2) and denote

$$\mathcal{A}_q := \mathbb{Z}[q^{\pm 1}].$$

The divided power integral forms $U_q(\mathfrak{n}_+)_{\mathcal{A}_q}$ and $U_q(\mathfrak{n}_-)_{\mathcal{A}_q}$ of $U_q(\mathfrak{n}_\pm)$ are the $\mathcal{A}_q$-subalgebras generated by

$$\{e_i^k/[k]_i! \mid i \in I, k \in \mathbb{Z}_+\} \quad \text{and} \quad \{f_i^k/[k]_i! \mid i \in I, k \in \mathbb{Z}_+\}.$$

The dual integral form $U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}_q}$ of $U_q(\mathfrak{n}_-)$ is defined as

$$U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}_q} := \{x \in U_q(\mathfrak{n}_-) \mid (x, U_q(\mathfrak{n}_+)_{\mathcal{A}_q})_{RT} \subset \mathcal{A}_q\}.$$

7.3 | Quantum Schubert cells

Fixing a Weyl group element and a reduced expression $w = s_{i_1} \dots s_{i_N}$, we denote the following elements of W:

$$w_{\leqslant k} := s_{i_1} \dots s_{i_k},\ w_{[j,k]} := s_{i_j} \dots s_{i_k},\ w^{-1}_{\leqslant k} := (w_{\leqslant k})^{-1},\ \text{and}\ w^{-1}_{[j,k]} := (w_{[j,k]})^{-1}$$

where $0 \leqslant j \leqslant k \leqslant N$. To each root $\beta_k := w_{\leqslant k-1}(\alpha_{i_k}) \in Q_+$ for $k \in [1, N]$, associate the root vectors

$$e_{\beta_k} := T^{-1}_{w^{-1}_{\leqslant k-1}}(e_{i_k}) \in U_q(\mathfrak{n}_+)_{\mathcal{A}_q} \quad \text{and} \quad f_{\beta_k} := T^{-1}_{w^{-1}_{\leqslant k-1}}(f_{i_k}) \in U_q(\mathfrak{n}_-)_{\mathcal{A}_q}.$$

The *quantum Schubert cells* $U_q(\mathfrak{n}_+(w))$ and $U_q(\mathfrak{n}_-(w))$ are defined to be the unital $\mathbb{Q}(q)$-subalgebras of $U_q(\mathfrak{n}_\pm)$ generated by $e_{\beta_1}, \dots, e_{\beta_N}$ and $f_{\beta_1}, \dots, f_{\beta_N}$, respectively. They were defined by De Concini–Kac–Procesi [12] and Lusztig [32], who considered the anti-isomorphic algebras $U_q^\pm[w] =* \left(A_q(\mathfrak{n}_\pm(w))\right)$. It was proved in [2, 30, 39] that

$$U_q\left(\mathfrak{n}_\pm(w)\right) = U_q\left(\mathfrak{n}_\pm\right) \cap T^{-1}_{w^{-1}}\left(U_q\left(\mathfrak{n}_\mp\right)\right).$$

The dual integral form of $U_q(\mathfrak{n}_-(w))$ is the $\mathcal{A}_q$-algebra

$$U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q} := U_q(\mathfrak{n}_-(w)) \cap U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}_q}.$$

The dual PBW generators of $U_q(\mathfrak{n}_-(w))$ are given by

$$f'_{\beta_k} := \frac{1}{(f_{\beta_k}, e_{\beta_k})_{RT}} f_{\beta_k} = (q^{-1}_{i_k} - q_{i_k}) f_{\beta_k} \in U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q}$$

for $k \in [1, N]$. Kimura proved [30, Proposition 4.26, Theorems 4.25 and 4.27] that

$$U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q} = \oplus_{m_1,\dots,m_N \in \mathbb{N}} \mathcal{A}_q \cdot (f'_{\beta_1})^{m_1} \cdots (f'_{\beta_N})^{m_N}. \tag{7.2}$$

7.4 | Quantum unipotent cells

Let $A_q(\mathfrak{n}_+)$, as in [21], denote the subalgebra of the full dual $U_q(\mathfrak{b}_+)^*$ of elements f that satisfy the following conditions.

(1) $f(yq^h) = f(y)$ for any $y \in U_q(\mathfrak{n}_+)$ and $h \in P^\vee$.

(2) There is a finite subset $S \subseteq Q_+$, such that $f(x) = 0$ for all $x \in U_q(\mathfrak{n}_+)_\gamma$ for $\gamma \in Q_+ \setminus S$.

The map $\iota : U_q(\mathfrak{n}_-) \to U_q(\mathfrak{b}_+)^*$ given by

$$\langle \iota(x), y\rangle = (x, y)_{RT} \quad \text{for all} \quad x \in U_q(\mathfrak{n}_-),\ y \in U_q(\mathfrak{b}_+)$$

is an algebra homomorphism because the Rosso–Tanisaki form is a Hopf pairing. The image of ι is contained in $A_q(\mathfrak{n}_+)$ by the properties listed in (7.1). As the Rosso–Tanisaki form is nondegenerate, ι is an isomorphism onto $A_q(\mathfrak{n}_+)$,

$$\iota : U_q(\mathfrak{n}_-) \xrightarrow{\simeq} A_q(\mathfrak{n}_+).$$

Following Geiß–Leclerc–Schröer [21], define the *quantum unipotent cell* $A_q(\mathfrak{n}_+(w)) \subseteq A_q(\mathfrak{n}_+)$ as the image of $U_q(\mathfrak{n}_-(w)) \subseteq U_q(\mathfrak{n}_-)$ under ι,

$$\iota : U_q(\mathfrak{n}_-(w)) \xrightarrow{\simeq} A_q(\mathfrak{n}_+(w)) \subset A_q(\mathfrak{n}_+).$$

The images of the elements of $U_q(\mathfrak{n}_-(w))$ in $A_q(\mathfrak{n}_+(w))$ will be denoted by the same symbols. We transport the automorphisms T_i via ι to a partial braid group action on $A_q(\mathfrak{n}_+(w))$. Quantum unipotent cells also inherit a Q_+-grading

$$A_q(\mathfrak{n}_+(w))_\gamma := \iota\big(U_q(\mathfrak{n}_-(w))_{-\gamma}\big) \quad \text{for all} \quad \gamma \in Q_+. \tag{7.3}$$

Finally, the dual integral form of $U_q(\mathfrak{n}_-(w))$ gives rise to an $\mathcal{A}_q$-integral form of the quantum unipotent cell $A_q(\mathfrak{n}_+(w))$,

$$A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q} := \iota\Big(U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q}\Big).$$

The restriction of ι gives rise to the $\mathcal{A}_q$-algebra isomorphism

$$\iota : U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q} \xrightarrow{\simeq} A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q}. \tag{7.4}$$

The integral forms $U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}_q}$, $U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}_q}$ and $A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q}$ are often defined by using the Kashiwara [28] and Lusztig [32] bilinear forms on $U_q(\mathfrak{n}_-)$ instead of the Rosso–Tanisaki form. However, the corresponding $\mathcal{A}_q$-algebras are isomorphic [26, Remark 5.3].

Following [21], define the *unipotent quantum minors* of $A_q(\mathfrak{n}_+(w))$ for $u \in W$, $\mu \in P_+$ as the elements of $A_q(\mathfrak{n}_+(w))_{(u-w)\mu}$ such that

$$\langle D_{u\mu,w\mu}, yq^h\rangle := \langle \xi_{u\mu}, yv_{w\mu}\rangle$$

for all $y \in U_q(\mathfrak{n}_+)$ and $h \in P^\vee$. The quantum minors $\Delta_{u\mu,w\mu} \in A_q(\mathfrak{g})$ only depend on $u\mu$ and $w\mu$ but not on the individual choice of w, u and μ, [4, section 9.3]. As the unipotent minors $D_{u\mu,w\mu}$ can be realized as homomorphic images of them [26, section 6.3], the same is true for them. The minors $D_{\mu,w\mu}$ q-commute with homogeneous elements with respect to the Q_+-grading [26, eq. (6.9)]:

$$D_{\mu,w\mu}x = q^{((w+1)\mu,\gamma)}xD_{\mu,w\mu}, \quad \forall \mu \in P_+, x \in A_q(\mathfrak{n}_+(w))_\gamma, \gamma \in Q_+. \tag{7.5}$$

7.5 | Specialization to roots of unity

Recall (3.1) and denote

$$\mathcal{A}_\varepsilon := \mathbb{Z}[\varepsilon].$$

For every symmetrizable Kac–Moody algebra $\mathfrak{g}$ and Weyl group element $w \in W$, define the (*integral*) *quantum unipotent cell at root of unity* to be the $\mathcal{A}_\varepsilon$-algebra

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} := A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q}/(\Phi_\ell(q)).$$

Denote the canonical projection

$$\eta_\varepsilon : A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q} \twoheadrightarrow A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} \tag{7.6}$$

and for $j \in [1, N]$ set

$$f''_{\beta_j} := \eta_\varepsilon \iota(f'_{\beta_j}) \in A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}. \tag{7.7}$$

By Kimura's result in (7.2) and the isomorphism (7.4), we have

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} = \oplus_{m_1,\dots,m_N \in \mathbb{N}} \mathcal{A}_\varepsilon \cdot (f''_{\beta_1})^{m_1} \cdots (f''_{\beta_N})^{m_N}. \tag{7.8}$$

Theorem 7.1 (De Concini–Kac–Procesi [12]). *For every symmetrizable Kac–Moody algebra $\mathfrak{g}$, Weyl group element $w \in W$, and primitive ℓth root of unity ε such that ℓ is coprime to $\{d_i \mid i \in I\}$,*

$$C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} = \oplus_{m_1,\dots,m_N \in \mathbb{N}} \mathcal{A}_\varepsilon \cdot (f''_{\beta_1})^{m_1\ell} \cdots (f''_{\beta_N})^{m_N\ell}$$

is a central $\mathcal{A}_\varepsilon$-subalgebra of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$.

The theorem was proved in [12] in the case when $\mathfrak{g}$ is finite-dimensional, but the same proof works for general symmetrizable Kac–Moody algebras. Alternatively, in the case when ℓ is odd, this theorem also follows by combining Proposition 4.4 and Theorem 8.5 (we note that the proof of Theorem 8.5 does not use Theorem 7.1).

8 | DISCRIMINANTS OF QUANTUM UNIPOTENT CELLS AT ROOTS OF UNITY

In this section, we obtain an explicit formula for the discriminant of each (integral) quantum unipotent cell $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ over the central subalgebra $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ for every symmetrizable Kac–Moody algebra $\mathfrak{g}$ and Weyl group element w. It is also proved that the algebras $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ possess a strict root of unity quantum cluster algebra structure. In this picture, we give an intrinsic interpretation of the central subalgebras $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ in cluster algebra terms.

8.1 | Theorem on discriminants of quantum unipotent cells

For a Weyl group element w denote its support $\mathcal{S}(w) := \{i \in I \mid s_i \leqslant w\}$.

It follows from (7.8) and the definition of $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ that $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ is a free module over $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ of rank ℓ^N with basis

$$\{(f''_{\beta_1})^{m_1} \dots (f''_{\beta_N})^{m_N} \mid m_1, \dots, m_N \in [0, \ell - 1]\}. \tag{8.1}$$

The corresponding discriminant is given by:

Theorem 8.1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, w be a Weyl group element with a reduced expression $w = s_{i_1} \dots s_{i_N}$, and $\ell > 2$ be an odd integer which is coprime to d_i for all $i \in \mathcal{S}(w)$. Let ε be a primitive ℓth root of unity. Then*

$$d\Big(A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}/C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}\Big) =_{\mathcal{A}_\varepsilon^\times} \ell^{(N\ell^N)} \prod_{i\in\mathcal{S}(w)} \eta_\varepsilon(D_{\varpi_i, w\varpi_i})^{\ell^N(\ell-1)}.$$

Note that, as $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ is a polynomial algebra over $\mathcal{A}_\varepsilon$, $C_\varepsilon(\mathfrak{n}_+(w))^\times_{\mathcal{A}_\varepsilon} = \mathcal{A}_\varepsilon^\times$. The theorem is proved in Subsection 8.5.

8.2 | Cluster structures of the integral forms of quantum unipotent cells

For the construction of strict root of unity quantum cluster structure on $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$, we will use results from [26, 29] on a quantum cluster algebra structure on

$$A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}} := A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q} \otimes_{\mathcal{A}_q} \mathcal{A}_q^{1/2}.$$

Fix a reduced expression $w = s_{i_1} \dots s_{i_N}$. In terms of the support of w, it is given by $\mathcal{S}(w) = \{t \in I \mid t = i_k \text{ for some } k\}$. Let $p : [1, N] \to [1, N-1] \cup \{-\infty\}$ and $s : [1, N] \to [2, N] \cup \{\infty\}$ be the predecessor and successor maps given by

$$p(k) = \max\{j < k \mid i_j = i_k\} \text{ where } \max \varnothing := -\infty,$$

$$s(k) = \min\{j > k \mid i_j = i_k\} \text{ where } \min \varnothing := \infty.$$

The mutable directions in the cluster structure will be given by the subset

$$\mathbf{ex}(w) := \{k \in [1, N] \mid i_j = i_k \text{ for } j > k\}.$$

It has cardinality $|\mathbf{ex}(w)| = N - |\mathcal{S}(w)|$ as each $t \in \mathcal{S}(w)$ in the support will have only one $j \in [1, N]$ such that $i_j = t$ and $s(j) = \infty$. Let $\widetilde{B}^w$ be the $N \times \mathbf{ex}(w)$ matrix with entries

$$(\widetilde{B}^w)_{j,k} = \begin{cases} 1, & \text{if } j = p(k) \\ -1, & \text{if } j = s(k) \\ a_{i_j i_k} & \text{if } j < k < s(j) < s(k) \\ -a_{i_j i_k} & \text{if } k < j < s(k) < s(j) \\ 0, & \text{otherwise.} \end{cases}$$

The principal part B^w is skew-symmetrizable by the matrix $D := \operatorname{diag}(d_{i_j}, j \in \mathbf{ex}(w))$. Moreover, $\widetilde{B}^w$ is compatible with the skew-symmetric $N \times N$ matrix

$$(\Lambda_w)_{j,k} := -\big((w_{\leqslant j} + 1)\varpi_{i_j}, (w_{\leqslant k} - 1)\varpi_{i_k}\big), \quad \text{for } 1 \leqslant j < k \leqslant N,$$

see [26, Proposition 7.2]. By (7.5), the unipotent quantum minors $D_{\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}}$, with weight $(1 - w_{\leqslant k})\varpi_{i_k}$, q-commute among themselves:

$$D_{\varpi_{i_j}, w_{\leqslant j}\varpi_{i_j}} D_{\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}} = q^{(\Lambda_w)_{j,k}} D_{\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}} D_{\varpi_{i_j}, w_{\leqslant j}\varpi_{i_j}}, \quad 1 \leqslant j < k \leqslant N.$$

There is a unique toric frame $M_q^w : \mathbb{Z}^N \to \operatorname{Fract}(A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}}) \simeq \operatorname{Fract}(\mathcal{T}_q(\Lambda_w))$, with corresponding skew-symmetric matrix Λ_w, given by

$$M_q^w(e_k) = q^{a[1,k]} D_{\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}} \quad \text{for any } k \in [1, N]$$

where

$$a[j,k] = \|(w_{[j,k]} - 1)\varpi_{i_k}\|^2/4 \in \tfrac{1}{2}\mathbb{Z}. \tag{8.2}$$

The above facts show that $(\Lambda_w, \widetilde{B}^w)$ is a compatible pair and that $(M_q^w, \widetilde{B}^w)$ is a quantum seed. The following theorem is proved in [26] and in [29] in the case of symmetric Kac–Moody algebras.

Theorem 8.2. *Let $\mathfrak{g}$ be any symmetrizable Kac–Moody algebra and $w \in W$ a Weyl element with a reduced expression $w = s_{i_1} \dots s_{i_N}$. Then the integral form of the corresponding quantum unipotent cells has a cluster structure, $A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}} \simeq \mathbf{A}_q(M_q^w, \widetilde{B}^w, \varnothing)$.*

Denote by Ξ_N the subset of the symmetric group S_N consisting of permutations σ such that $\sigma([1,k])$ is an interval for $1 \leqslant k \leqslant N$. We can combinatorially describe this subset in terms of one-line notation for the elements of S_N: first move 1 as far right as desired, then move 2 as far right as desired up to where 1 now is, then moving 3 right possibly up to 2, and so on. The elements of S_N obtained in this way are precisely those of Ξ_N. The following diagram illustrates this with arrows denoting pairs of elements of Ξ_N obtained from each other by a transposition:

$$\begin{array}{ccccccccc}
[1\,2\,3\,4 \dots N] & \to & [2\,1\,3\,4 \dots N] & \to & [2\,3\,1\,4 \dots N] & \to & [2\,3\,4\,1 \dots N] & \to & \dots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & [3\,2\,1\,4 \dots N] & \to & [3\,2\,4\,1 \dots N] & \to & \dots \\
 & & & & & & \downarrow & \searrow & \\
 & & & & & & \vdots & & \ddots
\end{array}$$

For each $\sigma \in \Xi_N$, [26, Theorem 7.3(b)] constructs a quantum seed of $\mathbf{A}_q(M_q^w, \widetilde{B}^w, \varnothing)$. Their toric frames (up to a permutation of the basis as below) have cluster variables

$$M_{q,\sigma}^w(e_l) = q^{a[j,k]} D_{w_{\leqslant j-1}\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}} = q^{a[j,k]} T_{w_{\leqslant j-1}} D_{\varpi_{i_k}, w_{[j,k]}\varpi_{i_k}}, \tag{8.3}$$

where $j = \min\{m \in \sigma([1,l]) \mid i_m = i_{\sigma(l)}\}$, $k = \max\{m \in \sigma([1,l]) \mid i_m = i_{\sigma(l)}\}$ and $a[j,k]$ are given by (8.2). In particular, $M_{q,\mathrm{id}}^w = M_q^w$. The exchange matrices of these seeds will not play a role in this paper. By abuse of notation, we will denote by Ξ_N this collection of quantum seeds of $\mathbf{A}_q(M_q^w, \widetilde{B}^w, \varnothing)$.

By [26, Theorem 7.3(c)], this collection of quantum seeds of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}}$ is linked by mutations as follows. Let $\sigma, \sigma' \in \Xi_N$ be such that $\sigma' = (\sigma(k), \sigma(k+1)) \circ \sigma = \sigma \circ (k, k+1)$ for $k \in [1, N-1]$.

$$\begin{aligned} &\text{If } i_{\sigma(k)} \neq i_{\sigma(k+1)}, \text{ then } M^w_{q,\sigma'} = M^w_{q,\sigma} \cdot (k, k+1); \\ &\text{If } i_{\sigma(k)} = i_{\sigma(k+1)}, \text{ then } M^w_{q,\sigma'} = \mu_k(M^w_{q,\sigma}), \end{aligned} \tag{8.4}$$

where we use the canonical action of S_N on quantum seeds and toric frames by reordering of basis elements given by $M_q \cdot \sigma(e_j) := M_q(e_{\sigma(j)})$ for $\sigma \in S_N$ and $1 \leqslant j \leqslant N$.

The following lemma is simple and is left to the reader:

Lemma 8.3. *The collection of quantum seeds Ξ_N of $\mathbf{A}_q(M_q^w, \widetilde{B}^w, \varnothing)$ is a nerve.*

8.3 | Root of unity quantum cluster structure on integral quantum unipotent cells

Assume that $\varepsilon^{1/2}$ is a primitive ℓth root of unity. Denote

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} := A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} \otimes_{\mathcal{A}_\varepsilon} \mathcal{A}_\varepsilon^{1/2}$$

In the case when ℓ is odd, ε is also a primitive ℓth root of of unity, $\mathcal{A}_\varepsilon^{1/2} = \mathcal{A}_\varepsilon$, and $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} \cong A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$. In the case when ℓ is even, ε is a primitive $(\ell/2)$th root of unity. Consider the canonical extension of the specialization (7.6) to a specialization map

$$\eta_\varepsilon : A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}} \twoheadrightarrow A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} \simeq A_q(\mathfrak{n}_+(w))_{\mathcal{A}_q^{1/2}} / (\Phi_\ell(q^{1/2}))$$

such that $q^{1/2} \mapsto \varepsilon^{1/2}$. By [26, Theorem 7.3(a)]

$$\mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing) = \mathbf{U}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing),$$

so we are in a position to apply Theorem 5.11. First, this gives that the maps

$$M^w_{\varepsilon,\sigma} := \eta_\varepsilon \circ M^w_{q,\sigma} : \mathbb{Z}^N \to A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}}$$

are toric frames for all $w \in W$ and $\sigma \in \Xi_N$. Second, we obtain that

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} \simeq \mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing).$$

This leads to the following theorem:

Theorem 8.4. *For every symmetrizable Kac–Moody algebra $\mathfrak{g}$, a Weyl group element w with a reduced expression $w = s_{i_1} \dots s_{i_N}$, and a primitive ℓth root of unity $\varepsilon^{1/2}$ for $\ell \in \mathbb{Z}_+$, the following hold.*

(1) *$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}}$ has the structure of strict root of unity quantum cluster algebra and is isomorphic to $\mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing)$.*
(2) *The root of unity quantum cluster algebra in part (1) has seeds indexed by $\sigma \in \Xi_N$ with toric frames $M^w_{\varepsilon,\sigma}$. By abuse of notation, this collection of seeds will be denoted by Ξ_N.*
(3) *The collection of seeds Ξ_N is a nerve and we have the mutation formulae (8.4) between them with $M^w_{q,\sigma}$ replaced by $M^w_{\varepsilon,\sigma}$.*
(4) *Under the isomorphism in part (1), $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} = \mathbf{A}_\varepsilon(\Xi_N, \varnothing)$.*

Proof. Parts (1) and (2) are established above.

(3) The mutation formulae (8.4) with $M^w_{q,\sigma}$ replaced by $M^w_{\varepsilon,\sigma}$ follow from the original formulae (8.4) by applying the ring homomorphism η_ε. It follows from Lemma 8.3 that Ξ_N is a nerve.

(4) It is clear that, under the isomorphism in part (1), $\mathbf{A}_\varepsilon(\Xi_N, \varnothing) \subseteq \mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing) = A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}}$. For the inverse inclusion, note that for each $k \in [1, N]$, there exists $\sigma \in \Xi_N$ such that $\sigma(1) = k$. For that σ we have

$$M^w_{q,\sigma}(e_1) = q_{i_k}^{1/2} \iota(f'_{\beta_k})$$

by combining [26, eq. (3.6), (7.2) and Theorem 7.1(c)], and thus

$$M^w_{\varepsilon,\sigma}(e_1) = \varepsilon_{i_k}^{d_{i_k}/2} f''_{\beta_k}. \tag{8.5}$$

Hence, under the isomorphism in part (1), $\mathbf{A}_\varepsilon(\Xi_N, \varnothing) \supseteq A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}}$, which completes the proof of the theorem. □

8.4 | Identification of central subalgebras

Let ε be a primitive ℓth root of unity such that ℓ is odd and coprime to the symmetrizing integers d_i for the Kac–Moody algebra $\mathfrak{g}$ and $i \in \mathcal{S}(w)$, $w \in W$. Choose a square root $\varepsilon^{1/2}$ of ε such that $\varepsilon^{1/2}$ is also a primitive ℓth root of unity. Then $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^{1/2}$. By Theorem 8.4, we have the identifications

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} = A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} = \mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing) = \mathbf{A}_\varepsilon(\Xi_N, \varnothing).$$

On the one hand, we have the central subalgebra $\mathbf{C}_\varepsilon(\Xi_N, \varnothing)$ of $\mathbf{A}_\varepsilon(\Xi_N, \varnothing)$ constructed by cluster theoretic methods, see Section 6.2. On the other hand, we have the De Concini–Kac–Procesi central subalgebra $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$, see Subsection 7.5.

Theorem 8.5. *In the setting of Theorem 8.1, the canonical central subalgebra $\mathbf{C}_\varepsilon(\Xi_N, \varnothing)$ of $\mathbf{A}_\varepsilon(\Xi_N, \varnothing) = A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ coincides with the De Concini–Kac–Procesi central subalgebra $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$.*

Proof. It follows from (8.5) that $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} \subseteq \mathbf{C}_\varepsilon(\Xi_N, \varnothing)$. To show the reverse inclusion, we need to show that for all $\sigma \in \Xi_N$ and $j \in [1, N]$, $M^w_{\varepsilon,\sigma}(j)^\ell \in C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$. By (8.3), this is equivalent to

$$\eta_\varepsilon\left(D^\ell_{w_{\leqslant j-1}\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}}\right) \in C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}, \quad \forall 1 \leqslant j \leqslant k \leqslant N \ \text{ with } \ i_j = i_k. \tag{8.6}$$

We prove (8.6) by induction on $k-j$. The case $k-j=0$ is trivial because

$$\eta_\varepsilon(D_{w_{\leqslant k-1}\varpi_{k_k},w_{\leqslant k}\varpi_{i_k}}) = \varepsilon_{i_k}^{d_{i_k}/2} f''_{\beta_k}$$

by (8.5). Now assume that $k-j=t$ for some $t\in\mathbb{Z}_+$ and that the statement holds for pairs $1\leqslant j'\leqslant k'\leqslant N$ with $k'-j'<t$. As $i_j=i_k$, $j\leqslant p(k)$ and $s(j)\leqslant k$. Consider the following elements of Ξ_N:

$$\sigma = [j+1,\dots,k-1,j,k,k+1,\dots,N,1\dots,j-1] \quad \text{and}$$

$$\sigma' = [j+1,\dots,k-1,k,j,k+1,\dots N,1,\dots,j-1] = \sigma(k-j,k-j+1)$$

in the two line notation for elements of S_N. By (8.4), $M^w_{\varepsilon,\sigma'} = \mu_{k-j}M^w_{\varepsilon,\sigma}$. From [25, Theorem 6.6] we have that the $(k-j)$th column of the exchange matrix of the root of unity quantum seed of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}}$ corresponding to σ has the form $(b_1,\dots,b_N)^\top$ with

$$b_{k-j+1}=-1,\quad b_{p(k)-j}=-1 \ \text{ if } \ j\leqslant p(k),$$

$$b_i\geqslant 0 \quad \text{for } \ i<k-j, i\neq p(k)-j,$$

$$b_i=0 \quad \text{otherwise.}$$

Combining this with Proposition 4.4 gives

$$M^w_{\varepsilon,\sigma'}(e_{k-j})^\ell = (\mu_{k-j}M^w_{\varepsilon,\sigma}(e_{k-j}))^\ell = M^w_{\varepsilon,\sigma}(e_{k-j})^{-\ell}\left(M^w_{\varepsilon,\sigma}(e_{k-j+1})^\ell M^\ell + \prod_{i<k-j,b_i>0} M^w_{\varepsilon,\sigma}(e_i)^\ell\right)$$

where

$$M^\ell := \begin{cases} M^w_{\varepsilon,\sigma}(e_{p(k)-j})^\ell, & \text{if } \ j\leqslant p(k)\\ 1, & \text{otherwise.}\end{cases}$$

It follows from (8.3) that $M^w_{\varepsilon,\sigma}(e_{k-j+1})^\ell = \eta_\varepsilon(D^\ell_{w_{\leqslant j-1}\varpi_{i_k},w_{\leqslant k}\varpi_{i_k}})$ and that $M^w_{\varepsilon,\sigma'}(e_{k-j})^\ell$ and $M^w_{\varepsilon,\sigma}(e_i)^\ell$ for $i\leqslant k-j$ are of the form $\eta_\varepsilon(D^\ell_{w_{\leqslant j'-1}\varpi_{i_{k'}},w_{\leqslant k'}\varpi_{i_{k'}}})$ for pairs $1\leqslant j'\leqslant k'\leqslant N$ with $k'-j'<k-j$. The induction assumption implies that

$$\eta_\varepsilon\left(D^\ell_{w_{\leqslant j-1}\varpi_{i_k},w_{\leqslant k}\varpi_{i_k}}\right) \in \mathrm{Fract}(C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}) \cap A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}.$$

It remains to prove that

$$\mathrm{Fract}(C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}) \cap A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} = C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}. \tag{8.7}$$

Let

$$P=\sum p_{m_1,\dots,m_N}(f''_{\beta_1})^{m_1\ell}\dots(f''_{\beta_N})^{m_N\ell},\quad Q=\sum q_{m_1,\dots,m_N}(f''_{\beta_1})^{m_1\ell}\dots(f''_{\beta_N})^{m_N\ell}$$

and

$$R=\sum r_{n_1,\dots,n_N}(f''_{\beta_1})^{n_1}\dots(f''_{\beta_N})^{n_N},$$

be such that $P = RQ$. As f''_{β_i} are in the center of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$,

$$QR = \sum r_{n_1,\dots,n_N} q_{m_1,\dots,m_N} (f''_{\beta_1})^{n_1+m_1\ell} \dots (f''_{\beta_N})^{n_m+m_N\ell}$$

In light of the PBW basis (7.8), the identity $P = QR$ implies that $r_{n_1,\dots,n_N} = 0$ unless $n_1, \dots, n_N$ are divisible by ℓ. This proves (8.7). □

8.5 | Proof of Theorem 8.1

As in the previous subsection, we chose a square root $\varepsilon^{1/2}$ of ε such that $\varepsilon^{1/2}$ is also a primitive ℓth root of unity. In particular, $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^{1/2}$. By Theorems 8.4 and 8.5, we have the identifications

$$A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} = A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon^{1/2}} = \mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing) = \mathbf{A}_\varepsilon(\Xi_N, \varnothing) \quad \text{and}$$

$$\mathbf{C}_\varepsilon(\Xi_N, \varnothing) = C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}.$$

As we are requiring that ℓ is coprime to all d_{i_k} for $1 \leqslant k \leqslant N$, the root of unity quantum seeds of $\mathbf{A}_\varepsilon(M_\varepsilon^w, \Lambda_w, \widetilde{B}^w, \varnothing)$ satisfy condition (**Coprime**). Its frozen variables are

$$M_\varepsilon^w(e_k) = \varepsilon^{a[1,k]}\eta_\varepsilon(D_{\varpi_{i_k}, w_{\leqslant k}\varpi_{i_k}}) = \varepsilon^{a[1,k]}\eta_\varepsilon(D_{\varpi_{i_k}, w\varpi_{i_k}}) \quad \text{for} \quad k \in [1,N]\backslash \mathbf{ex},$$

where the last equality holds because $w_{\leqslant k}\varpi_{i_k} = w\varpi_{i_k}$ for $k \in [1,N]\backslash\mathbf{ex}$. By the definitions of the sets $\mathbf{ex}$ and $\mathcal{S}(w)$, up to terms in $\mathcal{A}_\varepsilon^\times$, the frozen variables are

$$\eta_\varepsilon(D_{\varpi_i, w\varpi_i}) \quad \text{for} \quad i \in \mathcal{S}(w).$$

Theorem 6.6 implies that

$$d\Big(A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} / C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}\Big) =_{\mathcal{A}_\varepsilon^\times} \ell^{(N\ell^N)} \prod_{i\in\mathcal{S}(w)} \eta_\varepsilon(D_{\varpi_i, w\varpi_i})^{n_i} \tag{8.8}$$

for some $n_i \in \mathbb{N}$. Equation (6.1) and the fact that (8.1) is a basis of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ over $C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$ imply that with respect to the Q_+-grading (7.3) of $A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}$,

$$\begin{aligned}\deg d\Big(A_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon} / C_\varepsilon(\mathfrak{n}_+(w))_{\mathcal{A}_\varepsilon}\Big) &= 2 \sum_{0\leqslant m_k \leqslant \ell-1} \deg\Big((f''_{\beta_1})^{m_1} \dots (f''_{\beta_N})^{m_N}\Big) \\ &= \ell^N(\ell-1)(\beta_1 + \dots + \beta_N).\end{aligned} \tag{8.9}$$

For $k \in [1,N]\backslash\mathbf{ex}$, let r_k is the maximal integer such that $p^{r_k}(k) \neq -\infty$. Iterating the identity $w_{\leqslant j}\varpi_{i_j} = w_{\leqslant j-1}(\varpi_{i_j} - \alpha_{i_j}) = w_{\leqslant p(j)}\varpi_{i_j} - \beta_j, \forall j \in [1,N]$ gives

$$\beta_{p^{m_k}(k)} + \dots + \beta_k = (1 - w_{\leqslant k})\varpi_{i_k} = (1-w)\varpi_{i_k}.$$

Therefore,

$$\beta_1 + \dots + \beta_N = \sum_{k\in[1,N]\backslash\mathbf{ex}} (\beta_{p^{r_k}(k)} + \dots + \beta_k) = \sum_{i\in\mathcal{S}(w)} (1-w)\varpi_i. \tag{8.10}$$

Combining (8.8)–(8.10) and using that $\deg \eta_\varepsilon(D_{\varpi_i, w\varpi_i}) = (1-w)\varpi_i$ leads to

$$(1-w) \sum_{i \in \mathcal{S}(w)} (n_i - (\ell - 1)\ell^N) = 0.$$

This implies that $n_i = (\ell - 1)\ell^N$ for all $i \in \mathcal{S}(w)$ because $(1-w)$ is nondegenerate on $\mathrm{Span}\{\varpi_i \mid i \in \mathcal{S}(w)\}$.

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