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Integral Equations and Operator Theory

The Square Root Problem and Subnormal Aluthge Transforms of Recursively Generated Weighted Shifts

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Abstract. For recursively generated shifts, we provide definitive answers to two outstanding problems in the theory of unilateral weighted shifts: the Subnormality Problem (**SP**) (related to the Aluthge transform) and the Square Root Problem (**SRP**) (which deals with Berger measures of subnormal shifts). We use the Mellin Transform and the theory of exponential polynomials to establish that (**SP**) and (**SRP**) are equivalent if and only if a natural functional equation holds for the canonically associated Mellin transform. For p-atomic measures with $p \leq 6$, our main result provides a new and simple proof of the above-mentioned equivalence. Subsequently, we obtain an example of a 7-atomic measure for which the equivalence fails. This provides a negative answer to a problem posed by Exner (J Oper Theory 61:419–438, 2009), and to a recent conjecture formulated by Curto et al. (Math Nachr 292:2352–2368, 2019).

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1. Introduction

Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is normal if $TT^* = T^*T$, and subnormal if it is the restriction of a normal operator to an

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invariant subspace. (Here T^* stands for the usual adjoint operator of T.) The polar decomposition of T is given by the unique representation $T = V|T|$, where $|T| := (T^*T)^{\frac{1}{2}}$ and V is a partial isometry satisfying $\ker V = \ker T$. The Aluthge transform is then given by the expression

$$\tilde{T} := |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}.$$

The Aluthge transform was introduced in [1], in order to extend several inequalities valid for hyponormal operators, and has received ample attention in the last decades.

We consider below the Hilbert space $\mathcal{H} = l^2(\mathbb{Z}_+)$, endowed with the canonical orthonormal basis $\{e_n\}_{n\in\mathbb{Z}_+}$. The unilateral (forward) shift operator W_α is defined on the canonical basis by $W_\alpha e_n := \alpha_n e_{n+1}$, where $\alpha = (\alpha_n)_{n\geq 0}$ is a given sequence of positive real numbers (called *weights*). It is well known that W_α is bounded if and only if the sequence of weights is bounded, and $\|W_\alpha\| = \sup_{n\geq 0} \alpha_n < +\infty$. Clearly, W_α is never normal.

We associate with W_α the sequence defined by

$$\gamma_0 := 1 \text{ and } \gamma_k \equiv \gamma_k(\alpha) := \alpha_0^2 \alpha_1^2 \ldots \alpha_{k-1}^2 \text{ for } k \geq 1.$$

We will say that a sequence $\gamma = (\gamma_n)_{n\geq 0}$ is a moment sequence on $K \subseteq \mathbb{R}$, or that it admits a representing measure μ supported in K, if

$$\gamma_n = \gamma_n(\mu) := \int_K t^n d\mu(t) \text{ for every } n \geq 0 \text{ and } \operatorname{supp}(\mu) \subseteq K. \tag{1.1}$$

The Berger–Gellar–Wallen Theorem states that W_α is subnormal if and only if there exists a positive Borel measure μ (called *a Berger measure*), representing for γ and such that $\operatorname{supp}(\mu) \subseteq [0, \|W_\alpha\|^2]$ [4, III.8.16]. In the sequel, when such μ exists, we will also write $W_\alpha = W_\mu$, and identify the weighted shift and its Berger measure.

In the literature, it is common to refer to γ as the sequence of moments arising from the weight sequence α. Consequently, the Berger–Gellar–Wallen characterization is usually described as "W_α is subnormal if and only if the sequence γ of moments of α corresponds to the sequence of moments of a positive Borel measure μ." To avoid any possible confusion, in this paper we will reserve the phrase "moment sequence" for the sequence of moments of the measure μ. With the exception of the discussion in Sect. 6, throughout the rest of the paper our basic weighted shift W_α will be subnormal, and we will seek necessary and sufficient conditions for the subnormality of the square root shift $W_{\sqrt{\alpha}}$ and of the Aluthge transform $\widetilde{W_\alpha}$. (Here $(\sqrt{\alpha})_k := \sqrt{\alpha_k}$ $(k \geq 0)$).

In the case where μ is finitely atomic (that is, $\operatorname{supp}(\mu)$ is a finite set), there exists a nonzero polynomial P such that $P(\mu) = 0$. In particular, the sequence $(\gamma_n)_{n\geq 0}$ satisfies a recursive relation, and the weighted shift W_μ is said to be *recursively generated*. Conversely, if a subnormal weighted shift is recursively generated, then its Berger measure is finitely atomic [6, Remark 3.10(i)].

The Aluthge transform $\widetilde{W_\alpha}$ of a weighted shift W_α is also a weighted shift, associated with the sequence $\tilde{\alpha}_n = \sqrt{\alpha_n \alpha_{n+1}}$, $n \geq 0$. Indeed, it is easy to check that $|W_\alpha|e_n = \alpha_n e_n$ and that $Ve_n = e_{n+1}$. It follows that

$$\widetilde{W_\alpha} e_n = \sqrt{\alpha_n \alpha_{n+1}} e_{n+1} = W_{\tilde{\alpha}} e_n.$$

Notice also that $\tilde{\gamma}_n^2 = \frac{1}{\alpha_0^2}\gamma_n\gamma_{n+1}$. The problem of detecting the subnormality of the Aluthge transform of a weighted shift has been considered by several authors in the two last decades; see, for example [9]. As observed above, and previously in [5], the subnormality of the square root shift $W_{\sqrt{\alpha}}$ implies both the subnormality of W_α and the subnormality of the Aluthge transform $\widetilde{W_\alpha}$. On the other hand, it is possible to find a weight sequence α such that $W_{\sqrt{\alpha}}$ is *not subnormal*, while both W_α and $\widetilde{W_\alpha}$ are subnormal (Example 5.1). This is the first occurrence in the literature of such a shift, and it exemplifies the significance and usefulness of identifying non-positive charges to represent weighted shifts in a manner resembling (1.1).

The next question, which we call the Subnormality Problem **(SP)**, has been considered in recent papers (cf. [9, Question 4.1] and [2,7,10,11]).

(SP) Under what conditions is the subnormality of a weighted shift preserved under the Aluthge transform?

The general case remains open and only some partial affirmative results have been obtained in [2,7,10,11]. For the class of moment infinitely divisible ($\mathcal{MID}$) weighted shifts (i.e., subnormal shifts W_α for which W_α^t remains subnormal for all $0 < t < 1$), it was proved in [3] that the Aluthge transform maps $\mathcal{MID}$ bijectively onto $\mathcal{MID}$, and that W_α, its Aluthge transform, and the square root shift are simultaneously either $\mathcal{MID}$ or not $\mathcal{MID}$ (cf. [3, Corollary 4.7 and Theorem 4.10]).

(SP) can be reformulated in terms of moment sequences as follows:

Given a moment sequence $(\gamma_n)_n$ on $K := [0, M]$, under what conditions is $(\sqrt{\gamma_n\gamma_{n+1}})_n$ also a moment sequence?

A direct application of Schur's product theorem implies that if $(\sqrt{\gamma_n})_n$ is a moment sequence on K, then $(\sqrt{\gamma_n\gamma_{n+1}})_n$ is also a moment sequence on K; this gives, in particular, a sufficient condition to solve **(SP)**. The question of whether the reverse implication holds is stated in several recent papers. The next conjecture appears in [7, Conjecture 4.6]:

Conjecture A. Let W_μ be a recursively generated (i.e., $\operatorname{card}\operatorname{supp}\mu < \infty$) subnormal weighted shift. Then the following statements are equivalent:

(i) $W_{\sqrt{\alpha}}$ is subnormal;
(ii) $\widetilde{W_\mu}$ is subnormal.

Until now, the treatment of **(SP)** and of Conjecture A has focused on the number of atoms in the $\operatorname{supp}\mu$; let $p := \operatorname{card}\operatorname{supp}\mu$. Using algebraic proofs, affirmative answers to Conjecture A have been obtained for $p \le 6$. The case $p = 2$ was treated in [10], and the case $p = 3, 4$ in [11] where the conjecture appears for the first time; the case $p = 5$ was answered in [7], and finally the case $p = 6$ was solved in [8].

In this paper, we provide concrete examples of p-atomic measures disproving Conjecture A, with $p \in \{7, 8, 9\}$. We also recover the case $p \le 6$ as a simple consequence of a new *purely analytic* approach. The main tools are the Mellin transform and the theory of exponential polynomials developed by J. F. Ritt in the 1920–1930s.

The organization of the paper is as follows. In the next section, we state the definition of the convolution of measures and the main conjecture related to the so-called Square Root Problem **(SRP)**: Given a positive measure μ, under what conditions does there exist a positive measure ν such that $\nu * \nu = \mu$? In Sect. 3, we exhibit the role of the Mellin transform and the main properties of exponential polynomials as central tools for **(SP)** and **(SRP)**. In Sect. 4, we apply the results obtained in Sect. 3 to the **(SP)** and **(SRP)** in the case of finitely atomic measures. Finally, Sects. 5 and 6 include some illustrative examples.

2. Square Roots of Measures

Given two positive finite measures ν and μ, let $*$ denote the multiplicative convolution, defined as follows:

$$[\nu * \mu](E) := \int_{\mathbb{R}^2} \chi_E(xy)d\nu(x)d\mu(y),$$

where χ_E denotes the characteristic function of the Borel set E.

It is easy to check that, for any $n \in \mathbb{Z}_+$,

$$\int_{\mathbb{R}} t^n d(\mu * \nu)(t) = \int_{\mathbb{R}^2} (st)^n d\mu(t)d\nu(s) = \left(\int_{\mathbb{R}} t^n d\mu(t)\right)\left(\int_{\mathbb{R}} s^n d\nu(s)\right). \quad (2.1)$$

In particular, we get $\gamma_n(\mu * \mu) = \gamma_n^2(\mu)$, where $\gamma_n(\mu) := \int_{\mathbb{R}} t^n d\mu(t)$ is the moment of μ of order n. The square root problem is usually written as follows:

(SRP): Given a positive measure μ, under what conditions does there exist a positive measure ν such that $\nu * \nu = \mu$?

In the case of compactly supported measures, and thanks to the Weierstrass density theorem and Eq. (2.1), the **(SRP)** can be stated in the next simple form:

Let $(\gamma_n)_n$ be a moment sequence. Under what conditions is $(\sqrt{\gamma_n})_n$ also a moment sequence?

The close relationship between **(SRP)** and **(SP)** has already been observed in the following proposition from [7].

Proposition 2.1. *Let W_μ be a subnormal weighted shift with associated Berger measure μ. Then $\widetilde{W_\mu}$ is subnormal if and only if there exists a $\mathbb{R}^+$-supported probability measure ν such that $\nu * \nu = \mu * t\mu$.*

It follows from the previous proposition that

$$\mu \text{ has a square root } \Rightarrow \widetilde{W_\mu} \text{ is subnormal.}$$

The question of whether the reverse implication holds is our main motivation. We are naturally led to the following recent conjecture from [7].

Conjecture A. (cf. [7, Conjecture 4.6]). Let μ be a finitely atomic Berger measure with support in $\mathbb{R}_+$. Then the following statements are equivalent:

(i) μ has a square root;
(ii) $\mu * t\mu$ has a square root.

We will now use the next technical results from [8], associated with the square root problem of measures; this will allow us to simplify our proof.

Lemma 2.2. *Let μ be a positive measure such that $0 \notin \operatorname{supp}(\mu)$ and let $s > 0$. Also, let $\mu_s(t) := \mu(st)$ be the image measures by the mapping $t \to st$, and let $\nu := a\delta_0 + \mu$, where δ_α denotes the Dirac measure with atom α. Then the following statements are equivalent:*

1. *μ (resp. $\mu * t\mu$) admits a square root;*
2. *ν (resp. $\nu * t\nu$) admits a square root;*
3. *μ_s (resp. $\mu_s * t(\mu_s)$) admits a square root.*

Without loss of generality, hereafter we will assume that $x_0 := \min(\operatorname{supp}(\mu)) = 1$; in particular, the support of μ will be contained in $[1, +\infty)$.

We conclude this section with a diagram that illustrates how various conditions for measures and charges are related to the subnormality of W_α, $W_{\sqrt{\alpha}}$ and $\widetilde{W_\alpha}$.

($\mathcal{MID}$ level):

$$W_{\sqrt{\alpha}} \text{ is } \mathcal{MID} \Longleftrightarrow W_\alpha \text{ is } \mathcal{MID} \Longleftrightarrow W_{\widetilde{\alpha}} \text{ is } \mathcal{MID}.$$

(Subnormal level: $W_\alpha \sim \mu$)

$$\boxed{W_{\sqrt{\alpha}} \text{ is subnormal}} \quad \Longleftrightarrow \quad \boxed{\mu = \nu * \nu, \text{ with } \nu \geq 0}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{\widetilde{W_\alpha} \text{ is subnormal}} \quad \overset{\text{Prop. 2.1}}{\Longleftrightarrow} \quad \boxed{\mu * t\mu = \xi * \xi, \text{ with } \xi \geq 0}$$

$\not\Downarrow$ (Ex. 5.1 (resp. Ex. 5.2))

$$\boxed{W_{\sqrt{\alpha}} \text{ is subnormal}}$$

Example 5.1 (resp. Example 5.2): There exists a 9-atomic (resp. 7-atomic) positive measure μ supported in $\mathbb{R}_+$ such that $\mu * t\mu$ has a positive square root, while μ has no positive square root. Examples 5.1 and 5.2 are the first such example known in the literature.

3. The Mellin Transform and Its Relationship to the Aluthge Transform

Let μ be a finite positive Radon measure. The *Mellin transform* $\mathcal{M}_\mu$ is defined as

$$\mathcal{M}_\mu(z) := \int_{\mathbb{R}_+^*} t^z \, d\mu(t). \tag{3.1}$$

We now let

$$\mathfrak{M}_+(\mathbb{R}_+) := \{\mu : \mu \text{ is a finite positive Radon measure supported in } \mathbb{R}_+\}.$$

Using the Perron inversion formula, as in, e.g., [13, VI Theorem 9b], one can establish that the Mellin transform is one-to-one, and thus it characterizes the measure. We also have

$$\mathcal{M}_{\mu*\nu}(z) = \int_{\mathbb{R}_+^2} (uv)^z d\mu(u)d\nu(v) = \int_{\mathbb{R}_+} u^z d\mu(u) \int_{\mathbb{R}_+} v^z d\nu(v) = \mathcal{M}_\mu(z)\mathcal{M}_\nu(z).$$

Conjecture A is then equivalent to:

Conjecture B.

$$\begin{gathered} \mathcal{M}_{\mu*t\mu}(z) = [\mathcal{M}_\nu(z)]^2 \\ \Updownarrow \\ \text{there exists } \xi \in \mathfrak{M}_+(\mathbb{R}_+) \text{ with } \mathcal{M}_\mu(z) = [\mathcal{M}_\xi(z)]^2. \end{gathered} \tag{3.2}$$

In addition, we remark that

$$\mathcal{M}_{t\mu}(z) = \int_{\mathbb{R}_+} t^{z+1} d\mu(t) = \mathcal{M}_\mu(z+1),$$

for every $z \in \mathbb{C}$ and hence

$$\mathcal{M}_{\mu*t\mu}(z) = \mathcal{M}_\mu(z)\mathcal{M}_\mu(z+1).$$

Finally, Conjecture A is also equivalent to:

Conjecture C.

$$\begin{gathered} \mathcal{M}_\mu(z)\mathcal{M}_\mu(z+1) = [\mathcal{M}_\nu(z)]^2 \\ \Updownarrow \\ \text{there exists } \xi \in \mathfrak{M}_+(\mathbb{R}_+) \text{ such that } \mathcal{M}_\mu(z) = [\mathcal{M}_\xi(z)]^2. \end{gathered} \tag{3.3}$$

For $\mu = \sum_{k\geq 0} a_k \delta_{x_k}$ a finite discrete (positive) measure with compact support in the interval $(0, +\infty)$, the Mellin transform of μ is the Dirichlet series

$$\mathcal{M}_\mu(z) = \sum_{k\geq 0} a_k {x_k}^z = \sum_{k\geq 0} a_k e^{z\ln(x_k)}, \tag{3.4}$$

which converges uniformly on every compact set of the complex plane. Indeed, for every $R > 0$ we have

$$\sum_{k\geq 0} \sup_{z\in D(0,R)} |a_k {x_k}^z| = \sum_{k\geq 0} a_k \sup_{z\in D(0,R)} e^{\Re(z)\ln x_k} \leq \sum_{k\geq 0} a_k e^{R|\ln x_k|} \leq \|\mu\| e^{R\ln M},$$

where $\|\mu\| := \sum_{k\geq 0} a_k$ stands for the total variation of μ and M is a positive number such that $|\ln x_k| \leq \ln M$, for every $k \in \mathbb{Z}_+$. In particular $\mathcal{M}_\mu(z)$ is an entire function.

To deal with our main problem, we study assertion (3.2). To this end, we need two auxiliary results.

Lemma 3.1. *Let* $\mu = \sum_{k=0}^\infty a_k \delta_{x_k}$ *be a positive compactly supported measure in* $\mathbb{R}_+$ *such that* $x_{i_0} = x_{min} := \inf(\operatorname{supp}(\mu))$ *and* $x_{i_1} = x_{max} := \sup(\operatorname{supp}(\mu))$ *are isolated in* $\operatorname{supp}(\mu)$. *Then* $\mathcal{Z}(\mathcal{M}_\mu)$, *the zero set of* $\mathcal{M}_\mu$, *has a bounded real part.*

Proof. We have

$$\mathcal{M}_\mu(z) = \sum_{k\geq 0} a_k x_k^z = a_{i_1} x_{i_1}^z \left[1 + \sum_{k\neq i_1} \frac{a_k}{a_{i_1}} \left(\frac{x_k}{x_{i_1}}\right)^z \right],$$

and since

$$\lim_{\Re(z)\to+\infty} \sum_{k\neq i_1} \left| \frac{a_k}{a_{i_1}} \left(\frac{x_k}{x_{i_1}}\right)^z \right| = \lim_{\Re(z)\to+\infty} \sum_{k\neq i_1} \frac{a_k}{a_{i_1}} e^{\ln\left(\frac{x_k}{x_{i_1}}\right)\Re(z)} = 0,$$

we deduce that $\mathcal{M}_\mu(z) \neq 0$ for $\Re(z)$ large enough, where $\Re(\cdot)$ denotes real part. Using x_{min} instead of x_{max}, we obtain similarly that $\mathcal{M}_\mu(z) \neq 0$ for $-\Re(z)$ large enough. This completes the proof. □

Proposition 3.2. *Let W_μ be a subnormal weighted shift, with μ a discrete Berger measure as in the previous lemma, and assume that $\widetilde{W_\mu}$ is also subnormal. Let $(z_k, m_k)_k$ the family of zeros, and respective multiplicities, of $\mathcal{M}_\mu(z)$. Then m_k is even for every k.*

Proof. From Eq. (3.2), we have $\mathcal{M}_\mu(z)\mathcal{M}_\mu(z+1) = [\mathcal{M}_\nu(z)]^2$. On the set $\Omega \subseteq \mathbb{C}$ where $\mathcal{M}_\mu(z)$ is holomorphic, we obtain:

$$2\frac{\mathcal{M}_\nu'(z)}{\mathcal{M}_\nu(z)} = \frac{\mathcal{M}_\mu'(z)}{\mathcal{M}_\mu(z)} + \frac{\mathcal{M}_\mu'(z+1)}{\mathcal{M}_\mu(z+1)}.$$

Using Cauchy's argument principle, we derive that

$$2m(z, \mathcal{M}_\nu) = m(z, \mathcal{M}_\mu) + m(z+1, \mathcal{M}_\mu), \tag{3.5}$$

where $m(z, f)$ is the multiplicity of the zero z in f (with $m(z, f) = 0$ if z is not a zero of f).

Seeking a contradiction, assume that the zero set of odd multiplicity $\mathcal{Z}_{odd}(\mathcal{M}_\mu)$ is nonempty and let $z \in \mathcal{Z}_{odd}(\mathcal{M}_\mu)$. From Eq. (3.5), we derive that $\{z-1, z+1\} \subset \mathcal{Z}_{odd}(\mathcal{M}_\mu)$ and thus, by induction, $z + \mathbb{Z} \subseteq \mathcal{Z}_{odd}(\mathcal{M}_\mu)$. This last statement is false, using Lemma 3.1. This completes the proof. □

We now derive the next preparatory result.

Proposition 3.3. *Let W_μ be a subnormal weighted shift, with μ a discrete Berger measure as in the previous lemma, and assume that $\widetilde{W_\mu}$ is also subnormal. Assume also that the assumptions of Lemma 3.1 are satisfied. Then*

$$\mathcal{M}_\mu(z) = [H(z)]^2, \qquad z \in \mathbb{C},$$

for some entire function H.

Proof. Given a set $I \subseteq \mathbb{Z}_+$, let $\mathcal{Z}(\mathcal{M}_\mu) := \{(z_k, m_k),\ \ k \in I\}$. Since the multiplicities of all zeros of the entire function $\mathcal{M}_\mu(z)$ are even (using a simple factorization by $(z - z_k)^{m_k}$ in the finite case, or using the Weierstrass factorization theorem in the infinite case), we obtain the desired result. □

To reach our main theorem, we need one more auxiliary result. Consistent with the prevailing terminology, we will refer to signed measures as *charges*; these are Borel measures that are not necessarily positive. Thus, a charge ξ typically admits either an atom with negative density or a Borel set E for which $\xi(E) < 0$.

Theorem 3.4. *Let W_μ be a subnormal weighted shift, with μ a discrete Berger measure, and assume that $\widetilde{W_\mu}$ is also subnormal. Then there exists a finitely atomic charge ξ supported in $[1, +\infty)$ such that*

$$H(z) = \mathcal{M}_\xi(z),$$

where H is an entire function satisfying the equation $\mathcal{M}_\mu(z) = [H(z)]^2$, given by Proposition 3.3. In particular, $\xi * \xi = \mu$.

Proof. Write $\mu = \sum_{k=0}^{p} a_k \delta_{x_k}$, with $\operatorname{supp}(\mu) = \{1 = x_0 < x_1 < \cdots < x_p\}$.

The Mellin transform $\mathcal{M}_\mu$ is an exponential polynomial with nonnegative exponents:

$$\mathcal{M}_\mu(z) = \sum_{k=0}^{p} a_k {x_k}^z = \sum_{k=0}^{p} a_k e^{z \ln(x_k)}.$$

From the previous discussion, there exists an entire function H satisfying $H(z)^2 = \sum_{k=0}^{p} a_k {x_k}^z$, for all $z \in \mathbb{C}$.

Next, we use a suitable version of a theorem due to J.F. Ritt. As a consequence, we prove that the square root of a positive Borel measure always exists, if we allow charges as solutions. We briefly pause the proof to state this result.

Ritt's Theorem ([12, Theorems I and II]) Let P_k be exponential polynomials and f be an analytic solution, in a sector with opening greater than π, of the equation

$$f^n + P_{n-1} f^{n-1} + \cdots + P_0 = 0.$$

Then f is also an exponential polynomial, whose exponents are linear combinations of the exponents in the P_k's, and with rational coefficients.

From Ritt's Theorem, and from the equation

$$H(z)^2 = \sum_{k=0}^{p} a_k {x_k}^z := -P_0,$$

it follows that $H(z)$ is also an exponential polynomial. That is, there exist $b_i \in \mathbb{C}$ and $y_i \in \mathbb{R}$, such that $H(z) = \sum_{k=0}^{q} b_k e^{z y_k}$. (Here $\{y_0 < y_1 < \cdots < y_q\}$ are linear combinations of $\ln(x_k)$ with rational coefficients; in particular, all y_k's are real numbers). Moreover, using the uniqueness of the representation of exponential polynomials, we get

$$\{y_k + y_l,\ 0 \le k, l \le q\} = \{\ln(x_0) < \ln(x_1)) < \cdots < \cdots < \ln(x_p)\} \subseteq \mathbb{R}_+.$$

Since $2y_0 = \ln(x_0) = 0$, we obtain $\{y_0 < y_1 < \cdots < y_q\} \subset \mathbb{R}_+$.

Finally, $\xi = \sum_{k=0}^{q} b_k \delta_{e^{y_k}}$ is a charge satisfying $\xi * \xi = \mu$, and such that *supp*$(\xi) = \{e^{y_0}, \ldots, e^{y_q}\} \subset [1, +\infty)$. □

4. Applications to (SRP)

We begin with the following observation.

Proposition 4.1. *(i) Under the notations above, if ξ_1 and ξ_2 are square roots of μ, then $\xi_1 = \pm\xi_2$. (ii) Let ξ be a signed square root of a finitely atomic measure μ. Then μ has a positive square root if and only if the coefficients in ξ have constant sign.*

Proof. (i) If $\xi_1 * \xi_1 = \xi_2 * \xi_2$, then $\mathcal{M}_{\xi_1}(z)^2 = \mathcal{M}_{\xi_2}(z)^2$ for all $z \in \mathbb{C}$. Since $\mathcal{M}_{\xi_1}$ and $\mathcal{M}_{\xi_2}$ are entire functions, we deduce that $\mathcal{M}_{\xi_1}(z) = \pm\mathcal{M}_{\xi_2}(z) = \mathcal{M}_{\pm\xi_2}(z)$, and since the Mellin transform is one-to-one, we get $\xi_1 = \pm\xi_2$.

(ii) From Theorem 3.4, μ always admits a charge ξ as a square root. Because of (i), μ has, as two square roots, i.e., $\pm\xi$. Thus, μ has a positive square root ($\xi \geq 0$ or $-\xi \geq 0$) if and only if the densities in ξ have constant sign. □

In the sequel, we focus on positive measures μ such that $\mu * t\mu$ has a positive square root. Let us first consider ν a signed (i.e., not necessarily positive) square root of $\mu * t\mu$. Taking into account the previous proposition, we investigate when the coefficients in ν have a constant sign. Since $\mathrm{supp}(\mu) \subseteq [1, +\infty)$, we get $\mathrm{supp}(\mu * t\mu) \subseteq [1, +\infty)$ and then $\mathrm{supp}(\nu) \subseteq [1, +\infty)$. Using the identity

$$\mathcal{M}_\nu^2(z) = \mathcal{M}_\mu(z)\mathcal{M}_\mu(z+1) = H(z)^2 H(z+1)^2 \Rightarrow \mathcal{M}_\nu(z) = \pm H(z)H(z+1),$$

we get

$$\begin{aligned} \mathcal{M}_\nu(z) &= \pm\left(\textstyle\sum_{k=0}^q b_k e^{zy_k}\right)\left(\textstyle\sum_{k=0}^q b_k e^{y_k} e^{zy_k}\right) \\ &= \pm\textstyle\sum_{k,l=0}^q b_k b_l e^{y_k} e^{z(y_k+y_l)} \\ &= \textstyle\sum_k \left(\textstyle\sum_{(i,j)\in\Gamma(\gamma_k)} b_i b_j e^{y_i}\right) e^{z\gamma_k}, \end{aligned}$$

where $\Gamma(\gamma_k) = \big\{(i,j) \in (\mathbb{Z}_+)^2 \;\; 0 \leq i,j \leq q \text{ and } y_i + y_j = ln(\gamma_k)\big\}$.

Now, writing $\lambda_i = e^{y_i}$, we get

$$\nu = \sum_k \left(\sum_{(i,j)\in\Gamma(\gamma_k)} b_i b_j \lambda_i \right) \delta_{\gamma_k}. \tag{4.1}$$

We will now use the following useful observation.

Remark 4.2. In Eq. 4.1, the atom $\gamma_k := \lambda_i\lambda_j$ is said to be uniquely represented (in symbols, $\gamma_k \in \mathcal{UR}$) if $\mathrm{card}\,\Gamma(\gamma_k) \leq 2$. In this case, when γ_k is nonnegative, we readily get that b_i and b_j are of the same sign. The use of uniquely represented elements in *supp*(μ) will be helpful in the sequel.

Our strategy now is to consider a charge ν, such that both $\mu = \nu * \nu$ and $\nu * t\nu$ are positive. It will follow in particular that W_μ and $\widetilde{W_\mu}$ must be subnormal. We will then show that if ν has at most six atoms, it is necessarily positive, and that if ν has more than six atoms, then it is not necessarily positive. This will provide an affirmative answer to Conjecture A for $p \leq 6$ and a negative answer for $p \geq 7$.

We begin with the next two auxiliary lemmas.

Lemma 4.3. *Let* $\nu = \sum_{k=0}^{q} a_k \delta_{\lambda_k}$ *be a charge such that* $\nu * \nu = \sum_{k=0}^{p} b_k \delta_{\gamma_k}$ *and* $\nu * t\nu$ *are positive measures. If* $\operatorname{card}\Gamma(\lambda_k^2) \leq 3$ *for some* k*, then* $b_k \neq 0$.

Proof. The case $\operatorname{card}\Gamma(\lambda_k^2) = 1$ is trivial since it corresponds to a uniquely represented atom. Suppose $\Gamma(\lambda_k^2) = \{(k,k),(i,j),(j,i)\}$: we get $\lambda_k^2 = \lambda_i\lambda_j$ and if the coefficient $b_k = a_k^2 + 2a_ia_j = 0$, we will get for the coefficient of λ_k^2 in $\nu * t\nu$,

$$\begin{aligned} \lambda_k a_k^2 + (\lambda_i + \lambda_j)a_ia_j &= \sqrt{\lambda_i\lambda_j}a_k^2 + (\lambda_i + \lambda_j)a_ia_j \\ &< (\lambda_i + \lambda_j)\left(\frac{a_k^2}{2} + a_ia_j\right) \\ &= 0. \end{aligned}$$

This is a contradiction. □

Lemma 4.4. *Let* $\nu = \sum_{k=0}^{q} a_k \delta_{\lambda_k}$ *be a charge such that* $\nu * \nu = \sum_{k=0}^{p} b_k \delta_{\gamma_k}$ *and* $\nu * t\nu$ *are positive measures. Then (i) If* $q \geq 4$*, then* $p \geq 6$*. (ii) If* $q \geq 5$*, then* $p \geq 7$*. To list items (i) and (ii), please use the itemize environment, as follows:*

Proof. (i) Suppose $q \geq 4$. Since $\operatorname{card}\Gamma(\lambda_2^2) \leq 3$ and $\operatorname{card}\Gamma(\lambda_{q-1}^2) \leq 3$, we obtain $\{\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2, \lambda_{q-1}^2, \lambda_{q-1}\lambda_q, \lambda_q^2\} \subseteq supp(\mu)$ and hence $p \geq 6$.

(ii) From the previous item, $p \geq 6$. To show that $p \geq 7$, it suffices to exhibit a new atom.

First, if λ_2^2 is $\mathcal{UR}$ or if $\lambda_2^2 = \lambda_1\lambda_k$ with some $k > 4$, $\lambda_1\lambda_3$ becomes an $\mathcal{UR}$ and hence provides an additional atom in $\nu*\nu$. We write then $\lambda_2^2 = \lambda_1\lambda_3$ and we show that either λ_3^2 or $\lambda_2\lambda_3$ is the additional atom or produce a new one. To this goal, we suppose that neither λ_3^2 nor $\lambda_2\lambda_3$ is $\mathcal{UR}$. In this case, necessarily $\lambda_2\lambda_3 = \lambda_1\lambda_4$ (otherwise $\lambda_1\lambda_4$ will be the new atom as an $\mathcal{UR}$.) We write $\lambda_3^2 = \lambda_1\lambda_k$, $\lambda_3^2 = \lambda_2\lambda_l$ or $\lambda_3^2 = \lambda_1\lambda_k = \lambda_2\lambda_l$ with zero as corresponding coefficient. Since the two first situations will provide a new atom because of Lemma 4.3, we can assume that $\lambda_3^2 = \lambda_1\lambda_k = \lambda_2\lambda_l$. Now, multiplying $\lambda_2\lambda_3 = \lambda_1\lambda_4$ with λ_3 gives $l = 4$.

Now, from the identity $\lambda_1\lambda_k = \lambda_2\lambda_l$, we derive that $k \geq 5$.

1) In the case where $k > 5$ and $\lambda_1\lambda_5 = \lambda_3\lambda_4$, then by multiplying with λ_3, we get $\lambda_1\lambda_3\lambda_5 = \lambda_3^2\lambda_4 = \lambda_1\lambda_k\lambda_4$. It follows that $\lambda_3\lambda_5 = \lambda_k\lambda_4$ for some $k > 5$, which is impossible. Then if $k \neq 5$ $\lambda_1\lambda_5$ will give additional atom as an $\mathcal{UR}$ element.

2) $k = 5$. That is, $\lambda_1\lambda_4 = \lambda_2\lambda_3$, and $\lambda_3^2 = \lambda_1\lambda_5 = \lambda_2\lambda_4$. For $r = \frac{\lambda_2}{\lambda_1}$, we get $\lambda_k = \lambda_1 r^{k-1}$ for every $k \leq 5$. Now, to provide the 7^{th} atom, it suffices to show that either $a_2a_3 + a_1a_4 \neq 0$ or $a_3^2 + 2a_1a_5 + 2a_2a_4 \neq 0$. Seeking a contradiction, suppose that $a_2a_3 + a_1a_4 = a_3^2 + 2a_1a_5 + 2a_2a_4 = 0$. From the inequality $(r + r^2)a_2a_3 + (1 + r^3)a_1a_4 \geq 0$, we derive that $a_2a_3 < 0$. Otherwise,

$$0 \leq (r + r^2)a_2a_3 + (1 + r^3)a_1a_4 < (1 + r^3)(a_2a_3 + a_1a_4) = 0.$$

It follows also that $a_1a_4 > 0$ and then $a_2a_4 > 0$. Now, from $a_3^2 + 2a_1a_5 + 2a_2a_4 = 0$, we derive that $a_1a_5 < a_1a_5 + a_2a_4 = -a_3^2 < 0$ and since

$(1-r^2)^2 \ > \ r(1-r)^2$ we obtain the following contradiction:

$$\begin{aligned}0 &\le r^2a_3^2 + (1+r^4)a_1a_5 + (r+r^3)a_2a_4 \\ &= -r^2(2a_1a_5 + 2a_2a_4) + (1+r^4)a_1a_5 + (r+r^3)a_2a_4 \\ &= (1-r^2)^2a_1a_5 + r(1-r)^2a_2a_4 \\ &\le r(1-r)^2(a_1a_5 + a_2a_4) < 0.\end{aligned}$$

The proof is complete. □

We now state and prove our main result.

Theorem 4.5. *Let q, p be integers and $\nu = \sum_{k=0}^{q} b_k\delta_{\lambda_k}$ be a charge such that $\mu = \nu * \nu$ is a positive $p-$atomic measure with $p \le 6$. If $\mu = \nu * t\nu \ge 0$, then the coefficients b_k have a constant sign. In particular, μ admits a positive square root.*

The proof below provides a new and simple way to recover a solution to the square root problem in the case $p \le 6$.

Proof.

- $p = 2$. This is a trivial case, since a 2-atomic measure μ has a square root if and only if $\mu = a\delta_0 + b\delta_\lambda$, with $a, b, \lambda > 0$. As a result, there is no 2-atomic positive measure supported in $[1, +\infty)$ such that $\widetilde{W_\mu}$ is subnormal.
- $p = 3$. In this case $q = 2$, $\nu = b_1\delta_{\lambda_1} + b_2\delta_{\lambda_2}$ and $\mu = \nu * \nu = b_1^2\delta_{\lambda_1^2} + b_1b_2\delta_{\lambda_1\lambda_2} + b_2^2\delta_{\lambda_2^2}$ with b_1, b_2 real numbers. Since b_1b_2 is uniquely represented, it follows that b_1 and b_2 have the same sign.
- $p = 4$. A 4-atomic measure has no square root. Indeed, assume ν exists. Then, necessarily $q \ge 3$. Now write

$$\nu = b_1\delta_{\lambda_1} + b_2\delta_{\lambda_1} + \cdots + b_{q-1}\delta_{\lambda_{q-1}} + b_q\delta_{\lambda_q},$$

and therefore,

$$\nu * \nu = b_1^2\delta_{\lambda_1^2} + 2b_1b_2\delta_{\lambda_1\lambda_2} + b_2^2\delta_{\lambda_2^2} + \cdots + b_{q-1}^2\delta_{\lambda_{q-1}^2} + 2b_{q-1}b_q\delta_{\lambda_{q-1}\lambda_q} + b_q^2\delta_{\lambda_q^2}.$$

It is then clear that p should be at least 5, a contradiction.
- $p \in \{5, 6\}$. From Lemma 4.4, we obtain $q < 5$. Thus either $q = 3$ or $q = 4$.

  - $q = 3$. We put $\nu = b_1\delta_{\lambda_1} + b_2\delta_{\lambda_2} + b_3\delta_{\lambda_3}$ and $\mu = \nu * \nu = b_1^2\delta_{\lambda_1^2} + 2b_1b_2\delta_{\lambda_1\lambda_2} + b_2^2\delta_{\lambda_2^2} + 2b_1b_3\delta_{\lambda_1\lambda_3} + 2b_2b_3\delta_{\lambda_2\lambda_3} + b_3^2\delta_{\lambda_1^3} \ge 0$, with b_1, b_2 and b_3 real numbers. Since $\lambda_1\lambda_2$ and $\lambda_2\lambda_3$ are uniquely represented, it follow that $b_1b_2 > 0$ and $b_2b_3 > 0$. This gives as above b_1, b_2 and b_3 have the same sign.

  - For $q = 4$, we write $\nu = b_1\delta_{\lambda_1} + \cdots + b_4\delta_{\lambda_4}$ and

$$\begin{aligned}\mu = \nu * \nu = {} & b_1^2\delta_{\lambda_1^2} + b_2^2\delta_{\lambda_2^2} + b_3^2\delta_{\lambda_3^2} + b_4^2\delta_{\lambda_4^2} + 2(b_1b_2\delta_{\lambda_1\lambda_2} + b_1b_3\delta_{\lambda_1\lambda_3} \\ & + b_1b_4\delta_{\lambda_1\lambda_4} + b_2b_3\delta_{\lambda_2\lambda_3} + b_2b_4\delta_{\lambda_2\lambda_4} + b_3b_4\delta_{\lambda_3\lambda_4}) \ge 0.\end{aligned}$$

  As before, $b_1b_2 \ge 0$ and $b_3b_4 \ge 0$. If, moreover $\lambda_1\lambda_3 \in \mathcal{UR}$ or $\lambda_2\lambda_4 \in \mathcal{UR}$, we get $b_1b_3 \ge 0$ or $b_2b_4 \ge 0$ and then b_1, b_2, b_3 and b_4 have a constant sign.

If not, $\lambda_2^2 = \lambda_1\lambda_3$, and $\lambda_3^2 = \lambda_2\lambda_4$, we get $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_2} = \frac{\lambda_4}{\lambda_3}(= r)$ which corresponds to the case when the support is contained in a geometric sequence $\lambda_k = r^k$ with $a = \lambda_1 > 1$. Then

$$\mu = b_1^2\delta_{a^2r^2} + 2b_1b_2\delta_{a^2r^3} + (2b_1b_3 + b_2^2)\delta_{a^2r^4} + 2(b_1b_4 + b_2b_3)\delta_{a^2r^5}$$
$$+(2b_2b_4 + b_3^2)\delta_{a^2r^6} + 2b_3b_4\delta_{a^2r^7} + b_4^2\delta_{a^2r^8}.$$

It follows that $b_1b_4 + b_2b_3 \geq 0$ and thus b_1, b_2, b_3 and b_4 have constant sign.

5. Conjecture A Settled in the Negative

From the previous section if ν and $-\nu$ are both q atomic non-positive charges, then $\mu = \nu*\nu$ has no positive square root. Also, $\nu*t\nu$ is a square root of $\mu*t\mu$. Hence, if $\nu*t\nu$ is positive, we will have M_μ and $M_{\tilde{\mu}}$ are subnormal. This will provide a counter-example to Conjecture A. Since for $p \leq 6$, the conjecture is valid, we take $p = \operatorname{card}\operatorname{supp}(\mu) \geq 7$ and hence $q = \operatorname{card}\operatorname{supp}(\nu) \geq 5$.

Let $\lambda \in (1, +\infty)$ and consider the 5-atomic charge given by

$$\nu = b_1\delta_\lambda + b_2\delta_{\lambda^2} + b_3\delta_{\lambda^3} + b_4\delta_{\lambda^4} + b_5\delta_{\lambda^5}.$$

Assume that $\nu * \nu$ and $\nu * t\nu$ are both positive. We will have

$$\begin{aligned}\nu * \nu =& b_1^2\delta_{\lambda^2} + 2b_1b_2\delta_{\lambda^3} + (2b_1b_3 + b_2^2)\delta_{\lambda^4} + 2(b_1b_4 + b_2b_3)\delta_{\lambda^5}\\ &+ (b_3^2 + 2(b_1b_5 + b_2b_4))\delta_{\lambda^6} + 2(b_2b_5 + b_3b_4)\delta_{\lambda^7} + (2b_3b_5 + b_4^2)\delta_{\lambda^8}\\ &+ 2b_4b_5\delta_{\lambda^9} + b_5^2\delta_{\lambda^{10}},\\ \nu * t\nu =& \lambda b_1^2\delta_{\lambda^2} + (\lambda + \lambda^2)b_1b_2\delta_{\lambda^3} + (\lambda^2b_2^2 + (\lambda + \lambda^3)b_1b_3)\delta_{\lambda^4} + ((\lambda + \lambda^4)b_1b_4\\ &+ (\lambda^2 + \lambda^3)b_2b_3)\delta_{\lambda^5} + (\lambda^3b_3^2 + (\lambda + \lambda^5)b_1b_5 + (\lambda^2 + \lambda^4)b_2b_4))\delta_{\lambda^6}\\ &+ ((\lambda^2 + \lambda^5)b_2b_5 + (\lambda^3 + \lambda^4)b_3b_4)\delta_{\lambda^7} + ((\lambda^3 + \lambda^5)b_3b_5 + \lambda^4b_4^2)\delta_{\lambda^8}\\ &+ (\lambda^4 + \lambda^5)b_4b_5\delta_{\lambda^9} + \lambda^5b_5^2\delta_{\lambda^{10}}.\end{aligned}$$

As before $b_1b_2 > 0$ and $b_4b_5 > 0$. Since $b_1b_4+b_2b_3 \geq 0$ and $b_2b_5+b_3b_4 \geq 0$ we derive that b_1, b_2, b_4 and b_5 have constant sign. Otherwise $b_2b_5 < 0$ and $b_1b_4 < 0$ and both signs of b_3 will give a contradiction with $b_1b_4 + b_2b_3 \geq 0$ and $b_2b_5 + b_3b_4 \geq 0$.

Without loss of generality, we can assume b_1, b_2, b_4 and b_5 are nonnegative. Denote $p \in \{7, 8, 9\}$ for the number of atoms in $\nu * \nu$. If $b_3 > 0$, then $p = 9$ and in the case where $b_3 < 0$ the possible zero coefficients are

$$b_2b_5 + b_3b_4, \quad \text{and} \quad b_1b_4 + b_2b_3.$$

Clearly $p = 7 \iff b_2b_5 + b_3b_4 = b_1b_4 + b_2b_3 = 0$, $p = 8 \iff$ either $b_1b_4 + b_2b_3 = 0$ or $b_2b_5 + b_3b_4 = 0$ and $p = 9$ otherwise.

Let us now study those instances when $\mu * t\mu$ is positive. Since $\lambda > 1$, we have $(\lambda^2 + \lambda^5) > (\lambda + \lambda^4) > (\lambda^2 + \lambda^3)$, and we drive that

$$(\lambda^2 + \lambda^5)b_2b_5 + (\lambda^3 + \lambda^4)b_3b_4 > (\lambda^3 + \lambda^4)(b_2b_5 + b_3b_4) \geq 0.$$

and

$$(\lambda + \lambda^4)b_1b_4 + (\lambda^2 + \lambda^3)b_2b_3 > (\lambda^2 + \lambda^3)(b_1b_4 + b_2b_3) \geq 0.$$

Thus, $\mu * t\mu \geq 0$ if and only if

$$\lambda b_2^2 + (1+\lambda^2) b_1 b_3 \geq 0 \text{ and } (1+\lambda^2) b_3 b_5 + \lambda b_4^2 \geq 0,$$

equivalently

$$\frac{\lambda}{1+\lambda^2} \geq max\left(\frac{-b_1 b_3}{b_2^2}, \frac{-b_3 b_5}{b_4^2}\right) \tag{5.1}$$

Example 5.1. There exists a 9-atomic positive measure μ supported in $\mathbb{R}_+$ such that: $\mu * t\mu$ has a positive square root, while μ has no positive square root.

Proof. Let x be a positive real number and $\lambda > 1$. Consider the $5-$atomic charge ξ_x given by

$$\xi_x := \delta_\lambda + \delta_{\lambda^2} - x\delta_{\lambda^3} + \delta_{\lambda^4} + \delta_{\lambda^5}.$$

The coefficients of ξ_x are: $(b_1, b_2, b_3, b_4, b_5) = (1, 1, -x, 1, 1)$. The finite atomic measure $\mu_x = \xi_x * \xi_x$ given by

$$\begin{aligned}\mu_x = {} & \delta_{\lambda^2} + 2\delta_{\lambda^3} + (1-2x)\delta_{\lambda^4} + (2-2x)\delta_{\lambda^5} + (4+x^2)\delta_{\lambda^6} + (2-2x)\delta_{\lambda^7} \\ & + (1-2x)\delta_{\lambda^8} + 2\delta_{\lambda^9} + \delta_{\lambda^{10}},\end{aligned}$$

has no positive square root.

It is also clearly a positive 9-atomic measure if and only if $0 < x < \frac{1}{2}$. On the other hand, $\mu_x * t\mu_x$ possesses as square root $\nu_x = \xi_x * t\xi_x$ that is positive (because of (5.1)) for any λ satisfying

$$x \leq \frac{\lambda}{1+\lambda^2}.$$

One can take, for instance, $\lambda = 2$ and $x = \frac{1}{5}$. The measure

$$\mu = \delta_4 + 2\delta_8 + \frac{3}{5}\delta_{16} + \frac{8}{5}\delta_{32} + \frac{101}{25}\delta_{64} + \frac{8}{5}\delta_{128} + \frac{3}{5}\delta_{256} + 2\delta_{512} + \delta_{1024}$$

has no positive square root, and satisfies $\mu * t\mu \geq 0$. □

Example 5.2. For $p = 7$ and $p = 8$, there exists a p-atomic positive measure supported in $\mathbb{R}_+$ such that: $\mu * t\mu$ has a positive square root measure, but μ has no positive square root.

Proof. Let

$$\xi = \delta_\lambda + \alpha\delta_{\lambda^2} - \delta_{\lambda^3} + \alpha\delta_{\lambda^4} + \beta\delta_{\lambda^5}$$

where α, β, and $\lambda \neq 1$ are positive numbers. For $\mu = \xi * \xi$, we have

$$\begin{aligned}\mu = {} & \delta_{\lambda^2} + 2\alpha\delta_{\lambda^3} + (\alpha^2 - 2)\delta_{\lambda^4} + (1 + 2\alpha^2 + 2\beta)\delta_{\lambda^6} + (2\alpha\beta - 2\alpha)\delta_{\lambda^7} \\ & + (\alpha^2 - 2\beta)\delta_{\lambda^8} + 2\alpha\beta\delta_{\lambda^9} + \beta^2\delta_{\lambda^{10}}\end{aligned}$$

The measure μ is positive if and only if:

$$\alpha^2 \geq 2\beta \geq 2.$$

On the other hand, the coefficients of ξ are: $(b_1, b_2, b_3, b_4, b_5) = (1, \alpha, -1, \alpha, \beta)$. Again, because of (5.1), $\nu = \xi * t\xi \geq 0$, if and only if

$$\frac{\lambda}{1+\lambda^2} \geq max\left(\frac{1}{\alpha^2}, \frac{\beta}{\alpha^2}\right) = \frac{\beta}{\alpha^2}.$$

Thus, for $\beta = 2$, $\lambda = 2$ and $\alpha = 3$ (resp. $\beta = 1$, $\lambda = 2$ and $\alpha = 2$), the 8-atomic measure

$$\mu = \delta_4 + 6\delta_8 + 7\delta_{16} + 23\delta_{64} + 6\delta_{128} + 5\delta_{256} + 12\delta_{512} + 4\delta_{1024}$$

(resp. the 7-atomic measure

$$\mu = \delta_4 + 4\delta_8 + 2\delta_{16} + 11\delta_{64} + 2\delta_{256} + 4\delta_{512} + \delta_{1024}),$$

is positive, without a positive square root, and such that $\mu * t\mu$ has a positive square root. □

6. An Additional Example

We present a concrete example of a non-subnormal weighted shift W_α such that $\widetilde{W_\alpha}$ is subnormal.

Example 6.1. We now exhibit a weighted shift W_μ, where μ is a non-positive charge and such that $\mu * t\mu$ admits a positive square root. Taking into account the computations in the previous section, it suffices to find a non-positive charge ν such that $\mu = \nu * \nu$ is a non-positive charge and $\sqrt{\mu * t\mu} = \nu * t\nu$ is positive.

Proof. Let $\lambda \in (1, +\infty)$ and consider a 5-atomic charge given by

$$\nu = b_1\delta_\lambda + b_2\delta_{\lambda^2} - \delta_{\lambda^3} + b_4\delta_{\lambda^4} + b_5\delta_{\lambda^5},$$

where b_1, b_2, b_4 and b_5 are positive numbers. We have

$$\begin{aligned}\nu * \nu = & \; b_1^2\delta_{\lambda^2} + 2b_1b_2\delta_{\lambda^3} + (b_2^2 - 2b_1)\delta_{\lambda^4} + 2(b_1b_4 - b_2)\delta_{\lambda^5} \\ & + (1 + 2(b_1b_5 + b_2b_4))\delta_{\lambda^6} + 2(b_2b_5 - b_4)\delta_{\lambda^7} \\ & + (b_4^2 - 2b_5)\delta_{\lambda^8} + 2b_4b_5\delta_{\lambda^9} + b_5^2\delta_{\lambda^{10}},\end{aligned}$$

and

$$\begin{aligned}\nu * t\nu = & \; \lambda b_1^2\delta_{\lambda^2} + (\lambda + \lambda^2)b_1b_2\delta_{\lambda^3} + (\lambda^2b_2^2 - (\lambda + \lambda^3)b_1)\delta_{\lambda^4} \\ & + ((\lambda + \lambda^4)b_1b_4 - (\lambda^2 + \lambda^3)b_2)\delta_{\lambda^5} \\ & + (\lambda^3 + (\lambda + \lambda^5)b_1b_5 + (\lambda^2 + \lambda^4)b_2b_4)\delta_{\lambda^6} \\ & + ((\lambda^2 + \lambda^5)b_2b_5 - (\lambda^3 + \lambda^4)b_4)\delta_{\lambda^7} + (\lambda^4b_4^2 - (\lambda^3 + \lambda^5)b_5)\delta_{\lambda^8} \\ & + (\lambda^4 + \lambda^5)b_4b_5\delta_{\lambda^9} + \lambda^5b_5^2\delta_{\lambda^{10}}.\end{aligned}$$

It follows that

$$\nu * t\nu \geq 0 \iff \begin{cases} \lambda^2b_2^2 - (\lambda + \lambda^3)b_1 \geq 0 & (\lambda + \lambda^4)b_1b_4 - (\lambda^2 + \lambda^3)b_2 \geq 0 \\ \lambda^4b_4^2 - (\lambda^3 + \lambda^5)b_5 \geq 0 & (\lambda^2 + \lambda^5)b_2b_5 - (\lambda^3 + \lambda^4)b_4 \geq 0. \end{cases}$$

Now, using

$$\lambda^2b_2^2 - (\lambda + \lambda^3)b_1 \leq \lambda^2(b_2^2 - 2b_1)$$

and

$$\lambda^4b_4^2 - (\lambda^3 + \lambda^5)b_5 \leq \lambda^4(b_4^2 - 2b_5),$$

it follows that $b_2^2 - 2b_1 \geq 0$ and $b_4^2 - 2b_5 \geq 0$. In this case,

$$\mu := \nu * \nu \geq 0 \iff b_1 b_4 - b_2 \geq 0 \text{ and } b_2 b_5 - b_4 \geq 0,$$

and as above,

$$\nu * t\nu \geq 0 \iff \frac{\lambda}{1+\lambda^2} \geq max(\frac{b_1}{b_2^2}, \frac{b_5}{b_4^2}).$$

Taking $b_1 = b_5 = 1$, $b_2 = 2$ and $b_4 = 3$, we obtain that

$$\mu = \delta_{\lambda^2} + 4\delta_{\lambda^3} + 2\delta_{\lambda^4} + 2\delta_{\lambda^5} + 15\delta_{\lambda^6} - 2\delta_{\lambda^7} + 7\delta_{\lambda^8} + 6\delta_{\lambda^9} + \delta_{\lambda^{10}},$$

is a non-positive charge for every λ and

$$\nu * t\nu \geq 0 \iff \frac{\lambda}{1+\lambda^2} \geq \frac{1}{4} \iff \lambda \in (1, 2+\sqrt{3}].$$

□

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