Mathematische Zeitschrift (2024) 308:1
https://doi.org/10.1007/s00209-024-03559-9

Mathematische Zeitschrift

Central elements in the SL_d-skein algebra of a surface

Francis Bonahon[1,2] · Vijay Higgins[2]

Received: 15 January 2024 / Accepted: 28 May 2024 / Published online: 26 July 2024

Abstract
The SL_d-skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ of a surface S is a certain deformation of the coordinate ring of the character variety consisting of flat SL_d-local systems over the surface. As a quantum topological object, $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ is also closely related to the HOMFLYPT polynomial invariant of knots and links in $\mathbb{R}^3$. We exhibit a rich family of central elements in $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ that appear when the quantum parameter q is a root of unity. These central elements are obtained by threading along framed links certain polynomials arising in the elementary theory of symmetric functions, and related to taking powers in the Lie group SL_d.

Keywords Skein algebras · Skein modules

Mathematics Subject Classification 17K10 · 17K16 · 20G42 · 57R56

Let GL_d denote the general linear group of invertible d-by-d matrices, with entries in an irrelevant and therefore unspecified field, and let the special linear group SL_d consist of those matrices that have determinant 1. The SL_d-*skein module* $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ of an oriented 3-dimensional manifold M is a certain deformation of the coordinate ring of the character variety

$$\mathcal{X}_{\mathrm{SL}_d}(M) = \{\text{homomorphisms } r\colon \pi_1(M) \to \mathrm{SL}_d\} /\!\!/ \mathrm{GL}_d,$$

where GL_d acts on the set of group homomorphisms $r\colon \pi_1(M) \to \mathrm{SL}_d$ by conjugation. This quantum deformation depends on a nonzero quantum parameter q, and more precisely on a d-root $q^{\frac{1}{d}}$. In its current incarnations, the motivation for this mathematical object arises from Witten's topological quantum field theory interpretation of the Jones polynomials and other knot invariants [22], where the elements of $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ occur as morphisms. In particular, it is closely related to the HOMFLYPT invariant of knots and links in $\mathbb{R}^3$ [20].

This work was developed under the auspices of the Research Training Grant DMS-2135960, *RTG: Algebraic and Geometric Topology at Michigan State*, from the U.S. National Science Foundation.

✉ Francis Bonahon
fbonahon@usc.edu; bonahonf@msu.edu

Vijay Higgins
higgi231@msu.edu

1 Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA

2 Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

The elements of $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ can be represented by linear combinations of framed links in M where each component carries an integer weight $i \in \{1, 2, \dots, d-1\}$, standing for the i-th exterior power of the defining representation of the quantum group $\mathrm{U}_q(\mathfrak{sl}_d)$. The relations satisfied by these generators correspond to the full set of relations between tensor products of these representations in the braided tensor category of representations of $\mathrm{U}_q(\mathfrak{sl}_d)$, as determined by Cautis–Kamnitzer–Morrison [4]. When $d > 2$, these relations are better expressed in terms of more complicated objects called SL_d-webs; see Sect. 1.1.

A special case of interest is the one where M is equal to the thickening $S \times [0, 1]$ of an oriented surface S of finite topological type, in which case the resulting skein module $\mathcal{S}^q_{\mathrm{SL}_d}(S) = \mathcal{S}^q_{\mathrm{SL}_d}\big(S \times [0, 1]\big)$ is endowed with a natural multiplication by superposition, which also corresponds to the composition of morphisms in the topological quantum field theory framework; see Sect. 1.2. The viewpoint of [22] involves representations of this algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ and, if we want these representations to have finite dimension, it is natural to require that the quantum parameter q be a root of unity.

In the special case where $d = 2$ and q is a primitive n-root of unity with n odd, Helen Wong and the first author [2] discovered unexpected central elements in the skein algebra $\mathcal{S}^q_{\mathrm{SL}_2}(S)$, based on the Chebyshev polynomial of the first type $T_n \in \mathbb{Z}[e]$; see [12] for versions when n is even. Frohman, Kania–Bartoszyńska and Lê [5] later proved that these elements, together with the more obvious central elements associated to punctures that occur for all q, generate the whole center of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$. This, together with a combination of results from [2, 5, 6], led to a classification of "most" irreducible finite-dimensional representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$, in terms of points in a certain finite branched cover of the character variety $\mathcal{X}_{\mathrm{SL}_2}(M)$.

The current article is devoted to the development of similar central elements in the SL_d-skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ with $d \geqslant 2$, still when q is a root of unity. In particular, it provides a broader context explaining the occurrence of Chebyshev polynomials of the first type in the case of SL_2.

The regular functions on SL_2 that are invariant under conjugation by elements of GL_2 form a polynomial algebra generated by the trace function Tr, and the Chebyshev polynomial $T_n \in \mathbb{Z}[e]$ is determined by the property that $\mathrm{Tr}\, A^n = T_n(\mathrm{Tr}\, A)$ for every $A \in \mathrm{SL}_2$. For SL_d, the algebra of GL_d–invariant regular functions on SL_d is a polynomial algebra in $d-1$ variables, corresponding to the elementary symmetric polynomials $E_d^{(1)}, E_d^{(2)}, \dots, E_d^{(d-1)}$ in the eigenvalues. These are also related to the coefficients of the characteristic polynomial by the property that

$$\det(A + t\,\mathrm{Id}_d) = t^d + t^{d-1}E_d^{(1)}(A) + t^{d-2}E_d^{(2)}(A) + \cdots + t\,E_d^{(d-1)}(A) + 1$$

for every $A \in \mathrm{SL}_d$. An immediate consequence of the elementary theory of symmetric functions is that, for every $n \geqslant 1$ and for every $i \in \{1, 2, \dots, d-1\}$, there is a unique polynomial $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$ such that

$$E_d^{(i)}(A^n) = \widehat{P}_d^{(n,i)}\big(E_d^{(1)}(A), E_d^{(2)}(A), \dots, E_d^{(d-1)}(A)\big)$$

for every $A \in \mathrm{SL}_d$; see Sect. 3. We call these polynomials $\widehat{P}_d^{(n,i)}$ the *reduced power elementary polynomials*. For instance, when $d = 2$, there is only one such polynomial $\widehat{P}_2^{(n,1)}$ for every n, and this polynomial is just the Chebyshev polynomial T_n. See the Appendix for a method to explicitly compute the polynomials $\widehat{P}_d^{(n,i)}$.

Our new central elements in $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ are based on the threading operation along polynomials that was already at the basis of [2]. For a framed knot L in a 3-manifold M, the threading operation along a polynomial

$$P = \sum_{i_1, i_2, \ldots, i_{d-1}=0}^{i_{\max}} a_{i_1 i_2 \ldots i_{d-1}} e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$$

associates to L the skein

$$L^{[P]} = \sum_{i_1, i_2, \ldots, i_{d-1}=0}^{i_{\max}} a_{i_1 i_2 \ldots i_{d-1}} L^{[e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M),$$

where $L^{[e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M)$ is represented by the union of $i_1 + i_2 + \ldots i_{d-1}$ disjoint parallel copies of the knot L, taken in the direction of the framing, and with i_1 of these copies carrying the weight 1, i_2 carrying the weight 2, …, and i_{d-1} carrying the weight $d-1$. A similar construction applies to links L with several components. See Sect. 2 for details.

Theorem 1 *Suppose that the d-root $q^{\frac{1}{d}}$ occurring in the definition of skein modules $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ is such that $q^{\frac{2n}{d}} = 1$, and that $q^{2i} \neq 1$ for every integer i with $2 \leqslant i \leqslant \frac{d}{2}$. In a thickened surface $S \times [0,1]$, let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_c$ be a framed link in which each component L_j carries a weight $i_j \in \{1, 2, \ldots, d-1\}$. Then the skein $L^{[\widehat{P}_d^{(n,\bullet)}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$ obtained by threading the reduced power elementary polynomial $\widehat{P}_d^{(n,i_j)} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ along each component L_j is central in the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ of the surface S.*

Theorem 1 is based on a more general property for skein modules $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ of 3-manifolds which, borrowing terminology from [12], is a certain transparency property for threading operations along the reduced power polynomial $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$. This property states that, if L_0 is a framed link in a 3-manifold M carrying component weights in $\{1, 2, \ldots, d-1\}$ and if L is a framed knot disjoint from L_0, then the skein $L_0 \sqcup L^{[\widehat{P}_d^{(n,i)}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M)$ obtained by threading $\widehat{P}_d^{(n,i)}$ along L is invariant under any isotopy of L in M that is allowed to cross L_0.

Theorem 2 *Suppose that the d-root $q^{\frac{1}{d}}$ occurring in the definition of skein modules $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ is such that $q^{\frac{2n}{d}} = 1$, and that $q^{2i} \neq 1$ for every integer i with $2 \leqslant i \leqslant \frac{d}{2}$. Then, for every $i = 1, 2, \ldots, d-1$, the threading operation along the reduced power elementary polynomial $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ is transparent in the skein module $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ of any oriented 3-manifold M.*

As indicated in Remark 15, the hypothesis in Theorems 1 and 2 that $q^{2i} \neq 1$ for every integer i with $2 \leqslant i \leqslant \frac{d}{2}$ is probably unnecessary.

Similar results for G_2-skeins, where G_2 is the exceptional Lie group of rank 2, appear in [1].

1 SL_d-webs and skein relations

1.1 The SL_d-skein module of a 3-dimensional manifold

Throughout the article, SL_d will denote the Lie group of d-by-d matrices with determinant 1. Because the coefficient field of this algebraic group is irrelevant for our purposes, we will systematically omit it.

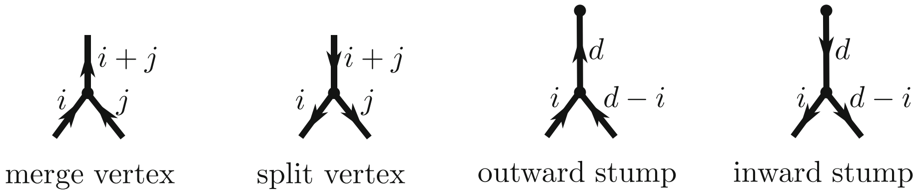


Fig. 1 Vertices of a web

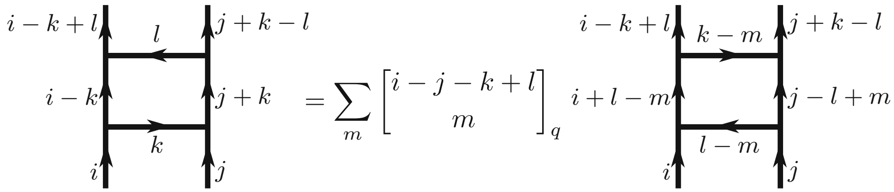


Fig. 2 A typical skein relation

We are here using the version of SL_d-skein modules that uses the webs developed by Cautis–Kamnitzer–Morrison in [4, 16]. There is another well-known alternative based on Kuperberg–Sikora spiders [11, 13, 19]. See [17] for the equivalence between the two viewpoints.

An SL_d-*web* in an oriented 3-dimensional manifold M is a graph W embedded in M endowed with additional data satisfying the following conditions:

(1) the graph W is endowed with a *ribbon structure* consisting of a thin oriented surface embedded in M that contains W and deformation retracts onto it;
(2) each edge of W carries an orientation and a weight $i \in \{1, 2, \dots, d\}$;
(3) each vertex of W is of one of the following three types:

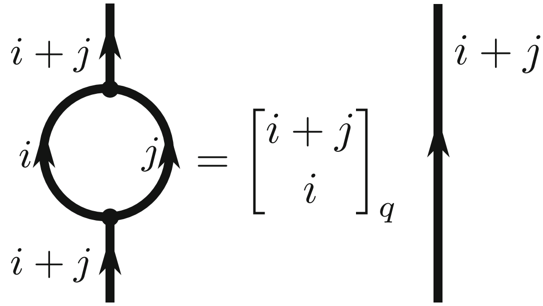
   (a) a vertex of type "merge" with two incoming edges of weights i and j and one outgoing edge of weight $i + j$, as in the first picture of Fig. 1;
   (b) a vertex of type "split" with one incoming edge of weight $i + j$ and two outgoing edges of weights i and j, as in the second picture of Fig. 1;
   (c) a vertex of type "stump" (also called "tag" in [4]) adjacent to exactly one edge of W, which carries weight d, as in the last two pictures of Fig. 1;

(4) the only edges that are allowed to carry weight d are those adjacent to a stump;
(5) W can have components that are closed loops, with no vertices, but no component can be the graph with exactly one edge and two stumps.

Along the components of W that are closed loops, the ribbon structure is equivalent to the very classical notion of *framing*, namely the data of a vector field that is everywhere transverse to the loop (or, equivalently, with a trivialization of the normal bundle of that loop). In particular, framed (oriented) links where each component carries a weight $i \in \{1, 2, \dots, d-1\}$ are fundamental examples of webs.

The SL_d-*skein module* $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ of the oriented 3-manifold M is obtained from the vector space over $\mathbb{C}$ (say) freely generated by the set of isotopy classes of SL_d-webs in M under a set of *skein relations* that are explicitly listed in [4]. Since we will not need most of them, we are only listing a few in Figs. 2–5 and refer to [4] for the full list.

Fig. 3 Another skein relation



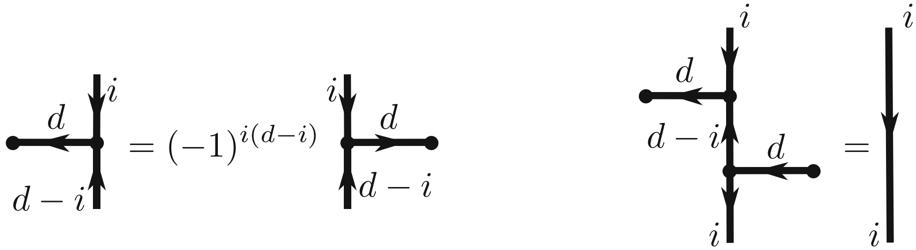


Fig. 4 Two skein relations involving stumps

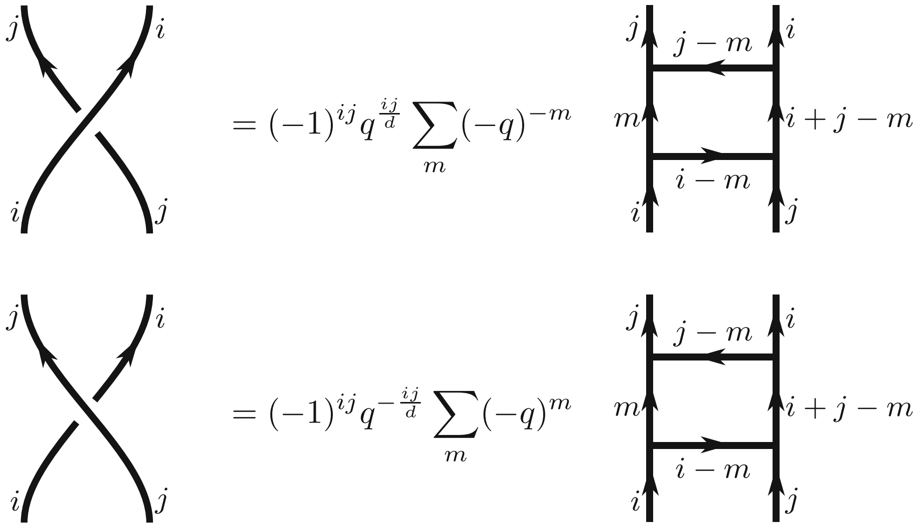


Fig. 5 Braiding relations

In these figures, each skein relation should be seen as occurring in a neighborhood of a disk embedded in M, in such a way that the ribbon structures of each web represented are horizontal for the projection to that disk. The sums are over indices $m \in \mathbb{Z}$, with the following conventions:

(1) the sum is limited to those values of m that lead to edge weights in $\{0, 1, \ldots, d\}$;
(2) an edge carrying weight 0 and its end vertices should be erased;

(3) an edge carrying weight d should be split into two edges with stumps, with a convention that will be more precisely described when we need it in our proof of Lemma 9.

Also, the symbols $\begin{bmatrix} i \\ j \end{bmatrix}_q$ represent the *quantum binomials*

$$\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{[i]_q\,[i-1]_q\dots[i-j+1]_q}{[j]_q\,[j-1]_q\dots[2]_q\,[1]_q} = \frac{[i]_q!}{[j]_q!\,[n-j]_q!}$$

with the *quantum integers*

$$[i]_q = \frac{q^i - q^{-i}}{q - q^{-1}}$$

and the *quantum factorials*

$$[i]_q! = [i]_q\,[i-1]_q\dots[2]_q\,[1]_q\,.$$

We will not need the skein relation of Fig. 2, which is shown here only to give the flavor of typical skein relations. However, we will make use of the relations of Figs. 3–5.

Note that the *braiding relations* of Fig. 5 require us to fix a d-root $q^{\frac{1}{d}}$ of the quantum parameter $q \in \mathbb{C} - \{0\}$. As a consequence, the skein module $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ depends on this choice of $q^{\frac{1}{d}}$ in spite of the fact that this is not reflected in the notation, which would otherwise be too cumbersome.

These skein relations originate from the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_d)$. The skein relations other than the braiding relations of Fig. 5 describe all the relations that occur between tensor products of the quantum exterior power representations $\Lambda^i_q\mathbb{C}^d$ of $\mathrm{U}_q(\mathfrak{sl}_d)$. The braiding relations reflect the braiding of the representation category of $\mathrm{U}_q(\mathfrak{sl}_d)$. See [4] for details.

1.2 The $\mathbf{SL}_d$-skein algebra of a surface

An important special case is provided by the thickening $M = S \times [0, 1]$ of an oriented surface S. In this case the skein module $\mathcal{S}^q_{\mathrm{SL}_d}(S \times [0, 1])$ admits a natural algebra structure where the multiplication is defined as follows. If $[W_1], [W_2] \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$ are respectively represented by webs W_1, W_2 in $S \times [0, 1]$, the product $[W_1] \bullet [W_2]$ is represented by the web $W_1' \cup W_2'$ where W_1' is obtained by rescaling W_1 inside $S \times [0, \frac{1}{2}]$ and W_2' is obtained by rescaling W_2 inside $S \times [\frac{1}{2}, 1]$. In practice if, by projection to S, we represent each W_i by the picture of a possibly knotted graph in S, $[W_1] \bullet [W_2]$ is obtained by placing W_2 on top of W_1.

The algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S \times [0, 1])$, denoted as $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ for short, is the SL_d*-skein algebra* of the oriented surface S.

2 Threading a polynomial along a framed link

Let L be an oriented framed knot in the 3-manifold M, namely a 1-dimensional oriented closed submanifold of M that is endowed with a nonzero section of its normal bundle. This framing can also be used to define a ribbon structure along L.

Given a polynomial

$$P = \sum_{i_1, i_2, \ldots, i_{d-1}=0}^{i_{\max}} a_{i_1 i_2 \ldots i_{d-1}} e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$$

in $(d-1)$ variables e_1, e_2, …, e_{d-1} with coefficients $a_{i_1 i_2 \ldots i_{d-1}} \in \mathbb{Z}$, the *skein obtained by threading P along L* is defined as the linear combination

$$L^{[P]} = \sum_{i_1, i_2, \ldots, i_{d-1}=0}^{i_{\max}} a_{i_1 i_2 \ldots i_{d-1}} L^{[e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M),$$

where $L^{[e_1^{i_1} e_2^{i_2} \ldots e_{d-1}^{i_{d-1}}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M)$ is represented by the union of $i_1 + i_2 + \ldots i_{d-1}$ disjoint parallel copies of the knot L, taken in the direction of the framing, and with i_1 of these copies carrying the weight 1, i_2 carrying the weight 2, …, and i_{d-1} carrying the weight $d-1$. In particular, $L^{[e_1^0 e_2^0 \ldots e_{d-1}^0]}$ is represented by the empty link.

More generally, if L is an oriented framed link with components L_1, L_2, $\ldots L_c$, the *skein obtained by threading the polynomials P_j along the components L_j of L* is defined as the disjoint union

$$L^{[P_1, P_2, \ldots, P_c]} = L_1^{[P_1]} \sqcup L_2^{[P_2]} \sqcup \cdots \sqcup L_c^{[P_c]} \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$$

where the parallel copies used to define each $L_j^{[P_j]}$ are chosen in disjoint tubular neighborhoods of the L_j. Note that, because each $L_j^{[P_j]}$ is represented by a linear combination of weighted links, the disjoint union $L^{[P_1, P_2, \ldots, P_c]} \in \mathcal{S}^q_{\mathrm{SL}_d}(M)$ is also defined by a linear combination of disjoint unions of those links.

Threading a polynomial $P \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ is *transparent* if, for every oriented framed knot L in a 3–manifold M and every web $W \subset M$ that is disjoint from L, the element of $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ that is represented by $L^{[P]} \sqcup W$ is invariant under any isotopy of the knot L that allows it to cross W. Because every skein module is spanned by weighted links, this is equivalent to the version given in the Introduction, where the web W was restricted to be a weighted link.

Lemma 3 *If threading each of the polynomials P_1, P_2, …, $P_c \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ is transparent then, for every surface S and every oriented framed link $L \subset S \times [0, 1]$ with components L_1, L_2, $\ldots L_c$, the skein $L^{[P_1 P_2 \ldots P_c]}$ obtained by threading the polynomials P_j along the components L_j of L is central in the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$.*

Proof If $[W] \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$ is represented by a web $W \subset S \times [\frac{1}{3}, \frac{2}{3}]$, then $[L^{[P]}] \bullet [W]$ is represented by $L_1^{[P]} \sqcup W$ where L_1 is obtained by rescaling L inside $S \times [0, \frac{1}{3}]$, while $[W] \bullet [L^{[P]}]$ is represented by $L_2^{[P]} \sqcup W$ with L_2 obtained by rescaling L inside $S \times [\frac{2}{3}, 1]$. Applying the transparency property to an isotopy moving L_1 to L_2 shows that $[L^{[P]}] \bullet [W] = [W] \bullet [L^{[P]}]$. □

3 Power elementary polynomials

In the ring $\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d]$ of polynomials with integer coefficients in d variables λ_1, λ_2, …, λ_d, recall that a polynomial is *symmetric* if it is invariant under all permutations of the variables λ_1, λ_2, ..., λ_d. Fundamental examples include the *elementary symmetric polynomials*

$$E_d^{(i)} = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant d} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_i},$$

defined for $1 \leqslant i \leqslant d$.

There is a well-known connection between the elementary symmetric polynomials $E_d^{(i)}$ and the Lie group GL_d. Namely, if $A \in \mathrm{GL}_d(\mathbb{K})$ is a matrix with coefficients in the field $\mathbb{K}$, with eigenvalues λ_1, λ_2, ..., λ_d in the algebraic closure of $\mathbb{K}$, the coefficient of the term of degree $d-i$ in the characteristic polynomial of A is equal to $(-1)^i E_d^{(i)}$. In this situation, we will also write

$$E_d^{(i)}(A) = E_d^{(i)}(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{K}.$$

A more intrinsic interpretation is that $E_d^{(i)}(A)$ is the trace of the action $\Lambda^i A \colon \Lambda^i \mathbb{K}^d \to \Lambda^i \mathbb{K}^d$ of A on the exterior power $\Lambda^i \mathbb{K}^d$.

If we are interested in the characteristic polynomial of a power A^n, whose eigenvalues are λ_1^n, λ_2^n, ..., λ_d^n, it makes sense to consider, for $n \geqslant 1$ and $1 \leqslant k \leqslant d$, the *power elementary symmetric polynomials*

$$E_d^{(n,i)} = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant d} \lambda_{j_1}^n \lambda_{j_2}^n \dots \lambda_{j_i}^n$$

obtained from $E_d^{(i)}$ by replacing each occurrence of the variable λ_j with its power λ_j^n. For instance, the case $n = 1$ gives the original elementary symmetric polynomial $E_d^{(1,i)} = E_d^{(i)}$, while the case $i = 1$ corresponds to the well-known family of *power sum polynomials* $E_d^{(n,1)} = \sum_{i=1}^d \lambda_i^n$.

Lemma 4 *There exists a unique polynomial $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_d]$ such that $E_d^{(n,i)} \in \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]$ is obtained from $P_d^{(n,i)}$ by replacing each variable e_j with the elementary symmetric polynomial $E_d^{(j)} \in \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]$.*

Proof This is an immediate consequence of the very classical property that the subring of symmetric polynomials in $\mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]$ is itself isomorphic to the polynomial ring $\mathbb{Z}[e_1, e_2, \dots, e_d]$, by an isomorphism sending each elementary symmetric polynomial $E_d^{(i)}$ to the variable e_i. See for instance [14, Sect. I.2]. □

We call these $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_d]$ the *power elementary polynomials*, not to be confused with the closely connected but formally different power elementary *symmetric* polynomials $E_d^{(n,i)} \in \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]$, which involve different variables.

Simple considerations show that $P_d^{(1,i)} = e_i$ when $n = 1$, and $P_d^{(n,d)} = e_d^n$ when $i = d$. See Proposition 19 in the Appendix for a method to explicitly compute the power elementary polynomials $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_d]$.

We are interested in the Lie group SL_d rather than GL_d. For a matrix $A \in \mathrm{SL}_d(\mathbb{K})$ with eigenvalues λ_1, λ_2, ..., λ_d in the algebraic closure of the field $\mathbb{K}$, we have that

$$E_d^{(d)}(A) = E_d^{(d)}(\lambda_1, \lambda_2, \dots, \lambda_d) = \lambda_1 \lambda_2 \dots \lambda_d = \det A = 1.$$

It is therefore natural to specialize the polynomial $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_d]$ by setting $e_d = 1$, and to consider the *reduced power elementary polynomial* $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$

defined by

$$\widehat{P}_d^{(n,i)}(e_1, e_2, \ldots, e_{d-1}) = P_d^{(n,i)}(e_1, e_2, \ldots, e_{d-1}, 1).$$

Lemma 5 *The power elementary polynomial $P_d^{(n,i)}$ is the unique polynomial in $\mathbb{Z}[e_1, e_2, \ldots, e_d]$ such that*

$$E_d^{(i)}(A^n) = P_d^{(n,i)}\left(E_d^{(1)}(A), E_d^{(2)}(A), \ldots, E_d^{(d)}(A)\right)$$

for every $A \in \mathrm{GL}_d$, where $E_d^{(i)}(A)$ is the i-th elementary symmetric polynomial in the eigenvalues of A.

The reduced power elementary polynomial $\widehat{P}_d^{(n,i)}$ is the unique polynomial in $\mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ such that

$$E_d^{(i)}(A^n) = \widehat{P}_d^{(n,i)}\left(E_d^{(1)}(A), E_d^{(2)}(A), \ldots, E_d^{(d-1)}(A)\right)$$

for every $A \in \mathrm{SL}_d$.

Proof If a matrix $A \in \mathrm{GL}_d$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$, its n-th power A^n has eigenvalues $\lambda_1^n, \lambda_2^n, \ldots, \lambda_d^n$. The fact that $P_d^{(n,i)}$ and $\widehat{P}_d^{(n,i)}$ satisfy the relations indicated then follows from their definitions, noting that $E_d^{(d)}(A) = 1$ for every $A \in \mathrm{SL}_d$. The uniqueness property immediately follows from the fact that the polynomials $E_d^{(1)}, E_d^{(2)}, \ldots, E_d^{(d)}$ are algebraically independent in $\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d]$ (see [14, Sect. I.2]). □

For future reference, we note the following elementary homogeneity property of the power elementary polynomials $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_d]$.

Lemma 6 *For an additional variable θ,*

$$P_d^{(n,i)}(\theta e_1, \theta^2 e_2, \ldots, \theta^d e_d) = \theta^{ni} P_d^{(n,i)}(e_1, e_2, \ldots, e_d)$$

as polynomials in $\mathbb{Z}[\theta, e_1, e_2, \ldots, e_d]$.

Proof This is an immediate consequence of the property that each elementary symmetric polynomial $E_d^{(i)} \in \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d]$ is homogeneous of degree i, while the power elementary symmetric polynomial $E_d^{(n,i)}$ is homogeneous of degree ni. □

The following result is much less natural, but it will play an essential role in the proof of the main result of this article.

Proposition 7 *Given commuting variables $x_1, x_2, \ldots, x_{d-1}$ with x_{d-1} invertible, define*

$$y_j = \begin{cases} x_{d-1}^{-1} + x_1 & \text{if } j = 1 \\ x_{j-1}x_{d-1}^{-1} + x_j & \text{if } 2 \leqslant j \leqslant d-1. \end{cases}$$

Then, for every n and every i with $1 \leqslant i \leqslant d-1$, we have the following equality

$$\widehat{P}_d^{(n,i)}(y_1, y_2, \ldots, y_{d-1}) = x_{d-1}^{-n} P_{d-1}^{(n,i-1)}(x_1, x_2, \ldots, x_{d-1}) + P_{d-1}^{(n,i)}(x_1, x_2, \ldots, x_{d-1})$$

of Laurent polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_{d-2}, x_{d-1}^{\pm 1}]$.

Proof The proof should make the statement less mysterious. Consider the ring homomorphism

$$\varphi\colon \mathbb{Z}[x_1, x_2, \dots, x_{d-2}, x_{d-1}^{\pm 1}] \to \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]/(\lambda_1\lambda_2\dots\lambda_d = 1)$$

sending each x_j to the elementary symmetric polynomial $E_{d-1}^{(j)} \in \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_{d-1}]$ in the first $d-1$ variables, and sending x_{d-1}^{-1} to λ_d. Note that φ is well-defined since, in the target space,

$$\varphi(x_{d-1}^{-1}) = \lambda_d = (\lambda_1\lambda_2\dots\lambda_{d-1})^{-1} = (E_{d-1}^{(d-1)})^{-1} = \varphi(x_{d-1})^{-1}.$$

Using the fact that the $E_{d-1}^{(i)}$ are algebraically independent in $\mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_{d-1}]$, a simple argument shows that φ is injective. To prove the proposed relation, we therefore only need to show that the two sides have the same image under φ.

The key property underlying the whole result is that, for $2 \leqslant j \leqslant d-1$,

$$\begin{aligned}
\varphi(y_j) &= \varphi(x_{j-1}x_{d-1}^{-1} + x_j) = E_{d-1}^{(j-1)}\lambda_d + E_{d-1}^{(j)}\\
&= \lambda_d \sum_{1\leqslant i_1<\dots<i_{j-1}\leqslant d-1} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_{j-1}} + \sum_{1\leqslant i_1<\dots<i_j\leqslant d-1} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_j}\\
&= \sum_{1\leqslant i_1<\dots<i_j\leqslant d} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_j} = E_d^{(j)}.
\end{aligned}$$

A similar argument shows that $\varphi(y_1) = E_d^{(1)}$.

Then, for the left-hand side of the proposed equality,

$$\begin{aligned}
\varphi\big(\widehat{P}_d^{(n,i)}(y_1, y_2, \dots, y_{d-1})\big) &= \widehat{P}_d^{(n,i)}\big(\varphi(y_1), \varphi(y_2), \dots, \varphi(y_{d-1})\big)\\
&= \widehat{P}_d^{(n,i)}(E_d^{(1)}, E_d^{(2)}, \dots, E_d^{(d-1)})\\
&= P_d^{(n,i)}(E_d^{(1)}, E_d^{(2)}, \dots, E_d^{(d-1)}, 1)\\
&= P_d^{(n,i)}(E_d^{(1)}, E_d^{(2)}, \dots, E_d^{(d-1)}, E_d^{(d)}) = E_d^{(n,i)}
\end{aligned}$$

using the properties that φ is a ring homomorphism and that $E_d^{(d)} = \lambda_1\lambda_2\dots\lambda_d = 1$ in the target space of φ.

For the right-hand side,

$$\begin{aligned}
&\varphi\big(x_{d-1}^{-n}P_{d-1}^{(n,i-1)}(x_1, x_2, \dots, x_{d-1}) + P_{d-1}^{(n,i)}(x_1, x_2, \dots, x_{d-1})\big)\\
&\quad= \varphi(x_{d-1}^{-1})^n P_{d-1}^{(n,i-1)}\big(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{d-1})\big)\\
&\qquad\qquad + P_{d-1}^{(n,i)}\big(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{d-1})\big)\\
&\quad= \lambda_d^n P_{d-1}^{(n,i-1)}(E_{d-1}^{(1)}, E_{d-1}^{(2)}, \dots, E_{d-1}^{(d-1)}) + P_{d-1}^{(n,i)}(E_{d-1}^{(1)}, E_{d-1}^{(2)}, \dots, E_{d-1}^{(d-1)})\\
&\quad= \lambda_d^n E_{d-1}^{(n,i-1)} + E_{d-1}^{(n,i)}\\
&\quad= \lambda_d^n \sum_{1\leqslant j_1<\dots<j_{i-1}\leqslant d-1} \lambda_{j_1}^n\lambda_{j_2}^n\dots\lambda_{j_{i-1}}^n + \sum_{1\leqslant j_1<\dots<j_i\leqslant d-1} \lambda_{j_1}^n\lambda_{j_2}^n\dots\lambda_{j_i}^n\\
&\quad= \sum_{1\leqslant j_1<\dots<j_i\leqslant d} \lambda_{j_1}^n\lambda_{j_2}^n\dots\lambda_{j_i}^n = E_d^{(n,i)} = \varphi\big(\widehat{P}_d^{(n,i)}(y_1, y_2, \dots, y_{d-1})\big).
\end{aligned}$$

Since φ is injective, this concludes the proof. □

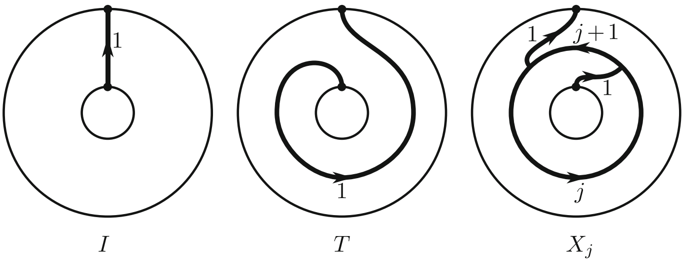


Fig. 6 A few webs, representing the elements $I, T, X_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$

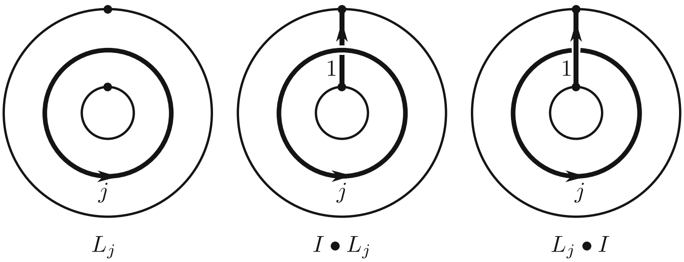


Fig. 7 The skeins $L_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)$, $I \bullet L_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ and $L_j \bullet I \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$

4 Computations in the annulus

Inspired by earlier constructions of Morton [15], Lê [12] and Queffelec-Wedrich [18], we let $A = S^1 \times [0, 1]$ be the annulus with two marked points $x_0 = (x, 0)$ and $x_1 = (x, 1)$ on the boundary (for an arbitrary $x \in S^1$). Let $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ be the vector space generated by webs in A with boundary $\{x_0, x_1\}$, where the edges containing these boundary points carry the weight 1 and are oriented inward at x_0 and outward at x_1, and quotiented by the skein relations of [4]. (The subscript io stands for "in-out".)

Figure 6 offers a few examples of webs representing elements of $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ in A. In particular, let $I \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ be represented by the arc $x \times [0, 1]$ of the first picture of Fig. 6, endowed with weight 1, and let the *twist element* $T \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ be the arc of the second diagram if Fig. 6, also endowed with weight 1. A more elaborate element $X_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$, with $1 \leqslant j \leqslant d-2$, is represented by the third web of Fig. 6.

The space $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ comes with a multiplication

$$\circ : \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}} \otimes \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}} \to \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$$

by concatenation, where the skein $W_1 \circ W_2$ is defined by placing W_1 in $S^1 \times [0, \frac{1}{2}]$ and W_2 in $S^1 \times [\frac{1}{2}, 1]$.

It also comes with left and right actions of the usual skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(A)$ by superposition where, if $[W_0] \in \mathcal{S}^q_{\mathrm{SL}_d}(A)$ and $[W_1] \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$, $[W_0] \bullet [W_1]$ is obtained by placing $[W_0]$ below $[W_1]$ and $[W_1] \bullet [W_0]$ is obtained by placing $[W_0]$ on top of $[W_1]$. We are particularly interested in the elements $I \bullet L_j$ and $L_j \bullet I \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ illustrated in the last two pictures of Fig. 7, where $L_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)$ is represented by a simple loop $L = S^1 \times \{\frac{1}{2}\}$ going counterclockwise around the annulus and carrying weight j.

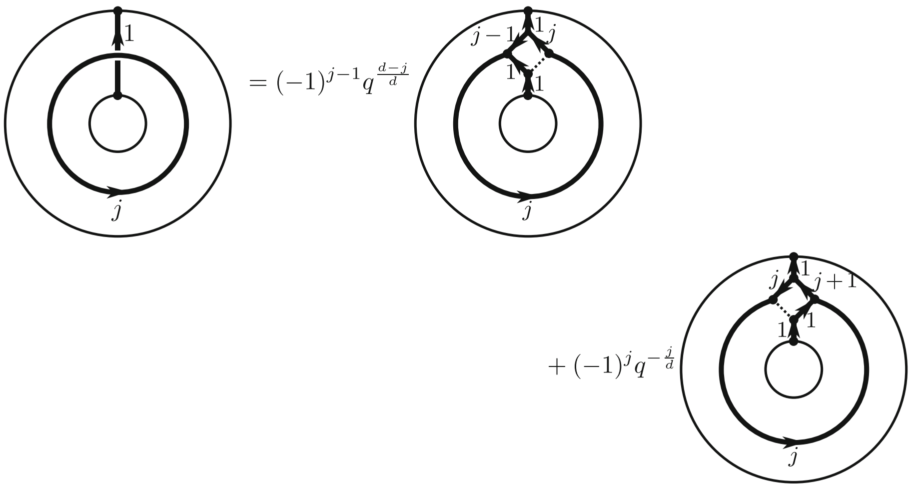


Fig. 8 The proof of Lemma 9 when $2 \leqslant j \leqslant d-2$

The following lemma states that the elements I, T, $I \bullet L_j$ and $L_j \bullet I$ are central in $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$, for the multiplication by concatenation $\circ$.

Lemma 8 *For every* $X \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$,

$$X \circ I = I \circ X = X \qquad X \circ T = T \circ X$$
$$X \circ (I \bullet L_j) = (I \bullet L_j) \circ X \qquad X \circ (L_j \bullet I) = (L_j \bullet I) \circ X.$$

Proof These properties are easily checked by elementary isotopies in the thickened annulus $A \times [0, 1]$. □

A less immediate relation between the skeins of Figs. 6–7 is provided by the skein relations of Sect. 1.1.

Lemma 9 *For* $1 \leqslant j \leqslant d-1$,

$$I \bullet L_j = \begin{cases} q^{\frac{d-1}{d}} T - q^{-\frac{1}{d}} X_1 & \text{if } j = 1 \\ (-1)^{j-1} q^{\frac{d-j}{d}} X_{j-1} \circ T + (-1)^j q^{-\frac{j}{d}} X_j & \text{if } 2 \leqslant j \leqslant d-2 \\ (-1)^{d-2} q^{\frac{1}{d}} X_{d-2} \circ T + q^{\frac{1-d}{d}} T^{-1} & \text{if } j = d-1 \end{cases}$$

$$L_j \bullet I = \begin{cases} q^{\frac{1-d}{d}} T - q^{\frac{1}{d}} X_1 & \text{if } j = 1 \\ (-1)^{j-1} q^{\frac{j-d}{d}} X_{j-1} \circ T + (-1)^j q^{\frac{j}{d}} X_j & \text{if } 2 \leqslant j \leqslant d-2 \\ (-1)^{d-2} q^{-\frac{1}{d}} X_{d-2} \circ T + q^{\frac{d-1}{d}} T^{-1} & \text{if } j = d-1 \end{cases}$$

where T^{-1} is the inverse of T for the composition operation $\circ$ (which is also its mirror image).

Proof This follows from an application of the braiding relations of Fig. 5, which express $L_j \bullet I$ and $I \bullet L_j$ as a linear combination of two webs.

When $2 \leqslant j \leqslant d-2$, the computation for $I \bullet L_j$ is illustrated in Fig. 8. On the right hand side of the equation, the webs represented each have one edge carrying weight 0 (represented

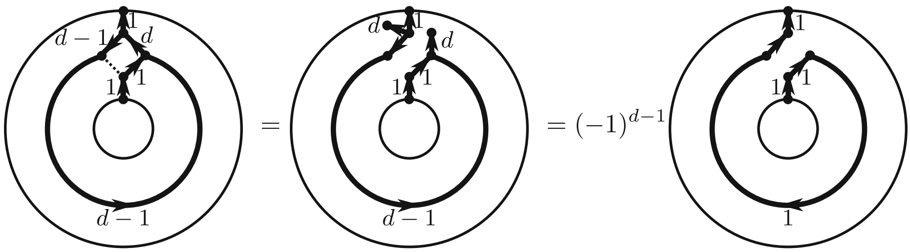


Fig. 9 The proof of Lemma 9 when $j = d - 1$

by a dotted line in the pictures) which must be erased by the conventions stated in Sect. 1.1. The first web is easily seen to be isotopic to $X_{j-1} \circ T$, while the second web is isotopic to X_j.

When $j = 1$, the first web occurring in the same computation now has two edge weights equal to 0. After erasing the corresponding two edges, the resulting web is isotopic to T. The second web is still isotopic to X_1.

When $j = d - 1$, the first web is still $X_{d-2} \circ T$ but the second web has an edge weight equal to d. We now need to use the conventions of [4] for this case, which we had skipped in our discussion in Sect. 1.1. These involve a two-step process, first splitting the weight d edge into two stumps and then flipping the resulting inward stump to the other side of the split vertex at which it is attached (see the top of Page 358 of [4]). After applying the second and third skein relations of Fig. 4 followed by an isotopy, we obtain the mirror image of T, which is also T^{-1} in $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$. See Fig. 9.

This completes the proof of the statement of Lemma 9 for $I \bullet L_j$. The proof for $L_j \bullet I$ is essentially identical. □

Lemma 10 *For every $X \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ and every j with $1 \leqslant j \leqslant d - 2$,*

$$X \circ X_j = X_j \circ X.$$

Proof By induction on j, the formulas of Lemma 9 show that, for the multiplication by concatenation $\circ$, the skein $X_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ can be expressed as a polynomial in the skeins I, T and $L_j \bullet I$. Since these skeins are central in $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ by Lemma 8, so is X_j. □

In the annulus $A = S^1 \times [0, 1]$, let $L = S^1 \times \{\frac{1}{2}\}$ be the loop that we used to define the skeins $L_j \in \mathcal{S}^q_{\mathrm{SL}_d}(A)$.

Proposition 11 *Suppose that the d-root $q^{\frac{1}{d}}$ occurring in the braiding relations of Fig. 5 is a $2n$–root of unity, and let $\widehat{P}^{(n,i)}_d \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$ be the reduced power elementary polynomial of* Sect. 3. *Then, for the framed link $L \subset A$ and the skein $I \in \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ of Fig. 7,*

$$L^{[\widehat{P}^{(n,i)}_d]} \bullet I = I \bullet L^{[\widehat{P}^{(n,i)}_d]}.$$

Proof Consider $\mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$ as a ring for the multiplication by concatenation $\circ$. Then, the commutativity property of Lemma 10 shows that there is a unique ring homomorphism

$$\psi\colon \mathbb{Z}[x_1, x_2, \dots, x_{d-2}, x_{d-1}^{\pm 1}] \to \mathcal{S}^q_{\mathrm{SL}_d}(A)_{\mathrm{io}}$$

such that $\psi(x_{d-1}) = q^{\frac{d-1}{d}} T^{-1}$ and $\psi(x_j) = (-1)^j q^{\frac{j}{d}} X_j$ for every j with $1 \leqslant j \leqslant d - 2$.

If we set

$$y_j = \begin{cases} x_{d-1}^{-1} + x_1 & \text{if } j = 1 \\ x_{j-1}x_{d-1}^{-1} + x_j & \text{if } 2 \leqslant j \leqslant d-1 \end{cases}$$

as in Proposition 7, the first batch of computations in Lemma 9 show that $\psi(y_j) = L_j \bullet I$ for every j. Applying the ring homomorphism ψ to both sides of the conclusion

$$\widehat{P}_d^{(n,i)}(y_1, y_2, \dots, y_{d-1}) = x_{d-1}^{-n} P_{d-1}^{(n,i-1)}(x_1, x_2, \dots, x_{d-1}) + P_{d-1}^{(n,i)}(x_1, x_2, \dots, x_{d-1})$$

of Proposition 7, we conclude that

$$\begin{aligned} \widehat{P}_d^{(n,i)}&(L_1 \bullet I, L_2 \bullet I, \dots, L_{d-1} \bullet I) \\ &= q^{\frac{n(1-d)}{d}} T^n \circ P_{d-1}^{(n,i-1)}\big(-q^{\frac{1}{d}} X_1, +q^{\frac{2}{d}} X_2, \dots, (-1)^{d-1} q^{\frac{d-1}{d}} X_{d-1}\big) \\ &\qquad + P_{d-1}^{(n,i)}\big(-q^{\frac{1}{d}} X_1, +q^{\frac{2}{d}} X_2, \dots, (-1)^{d-1} q^{\frac{d-1}{d}} X_{d-1}\big) \\ &= (-1)^{n(i-1)} q^{\frac{n(i-d)}{d}} T^n \circ P_{d-1}^{(n,i-1)}\big(X_1, X_2, \dots, X_{d-1}\big) \\ &\qquad + (-1)^{ni} q^{\frac{ni}{d}} P_{d-1}^{(n,i)}\big(X_1, X_2, \dots, X_{d-1}\big), \end{aligned}$$

using the specialization of Lemma 6 at $\theta = -q^{\frac{1}{d}}$ for the second equality.

When evaluating a polynomial on elements of $\mathcal{S}_{\mathrm{SL}_d}^q(A)_{\mathrm{io}}$, we used the multiplication by concatenation $\circ$. However, in the case of the skeins $L_j \bullet I$, this evaluation is also closely related to the multiplication by superposition $\bullet$ and to the threading operation. Indeed, by inspection of the definitions,

$$P(L_1 \bullet I, L_2 \bullet I, \dots, L_{d-1} \bullet I) = L^{[P]} \bullet I$$

for every polynomial $P \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$. In particular, we now conclude that

$$\begin{aligned} L^{[\widehat{P}_d^{(n,i)}]} \bullet I = (-1)^{n(i-1)} q^{\frac{n(i-d)}{d}} T^n \circ P_{d-1}^{(n,i-1)}\big(X_1, X_2, \dots, X_{d-1}\big) \\ + (-1)^{ni} q^{\frac{ni}{d}} P_{d-1}^{(n,i)}\big(X_1, X_2, \dots, X_{d-1}\big). \end{aligned}$$

If we now use the second batch of computations in Lemma 9, where q is replaced by q^{-1}, the same arguments show that

$$\begin{aligned} I \bullet L^{[\widehat{P}_d^{n,k}]} = (-1)^{n(i-1)} q^{-\frac{n(i-d)}{d}} T^n \circ P_{d-1}^{(n,i-1)}\big(X_1, X_2, \dots, X_{d-1}\big) \\ + (-1)^{ni} q^{-\frac{ni}{d}} P_{d-1}^{(n,i)}\big(X_1, X_2, \dots, X_{d-1}\big). \end{aligned}$$

We are now ready to use our hypothesis that $q^{\frac{1}{d}}$ is a $2n$–root of unity, which means that $q^{\frac{n}{d}} = q^{-\frac{n}{d}}$. The above computations then show that $L^{[\widehat{P}_d^{(n,i)}]} \bullet I = I \bullet L^{[\widehat{P}_d^{(n,i)}]}$. □

5 Central and transparent skeins from power elementary polynomials

We now use Proposition 11 to construct transparent elements in the skein module $\mathcal{S}_{\mathrm{SL}_d}^q(M)$. The following lemma will enable us to limit our argument to web edges that carry weight 1.

Lemma 12 *Let W be a web in the 3-manifold M, and let $B \subset M$ be a ball meeting W along an arc contained in the interior of an edge e of W carrying weight $i \in \{1, 2, \dots, d-1\}$.*

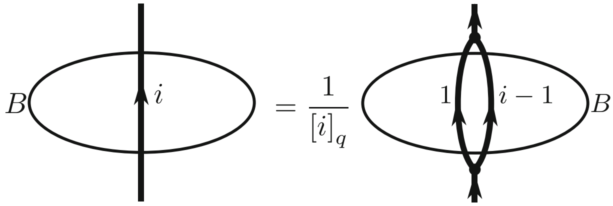


Fig. 10 The proof of Lemma 12

Suppose that $q^{2j} \neq 1$ *for every* $j \in \{2, 3, \dots, i\}$*, so that the quantum factorial* $[i]_q!$ *is nonzero. Then there exists a web* W' *in* M *such that*

(1) $[W] = \frac{1}{[i]_q!}[W']$ *in the skein module* $\mathcal{S}^q_{\mathrm{SL}_d}(M)$*;*
(2) *every point of* $B \cap W'$ *is contained in a weight* 1 *edge of* W'*;*
(3) W' *is contained in an arbitrarily small neighborhood of* W*.*

Proof The skein relation of Fig. 3 gives us the relation of Fig. 10. The result then follows by repeated application of this property. □

Theorem 13 *Suppose that the* d*-root* $q^{\frac{1}{d}}$ *occurring in the definition of skein modules* $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ *is such that* $q^{\frac{2n}{d}} = 1$*, and that* $q^{2j} \neq 1$ *for every integer* j *with* $2 \leqslant j \leqslant \frac{d}{2}$*. Then, for every* $i = 1, 2, \dots, d-1$*, threading the reduced power elementary polynomial* $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$ *is transparent in the skein module* $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ *of any oriented* 3*–manifold* M*.*

Proof Let W be a web in a oriented 3-manifold M, and let L_1 and L_2 be two framed knots in M that are disjoint from W and isotopic to each other by an isotopy that is allowed to cross W. We want to show that $L_1^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ and $L_2^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ represent the same element in $\mathcal{S}^q_{\mathrm{SL}_d}(M)$.

By decomposing the isotopy into little steps, we can clearly restrict attention to the case where it crosses W in exactly one point, located in an edge of W carrying weight i. If $i > \frac{d}{2}$, we can use the second skein relation of Fig. 4 to replace i by $d-i$; we can therefore assume that $i \leqslant \frac{d}{2}$, and in particular that $[i]_q! \neq 0$ by our hypotheses on q. Then, applying Lemma 12 to a small ball around the crossing point enables us to restrict attention to the case where the isotopy crosses W transversely in one point contained in an edge e with weight 1.

By transversality, we can further choose the isotopy so that, for the standard annulus A, there is an embedding of $A \times [0, 1]$ in M such that:

(1) the intersection of the edge e with $A \times [0, 1]$ is equal to $I \times \frac{1}{2}$ for the arc I of Fig. 6, and the ribbon structure near e is horizontal for the projection to A;
(2) shortly around the time when the isotopy crosses e, the link is contained in $A \times [0, 1]$, its projection to A is equal to the knot L of Fig. 7, and its ribbon structure is horizontal;
(3) the knot is contained in $A \times [\frac{1}{2}, 1]$ shortly before the isotopy crosses W, and in $A \times [0, \frac{1}{2}]$ shortly after that.

See Fig. 11. Restricting the isotopy to times near the crossing time, we can even assume that the knot L stays as in (2) throughout the isotopy, that L_1 is contained in $A \times [0, \frac{1}{2}]$, and that L_2 is contained in $A \times [\frac{1}{2}, 1]$

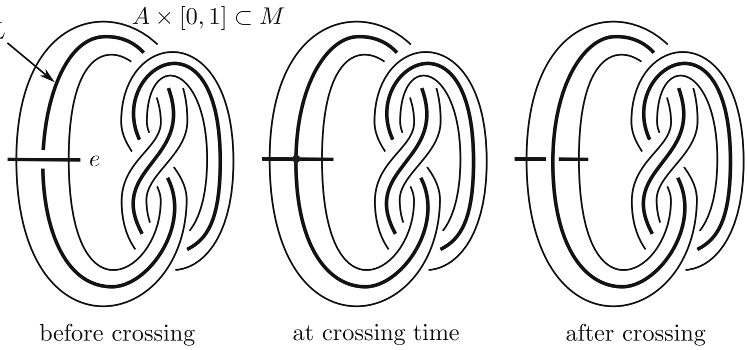


Fig. 11 The link L crossing the web W

We can then apply Proposition 11 to conclude that the intersections of $L_1^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ and $L_2^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ with $A \times [0, 1]$ differ by a sequence of isotopies and skein relations supported in the interior of $A \times [0, 1]$. Since $L_1^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ and $L_2^{[\widehat{P}_d^{(n,i)}]} \sqcup W$ coincide outside of $A \times [0, 1]$, we conclude that $[L_1^{[\widehat{P}_d^{(n,i)}]} \sqcup W] = [L_2^{[\widehat{P}_d^{(n,i)}]} \sqcup W]$ in $\mathcal{S}^q_{\mathrm{SL}_d}(M)$. □

Applying Lemma 3, an immediate corollary is that Theorem 13 provides central elements in the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$.

Corollary 14 *Suppose that the d-root $q^{\frac{1}{d}}$ occurring in the definition of the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(M)$ is such that $q^{\frac{2n}{d}} = 1$, and that $q^{2i} \neq 1$ for every integer i with $2 \leqslant i \leqslant \frac{d}{2}$. In a thickened surface $S \times [0, 1]$, let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_c$ be a framed link in which each component L_j carries a weight $i_j \in \{1, 2, \dots, d-1\}$. Then the skein $L^{[\widehat{P}_d^{(n,\bullet)}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$ obtained by threading the reduced power elementary polynomial $\widehat{P}_d^{(n,i_j)} \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$ along each component L_j is central in the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ of the surface S.* □

Remark 15 In the statements of Theorem 13 and Corollary 14, the condition that $q^{2i} \neq 1$ for every i with $2 \leqslant i \leqslant \frac{d}{2}$ is an artifact of our use of Lemma 12 in the proof, and is probably unnecessary.

6 Two conjectures

We conclude with two conjectures.

The first conjecture is the obvious one regarding the center of the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$. In addition to the elements exhibited in this article, the center of $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ admits more obvious elements associated to the punctures of the surface S. Indeed, if $[P_i] \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$ is represented by a small loop going around one of the punctures of the surface S, endowed with a weight $i \in \{1, 2, \dots, d-1\}$, a simple isotopy shows that $[P_i]$ in central in $\mathcal{S}^q_{\mathrm{SL}_d}(S)$, and this for any value of q.

Conjecture 16 *Suppose that the d-root $q^{\frac{1}{d}}$ occurring in the definition of the skein algebra $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ is such that $q^{\frac{2}{d}}$ is a primitive n-root of unity. Then, for every oriented surface S of* *finite topological type, the center of* $\mathcal{S}^q_{\mathrm{SL}_d}(S)$ *is generated (as a subalgebra) by the skeins* $[P_i]$ *associated to punctures as above, as well as by the skeins* $L^{[\widehat{P}_d^{(n,i)}]}$ *obtained by threading reduced power elementary polynomials* $\widehat{P}_d^{(n,i)}$ *around framed knots* $L \subset S \times [0, 1]$.

See [5] for a proof of this conjecture in the case where $d = 2$.

The second conjecture is the full SL_d analogue of the main statement underlying the results of [2] for $\mathcal{S}^q_{\mathrm{SL}_2}(S)$. It essentially asserts that the central skeins $L^{[\widehat{P}_d^{(n,i)}]} \in \mathcal{S}^q_{\mathrm{SL}_d}(S)$, obtained by threading reduced power elementary polynomials along framed knots, satisfy the skein relations corresponding to $q^{\frac{1}{d}} = 1$.

Conjecture 17 *Let* S *be an oriented surface of finite topological type. If the* d*-root* $q^{\frac{1}{d}}$ *occurring in the definition of the* SL_d*-skein algebra is a root of unity of order* n *coprime with* $2d$*, and if the commutative skein algebra* $\mathcal{S}^1_{\mathrm{SL}_d}(S)$ *is defined with the convention that* $1^{\frac{1}{d}} = 1$*, there exists an algebra homomorphism*

$$\Phi\colon \mathcal{S}^1_{\mathrm{SL}_d}(S) \to \mathcal{S}^q_{\mathrm{SL}_d}(S)$$

with central image such that, for every skein $[L] \in \mathcal{S}^1_{\mathrm{SL}_d}(S)$ *represented by a framed knot* L *carrying weight* $i \in \{1, 2, \dots, d-1\}$*, the image* $\Phi\big([L]\big) = L^{[\widehat{P}_d^{(n,i)}]}$ *is obtained by threading the reduced power elementary polynomial* $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_{d-1}]$ *along* L*, in the sense defined in Sect. 2.*

Remark 18 It easily follows from the skein relations that the algebra $\mathcal{S}^1_{\mathrm{SL}_d}(S)$ is generated by knots carrying a weight $i \in \{1, 2, \dots, d-1\}$. So the homomorphism $\Phi\colon \mathcal{S}^1_{\mathrm{SL}_d}(S) \to \mathcal{S}^q_{\mathrm{SL}_d}(S)$ is unique if it exists.

The case $d = 2$ of this Conjecture 17 was proved in [2] when n is odd. See also [2, 12] for related statements with other conditions on $q^{\frac{1}{2}}$. These properties played a fundamental role in the study of the finite-dimensional representation theory [2, 5, 6] of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$.

See [8] for a proof of Conjecture 17 when $d = 3$.

For general d, the homomorphism predicted by Conjecture 17 is likely to be the Frobenius homomorphism $\Phi\colon \mathcal{S}^1_{\mathrm{SL}_3}(S) \to \mathcal{S}^q_{\mathrm{SL}_3}(S)$ constructed for $d = 3$ in [7] (see also [10] for $d = 2$), and conjectured to exist for all d. See [21] for an explicit construction of this Frobenius homomorphism when the surface has nonempty boundary, and [9] for a related construction. Also see [3, 6] for more general developments.

Appendix A. Power elementary polynomials

Recall that the *power elementary symmetric polynomial* $E_d^{(n,i)} \in \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_d]$ is obtained from the elementary symmetric polynomial

$$E_d^{(i)} = \sum_{1 \leqslant j_1 < j_2 < \dots < j_i \leqslant d} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_i}$$

by replacing each variable λ_j by its power λ_j^n, and that the *power elementary polynomial* $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \dots, e_d]$ is the unique polynomial such that

$$E_d^{(n,i)} = P_d^{(n,i)}(E_d^{(1)}, E_d^{(2)}, \dots, E_d^{(d)})$$

in $\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d]$.

The following generating series property gives a method to explicitly compute the polynomials $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_d]$. As a consequence, it also provides an algorithmic computation of the reduced power elementary polynomials $\widehat{P}_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_{d-1}]$ that play a critical role in this article, since these are defined by

$$\widehat{P}_d^{(n,i)}(e_1, e_2, \ldots, e_{d-1}) = P_d^{(n,i)}(e_1, e_2, \ldots, e_{d-1}, 1).$$

Proposition 19 *For every $i = 1, 2, \ldots, d$, the power elementary polynomial $P_d^{(n,i)} \in \mathbb{Z}[e_1, e_2, \ldots, e_d]$ is the coefficient of t^i in the expansion of the power series*

$$\exp\left(\sum_{j=1}^{\infty} (-1)^{j(n+1)} \sum_{\substack{i_1+2i_2+\cdots+di_d=jn \\ i_1, i_2, \ldots, i_d \geqslant 0}} A_{i_1 i_2 \ldots i_d} \quad t^j \right) \in \mathbb{Z}[e_1, e_2, \ldots, e_d][[t]]$$

where

$$A_{i_1 i_2 \ldots i_d} = (-1)^{i_1+i_2+\cdots+i_d-1} n \frac{(i_1+i_2+\cdots+i_d-1)!}{i_1! i_2! \ldots i_d!} e_1^{i_1} e_2^{i_2} \ldots e_d^{i_d}.$$

In particular, considering the coefficient of the term of degree 1 yields the following well-known expression of the n–th power sum symmetric polynomial in terms of elementary symmetric polynomials.

Corollary 20

$$P_d^{(n,1)} = (-1)^n n \sum_{\substack{i_1+2i_2+\cdots+di_d=n \\ i_1, i_2, \ldots, i_d \geqslant 0}} (-1)^{i_1+i_2+\cdots+i_d} \frac{(i_1+i_2+\cdots+i_d-1)!}{i_1! i_2! \ldots i_d!} e_1^{i_1} e_2^{i_2} \ldots e_d^{i_d}.$$

□

Remark 21 Proposition 19 shows that the power series expansion of the exponential that occurs there is finite, which reflects hidden relations between the coefficients of the monomials $A_{i_1 i_2 \ldots i_d}$. It is possible that similar relations can be used to simplify the formula of Proposition 19. As is, this statement is already reasonably effective for computations.

Proof of Proposition 19 Consider the generating series

$$\Sigma^{(n)}(t) = 1 + \sum_{j=1}^{d} E_d^{(n,j)} t^j = \prod_{i=1}^{d} (1 + \lambda_i^n t) \in \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d, t],$$

obtained from the classical generating series (see for instance [14, Sect. I.2])

$$\Sigma^{(1)}(t) = 1 + \sum_{j=1}^{d} E_d^{(j)} t^j = \prod_{i=1}^{d} (1 + \lambda_i t)$$

for elementary symmetric polynomials by replacing each λ_i with λ_i^n.

As a preliminary computation, let us expand

$$\log \Sigma^{(1)}(t) = \log\left(1 + \sum_{j=1}^{d} E_d^{(j)} t^j\right) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\sum_{j=1}^{d} E_d^{(j)} t^j\right)^i$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \sum_{j_1+j_2+\cdots+j_d=i} \frac{i!}{j_1!j_2!\dots j_d!} E_d^{(1)j_1} E_d^{(2)j_2} \dots E_d^{(d)j_d}\, t^{j_1+2j_2+\cdots+dj_d} \\
&= \sum_{k=1}^{\infty} \sum_{j_1+2j_2+\cdots+dj_d=k} (-1)^{j_1+j_2+\cdots+j_d-1} \frac{(j_1+j_2+\cdots+j_d-1)!}{j_1!j_2!\dots j_d!} \\
&\qquad\qquad E_d^{(1)j_1} E_d^{(2)j_2} \dots E_d^{(d)j_d}\, t^k \\
&= \sum_{j=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=j} a_{i_1i_2\dots i_d} E_d^{(1)i_1} E_d^{(2)i_2} \dots E_d^{(d)i_d}\, t^j
\end{aligned}
$$

with

$$
a_{i_1i_2\dots i_d} = (-1)^{i_1+i_2+\cdots+i_d-1} \frac{(i_1+i_2+\cdots+i_d-1)!}{i_1!i_2!\dots i_d!}.
$$

To compute the generating series $\Sigma^{(n)}(t)$, let $\omega \in \mathbb{C}$ be a primitive n-root of unity, and let $\theta \in \mathbb{C}$ be any number such that $\theta^n = -1$. Then,

$$
\begin{aligned}
\Sigma^{(n)}(t) &= \prod_{i=1}^{d} (1+\lambda_i^n t) = \prod_{i=1}^{d} (1-\lambda_i^n \theta^n t) \\
&= \prod_{i=1}^{d} \prod_{j=1}^{n} \big(1-\omega^j \lambda_i \theta t^{\frac{1}{n}}\big) = \prod_{j=1}^{n} \Sigma^{(1)}\big(-\omega^j \theta t^{\frac{1}{n}}\big).
\end{aligned}
$$

Using our earlier computation,

$$
\begin{aligned}
\log \Sigma^{(n)}(t) &= \sum_{i=1}^{n} \log \Sigma^{(1)}\big(-\omega^i \theta t^{\frac{1}{n}}\big) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=j} a_{i_1i_2\dots i_d} E_d^{(1)i_1} E_d^{(2)i_2} \dots E_d^{(d)i_d} (-1)^j \omega^{ij} \theta^j t^{\frac{j}{n}} \\
&= n \sum_{k=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=kn} a_{i_1i_2\dots i_d} E_d^{(1)i_1} E_d^{(2)i_2} \dots E_d^{(d)i_d} (-1)^{kn} \theta^{kn} t^k \\
&= n \sum_{j=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=jn} a_{i_1i_2\dots i_d} E_d^{(1)i_1} E_d^{(2)i_2} \dots E_d^{(d)i_d} (-1)^{j(n+1)} t^j
\end{aligned}
$$

since, as ω is a primitive n–root of unity,

$$
\sum_{i=1}^{n} \omega^{ij} = \begin{cases} n & \text{if } n \text{ divides } j \\ 0 & \text{otherwise} \end{cases}
$$

and $\theta^n = -1$.

As a consequence,

$$
\Sigma^{(n)}(t) = \exp\left(n \sum_{j=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=jn} a_{i_1i_2\dots i_d} E_d^{(1)i_1} E_d^{(2)i_2} \dots E_d^{(d)i_d} (-1)^{j(n+1)} t^j \right)
$$

in $\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_d][[t]]$, and

$$1 + \sum_{i=1}^{d} P_d^{(n,i)} t^i = \exp\left(n \sum_{j=1}^{\infty} \sum_{i_1+2i_2+\cdots+di_d=jn} a_{i_1 i_2 \ldots i_d} e_1^{i_1} e_2^{i_2} \ldots e_d^{i_d} (-1)^{j(n+1)} t^j \right)$$

in $\mathbb{Z}[e_1, e_2, \ldots, e_d][[t]]$. This proves the statement of Proposition 19, by setting

$$A_{i_1 i_2 \ldots i_d} = n a_{i_1 i_2 \ldots i_d} e_1^{i_1} e_2^{i_2} \ldots e_d^{i_d}.$$

□

Finally, the following elementary symmetry property is probably worth mentioning.

Proposition 22 *For every i,*

$$P_d^{(n,d-i)}(e_1, e_2, \ldots, e_{d-1}, e_d) = e_d^n P_d^{(n,i)}(e_d^{-1} e_{d-1}, e_d^{-1} e_{d-2}, \ldots, e_d^{-1} e_1, e_d^{-1}).$$

Proof If one replaces each λ_j by λ_j^{-1}, the power elementary symmetric polynomial $E_d^{(n,i)}$ gets replaced by

$$\left(E_d^{(n,d)}\right)^{-1} E_d^{(n,d-i)} = \left(E_d^{(d)}\right)^{-n} E_d^{(n,d-i)}$$

and, in particular, each $E_d^{(i)}$ gets replaced by $\left(E_d^{d}\right)^{-1} E_d^{(d-i)}$. The property follows. □

Funding Open access funding provided by SCELC, Statewide California Electronic Library Consortium.

References

1. Beaumont-Gould, B., Brodsky, E., Hogan, A., Higgins, V., Melby, J., Piazza, J.: Power sum elements in the G_2 skein algebra, 2023, to appear in Algebr. Geom. Top., arXiv:2310.01773
2. Bonahon, F., Wong, H.: Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations. Invent. Math. **204**(1), 195–243 (2016)
3. Costantino, F., Korinman, J., Lê, T.T.Q.: (2024), in preparation
4. Cautis, S., Kamnitzer, J., Morrison, S.: Webs and quantum skew Howe duality. Math. Ann. **360**(1–2), 351–390 (2014)
5. Frohman, C., Kania-Bartoszynska, J., Lê, T.T.Q.: Unicity for representations of the Kauffman bracket skein algebra. Invent. Math. **215**(2), 609–650 (2019)
6. Ganev, I., Jordan, D., Safronov, P.: The quantum Frobenius for character varieties and multiplicative quiver varieties, 2023, to appear in J. Eur. Math. Soc., arXiv:1901.11450
7. Higgins, V.: Triangular decomposition of SL_3-skein algebras. Quantum Topol. **14**(1), 1–63 (2023)
8. Higgins, V.: Miraculous cancellations and Frobenius homomorphism for SL_3-skein modules (2024), in preparation
9. Kim, H.K., Lê, T.T.Q.: (2024), in preparation
10. Korinman, J., Quesney, A.: Classical shadows of stated skein representations at roots of unity, 2019, to appear in Algebr. Geom. Topol., arXiv:1905.03441

11. Kuperberg, G.: Spiders for rank 2 Lie algebras. Comm. Math. Phys. **180**(1), 109–151 (1996)
12. Lê, T.T.Q.: On Kauffman bracket skein modules at roots of unity. Algebr. Geom. Topol. **15**(2), 1093–1117 (2015)
13. Lê, T.T.Q., Sikora, A.: Stated $SL(n)$-skein modules and algebras (2022), preprint, arXiv:2201.00045
14. Macdonald, I.G.: Symmetric functions and Hall polynomials, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York (2015), With contribution by A. V. Zelevinsky and a foreword by Richard Stanley
15. Morton, H.R.: Skein theory and the Murphy operators, J. Knot Theory Ramifications **11** (2002), (4), 475–492, Knots 2000 Korea, Vol. 2 (Yongpyong)
16. Morrison, S.: A diagrammatic category for the representation theory of $U_q(\mathfrak{sl}_n)$ (2007), PhD thesis, UC Berkeley, arXiv:0704.1503
17. Poudel, A.: A comparison between SL_n spider categories (2022), preprint, arXiv:2210.09289
18. Queffelec, H., Wedrich, P.: Extremal weight projectors II, $\mathfrak{gl}_N$ case. Algebr. Comb. **7**(1), 187–223 (2024)
19. Sikora, A.S.: Skein theory for $SU(n)$-quantum invariants. Algebr. Geom. Topol. **5**, 865–897 (2005)
20. Turaev, V.G.: The Yang-Baxter equation and invariants of links. Invent. Math. **92**(3), 527–553 (1988)
21. Wang, Z.: On stated $SL(n)$–skein modules (2023), preprint, arXiv:2307.10288
22. Witten, E.: Quantum field theory and the Jones polynomial. Comm. Math. Phys. **121**(3), 351–399 (1989)