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Slow Propagation of Information on the Random XXZ Quantum Spin Chain

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Abstract: The random XXZ quantum spin chain manifests localization (in the form of quasi-locality) in any fixed energy interval, as previously proved by the authors. In this article it is shown that this property implies slow propagation of information, one of the putative signatures of many-body localization (MBL), in the same energy interval.

Contents

1. Introduction

The folk wisdom in physics is that particle interactions tend to delocalize (or, more precisely, dynamically thermalize) an isolated quantum system. In contrast, the presence of disorder in the single particle context leads to the emergence of localization. It is, therefore, an interesting question of how such systems behave in the presence of both disorder and interactions. It has been proposed in the physics literature that in dimension one strong disorder leads to the so-called many-body localized (MBL) phase, presumed to be characterized by several exotic properties, such as the absence of thermalization, slow propagation of information, zero-velocity Lieb–Robinson bound, Poisson distribution for level statistics, and area-law entanglement of eigenstates. This led to significant theoretical and experimental work in condensed matter physics over the last decade that focused on this phenomenon and its implications (see the physics reviews [2,4,20]).

Let us stress that, as of today, there is no unifying physics theory for MBL as well as no clear consensus among the physics community on the existence and stability of

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an MBL phase in the thermodynamic limit, even in the strong disorder regime, due to new numerical evidence and some theoretical work [1,14,17,21–25]. Moreover, the hierarchical relationship between the proposed properties (i.e., whether one of them implies others) is also not clear. The most significant difficulty in analyzing such models is that the dimension of the underlying Hilbert space grows exponentially fast with the system size L. Such growth limits reliable numerics to small L and makes it exceedingly hard to capture the rare but potentially critical events (such as resonances) that inevitably occur as the system size increases.

One of the central questions among the physics community is what a suitable (and malleable) definition of MBL should be. A popular choice there is the existence of Local Integrals of Motion (LIOM), which in particular implies a form of dynamical localization [19]. However, the existence of LIOM is based on the exact diagonalization of the entire Hamiltonian, a very strong assumption (it implies, in particular, the absence of a phase transition for the infinite system—a debatable assertion even among physicists).

In [7], we introduced and proved a suitably defined notion of quasi-locality associated with the finite XXZ spin-$\frac{1}{2}$ random chain in any fixed energy interval in a certain parameter region, that includes the limiting cases of strong disorder and weak interactions. The disordered XXZ model is one of the most common models used in the physics and mathematics literature for the study of MBL (e.g., [2]). We consider finite volume Hamiltonians, which are what is typically discussed in the physics literature (e.g., [2]). An important feature of our result is that while the parameter region depends on the energy interval, it is independent of system size.

A fixed energy interval is sometimes referred to in physics as the zero temperature regime and has to be contrasted with the infinite-temperature regime, that is, the whole energy spectrum.[1] While we argued in [7] that our quasi-locality property (different from any of the physics signatures of MBL mentioned above) is very natural from a mathematical point of view, we do not expect it to be useful for studying the infinite-temperature MBL.

While the quasi-locality property is amenable to rigorous analysis, we did not attempt in [7] to explore its connection to the putative manifestations of MBL proposed in the physics literature and mentioned above. The current work shows that the quasi-locality property implies slow propagation of information (one of the aforementioned signature properties) in the same energy interval on which the quasi-locality holds. This implication was not obvious to us when [7] was completed, as it addresses a different object and its proof required a new set of ideas.

Let us mention that the other proposed indicators of MBL-type localization (besides slow information propagation) seem to be either unaccessible or significantly harder to reach:

(i) It is expected that generic quantum many-body systems exhibit thermalization or even satisfy the eigenstate thermalization hypothesis. However, up to now mathematically tangible arguments have not been found for proving thermalization or its failure outside the realm of exactly solvable systems.
(ii) As we already mentioned, the LIOMs' approach hinges on a complete (for all energies) localization of the underlying Hamiltonian. Such property has been only achieved for some exactly solvable models. Similarly, the proposal that the zero velocity Lieb–Robinson bound holds for random systems (see [10]) relies on complete localization. A modified analogue of the Lieb–Robinson bound was proven

[1] Note that the diameter of the spectrum of the interacting system grows with the system size.

in the droplet spectrum (the special interval at the bottom of the spectrum) of the random XXZ spin chain [8, Theorem 3]. However, the form of the resulting bound is interval-dependent, making it unlikely to be a suitable candidate for an MBL characteristic for systems where a phase transition could potentially occur.

(iii) In [6], it was shown that, even without disorder, the excited states of the XXZ spin chain (with pretty much any choice for the background potential) satisfy the area law with logarithmic corrections for any fixed energy interval. While the area law without these corrections holds in the droplet spectrum of the random XXZ spin chain [6], it is not expected to persist beyond this interval on physical grounds [5,10]. Thus it is hard to probe localization at higher energies using an area law-type criterion alone.

(iv) Another proposal is to link localization of the interacting ground state with exponential decay of the zero temperature grand-canonical truncated correlations of local operators [16]. This result involves multiple limits (including zero temperature and thermodynamic ones), so it is not clear how to formulate it as a statement that holds for a finite system. In addition, it appears that the result is highly sensitive to the order in which these limits are taken.

Let us give an informal account of our result (the formal statement can be found in Sect. 2 below). The random XXZ quantum spin-$\frac{1}{2}$ chain on the finite discrete interval (i.e., an interval in $\mathbb{Z}$) $\Lambda_L = [1, L]$ is given by the Hamiltonian $H_\omega^L = H_0^L + \lambda V_\omega^L$ acting on $\bigotimes_{i\in\Lambda_L} \mathbb{C}_i^2$ (here $\mathbb{C}_i^2$ is a copy of $\mathbb{C}^2$), where

$$H_0 = \sum_{i=1}^{L-1} \tfrac{1}{4}\left(I - \sigma_i^z \sigma_{i+1}^z\right) - \tfrac{1}{4\Delta}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)$$

(here $\sigma^{x,y,z}$ are the standard Pauli matrices and $\Delta > 1$ is the anisotropy parameter), and $V_\omega^L = \sum_{i=1}^L \omega_i \mathcal{N}_i$ is the random field (here $\mathcal{N} = \frac{1}{2}(I - \sigma^z)$, $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ is a family of independent identically distributed positive random variables with sufficiently regular randomness, and $\lambda > 0$ is the disorder parameter).

The following theorem is an informal statement of our main result, Theorem 2.6.

Theorem (Slow propagation of information, informal). *For a given energy $E > 0$, there exists a non-trivial region in the (Δ, λ) parameter space, such that for any fixed point in this region, scales $L, \ell \in \mathbb{N}$, and all $t \in \mathbb{R}$, the following holds: For every observable $\mathcal{O}$ supported on a discrete interval $[a, b] \subset \Lambda_L$, there exists an observable $\mathcal{O}_t = \mathcal{O}(t, E, \ell, L)$, supported on $[a - c_E\ell, b + c_E\ell] \cap \Lambda_L$, such that*

$$\mathbb{E}\left\| P_{[0,E]}\left(\mathrm{e}^{itH_\omega^L}\mathcal{O}\mathrm{e}^{-itH_\omega^L} - \mathcal{O}_t\right)P_{[0,E]}\right\| \le C_E \|\mathcal{O}\| (|t|+1)^{q_E} L^{\xi_E} \mathrm{e}^{-\theta_E \ell}, \quad (1.1)$$

where $\mathbb{E}$ stands for the expectation with respect to ω, $P_{[0,E]}$ is the spectral projection of H_ω^L onto $[0, E]$, and $c_E, \xi_E, \theta_E > 0$.

In general, the Lieb–Robinson bound for local spin Hamiltonians [15,18] implies that there is an effective light cone for two-point dynamical correlations for such systems, meaning that these correlations propagate no faster than linearly in time, up to exponentially small corrections. As a consequence of this bound, if $\mathcal{O}$ is a local observable supported on the discrete interval $[a, b]$, given $\ell \in \mathbb{N}$ there exists an observable $\mathcal{O}_{t,\ell}$, supported on the discrete interval $[a-\ell, b+\ell]$, that approximates the (full) Heisenberg evolution $\mathrm{e}^{itH}\mathcal{O}\mathrm{e}^{-itH}$:

$$\left\| \mathrm{e}^{itH}\mathcal{O}\mathrm{e}^{-itH} - \mathcal{O}_{t,\ell} \right\| \leq C \left\| \mathcal{O} \right\| e^{-m(\ell - v|t|)},$$

where $m > 0$ and $v > 0$ is the velocity in the Lieb–Robinson bound (see [11]).

For translation invariant systems, it is expected that information can indeed spread within the light cone. This should be contrasted with the theorem above, that indicates a much slower rate at which the information spreads for the random system: For a given value of $\ell \gg \ln L$, it takes time $t \sim e^{c\ell}$ rather than $t \sim \ell$ until information can potentially escape the corresponding cone. We note that our result considers propagation of observables within the energy window $[0, E]$, and, in particular, is fully compatible with a possible phase transition for higher energies.

Rigorous results of this kind have been previously obtained for exactly solvable systems (in fact, with no propagation at all), see, e.g., [3], and for the XXZ ferromagnetic spin chain studied here, but restricted to the energy interval $I_{\leq 1}$ (introduced in Sect. 2) corresponding to so called droplet spectrum [8, Theorem 2]. Albeit this result provides a strong bound on the information propagation speed, it is tailored for $I_{\leq 1}$, and the method developed there cannot be adapted to the larger intervals $I_{\leq q}$ that are handled in Theorem 2.6. While the dependence of our bound on both the system size and time is by no means optimal, it is generally expected that the random XXZ spin chain should exhibit some form of slow propagation (e.g., [2]).

We now want to address the presence of the polynomial pre-factor in the volume size in (1.1). It is not unusual to have a volume dependence in local results concerning random systems (e.g., the multiscale analysis for random Schrödinger operators yields decay in a box of size L for distances $\geq L^{\zeta}$, $\zeta \in (0, 1)$), as in the localization phenomenon there are two competing effects: A natural tendency for eigenstates to localize versus the small denominator problem, coming from resonances. The former is responsible for the exponential decay in (1.1), whereas the appearance of the volume prefactor is a manifestation of the latter. Indeed, the number of resonances is directly related to the density of states, which, for the ferromagnetic XXZ spin system studied here, grows as a power of the volume (with the power increasing with the energy). For this reason, we do not expect that the bound in (1.1) can be significantly improved.

Since the physics literature mainly considers *finite* spin systems (e.g., [2]), the polynomial pre-factor is not a real issue in the presence of the exponential decay in ℓ, as long as $\ell \geq C \ln L$. Let us also mention that in (reliable) numerical experiments on which physicists base their conclusions about the system's behavior, the typical value of L does not exceed a few dozen, in which case our results fit in rather well with the physics picture for *all* energies.

Nonetheless, the presence of the volume factor has the unfortunate side effect that it is not clear whether any conclusions can be drawn about the infinite volume XXZ model from this estimate. This should be contrasted with the random Schrödinger operator case, where the volume dependence in the decay estimates can be overcome to yield Anderson localization in the infinite volume. This disparity can be traced to the radical difference in the rank of the perturbation needed to decouple the Hamiltonian into two (spatially non-interacting) parts in a discrete Schrödinger operator and in a spin chain. In the former case, the rank is comparable with the size of the boundary between these two parts (rank 2 in one dimension), while in the latter case the rank is comparable with the dimension of the full Hilbert space, due to its tensor product nature.

One way to mitigate the influence of resonances is to consider a matrix element analogue of (1.1), that is, to consider $\left|\langle\psi, P_{[0,E]}\left(\mathrm{e}^{itH_\omega^L}\mathcal{O}\mathrm{e}^{-itH_\omega^L}-\mathcal{O}_t\right)P_{[0,E]}\phi\rangle\right|$ for a pair of states $\psi,\phi\in\mathcal{H}$ instead of $\left\|P_{[0,E]}\left(\mathrm{e}^{itH_\omega^L}\mathcal{O}\mathrm{e}^{-itH_\omega^L}-\mathcal{O}_t\right)P_{[0,E]}\right\|$. As a consequence of (1.1), we obtain a bound on the expectation of this object where the dependence on the volume is reduced to powers of $\ln L$ instead of L, see Corollary 2.7 below. The price one has to pay here is that a priori the operator $\mathcal{O}_t$ constructed this way depends also on the states ψ,ϕ. We expect that this result may shed light on properties of the infinite volume system.

This article is organized as follows: Sect. 2 starts with the introduction of the XXZ quantum spin-$\frac{1}{2}$ chain in a random field, followed by a short summary (Theorem 2.2) of our localization results in [7], which serves as the starting point for the statement of our main result, Theorem 2.6, exhibiting slow propagation of information under localization. The statement for matrix elements is given in Corollary 2.7. In Sect. 3 we introduce important ingredients for the proof of Theorem 2.6. Section 4 contains the proof of Theorem 2.6. Corollary 2.7 is proven in Sect. 5.

Throughout the paper, we will use generic constants C, c, etc., whose values will be allowed to change from line to line, even in a displayed equation. These constants will not depend on subsets of $\mathbb{Z}$, but they will, in general depend on parameters of the model such as μ, Δ_0, and λ_0. When necessary, we will indicate the dependence of a constant on other parameters, say q, explicitly by writing the constant as C_q, etc. These constants can always be estimated from the arguments, but we will not track the changes to avoid complicating the arguments.

2. The Model, Localization, and the Main Result

2.1. Model description. Let $\uparrow\rangle:=\begin{pmatrix}1\\0\end{pmatrix}$ and $\downarrow\rangle=\begin{pmatrix}0\\1\end{pmatrix}$ denote the elements of the canonical basis of $\mathbb{C}^2$, called spin-up and spin-down, respectively. Let $\sigma^{x,y,z}$ be the standard Pauli matrices, $\sigma^{\pm}=\frac{1}{2}(\sigma^x\pm i\sigma^y)$. Set $\mathcal{N}=\frac{1}{2}(I-\sigma^z)$, an operator on $\mathbb{C}^2$, and note that $\mathcal{N}\uparrow\rangle=0$ and $\mathcal{N}\downarrow\rangle=\downarrow\rangle$. We interpret $\downarrow\rangle$ as a particle, so $\mathcal{N}$ is the projection onto the spin-down state (or local number operator).

Let $\mathcal{H}_i=\mathcal{H}_{\{i\}}=\mathbb{C}_i^2$ for $i\in\mathbb{Z}$. Given a vector $v\in\mathbb{C}^2$, we denote by v_i its copy in $\mathcal{H}_i$. If T is an observable (i.e., operator) on $\mathbb{C}^2$, we denote by T_i the observable T acting on $\mathcal{H}_i$.

The (infinite volume) XXZ quantum spin-$\frac{1}{2}$ chain in a random field is informally given by the Hamiltonian

$$H_\omega=H_0+\lambda V_\omega, \tag{2.1}$$

acting on $\bigotimes_{i\in\mathbb{Z}}\mathcal{H}_i$, where:

(i) The (disorder) free Hamiltonian H_0 is given by

$$\begin{aligned}H_0&=\sum_{i\in\mathbb{Z}}\left(\tfrac{1}{4}\left(I-\sigma_i^z\sigma_{i+1}^z\right)-\tfrac{1}{4\Delta}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y\right)\right)\\&=\sum_{i\in\mathbb{Z}}\left(\tfrac{1}{4}\left(I-\sigma_i^z\sigma_{i+1}^z\right)-\tfrac{1}{2\Delta}\left(\sigma_i^+\sigma_{i+1}^-+\sigma_i^-\sigma_{i+1}^+\right)\right),\end{aligned} \tag{2.2}$$

where $\Delta>1$ is the anisotropy parameter, specifying the Ising phase ($\Delta=1$ selects the Heisenberg chain and $\Delta=\infty$ corresponds to the the Ising chain).

(ii) $V_\omega = \sum_{i\in\mathbb{Z}} \omega_i \mathcal{N}_i$ is the random field, where $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ is a family of independent identically distributed random variables, whose common probability distribution μ is absolutely continuous with a bounded density and satisfies

$$\{0, 1\} \subset \operatorname{supp} \mu \subset [0, 1], \tag{2.3}$$

and $\lambda > 0$ is the disorder parameter.

We set $\omega_S = \{\omega_i\}_{i\in S}$ for $S \subset \mathbb{Z}$, and denote the corresponding expectation and probability by $\mathbb{E}_S$ and $\mathbb{P}_S$. (We will mostly omit the subscript and just write $\mathbb{E}$ and $\mathbb{P}$ when the choice of S is clear from the context.)

The (infinite volume) Hamiltonian H_ω in (2.1) can be rigorously defined on an appropriately defined Hilbert space, but in this work we only consider finite volume Hamiltonians, since that is what is typically discussed in the physics literature.

Given a finite subset Λ of $\mathbb{Z}$ (Λ will always denote a finite subset), we consider the finite dimensional Hilbert space $\mathcal{H}_\Lambda = \otimes_{i\in\Lambda}\mathcal{H}_i$. If $A \subset \Lambda$ and T is an operator on $\mathcal{H}_A$, we consider T as an operator on $\mathcal{H}_\Lambda$ by identifying it with the operator $T \otimes I_{\mathcal{H}_{\Lambda\setminus A}}$ acting on $\mathcal{H}_\Lambda = \mathcal{H}_A \otimes \mathcal{H}_{\Lambda\setminus A}$. (For a fixed Λ we will often omit Λ from the notation, e.g., $A^c = \Lambda \setminus A$.) For $S \subset \mathbb{Z}$ we let $|S|$ denote the cardinality of the set S.

Since

$$\tfrac{1}{4}\left(I - \sigma_i^z \sigma_{i+1}^z\right) = \tfrac{1}{2}\left(\mathcal{N}_i + \mathcal{N}_{i+1}\right) - \mathcal{N}_i\mathcal{N}_{i+1}, \tag{2.4}$$

we set

$$h_{i,i+1} = -\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{1}{2\Delta}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right), \tag{2.5}$$

a self-adjoint operator $h_{i,i+1}$ on the four-dimensional Hilbert space $\mathcal{H}_{\{i,i+1\}} = \mathcal{H}_i \otimes \mathcal{H}_{i+1}$. An explicit calculation shows

$$\left\| h_{i,i+1} \right\| = 1. \tag{2.6}$$

We can rewrite H_0 as

$$H_0 = \sum_{i\in\mathbb{Z}} \left(h_{i,i+1} + \tfrac{1}{2}(\mathcal{N}_i + \mathcal{N}_{i+1})\right) = \sum_{i\in\mathbb{Z}} h_{i,i+1} + \mathcal{N}^{\mathbb{Z}}, \quad \text{where} \quad \mathcal{N}^{\mathbb{Z}} = \sum_{i\in\mathbb{Z}} \mathcal{N}_i, \tag{2.7}$$

which leads naturally to our definition of finite volume Hamiltonians.

Definition 2.1. The random XXZ quantum spin-$\frac{1}{2}$ chain on a finite subset Λ of $\mathbb{Z}$ is given by the self-adjoint Hamiltonian

$$H^\Lambda = H_0^\Lambda + \lambda V_\omega^\Lambda \quad \text{acting on} \quad \mathcal{H}_\Lambda, \tag{2.8}$$

where

$$H_0^\Lambda = \sum_{\{i,i+1\}\subset\Lambda} h_{i,i+1} + \mathcal{N}^\Lambda, \quad \text{with} \quad \mathcal{N}^\Lambda = \sum_{i\in\Lambda} \mathcal{N}_i \quad \text{and} \quad V_\omega^\Lambda = \sum_{i\in\Lambda} \omega_i \mathcal{N}_i. \tag{2.9}$$

We set $R_z^\Lambda = (H^\Lambda - z)^{-1}$ for $z \notin \sigma(H^\Lambda)$, the resolvent of H^Λ.

The free Hamiltonian H_0^Λ can be rewritten as

$$H_0^\Lambda = -\tfrac{1}{2\Delta}\mathbf{\Delta}^\Lambda + \mathcal{W}^\Lambda \quad \text{on} \quad \mathcal{H}_\Lambda, \tag{2.10}$$

where

$$\mathbf{\Delta}^\Lambda = \sum_{\{i,i+1\}\subset\Lambda} \left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) \quad \text{and} \quad \mathcal{W}^\Lambda = \mathcal{N}^\Lambda - \sum_{\{i,i+1\}\subset\Lambda} \mathcal{N}_i\mathcal{N}_{i+1}. \tag{2.11}$$

The canonical (orthonormal) basis Φ_Λ for $\mathcal{H}_\Lambda$ is constructed as follows: Let $\phi_\emptyset = \Omega_\Lambda = \otimes_{i\in\Lambda} \uparrow\rangle_i$ be the vacuum state. Then

$$\Phi_\Lambda = \left\{\phi_A = \left(\prod_{i\in A}\sigma_i^-\right)\Omega_\Lambda : \ A\subset\Lambda\right\} = \bigcup_{N=0}^{|\Lambda|}\Phi_\Lambda^{(N)}, \tag{2.12}$$

where $\Phi_\Lambda^{(N)} = \{\phi_A : A\subset\Lambda,\ |A| = N\}$. We remark that $\Phi_\Lambda^{(0)} = \{\Omega_\Lambda\}$.

The total (spin-down) number operator $\mathcal{N}^\Lambda$ on Λ is diagonalized by the canonical basis: $\mathcal{N}^\Lambda\phi_A = |A|\,\phi_A$ for $A\subset\Lambda$, and hence has eigenvalues $0, 1, 2, \ldots, |\Lambda|$. We set $\mathcal{H}_\Lambda^{(N)} = \operatorname{Ran}\left(\chi_N(\mathcal{N}^\Lambda)\right)$, obtaining the Hilbert space decomposition $\mathcal{H}_\Lambda = \bigoplus_{N=0}^{|\Lambda|}\mathcal{H}_\Lambda^{(N)}$.

The operators $\mathcal{W}^\Lambda$ and V_ω^Λ are also diagonalized by the canonical basis, and hence the operators $\mathcal{N}^\Lambda$, $\mathcal{W}^\Lambda$, and V_ω^Λ commute. $\mathcal{W}^\Lambda$ is the number of clusters operator: $\mathcal{W}^\Lambda\phi_A = W_A\phi_A$ for $A\subset\Lambda$, where W_A is the number of connected components (clusters) of A as a subset of Λ, so $\sigma\left(\mathcal{W}^\Lambda\right)\subset\{0, 1, 2, \ldots, |\Lambda|\}$. V_ω^Λ is the random field: $V_\omega^\Lambda\phi_A = \omega_A\phi_A$ for $A\subset\Lambda$, where $\omega_A = \sum_{i\in A}\omega_i$.

The Hamiltonian H^Λ preserves the total particle number,

$$[H^\Lambda, \mathcal{N}^\Lambda] = -\tfrac{1}{2\Delta}[\mathbf{\Delta}^\Lambda, \mathcal{N}^\Lambda] = 0, \tag{2.13}$$

a feature that makes the XXZ model especially amenable to analysis.

It can be verified (e.g., [7]), that

$$\left(1 - \tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda \le H_0^\Lambda, \quad \text{so} \quad \left(1 - \tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda \le H^\Lambda, \tag{2.14}$$

and the spectrum of H^Λ is of the form

$$\sigma(H^\Lambda) = \{0\}\cup\left(\left[1 - \tfrac{1}{\Delta}, \infty\right)\cap\sigma(H^\Lambda)\right). \tag{2.15}$$

Moreover, the lower bound in (2.14) suggests the introduction of the energy thresholds $k\left(1 - \frac{1}{\Delta}\right)$, $k = 0, 1, 2\ldots$.

2.2. Localization as quasi-locality. Henceforth, by a subset of $\mathbb{Z}$ we will always mean a finite subset and by an interval in $\mathbb{Z}$ a connected nonempty subset of $\mathbb{Z}$. The observable $\mathcal{O}$ is said to have support in $A \subset \Lambda$ (we write $\operatorname{supp} \mathcal{O} = A$) if $\mathcal{O}$ acts trivially on $\mathcal{H}_{A^c}$, that is, $\mathcal{O} = \mathcal{O}_A \otimes I_{\mathcal{H}_{A^c}}$ where $\mathcal{O}_A$ is an observable on A. (We will identify $\mathcal{O}$ with $\mathcal{O}_A$.) Note that the support of an operator is not uniquely defined.

Given $\emptyset \neq S \subset \mathbb{Z}$, we define the orthogonal projections $P_\pm^S$ on $\mathcal{H}_S$ by

$$P_+^S = \bigotimes_{i \in S} \left(I_{\mathcal{H}_i} - \mathcal{N}_i\right) = \chi_{\{0\}}\left(\mathcal{N}^S\right) \quad \text{and} \quad P_-^S = I_{\mathcal{H}_S} - P_+^S = \chi_{[1,\infty)}\left(\mathcal{N}^S\right). \tag{2.16}$$

P_+^S is the orthogonal projection onto states with no particles in the set S; P_-^S is the orthogonal projection onto states with at least one particle in S. (Note that $P_-^{\{i\}} = \mathcal{N}_i$ for $i \in \mathbb{Z}$.) We also set

$$P_+^{\emptyset} = I_{\mathcal{H}_S} \quad \text{and} \quad P_-^{\emptyset} = 0. \tag{2.17}$$

Given $J \subset \mathbb{R}$ measurable, $B(J)$ denotes the collection of Borel measurable functions that vanish outside J; we set $B_1(J) = \{f \in B(J) : \sup |f| \leq 1\}$.

In [7] we interpreted localization for the random XXZ quantum spin-$\frac{1}{2}$ chain as a form of quasi-locality. The following theorem follows immediately from [7, Theorem 2.4 and Corollary 2.6].

Theorem 2.2 (Quasi-locality). *Fix $\Delta_0 > 1$ and $\lambda_0 > 0$. Then for all $R \geq 0$ there exist constants $D_R, F_R, \widetilde{\xi}_R, \widetilde{\theta}_R > 0$ (depending on R, Δ_0, λ_0) such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ with $\lambda\Delta^2 \geq D_R$, finite interval $\Lambda \subset \mathbb{Z}$, and $A \subset B \subset \Lambda$ with A connected in Λ, we have the following:*

(i) For all $z \in \mathbb{C}$ with $\operatorname{Re} z \leq R\left(1 - \frac{1}{\Delta}\right)$ *we have*

$$\mathbb{E}_\Lambda \left\{ \left\| P_-^A R_z^\Lambda P_+^B \right\|^{\frac{1}{4}} \right\} \leq F_R \, |\Lambda|^{\widetilde{\xi}_R} \, \mathrm{e}^{-\widetilde{\theta}_R \operatorname{dist}_\Lambda(A, B^c)}. \tag{2.18}$$

(ii)

$$\mathbb{E}_\Lambda \left(\sup_{f \in B_1\left(\left(-\infty, R\left(1 - \frac{1}{\Delta}\right)\right]\right)} \left\| P_-^A f(H^\Lambda) P_+^B \right\| \right) \leq F_R \, |\Lambda|^{\widetilde{\xi}_R} \, \mathrm{e}^{-\widetilde{\theta}_R \operatorname{dist}_\Lambda(A, B^c)}. \tag{2.19}$$

Remark 2.3. [7, Theorem 2.4] is stated and proved for $R = k + \frac{3}{4}$, where $k \in \mathbb{N}^0$, and real energies $E \leq (k + \frac{3}{4})\left(1 - \frac{1}{\Delta}\right)$. However, the proof of [7, Theorem 2.4] is also valid for complex energies z with $\operatorname{Re} z \leq (k + \frac{3}{4})\left(1 - \frac{1}{\Delta}\right)$, with the same constants. Picking $k \in \mathbb{N}^0$ so that $R \leq k + \frac{3}{4}$ yields the result stated above. (As an alternative, the proof of [7, Theorem 2.4] can be adapted for the case $R = k + \beta$ with $\beta \in (0, 1)$; we fixed $\beta = \frac{3}{4}$ in [7] for simplicity.)

Remark 2.4. If A is not connected in Λ, the theorem still holds with (2.18) replaced by

$$\mathbb{E}_\Lambda \left\{ \left\| P_-^A R_z^\Lambda P_+^B \right\|^{\frac{1}{4}} \right\} \le F_R \Upsilon_A^\Lambda \, |\Lambda|^{\widetilde{\xi}_R} \, \mathrm{e}^{-\widetilde{\theta}_R \operatorname{dist}_\Lambda(A, B^c)}, \tag{2.20}$$

where Υ_A^Λ denotes the number of connected components of A in Λ. This follows from (2.18) and

$$P_-^A = \sum_{j=1}^{\Upsilon_A^\Lambda} P_+^{\bigcup_{i=i}^{j-1} A_i} P_-^{A_j}, \tag{2.21}$$

where A_j, $j = 1, 2, \ldots, \Upsilon_A^\Lambda$, are the connected components of A in Λ.

2.3. Slow propagation of information. Our main result shows that localization of the XXZ random spin chain in a fixed energy interval implies slow propagation of information in this interval.

In this article we will assume that $\Delta_0 > 9$ in Theorem 2.2. This is done to simplify our analysis, but in fact the result holds for arbitrary $\Delta_0 > 1$ with minor modifications of the proofs.[2]

Given $q \in \frac{1}{2}\mathbb{Z}$, we consider the energy intervals

$$\begin{aligned} I_{\le q} &= \left(-\infty, (q + \tfrac{3}{4})\left(1 - \tfrac{1}{\Delta}\right)\right], \quad & I_q &= \left[1 - \tfrac{1}{\Delta}, (q + \tfrac{3}{4})\left(1 - \tfrac{1}{\Delta}\right)\right], \\ \check{I}_{\le q} &= \left(-\infty, (q + \tfrac{7}{8})\left(1 - \tfrac{1}{\Delta}\right)\right], \quad & \check{I}_q &= \left[1 - \tfrac{1}{\Delta}, (q + \tfrac{7}{8})\left(1 - \tfrac{1}{\Delta}\right)\right]. \end{aligned} \tag{2.22}$$

These intervals are increasing with q. We also note the relation

$$\check{I}_{\le q-1} \subset I_{\le q-\frac{1}{2}} \subset \check{I}_{\le q-\frac{1}{2}} \quad \text{for} \quad q \in \tfrac{1}{2}\mathbb{N}. \tag{2.23}$$

Let Λ be a finite subset of $\mathbb{Z}$. Given an interval $I \subset \mathbb{R}$, we set $P_I = P_I^\Lambda = \chi_I(H^\Lambda)$. If T and Y are observables on $\mathcal{H}_\Lambda$, we define

$$(T)_Y = YTY^*. \tag{2.24}$$

We also consider the Heisenberg time evolution of observables:

$$\tau_t^\Lambda(T) = \mathrm{e}^{itH^\Lambda} T \mathrm{e}^{-itH^\Lambda} \quad \text{for} \quad t \in \mathbb{R}. \tag{2.25}$$

Given $M \subset \Lambda \subset \mathbb{Z}$, we let

$$[M]_s^\Lambda := \begin{cases} \{x \in \Lambda : \operatorname{dist}_\Lambda(x, M) \le s\} & \text{if } s \in \mathbb{N}^0 = \{0\} \cup \mathbb{N} \\ \left\{x \in \Lambda : \operatorname{dist}_\Lambda\left(x, M^c\right) \ge 1 - s\right\} = M \setminus [M^c]_{-s}^\Lambda & \text{if } s \in -\mathbb{N} \end{cases}. \tag{2.26}$$

If $M = \{j\}$ we write $[j]_q^\Lambda = [\{j\}]_q^\Lambda$.

[2] For $1 < \Delta_0 \le 9$ we need to improve the decay rate m_0 in (3.11), which is derived from the lower bound in (3.7). If $1 < \delta_0 < \Delta_0 \le 9$, we would have to replace $\widehat{H}_k^\Lambda$ in the proof by $\widehat{H}_{k+r}^\Lambda$, where $r = \lceil \frac{1}{\delta_0 - 1} - \frac{1}{8} \rceil$, leading to $m_0 = \ln\left((r + \frac{1}{8})(\delta_0 - 1)\right) > 0$.

Definition 2.5. Given $q \in \frac{1}{2}\mathbb{N}^0$, we say that Condition $\mathcal{L}_q$ is satisfied if the parameters Δ and λ satisfy the hypotheses of Theorem 2.2 for $R = q + \frac{7}{8}$, so the conclusions of the theorem are valid for $R = q + \frac{7}{8}$, that is, on the energy interval $\check{I}_{\le q}$. We set $\xi_q = \widetilde{\xi}_{q+\frac{7}{8}}$ and $\theta_q = \widetilde{\theta}_{q+\frac{7}{8}}$.

We use the notation $\langle t \rangle = \left(1 + t^2\right)^{\frac{1}{2}}$ for $t \in \mathbb{R}$. For $q \in \frac{1}{2}\mathbb{N}^0$ we set $\hat{q} = \lceil q \rceil$ and define β_q recursively by

$$\beta_0 = 0, \quad \beta_q = \beta_{q-\frac{1}{2}} + 9\hat{q} + 13. \tag{2.27}$$

Note that $9q^2 + \frac{61}{2}q \le \beta_q \le 9q^2 + \frac{79}{2}q$.

Theorem 2.6 (Slow propagation of information). *Let $q \in \frac{1}{2}\mathbb{N}$, and assume Condition $\mathcal{L}_{q+\frac{1}{2}}$ is satisfied. Then there exists a constant C_q such that for any given finite interval $\Lambda \subset \mathbb{Z}$, scale $\ell \in \mathbb{N}$, and all $t \in \mathbb{R}$, the following holds: For every observable T supported on an interval $\mathcal{X} \subset \Lambda$ with $\|T\| \le 1$ there exists an observable $T_t = T(t, q, \ell, \Lambda)$, supported in $[\mathcal{X}]^{\Lambda}_{(13+\beta_{q+\frac{1}{2}})\ell}$, such that*

$$\mathbb{E} \left\| \left(\tau_t^{\Lambda}(T) - T_t\right)_{P^{\Lambda}_{I_{\le q}}} \right\| \le C_q \langle t \rangle^{2q+4} |\Lambda|^{\xi_{q+\frac{1}{2}}} \mathrm{e}^{-\theta_{q+\frac{1}{2}} \ell}. \tag{2.28}$$

This theorem is proved in Sect. 4.

As discussed in the introduction, we can improve the dependence on the volume by considering matrix elements instead of the norm. The constant $c_\mu > 0$ in the estimate comes from the large deviation estimate [7, Eq. (3.50]) and depends only on the probability distribution μ.

Corollary 2.7 (Slow propagation of information, matrix elements version). *Let $q \in \frac{1}{2}\mathbb{N}$, and assume Condition $\mathcal{L}_{q+1}$ is satisfied. There exist constants C_q and Y_q such that given a finite interval $\Lambda \subset \mathbb{Z}$, scale $\ell \in \mathbb{N}$, $t \in \mathbb{R}$, the following holds: Given subsets $M_i \subset \Lambda$, $i = 1, 2$, with $|M_1| = |M_2|$, then for every observable T supported on an interval $\mathcal{X} \subset \Lambda$, with $\|T\| \le 1$ and $|\mathcal{X}| \le \ln \Lambda$, there exists an observable $T_t = T(t, q, \ell, \Lambda, M_1 \cup M_2)$, supported on $[\mathcal{X}]^{\Lambda}_{(13+\beta_{q+1})\ell}$, such that*

$$\mathbb{E} \left\| \pi_{M_1} \left(\tau_t^{\Lambda}(T) - T_t\right)_{P^{\Lambda}_{I_{\le q}}} \pi_{M_2} \right\| \le C_q \langle t \rangle^{2q+5} (\ln |\Lambda|)^{\xi_{q+1}} \mathrm{e}^{-\frac{1}{2}\min\left\{\theta_q, \theta_{q+\frac{1}{2}}, c_\mu\right\}\ell}, \tag{2.29}$$

provided $|\Lambda| \ge Y_q$.

For fixed $M_i \subset \Lambda$, $i = 1, 2$, with $|M_1| = |M_2|$, the bound (2.29) improves on the dependence on $|\Lambda|$ in (2.28), but the observable T_t in (2.29) a-priori depends on $M_1 \cup M_2$. Note also that if $|M_1| \ne |M_2|$ the left hand side of (2.29) equals 0.

The proof of this corollary is given in Sect. 5.

3. Key Proof Ingredients

In this section we collect a number of definitions, statements and lemmas that, in conjunction with Theorem 2.2, will facilitate the proof of Theorem 2.6.

3.1. Preliminaries. Λ will always denote a finite subset of $\mathbb{Z}$ and $A \subset \Lambda$ will always denote an interval.

Given $M \subset \Lambda$ and $s \in \mathbb{N}$, we set

$$\begin{aligned} \partial_s^{\Lambda,out} M &:= \{x \in \Lambda : \ \mathrm{dist}_\Lambda(x, M) = s\} = [M]_s^\Lambda \setminus M, \\ \partial_s^{\Lambda,in} M &:= \left\{x \in \Lambda : \ \mathrm{dist}_\Lambda\left(x, M^c\right) = s\right\} = M \setminus [M]_{-s}^\Lambda, \\ \partial_s^\Lambda M &:= \partial_s^{\Lambda,in} M \cup \partial_{ex}^{\Lambda,out} M = [M]_s^\Lambda \setminus [M]_{-s}^\Lambda, \\ \boldsymbol{\partial}^\Lambda M &:= \left\{\{x,y\} \subset \Lambda : \ (x,y) \in \left(\partial_1^{\Lambda,in} M \times \partial_1^{\Lambda,out} M\right) \cup \left(\partial_1^{\Lambda,out} M \times \partial_1^{\Lambda,in} M\right)\right\}. \end{aligned} \tag{3.1}$$

If $s = 1$, we occasionally omit it from the notation altogether.

Given $B \subset \mathbb{N}^0$, we set $Q_B^\Lambda = \chi_B\left(\mathcal{W}^\Lambda\right)$, $Q_m^\Lambda = Q_{\{m\}}^\Lambda$ for $m \in \mathbb{N}^0$, and note that $Q_0^\Lambda = P_+^\Lambda$ and $Q_\mathbb{N}^\Lambda = \chi_\mathbb{N}(\mathcal{N}^\Lambda)$. For $k \in \mathbb{N}$, we set

$$Q_{\le k}^\Lambda = Q_{\{1,2,\ldots,k\}}^\Lambda = \sum_{m=1}^k Q_m^\Lambda \quad \text{and} \quad \widehat{Q}_{\le k}^\Lambda = Q_{\le k}^\Lambda + \tfrac{k+1}{k} Q_0^\Lambda. \tag{3.2}$$

We also set $Q_{>k}^\Lambda = I - \widehat{Q}_{\le k}^\Lambda = \chi_{[k+1,\Lambda]}\left(\mathcal{W}^\Lambda\right)$. For $k \in \mathbb{N}$ we have (see [7, Lemma 3.5])

$$\mathrm{tr}\, Q_{\le k}^\Lambda \le k\,|\Lambda|^{2k} \quad \text{and} \quad \mathrm{tr}\, \chi_{\check{I}_{\le k}}(H^\Lambda) \le k\,|\Lambda|^{2k} + 1. \tag{3.3}$$

We also set

$$\begin{aligned} \widehat{H}_0^\Lambda &= H^\Lambda + \left(1 - \tfrac{1}{\Delta}\right) Q_0^\Lambda, \\ \widehat{H}_k^\Lambda &= H^\Lambda + k\left(1 - \tfrac{1}{\Delta}\right) \widehat{Q}_{\le k}^\Lambda \quad \text{for} \quad k \in \mathbb{N}. \end{aligned} \tag{3.4}$$

We use the notation

$$\widehat{R}_{k,z}^\Lambda = \left(\widehat{H}_k^\Lambda - z\right)^{-1} \quad \text{for} \ \ z \notin \sigma(\widehat{H}_k^\Lambda) \quad \text{for} \quad k \in \mathbb{N}^0, \tag{3.5}$$

and recall the resolvent identity

$$R_z^\Lambda = \widehat{R}_{k,z}^\Lambda + k\left(1 - \tfrac{1}{\Delta}\right) R_z^\Lambda \widehat{Q}_{\le k}^\Lambda \widehat{R}_{k,z}^\Lambda = \widehat{R}_{k,z}^\Lambda + k\left(1 - \tfrac{1}{\Delta}\right) \widehat{R}_{k,z}^\Lambda \widehat{Q}_{\le k}^\Lambda R_z^\Lambda. \tag{3.6}$$

It follows from (2.14) and (2.22) that for $k \in \mathbb{N}^0$ we have

$$\widehat{H}_k^\Lambda \ge (k+1)\left(1 - \tfrac{1}{\Delta}\right) I \quad \text{and} \quad \left(\widehat{H}_k^\Lambda - E\right) \ge \tfrac{1}{8}\left(1 - \tfrac{1}{\Delta}\right) I \ \ \text{for} \ \ E \in \check{I}_{\le k} \tag{3.7}$$

and

$$\left\|\widehat{R}_{k,z}^\Lambda\right\| \le \left\|\widehat{R}_{k,\mathrm{Re}\, z}^\Lambda\right\| \le 8\left(1 - \tfrac{1}{\Delta}\right)^{-1} \quad \text{for} \quad \mathrm{Re}\, z \in \check{I}_{\le k}. \tag{3.8}$$

For $q \in \frac{1}{2}\mathbb{N}^0$, we set

$$\widehat{H}_q^\Lambda = \widehat{H}_{\widehat{q}}^\Lambda \quad \text{and} \quad \widehat{R}_{q,z}^\Lambda = \widehat{R}_{\widehat{q},z}^\Lambda. \tag{3.9}$$

3.2. Quasi-locality (deterministic).

Lemma 3.1 ([7, Lemma 3.1]). *Let T be an operator on the Hilbert space $\mathcal{H}_\Lambda$, and let Y be a projection on $\mathcal{H}_\Lambda$ such that $[Y, T] = 0$ and $[Y, P_\pm^K] = 0$ for all $K \subset \Lambda$.*

Suppose

(i) For all $K \subset \Lambda$ we have $[P_-^K, T] P_+^{[K]_1^\Lambda} = 0$.
(ii) For all connected $K \subset \Lambda$ we have $\left\| [P_-^K, T] \right\| \leq \gamma$.
(iii) T_Y, the restriction of the operator T to $\operatorname{Ran} Y$, is invertible with $\left\| T_Y^{-1} \right\|_{\operatorname{Ran} Y} \leq \eta^{-1}$, where $\eta > 0$.

Then for all $A \subset B \subset \Lambda$, we have

$$\left\| P_-^A \, T_Y^{-1} \, P_+^B \right\|_{\operatorname{Ran} Y} \leq \gamma^{-1} \mathrm{e}^{-m \operatorname{dist}_\Lambda (A, B^c)} = \eta^{-1} \mathrm{e}^{-m (\operatorname{dist}_\Lambda (A, B^c) - 1)}, \text{ with } m = \ln \left(\gamma^{-1} \eta \right). \tag{3.10}$$

This lemma yields quasi-locality for the resolvent of the operators H^Λ and $\widehat{H}_q^\Lambda$, as discussed in [7, Section 3.2]. The operator $T = \widehat{H}_q^\Lambda - z$ satisfies the hypotheses of Lemma 3.1 for $q \in \frac{1}{2}\mathbb{N}$ and $\operatorname{Re} z \in \check{I}_{\leq \widehat{q}}$, with $\gamma = \frac{1}{\Delta} \leq \frac{1}{\Delta_0}$, $Y = I_{\mathcal{H}_\Lambda}$, and $\eta = \operatorname{dist}(z, \sigma(\widehat{H}_q^\Lambda)) \geq \frac{1}{8}\left(1 - \frac{1}{\Delta}\right) \geq \frac{1}{8}\left(1 - \frac{1}{\Delta_0}\right)$, and hence for $A \subset B \subset \Lambda$ the estimate (3.10) yields (recall we assumed $\Delta_0 > 9$)

$$\left\| P_-^A \widehat{R}_{q,z}^\Lambda P_+^B \right\| \leq \tfrac{1}{\Delta_0} \mathrm{e}^{-m_0 \operatorname{dist}_\Lambda (A, B^c)}, \quad \text{where} \;\; m_0 = \ln \tfrac{\Delta_0 - 1}{8} > 0. \tag{3.11}$$

3.3. Consequences of quasi-locality.

Lemma 3.2. *Fix $k \in \mathbb{N}$. Given a collection $\{S_i\}_{i=1}^{k+1}$ of nonempty subsets of Λ with*

$$\min_{i \neq j} \operatorname{dist}_\Lambda \left(S_i, S_j \right) \geq 2\ell + 1, \quad \textit{where} \quad \ell \in \mathbb{N}, \tag{3.12}$$

we have

$$\mathbb{E} \left\| P_{I_{\leq k}} \prod_{i=1}^{k+1} P_-^{S_i} \right\| \leq C_k \Upsilon_{max}^\Lambda \, |\Lambda|^{2k+1} \, \mathrm{e}^{-m_0 \ell}, \tag{3.13}$$

where $\Upsilon_{max}^\Lambda = \max_{i=1}^{k+1} \Upsilon_{S_i}^\Lambda$ (see Remark 2.4).

Proof. Note that $P_{I_{\leq k}} \prod_{i=1}^{k+1} P_-^{S_i} = P_{I_k} \prod_{i=1}^{k+1} P_-^{S_i}$. We can represent P_{I_k} as a contour integral

$$P_{I_k} = \frac{1}{2\pi i} P_{I_k} \oint_\Gamma R_z^\Lambda dz, \tag{3.14}$$

where Γ is defined by $\Gamma = \left\{ z \in \mathbb{C} : \; \min_{x \in I_k} |x - z| = \frac{1}{8}\left(1 - \frac{1}{\Delta}\right) \right\}$. Note that $\left\| P_{I_k} R_z^\Lambda \right\| \leq \frac{8}{1 - \frac{1}{\Delta}}$ for any $z \in \Gamma$.

Using (3.6) and (3.7), we deduce that

$$P_{I_k} = \frac{k\left(1-\frac{1}{\Delta}\right)}{2\pi i} P_{I_k} \oint_\Gamma R_z^\Lambda \widehat{Q}_{\leq k}^\Lambda \widehat{R}_{k,z}^\Lambda dz. \tag{3.15}$$

Let $\Theta_{k,\ell} = \chi_{[0,2\ell+k]}\left(\mathcal{N}^\Lambda\right)$. Note that $[H^\Lambda, \Theta_{k,\ell}] = 0$ and $[P_\pm^B, \Theta_{k,\ell}] = 0$ for $B \subset \Lambda$. Moreover, it follows from (3.12) that

$$\Theta_{k,\ell} \prod_{i=1}^{k+1} P_-^{[S_i]_\ell^\Lambda} = \Theta_{k,\ell} Q_{>k}^\Lambda \prod_{i=1}^{k+1} P_-^{[S_i]_\ell^\Lambda}. \tag{3.16}$$

Since (3.11) and (2.21) yield

$$\left\| P_+^{[S_i]_\ell^\Lambda} \widehat{R}_{k,z}^\Lambda P_-^{S_i} \right\| \leq C \Upsilon_{S_i}^\Lambda \mathrm{e}^{-m_0 \ell} \quad \text{for} \quad i = 1, 2, \ldots k+1, \tag{3.17}$$

we have, using $P_+^{[S_i]_\ell^\Lambda} + P_-^{[S_i]_\ell^\Lambda} = I_{\mathcal{H}_\Lambda}$, that

$$\begin{aligned} \left\| \widehat{Q}_{\leq k}^\Lambda \widehat{R}_{k,z}^\Lambda \prod_{i=1}^{k+1} P_-^{S_i} \Theta_{k,\ell} \right\| &\leq C(k+1)\Upsilon_{max}^\Lambda \mathrm{e}^{-m_0\ell} + \left\| Q_{\leq k}^\Lambda Q_{>k} \Theta_{k,\ell} \prod_{i=1}^{k+1} P_-^{[S_i]_\ell^\Lambda} \widehat{R}_{k,z}^\Lambda \right\| \\ &= C(k+1)\Upsilon_{max}^\Lambda \mathrm{e}^{-m_0\ell} = C_k \Upsilon_{max}^\Lambda \mathrm{e}^{-m_0\ell}, \end{aligned} \tag{3.18}$$

where we used (3.16) and $Q_{\leq k}^\Lambda Q_{>k} = 0$.

To prove (3.13), by a large deviation estimate (see [7, Eqs (5.18)–(5.23)]) we have

$$\begin{aligned} &\mathbb{P}\left\{P_{I_{\leq k}} \neq P_{I_{\leq k}} \Theta_{k\ell}\right\} = \mathbb{P}\left\{P_{I_{\leq k}} \chi_{(2\ell+k,|\Lambda|]}\left(\mathcal{N}^\Lambda\right) \neq 0\right\} \\ &\quad = \mathbb{P}\left\{\sigma\left(H^\Lambda \chi_{(2\ell+k,|\Lambda|]}\left(\mathcal{N}^\Lambda\right)\right) \cap I_{\leq k} \neq \emptyset\right\} \leq C_k \left|\Lambda\right|^{2k+1} \mathrm{e}^{-d_\mu \ell}, \end{aligned} \tag{3.19}$$

where d_μ depends only on the probability distribution μ. The estimate (3.13) follows by assuming, without loss of generality, that $m_0 \leq d_\mu$. □

Remark 3.3. We have the following consequence of Lemma 3.2 and (2.19). Consider $i, j_1, j_2, \ldots, j_k \in \Lambda$ such that $\min_{s \neq r} |j_r - j_s| \geq 2\ell + 1$ and $\min_s |i - j_s| \geq 3\ell + 1$. Then

$$\mathbb{E}\left(\sup_{\substack{f \in B(I_k): \\ \|f\|_\infty \leq 1}} \left\|\mathcal{N}_i f(H^\Lambda) \mathcal{N}_{j_1} \mathcal{N}_{j_2} \ldots \mathcal{N}_{j_k}\right\|\right) \leq C_k \left|\Lambda\right|^{\max\{\xi_k, 2k+1\}} \mathrm{e}^{-\theta_k \ell}. \tag{3.20}$$

For $k = 1$ this bound has been established in [9]. It is proved as follows:

$$\begin{aligned} \mathcal{N}_i f(H^\Lambda) \mathcal{N}_{j_1} \mathcal{N}_{j_2} \ldots \mathcal{N}_{j_k} &= \mathcal{N}_i f(H^\Lambda) P_+^{[i]_\ell} \mathcal{N}_{j_1} \mathcal{N}_{j_2} \ldots \mathcal{N}_{j_k} \\ &\quad + \mathcal{N}_i f(H^\Lambda) P_-^{[i]_\ell} \mathcal{N}_{j_1} \mathcal{N}_{j_2} \ldots \mathcal{N}_{j_k}. \end{aligned}$$

The expectation of the first term (with the sup inside) is estimated by (2.19). The second term is estimated by (3.13) using $f(H^\Lambda) = f(H^\Lambda) P_{I_k}$ for $f \in B(I_k)$.

Lemma 3.4. *Let T be an observable supported on an interval $\mathcal{X} \subset \Lambda$ with $\|T\| \leq 1$, let $k, \ell \in \mathbb{N}$, and assume* $\operatorname{dist}(\mathcal{X}, \mathbb{Z} \setminus \Lambda) > 9(k+1)\ell + 1$. *Consider the observables*

$$T_j = T\, P_+^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} \prod_{i=1}^{j-1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}} \text{ for } j = 1, 2, \ldots, k+1, \text{ with } \prod_{i=1}^{0} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}} = I. \tag{3.21}$$

Then

$$T_j = P_+^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} T_j P_+^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} = P_+^{\partial^{out}_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} \otimes P_+^{\partial^{in}_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} \otimes T \prod_{i=1}^{j-1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}}, \tag{3.22}$$

where $\operatorname{supp} T_j = [\mathcal{X}]^\Lambda_{(9j+3)\ell}$ *and* $\operatorname{supp}\left(T \prod_{i=1}^{j-1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}}\right) = [\mathcal{X}]^\Lambda_{(9j-6)\ell}$, *and such that*

$$\mathbb{E} \left\| \left(T - \sum_{j=1}^{k+1} T_j\right)_{P_{I_{\leq k}}} \right\| \leq C_k\, |\Lambda|^{2k+1}\, \mathrm{e}^{-m_0 \ell}. \tag{3.23}$$

Proof. We decompose

$$I = \prod_{i=1}^{k+1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}} + \sum_{j=1}^{k+1} P_+^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9j\ell}} \prod_{i=1}^{j-1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}}. \tag{3.24}$$

It follows from Lemma 3.2, using $3\ell > 2\ell + 1$, $\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell} \neq \emptyset$ for $i = 1, 2, \ldots, k+1$ since $\operatorname{dist}(\mathcal{X}, \mathbb{Z}\setminus\Lambda) > 9(k+1)\ell + 1$, and $\Upsilon^\Lambda_{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}} \leq 2$, that

$$\mathbb{E} \left\| \left(T \prod_{i=1}^{k+1} P_-^{\partial_{3\ell}[\mathcal{X}]^\Lambda_{9i\ell}}\right)_{P_{I_{\leq k}}} \right\| \leq C_k\, |\Lambda|^{2k+1}\, e^{-m_0 \ell}. \tag{3.25}$$

The estimate (3.23) follows. □

3.4. Decoupling. Let $A \subset \Lambda$ be an interval. We consider the Hamiltonian

$$H^{A,A^c} = H^A + H^{A^c} \quad \text{on} \quad \mathcal{H}_\Lambda, \tag{3.26}$$

set $R_z^{A,A^c} = \left(H^{A,A^c} - z\right)^{-1}$, and let

$$\Gamma^A = H^\Lambda - H^{A,A^c} = \sum_{\{i,i+1\} \in \boldsymbol{\partial}^\Lambda A} h_{i,i+1}. \tag{3.27}$$

It follows from (2.5) that

$$\left\| P_+^{\{i\}} h_{i,i+1} \right\| = \left\| P_+^{\{i+1\}} h_{i,i+1} \right\| = \tfrac{1}{2\Delta}, \tag{3.28}$$

so

$$\left\| P_+^A \Gamma^A \right\| \leq \tfrac{1}{\Delta} \quad \text{and} \quad \left\| P_+^{A^c} \Gamma^A \right\| \leq \tfrac{1}{\Delta}. \tag{3.29}$$

We also set $\tau_t^{A,A^c}(M) = \mathrm{e}^{itH^{A,A^c}} M e^{-itH^{A,A^c}}$, where M is an observable on $\mathcal{H}_\Lambda$.
We fix an infinitely differentiable function $\widetilde{\Psi} : \mathbb{R} \to [0, 1]$ such that

$$\widetilde{\Psi}(u) = \begin{cases} 0 & \text{for} \quad u \in (-\infty, -1] \\ 1 & \text{for} \quad u \in [0, \frac{3}{4}\left(1 - \frac{1}{\Delta}\right)] \\ 0 & \text{for} \quad u \in [\frac{7}{8}\left(1 - \frac{1}{\Delta}\right), \infty) \end{cases}, \tag{3.30}$$

and set

$$\Psi_q(u) = \begin{cases} \widetilde{\Psi}(u) & \text{for} \quad u \in (-\infty, 0] \\ 1 & \text{for} \quad u \in [0, \left(q + \frac{3}{4}\right)\left(1 - \frac{1}{\Delta}\right)] \\ \widetilde{\Psi}(u - q) & \text{for} \quad u \in [\left(q + \frac{3}{4}\right)\left(1 - \frac{1}{\Delta}\right), \infty) \end{cases} \quad \text{for} \quad q \in \tfrac{1}{2}\mathbb{N}. \tag{3.31}$$

Note that Ψ_q is an infinitely differentiable function on the real line such that $0 \leq \Psi_q \leq 1$, and

$$\Psi_q(u) = \begin{cases} 0 & \text{for} \quad u \in (-\infty, -1] \\ 1 & \text{for} \quad u \in [0, \left(q + \frac{3}{4}\right)\left(1 - \frac{1}{\Delta}\right)] \\ 0 & \text{for} \quad u \in [\left(q + \frac{7}{8}\right)\left(1 - \frac{1}{\Delta}\right), \infty) \end{cases}. \tag{3.32}$$

We also define $\Phi_{q,t}(u) = \Psi_q(u)\mathrm{e}^{itu}$ for $t \in \mathbb{R}$, so $\Psi_q = \Phi_{q,0}$. Note that $\operatorname{supp} \Phi_{q,t} \subset [-1, \left(q + \frac{7}{8}\right)\left(1 - \frac{1}{\Delta}\right)]$.

We have

$$P_{I_{\leq q}} = P_{I_{\leq q}} \Phi_q(H^\Lambda) \leq \Phi_q(H^\Lambda) \leq P_{\check{I}_{\leq q}}. \tag{3.33}$$

Thus, if M is an observable on $\mathcal{H}_\Lambda$, we have

$$\left\| (M)_{P_{I_{\leq q}}} \right\| \leq \left\| (M)_{\Psi_q} \right\| \leq \left\| (M)_{P_{\check{I}_{\leq q}}} \right\|. \tag{3.34}$$

Lemma 3.5. *Let $q \in \frac{1}{2}\mathbb{N}$, and assume Condition $\mathcal{L}_q$. Then for $t \in \mathbb{R}$, $\ell \in \mathbb{N}$, $b \in \mathbb{N}$, and an interval $A \subset \Lambda$, we have*

$$\mathbb{E} \left\| \left(\Phi_{q,t}(H^\Lambda) - \Phi_{q,t}(H^{A,A^c}) \right) P_+^{\partial_{b\ell} A} \right\| \leq C_q \langle t \rangle^3 |\Lambda|^{\xi_q} \mathrm{e}^{-b\theta_q \ell}. \tag{3.35}$$

Proof. We use the Helffer–Sjöstrand formula for smooth functions f of self-adjoint operators [12,13]. We consider the norms

$$\{\!\{ f \}\!\}_m := \sum_{r=0}^{m} \int_{\mathbb{R}} \mathrm{d}u \, |f^{(r)}(u)| \, (1 + |u|^2)^{\frac{r-1}{2}}, \quad m = 1, 2, \ldots. \tag{3.36}$$

If $\{\!\{ f \}\!\}_m < \infty$ with $m \geq 2$, then for any self-adjoint operator K we have

$$f(K) = \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}(z) \, (K - z)^{-1}, \tag{3.37}$$

where the integral converges absolutely in operator norm. Here $z = x + iy$, $\tilde{f}(z)$ is an *almost analytic extension* of f to the complex plane, $\mathrm{d}\tilde{f}(z) := \frac{1}{2\pi}\partial_{\bar{z}}\tilde{f}(z)\,\mathrm{d}x\,\mathrm{d}y$, with $\partial_{\bar{z}} = \partial_x + i\partial_y$, and $|\mathrm{d}\tilde{f}(z)| := (2\pi)^{-1}|\partial_{\bar{z}}\tilde{f}(z)|\,\mathrm{d}x\,\mathrm{d}y$. Moreover, for all $p \geq 0$ we have

$$\int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)|\,\frac{1}{|\mathrm{Im}\, z|^p} \leq c_p\ \{\{f\}\}_m < \infty \quad \text{for} \quad m \geq p+1 \tag{3.38}$$

with a constant c_p (see [13, Appendix B] for details).

By the Helffer–Sjöstrand formula we have (recall (3.27))

$$\begin{aligned}\Phi_{q,t}(H^\Lambda) - \Phi_{q,t}(H^{A,A^c}) &= \int_{\mathbb{R}^2} \mathrm{d}\widetilde{\Phi}_{q,t}(z)\left((H^\Lambda - z)^{-1} - (H^{A,A^c} - z)^{-1}\right) \\ &= -\int_{\mathbb{R}^2} \mathrm{d}\widetilde{\Phi}_{q,t}(z)(H^{A,A^c} - z)^{-1}\Gamma^A(H^\Lambda - z)^{-1}.\end{aligned} \tag{3.39}$$

Thus

$$\begin{aligned}&\mathbb{E}\left\|\left(\Phi_{q,t}(H^\Lambda) - \Phi_{q,t}(H^{A,A^c})\right)P_+^{\partial_{b\ell}A}\right\| \\ &\leq \int_{\mathbb{R}^2}\left|\mathrm{d}\widetilde{\Phi}_{q,t}(z)\right|\mathbb{E}\left\|(H^{A,A^c} - z)^{-1}\Gamma^A(H^\Lambda - z)^{-1}P_+^{\partial_{b\ell}A}\right\| \\ &\leq C\int_{[-1,\left(q+\frac{7}{8}\right)\left(1-\frac{1}{\Delta}\right)]\times\mathbb{R}}\left|\mathrm{d}\widetilde{\Phi}_{q,t}(z)\right|\ |\mathrm{Im}\, z|^{-\frac{7}{4}}\ \mathbb{E}\left\|P_-^{\partial_1 A}(H^\Lambda - z)^{-1}P_+^{\partial_{b\ell}A}\right\|^{\frac{1}{4}} \\ &\leq C\left\{\!\left\{\Phi_{q,t}\right\}\!\right\}_3|\Lambda|^{\xi_q}\,\mathrm{e}^{-b\theta_q\ell} \leq Cq\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-b\theta_q\ell},\end{aligned} \tag{3.40}$$

where we used $\Gamma^A = \Gamma^A P_-^{\partial_1 A}$, (3.38), the fact that $\left\{\!\left\{\Phi_{q,t}\right\}\!\right\}_3 \leq Cq\,\langle t\rangle^3$ with a constant independent of q by its construction, and Theorem 2.2. □

Lemma 3.6. *Let $q \in \frac{1}{2}\mathbb{N}$, and assume Condition $\mathcal{L}_q$. Let $\ell \in \mathbb{N}$, and consider an interval $A \subset \Lambda$. Let T be an arbitrary observable with $\|T\| \leq 1$. Then for all $b \in \mathbb{N}$ we have*

$$\begin{aligned}&\mathbb{E}\left\|\left(\tau_t\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right) - \Psi_q(H^{A,A^c})\tau_t^{A,A^c}\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Psi_q(H^{A,A^c})\right)_{P_{I_{\leq q}}}\right\| \\ &\leq Cq\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-b\theta_q\ell}.\end{aligned} \tag{3.41}$$

Proof. We have, recalling $P_{I_{\leq q}} = \Phi_q(H^\Lambda)P_{I_{\leq q}}$,

$$\begin{aligned}&\left\|\left(\tau_t\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right) - \Psi_q(H^{A,A^c})\tau_t^{A,A^c}\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Psi_q(H^{A,A^c})\right)_{P_{I_{\leq q}}}\right\| \\ &= \left\|\left(\Psi_q(H^\Lambda)\tau_t\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Psi_q(H^\Lambda)\right.\right. \\ &\quad\left.\left. -\ \Psi_q(H^{A,A^c})\tau_t^{A,A^c}\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Psi_q(H^{A,A^c})\right)_{P_{I_{\leq q}}}\right\| \\ &\leq \left\|\Phi_{q,t}(H^\Lambda)\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Phi_{q,-t}(H^\Lambda) - \Phi_{q,t}(H^{A,A^c})\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Phi_{q,-t}(H^{A,A^c})\right\|.\end{aligned} \tag{3.42}$$

Using Lemma 3.5, we conclude that

$$\begin{aligned}&\mathbb{E}\left\|\left(\tau_t\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right) - \Psi_q(H^{A,A^c})\tau_t^{A,A^c}\left(P_+^{\partial_{b\ell}A}TP_+^{\partial_{b\ell}A}\right)\Psi_q(H^{A,A^c})\right)_{P_{I_{\leq q}}}\right\| \\ &\qquad \leq C_q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-b\theta_q\ell}.\end{aligned} \tag{3.43}$$

□

Note that

$$\Psi_q(H^{A,A^c}) = \Psi_q(H^{A,A^c}) P_{\check{I}_{\le q}}(H^{A,A^c}), \quad \text{so}$$
$$\left\| \left((M)_{\Psi_q(H^{A,A^c})} \right)_{P_{I_{\le q}}} \right\| \le \left\| (M)_{P_{\check{I}_{\le q}}(H^{A,A^c})} \right\|. \tag{3.44}$$

4. Proof of Slow Propagation of Information

In this section we prove Theorem 2.6. We start with the following lemma.

Lemma 4.1. *Let $q \in \frac{1}{2}\mathbb{N}^0$, and assume Condition $\mathcal{L}_q$ is satisfied. Then there exists a constant C_q such that, for any given finite interval $\Lambda \subset \mathbb{Z}$, scale $\ell \in \mathbb{N}$, and all $t \in \mathbb{R}$, the following holds:*

(i) Let T be an observable supported on an interval $\mathcal{X} \subset \Lambda$ with $\|T\| \le 1$, such that

$$T = P_-^{\mathcal{X}} T P_-^{\mathcal{X}}. \tag{4.1}$$

Then there exists an observable $T_t = T(t, q, \ell)$, supported in $[\mathcal{X}]^{\Lambda}_{\beta_q \ell}$, such that

$$\mathbb{E} \left\| \left(\tau_t^{\Lambda}(T) - T_t \right)_{P^{\Lambda}_{I_{\le q}}} \right\| \le C_q \langle t \rangle^{p_q} \, |\Lambda|^{\xi_q} \, \mathrm{e}^{-\theta_q \ell}. \tag{4.2}$$

where $p_0 = 0$, $p_q = 2q + 2$ for $q \in \frac{1}{2}\mathbb{N}$, and β_q is defined in (2.27).

(ii) If T is is an observable on Λ with $\|T\| \le 1$ of the form

$$T = P_+^{[\mathcal{Y}]^{\Lambda}_{2\ell-1}} P_-^{\partial^{\Lambda}[\mathcal{Y}]^{\Lambda}_{2\ell}} \widetilde{T} P_-^{\partial^{\Lambda}[\mathcal{Y}]^{\Lambda}_{2\ell}} P_+^{[\mathcal{Y}]^{\Lambda}_{2\ell-1}}, \quad \text{with} \quad \operatorname{supp} \widetilde{T} = \partial^{\Lambda}[\mathcal{Y}]^{\Lambda}_{2\ell}, \tag{4.3}$$

where $\mathcal{Y} \subset \Lambda$ is an interval, we can choose T_t, supported on $[\mathcal{Y}]^{\Lambda}_{(\beta_q+2)\ell+1}$ and satisfying (4.2), such that

$$T_t = P_+^{\mathcal{Y}} T_t P_+^{\mathcal{Y}}. \tag{4.4}$$

Proof. The lemma is proved by induction on $q \in \frac{1}{2}\mathbb{N}^0$. The lemma is obviously true for $q = 0$ with $M_t = M$, $C_0 = 0$, $p_0 = 0$, $\xi_0 = 0$, $\theta_0 = 0$. Given $q \in \frac{1}{2}\mathbb{N}$, we assume the lemma is true for $q - \frac{1}{2}$ and prove the lemma also holds for q.

So let $\Lambda \subset \mathbb{Z}$ be a finite interval. (We will often omit Λ from the notation.) We first consider an observable M on Λ with $\|M\| \le 1$ of the form

$$M = P_+^{\partial^{out}_{3\ell} A} \otimes \widetilde{M}, \tag{4.5}$$

where $A \subset \Lambda$ is an interval and $\widetilde{M}$ is an observable such that

$$\operatorname{supp} \widetilde{M} = A, \quad \widetilde{M} = P_+^{\partial^{in}_{3\ell} A} \widetilde{M} P_+^{\partial^{in}_{3\ell} A}, \tag{4.6}$$

and

$$\widetilde{M} = P_-^{A} \widetilde{M} P_-^{A}. \tag{4.7}$$

Note that $M = P_+^{\partial_{3\ell} A} M P_+^{\partial_{3\ell} A}$.

Lemma 3.6 gives

$$\mathbb{E}\left\| \left(\tau_t^{\Lambda}(M) - \Psi_q(H^{A,A^c})\tau_t^{A,A^c}(M)\,\Psi_q(H^{A,A^c})\right)_{P^{\Lambda}_{I_{\leq q}}} \right\| \leq C q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-3\theta_q \ell}. \tag{4.8}$$

It follows from (4.5), (4.6) and (4.7) that

$$\tau_t^{A,A^c}(M) = \tau_t^{A}(\widetilde{M})\tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) = P_-^{A}\tau_t^{A}(\widetilde{M})\tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A})P_-^{A}. \tag{4.9}$$

Using

$$P^{A,A^c}_{\check{I}_{\leq q}} P_-^{A} = P^{A,A^c}_{\check{I}_{q}} P^{A}_{\check{I}_q} P^{A^c}_{\check{I}_{\leq q-1}} P_-^{A} = P^{A,A^c}_{\check{I}_{\leq q}} P^{A}_{\check{I}_{\leq q}} P^{A^c}_{\check{I}_{\leq q-1}} P_-^{A}, \tag{4.10}$$

we get

$$\begin{aligned} &P^{A,A^c}_{\check{I}_{\leq q}} \tau_t^{A}(\widetilde{M})\tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P^{A,A^c}_{\check{I}_{\leq q}} = P^{A,A^c}_{\check{I}_{\leq q}} P_-^{A}\tau_t^{A}(\widetilde{M})\tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P_-^{A} P^{A,A^c}_{\check{I}_{\leq q}} \\ &= P^{A,A^c}_{\check{I}_{\leq q}} P^{A}_{\check{I}_{\leq q}} \tau_t^{A}(\widetilde{M}) P^{A}_{\check{I}_{\leq q}} P^{A^c}_{\check{I}_{\leq q-1}} \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P^{A^c}_{\check{I}_{\leq q-1}} P^{A,A^c}_{\check{I}_{\leq q}}. \end{aligned} \tag{4.11}$$

We set

$$\widetilde{M}_t = P_+^{\partial_\ell^{in} A} P^{A}_{\check{I}_{\leq q}} \tau_t^{A}(\widetilde{M}) P^{A}_{\check{I}_{\leq q}} P_+^{\partial_\ell^{in} A}, \tag{4.12}$$

an observable with $\operatorname{supp}\widetilde{M}_t = A$. Using $\widetilde{M} = P_+^{\partial_{3\ell}^{in} A}\widetilde{M}P_+^{\partial_{3\ell}^{in} A}$, $I - P_+^{\partial_{2\ell}^{in} A} = P_-^{\partial_{2\ell}^{in} A}$, and (2.19), we get

$$\mathbb{E}\left\| P^{A}_{\check{I}_{\leq q}} \tau_t^{A}(\widetilde{M}) P^{A}_{\check{I}_{\leq q}} - P^{A}_{\check{I}_{\leq q}} \widetilde{M}_t P^{A}_{\check{I}_{\leq q}} \right\| \leq C_q\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-2\theta_q \ell}. \tag{4.13}$$

To treat the term in A^c in (4.11), note that $A^c = A^c_L \cup A^c_R$, where A^c_L and A^c_R are subintervals of Λ (possibly empty, in each case they do not have to be considered); A^c_L is the interval to the left of A and A^c_R the interval to the right of A. Since $\operatorname{dist}_\Lambda\left(A^c_L, A^c_R\right) \geq |A| + 1 \geq 2$, we have $H^{A^c} = H^{A^c_L} + H^{A^c_R}$, and $P_+^{\partial_{3\ell}^{out} A} = P_+^{\partial_{3\ell}^{out,L} A} P_+^{\partial_{3\ell}^{out,R} A}$, where $\partial_{3\ell}^{out,\#} A = \partial_{3\ell}^{out} A \cap A^c_\#$ for $\# = L, R$. Thus we have

$$\begin{aligned} &P^{A^c}_{\check{I}_{\leq q-1}} \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P^{A^c}_{\check{I}_{\leq q-1}} \\ &= P^{A^c}_{\check{I}_{\leq q-1}} \left(P^{A^c_L}_{\check{I}_{\leq q-1}} \tau_t^{A^c_L}(P_+^{\partial_{3\ell}^{out,L} A}) P^{A^c_L}_{\check{I}_{\leq q-1}} \right) \left(P^{A^c_R}_{\check{I}_{\leq q-1}} \tau_t^{A^c_R}(P_+^{\partial_{3\ell}^{out,R} A}) P^{A^c_R}_{\check{I}_{\leq q-1}} \right) P^{A^c}_{\check{I}_{\leq q-1}}. \end{aligned} \tag{4.14}$$

The two expressions in parenthesis in (4.14) are treated the same way. So let us consider the expression on the right. Since $\check{I}_{\leq q-1} \subset I_{\leq q-\frac{1}{2}}$, we can use the induction hypothesis on the interval A^c_R. But it does not suffice to use it directly for the observable $P_+^{\partial_{3\ell}^{out,R} A}$ because we seek an observable $Y_t = Y(t, q, \ell)$ on A^c (we now drop the R from the notation to simplify the exposition: A^c will stand for A^c_R, $P_+^{\partial_{3\ell}^{out,}A}$ for $P_+^{\partial_{3\ell}^{out,R}A}$, etc.) that not only satisfies an estimate like

$$\mathbb{E}\left\| P^{A^c}_{\check{I}_{\le q-1}} \left(\tau_t^{A^c}(P_+^{\partial_{3\ell}^{out}A}) - Y_t\right) P^{A^c}_{\check{I}_{\le q-1}} \right\| \le C_{q-\frac{1}{2}} \langle t \rangle^{p_{q-\frac{1}{2}}} \left|A^c\right|^{\xi_{q-\frac{1}{2}}} \mathrm{e}^{-\theta_{q-\frac{1}{2}}\ell}, \tag{4.15}$$

but also satisfies

$$Y_t = P_+^{\partial_\ell^{out}A} Y_t P_+^{\partial_\ell^{out}A}. \tag{4.16}$$

Let $K(t) = \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out}A})$. Then

$$\dot{K}(t) = \tau_t^{A^c}(D), \quad \text{where} \quad D := i[H^{A^c}, P_+^{\partial_{3\ell}^{out}A}], \tag{4.17}$$

and (for $t \ge 0$; the case $t \le 0$ can be treated in the same way),

$$K(t) - K(0) = \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out}A}) - P_+^{\partial_{3\ell}^{out}A} = \int_0^t \tau_s^{A^c}(D)\, ds. \tag{4.18}$$

Since $P_+^{\partial_{3\ell}^{out}A} = P_+^{\partial_{3\ell-1}^{out}A} P_+^{\partial_{3\ell}^{out}A}$, we have

$$D = P_+^{\partial_{3\ell-1}^{out}A} \widetilde{D} P_+^{\partial_{3\ell-1}^{out}A}, \text{ where } \operatorname{supp} \widetilde{D} = \partial[A]_{3\ell}^{\Lambda} \text{ and } \widetilde{D} = P_-^{\partial[A]_{3\ell}} \widetilde{D} P_-^{\partial[A]_{3\ell}}. \tag{4.19}$$

Let us set $B = \partial_\ell^{\Lambda,out}A \subset A^c$, so $\partial_{3\ell-1}^{\Lambda,out}A = [B]_{2\ell-1}^{A^c}$, $\partial^\Lambda[A]_{3\ell}^\Lambda = \partial^{A^c}[B]_{2\ell}^{A^c}$. (Note one endpoint of B is an endpoint of A^c. We are ignoring that A^c consists of possibly two intervals, we do the procedure separately on each one.) We can write

$$D = P_+^{[B]_{2\ell-1}^{A^c}} P_-^{\partial^{A^c}[B]_{2\ell}^{A^c}} \widetilde{D} P_-^{\partial^{A^c}[B]_{2\ell}^{A^c}} P_+^{[B]_{2\ell-1}^{A^c}}, \quad \text{with} \quad \operatorname{supp} \widetilde{D} = \partial^{A^c}[B]_{2\ell}^{A^c}. \tag{4.20}$$

Since D is an observable on A^c of the form given in (4.3), and by the induction hypothesis the lemma is true for $q - \frac{1}{2}$, it follows from Part (ii) of the theorem that there exists an observable $D_t = D(t, q - \frac{1}{2}, \ell)$ on A^c, supported on

$$[B]^{A^c}_{(\beta_{q-\frac{1}{2}}+2)\ell+1} = \partial^{\Lambda,out}_{(\beta_{q-\frac{1}{2}}+3)\ell+1}A, \tag{4.21}$$

such that

$$\begin{aligned} \mathbb{E}\left\| P^{A^c}_{\check{I}_{\le q-1}} \left(\tau_t^{A^c}(D) - D_t\right) P^{A^c}_{\check{I}_{\le q-1}} \right\| &\le \mathbb{E}\left\| P^{A^c}_{I_{\le q-\frac{1}{2}}} \left(\tau_t^{A^c}(D) - D_t\right) P^{A^c}_{I_{\le q-\frac{1}{2}}} \right\| \\ &\le C_{q-\frac{1}{2}} \langle t \rangle^{p_{q-\frac{1}{2}}} \left|A^c\right|^{\xi_{q-\frac{1}{2}}} \mathrm{e}^{-\theta_{q-\frac{1}{2}}\ell}, \end{aligned} \tag{4.22}$$

and

$$D_t = P_+^B D_t P_+^B = P_+^{\partial_\ell^{\Lambda,out}A} D_t P_+^{\partial_\ell^{\Lambda,out}A}. \tag{4.23}$$

It follows, using (4.18), that the observable on A^c given by (we now bring back the R index)

$$Y_t^{(R)} = P_+^{\partial_{3\ell}^{\Lambda,out,R} A} + \int_0^t \tau_s^{A_R^c}(D)\, ds, \tag{4.24}$$

is supported on $[\partial_\ell^{\Lambda,out,R} A]^{A_R^c}_{(\beta_{q-\frac{1}{2}}+3)\ell+1}$, and satisfies

$$\mathbb{E}\left\| P^{A_R^c}_{\check{I}_{\le q-1}} \left(\tau_t^{A_R^c}(P_+^{\partial_{3\ell}^{out,R} A}) - Y_t \right) P^{A_R^c}_{\check{I}_{\le q-1}} \right\| \le C'_{q-\frac{1}{2}} \langle t \rangle^{p_{q-\frac{1}{2}}+1} \left| A_R^c \right|^{\xi_{q-\frac{1}{2}}} \mathrm{e}^{-\theta_{q-\frac{1}{2}}\ell}, \tag{4.25}$$

and

$$Y_t^{(R)} = P_+^{\partial_\ell^{\Lambda,out,R} A} Y_t^{(R)} P_+^{\partial_\ell^{\Lambda,out,R} A}. \tag{4.26}$$

We construct $Y_t^{(L)}$ in a similar way for the interval A_L^c, and define

$$M_t = M(t, q, \ell) = \widetilde{M}_t Y_t^{(L)} Y_t^{(R)}. \tag{4.27}$$

It follows that

$$\operatorname{supp} M_t = A \cup \partial^{\Lambda,out}_{(\beta_{q-\frac{1}{2}}+3)\ell+1} A = [A]^\Lambda_{(\beta_{q-\frac{1}{2}}+3)\ell+1} \subset [A]^\Lambda_{(\beta_{q-\frac{1}{2}}+4)\ell}, \tag{4.28}$$

and

$$M_t = P_+^{\partial_\ell^\Lambda A} M_t P_+^{\partial_\ell^\Lambda A}. \tag{4.29}$$

We have, using (4.13) and (4.25),

$$\begin{aligned} \mathbb{E}\left\| \Psi_q(H^{A,A^c}) \left(\tau_t^{A,A^c}(M) - M_t \right) \Psi_q(H^{A,A^c}) \right\| &\le \mathbb{E}\left\| P^{A,A^c}_{\check{I}_{\le q}} \left(\tau_t^{A,A^c}(M) - M_t \right) P^{A,A^c}_{\check{I}_{\le q}} \right\| \\ &\le C_q \, |\Lambda|^{\xi_q} \mathrm{e}^{-2\theta_q \ell} + 2C'_{q-\frac{1}{2}} \langle t \rangle^{p_{q-\frac{1}{2}}+1} \left| A^c \right|^{\xi_{q-\frac{1}{2}}} \mathrm{e}^{-\theta_{q-\frac{1}{2}}\ell} \\ &\le C_q \langle t \rangle^{p_{q-\frac{1}{2}}+1} |\Lambda|^{\xi_q} e^{-\theta_{q-\frac{1}{2}}\ell}. \end{aligned} \tag{4.30}$$

In addition, it follows from Lemma 3.6 and (4.29) that (recall $\Psi_q = \Psi_{q,0}$)

$$\begin{aligned} &\mathbb{E}\left\| \left(\Psi_q(H^{A,A^c}) M_t \Psi_q(H^{A,A^c}) - M_t \right)_{P^\Lambda_{I_{\le q}}} \right\| \\ &= \mathbb{E}\left\| \left(\Psi_q(H^{A,A^c}) P_+^{\partial_\ell^\Lambda A} M_t P_+^{\partial_\ell^\Lambda A} \Psi_q(H^{A,A^c}) - P_+^{\partial_\ell^\Lambda A} M_t P_+^{\partial_\ell^\Lambda A} \right)_{P^\Lambda_{I_{\le q}}} \right\| \\ &\le C_q \, \langle t \rangle^3 \, |\Lambda|^{\xi_q} \, \mathrm{e}^{-\theta_q \ell}. \end{aligned} \tag{4.31}$$

Using (4.8), (4.30), and (4.31), we get

$$\begin{aligned}
\mathbb{E}\left\|\left(\tau_t^{\Lambda}(M)-M_t\right)_{P^{\Lambda}_{I_{\le q}}}\right\| &\le \mathbb{E}\left\|\left(\tau_t^{\Lambda}(M)-\Psi_q(H^{A,A^c})\tau_t^{A,A^c}(M)\,\Psi_q(H^{A,A^c})\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\quad+\mathbb{E}\left\|\left(\Psi_q(H^{A,A^c})\tau_t^{A,A^c}(M)\,\Psi_q(H^{A,A^c})-\Psi_q(H^{A,A^c})M_t\Psi_q(H^{A,A^c})\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\quad+\mathbb{E}\left\|\left(\Psi_q(H^{A,A^c})M_t\Psi_q(H^{A,A^c})-M_t\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\le C_q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-3\theta_q\ell}+C_q\langle t\rangle^{p_{q-\frac{1}{2}}+1}\,|\Lambda|^{\xi_q}\,e^{-\theta_{q-\frac{1}{2}}\ell}+C_q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-\theta_q\ell} \\
&\le C_q\langle t\rangle^{\max\left\{p_{q-\frac{1}{2}}+1,3\right\}}\,|\Lambda|^{\xi_q}\,e^{-\theta_q\ell}.
\end{aligned} \tag{4.32}$$

We can now prove Part (i). Let T be an observable supported on an interval $\mathcal{X}\subset\Lambda$ with $\|T\|\le 1$ satisfying (4.1). If $\operatorname{dist}(\mathcal{X},\mathbb{Z}\setminus\Lambda)\le\beta_q\ell$, there is nothing to prove since $[\mathcal{X}]^{\Lambda}_{\beta_q\ell}=\Lambda$, just take $T_t=\tau_t^{\Lambda}(T)$. So assume $\operatorname{dist}(\mathcal{X},\mathbb{Z}\setminus\Lambda)>\beta_q\ell$. As $\beta_q-9(\hat{q}+1)=\beta_{q-\frac{1}{2}}+4\ge 4$, we can use Lemma 3.4, and let $\widehat{T}=\sum_{j=1}^{\hat{q}+1}T_j$ where T_j is given in (3.22), so $\left\|T_j\right\|\le 1$, and observe that in view of (3.23) it suffices to prove Part (i) for each T_j, $j=1,2,\ldots,\hat{q}+1$.

Let $j=1,2,\ldots,\hat{q}+1$ and set $A_j=[\mathcal{X}]_{9j\ell}$. It follows from (3.22) that

$$T_j=P_+^{\partial_{3\ell}^{out}A_j}\otimes\widetilde{T}_j=P_+^{\partial_{3\ell}A_j}T_jP_+^{\partial_{3\ell}A_j},\ \operatorname{supp}\widetilde{T}_j=A_j,\ \widetilde{T}_j=P_+^{\partial_{3\ell}^{in}A_j}\widetilde{T}_jP_+^{\partial_{3\ell}^{in}A_j}. \tag{4.33}$$

Moreover, it follows from (4.1) that

$$\widetilde{T}_j=P_-^{A_j}\widetilde{T}_jP_-^{A_j}\quad\text{for}\quad j=1,2,\ldots,\hat{q}+1. \tag{4.34}$$

Thus T_j is an observable satisfying (4.5) and (4.7) with $M=T_j$ and $A=A_j$. Thus there exists an observable $(T_j)_t$ with support

$$^{\Lambda}_{(\beta_{q-\frac{1}{2}}+4)\ell}=[\mathcal{X}]^{\Lambda}_{(\beta_{q-\frac{1}{2}}+4+9j)\ell}, \tag{4.35}$$

satisfying (4.32) for T_j.

Defining

$$T_t=\sum_{j=1}^{\hat{q}+1}(T_j)_t, \tag{4.36}$$

an observable on Λ with

$$\operatorname{supp}T_t=[\mathcal{X}]^{\Lambda}_{(\beta_{q-\frac{1}{2}}+4+9(\hat{q}+1))\ell}=[\mathcal{X}]^{\Lambda}_{\beta_q\ell},\quad\text{where}\ \ \beta_q=(\beta_{q-\frac{1}{2}}+9\hat{q}+13), \tag{4.37}$$

we have, using also (3.23),

$$\mathbb{E}\left\|\left(\tau_t^{\Lambda}(T)-T_t\right)_{P^{\Lambda}_{I_{\leq q}}}\right\| \leq C_q \langle t\rangle^{p_q}\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-\theta_q \ell}, \quad \text{with} \quad p_q=\max\left\{p_{q-\frac{1}{2}}+1,3\right\}. \tag{4.38}$$

Since $p_0=0$, we have $p_{\frac{1}{2}}=3$, so $p_q=p_{q-\frac{1}{2}}+1>3$ for $q\geq 1$. It follows that $p_q=3+2(q-\frac{1}{2})=2q+2$ for $q\geq 1$.

Using (2.27), we have

$$\beta_{q-\frac{1}{2}}+9q+13\leq\beta_q\leq\beta_{q-\frac{1}{2}}+9(q+\tfrac{1}{2})+13=\beta_{q-\frac{1}{2}}+9q+\tfrac{35}{2}. \tag{4.39}$$

Letting $\gamma_s=\beta_{\frac{s}{2}}$ for $s\in\mathbb{N}^0$, we have $\gamma_0=0$ and $\gamma_{s-1}+\frac{9}{2}s+13\leq\gamma_s\leq\gamma_{s-1}+\frac{9}{2}s+\frac{35}{2}$ for $s\in\mathbb{N}$. It follows that for $s\in\mathbb{N}$ we have

$$\tfrac{9}{2}\tfrac{s(s+1)}{2}+13s\leq\sum_{r=1}^{s}\left(\tfrac{9}{2}r+13\right)\leq\gamma_s\leq\sum_{r=1}^{s}\left(\tfrac{9}{2}r+\tfrac{35}{2}\right)=\tfrac{9}{2}\tfrac{s(s+1)}{2}+\tfrac{35}{2}s, \tag{4.40}$$

so $\frac{9}{4}s^2+\frac{61}{4}s\leq\gamma_s\leq\frac{9}{4}s^2+\frac{79}{4}s$. It follows that for $q\in\frac{1}{2}\mathbb{N}$ we have $9q^2+\frac{61}{2}q\leq\beta_q\leq 9q^2+\frac{79}{2}q$.

We now turn to Part (ii). If T is is an observable on Λ with $\|T\|\leq 1$ of the form given in (4.3), T satisfies the hypothesis of Part (i) with $\mathcal{X}=\operatorname{supp}T=[\mathcal{Y}]^{\Lambda}_{2\ell+1}$. Thus the proof of Part (i) applies to this observable T, but we modify it as follows. We observe that since $T=P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}\,T\,P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}$ in view of (4.3), when we apply Lemma 3.4 the observable T_j also satisfies $T_j=P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}\,T_j\,P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}$ in view of (3.22). Moreover, the corresponding observable $\widetilde{T}_j$ given in (4.5) also satisfies $\widetilde{T}_j=P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}\,\widetilde{T}_j\,P_+^{[\mathcal{Y}]^{\Lambda}_{\ell}}$. We define $(\widetilde{T}_j)_t$ as in (4.12), and also define $(\widehat{T}_j)_t=P_+^{\mathcal{Y}}(\widetilde{T}_j)_t P_+^{\mathcal{Y}}$. Since $I-P_+^{\mathcal{Y}}=P_-^{\mathcal{Y}}$ (note $A_j=[[\mathcal{Y}]^{\Lambda}_{2\ell+1}]^{\Lambda}_{9j\ell}=[\mathcal{Y}]^{\Lambda}_{(9j+2)\ell+1}$), we have

$$\begin{aligned}\mathbb{E}\left\|(\widetilde{T}_j)_t-(\widehat{T}_j)_t\right\| &\leq\mathbb{E}\left\|P^{A_j}_{\check{I}_{\leq q}}\tau_t^{A_j}(\widetilde{T}_j)P^{A_j}_{\check{I}_{\leq q}}-P_+^{\mathcal{Y}}P^{A_j}_{\check{I}_{\leq q}}\tau_t^{A_j}(\widetilde{T}_j)P^{A_j}_{\check{I}_{\leq q}}P_+^{\mathcal{Y}}\right\| \\ &\leq C_q\,|\Lambda|^{\xi_q}\,e^{-\theta_q\ell}\end{aligned} \tag{4.41}$$

by (2.19). As a result, we can replace $(\widetilde{T}_j)_t$ by $(\widehat{T}_j)_t$ for $j=1,2,\ldots,\hat{q}+1$ in the definition of $(T_j)_t$ in (4.27), so now we have $(T_j)_t=P_+^{\mathcal{Y}}(T_j)_t P_+^{\mathcal{Y}}$. As a consequence, T_t defined as in (4.36) satisfies (4.4), and Part (ii) is proven. □

Proof of Theorem 2.6. Let $q\in\frac{1}{2}\mathbb{N}^0$, and assume Condition $\mathcal{L}_{q+\frac{1}{2}}$ is satisfied. Let T be an observable supported on an interval $\mathcal{X}\subset\Lambda$ with $\|T\|\leq 1$. As in the proof of Part (i) of Lemma 4.1, we may assume $\operatorname{dist}(\mathcal{X},\mathbb{Z}\backslash\Lambda)>(\beta_{q+\frac{1}{2}}+13)\ell$, and use Lemma 3.4. Let $\widehat{T}=\sum_{j=1}^{\hat{q}+1}T_j=T_1+\check{T}$, where $\check{T}=\sum_{j=2}^{\hat{q}+1}T_j$ with T_j given in (3.22), so $\|T_j\|\leq 1$, and, using (3.23), it suffices to prove the theorem for T_1 and $\check{T}$.

Each T_j, $j=2,3,\ldots,\hat{q}+1$, T can be treated as in the proof of Part (i) of Lemma 4.1, see (4.33)–(4.38). Setting $\check{T}_t=\sum_{j=2}^{\hat{q}+1}(T_j)_t$, we get (4.37) and (4.38) with $\check{T}_t$ substituted for T_t. Note that $\operatorname{supp}\check{T}=[\mathcal{X}]^{\Lambda}_{(9(\hat{q}+1)+3)\ell}$.

Thus it only remains to prove the theorem for $T_1 = T P_+^{\partial_{3\ell}[\mathcal{X}]_{9\ell}} = P_+^{\partial_{3\ell}[\mathcal{X}]_{9\ell}} T P_+^{\partial_{3\ell}[\mathcal{X}]_{9\ell}}$. Note that $M = T_1$ satisfies (4.5)–(4.6) with $A = [\mathcal{X}]_{9\ell}$, but (4.7) may not hold. We proceed as in the proof for M in (4.5) but without (4.7), so (4.11) is replaced by

$$\begin{aligned} &P_{\check{I}_{\leq q}}^{A,A^c} \tau_t^A(\widetilde{M}) \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P_{\check{I}_{\leq q}}^{A,A^c} \\ &\quad = P_{\check{I}_{\leq q}}^{A,A^c} P_{\check{I}_{\leq q}}^{A} \tau_t^A(\widetilde{M}) P_{\check{I}_{\leq q}}^{A} P_{\check{I}_{\leq q}}^{A^c} \tau_t^{A^c}(P_+^{\partial_{3\ell}^{out} A}) P_{\check{I}_{\leq q}}^{A^c} P_{\check{I}_{\leq q}}^{A,A^c}. \end{aligned} \tag{4.42}$$

We continue the proof as before up to (4.20). D, defined in (4.20), is an observable on A_R^c of the form given in (4.3). We now use that Condition $\mathcal{L}_{q+\frac{1}{2}}$ is satisfied, and hence we can apply Part (ii) of Lemma 4.1 to the observable D in the energy interval $I_{\leq q+\frac{1}{2}}$. It follows that there exists an observable $D_t = D(t, q + \frac{1}{2}, \ell)$ on A_R^c, supported on

$$[B]^{A_R^c}_{(\beta_{q+\frac{1}{2}}+2)\ell+1} = \partial^{\Lambda,out,R}_{(\beta_{q+\frac{1}{2}}+3)\ell+1} A, \tag{4.43}$$

such that

$$\begin{aligned} \mathbb{E} \left\| \left(\tau_t^{A_R^c}(D) - D_t \right)_{P^{A^c}_{\check{I}_{\leq q}}} \right\| &\leq \mathbb{E} \left\| \left(\tau_t^{A_R^c}(D) - D_t \right)_{P^{A_R^c}_{I_{\leq q+\frac{1}{2}}}} \right\| \\ &\leq C_{q+\frac{1}{2}} \langle t \rangle^{p_{q+\frac{1}{2}}} \left| A^c \right|^{\xi_{q+\frac{1}{2}}} \mathrm{e}^{-\theta_{q+\frac{1}{2}}\ell}, \end{aligned} \tag{4.44}$$

where we used $\check{I}_{\leq q} \subset I_{\leq q+\frac{1}{2}}$, and

$$D_t = P_+^B D_t P_+^B = P_+^{\partial_\ell^{\Lambda,out,R} A} D_t P_+^{\partial_\ell^{\Lambda,out,R} A}. \tag{4.45}$$

Letting $Y_t^{(R)}$ be as in (4.24), we have $\operatorname{supp} Y_t = [\partial_\ell^{\Lambda,out,R} A]^{A_R^c}_{(\beta_{q+\frac{1}{2}}+3)\ell+1}$,

$$\mathbb{E} \left\| P^{A_R^c}_{\check{I}_{\leq q}} \left(\tau_t^{A_R^c}(P_+^{\partial_{3\ell}^{out,R} A}) - Y_t^{(R)} \right) P^{A_R^c}_{\check{I}_{\leq q}} \right\| \leq C'_{q+\frac{1}{2}} \langle t \rangle^{p_{q+\frac{1}{2}}+1} \left| A^c \right|^{\xi_{q+\frac{1}{2}}} \mathrm{e}^{-\theta_{q+\frac{1}{2}}\ell}, \tag{4.46}$$

and we have (4.26).

We construct $Y_t^{(L)}$ in a similar way for the interval A_L^c, and define

$$M_t = M(t, q, \ell) = \widetilde{M}_t Y_t^{(L)} Y_t^{(R)}, \tag{4.47}$$

as in (4.27), so

$$\operatorname{supp} M_t = A \cup \partial^{\Lambda,out}_{(\beta_{q+\frac{1}{2}}+3)\ell+1} A = [A]^{\Lambda}_{(\beta_{q+\frac{1}{2}}+3)\ell+1} \subset [A]^{\Lambda}_{(\beta_{q+\frac{1}{2}}+4)\ell} = [\mathcal{X}]^{\Lambda}_{(\beta_{q+\frac{1}{2}}+13)\ell}, \tag{4.48}$$

and we have (4.29).

We have, using (4.13) and (4.46),

$$\begin{aligned}
&\mathbb{E}\left\|\Psi_q(H^{A,A^c})\left(\tau_t^{A,A^c}(M)-M_t\right)\Psi_q(H^{A,A^c})\right\| \le \mathbb{E}\left\|P^{A,A^c}_{\check{I}_{\le q}}\left(\tau_t^{A,A^c}(M)-M_t\right)P^{A,A^c}_{\check{I}_{\le q}}\right\| \\
&\le C_q\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-2\theta_q\ell}+2C'_{q+\frac12}\langle t\rangle^{p_{q+\frac12}+1}\left|A^c\right|^{\xi_{q+\frac12}}\mathrm{e}^{-\theta_{q+\frac12}\ell} \\
&\le C_{q+\frac12}\langle t\rangle^{p_{q+\frac12}+1}|\Lambda|^{\max\left\{\xi_q,\xi_{q+\frac12}\right\}}e^{-\min\left\{2\theta_q,\theta_{q+\frac12}\right\}\ell} \\
&\le C_{q+\frac12}\langle t\rangle^{p_{q+\frac12}+1}|\Lambda|^{\xi_{q+\frac12}}e^{-\theta_{q+\frac12}\ell}.
\end{aligned} \tag{4.49}$$

In addition, (4.31) holds. Using (4.8), (4.49), and (4.31), we get

$$\begin{aligned}
&\mathbb{E}\left\|\left(\tau_t^{\Lambda}(M)-M_t\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \le \mathbb{E}\left\|\left(\tau_t^{\Lambda}(M)-\Psi_q(H^{A,A^c})\tau_t^{A,A^c}(M)\,\Psi_q(H^{A,A^c})\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\quad+\mathbb{E}\left\|\left(\Psi_q(H^{A,A^c})\tau_t^{A,A^c}(M)\,\Psi_q(H^{A,A^c})-\Psi_q(H^{A,A^c})M_t\Psi_q(H^{A,A^c})\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\quad+\mathbb{E}\left\|\left(\Psi_q(H^{A,A^c})M_t\Psi_q(H^{A,A^c})-M_t\right)_{P^{\Lambda}_{I_{\le q}}}\right\| \\
&\le C_q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-3\theta_q\ell}+C_{q+\frac12}\langle t\rangle^{p_{q+\frac12}+1}|\Lambda|^{\xi_{q+\frac12}}e^{-\theta_{q+\frac12}\ell}+C_q\,\langle t\rangle^3\,|\Lambda|^{\xi_q}\,\mathrm{e}^{-\theta_q\ell} \\
&\le C_{q+\frac12}\langle t\rangle^{\max\left\{p_{q+\frac12}+1,3\right\}}|\Lambda|^{\max\left\{\xi_q,\xi_{q+\frac12}\right\}}\mathrm{e}^{-\theta_{q+\frac12}\ell} \\
&= C_{q+\frac12}\langle t\rangle^{p_{q+\frac12}+1}|\Lambda|^{\xi_{q+\frac12}}\mathrm{e}^{-\theta_{q+\frac12}\ell}=C_{q+\frac12}\langle t\rangle^{2q+4}|\Lambda|^{\xi_{q+\frac12}}\mathrm{e}^{-\theta_{q+\frac12}\ell}.
\end{aligned} \tag{4.50}$$

Setting $T_t=\widehat{T}_t+M_t$, then T_t is supported in $[\mathcal{X}]^{\Lambda}_{(\beta_{q+\frac12}+13)\ell}$ and, using now (3.23), T_t satisfies (2.28). □

5. Proof of Slow Propagation of Information, Matrix Elements Version

Proof of Corollary 2.7. Let $q\in\frac12\mathbb{N}$, and assume Condition $\mathcal{L}_{q+1}$ is satisfied. Set

$$\alpha=\max\left\{2\frac{\xi_{q+\frac12}}{\theta_{q+\frac12}},\ \frac{4\lceil q\rceil}{c_\mu}+1\right\}, \tag{5.1}$$

and consider a finite interval $\Lambda\subset\mathbb{Z}$ with $|\Lambda|$ sufficiently large so

$$r+2(4r+2)r=8r^2+5r<|\Lambda|\,,\quad\text{where}\quad r=\lceil\alpha\ln|\Lambda|\rceil. \tag{5.2}$$

Let $\ell\in\mathbb{N}$, $t\in\mathbb{R}$, and $M_1,M_2\subset\Lambda$ with $|M_1|=|M_2|=\widehat{N}\in[1,|\Lambda|]\cap\mathbb{N}$.

Suppose $\ell\ge r$. In this case we pick T_t as in Theorem 2.6, so it follows from (2.28) and (5.1) that

$$\mathbb{E}\left\|\pi_{M_1}\left(\tau_t^{\Lambda}(T)-T_t\right)_{P^{\Lambda}_{I_{\le q}}}\pi_{M_2}\right\|\le\mathbb{E}\left\|\left(\tau_t^{\Lambda}(T)-T_t\right)_{P^{\Lambda}_{I_{\le q}}}\right\|\le C_q\langle t\rangle^{2q+4}e^{-\frac12\theta_{q+\frac12}\ell}. \tag{5.3}$$

Thus we only need to consider the case $\ell < r$. Suppose first that $\widehat{N} \geq r$. In this case we use a large deviation argument. On the complement of the event $\mathcal{B}_{\lceil q \rceil}^{\widehat{N}}$ we have $\chi_{\Lambda}^{\widehat{N}} P_{I_{\leq q}}^{\Lambda} = 0$ (see [7, Eqs. (3.52 and (3.55)]), and hence, taking $T_t = T$ we have

$$\begin{aligned} \mathbb{E} \left\| \pi_{M_1} \left(\tau_t^{\Lambda}(T) - T_t \right)_{P_{I_{\leq q}}^{\Lambda}} \pi_{M_2} \right\| &\leq 2\mathbb{P}\left(\mathcal{B}_{\lceil q \rceil}^{\widehat{N}} \right) \leq C_q \, |\Lambda|^{2\lceil q \rceil} \, \mathrm{e}^{-c_\mu \widehat{N}} \\ &\leq C_q \mathrm{e}^{-\frac{1}{2} c_\mu r} \leq C_q \mathrm{e}^{-\frac{1}{2} c_\mu \ell}, \end{aligned} \tag{5.4}$$

where in the second step we used the large deviation estimate [7, Eq. (3.53)] and (5.1).

It remains to consider the case $\widehat{N} < r$ and $\ell < r$. It follows that $|M_1 \cup M_2| \leq 2(r-1)$. We assume $|\mathcal{X}| \leq \ln |\Lambda| < r$ (note $\alpha > 1$ by (5.1)). It follows from (5.2) that there exists $j \in [0, 2r] \cap \mathbb{N}^0$ such that

$$\left([\mathcal{X}]_{(2j+2)r}^{\Lambda} \setminus [\mathcal{X}]_{2jr}^{\Lambda} \right) \cap (M_1 \cup M_2) = \emptyset, \tag{5.5}$$

and set $\mathcal{X}_r := [\mathcal{X}]_{(2j+1)\ell}^{\Lambda}$, and observe that

$$|\mathcal{X}_r| \leq |\mathcal{X}| + 2(4r+1)r \leq r + 2(4r+1)r = 8r^2 + 3r \leq 11r^2. \tag{5.6}$$

Since T is supported by $\mathcal{X} \subset \mathcal{X}_r$, we use Theorem 2.6 with $\Lambda = \mathcal{X}_r$ and $q + \frac{1}{2}$ instead of q, concluding that there exists an observable $T_\ell = T(t, q + \frac{1}{2}, \ell, \mathcal{X}_r)$ supported by $[\mathcal{X}]_{(13+\beta_{q+1})\ell}^{\mathcal{X}_r} \subset [\mathcal{X}]_{(13+\beta_{q+1})\ell}^{\Lambda}$, such that

$$\begin{aligned} \mathbb{E} \left\| \left(\tau_t^{\mathcal{X}_r}(T) - T_t \right)_{P_{\check{I}_{\leq q}}^{\mathcal{X}_r}} \right\| &\leq \mathbb{E} \left\| \left(\tau_t^{\mathcal{X}_r}(T) - T_t \right)_{P_{I_{\leq q+\frac{1}{2}}}^{\mathcal{X}_r}} \right\| \\ &\leq C_{q+\frac{1}{2}} \langle t \rangle^{2q+5} |\mathcal{X}_r|^{\xi_{q+1}} \mathrm{e}^{-\theta_{q+1}\ell} \leq C_{q+\frac{1}{2}} \langle t \rangle^{2q+5} \left(\ln(11r^2) \right)^{\xi_{q+1}} \mathrm{e}^{-\theta_{q+1}\ell} \\ &\leq C_q' \langle t \rangle^{2q+5} (\ln r)^{\xi_{q+1}} \mathrm{e}^{-\theta_{q+1}\ell} \leq C_q'' \langle t \rangle^{2q+5} (\ln |\Lambda|)^{\xi_{q+1}} \mathrm{e}^{-\theta_{q+1}\ell}. \end{aligned} \tag{5.7}$$

Since $\pi_{M_i} = \pi_{M_i} P_+^{\partial_r^\Lambda \mathcal{X}_r} = P_+^{\partial_r^\Lambda \mathcal{X}_r} \pi_{M_i}$, $i = 1, 2$, we use Lemma 3.5 and an argument similar to the proof of Lemma 3.6 to deduce that

$$\begin{aligned} &\mathbb{E} \left\| \pi_{M_1} \left(\tau_t^{\Lambda}(T) - \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \tau_t^{\mathcal{X}_r, \mathcal{X}_r^c}(T) \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \right)_{P_{I_{\leq q}}^{\Lambda}} \pi_{M_2} \right\| \\ &= \mathbb{E} \left\| \pi_{M_1} P_+^{\partial_r^\Lambda \mathcal{X}_r} \left(\tau_t^{\Lambda}(T) - \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \tau_t^{\mathcal{X}_r, \mathcal{X}_r^c}(T) \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \right)_{P_{I_{\leq q}}^{\Lambda}} P_+^{\partial_r^\Lambda \mathcal{X}_r} \pi_{M_2} \right\| \\ &\leq \mathbb{E} \left\| P_+^{\partial_r^\Lambda \mathcal{X}_r} \left(\tau_t^{\Lambda}(T) - \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \tau_t^{\mathcal{X}_r, \mathcal{X}_r^c}(T) \Psi_q(H^{\mathcal{X}_r, \mathcal{X}_r^c}) \right)_{P_{I_{\leq q}}^{\Lambda}} P_+^{\partial_r^\Lambda \mathcal{X}_r} \right\| \\ &\leq C_q \, \langle t \rangle^3 \, |\Lambda|^{\xi_q} \, \mathrm{e}^{-\theta_q r} \leq C_q \, \langle t \rangle^3 \, \mathrm{e}^{-\frac{1}{2}\theta_q r}, \end{aligned} \tag{5.8}$$

where we used (5.1) and $\frac{\xi_q}{\theta_q} \leq \frac{\xi_{q+\frac{1}{2}}}{\theta_{q+\frac{z}{z}12}}$.

We note that $\tau_t^{\mathcal{X}_r,\mathcal{X}_r^c}(T)=\tau_t^{\mathcal{X}_r}(T)$ on $\mathcal{H}_\Lambda$ since T is supported by $\mathcal{X}\subset\mathcal{X}_r$. Thus

$$\begin{aligned}
&\mathbb{E}\left\|\pi_{M_1}\left(\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\left(\tau_t^{\mathcal{X}_r,\mathcal{X}_r^c}(T)-T_t\right)\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\right)_{P^\Lambda_{I_{\le q}}}\pi_{M_2}\right\|\\
&\quad\le\mathbb{E}\left\|\left(\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\left(\tau_t^{\mathcal{X}_r,\mathcal{X}_r^c}(T)-T_t\right)\Psi_q(H^{\mathcal{X}_r\mathcal{X}_r^c})\right)_{P^{\mathcal{X}_r,\mathcal{X}_r^c}_{\check{I}_{\le q}}}\right\| \qquad (5.9)\\
&\quad\le\mathbb{E}\left\|\left(\tau_t^{\mathcal{X}_r,}(T)-T_t\right)_{P^{\mathcal{X}_r}_{\check{I}_{\le q}}}\right\|\le C_q''\langle t\rangle^{2q+5}\left(\ln|\Lambda|\right)^{\xi_{q+1}}\mathrm{e}^{-\theta_{q+1}\ell},
\end{aligned}$$

where we used $P^{\mathcal{X}_r,\mathcal{X}_r^c}_{\check{I}_{\le q}}=P^{\mathcal{X}_r}_{\check{I}_{\le q}}P^{\mathcal{X}_r,\mathcal{X}_r^c}_{\check{I}_{\le q}}$ and (5.7)

It follows from (5.8) and (5.9) that

$$\begin{aligned}
&\mathbb{E}\left\|\pi_{M_1}\left(\tau_t^\Lambda(T)-T_t\right)_{P^\Lambda_{I_{\le q}}}\pi_{M_2}\right\|\\
&\le\mathbb{E}\left\|\pi_{M_1}\left(\tau_t^\Lambda(T)-\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\tau_t^{\mathcal{X}_r,\mathcal{X}_r^c}(T)\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\right)_{P^\Lambda_{I_{\le q}}}\pi_{M_2}\right\|\\
&\quad+\mathbb{E}\left\|\pi_{M_1}\left(\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\left(\tau_t^{\mathcal{X}_r,\mathcal{X}_r^c}(T)-T_t\right)\Psi_q(H^{\mathcal{X}_r,\mathcal{X}_r^c})\right)_{P^\Lambda_{I_{\le q}}}\pi_{M_2}\right\| \qquad (5.10)\\
&\le C_q\,\langle t\rangle^3\,\mathrm{e}^{-\frac{1}{2}\theta_q r}+C_q''\langle t\rangle^{2q+5}\left(\ln|\Lambda|\right)^{\xi_{q+1}}\mathrm{e}^{-\theta_{q+1}\ell}\\
&\le C_q'''\langle t\rangle^{2q+5}\left(\ln|\Lambda|\right)^{\xi_{q+1}}\mathrm{e}^{-\min\left\{\frac{1}{2}\theta_q,\theta_{q+1}\right\}\ell}.
\end{aligned}$$

We conclude from (5.3), (5.4), and (5.10) that for all $\ell\in\mathbb{N}$ there exists an observable $T_t=T(t,q,\ell,\Lambda,M_1\cup M_2)$, supported in $[\mathcal{X}]^\Lambda_{(13+\beta_{q+1})\ell}$, such that we have

$$\mathbb{E}\left\|\pi_{M_1}\left(\tau_t^\Lambda(T)-T_t\right)_{P^\Lambda_{I_{\le q}}}\pi_{M_2}\right\|\le C_q\langle t\rangle^{2q+5}\left(\ln|\Lambda|\right)^{\xi_{q+1}}\mathrm{e}^{-\frac{1}{2}\min\left\{\theta_q,\theta_{q+\frac{1}{2}},c_\mu\right\}\ell}. \qquad (5.11)$$

□

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Data availability We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach. One can obtain the relevant materials from the references below.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

1. Abanin, D., Bardarson, J., De Tomasi, G., Gopalakrishnan, S., Khemani, V., Parameswaran, S., Pollmann, F., Potter, A., Serbyn, M., Vasseur, R.: Distinguishing localization from chaos: challenges in finite-size systems. Ann. Phys. **427**, 168415 (2021)
2. Abanin, D.A., Altman, E., Bloch, I., Serbyn, M.: Colloquium: Many-body localization, thermalization, and entanglement. Rev. Mod. Phys. **91**(2), 021001 (2019)
3. Abdul-Rahman, H., Nachtergaele, B., Sims, R., Stolz, G.: Localization properties of the disordered XY spin chain: A review of mathematical results with an eye toward many-body localization. Ann. Phys. **529**(7), 1600280 (2017)
4. Alet, F., Laflorencie, N.: Many-body localization: an introduction and selected topics. C. R. Phys. **19**(6), 498–525 (2018)
5. Bauer, B., Nayak, C.: Area laws in a many-body localized state and its implications for topological order. J. Stat. Mech. **2013**(09), P09005 (2013)
6. Beaud, V., Warzel, S.: Bounds on the entanglement entropy of droplet states in the XXZ spin chain. J. Math. Phys. **59**, 1 (2018)
7. Elgart, A., Klein, A.: Localization in the random XXZ quantum spin chain (2024). https://doi.org/10.48550/arXiv.2210.14873v3
8. Elgart, A., Klein, A., Stolz, G.: Manifestations of dynamical localization in the disordered XXZ spin chain. Commun. Math. Phys. **361**(3), 1083–1113 (2018)
9. Elgart, A., Klein, A., Stolz, G.: Many-body localization in the droplet spectrum of the random XXZ quantum spin chain. J. Funct. Anal. **275**(1), 211–258 (2018)
10. Gogolin, C., Eisert, J.: Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems. Rep. Prog. Phys. **79**(5), 056001 (2016)
11. Hastings, M.B.: Locality in quantum systems. Quantum Theory Small Large Scales **95**, 171–212 (2010)
12. Helffer, B., Sjöstrand, J.: Équation de Schrödinger avec champ magnétique et équation de Harper Schr ö dinger operators (Sønderborg,: Lecture Notes in Phys. 345. Springer, Berlin 1989, pp. 118–197 (1988)
13. Hunziker, W., Sigal, I.M.: Time-dependent scattering theory of N-body quantum systems. Rev. Math. Phys. **12**(8), 1033–1084 (2000)
14. Kiefer-Emmanouilidis, M., Unanyan, R., Fleischhauer, M., Sirker, J.: Evidence for unbounded growth of the number entropy in many-body localized phases. Phys. Rev. Lett. **124**(24), 243601 (2020)
15. Lieb, E.H., Robinson, D.W.: The finite group velocity of quantum spin systems. Commun. Math. Phys. 251–257 (1972)
16. Mastropietro, V.: Localization in interacting fermionic chains with quasi-random disorder. Commun. Math. Phys. **351**(1), 283–309 (2017)
17. Morningstar, A., Colmenarez, L., Khemani, V., Luitz, D.J., Huse, D.A.: Avalanches and many-body resonances in many-body localized systems. Phys. Rev. B **105**(17), 174205 (2022)
18. Nachtergaele, B., Sims, R.: Lieb–Robinson bounds in quantum many-body physics. Entropy and the quantum. Contemp. Math. 529. Amer. Math. Soc., Providence, RI, pp.141–176 (2010)
19. Nachtergaele, B., Reschke, J.: Slow propagation in some disordered quantum spin chains. J. Stat. Phys. **182**, 1–28 (2021)
20. Nandkishore, R., Huse, D.A.: Many-body localization and thermalization in quantum statistical mechanics. Annu. Rev. Condens. Matter Phys. **6**(1), 15–38 (2015)
21. Sels, D., Polkovnikov, A.: Dynamical obstruction to localization in a disordered spin chain. Phys. Rev. E **104**, 054105 (2021)
22. Sierant, P., Delande, D., Zakrzewski, J.: Thouless time analysis of Anderson and many-body localization transitions. Phys. Rev. Lett. **124**(18), 186601 (2020)
23. Sierant, P., Zakrzewski, J.: Challenges to observation of many-body localization. Phys. Rev. B **105**(22), 224203 (2022)
24. Šuntajs, J., BonÄča, J., Prosen, T., Vidmar, L.: Ergodicity breaking transition in finite disordered spin chains. Phys. Rev. B **102**, 064207 (2020)
25. Šuntajs, J., BonÄča, J., Prosen, T., Vidmar, L.: Quantum chaos challenges many-body localization. Phys. Rev. E **102**, 062144 (2020)

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