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Commutativity of Hankel and Toeplitz operators on the Hardy space of the n-torus

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We consider Hankel and Toeplitz operators on $H^2(\mathbb{T}^n)$, the Hardy space of the n-torus $\mathbb{T}^n$. Given symbols φ and ψ in $L^\infty(\mathbb{T}^n)$ with suitable properties, we obtain necessary and sufficient conditions for the Hankel operator $H_{\psi,n}$ and the Toeplitz operator $T_{\varphi,n}$ to commute. We then extend the study to the more general situation where no assumptions are imposed on φ, and provide new, non-trivial necessary conditions for the commutativity of $H_{\psi,n}$ and $T_{\varphi,n}$. We also show that certain well known commutativity results between Hankel and Toeplitz operators in the one-variable case do not extend to the multivariable setting.

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1. Introduction

In 1963, A. Brown and P.R. Halmos characterized all pairs of commuting Toeplitz operators on the Hardy space over the unit disc [3]. In the literature, the result is referred to as the Brown-Halmos Theorem. In the ensuing decades, providing suitable extensions of this result to the case of Hilbert spaces of holomorphic functions on general domains in several complex variables, and studying the corresponding Brown-Halmos-type theorems has been a central theme of research in Toeplitz operator theory. In particular, the (essentially) commuting problem for (small) Hankel and Toeplitz operators on the Hardy/Bergman space of several variables is quite important and interesting. In the present paper, we attempt to contribute to this fascinating area of research with a number of necessary and sufficient conditions that guarantee the commutativity of certain Hankel and Toeplitz operators on the Hardy space of the n–torus.

Throughout the paper, we will use the symbol $\mathbb{D}$ to denote the open unit disc and $\mathbb{T}$ to denote the unit circle in the complex plane $\mathbb{C}$. For a positive integer n, the open unit polydisc and n-torus in $\mathbb{C}^n$ are denoted by $\mathbb{D}^n$ and $\mathbb{T}^n$, respectively. Although the function theory on the polydisc differs significantly from that on the unit disc, we will utilize the available theory related with multiple Fourier series on the n-dimensional torus. By $L^2(\mathbb{T}^n)$ $(= (L^2(\mathbb{T}^n), d\mu))$ we will denote the Lebesgue space of measurable and square integrable functions defined on $\mathbb{T}^n$, with $d\mu$ the normalized Lebesgue measure on $\mathbb{T}^n$. The Hardy space $H^2(\mathbb{T}^n)$ is a closed subspace of $L^2(\mathbb{T}^n)$. As usual, we will denote elements of $\mathbb{T}^n$ by $\boldsymbol{z} = (z_1, \dots, z_n)$, elements of $\mathbb{Z}^n$ by $\boldsymbol{k}$, and $\boldsymbol{z}^{\boldsymbol{k}} := z_1^{k_1} \cdots z_n^{k_n}$. With the help of multivariable Fourier series [15], we have

$$L^2(\mathbb{T}^n) = \{f |\ f : \mathbb{T}^n \mapsto \mathbb{C} \ \text{ with } f(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} f_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$$

$$\text{with} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |f_{\boldsymbol{k}}|^2 < \infty\},$$

and

$$H^2(\mathbb{T}^n) = \{f \in L^2(\mathbb{T}^n) |\ f_{\boldsymbol{k}} = 0 \text{ whenever } \boldsymbol{k} \notin \mathbb{Z}_+^n\}$$

(Throughout this paper, the sets $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of all integers and the set of all nonnegative integers, respectively.) It is straightforward to verify that the sets of monomials $\{\boldsymbol{z}^{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n}$ and $\{\boldsymbol{z}^{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}$ form orthonormal bases of $L^2(\mathbb{T}^n)$ and $H^2(\mathbb{T}^n)$, respectively. By $L^\infty(\mathbb{T}^n)$ we denote the space of essentially bounded measurable functions defined on $\mathbb{T}^n$. By an operator we mean a bounded linear transformation on a Hilbert space $\mathcal{H}$, and the symbol $\mathcal{B}(\mathcal{H})$ is used to denote the space of all bounded operators on $\mathcal{H}$.

For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ acting on $H^2(\mathbb{T})$ (introduced by O. Toeplitz), is defined as

$$T_\varphi := PM_\varphi,$$

where M_φ denotes the multiplication operator, induced by φ and P is the orthogonal projection of $L^2(\mathbb{T})$ onto the subspace $H^2(\mathbb{T})$. The Hankel operators, which are characterized in terms of Hankel matrices, are defined as

$$H_\psi(f) = PJM_\psi(f) \quad (f \in H^2(\mathbb{T})),$$

where $\psi \in L^\infty(\mathbb{T})$. Here J is the flip operator defined as $J(f)(z) := f(\overline{z})$ for $f \in L^2(\mathbb{T})$.

Due to its vast applicability, the theory of Hankel and Toeplitz operators on $H^2(\mathbb{T})$ possesses extensive literature. The appearance of Hankel operators and Hankel matrices is seen in diverse areas such as control theory, approximation theory, Wiener-Hopf factorizations, interpolation problem and perturbation theory. (See [2,6,11,13,16] and the references therein). In this paper, we study the commutativity between Hankel and Toeplitz operators defined on the space $H^2(\mathbb{T}^n)$. For $n = 1$, R.A. Martínez-Avendaño [12], in 2000, classified commuting Hankel and Toeplitz operators; in 2003, Guo and Zheng [9] classified when a Hankel and a Toeplitz operator have a compact commutator.

The situation in the multivariable setting requires special techniques. C. Gu (see [7,8]) discussed some algebraic properties of Hankel and Toeplitz operators on the Hardy space of the polydisc, and some conditions for the product of Hankel and Toeplitz operators to be of finite rank operator were derived. Though commutativity and essential commutativity between slant Hankel and slant Toeplitz operators on the space $L^2(\mathbb{T}^n)$ has been studied in [5], not much is known regarding commutativity between Hankel and Toeplitz operators in the multivariable case. In the present paper we are able to obtain a necessary condition for commutativity between Hankel and Toeplitz operators in those instances when the Hankel operator is induced by a specific kind of symbol. Further, we show that certain results concerning commutativity between these operators which hold in the one-variable case may not hold in the multivariable case.

2. Hankel operators on $H^2(\mathbb{T}^n)$

We begin with the definition of Toeplitz operator, a formal companion of Hankel operator, on the space $H^2(\mathbb{T}^n)$ [10]. For φ in $L^\infty(\mathbb{T}^n)$ $T_{\varphi,n}$, the Toeplitz operator on $H^2(\mathbb{T}^n)$, is defined as $T_{\varphi,n} = PM_\varphi\big|_{H^2(\mathbb{T}^n)}$. Here M_φ is the multiplication operator defined on $L^2(\mathbb{T}^n)$ and P represents the orthogonal projection of $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$.

In the literature, Hankel operators have been defined in various forms. For instance, these operators are considered in the form $H_\psi(f) = (I - P)M_\psi f$ over the Hardy-Sobolev spaces in [1] and in the same form on the Bergman spaces of the polydisc in $\mathbb{C}^n$ in [14] (where ψ an essentially bounded function). K. Guo and D. Zheng in [9] considered the Hankel operator defined on $H^2(\mathbb{T})$ as $H_\psi(f) := PUM_\psi f$, (ψ in $L^\infty(\mathbb{T})$), where U is defined on $L^2(\mathbb{T})$ as $Uf(z) := \overline{z}f(\overline{z})$. C. Gu in [7] considered $H_\psi(f) := VPM_\psi(\overline{f})$ for $f \in H^2(\mathbb{D}^n)$, with V being the anti-unitary operator defined on $H^2(\mathbb{D}^n)$ as $V(f)(z_1, \ldots, z_n) := \overline{f(\overline{z_1}, \ldots, \overline{z_n})}$.

In this paper, we extend the definition of Hankel operator taken up by Martínez-Avendaño in [12] to the space $H^2(\mathbb{T}^n)$ and study some of the properties of this operator in this section.

Definition 2.1. For ψ in $L^\infty(\mathbb{T}^n)$, the Hankel operator on $H^2(\mathbb{T}^n)$ with symbol ψ is denoted by $H_{\psi,n}$ and is defined as $H_{\psi,n} := PJ_nM_\psi\big|_{H^2(\mathbb{T}^n)}$. Here J_n is the flip operator [4] defined on $L^2(\mathbb{T}^n)$ as $J_n(f)(\boldsymbol{z}) := f(\overline{\boldsymbol{z}})$ for each $f \in L^2(\mathbb{T}^n)$.

It is easy to see that J_n is a unitary operator on $L^2(\mathbb{T}^n)$. (For more properties of the operator J_n, we refer the reader to [4].) The boundedness of ψ provides the boundedness of $H_{\psi,n}$ with

$$\|H_{\psi,n}\| \leqslant \|P\|\|J_n\|\|M_\psi\| \leqslant \|\psi\|_\infty.$$

Further, for ψ in $L^\infty(\mathbb{T}^n)$ and $\boldsymbol{m} \in \mathbb{Z}^n_+$, we have

$$\begin{aligned} H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}) &= PJ_n(\psi \cdot \boldsymbol{z}^{\boldsymbol{m}}) \\ &= PJ_n\left(\sum_{\boldsymbol{i}\in\mathbb{Z}^n} \psi_{\boldsymbol{i}}\ \boldsymbol{z}^{\boldsymbol{m}+\boldsymbol{i}}\right) \\ &= \sum_{\boldsymbol{i}\in\mathbb{Z}^n_+} \psi_{-\boldsymbol{i}-\boldsymbol{m}}\ \boldsymbol{z}^{\boldsymbol{i}}. \end{aligned}$$

As a result, if $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^n_+$ we obtain

$$\begin{aligned} \left\langle H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{i}}),\ \boldsymbol{z}^{\boldsymbol{j}} \right\rangle &= \left\langle \sum_{\boldsymbol{m}\in\mathbb{Z}^n_+} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \cdot \boldsymbol{z}^{\boldsymbol{m}},\ \boldsymbol{z}^{\boldsymbol{j}} \right\rangle \\ &= \psi_{-\boldsymbol{i}-\boldsymbol{j}} \\ &= \left\langle \boldsymbol{z}^{\boldsymbol{i}}, \sum_{\boldsymbol{m}\in\mathbb{Z}^n_+} \overline{\psi}_{-\boldsymbol{j}-\boldsymbol{m}} \cdot \boldsymbol{z}^{\boldsymbol{m}} \right\rangle. \end{aligned}$$

Thus,

$$H^*_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^n_+} \overline{\psi}_{-\boldsymbol{m}-\boldsymbol{j}}\ \boldsymbol{z}^{\boldsymbol{j}}, \tag{2.1}$$

for each $\boldsymbol{m}$ in $\mathbb{Z}^n_+$.

We now discuss some notations which will help us derive certain properties of Hankel operators.

Definition 2.2. For f in $L^2(\mathbb{T}^n)$, $\overline{f}$ and $\widetilde{f}$ are defined respectively as

$$\overline{f}(\boldsymbol{z}) := \overline{f(\boldsymbol{z})}$$

and

$$\widetilde{f}(\boldsymbol{z}) := \overline{f(\overline{\boldsymbol{z}})} \quad .$$

It is plain to see that $f \in L^2(\mathbb{T}^n)$ if and only if $\widetilde{f} \in L^2(\mathbb{T}^n)$. In fact, $f \in H^2(\mathbb{T}^n)$ if and only if $\widetilde{f} \in H^2(\mathbb{T}^n)$ and $f \in L^\infty(\mathbb{T}^n)$ if and only if $\widetilde{f} \in L^\infty(\mathbb{T}^n)$. Likewise, $f \in L^2(\mathbb{T}^n)$ if and only if $\overline{f} \in L^2(\mathbb{T}^n)$ and $f \in L^\infty(\mathbb{T}^n)$ if and only if $\overline{f} \in L^\infty(\mathbb{T}^n)$. For f in $L^2(\mathbb{T}^n)$, the flip of f, denoted by f^*, is defined as

$$f^*(\boldsymbol{z}) := f(\overline{\boldsymbol{z}}).$$

As a result,

$$f^*(\boldsymbol{z}) = \sum_{\boldsymbol{j} \in \mathbb{Z}^n} f_{-\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}}.$$

Again $f \in L^2(\mathbb{T}^n)$ if and only if $f^* \in L^2(\mathbb{T}^n)$ and $f \in L^\infty(\mathbb{T}^n)$ if and only if $f^* \in L^\infty(\mathbb{T}^n)$ hold.
Using the above notation and equation (2.1), we see that

$$H^*_{\psi,n} = H_{\widetilde{\psi},n} \ ,$$

for ψ in $L^\infty(\mathbb{T}^n)$. This shows that the collection $\{H_{\psi,n} : \psi \in L^\infty(\mathbb{T}^n)\}$ of Hankel operators on $H^2(\mathbb{T}^n)$ is self-adjoint.

Theorem 2.3. *The map $\psi \mapsto H_{\psi,n}$ from $L^\infty(\mathbb{T}^n)$ to $\mathcal{B}(H^2(\mathbb{T}^n))$ is linear but not one-to-one.*

Proof. Linearity of the given mapping is evident from the fact that the mapping $\psi \mapsto M_\psi$ from $L^\infty(\mathbb{T}^n)$ to $\mathcal{B}(H^2(\mathbb{T}^n))$ is linear. To investigate injectivity, we consider the function $\psi(\boldsymbol{z}) := \boldsymbol{z}^{\mathbf{1}}$, where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{Z}^n$. Then $\psi \in L^\infty(\mathbb{T}^n)$ and $H_{\psi,n} = 0$, the zero operator. However $\psi \neq 0$. $\square$

Recall now that $H^\infty(\mathbb{T}^n) = L^\infty(\mathbb{T}^n) \cap H^2(\mathbb{T}^n)$. From the explicit expression of $H_{\psi,n}$, we observe that the action of $H_{\psi,n}$ on the canonical basis elements involves only those Fourier coefficients $\psi_{\boldsymbol{j}}$ of ψ (in $L^\infty(\mathbb{T}^n)$) for which each $j_t \leqslant 0$, $1 \leqslant t \leqslant n$. Hence, it is easy to see that the operator $H_{\psi,n} = 0$, if $\psi \in \boldsymbol{z}^{\boldsymbol{i}} H^\infty(\mathbb{T}^n)$, where $i_j > 0$ for at least one j $(1 \leqslant j \leqslant n)$. However, the converse, which is true for $n = 1$, is not true in case of $n > 1$. To see this, consider $n = 2$ and $\psi(z_1, z_2) := z_1^{-2} z_3^2 + z_1^3 z_2^{-2}$. Clearly $\psi \in L^\infty(\mathbb{T}^n)$ and $H_{\psi,2} = 0$, but $\psi \notin z_1^{i_1} z_2^{i_2} H^2(\mathbb{T}^2)$ if $i_j > 0$ for $j = 1$ or 2. In order to obtain a characterization for symbols inducing zero Hankel operators, we now state the following definition.

Definition 2.4. For ψ in $L^2(\mathbb{T}^n)$ with Fourier series expansion given by $\psi(\boldsymbol{z}) = \sum_{\boldsymbol{i}\in\mathbb{Z}^n} \psi_{\boldsymbol{i}}\boldsymbol{z}^{\boldsymbol{i}}$, we define the coanalytic part of ψ by

$$\boldsymbol{z} \mapsto \sum_{i_j \leqslant 0, 1\leqslant j\leqslant n} \psi_{\boldsymbol{i}}\boldsymbol{z}^{\boldsymbol{i}}$$

and denote it by $(CAP)_\psi$.

The following result is straightforward.

Theorem 2.5. *For ψ in $L^\infty(\mathbb{T}^n)$, $H_{\psi,n} = 0$ if and only if $(CAP)_\psi$, the coanalytic part of ψ, is zero.*

Theorem 2.3 helps us reformulate Theorem 2.5 in the following way.

Corollary 2.6. *For ϕ and ψ in $L^\infty(\mathbb{T}^n)$, $H_{\phi,n} = H_{\psi,n}$ if and only if $(CAP)_\phi = (CAP)_\psi$.*

In view of Corollary 2.6 we see that the symbol of a Hankel operator on $H^2(\mathbb{T}^n)$ is not uniquely determined.

3. Commutativity between Hankel and Toeplitz operators

In this section we discuss the commutativity between Hankel and Toeplitz operators in several variable. From the definitions, it is easy to verify that for φ in $L^\infty(\mathbb{T}^n)$ the action of the Toeplitz operator $T_{\varphi,n}(= PM_\varphi)$ on any basis element of $H^2(\mathbb{T}^n)$ is given by

$$T_{\varphi,n}(\boldsymbol{z}^{\boldsymbol{m}}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^n_+} \varphi_{\boldsymbol{j}-\boldsymbol{m}}\boldsymbol{z}^{\boldsymbol{j}},$$

where $\boldsymbol{m} \in \mathbb{Z}^n_+$. The adjoint of $T_{\varphi,n}$ is $T^*_{\varphi,n} = PM_{\bar{\varphi}} = T_{\bar{\varphi},n}$.

A symbol φ in $L^2(\mathbb{T}^n)$ is said to be analytic if $\varphi \in H^2(\mathbb{T}^n)$, and φ is said to be coanalytic if $\bar{\varphi}$ is analytic. (For other relevant results on $T_{\varphi,n}$, we refer the reader to [11].)

In the one-variable setting, it is well known that Toeplitz and Hankel operators do not commute in general. This fact is also present in the multivariable setting, if we consider, for instance, the symbols $\varphi(z_1,\ldots,z_n) := z_1$ and $\psi(z_1,\ldots,z_n) := 1$ defined on $\mathbb{T}^n$ when $n > 1$. Then $T_{\varphi,n}H_{\psi,n}(1) = T_{\varphi,n}PJ_n(1) = T_{\varphi,n}(1) = P(\varphi) = \varphi = z_1$ and $H_{\psi,n}T_{\varphi,n}(1) = H_{\psi,n}P(\varphi) = H_{\psi,n}(\varphi) = PJ_n(\varphi) = P(\overline{z_1}) = 0$.

Our next aim is to obtain a natural and reasonable necessary condition for the commutativity between a Toeplitz operator and a Hankel operator in several variables. In this direction we first prove the following result.

Theorem 3.1. *Let $\varphi, \psi \in L^\infty(\mathbb{T}^n)$ and let $T_{\varphi,n}$ and $H_{\psi,n}$ be a nonzero Toeplitz and a nonzero Hankel operator, respectively, acting on the space $H^2(\mathbb{T}^n)$. Assume that φ is analytic. Then*

$$T^*_{\varphi,n} H_{\psi,n} = H_{\psi,n} T_{\widetilde{\varphi},n}.$$

Proof. For $\boldsymbol{m}, \boldsymbol{p} \in \mathbb{Z}^n_+$, we have

$$\begin{aligned}\left\langle T^*_{\varphi,n} H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \right\rangle &= \langle H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}), T_{\varphi,n}(\boldsymbol{z}^{\boldsymbol{p}}) \rangle \\ &= \left\langle \sum_{\boldsymbol{i} \in \mathbb{Z}^n_+} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{i}}, T_{\varphi,n}(\boldsymbol{z}^{\boldsymbol{p}}) \right\rangle \\ &= \sum_{\boldsymbol{i} \in \mathbb{Z}^n_+} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \overline{\phi}_{\boldsymbol{i}-\boldsymbol{p}}. \end{aligned} \tag{3.1}$$

Similarly,

$$\langle H_{\psi,n} T_{\widetilde{\varphi},n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \rangle = \sum_{\boldsymbol{i} \in \mathbb{Z}^n_+} \psi_{-\boldsymbol{i}-\boldsymbol{p}} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{m}}.$$

Due to the analyticity of φ, equation (3.1) reduces to the following:

$$\begin{aligned}\left\langle T^*_{\varphi,n} H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \right\rangle &= \sum_{\boldsymbol{i} \geqslant \boldsymbol{p}} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}} \\ &= \sum_{\boldsymbol{t} \in \mathbb{Z}^n_+} \psi_{-\boldsymbol{t}-\boldsymbol{m}-\boldsymbol{p}} \bar{\varphi}_{\boldsymbol{t}}.\end{aligned}$$

Note here that by $\boldsymbol{i} \geqslant \boldsymbol{p}$ we mean all those $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ in $\mathbb{Z}^n_+$ such that $i_j \geqslant p_j$ for each $1 \leqslant j \leqslant n$. Similarly,

$$\langle H_{\psi,n} T_{\widetilde{\varphi},n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \rangle = \sum_{\boldsymbol{s} \in \mathbb{Z}^n_+} \psi_{-\boldsymbol{s}-\boldsymbol{m}-\boldsymbol{p}} \bar{\varphi}_{\boldsymbol{s}}.$$

Thus,

$$T^*_{\varphi,n} H_{\psi,n} = H_{\psi,n} T_{\widetilde{\varphi},n},$$

as desired. □

It is worth noticing here that the converse of Theorem 3.1 is true when $n = 1$ (see [12]), but does not hold when $n > 1$. For, consider $\varphi(z_1, z_2, \ldots, z_n) := \overline{z_1} z_2 \ldots z_n$, $(n > 1)$ and $\psi(z_1, z_2, \ldots, z_n) := 1$ on $\mathbb{T}^n$. Then φ and ψ are in $L^\infty(\mathbb{T}^n)$. A simple computation reveals that for any $\boldsymbol{m} \in \mathbb{Z}^n_+$, we have

$$T_{\varphi,n}(z_1^{m_1}z_2^{m_2}\dots z_n^{m_n}) = \begin{cases} z_1^{m_1-1}z_2^{m_2+1}\dots z_n^{m_n+1} & \text{if } m_1 \geqslant 1 \\ 0 & \text{otherwise} \end{cases},$$

and

$$H_{\psi,n}(z_1^{m_1}z_2^{m_2}\dots z_n^{m_n}) = \begin{cases} 1 & \text{if } m_1 = m_2 = \dots = m_n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $T_{\varphi,n}$ and $H_{\psi,n}$ are nonzero operators. Moreover, we have

$$\begin{aligned} T^*_{\varphi,n}H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}) &= \begin{cases} T^*_{\varphi,n}(1) & \text{if } \boldsymbol{m} = \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} H_{\psi,n}T_{\widetilde{\varphi},n}(\boldsymbol{z}^{\boldsymbol{m}}) &= \begin{cases} H_{\psi,n}(z_1^{m_1-1}z_2^{m_2+1}\dots z_n^{m_n+1}) & \text{if } m_1 \geqslant 1 \\ 0 & \text{otherwise} \end{cases} \\ &= 0. \end{aligned}$$

It follows that $T^*_{\varphi,n}H_{\psi,n} = 0 = H_{\psi,n}T_{\widetilde{\varphi},n}$ but φ is not analytic.

We now derive a few corollaries from Theorem 3.1. In all these corollaries we will assume that the Toeplitz operator and the Hankel operator under consideration are nonzero and φ and ψ are in $L^\infty(\mathbb{T}^n)$.

Corollary 3.2. *If φ is analytic then $T_{\varphi^*,n}H_{\psi,n} = H_{\psi,n}T_{\varphi,n}$.*

Proof. If φ is analytic then $\widetilde{\varphi}$ is analytic. The proof follows using Theorem 3.1 (for $\widetilde{\varphi}$) and the identities $\overline{\widetilde{\varphi}} = \varphi^*$ and $\widetilde{\widetilde{\varphi}} = \varphi$. □

Corollary 3.3. *If φ is coanalytic then $T_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\varphi^*,n}$. Further,*

$$T_{\widetilde{\varphi},n}H_{\psi,n} = H_{\psi,n}T_{\bar{\varphi},n}.$$

Proof. First apply Theorem 3.1 on $\bar{\varphi}$, which is analytic because φ is coanalytic. Next, observe that φ^* is analytic, $\overline{\varphi^*} = \widetilde{\varphi}$ and $\widetilde{\varphi^*} = \bar{\varphi}$, which yield the desired result. □

Remark 3.4. It is worth noting again that the converse of Corollaries 3.2 and 3.3 hold in the case $n = 1$ (see [12]), but do not hold when $n > 1$. In fact, for $n > 1$ the functions $\varphi(z_1, z_2, \dots, z_n) := \overline{z_1}z_2 \dots z_n$ and $\psi(z_1, z_2, \dots, z_n) := 1$ serve as an example.

Our aim now is to discuss the converse of Theorem 3.1 in case $n > 1$. Not enough is known in this direction, however we are able to derive some information about the symbol φ when the Toeplitz and Hankel operators $T_{\varphi,n}$ and $H_{\psi,n}$ satisfy the desired commutator equation, and when the symbol ψ is in a specific format.

Theorem 3.5. *Let $n > 1$. Suppose that $T_{\varphi,n}$ and $H_{\psi,n}$ are a nonzero Toeplitz and a nonzero Hankel operator with symbols φ and ψ in $L^\infty(\mathbb{T}^n)$. Assume that*

$$\psi(z_1,\dots,z_n) = \left(\prod_{d\neq i=1}^{n} z_i^{-q_i}\right) f(z_d) = z_1^{-q_1} z_2^{-q_2} \dots z_{d-1}^{-q_{d-1}} f(z_d) z_{d+1}^{-q_{d+1}} \dots z_n^{-q_n},$$

for some function f of the variable z_d, $1 \leqslant d \leqslant n$, and $(n-1)$ nonnegative integers q_i, $1 \leqslant i \leqslant n$, $i \neq d$. If $T^*_{\varphi,n} H_{\psi,n} = H_{\psi,n} T_{\tilde{\varphi},n}$ then

(i) $\varphi_{m_1,\dots,m_n} = 0$ *if* $m_d < 0$ *and* $0 \leqslant m_i \leqslant q_i$ *for each* $1 \leqslant i \leqslant n$, $i \neq d$.

(ii) $\varphi_{m_1,\dots,m_n} = 0$ *if* $m_d < 0$ *and* $m_i \leqslant q_i$ *for each* $1 \leqslant i \leqslant n, i \neq d$, *and such that for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j < 0$.

(iii) $\sum_{i_d=0}^{\infty} (\bar{\varphi}_{q_1-m_1,\ q_2-m_2,\ \dots,\ q_{d-1}-m_{d-1},\ i_d,\ q_{d+1}-m_{d+1},\ q_{d+2}-m_{d+2},\ \dots,\ q_n-m_n})\cdot$
$(\psi_{-q_1,\ \dots,\ -q_{d-1},\ -i_d-p_d,\ -q_{d+1},\ \dots,\ -q_n}) = 0$, *for each integer* $p_d \geqslant 0$ *and* $m_i \geqslant 0$ *for each* $1 \leqslant i \leqslant n, i \neq d$, *and such that for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j > q_j$.

(For a visualization of conditions (i), (ii), and (iii) in the case $n = 2$, the reader is referred to Fig. 1 on page 19.)

Proof. For two n–tuples $\boldsymbol{i}$ and $\boldsymbol{j}$ in $\mathbb{Z}_+^n$ satisfying the inequalities $i_t \geqslant j_t$ $(1 \leqslant t \leqslant n)$, we will use the notation $\boldsymbol{i} \geqslant \boldsymbol{j}$. Let $\boldsymbol{m}, \boldsymbol{p}$ in $\mathbb{Z}_+^n$. Then

$$\begin{aligned}
\left\langle T^*_{\varphi,n} H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \right\rangle &= \langle H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}), T_{\varphi,n}(\boldsymbol{z}^{\boldsymbol{p}}) \rangle \\
&= \left\langle \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{i}}, \sum_{\boldsymbol{j}\in\mathbb{Z}_+^n} \varphi_{\boldsymbol{j}-\boldsymbol{p}} \boldsymbol{z}^{\boldsymbol{j}} \right\rangle \\
&= \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}} \\
&= \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\ \boldsymbol{i}\geqslant\boldsymbol{p}} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}} \\
&\quad + \sum_{\boldsymbol{i}-\boldsymbol{p}\notin\mathbb{Z}_+^n} \psi_{-\boldsymbol{i}-\boldsymbol{m}} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}} \\
&= \sum_{\boldsymbol{s}\in\mathbb{Z}_+^n} \psi_{-\boldsymbol{s}-\boldsymbol{m}-\boldsymbol{p}} \bar{\varphi}_{\boldsymbol{s}}
\end{aligned}$$

$$+ \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\, \boldsymbol{i}-\boldsymbol{p}\notin\mathbb{Z}_+^n} \psi_{-\boldsymbol{i}-\boldsymbol{m}}\bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}} \ . \tag{3.2}$$

Similarly,

$$\langle H_{\psi,n}T_{\widetilde{\varphi},n}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}}\rangle = \sum_{\boldsymbol{s}\in\mathbb{Z}_+^n} \psi_{-\boldsymbol{s}-\boldsymbol{m}-\boldsymbol{p}}\bar{\varphi}_{\boldsymbol{s}}$$

$$+ \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\, \boldsymbol{i}-\boldsymbol{m}\notin\mathbb{Z}_+^n} \psi_{-\boldsymbol{i}-\boldsymbol{p}}\bar{\varphi}_{\boldsymbol{i}-\boldsymbol{m}} \ . \tag{3.3}$$

Since $T^*_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\widetilde{\varphi},n}$, equations (3.2) and (3.3) yield

$$\sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\, \boldsymbol{i}-\boldsymbol{p}\notin\mathbb{Z}_+^n} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}}\psi_{-\boldsymbol{i}-\boldsymbol{m}} = \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\, \boldsymbol{i}-\boldsymbol{m}\notin\mathbb{Z}_+^n} \bar{\varphi}_{\boldsymbol{i}-\boldsymbol{m}}\psi_{-\boldsymbol{i}-\boldsymbol{p}}, \tag{3.4}$$

for all $\boldsymbol{p}, \boldsymbol{m} \in \mathbb{Z}_+^n$. Observe that when $p_j = 0$ (resp. $m_j = 0$) for each $1 \leqslant j \leqslant n$, then the sum on the left-hand side (resp. right-hand side) of equation (3.4) is zero. We now choose different values for $\boldsymbol{p}$ and $\boldsymbol{m}$ in equation (3.4) to derive conclusions (i), (ii) and (iii).

We begin by choosing $p_t = 0$ and $0 \leqslant m_t \leqslant q_t$ for each $1 \leqslant t \leqslant n, t \neq d$ then equation (3.4) gives

$$\sum_{\substack{\boldsymbol{i}\in\mathbb{Z}_+^n \\ \text{and} \\ 0\leqslant i_d\leqslant p_d-1}} \bar{\varphi}_{i_1,\ldots,i_{d-1},i_d-p_d,i_{d+1}\ldots,i_n}\psi_{-i_1-m_1,\ldots,-i_d-m_d,\ldots,-i_n-m_n}$$

$$= \sum_{\boldsymbol{i}\in\mathbb{Z}_+^n;\, \boldsymbol{i}-\boldsymbol{m}\notin\mathbb{Z}_+^n} \bar{\varphi}_{i_1-m_1,\ldots,i_d-m_d,\ldots,i_n-m_n}\psi_{-i_1,\ldots,-i_d-p_d,\ldots,-i_n}, \tag{3.5}$$

for each p_d and m_d in $\mathbb{Z}_+$. Due to the given form of ψ, in equation (3.5) we have fixed choices for $(i_1, \ldots, i_{d-1}, i_{d+1}, \ldots, i_n)$ and thus equation (3.5) reduces to

$$\sum_{i_d=0}^{p_d-1} (\bar{\varphi}_{q_1-m_1,\ q_2-m_2,\ \ldots,\ q_{d-1}-m_{d-1},\ i_d-p_d,\ q_{d+1}-m_{d+1},\ \ldots,\ q_n-m_n})\cdot$$

$$(\psi_{-q_1,\ldots,\ -q_{d-1},-i_d-m_d,-q_{d+1},\ldots,-q_n})$$

$$= \sum_{i_d=0}^{m_d-1} (\bar{\varphi}_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},i_d-m_d,q_{d+1}-m_{d+1}\ldots,q_n-m_n})\cdot$$

$$(\psi_{-q_1,\ldots,-q_{d-1},-i_d-p_d,-q_{d+1},\ldots,-q_n}), \tag{3.6}$$

for each p_d and m_d in $\mathbb{Z}_+$, keeping in mind that the sum on the left-hand side (resp. right-hand side) of equation (3.6) is zero if $p_d = 0$ (or $m_d = 0$). Without loss of generality, we assume $p_d > m_d \geqslant 0$ and rewrite the sum on the left-hand side of equation (3.6) as

$$\sum_{i_d=0}^{p_d-1}(\bar{\varphi}_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},i_d-p_d,q_{d+1}-m_{d+1},\ldots,q_n-m_n})\cdot(\psi_{-q_1,..,-q_{d-1},-i_d-m_d}$$

$$_{,-q_{d+1},..,-q_n})=\sum_{i_d=0}^{p_d-m_d-1}(\bar{\varphi}_{q_1-m_1,..,q_{d-1}-m_{d-1},i_d-p_d,q_{d+1}-m_{d+1},..,q_n-m_n})\cdot$$

$$(\psi_{-q_1,\ldots,-q_{d-1},-i_d-m_d,-q_{d+1},\ldots,-q_n})+\sum_{i_d=p_d-m_d}^{p_d-1}(\bar{\varphi}_{q_1-m_1,\ \ldots,\ q_{d-1}-m_{d-1},i_d-p_d}$$

$$_{,q_{d+1}-m_{d+1}\ldots,q_n-m_n})\cdot(\psi_{-q_1,\ldots,-q_{d-1},-i_d-m_d,-q_{d+1},\ldots,-q_n}). \tag{3.7}$$

Using a change of variable (that is, by substituting, say, $i_d-(p_d-m_d)=s_d$ for the second sum on the right-hand side of equation (3.7), we easily see that it is equal to the sum on the right-hand side of equation (3.6). As a result, equation (3.6) yields

$$\sum_{i_d=0}^{p_d-m_d-1}(\bar{\varphi}_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},i_d-p_d,q_{d+1}-m_{d+1}\ldots,q_n-m_n})\cdot$$
$$(\psi_{-q_1,\ldots,-q_{d-1},-i_d-m_d,-q_{d+1},\ldots,-q_n})=0, \tag{3.8}$$

for all integers $p_d>m_d\geqslant 0$. Since the Hankel operator $H_{\psi,n}$ is nonzero there must exist some nonnegative integer ℓ_d such that $\psi_{-q_1,\ldots,-q_{d-1},-\ell_d,-q_{d+1},\ldots,-q_n}\neq 0$. Fix $m_d=\ell_d$ and put $p_d=\ell_d+1,\ell_d+2,\ell_d+3,\ldots$ successively in equation (3.8) to obtain

$$\varphi_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},-(\ell_d+b),q_{d+1}-m_{d+1},\ldots,q_n-m_n}=0,\ \text{for all integers } b>0. \tag{3.9}$$

However, if we choose $p_d=\ell_d+1$ and put $m_d=\ell_d-1,\ell_d-2,\ldots,2,1,0$ successively in equation (3.8), we get

$$\varphi_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},-(\ell_d-s),q_{d+1}-m_{d+1},\ldots,q_n-m_n}=0 \tag{3.10}$$

for each integer s such that $0\leqslant s\leqslant \ell_d-1$.

Equations (3.9) and (3.10) lead to

$$\varphi_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},-b,q_{d+1}-m_{d+1},\ldots,q_n-m_n}=0, \tag{3.11}$$

for all integers $b>0$ and $0\leqslant m_t\leqslant q_t$ for all $1\leqslant t\leqslant n,\ t\neq d$. This proves part (i).
We now return to equation (3.4), applied to $\boldsymbol{p}$ and $\boldsymbol{m}$ in $\mathbb{Z}_+^n$, when $m_t=0$ for each $1\leqslant t\leqslant n, t\neq d$ and $p_t>q_t$ for at least one $1\leqslant t\leqslant n$, we have $t\neq d$. It is straightforward to note that, in this setting and using the given form of ψ, we have

$$\sum_{i_d\geqslant 0}(\bar{\varphi}_{q_1-p_1,\ q_2-p_2,\ \ldots,\ q_{d-1}-p_{d-1},\ i_d-p_d,\ q_{d+1}-p_{d+1},\ ..,\ q_n-p_n})\cdot(\psi_{-q_1,\ ..,\ -q_{d-1},}$$
$$_{-i_d-m_d,\ -q_{d+1},..,-q_n})=0, \tag{3.12}$$

for each p_d and m_d in $\mathbb{Z}_+$. There is no loss of generality if we further assume $p_d > m_d \geqslant 0$. This allows us to rewrite the sum on the left-hand side of equation (3.12) as

$$\sum_{i_d \geqslant 0} \bar{\varphi}_{q_1-p_1,..,q_{d-1}-p_{d-1},i_d-p_d,q_{d+1}-p_{d+1},..,q_n-p_n} \psi_{-q_1,..,-q_{d-1},-i_d-m_d,-q_{d+1},..,-q_n}$$
$$= \sum_{i_d=0}^{p_d-m_d-1} (\bar{\varphi}_{q_1-p_1,..,q_{d-1}-p_{d-1},i_d-p_d,q_{d+1}-p_{d+1},..,q_n-p_n}) \cdot (\psi_{-q_1,..,-q_{d-1},-i_d-m_d}$$
$$_{,-q_{d+1},..,-q_n}) + \sum_{i_d=p_d-m_d}^{\infty} (\bar{\varphi}_{q_1-p_1,..,q_{d-1}-p_{d-1},i_d-p_d,q_{d+1}-p_{d+1},..,q_n-p_n}) \cdot (\psi_{-q_1,..,}$$
$$_{-q_{d-1},-i_d-m_d,-q_{d+1},\ldots,-q_n}). \tag{3.13}$$

By substituting $i_d - (p_d - m_d) = s_d$ in the second sum on the right-hand side of equation (3.13), and then using equation (3.12), we obtain that

$$\sum_{i_d=0}^{p_d-m_d-1} (\bar{\varphi}_{q_1-p_1,..,q_{d-1}-p_{d-1},i_d-p_d,q_{d+1}-p_{d+1},..,q_n-p_n}) \cdot (\psi_{-q_1,..,-q_{d-1},-i_d-m_d,}$$
$$_{-q_{d+1},..,-q_n}) = 0, \tag{3.14}$$

for all integers $p_d > m_d \geqslant 0$ and $(p_1, \ldots, p_{d-1}, p_{d+1}, \ldots, p_n) \in \mathbb{Z}_+^{n-1}$ such that $p_t > q_t$ for at least one $1 \leqslant t \leqslant n$, we have $t \neq d$. If we now follow the steps used earlier (between equation (3.8) to equation (3.11)), we can conclude from equation (3.14) that

$$\varphi_{q_1-p_1,\ldots,q_{d-1}-p_{d-1},-b,q_{d+1}-p_{d+1},\ldots,q_n-p_n} = 0, \tag{3.15}$$

for all integers $b > 0$ and $(p_1, \ldots, p_{d-1}, p_{d+1}, \ldots, p_n) \in \mathbb{Z}_+^{n-1}$ such that $p_t > q_t$ for at least one $1 \leqslant t \leqslant n$, we have $t \neq d$. Equation (3.15) now establishes part (ii).
By applying again equation (3.4) with the substitutions $m_d = 0$ and $p_t = 0$ for each $1 \leqslant t \leqslant n, t \neq d$, we obtain

$$\sum_{\substack{\boldsymbol{i} \in \mathbb{Z}_+^n \\ \text{with} \\ 0 \leqslant i_d \leqslant p_d - 1}} \bar{\varphi}_{i_1,..,i_{d-1},i_d-p_d,..,i_n} \psi_{-i_1-m_1,..,-i_{d-1}-m_{d-1},-i_d,-i_{d+1}-m_{d+1},..,-i_n-m_n} =$$
$$\sum_{\substack{\boldsymbol{i} \in \mathbb{Z}_+^n; \\ \boldsymbol{i}-\boldsymbol{m} \notin \mathbb{Z}_+^n}} \bar{\varphi}_{i_1-m_1,..,i_{d-1}-m_{d-1},i_d,i_{d+1}-m_{d+1},..,i_n-m_n} \psi_{-i_1,..,-i_{d-1},-i_d-p_d,-i_{d+1},..,-i_n}, \tag{3.16}$$

for each integer $p_d \geqslant 0$ and $(m_1, \ldots, m_{d-1}, m_{d+1}, \ldots, m_n) \in \mathbb{Z}_+^{n-1}$. Using the given structure of ψ in equation (3.16) we obtain that the sum on the left hand side is zero for the choice of $(m_1, \dots, m_{d-1}, m_{d+1}, \dots, m_n) \in \mathbb{Z}_+^{n-1}$ such that $m_t > q_t$ for at least one $1 \leqslant t \leqslant n$, $t \neq d$. Thus equation (3.16) provides that

$$\sum_{i_d \geqslant 0} \bar{\varphi}_{q_1-m_1,..,q_{d-1}-m_{d-1},i_d,q_{d+1}-m_{d+1},..,q_n-m_n} \psi_{-q_1,..,-q_{d-1},-i_d-p_d,-q_{d+1},..,-q_n} = 0, \tag{3.17}$$

for each integer $p_d \geqslant 0$ and $(m_1, \dots, m_{d-1}, m_{d+1}, \dots, m_n) \in \mathbb{Z}_+^{n-1}$ such that $m_t > q_t$ for at least one $1 \leqslant t \leqslant n$, with $t \neq d$. This prove part (iii). The proof of the theorem is now complete. $\square$

We verify the proof of Theorem 3.5 by tracing through it for the following example in the setting $n = 2$.

Example 3.6. Consider $n = 2$ and $\psi(z_1, z_2) = \overline{z_1}(\overline{z_2} + \overline{z_2}^2)$. Suppose that $T^*_{\varphi,2} H_{\psi,2} = H_{\psi,2} T_{\widetilde{\varphi},2}$ for $\phi \in L^\infty(\mathbb{T}^2)$. In view of Theorem 3.5, we have, $q_1 = 1$, $d = 2$ and $f(z_2) = \overline{z_2} + \overline{z_2}^2$. Let $\mathbf{m} := (m_1, m_2)$ and $\mathbf{p} := (p_1, p_2)$ be in $\mathbb{Z}_+^2$. Consider

$$\begin{aligned}
\left\langle T^*_{\varphi,2} H_{\psi,2}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \right\rangle &= \left\langle T^*_{\varphi,2} H_{\psi,2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2} \right\rangle \\
&= \langle H_{\psi,2}(z_1^{m_1} z_2^{m_2}), T_{\varphi,2}(z_1^{p_1} z_2^{p_2}) \rangle \\
&= \sum_{\boldsymbol{i} := (i_1,i_2) \in \mathbb{Z}_+^2} \psi_{-i_1-m_1,-i_2-m_2} \bar{\varphi}_{i_1-p_1,i_2-p_2} \\
&= \sum_{\substack{(i_1,i_2) \in \mathbb{Z}_+^2 \\ \text{with} \\ i_1 \geqslant p_1, i_2 \geqslant p_2}} \psi_{-i_1-m_1,-i_2-m_2} \bar{\varphi}_{i_1-p_1,i_2-p_2} \\
&\quad + \sum_{\substack{(i_1,i_2) \in \mathbb{Z}_+^2 \\ \text{with} \\ i_1 \geqslant p_1, i_2 \geqslant p_2 \\ \text{does not hold}}} \psi_{-i_1-m_1,-i_2-m_2} \bar{\varphi}_{i_1-p_1,i_2-p_2} \\
&= \sum_{(s_1,s_2) \in \mathbb{Z}_+^2} \psi_{s_1-m_1-p_1,s_2-m_2-p_2} \bar{\varphi}_{s_1,s_2} \\
&\quad + \sum_{\substack{(i_1,i_2) \in \mathbb{Z}_+^2 \\ \text{with} \\ i_1 \geqslant p_1, i_2 \geqslant p_2 \\ \text{does not hold}}} \psi_{-i_1-m_1,-i_2-m_2} \bar{\varphi}_{i_1-p_1,i_2-p_2}.
\end{aligned} \tag{3.18}$$

Similarly,

$$\begin{aligned}
\langle H_{\psi,2} T_{\widetilde{\varphi},2}(\boldsymbol{z}^{\boldsymbol{m}}), \boldsymbol{z}^{\boldsymbol{p}} \rangle &= \langle H_{\psi,2} T_{\widetilde{\varphi},2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2} \rangle \\
&= \left\langle T_{\widetilde{\varphi},2}(z_1^{m_1} z_2^{m_2}), H_{\widetilde{\psi},2}(z_1^{p_1} z_2^{p_2}) \right\rangle
\end{aligned}$$

$$= \sum_{(s_1,s_2)\in\mathbb{Z}_+^2} \psi_{s_1-m_1-p_1,s_2-m_2-p_2}\bar{\varphi}_{s_1,s_2}$$
$$+ \sum_{\substack{(i_1,i_2)\in\mathbb{Z}_+^2 \\ \text{with} \\ i_1\geqslant m_1, i_2\geqslant m_2 \\ \text{does not hold}}} \psi_{-i_1-p_1,-i_2-p_2}\bar{\varphi}_{i_1-m_1,i_2-m_2}. \tag{3.19}$$

Since $T^*_{\varphi,2}H_{\psi,2} = H_{\psi,2}T_{\tilde{\varphi},2}$, equations (3.18) and (3.19) yield

$$\sum_{\substack{(i_1,i_2)\in\mathbb{Z}_+^2 \\ \text{with} \\ i_1\geqslant p_1, i_2\geqslant p_2 \\ \text{does not hold}}} \psi_{-i_1-m_1,-i_2-m_2}\bar{\varphi}_{i_1-p_1,i_2-p_2} = \sum_{\substack{(i_1,i_2)\in\mathbb{Z}_+^2 \\ \text{with} \\ i_1\geqslant m_1, i_2\geqslant m_2 \\ \text{does not hold}}} \psi_{-i_1-p_1,-i_2-p_2}\bar{\varphi}_{i_1-m_1,i_2-m_2} \tag{3.20}$$

for all $(p_1, p_2), (m_1, m_2) \in \mathbb{Z}_+^2$. Observe that when $p_1 = p_2 = 0$ (resp. $m_1 = m_2 = 0$) then the sum on the left-hand side (resp. right-hand side) of equation (3.20) is zero. Here onwards we divide the proof in steps.

Step 1: Choose $p_1 = 0$ and $0 \leqslant m_1 \leqslant 1$ in equation (3.20). Actual idea is to choose m_1 between 0 and q_1. We thus obtain

$$\sum_{\substack{i_1\in\mathbb{Z}_+ \\ \text{and} \\ 0\leqslant i_2\leqslant p_2-1}} \bar{\varphi}_{i_1,i_2-p_2}\psi_{-i_1-m_1,-i_2-m_2}$$
$$= \sum_{\boldsymbol{i}\in\mathbb{Z}_+^2;\ \boldsymbol{i}-\boldsymbol{m}\notin\mathbb{Z}_+^2} \bar{\varphi}_{i_1-m_1,i_2-m_2,}\psi_{-i_1,-i_2-p_2}, \tag{3.21}$$

for each p_2 and m_2 in $\mathbb{Z}_+$. Due to the given form of ψ, only $\psi_{-1,-1}$ and $\psi_{-1,-2}$ are nonzero (each equal to 1) and all other Fourier coefficients of ψ are zero, thus equation (3.21) reduces to

$$\sum_{i_2=0}^{p_2-1}(\bar{\varphi}_{1-m_1,\ i_2-p_2})\cdot(\psi_{-1,-i_2-m_2}) = \sum_{i_2=0}^{m_2-1}(\bar{\varphi}_{1-m_1,i_2-m_2})\cdot(\psi_{-1,-i_2-p_2}), \tag{3.22}$$

for each p_2 and m_2 in $\mathbb{Z}_+$, keeping in mind that the sum on the left-hand side (resp. right-hand side) of equation (3.22) is zero if $p_2 = 0$ (or $m_2 = 0$). Without loss of generality, we assume $p_2 > m_2 \geqslant 0$ and rewrite the sum on the left-hand side of equation (3.22) as

$$\sum_{i_2=0}^{p_2-1}(\bar{\varphi}_{1-m_1,i_2-p_2,})\cdot(\psi_{-1,-i_2-m_2}) = \sum_{i_2=0}^{p_2-m_2-1}(\bar{\varphi}_{1-m_1,i_2-p_2})\cdot(\psi_{-1,-i_2-m_2})$$

$$+\sum_{i_2=p_2-m_2}^{p_2-1}(\bar{\varphi}_{1-m_1,i_2-p_2})\cdot(\psi_{-1,-i_2-m_2}). \quad (3.23)$$

Using a change of variable (put $i_2-(p_2-m_2)=s_2$ for the second sum on the right-hand side of equation (3.23), we easily see that it is equal to the sum on the right-hand side of equation (3.22). As a result, equation (3.23) reduces to

$$\sum_{i_2=0}^{p_2-m_2-1}(\bar{\varphi}_{q_1-m_1,i_2-p_2})\cdot(\psi_{-q_1,-i_2-m_2})=0, \quad (3.24)$$

for all integers $p_2>m_2\geqslant 0$. Since the Hankel operator $H_{\psi,2}$ is nonzero there must exist some nonnegative integer ℓ_2 such that $\psi_{-q_1,-\ell_2}\neq 0$ (here in this case we can take $\ell_2=1$ or 2). Let us fix $m_2=\ell_2=1$ and put $p_2=2,3,\ldots$ successively in equation (3.24) to obtain

$$\varphi_{1-m_1,-(1+b)}=0, \text{ for all integers } b>0. \quad (3.25)$$

Next, choose $p_2=\ell_2+1=2$ and put $m_2=\ell_2-1,\ell_2-2,\ldots,2,1,0$ successively, that is, in this case $m_2=0$ in equation (3.24), we get

$$\sum_{i_2=0}^{1}(\bar{\varphi}_{1-m_1,i_2-p_2})\cdot(\psi_{-1,-i_2-m_2})=0, \quad (3.26)$$

which further provides

$$\varphi_{1-m_1,-1}=0 \quad (3.27)$$

Equations (3.25) and (3.27) lead to

$$\varphi_{1-m_1,-b}=0, \quad (3.28)$$

for all integers $b>0$ and $0\leqslant m_1\leqslant 1$. This proves part (i).
Step 2: We now return to equation (3.20), applied to $\boldsymbol{p}$ and $\boldsymbol{m}$ in $\mathbb{Z}_+^2$ and choose $m_1=0$ and $p_1>1(=q_1)$. It is straightforward to note that, in this setting and using the given form of ψ, we have

$$\sum_{i_2\geqslant 0}(\bar{\varphi}_{1-p_1,\ i_2-p_2})\cdot(\psi_{-1,-i_2-m_2})=0, \quad (3.29)$$

for each p_2 and m_2 in $\mathbb{Z}_+$. There is no loss of generality if we further assume $p_2>m_2\geqslant 0$. This allows us to rewrite the sum on the left-hand side of equation (3.29) as

$$\sum_{i_2 \geqslant 0} \bar{\varphi}_{1-p_1, i_2-p_2} \psi_{-1,-i_2-m_2} = \sum_{i_2=0}^{p_2-m_2-1} (\bar{\varphi}_{1-p_1, i_2-p_2}) \cdot (\psi_{-1,-i_2-m_2}) + \sum_{i_2=p_2-m_2}^{\infty} (\bar{\varphi}_{1-p_1, i_2}) \cdot (\psi_{-1,-i_2-m_2}). \tag{3.30}$$

By substituting $i_2 - (p_2 - m_2) = s_2$ in the second sum on the right-hand side of equation (3.30), and then using equation (3.29), we obtain that

$$\sum_{i_2=0}^{p_2-m_2-1} (\bar{\varphi}_{1-p_1, i_2-p_2}) \cdot (\psi_{-1,-i_2-m_2}) = 0, \tag{3.31}$$

for all integers $p_2 > m_2 \geqslant 0$ and $p_1 \in \mathbb{Z}_+$ such that $p_1 > q_1$. If we now follow the steps used earlier (between equation (3.24) to equation (3.28)), we can conclude from equation (3.31) that

$$\varphi_{1-p_1,-b} = 0, \tag{3.32}$$

for all integers $b > 0$ and $p_1 \in \mathbb{Z}_+$ such that $p_1 > q_1$. Equation (3.32) now establishes part (ii).

Step 3: By applying again equation (3.20) with the substitutions $m_2 = 0$ and $p_1 = 0$, we obtain

$$\sum_{\substack{i_1 \in \mathbb{Z}_+ \\ \text{with} \\ 0 \leqslant i_2 \leqslant p_2 - 1}} \bar{\varphi}_{i_1, i_2-p_2} \psi_{-i_1-m_1,-i_2} = \sum_{\substack{i_2 \in \mathbb{Z}_+; \\ \text{with} \\ 0 \leqslant i_1 \leqslant m_1 - 1}} \bar{\varphi}_{i_1-m_1, i_2} \psi_{-i_1,-i_2-p_2}, \tag{3.33}$$

for each integer $p_2 \geqslant 0$ and $m_1 \in \mathbb{Z}_+$. Using the given structure of ψ, in the right-hand side of equation (3.33) only $i_1 = q_1 = 1$ will contribute, but for that we need $1 \leqslant m_1 - 1$, that is, $1 < m_1$. Thus we obtain

$$\sum_{i_2 \geqslant 0} \bar{\varphi}_{1-m_1, i_2} \psi_{-1,-i_2-p_2} = 0, \tag{3.34}$$

for each integer $p_2 \geqslant 0$ and $m_1 \in \mathbb{Z}_+$ such that $m_1 > 1$. This completes the illustration of the working of the Theorem 3.5. However, equation (3.34) can further be solved by taking different values of $p_2 \geqslant 0$ and using given information that $\psi_{-1,-1} = 1 = \psi_{-1,-2}$ and all other Fourier coefficients of ψ are zero. This provides $\phi_{m_1,0} = \phi_{m_1,1} = \phi_{m_1,2} = 0$ for all integers $m_1 < 0$.

We now provide an immediate consequence of Theorem 3.5.

Corollary 3.7. *If ψ in $L^\infty(\mathbb{T}^n)$ is such that the coanalytic part of ψ is a constant (nonzero) function then a necessary condition for a nonzero Toeplitz operator $T_{\varphi,n}$ ($\varphi \in L^\infty(\mathbb{T}^n)$)*

to satisfy $T^*_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\widetilde{\varphi},n}$ is that $(CAP)_\varphi$, the coanalytic part of φ, be a constant function.

Proof. Suppose that the coanalytic part of ψ in $L^\infty(\mathbb{T}^n)$ is a constant (nonzero) function, i.e., $(CAP)_\psi(z_1,\dots,z_n) = \psi_{0,\dots,0}$ for some $0 \neq \psi_{0,\dots,0} \in \mathbb{C}$. We can then take $f(z_d) = \psi_{0,\dots,0}$ for some $1 \leqslant d \leqslant n$ and $q_i = 0$ for each $1 \leqslant q_i \leqslant n, i \neq d$. If $T^*_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\widetilde{\varphi},n}$, then by using Theorem 3.5 we have

(a) $\varphi_{0,\dots,-b,\dots,0} = 0$ for all integers $b > 0$.
(b) $\varphi_{-p_1,\dots,-p_n} = 0$ for all $(p_1,\dots,p_n) \in \mathbb{Z}^n_+$ such that $p_t > 0$ for at least one t, we have $1 \leqslant t \leqslant n$.

Thus, $(CAP)_\varphi$ is also a constant function. □

We note here that the proof of Theorem 3.5 holds up to equation (3.11) for $n \geqslant 1$ and demands $n > 1$ to proceed ahead, so in the one-variable case (that is, $\psi = f(z)$ is any function in $L^\infty(\mathbb{T})$) we obtain that a necessary condition for $T^*_\varphi H_\psi = H_\psi T_{\widetilde{\varphi}}$ is that $\varphi_{-b} = 0$ for all integers $b > 0$, that is, φ is analytic. But when $n > 1$, the three conditions combined together do not provide the analyticity of φ.
We now see some applications of Theorem 3.5 via some examples. While conditions (i) and (ii) can be easily visualized, the same is not necessarily true of condition (iii). To provide further clarity on what condition (iii) entails, we present an example (Example 3.10) with a detailed analysis of the Toeplitz symbol φ that go along with a Hankel symbol of the form $\psi(z_1,z_2) = \bar{z}_1^{q_1}(z_2^2 + \bar{z}_2^3)$.

Example 3.8. Take $n = 2$ and let $H_{\psi,2}$ be a nonzero Hankel operator with symbol $\psi \in L^\infty(\mathbb{T}^2)$. Assume that ψ is of the form $\psi(z_1,z_2) = z_1^{-2}f(z_2)$ for any function f in the variable z_2. Suppose that $\varphi \in L^\infty(\mathbb{T}^2)$ is such that $T_{\varphi,2}$ is a nonzero Toeplitz operator on the space $H^2(\mathbb{T}^2)$ satisfying $T^*_{\varphi,2}H_{\psi,2} = H_{\psi,2}T_{\widetilde{\varphi},2}$. Keeping the notation used in Theorem 3.5 intact, here $d = 2$ and $q_1 = 2$. Then the information gathered from (i) and (ii) for the function φ yields the following information regarding the Fourier coefficients φ_{i_1,i_2}.

(a) $\varphi_{2,-s} = \varphi_{1,-s} = \varphi_{0,-s} = 0$, for each integer $s > 0$ and,
(b) $\varphi_{-r,-s} = 0$, for integers $r > 0$ and $s > 0$.

Also, the information in Theorem 3.5(iii) means that $\sum_{i_2 \geqslant 0} \bar{\varphi}_{2-m_1,i_2}\psi_{-2,-i_2-p_2} = 0$, for all integers $m_1 > 2$ and $p_2 \geqslant 0$. In particular, if $f(z_2) = 1$ for all $z_2 \in \mathbb{T}$, then in addition to (a) and (b) obtained above, we have

(c) $\varphi_{-r,0} = 0$, for each integer $r > 0$.

We note that when $f(z_2) = 1$ for all $z_2 \in \mathbb{T}$, we can also take $d = 1$ and $q_2 = 0$. With this setting, (i), (ii) and (iii) obtained for the function φ in Theorem 3.5 produce the same information regarding the Fourier coefficients of φ as obtained in (a), (b) and (c).

Example 3.9. Take $n = 3$ and define $\psi(z_1, z_2, z_3) = z_1^{-1} z_2^{-1} z_3^{-2}$; that is, $d = 3, q_1 = 1, q_2 = 1$ and $f(z_3) = z_3^{-2}$. Then $\psi \in L^\infty(\mathbb{T}^3)$. Let $\varphi \in L^\infty(\mathbb{T}^3)$ be such that $T^*_{\varphi,3} H_{\psi,3} = H_{\psi,3} T_{\tilde{\varphi},3}$, where $T_{\varphi,3}$ and $H_{\psi,3}$ are nonzero Toeplitz and Hankel operators on the space $H^2(\mathbb{T}^3)$. Using Theorem 3.5, we obtain the following information regarding the Fourier coefficients of φ.

(a) $\varphi_{1,1,-t} = \varphi_{1,0,-t} = \varphi_{0,1,-t} = \varphi_{0,0,-t} = 0$, for each integer $t > 0$.
(b) $\varphi_{-r,1,-t} = \varphi_{-r,0,-t} = \varphi_{-r,-s,-t} = \varphi_{1,-s,-t} = \varphi_{0,-s,-t} = 0$, for positive integers r, s and t,

along with

$$\sum_{i_3 \geqslant 0} \bar{\varphi}_{1-m_1, 1-m_2, i_3} \psi_{-1,-1,-i_3-p_3} = 0, \tag{3.35}$$

for all integers $p_3 \geqslant 0$ and for $(m_1, m_2) \in \mathbb{Z}_+^2$ such that either $m_1 > 1$ and $0 \leqslant m_2 \leqslant 1$, or $m_1 > 1$ and $m_2 > 1$, or $0 \leqslant m_1 \leqslant 1$ and $m_2 > 1$. Information obtained from the relation (3.35) by simple computations specifically provides the following.

(c) $\varphi_{-r,1,2} = \varphi_{-r,0,2} = \varphi_{1,-s,2} = \varphi_{0,-s,2} = \varphi_{-r,-s,2} = 0$, for positive integers r and s.
(d) $\varphi_{-r,1,1} = \varphi_{-r,0,1} = \varphi_{-r,-s,1} = \varphi_{1,-s,1} = \varphi_{0,-s,1} = 0$, for positive integers r and s.
(e) $\varphi_{-r,1,0} = \varphi_{-r,0,0} = \varphi_{-r,-s,0} = \varphi_{1,-s,0} = \varphi_{0,-s,0} = 0$, for positive integers r and s.

Likewise if we consider $d = 1, q_2 = 1, q_3 = 2, f(z_1) = z_1^{-1}$ or $d = 2, q_1 = 1, q_3 = 2, f(z_2) = z_2^{-1}$, conditions (i), (ii) and (iii) of Theorem 3.5 produces the same information as obtained above.

Example 3.10. Consider now a Hankel symbol of the form

$$\psi(z_1, z_2) = \bar{z_1}^{q_1} f(z_2) = \bar{z_1}^{q_1} (z_2^2 + \bar{z_2}^3).$$

It is straightforward to observe that, without loss of generality, we can erase the first term of $f(z_2)$. It follows that the region that provides information about $\varphi_{m_1,m_2} = 0$ with respect to condition (iii) is given by the inequalities $m_1 < 0$ and $0 \leqslant m_2 \leqslant 3$. In fact, a more general result holds. Consider a trigonometric polynomial $f(z_2) = \sum_{j=-t}^{s} a_j z_2^j$, with $a_{-t} \neq 0$. Then condition (iii) in Theorem 3.5 becomes

$$\sum_{i_2=0}^{\infty} \bar{\varphi}_{q_1-m_1, i_2} \psi_{-q_1, -i_2-p_2} = 0,$$

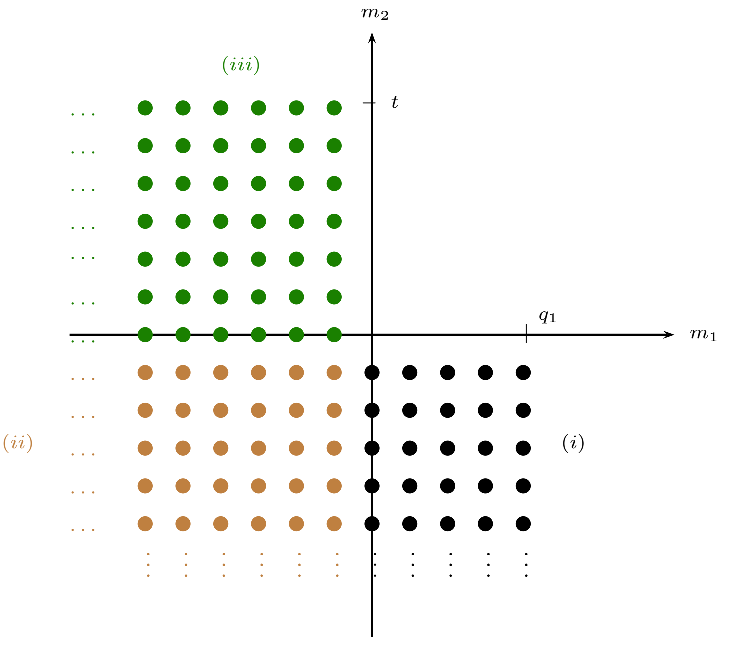

Fig. 1. The three colored regions represent the subsets of $\mathbb{Z}^2$ that satisfy conditions (i), (ii) and (iii) in Theorem 3.5. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

for all $p_2 \geqslant 0, m_1 \geqslant 0$ and $m_1 > q_1$, which readily implies

$$\sum_{i_2=0}^{\infty} \bar{\varphi}_{r,i_2} \psi_{-q_1,-i_2-p_2} = 0 \quad (\text{for all } r < 0, p_2 \geqslant 0). \tag{3.36}$$

We show that these equations will always yield $\varphi_{m_1,m_2} = 0$ for $m_1 < 0$ and $m_2 = 0, 1, \ldots, t$.

For, given $p_2 = t$, (3.36) implies $\bar{\varphi}_{r,0}\psi_{-q_1,-t} = 0$, and therefore $\bar{\varphi}_{r,0} a_{-t} = 0$, that is, $\bar{\varphi}_{r,0} = 0$ (all $r < 0$).

It follows that $\varphi_{m_1,m_2} = 0$ whenever $m_1 < 0$ and $m_2 = 0$.

For $p_2 = t - 1$, (3.36) yields

$$\sum_{i_2=0}^{\infty} \bar{\varphi}_{r,i_2} \psi_{-q_1,-i_2-t+1} = 0 \quad (\text{for all } r < 0),$$

which implies

$$\bar{\varphi}_{r,0}\psi_{-q_1,-(t-1)} + \bar{\varphi}_{r,1}\psi_{-q_1,-1-(t-1)} = 0,$$

and therefore

$$\bar{\varphi}_{r,1}\psi_{-q_1,-t}=0,$$

and $\bar{\varphi}_{r,1}=0$, as desired. In an entirely similar way one obtains the remaining conclusions. The regions described by conditions (i), (ii) and (iii) with respect to $\psi(z_1,z_2)=\bar{z_1}^{q_1}f(z_2)$ where $f(z_2)=\sum_{j=-t}^{s}a_j z_2^j$, with $a_{-t}\neq 0$ are shown in Fig. 1.

After illustrating the necessary conditions for $T^*_{\varphi,n}H_{\psi,n}=H_{\psi,n}T_{\widetilde{\varphi},n}$, we now prove that the conditions in Theorem 3.5 are also sufficient.

Theorem 3.11. *Suppose $n>1$. Let $T_{\varphi,n}$ and $H_{\psi,n}$ be a nonzero Toeplitz and a nonzero Hankel operator induced by φ and ψ in $L^\infty(\mathbb{T}^n)$ respectively, where ψ is of the form*

$$\psi(z_1,..,z_n)=\left(\prod_{d\neq i=1}^{n}z_i^{-q_i}\right)f(z_d)=z_1^{-q_1}z_2^{-q_2}\cdots z_{d-1}^{-q_{d-1}}f(z_d)z_{d+1}^{-q_{d+1}}\cdots z_n^{-q_n},$$

for some function f of the variable z_d, $1\leqslant d\leqslant n$, and $(n-1)$ nonnegative integers q_i, $1\leqslant i\leqslant n$, $i\neq d$. Then $T^*_{\varphi,n}H_{\psi,n}=H_{\psi,n}T_{\widetilde{\varphi},n}$ if and only if

(i) $\varphi_{m_1,\dots,m_n}=0$ *if* $m_d<0$ *and* $0\leqslant m_i\leqslant q_i$ *for each* $1\leqslant i\leqslant n$, $i\neq d$.

(ii) $\varphi_{m_1,\dots,m_n}=0$ *if* $m_d<0$ *and* $m_i\leqslant q_i$ *for each* $1\leqslant i\leqslant n$ $(i\neq d)$*, and such that for at least one* $1\leqslant j\leqslant n$, $j\neq d$*, we have* $m_j<0$.

(iii) $\sum_{i_d=0}^{\infty}(\bar{\varphi}_{q_1-m_1,\dots,q_{d-1}-m_{d-1},i_d,q_{d+1}-m_{d+1},\dots,q_n-m_n})\cdot(\psi_{-q_1,\dots,-q_{d-1},-i_d-p_d,-q_{d+1},\dots,-q_n})$ $=0$*, for each integer* $p_d\geqslant 0$ *and* $m_i\geqslant 0$ $(1\leqslant i\leqslant n,\ i\neq d)$*, and such that for at least one* $1\leqslant j\leqslant n$, $j\neq d$*, we have* $m_j>q_j$.

Proof. The necessity was already proved in Theorem 3.5. For the sufficiency, we assume that the relations $(i),(ii)$ and (iii) hold. For $\boldsymbol{m},\boldsymbol{p}\in\mathbb{Z}^n_+$ we have

$$\left\langle T^*_{\varphi,n}H_{\psi,n}(\boldsymbol{z}^{\boldsymbol{m}}),\boldsymbol{z}^{\boldsymbol{p}}\right\rangle=\sum_{\boldsymbol{i}\in\mathbb{Z}^n_+}\bar{\varphi}_{\boldsymbol{i}-\boldsymbol{p}}\psi_{-\boldsymbol{i}-\boldsymbol{m}} \tag{3.37}$$

and

$$\langle H_{\psi,n}T_{\widetilde{\varphi},n}(\boldsymbol{z}^{\boldsymbol{m}}),\boldsymbol{z}^{\boldsymbol{p}}\rangle=\sum_{\boldsymbol{i}\in\mathbb{Z}^n_+}\bar{\varphi}_{\boldsymbol{i}-\boldsymbol{m}}\psi_{-\boldsymbol{i}-\boldsymbol{p}}. \tag{3.38}$$

Using given conditions $(i),(ii),(iii)$ and form of ψ, the right hand side expression of (3.37) reduces to

$$\begin{cases} \sum_{i_d \geqslant p_d} (\bar{\varphi}_{q_1-p_1-m_1,..,q_{d-1}-p_{d-1}-m_{d-1},i_d-p_d,q_{d+1}-p_{d+1}-m_{d+1},..,q_n-p_n-m_n}) \cdot \\ \quad (\psi_{-q_1,..,-q_{d-1},-i_d-m_d,-q_{d+1},..,-q_n}), & \text{if } 0 \leqslant m_t \leqslant q_t \text{ for each } \\ & 1 \leqslant t \leqslant n,\ t \neq d \\ 0 & \text{otherwise} \end{cases}$$

which can be rewritten as

$$\begin{cases} \sum_{s_d \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1,\cdots,q_{d-1}-p_{d-1}-m_{d-1},s_d,q_{d+1}-p_{d+1}-m_{d+1},\cdots,q_n-p_n-m_n}) \cdot \\ (\psi_{-q_1,\cdots,-q_{d-1},-s_d-m_d-p_d,-q_{d+1},\cdots,-q_n}), & \text{if } 0 \leqslant m_t \leqslant q_t \text{ for each } \\ & 1 \leqslant t \leqslant n,\ t \neq d \\ 0 & \text{otherwise} \end{cases}$$

and is further same as,

$$\begin{cases} \sum_{s_d \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1,\cdots,q_{d-1}-p_{d-1}-m_{d-1},s_d,q_{d+1}-p_{d+1}-m_{d+1},\cdots,q_n-p_n-m_n}) \cdot \\ \quad (\psi_{-q_1,\cdots,-q_{d-1},-s_d-m_d-p_d,-q_{d+1},\cdots,-q_n}), & \text{if } 0 \leqslant m_t + p_t \leqslant q_t \\ & \text{for } 1 \leqslant t \leqslant n,\ t \neq d, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we find that the right hand side expression of (3.38) becomes

$$\begin{cases} \sum_{i_d \geqslant m_d} (\bar{\varphi}_{q_1-p_1-m_1,\cdots,q_{d-1}-p_{d-1}-m_{d-1},i_d-m_d,q_{d+1}-p_{d+1}-m_{d+1},\cdots,q_n-p_n-m_n}) \cdot \\ \quad (\psi_{-q_1,\cdots,-q_{d-1},-i_d-p_d,-q_{d+1},\cdots,-q_n}), & \text{if } 0 \leqslant p_t \leqslant q_t \text{ for each } \\ & 1 \leqslant t \leqslant n,\ t \neq d, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sum_{s_d \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1,\cdots,q_{d-1}-p_{d-1}-m_{d-1},s_d,q_{d+1}-p_{d+1}-m_{d+1},\cdots,q_n-p_n-m_n}) \cdot \\ (\psi_{-q_1,\cdots,-q_{d-1},-s_d-m_d-p_d,-q_{d+1},\cdots,-q_n}), & \text{if } 0 \leqslant p_t \leqslant q_t \text{ for each } \\ & 1 \leqslant t \leqslant n,\ t \neq d, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sum_{s_d \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1,\cdots,q_{d-1}-p_{d-1}-m_{d-1},s_d,q_{d+1}-p_{d+1}-m_{d+1},\cdots,q_n-p_n-m_n}) \cdot \\ (\psi_{-q_1,\cdots,-q_{d-1},-s_d-m_d-p_d,-q_{d+1},\cdots,-q_n}), & \text{if } 0 \leqslant m_t + p_t \leqslant q_t \text{ for } \\ & 1 \leqslant t \leqslant n,\ t \neq d, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we conclude that

$$\left\langle T^*_{\varphi,n} H_{\psi,n}(z_1^{m_1}\dots z_n^{m_n}), z_1^{p_1}\dots z_n^{p_n}\right\rangle = \left\langle H_{\psi,n} T_{\widetilde{\varphi},n}(z_1^{m_1}\dots z_n^{m_n}), z_1^{p_1}\dots z_n^{p_n}\right\rangle,$$

for each $(m_1,\dots,m_n)$ and $(p_1,\dots,p_n)$ in $\mathbb{Z}^n_+$. Thus $T^*_{\varphi,n} H_{\psi,n} = H_{\psi,n} T_{\widetilde{\varphi},n}$. □

We now present an illustrative example (for the case $n = 2$) that provides a better understanding of the proof of sufficiency in Theorem 3.11.

Example 3.12. Suppose $T_{\varphi,2}$ and $H_{\psi,2}$ are nonzero Toeplitz and nonzero Hankel operators induced by φ and ψ in $L^\infty(\mathbb{T}^2)$, respectively, where ψ is of the form

$$\psi(z_1,z_2) = z_1^{-q_1} f(z_2),$$

for some function f of the variable z_2 and nonnegative integer q_1. If the function φ satisfies the following conditions

(i) $\varphi_{m_1,m_2} = 0$ if $m_2 < 0$ and $m_1 \leqslant q_1$, and
(ii) $\displaystyle\sum_{i_2=0}^{\infty} \bar{\varphi}_{q_1-m_1,i_2} \cdot \psi_{-q_1,-i_2-p_2} = 0$, for integers $p_2 \geqslant 0$ and $m_1 > q_1$,

then we prove that $T^*_{\varphi,2} H_{\psi,2} = H_{\psi,2} T_{\widetilde{\varphi},2}$. To begin with, we take (m_1,m_2) and (p_1,p_2) in $\mathbb{Z}_2^+$, then,

$$\left\langle T^*_{\varphi,2} H_{\psi,2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2}\right\rangle = \sum_{(i_1,i_2)\in\mathbb{Z}^2_+} \bar{\varphi}_{i_1-p_1,i_2-p_2} \psi_{-i_1-m_1,-i_2-m_2}, \tag{3.39}$$

and

$$\left\langle H_{\psi,2} T_{\widetilde{\varphi},2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2}\right\rangle = \sum_{(i_1,i_2)\in\mathbb{Z}^2_+} \bar{\varphi}_{i_1-m_1,i_2-m_2} \psi_{-i_1-p_1,-i_2-p_2}. \tag{3.40}$$

Since $\psi_{j_1,j_2} = 0$ whenever $j_1 \neq -q_1$, thus in the summation in equation (3.39) we put $i_1 = q_1 - m_1$, provided $q_1 - m_1 \geqslant 0$. Thus equation (3.39) reduces to

$$\begin{cases} \displaystyle\sum_{i_2\geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1,i_2-p_2}) \cdot (\psi_{-q_1,-i_2-m_2}) & \text{provided } 0 \leqslant m_1 \leqslant q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varphi_{j_1,j_2} = 0$ whenever $j_2 < 0$ and $j_1 \leqslant q_1$, thus the above expression further reduces to,

$$\begin{cases} \sum\limits_{i_2 \geqslant p_2} (\bar{\varphi}_{q_1-p_1-m_1, i_2-p_2}) \cdot (\psi_{-q_1, -i_2-m_2}) & \text{provided } 0 \leqslant m_1 \leqslant q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Substitute $i_2 - p_2 = s_2$ in the above summation to obtain the following

$$\begin{cases} \sum\limits_{s_2 \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1, s_2}) \cdot (\psi_{-q_1,\ -s_2-m_2-p_2}) & \text{provided } 0 \leqslant m_1 \leqslant q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the given condition (ii) we finally obtain that $\left\langle T^*_{\varphi,2} H_{\psi,2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2} \right\rangle =$

$$\begin{cases} \sum\limits_{s_2 \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1, s_2}) \cdot (\psi_{-q_1,\ -s_2-m_2-p_2}) & \text{provided } 0 \leqslant m_1 + p_1 \leqslant q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we see that $\langle H_{\psi,2} T_{\tilde{\varphi},2}(z_1^{m_1} z_2^{m_2}), z_1^{p_1} z_2^{p_2} \rangle$ is also equal to

$$\begin{cases} \sum\limits_{s_2 \geqslant 0} (\bar{\varphi}_{q_1-p_1-m_1, s_2}) \cdot (\psi_{-q_1,\ -s_2-m_2-p_2}) & \text{provided } 0 \leqslant m_1 + p_1 \leqslant q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$T^*_{\varphi,2} H_{\psi,2} = H_{\psi,2} T_{\tilde{\varphi},2}.$$

As an application of Theorem 3.11 we now discuss the following example.

Example 3.13. Consider the case $n = 2$. Take $\psi(z_1, z_1) = {z_1}^{-1} {z_2}^{-1}$ and $\varphi(z_1, z_2) = \sum\limits_{\substack{(i_1,i_2) \in \mathbb{Z}^2 \\ \text{with} i_1 < 0 \\ i_2 \geqslant 2}} 2^{i_1} z_1^{i_1} z_2^{i_2}$. Then φ and ψ are in $L^\infty(\mathbb{T}^2)$. We can take $d = 1$ and $q_2 = 1$ or $d = 2$ and $q_1 = 1$. Let us take $q_1 = 1$ and $d = 2$. For the given function φ we verify the conditions (i), (ii) and (iii) of Theorem 3.11.

Since the Fourier coefficients φ_{m_1,m_2} are zero if $0 \leqslant m_1 \leqslant 1$ and $m_2 < 0$, condition (i) is verified. Next, we see that $\varphi_{m_1,m_2} = 0$ if $m_1 < 0$ and $m_2 < 0$ and thus condition (ii) is satisfied. Further we verify condition (iii), by considering the expression $\sum_{i_2=0}^{\infty} \bar{\varphi}_{1-m_1,i_2} \cdot \psi_{-1,-i_2-p_2}$, for all integers $p_2 \geqslant 0, m_1 > 1$. Since $\psi_{-1,-1} = 1$ and all other Fourier coefficients of the function ψ are zero, so in the above summation only those values of i_2 and p_2 will contribute for which $i_2 + p_2 = 1$.

Thus, for $p_2 = 0$, i_2 will be 1 and we see that $\varphi_{1-m_1,1}$ is zero for all integers $m_1 > 1$. For $p_2 = 1$, i_2 will be 0 and $\varphi_{1-m_1,0}$ is zero for all integers $m_1 > 1$. Hence φ satisfies conditions (i), (ii) and (iii) of Theorem 3.11 and the relationship $T^*_{\varphi,2}H_{\psi,2} = H_{\psi,2}T_{\widetilde{\varphi},2}$ holds.

Our next result provides a characterization for the commutativity of Toeplitz and Hankel operators induced by symbols of a specific type.

Theorem 3.14. *Let $n > 1$ and suppose that $H_{\psi,n}$ is a nonzero Hankel operator induced by ψ in $L^\infty(\mathbb{T}^n)$ of the form*

$$\psi(z_1, \ldots, z_n) = z_1^{-q_1} z_2^{-q_2} \ldots z_{d-1}^{-q_{d-1}} f(z_d) z_{d+1}^{-q_{d+1}} \ldots z_n^{-q_n},$$

for some function f of the variable z_d, $1 \leqslant d \leqslant n$, and $(n-1)$ nonnegative integers $q_i, 1 \leqslant q_i \leqslant n, i \neq d$. If $0 \neq \varphi$ in $L^\infty(\mathbb{T}^n)$ is such that $\varphi = \varphi^*$ then the Toeplitz operator $T_{\varphi,n}$ commutes with $H_{\psi,n}$ if and only if the following conditions hold:

(i) $\varphi_{m_1,\ldots,m_n} = 0$ *if* $m_d > 0$ *and* $-q_i \leqslant m_i \leqslant 0$ *for each* $1 \leqslant i \leqslant n$, $i \neq d$.
(ii) $\varphi_{m_1,\ldots,m_n} = 0$ *if* $m_d > 0$ *and* $-q_i \leqslant m_i$ *for each* $1 \leqslant i \leqslant n, i \neq d$, *and such that for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j > 0$.
(iii) $\sum_{i_d=0}^{\infty} (\varphi_{m_1-q_1,\ldots,m_{d-1}-q_{d-1},-i_d,m_{d+1}-q_{d+1},\ldots,m_n-q_n}) \cdot (\psi_{-q_1,\ldots,-q_{d-1},-i_d-p_d,-q_{d+1},\ldots,-q_n}) = 0$, *for each integers* $p_d \geqslant 0$ *and* $m_i \geqslant 0$ *such that* $1 \leqslant i \leqslant n, i \neq d$, *and for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j > q_j$.

Proof. Since $\varphi = \varphi^*$, $\varphi_{i_1,\ldots,i_n} = \varphi_{-i_1,\ldots,-i_n}$ for each $(i_1, \ldots, i_n) \in \mathbb{Z}^n$. Also, $\widetilde{\bar{\varphi}} = \varphi^* = \varphi$; thus, the commutativity of $T_{\varphi,n}$ and $H_{\psi,n}$ is equivalent to say that $T^*_{\bar{\varphi},n}H_{\psi,n} = H_{\psi,n}T_{\widetilde{\bar{\varphi}},n}$. Replacing φ by $\bar{\varphi}$ in Theorem 3.11, we obtain the desired result. □

Theorem 3.1 and the observation drawn immediately after Corollary 3.7 yield the result proved in [12] in case of one variable, which states that if φ in $L^\infty(\mathbb{T})$ is such that $\varphi = \varphi^*$ then $T_\varphi H = HT_\varphi$ for a nonzero Hankel operator H if and only if φ is a constant function. In the next example, we see that the same is not true in case $n > 1$. The example also illustrates an application of Theorem 3.14 to provide commuting Toeplitz and Hankel operators.

Example 3.15. Take $n = 2$ and define

$$\varphi(z_1, z_2) = \overline{z_1} z_2 + z_1 \overline{z_2}$$

and

$$\psi(z_1, z_2) = 1$$

for $(z_1, z_2) \in \mathbb{T}^2$. It is straightforward to see that φ and ψ are in $L^\infty(\mathbb{T}^2)$ and $\varphi = \varphi^*$. Here, $\varphi_{-1,1} = 1 = \varphi_{1,-1}$ and all other Fourier coefficients in the Fourier series expansion of φ are zero. It is computationally simple to see that conditions $(i), (ii)$ and (iii) in Theorem 3.14 are satisfied and $T_{\varphi,2}$ and $H_{\psi,2}$ are nonzero operators. Thus $T_{\varphi,2}H_{\psi,2} = H_{\psi,2}T_{\varphi,2}$ (in fact on computation we see that $T_{\varphi,2}H_{\psi,2}$ and $H_{\psi,2}T_{\varphi,2}$ both are zero operators). However, φ is not a constant function.

We now define an operator A on $H^2(\mathbb{T}^n)$ as $A(f) = \widetilde{f}$ for each f in $H^2(\mathbb{T}^n)$. Operator A helps us in further investigation of commutative properties between Hankel and Toeplitz operators on $H^2(\mathbb{T}^n)$. We first see the following properties of A which can be obtained with simple computations.

Proposition 3.16.

(i) *A is anti-linear and $A^2 = I$, the identity operator.*
(ii) *$\langle Af, Ag\rangle = \overline{\langle f, g\rangle}$ for each f and g in $H^2(\mathbb{T}^n)$.*
(iii) *For any φ in $L^\infty(\mathbb{T}^n)$, $AT_{\varphi,n}A = T_{\widetilde{\varphi},n}$ and $AH_{\varphi,n}A = H^*_{\varphi,n} = H_{\widetilde{\varphi},n}$.*

Lemma 3.17. *For φ and ψ in $L^\infty(\mathbb{T}^n)$, $H_{\psi,n}$ commutes with $T_{\varphi,n}$ if and only if it commutes with $T_{\varphi^*,n}$.*

Proof. Now it is direct to see that

$$\begin{aligned} T_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\varphi,n} &\Leftrightarrow AT_{\varphi,n}AAH_{\psi,n}A = AH_{\psi,n}AAT_{\varphi,n}A \\ &\Leftrightarrow T_{\widetilde{\varphi},n}H^*_{\psi,n} = H^*_{\psi,n}T_{\widetilde{\varphi},n} \\ &\Leftrightarrow H_{\psi,n}T^*_{\widetilde{\varphi},n} = T^*_{\widetilde{\varphi},n}H_{\psi,n} \\ &\Leftrightarrow T_{\varphi^*,n}H_{\psi,n} = H_{\psi,n}T_{\varphi^*,n}. \end{aligned}$$

Hence the result. □

Now we use our findings to prove a necessary condition for the commutativity of a Toeplitz and a nonzero Hankel operator on the space $H^2(\mathbb{T}^n)$.

Theorem 3.18. *Let $H_{\psi,n}$ be a nonzero Hankel operator induced by symbol ψ in $L^\infty(\mathbb{T}^n)$, $(n > 1)$, of the form*

$$\psi(z_1, \ldots, z_n) = z_1^{-q_1} z_2^{-q_2} \ldots z_{d-1}^{-q_{d-1}} f(z_d) z_{d+1}^{-q_{d+1}} \ldots z_n^{-q_n},$$

for some function f of the variable z_d, $(1 \leqslant d \leqslant n)$, and $(n-1)$ nonnegative integers $q_i, 1 \leqslant q_i \leqslant n, i \neq d$. If $\varphi \in L^\infty(\mathbb{T}^n)$ is given by $\varphi(z_1, \ldots, z_n) = \sum_{\boldsymbol{i}=(i_1,\ldots,i_n)\in\mathbb{Z}^n} \varphi_{i_1,\ldots,i_n} {z_1}^{i_1} \ldots {z_n}^{i_n}$, then the commutativity of $T_{\varphi,n}$ and $H_{\psi,n}$ implies the following:

(i) *$\varphi_{m_1,\ldots,m_{d-1},m_d,m_{d+1},\ldots,m_n} = -\varphi_{-m_1,\ldots,-m_{d-1},-m_d,-m_{d+1},\ldots,-m_n}$ if $m_d < 0$ and $0 \leqslant m_i \leqslant q_i$ for each $1 \leqslant i \leqslant n$, $i \neq d$.*

(ii) $\varphi_{m_1,\ldots,m_{d-1},m_d,m_{d+1},\ldots,m_n} = -\varphi_{-m_1,\ldots,-m_{d-1},-m_d,-m_{d+1},\ldots,-m_n}$ *if* $m_d < 0$ *and* $m_i \leqslant q_i$ *for each* $1 \leqslant i \leqslant n, i \neq d$, *and such that for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j < 0$.

(iii) $$\sum_{i_d=0}^{\infty}(\varphi_{m_1-q_1,\ldots,m_{d-1}-q_{d-1},-i_d,m_{d+1}-q_{d+1},\ldots,m_n-q_n})\cdot(\psi_{-q_1,\ldots,-q_{d-1},-i_d-p_d,-q_{d+1},\ldots,-q_n})= -\sum_{i_d=0}^{\infty}(\varphi_{q_1-m_1,\ldots,q_{d-1}-m_{d-1},i_d,q_{d+1}-m_{d+1},\ldots,q_n-m_n})\cdot(\psi_{-q_1,\ldots,-q_{d-1},-i_d-p_d,-q_{d+1},\ldots,-q_n}),$$
for each integer $p_d \geqslant 0$ *and* $m_i \geqslant 0$ *for each* $1 \leqslant i \leqslant n, i \neq d$, *and such that for at least one* $1 \leqslant j \leqslant n, j \neq d$, *we have* $m_j > q_j$.

Proof. Suppose that for given φ and ψ, $T_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\varphi,n}$. Using Lemma 3.17, we obtain $T_{\varphi^*,n}H_{\psi,n} = H_{\psi,n}T_{\varphi^*,n}$; thus $T_{\varphi+\varphi^*,n}H_{\psi,n} = H_{\psi,n}T_{\varphi+\varphi^*,n}$. Applying Theorem 3.14 and the fact that $\varphi^*_{i_1,\ldots,i_n} = \varphi_{-i_1,\ldots,-i_n}$ for any $(i_1,\ldots,i_n)$ in $\mathbb{Z}^n$, we obtain the desired conditions. □

The conditions obtained in Theorem 3.18 are only necessary, and not sufficient to ensure the commutativity between a Toeplitz operator and a nonzero Hankel operator in several variables. The following example illustrates this fact.

Example 3.19. Consider $n = 2$ and take $\varphi(z_1, z_2) = \overline{z_1}z_2 - z_1\overline{z_2}$ and $\psi(z_1, z_2) = \overline{z_1}\,\overline{z_2} + \overline{z_1}z_2^2$ on $\mathbb{T}^2$. Then φ and ψ are in $L^\infty(\mathbb{T}^2)$ and satisfy the conditions (i), (ii) and (iii) of Theorem 3.18, but $T_{\varphi,2}H_{\psi,2}(1) = z_2^2 - z_1^2$ and $H_{\psi,n}T_{\varphi,n}(1) = 0$.

We observe that when $n > 1$, the condition $T_{\varphi,n}H_{\psi,n} = H_{\psi,n}T_{\varphi,n}$ (for nonzero $H_{\varphi,n}$) need not provide that $\varphi+\varphi^*$ is a constant function, unlike in the single variable case. This can be seen by considering φ and ψ defined in Example 3.15, which satisfy $T_{\varphi,2}H_{\psi,2} = H_{\psi,2}T_{\varphi,2}$. But $(\varphi + \varphi^*)(z_1, z_2) = 2\varphi$ is not a constant function.

We conclude with the remark that all the results beginning with Theorem 3.5 onward do hold if we replace ψ by ζ in $L^\infty(\mathbb{T}^n)$ such that $(CAP)_\zeta = (CAP)_\psi$, which follows immediately using Corollary 2.6.

Declaration of competing interest

The submitted work is original. It has not been published elsewhere in any form or language (partially or in full), and it is not under simultaneous consideration or in press by another journal. The authors have no competing interests to declare that are relevant to the content of this article.

Data availability

No data was used for the research described in the article.

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