Received: 10 February 2024 | Revised: 4 June 2024 | Accepted: 4 July 2024

DOI: 10.1112/blms.13140

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Crossing numbers of cable knots

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Funding information
NSF, Grant/Award Numbers: DMS-2004155, DMS-2304033; NSF/RTG, Grant/Award Number: DMS-2135960

Abstract

We use the degree of the colored Jones knot polynomials to show that the crossing number of a (p, q)-cable of an adequate knot with crossing number c is larger than $q^2\, c$. As an application, we determine the crossing number of 2-cables of adequate knots. We also determine the crossing number of the connected sum of any adequate knot with a 2-cable of an adequate knot.

MSC 2020
57K10, 57K14, 57K16 (primary)

1 | INTRODUCTION

Given a knot K, we will use $c(K)$ to denote the crossing number of K, which is the smallest number of crossings over all diagrams that represent K. Crossing numbers are known to be notoriously intractable. For instance, their behavior under basic knot operations, such as connect sum of knots and satellite operations, is poorly understood. In particular, the basic conjecture that if K is a satellite knot with companion C, then $c(K) \geqslant c(C)$ is still open [11, Problem 1.68]. In this direction, Lackenby [13] proved that we have $c(K) \geqslant 10^{-13}\, c(C)$, for any satellite knot K with companion C. In this note, we prove a much stronger inequality for cables of adequate knots and we determine the exact crossing numbers of infinite families of such knots. Since alternating knots are known to be adequate, our results apply, in particular, to cables of alternating knots.

To state our results, for a knot K in the 3-sphere, let $N(K)$ denote a tubular neighborhood of K. Given coprime integers p, q, let $K_{p,q}$ denote the (p, q)-cable of K. In other words, $K_{p,q}$ is the simple closed curve on $\partial N(K)$ that wraps p times around the meridian and q-times around the canonical longitude of K. Recall that the writhe of an adequate diagram $D = D(K)$ is an invariant of the knot K [14]. We will use $\mathrm{wr}(K)$ to denote this invariant.

Theorem 1.1. *For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have $c(K_{p,q}) \geqslant q^2\, c(K) + 1$.*

Theorem 1.1, combined with the results of [8], has applications in determining crossing numbers of prime satellite knots. We have the following.

Corollary 1.2. *Let K be an adequate knot with crossing number $c(K)$ and writhe number $\mathrm{wr}(K)$. If $p = 2\,\mathrm{wr}(K) \pm 1$, then $K_{p,2}$ is nonadequate and $c(K_{p,2}) = 4\,c(K) + 1$.*

The proof of Corollary 1.2 shows that when $p = 2\,\mathrm{wr}(K) \pm 1$, if we apply the $(p, 2)$-cabling operation to an adequate diagram of K, the resulting diagram is a minimum crossing diagram of the knot $c(K_{p,2})$. It should be compared with other results in the literature, asserting that the crossing numbers of some important classes of knots are realized by a "special type" of knot diagrams. These classes include alternating and more generally adequate knots, torus knots, Montesinos knots [10, 17, 20], and untwisted Whitehead doubles of adequate knots with zero writhe number [8]. We note that these Whitehead doubles and the cables $c(K_{p,2})$ of Corollary 1.2 are the first infinite families of prime satellite knots for which the crossing numbers have been determined. In [1], Baker Motegi and Takata obtained lower bounds for crossing numbers of Mazur doubles of adequate knots. In particular, they show that if K is an adequate knot with $\mathrm{wr}(K) = 0$, then the crossing number of the Mazur double of K is either $9\,c(K) + 2$ or $9\,c(K) + 3$.

We note that a geometric lower bound that applies to crossing number of satellites of hyperbolic knots is given in [4].

Corollary 1.2 allows us to compute the crossing number of $(\pm 1, 2)$-cables of adequate knots that are equivalent to their mirror images (a.k.a. amphicheiral) since such knots are known have $\mathrm{wr}(K) = 0$. In particular, since for any adequate knot K with mirror image K^*, the connect sum $K\#K^*$ is adequate and amphicheiral, we have the following.

Corollary 1.3. *For any adequate knot K with crossing number $c(K)$ and mirror image K^*, let $K^2 := K\#K^*$. Then, $c(K^2_{\pm 1,2}) = 8\,c(K) + 1$.*

Our results also have an application to the open conjecture on the additivity of crossing numbers [11, Problem 1.68] under connect sums. Lower bounds for the connect sum of knots in terms of the crossing numbers of the summands that apply to all knots are obtained in [5, 12]. The conjecture has been proved in the cases where each summand is adequate [10, 17, 20] or a torus knots [3], and when one summand is adequate and the other an untwisted Whitehead doubles of adequate knots with zero writhe number [8]. To these, we add the following.

Theorem 1.4. *Suppose that K is an adequate knot and let $K_1 := K_{p,2}$, where $p = 2\,\mathrm{wr}(K) \pm 1$. Then, for any adequate knot K_2, the connected sum $K_1\#K_2$ is nonadequate and we have*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

It may be worth noting that out of the 2977 prime knots with up to 12 crossings, 1851 are listed as adequate on Knotinfo [16], and thus, our results above can be applied to them.

2 | CROSSING NUMBERS OF CABLES OF ADEQUATE KNOTS

2.1 | Preliminaries

A *Kauffman state* on a knot diagram D is a choice of either the A-resolution or the B-resolution for each crossing of D as shown in Figure 1. The result of applying σ to D is a collection $\sigma(D)$ of disjoint simple closed curves called *state circles*. The *all-A* (resp. *all-B*) *state*, denoted by σ_A (resp. σ_B), is the state where the A-resolution (resp. the B-resolution) is chosen at every crossing of D.

- For an oriented knot diagram D, with $c(D)$ crossings, $c_+(D)$ and $c_-(D)$ are, respectively, the number of positive crossings and negative crossings of D (see Figure 2). The *writhe* of D is given by $\mathrm{wr}(D) := c_+(D) - c_-(D)$.
- The graph $\mathbb{G}_A(D)$ (resp. $\mathbb{G}_B(D)$) has vertices the state circles of the all-A (resp. all-B state) and edges the segments recording the original location of the crossings (see Figure 1). We denote by $v_A(D)$ (resp. $v_B(D)$) the number of vertices of $\mathbb{G}_A(D)$ (resp. $\mathbb{G}_A(D)$).

Definition 2.1. A knot diagram $D = D(K)$ is called *A-adequate* (resp. *B-adequate*) if $\mathbb{G}_A(D)$ (resp. $\mathbb{G}_B(D)$) has no one-edged loops. A knot is *adequate* if it admits a diagram $D := D(K)$ that is both A- and B-adequate [14, 15].

If $D := D(K)$ is an adequate diagram, the quantities $c(D)$, $c_\pm(D)$, $\mathrm{wr}(D)$ are invariants of K [14], and will be denoted by $c(K)$, $c_\pm(K)$, $g_T(K)$, and $\mathrm{wr}(K)$, respectively.

Given a knot K, let $J_K(n)$ denote its nth unreduced colored Jones polynomial, which is a Laurent polynomial in a variable t. The value on the unknot U is given by

$$J_U(n)(t) = (-1)^{n-1}\frac{t^{-n/2} - t^{n/2}}{t^{-1/2} - t^{1/2}},$$

for $n \geqslant 2$. Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t, and set

$$d[J_K(n)] := 4d_+[J_K(n)] - 4d_-[J_K(n)].$$

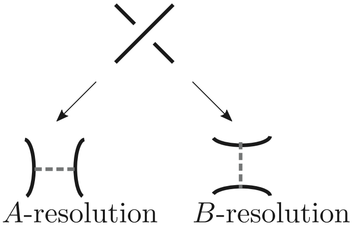

FIGURE 1 The A- and B-resolution and the corresponding edges of $\mathbb{G}_A(D)$ and $\mathbb{G}_B(D)$.

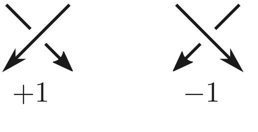

FIGURE 2 A positive crossing and a negative crossing.

For the purposes of this paper, we will assume that the set of cluster points

$$\{|n^{-2}\,d[J_K(n)]|\}'_{n\in\mathbb{N}},$$

consists of a single point and denoted by dj_K. This number is called the *Jones diameter* of K. We recall the following.

Theorem 2.2 [8]. *Let K be a knot with Jones diameter dj_K and crossing number $c(K)$. Then,*

$$dj_K \leqslant 2\,c(K),$$

with equality $dj_K = 2\,c(K)$ if and only if K is adequate.

In particular, if K is a nonadequate knot admitting a diagram D such that $dj_K = 2\,(c(D) - 1)$, then we have $c(D) = c(K)$.

Next, we recall a couple of results from the literature that give the extreme degrees of the colored Jones polynomials of the cables $K_{p,q}$ in the case where the degrees $d_\pm[J_K(n)]$ are quadratic polynomials.

Proposition 2.3 [2, 9]. *Suppose that K is a knot such that $d_+[J_K(n)] = a_2\,n^2 + a_1\,n + a_0$ and $d_-[J_K(n)] = a_2^*\,n^2 + a_1^*\,n + a_0^*$ are quadratic polynomials for all $n > 0$. Suppose, moreover, that $a_1 \leqslant 0$, $a_1^* \geqslant 0$ and that $\frac{p}{q} < 4\,a_2$, $\frac{-p}{q} < -4\,a_2^*$.*

Then, for n large enough, we have

$$4\,d_+[J_{K_{p,q}}(n)] = 4\,q^2\,a_2\,n^2 + (q\,4\,a_1 + 2\,(q-1)\,(p - 4\,q\,a_2))\,n + A,$$

$$4\,d_-[J_{K_{p,q}}(n)] = 4\,q^2\,a_2^*\,n^2 + (q\,4\,a_1^* + 2\,(q-1)\,(p - 4\,q\,a_2^*))\,n + A^*,$$

where $A, A^* \in \mathbb{Q}$ depend only on K and p, q.

Proof. The first equation is shown in [9] (see also [2]). To obtain the second equation, note that, since $K^*_{-p,q} = (K_{p,q})^*$, we have $d_-[J_{K_{p,q}}(n)] = -d_+[J_{K^*_{-p,q}}(n)]$. Since $d_+[J_{K^*}(n)] = -d_-[J_K(n)] = -a_2^*\,n^2 - a_1^*\,n - a_0^*$, the result follows by applying the first equation to $K^*_{-p,q}$. □

Now we recall the second result promised earlier.

Lemma 2.4 [2, 9]. *Let the notation and setting be as in Proposition 2.3.*

If $\frac{p}{q} > 4a_2$, then

$$4\,d_+[J_{K_{p,q}}(n)] = p\,q\,n^2 + B,$$

where $B \in \mathbb{Q}$ depends only on K and p, q.

Similarly, if $\frac{-p}{q} > -4a_2^*$, then

$$4\,d_-[J_{K_{p,q}}(n)] = p\,q\,n^2 + B',$$

where $B' \in \mathbb{Q}$ depends only on K and p, q.

Proof. The first equation is shown in [9] (see also [2]). As in the proof of Proposition 2.3, to see the second equation, we use the fact that $d_-[J_{K_{p,q}}(n)] = -d_+[J_{K^*_{-p,q}}(n)]$. Applying the first equation to $K^*_{-p,q}$, we get $4\,d_+[J_{K^*_{-p,q}}(n)] = -p\,q\,n^2 + B^*$, and hence $4\,d_-[J_{K_{p,q}}(n)] = p\,q\,n^2 - B^*$. Setting $B' := -B^*$, we obtain the desired result. □

2.2 | Lower bounds and admissible knots

We will say that a knot K is *admissible* if there is a diagram $D = D(K)$ such that we have $dj_K = 2\,(c(D) - 1)$. Our interest in admissible knots comes from the fact that if K is admissible and nonadequate, then by Theorem 2.2, D is a minimal diagram (i.e., $c(D) = c(K)$).

Theorem 2.5. *Let K be an adequate knot and let $c(K)$, $c_\pm(K)$ and $\mathrm{wr}(K)$ be as above.*

(a) *For any coprime integers p, q, we have*

$$c(K_{p,q}) \geqslant q^2\,c(K). \tag{1}$$

(b) *The cable $K_{p,q}$ is admissible if and only if $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$.*

Proof. Since K is adequate, we have

$$4\,d_+[J_K(n)] = 2\,c_+(K)\,n^2 + O(n) \ \text{ and } \ 4\,d_-[J_K(n)] = -2\,c_-(K)\,n^2 + O(n),$$

and hence,

$$4\,d_+[J_K(n)] - 4\,d_-[J_K(n)] = 2\,c(K)\,n^2 + O(n), \tag{2}$$

for every $n \geqslant 0$ [14]. We distinguish three cases.

Case 1. Suppose that $\frac{p}{q} < 2\,c_+(K)$ and $\frac{-p}{q} < 2\,c_-(K)$. Then, $d_+[J_K(n)]$ satisfies the hypothesis of Proposition 2.3 with $4\,a_2 = 2\,c_+(K) > 0$ and $d_-[J_K(n)] = -d_+[J_{K^*}(n)]$, where $d_+[J_{K^*}(n)]$ satisfies that hypothesis of Proposition 2.3 with $-4\,a_2^* = 2\,c_+(K^*) = 2\,c_-(K)$. The requirements that $a_1 \leqslant 0$ and $a_1^* \geqslant 0$ are satisfied since for adequate knots, the linear terms of the degree of $J_K^*(n)$ are multiples of Euler characteristics of spanning surfaces of K. Indeed, a_1 (resp. a_1^*) is equal to (resp. the opposite of) the Euler characteristic of a surface bounded by K. See [9, Lemmas 3.6, 3.7] or [6, 7]. Now Proposition 2.3 implies that, for sufficiently large n, the quadratic coefficient of $d_+[J_{K_{p,q}}(n)]$ (resp $d_-[J_{K_{p,q}}(n)]$) is equal to $4\,a_2 = 2\,c_+(K)$ (resp. $4\,a_2^* = -2\,c_-(K)$). Hence, the Jones diameter of $K_{p,q}$ is

$$dj_{K_{p,q}} = 2\,q^2\,c(K). \tag{3}$$

Now by Theorem 2.2, we get $c(K_{p,q}) \geqslant q^2\,c(K)$ which proves part (a) of Theorem 2.5 in this case.

For part (b), we recall that a diagram $D_{p,q}$ of $K_{p,q}$ is obtained as follows: Start with an adequate diagram $D = D(K)$ and take q parallel copies to obtain a diagram D^q. In other words, take the q-cabling of D following the blackboard framing. To obtain $D_{p,q}$ add t-*twists* to D^q, where $t := p - q\,\mathrm{wr}(K)$ as follows: If $t < 0$, then a twist takes the leftmost string in D^q and slides it over the $q - 1$ strings to the right; then we repeat the operation $|t|$-times. If $t > 0$, a twist takes the rightmost

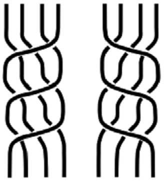

FIGURE 3 Three positive (left) and three negative (right) twists on four strands.

string in D^q and slides it over the $q-1$ strings to the left; then we repeat the operation $|t|$-times. See Figure 3. Now

$$c(D_{p,q}) = q^2\, c(K) + |t|(q-1) = q^2\, c(K) + |p - q\,\mathrm{wr}(K)|\,(q-1),$$

while $dj_{K_{p,q}} = 2\,q^2\, c(K)$. Now setting $2\,c(D_{p,q}) - 2 = dj_K$, we get $|p - q\,\mathrm{wr}(K)|\,(q-1) = 1$ which gives that $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$. Similarly, if we set $p = q\,\mathrm{wr}(K) \pm 1$ and $q = 2$, we find that $2\,c(D_{p,q}) - 2 = dj_{K_{p,q}}$ must also be true. Hence, in this case, both (a) and (b) hold.

Case 2. Suppose that $\frac{p}{q} > 2\,c_+(K)$. Then, by Lemma 2.4,

$$4\,d_+[J_{K_{p,q}}(n)] = p\,q\,n^2 + O(n). \tag{4}$$

Since $\frac{p}{q} > 2\,c_+(K)$, multiplying both sides by q^2, we get

$$p\,q > 2q^2\, c_+(K). \tag{5}$$

On the other hand, since $\frac{-p}{q} < 0$, we clearly have $\frac{-p}{q} < 2\,c_-(K)$, and Proposition 2.3 applies to give

$$4\,d_-[J_{K_{p,q}}(n)] = -2\,c_-(K)\,n^2 + O(n). \tag{6}$$

By Equations (4) and (6), we obtain

$$4\,d_+[J_K(n)] - 4\,d_-[J_K(n)] = (p\,q + 2\,q^2\, c_-(K))\,n^2 + O(n). \tag{7}$$

Now by Equations (7) and (5), we have

$$dj_{K_{p,q}} = p\,q + 2\,q^2\, c_-(K) > 2\,q^2\, c_+(K) + 2\,q^2\, c_-(K) = 2\,q^2\, c(K), \tag{8}$$

which finishes the proof for part (a) of the theorem in this case.

Next we argue that in this case, we do not get any admissible knots: First, note that

$$p > 2q\, c_+(K) > q\,\mathrm{wr}(K).$$

As in Case 1, we get a diagram $D_{p,q}$ of $K_{p,q}$ with

$$c(D_{p,q}) = q^2\, c(K) + (p - q\,\mathrm{wr}(K))\,(q-1),$$

while $dj_{K_{p,q}} = p\,q + 2\,q^2\,c_-(K)$. Now setting $2c(D_{p,q}) - 2 = dj_{K_{p,q}}$, and after some straightforward algebra, we find that in order for $K_{p,q}$ to be admissible, we must have

$$2\,(q^2 - q)\,c_-(K) + 2\,q\,c_+(K) + p\,(q-2) - 2 = 0.$$

However, since $p, c(K) > 0$ and $q \geqslant 2$, above equation is never satisfied.

Case 3. Finally, suppose that $\frac{-p}{q} > 2c_-(K) > 0$. By Lemma 2.4,

$$4\,d_-[J_{K_{p,q}}(n)] = p\,q\,n^2 + O(n). \tag{9}$$

Since $\frac{-p}{q} > 2\,c_-(K) > 0$, we conclude that

$$-p\,q > 2\,q^2\,c_-(K). \tag{10}$$

Since $\frac{p}{q} < 0$, we clearly have $\frac{p}{q} < 2\,c_+(K)$, and Proposition 2.3 applies to give

$$4\,d_+[J_{K_{p,q}}(n)] = 2\,c_+(K)\,n^2 + O(n). \tag{11}$$

By Equations (9) and (11), and using (10), we obtain

$$dj_{K_{p,q}} = 2\,q^2\,c_+(K) - p\,q > 2\,q^2\,c_+(K) + 2\,q^2\,c_-(K) = 2\,q^2\,c(K), \tag{12}$$

which finishes the proof for part (a) of the theorem. An argument similar to this of Case 2 above shows that we do not get any admissible knots in Case 3 as well. □

Remark 2.6. In [18], inequality (1) is also verified, for some choices of p and q, using crossing number bounds obtained from the ordinary Jones polynomial in [19] and also from the 2-variable Kauffman polynomial. Theorem 1.1 shows that the colored Jones polynomial and the results of [8] provide better bounds for crossing numbers of satellite knots, allowing in particular exact computations.

3 | NONADEQUACY RESULTS

To prove the stronger version of inequality (1), stated in Theorem 1.1, we need to know that the cables $K_{p,q}$ are not adequate. This is the main result in this section.

Theorem 3.1. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2\,c_+(K)$ and $\frac{-p}{q} < 2\,c_-(K)$. Then, the cable $K_{p,q}$ is nonadequate.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2\,c_+(K)$ and $\frac{-p}{q} < 2\,c_-(K)$. If $K_{p,q}$ is adequate, then $c(K_{p,q}) = q^2\,c(K)$.*

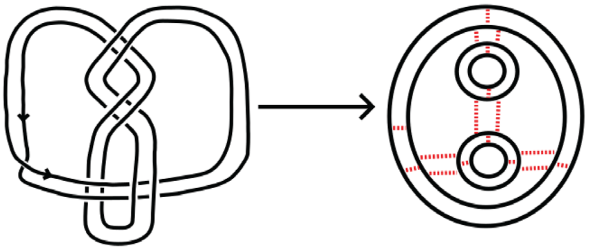

FIGURE 4 A diagram of the (−1,2)-cable of the figure eight knot and its all-B state graph.

Proof. By Proposition 2.3, for n large enough,

$$4\,d_+[J_{(K_{p,q}}(n)] - 4\,d_-[J_{K_{p,q}}(n)] = d_2\,n^2 + d_1\,n + d_0,$$

with $d_i \in \mathbb{Q}$. By Proposition 2.3, and the discussion in the beginning of the proof of Theorem 2.5, we compute $d_2 = q^2\,(4\,a_2 - 4\,a_2^*) = 2\,q^2\,c(K)$. Now if $K_{p,q}$ is adequate, since by applying Equation (2) to $K_{p,q}$ gives $d_2 = 2\,c(K_{p,q})$, we must have $c(K_{p,q}) = q^2\,c(K)$. □

We now give the proof of Theorem 3.1.

Proof. First, we let K, p, and q such that $t := p - q\,\mathrm{wr}(K) < 0$.

Recall that if K has an adequate diagram $D = D(K)$ with $c(D) = c_+(D) + c_-(D)$ crossings and the all-A (rep. all-B) resolution has $v_A = v_A(D)$ (resp. $v_B = v_B(D)$) state circles, then

$$4\,d_-[J_K(n)] = -2\,c_-(D)n^2 + 2\,(c(D) - v_A(D))\,n + 2\,v_A(D) - 2\,c_+(D), \tag{13}$$

$$4\,d_+[J_K(n)] = 2\,c_+(D)\,n^2 + 2\,(v_B(D) - c(D))\,n + 2\,c_-(D) - 2\,v_B(D). \tag{14}$$

Equation (13) holds for A-adequate diagrams $D = D(K)$. Thus, in particular, the quantities $c_-(D), v_A(D)$ are invariants of K (independent of the particular A-adequate diagram). Similarly, Equation (14) holds for B-adequate diagrams $D = D(K)$, and hence, $c_+(D), v_B(D)$ are invariants of K. Recall also that $c(D) = c(K)$ since D is adequate.

Now we start with a knot K that has an adequate diagram D. Since $\mathrm{wr}(D) = \mathrm{wr}(K)$, we have $c_+(D) = c_-(D) + \mathrm{wr}(K)$. Since D is B-adequate and $t < 0$, the cable $D_{p,q}$ is a B-adequate diagram of $K_{p,q}$, with $v_B(D_{p,q}) = q\,v_B(D)$ and $c_+(D_{p,q}) = q^2\,c_+(D)$. See Figure 4. Furthermore, since as said above these quantities are invariants of $K_{p,q}$, they remain the same for all B-adequate diagrams of $K_{p,q}$.

Now assume, for a contradiction, that $K_{p,q}$ is adequate: Then, it has a diagram $\bar{D}$ that is both A and B-adequate. By above observation, we must have $v_B(\bar{D}) = v_B(D_{p,q}) = q\,v_B(D)$ and $c_+(\bar{D}) = c_+(D_{p,q}) = q^2\,c_+(D)$.

By Lemma 3.2, $c(\bar{D}) = c(K_{p,q}) = q^2\,c(K)$. Write

$$4\,d_+[J_{K_{p,q}}(n)] = x\,n^2 + y\,n + z,$$

for some $x, y, z \in \mathbb{Q}$.

For sufficiently large n, we have two different expressions for x, y, z. On the one hand, because $\bar{D}$ is adequate, we can use Equation (14) to determine x, y, z. On the other hand, using

$4\,d_+[J_{K^*_{-p,q}}(n)]$, x, y, z can be determined using Proposition 2.3 with a_2 and a_1 coming from Equation (14).

We will use these two ways to find the quantity y. Applying Equation (14) to $\bar{D}$, we obtain

$$y = 2\,(v_B(\overline{D}) - c(\bar{D})) = 2q\,v_B(D) - 2\,q^2\,c(D). \tag{15}$$

On the other hand, using Proposition 2.3 with a_2 and a_1 coming from Equation (14), we have: $4\,a_2 = 2\,c_+(D) = c(D) + \mathrm{wr}(K)$. Also, we have $4\,a_1 = 2\,v_B(D) - 2\,c(D)$. We obtain

$$\begin{aligned} y &= q\,(4\,a_1) - 2\,q\,(q-1)\,(4\,a_2) + 2\,(q-1)\,p \\ &= 2\,q\,v_B(D) - 2\,q^2\,c(D) + 2\,(q-1)\,p - 2\,q\,(q-1)\mathrm{wr}(K). \end{aligned} \tag{16}$$

For the two expressions derived for y from Equations (15) and (16) to agree, we must have $2\,q\,((q-1)\,2\,\mathrm{wr}(K) + p) - 2\,p = 0$. However, this is impossible since $q > 1$ and p, q are coprime. This contradiction shows that $K_{p,q}$ is nonadequate.

To deduce the result for $K_{p,q}$, with $t(K, p, q) := p - q\,\mathrm{wr}(K) > 0$, let K^* denote the mirror image of K. Note that $(K_{p,q})^* = K^*_{-p,q}$ and since being adequate is a property that is preserved under taking mirror images, it is enough to show that $K^*_{-p,q}$ is nonadequate. Since $t(K^*, -p, q) := -p - q\,\mathrm{wr}(K^*) = -t(K, p, q) < 0$, the later result follows from the argument above. □

Now we are ready to give the proofs of Theorem 1.1 and Corollary 1.2 which we restate for the convenience of the reader.

Theorem 1.1. *For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have $c(K_{p,q}) \geqslant q^2\,c(K) + 1$.*

Proof. By Theorem 2.5, we have $c(K_{p,q}) \geqslant q^2\,c(K)$. We need to show that this inequality is actually strict. Following the proof of of Theorem 2.5, we distinguish three cases.

Case 1. Suppose that $\frac{p}{q} < 2\,c_+(K)$ and $\frac{-p}{q} < 2\,c_-(K)$. Then, by Equation (3), we have $dj_{K_{p,q}} = 2\,q^2\,c(K)$. By Theorem 3.1, $K_{p,q}$ is nonadequate and hence by Theorem 2.2 again we have $2\,c(K_{p,q}) > dj_{K_{p,q}}$, and the strict inequality follows.

Case 2. Suppose that $\frac{p}{q} > 2\,c_+(K)$. Then, by Equation (8), we have $c(K_{p,q}) > q^2\,c(K)$, and the result follows in this case

Case 3. Suppose that $\frac{-p}{q} > 2\,c_-(K)$. Then, by Equation (12) again, we have $c(K_{p,q}) > q^2\,c(K)$, as desired. □

Next we discuss how to deduce Corollary 1.2.

Corollary 1.2. *Let K be an adequate knot with crossing number $c(K)$ and writhe number* $\mathrm{wr}(K)$. *If $p = 2\,\mathrm{wr}(K) \pm 1$, then $K_{p,2}$ is nonadequate and $c(K_{p,2}) = 4\,c(K) + 1$.*

Proof. If $q = 2$ and $p = q\,\mathrm{wr}(K) \pm 1$, then, by Theorem 2.5, $K_{p,q}$ is admissible. Thus, by Theorem 2.2, the diagram $D_{p,2}$ constructed in the proof of Theorem 2.5 is minimal. That is, $c(K_{p,2}) = c(D_{p,2}) = 4\,c(K) + 1$. □

4 | COMPOSITE NONADEQUATE KNOTS

In this section, we prove Theorem 1.4.

Given a knot K, such that for n large enough the degrees of the colored Jones polynomials of K are quadratic polynomials with rational coefficients, we will write

$$4\,d_+[J_K(n)] - 4\,d_-[J_K(n)] = d_2(K)\,n^2 + d_1(K)\,n + d_0(K).$$

Lemma 4.1. *Let K be a nontrivial adequate knot, $p = 2\,\mathrm{wr}(K) \pm 1$ and let $K_1 := K_{p,2}$. Then, for any adequate knot K_2, the connected sum $K_1\#K_2$ is nonadequate.*

Proof. The claim is proven by applying the arguments applied to $K_1 = K_{p,2}$ in the proofs of Lemma 3.2 and Theorem 3.1 to $K_1\#K_2$ and properties of the degrees of colored Jones polynomial [8, Lemma 5.9].

First, we claim that if $K_1\#K_2$ were adequate, then we would have

$$c(K_1\#K_2) = 4\,c(K) + c(K_2). \tag{17}$$

Note that as $p = 2\,\mathrm{wr}(K) \pm 1$, we have $\frac{p}{2} < 2\,c_+(K)$ and $\frac{-p}{2} < 2\,c_-(K)$. Hence, Proposition 2.3 applies to K_1. Now write

$$4\,d_+[J_{K_1\#K_2}(n)] - 4\,d_-[J_{K_1\#K_2}(n)] = d_2(K_1\#K_2)\,n^2 + d_1(K_1\#K_2)\,n + d_0(K_1\#K_2).$$

Since we assumed that $K_1\#K_2$ is adequate, we have $d_2(K_1\#K_2) = 2\,c(K_1\#K_2)$ and by [8, Lemma 5.9], $d_2(K_1\#K_2) = d_2(K_1) + d_2(K_2) = 2\,4\,c(K) + 2\,c(K_2)$, which leads to (17).

Case 1. Suppose that $p - 2\,\mathrm{wr}(K) = -1 < 0$.

Start with $D = D(K)$ an adequate diagram and let $D_1 := D_{p,2}$ be constructed as in the proof of Theorem 2.5. Also let D_2 be an adequate diagram of K_2. As in the proof of Theorem 3.1, conclude that $D_1\#D_2$ is a B-adequate diagram for $K_1\#K_2$ and that the quantities $v_B(D_1\#D_2) = 2\,v_B(D) + v_B(D_2) - 1$ and $c_+(D_1\#D_2) = 4\,c_+(D) + c_+(D_2)$ are invariants of $K_1\#K_2$.

Let $\bar{D}$ be an adequate diagram. Then,

$$v_B(\bar{D}) = v_B(D_1\#D_2) = 2\,v_B(D) + v_B(D_2) - 1 \text{ and } c_+(\bar{D}) = 4\,c_+(D) + c_+(D_2).$$

Next, we will calculate the quantity $d_1(K_1\#K_2)$ in two ways: First, since we assumed that $\bar{D}$ is an adequate diagram for $K_1\#K_2$, applying Equation (14), we get

$$d_1(K_1\#K_2) = 2\,(v_B(\bar{D}) - c(\bar{D})) = 2\,(2\,v_B(D) + v_B(D_2) - 1 - 4\,c(D) - c(D_2)).$$

Second, using by Proposition 2.3, we get $d_1(K_1) = 2\,(2\,v_B(D) - 4\,c(D) + p + 2\,\mathrm{wr}(K))$. Thus, we get

$$d_1(K_1\#K_2) = d_1(K_1) + d_1(K_2) - 2 = 2\,(2\,v_B(D) - 4\,c(D) + p - 2\,\mathrm{wr}(K) + v_B(D_2) - c(D_2) - 1).$$

Now note that in order for the two resulting expressions for $d_1(K_1\#K_2)$ to be equal, we must have $(p - 2\,\mathrm{wr}(K)) = 0$ that contradicts our assumption that $p - 2\,\mathrm{wr}(K) = -1$. We conclude that $K_1\#K_2$ is nonadequate.

Case 2. Assume now that $p - 2\,\mathrm{wr}(K) = 1$. Since $(K_{p,2})^* = K^*_{-p,2}$ and being adequate is preserved under taking mirror images, it is enough to show that $K^*_{-p,2}\#K^*_2$ is nonadequate. Since $-p - 2\,\mathrm{wr}(K^*) = -(p - 2\,\mathrm{wr}(K))) = -1$, the later result follows from Case 1. □

Now we give the proof of Theorem 1.4, which we also restate here.

Theorem 1.4. *Suppose that K is an adequate knot and let $K_1 := K_{p,2}$, where $p = 2\,\mathrm{wr}(K) \pm 1$. Then, for any adequate knot K_2, the connected sum $K_1\#K_2$ is nonadequate and we have*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

Proof. Note that if K is the unknot, then so is $K_{p,2}$ and the result follows trivially. Suppose that K is a nontrivial knot. Then, by Lemma 4.1, we obtain that $K_1\#K_2$ is nonadequate. By Part (b) of Theorem 2.5, we have $dj_{K_1} = 2\,(c(D_{\pm1,2}) - 1)$ and $dj_{K_2} = 2\,c(D_2) = 2\,c(K)$ where D_2 is an adequate diagram for K_2. Hence, $dj_{K_1\#K_2} = 2\,(c(D_1\#D_2) - 1)$, where $D_1 = D_{\pm1,2}$ and by Theorem 2.2,

$$c(K_1\#K_2) = c(D_1\#D_2) = c(D_1) + c(D_2) = c(K_1) + c(K_2),$$

where the last equality follows since, by Corollary 1.2, we have $c(K_1) = c(D_1) = c(D_{p,2})$. □

ACKNOWLEDGMENTS

The authors acknowledge partial research support through NSF Grants DMS-2004155 and DMS-2304033 and from the NSF/RTG grant DMS-2135960.

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