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Hook formulae from Segre–MacPherson classes

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Abstract Nakada's colored hook formula is a vast generalization of many important formulae in combinatorics, such as the classical hook length formula and the Peterson's formula for the number of reduced expressions of minuscule Weyl group elements. In this paper, we use cohomological properties of Segre–MacPherson classes of Schubert cells and varieties to prove a generalization of a cohomological version of Nakada's formula, in terms of smoothness properties of Schubert varieties. A key ingredient in the proof is the study of a decorated version of the Bruhat graph. Weights of the paths in this graph give the terms in the generalized Nakada's formula, and the summation over all paths is equal to the equivariant multiplicity of the Chern–Schwartz–MacPherson class of a Richardson variety. Among the applications we mention an algorithm to calculate structure constants of multiplications of Segre–MacPherson classes of Schubert cells, and a skew version of Nakada–Peterson's formula.

1. Introduction

The goal of this paper is to reprove and generalize a version of Nakada's colored hook formula, using localization properties of the Segre–MacPherson classes of Schubert varieties. We recall first the version of Nakada's formula we generalize in this paper.

Let G be a complex semisimple Lie group and fix opposite Borel groups B, B^-, giving $B \cap B^- = T$, a maximal torus in G. Let R^+ be the set of positive roots in B, and let $W := N_G(T)/T$ be the Weyl group, endowed with the Bruhat order $<$, and its length function $\ell : W \to \mathbb{N}$. For a Weyl group element $w \in W$, define the set $S(w) = \{\beta \in R^+ : s_\beta w < w\}$.

Let $w \in W$ be a π-minuscule Weyl group element for an integral dominant weight π, in the sense of Peterson; see [40, 10, 42, 48, 49, 19] and §3.2 below. Nakada's colored hook formula [34] is the identity:

$$\sum \frac{1}{\beta_1} \cdot \frac{1}{\beta_1+\beta_2} \cdot \ldots \cdot \frac{1}{\beta_1+\beta_2+\cdots+\beta_r} = \prod_{\beta \in S(w)} \left(1+\frac{1}{\beta}\right). \tag{1}$$

Here the sum is over $r \geqslant 0$ and oriented paths $x_r \stackrel{\beta_r}{\to} x_{r-1} \to \ldots \to x_1 \stackrel{\beta_1}{\to} x_0 = w$ in the Bruhat graph of the flag manifold G/P, where P is the parabolic group with Weyl group $W_P = \mathrm{Stab}_W(\pi)$, and the notation $x \stackrel{\beta}{\to} y$ means that x, y are minimal length representatives such that $yW_P = s_\beta x W_P > xW_P$. See §3 below for details.

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Nakada's formula is a vast generalization of several remarkable combinatorial formulae. One is the classical Frame–Robinson–Thrall hook formula calculating the dimension of the irreducible representation of the symmetric group S_n indexed by the partition λ, or, equivalently, the number of standard tableaux of shape λ:

$$\chi_\lambda(1) = \#\,\mathrm{STab}(\lambda) = \frac{n!}{\prod_{\square\in\lambda} h_\square}.$$

Here $h_\square$ is the hook length of a cell $\square$ in the Young diagram of λ. Another special case is the Peterson formula counting the number $\#\,\mathrm{Red}(w)$ of reduced decompositions of a π-minuscule Weyl group element w:

$$\#\,\mathrm{Red}(w) = \frac{\ell(w)!}{\prod_{\beta\in S(w)} \mathrm{ht}(\beta)}. \tag{2}$$

Here $\mathrm{ht}(\alpha)$ denotes the height of the (positive) root α, i.e., the sum of the coefficients of α when expanded in simple roots; see §2. We refer to [34] or to §9.1 below for more details about how to obtain these specializations from Nakada's formula.

Let W^P be the set of minimal representatives for the quotient W/W_P. For each $w \in W^P$, denote by $Y(w) := \overline{B^- wP/P} \subset G/P$ the corresponding Schubert variety. Our main goal is to prove an identity generalizing Equation (1) in three directions:

(1) We remove the hypothesis that $w \in W$ is (Peterson) π-minuscule.
(2) We consider a 'skew-version', for paths $v \leqslant x_r \to \ldots \to x_0 = w$ where v and w are fixed, and such that the Schubert variety $Y(v)$ is smooth at w. Nakada's formula is obtained from specializing $v = id$, since $Y(id) = G/P$ is smooth at each w.
(3) Instead of fixing a single integral weight π, we consider an **admissible weight function** $\Lambda : [v,w]^P \to X^*(T)_P$ from the Bruhat interval $[v,w]^P$ to the set of integral weights which stabilize P.

The admissible functions are required to satisfy certain hypotheses spelled out in §8. An admissible function Λ leads to a decorated version of the usual Bruhat graph which we call the **Λ-Bruhat graph** of G/P. This graph was used in [28, 31] as part of an algorithm calculating the Schubert structure constants of the equivariant quantum cohomology ring of G/P.

A key observation going back to [31, 36, 37] is that the terms in special cases of Nakada's formula have geometric meaning, via localization properties of Schubert classes in equivariant cohomology and equivariant K-theory rings of flag manifolds. However, in order to obtain Nakada's formula (1), a deformation of Schubert classes is needed.

Let $c_{\mathrm{SM}}(Y(v)) \in H_T^*(G/P)$ be the equivariant Chern–Schwartz–MacPherson (CSM) class of the Schubert variety $Y(v)$. This class is defined using MacPherson's construction of characteristic classes of singular varieties [27], generalized to the equivariant case by Ohmoto [38]; see §4. Denote by

$$s_{\mathrm{M}}(Y(v)) = \frac{c_{\mathrm{SM}}(Y(v))}{c^T(T(G/P))} \in \widehat{H}_T^*(G/P)$$

the (equivariant) Segre–MacPherson (SM) class, which is an element in an appropriate completion of the equivariant cohomology ring. Let also $s_{\mathrm{M}}(Y(v))|_w \in H_T^*(pt)_{\mathrm{loc}}$ denote the localization of the SM class at the torus fixed point $w \in G/P$. (Here $H_T^*(pt)_{\mathrm{loc}}$ is the fraction field of $H_T^*(pt)$.) Our most general result is the following; see Theorem 7.5 and Corollary 8.13:

Theorem 1.1. *Let $v \leqslant w \in W^P$, and fix an admissible function $\Lambda : [v,w]^P \to X^*(T)_P$ with the associated Λ-Bruhat graph Γ. Then:*

$$\sum \frac{m_\Lambda(x_r, x_{r-1})}{\mathcal{W}_\Lambda(x_r)} \cdot \frac{m_\Lambda(x_{r-1}, x_{r-2})}{\mathcal{W}_\Lambda(x_{r-1})} \cdot \ldots \cdot \frac{m_\Lambda(x_1, x_0)}{\mathcal{W}_\Lambda(x_1)} = \frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w}. \tag{3}$$

The sum is over integers $r \geqslant 0$, and over all directed paths $v \leqslant x_r \to x_{r-1} \to \ldots \to x_0 = w$ in Γ; $m_\Lambda(x,y) \in \mathbb{Z}$ denotes the multiplicity of the edge $x \to y$; and $\mathcal{W}_\Lambda(x) \in X^*(T)_P$ denotes the Λ-weight of x; see Definition 8.6 below.

Furthermore, set $S(w/v) := \{\beta \in R^+ \mid v \leqslant s_\beta w < w\}$. Then:

$$Y(v) \subset G/P \text{ is smooth at } w \in G/P \iff \frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = \prod_{\beta \in S(w/v)} \left(1 + \frac{1}{\beta}\right).$$

As a corollary we obtain a generalized version of Nakada's formula; cf. Theorem 9.1:

Corollary 1.2. *With the notation and hypotheses from the above theorem,*

$Y(v) \subset G/P$ is smooth at $w \in G/P$ if and only if

$$\sum \frac{m_\Lambda(x_r, x_{r-1})}{\mathcal{W}_\Lambda(x_r)} \cdot \frac{m_\Lambda(x_{r-1}, x_{r-2})}{\mathcal{W}_\Lambda(x_{r-1})} \cdot \ldots \cdot \frac{m_\Lambda(x_1, x_0)}{\mathcal{W}_\Lambda(x_1)} = \prod_{\beta \in S(w/v)} \left(1 + \frac{1}{\beta}\right). \tag{4}$$

If w is π-minuscule for a dominant integral weight π, and if the admissible function is constant $\Lambda \equiv \pi$ on the interval $[v,w]^P$, then each multiplicity $m_\Lambda(x_i, x_{i-1}) = 1$, and one obtains a skew generalization of Nakada's identity (1); see Corollary 9.3 below. The proof of this ultimately requires a good understanding of the (paths in the) interval $[v,w]^P$ for π-minuscule elements v, w, in analogy to the intervals in the Young lattice; for instance, the intervals in the weak and strong Bruhat order coincide. We refer to §3.3 and §9 for more details. We encourage the reader to jump to §10, where we give some examples illustrating this theorem.

Equation (3) is proved by utilizing a Molev–Sagan type recursion [32], based on a Chevalley formula to multiply SM classes; cf. Proposition 5.3 and see also [50, 1]. Variants of this recursion have been successfully used in [23, 28, 31, 36, 8, 37] to study properties of the *equivariant* (quantum) cohomology or K-theory of flag manifolds. As a by-product of this study, and in the same spirit as the references above, we obtain an algorithm for the structure constants of the multiplication of SM classes; see Corollary 5.4 below. Different algorithms were also obtained in [51].

Once equation (3) is proved, the second part of Theorem 1.1 follows from a smoothness criterion of Schubert varieties in terms of localization of SM classes. More precisely, if R_w^v denotes the Richardson variety, the fraction

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = e_{w,G/P}(c_{\mathrm{SM}}(R_w^v))$$

is equal to Brion's equivariant multiplicity [7] of the CSM class of R_w^v; cf. Proposition 7.4. From this perspective, the (generalized) Nakada's formula calculates the equivariant multiplicity of a Richardson variety. The smoothness of $Y(v)$ at w ensures that the equivariant multiplicity is a product of factors corresponding to weights of the normal space of $Y(v)$ at w, and it leads to the right hand side of (4). Our smoothness criterion for $Y(v)$ at w in Theorem 7.5 generalizes similar results about smoothness of Schubert varieties by Kumar [24] and Brion [7], the latter in terms of equivariant multiplicities. Our criterion is also related to the one from [2] for motivic Chern classes of Schubert varieties, which has consequences in p-adic representation theory.

Among the consequences of Corollary 1.2 proved in section §9.1 we mention a skew version of the Peterson formula (2); cf. Corollary 9.4.

Corollary 1.3. *Take $v < w \in W$ be π-minuscule elements such that $Y(v)$ is smooth at w. Recall that $S(w/v) \coloneqq \{\beta \in R^+ \mid v \leqslant s_\beta w < w\}$. Then:*

$$\# \operatorname{Red}(wv^{-1}) = \frac{(\ell(w) - \ell(v))!}{\prod_{\beta \in S(w/v)} \operatorname{ht}(\beta)} \;. \tag{5}$$

When considered in this generality, this corollary seems to be new. In type A, this formula has an interpretation as counting standard Young tableaux on the skew shape w/v, see [36, 33, 39]. We will prove in a continuation of this work that this interpretation holds in other Lie types, and, furthermore, if W is simply laced, the 'skew' formula for $v < w$ is equal to the 'straight' formula from (2), applied to $id < wv^{-1}$. We also note that our general formula from Theorem 1.1 implies a variant of the Corollary above which does not require smoothness; see Remark 9.6 below.

Theorem 1.1 is the prototype of more general results. For instance, a similar theorem - with essentially the same proof - may be obtained if one replaces the (cohomological) SM classes with their K-theoretic versions, the *motivic Segre classes* of Schubert varieties [2, 29]. Because of the technical challenges (in geometry and combinatorics) when working in K-theory, this will be studied in separate work. Furthermore, there are versions of Nakada's colored hook formula [34] which hold in the Kac–Moody setting, suggesting that analogues of Theorem 1.1 might exist in that generality as well.[(1)]

2. Preliminaries

We start by fixing the notation used throughout the paper. Let G be a simply connected complex Lie group with Borel subgroup B and maximal torus $T \subset B$. Denote by $\operatorname{Lie}(G)$ and by $\operatorname{Lie}(T)$ be corresponding Lie algebras. Let $R^+ \subset \operatorname{Lie}(T)^* \coloneqq \operatorname{Lie}(T)^*_{\mathbb{Q}}$ denote the positive roots, i.e those roots in B, and by $\Sigma = \{\alpha_i : i \in I\}$ the set of simple roots. Let $\operatorname{ht} : R^+ \to \mathbb{Z}$ denote the height function, defined by $\operatorname{ht}(\sum a_i \alpha_i) = \sum a_i$. Let $R \coloneqq R^+ \sqcup -R^+$. We use $\alpha > 0$ (resp. $\alpha < 0$) to denote $\alpha \in R^+$ (resp. $\alpha \in -R^+$). For any root $\alpha \in R$, let $\alpha^\vee \subset \operatorname{Lie}(T)$ denote the corresponding coroot. Let $\langle \cdot, \cdot \rangle : \operatorname{Lie}(T)^* \times \operatorname{Lie}(T) \to \mathbb{Q}$ denote the usual pairing, and let $X^*(T) \subset \operatorname{Lie}(T)^*$ be the (integral) weight lattice. Let $\{\varpi_i \mid i \in I\} \subset X^*(T)$ be the fundamental weights; they satisfy $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$. The Weyl group $W = N_G(T)/T$ is generated by simple reflections $s_i = s_{\alpha_i}$ $(i \in I)$, and it is equipped with the Bruhat order $\leqslant$; we denote by w_0 the longest element. For any $w \in W$, let $\operatorname{Red}(w)$ denote the set of all the reduced expressions for w. For $v < w \in W$, define

$$S(w/v) \coloneqq \{\beta \in R^+ \mid v \leqslant s_\beta w < w\}. \tag{6}$$

If $v = id$, we denote $S(w/id)$ by $S(w)$.

Let $P(\supseteq B)$ be a parabolic subgroup with simple roots $\Sigma_P \subset \Sigma$ in P. This determines the set $R_P^+ \subset R^+$ of those positive roots spanned by Σ_P, and the subgroup $W_P \subset W$ generated by the simple reflections s_i where $\alpha_i \in \Sigma_P$. Denote by W^P the set of minimal length representatives in W/W_P. The elements $w \in W^P$ are characterized by the property that $w(R_P^+) \subset R^+$. The torus fixed points $(G/P)^T$ are $\{wP \mid w \in W^P\}$. For any $w \in W^P$, let $T_w(G/P)$ denote the tangent space at wP. For any $w \in W$, let $X(w)^\circ \coloneqq BwP/P \subset G/P$ (resp. $Y(w)^\circ \coloneqq B^- wP/P \subset G/P$) denote the Schubert cell with closure $X(w)$ (resp. $Y(w)$), where B^- is the opposite Borel subgroup. In particular, $X(w)^\circ = X(u)^\circ$ when the two cosets wW_P and uW_P are equal to each other. The Bruhat order restricts to the (Bruhat) order on the cosets $\{wW_P \mid w \in W^P\}$, and it characterized by $uW_P \leqslant wW_P$ if and only if $uP \in X(w)$.

(1) After this paper was written, a generalization to arbitrary Coxeter groups Nakada's formula from Theorem 3.11 was proved in [30].

Let $X^*(T)_P \coloneqq \{\lambda \in X^*(T) \mid \langle \lambda, \gamma^\vee \rangle = 0 \text{ for all } \gamma \in R_P^+\}$ be the set of integral weights which vanish on $(R_P^+)^\vee$. For any $\lambda \in X^*(T)_P$, let $\mathcal{L}_\lambda$ denote the line bundle $G \times^P \mathbb{C}_\lambda \in \mathrm{Pic}(G/P)$, which has fibre over $1.P$ the T-module of weight λ.

Let $H_T^*(G/P)$ denote the equivariant cohomology of the partial flag variety G/P. It has a basis of the Schubert classes:

$$H_T^*(G/P) = \bigoplus_{w \in W^P} H_T^*(pt)[X(w)] = \bigoplus_{w \in W^P} H_T^*(pt)[Y(w)],$$

where $[X(w)]$ and $[Y(w)]$ denote the Poincaré dual of the fundamental classes of the Schubert varieties. For any $\kappa \in H_T^*(G/P)$ and $w \in W^P$, let $\kappa|_w \in H_T^*(pt)$ denote the restriction of κ to the fixed point $wP \in G/P$. Let $H_T^*(G/P)_{\mathrm{loc}} \coloneqq H_T^*(G/P) \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$ be the localized equivariant cohomology of G/P, where $\mathrm{Frac}\, H_T^*(pt)$ denotes the fraction field of $H_T^*(pt)$. By the localization theorem, $H_T^*(G/P)_{\mathrm{loc}}$ has a basis formed by the classes of fixed points $\{[wP] \mid w \in W^P\}$; this is called the fixed point basis.

The Weyl group W acts on G/P by left multiplication. It induces an action of W on $H_T^*(G/P)$, which acts on the base ring $H_T^*(pt) = Sym_{\mathbb{Z}}(X^*(T))$ by the usual Weyl group action. Let φ_{w_0} denote the action induced by the longest Weyl group element. Then $\varphi_{w_0}([X(w)]) = [Y(w_0 w)]$.

This paper will focus on Peterson, or π-minuscule, elements, for some dominant integral weight π. Before giving the formal definition in Section 3.2, we note that the Schubert varieties indexed by π-minuscule elements share most geometric and combinatorial properties of the Schubert varieties in Grassmannians, or, more generally, in (co)minuscule Grassmannians.(2) In fact, any Schubert variety in a cominuscule Grassmannian G/P is indexed by a π-minuscule element, for π the fundamental weight associated to the parabolic subgroup P. As a first approximation, the reader may consider π-minuscule elements as generalizing the usual Young diagrams for Grassmannians, or the more general Young diagrams defined by Proctor [40, 42], and Stembridge [48, 49] (see also [19]) in other Lie types.

3. Nakada's colored hook formula

In this section, we review Nakada's colored hook formula from [34, 35]. This formula generalizes results of Proctor [42, 41] and D. Peterson (see e.g. [10]) about the combinatorics of complete d-posets, and the number of reduced decompositions of certain minuscule Weyl group elements; see also [47]. In order to be able to use geometric arguments, in this paper we require that the Lie algebra of G is of finite type, although Nakada's formula may be formulated for any Kac–Moody Lie algebra.

3.1. Nakada's formula and pre-dominant weights. Recall that the integral weights $\lambda \in \mathrm{Lie}(T)^*$ satisfy $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$, for each $\alpha_i \in \Sigma$; a weight λ is *dominant* if in addition $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$. Following [34], we say that an integral weight λ is *pre-dominant* if $\langle \lambda, \beta^\vee \rangle \geqslant -1$ for all positive roots $\beta \in R^+$. For a pre-dominant integral weight λ, the *diagram* $D(\lambda)$ of λ is defined by

$$D(\lambda) \coloneqq \{\beta \in R^+ \mid \langle \lambda, \beta^\vee \rangle = -1\}. \tag{7}$$

If λ is a dominant integral weight, it is pre-dominant, but $D(\lambda) = \varnothing$.

We recall some elementary facts about pre-dominant integral weights from [34].

(2)Aside from the ordinary Grassmannians, the list of minuscule Grassmannians includes the maximal orthogonal Grassmannians in Lie type D, even quadrics, the Cayley plane in type E_6, and the Freudenthal variety in type E_7. If, as in this paper, one distinguishes the data of torus actions, then the projective spaces $\mathbb{P}^{2n-1} \simeq \mathrm{IG}(1, 2n)$ in types A and C, and the maximal orthogonal Grassmannians $\mathrm{OG}(n, 2n+1) \simeq \mathrm{OG}(n+1, 2n+2)$ in types B and D are listed as separate varieties.

Lemma 3.1 ([34, Lemma 4.1]). *Let λ be a pre-dominant integral weight.*
(1) If $D(\lambda) \neq \varnothing$, then $D(\lambda) \cap \Sigma \neq \varnothing$.
(2) For $\beta \in D(\lambda)$, $s_\beta(\lambda)$ is a pre-dominant integral weight.
(3) In case (2), $D(s_\beta(\lambda)) = s_\beta(D(\lambda) \setminus S(s_\beta))$.

For integral weights $\mu, \nu \in \mathfrak{h}^*$ and $\beta \in R^+$, define

$$\mu \xrightarrow{\beta} \nu \iff \langle \mu, \beta^\vee \rangle = -1 \text{ and } \nu = s_\beta(\mu).$$

In particular, if $\mu \xrightarrow{\beta} \nu$ then $\nu = \mu + \beta$.

A λ-path of length r is a sequence $(\beta_1, \beta_2, \dots, \beta_r)$ where

$$\lambda = \lambda_0 \xrightarrow{\beta_1} \lambda_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} \lambda_r.$$

We denote by $\mathrm{Path}(\lambda)$ the set of all λ-paths. By Lemma 3.1(2), if λ is pre-dominant, all the weights λ_i in a λ-path are pre-dominant. At each step in a λ-path, $|D(\lambda_i)|$ strictly decreases, therefore the length of a λ-path for a pre-dominant weight must be at most the size of $D(\lambda)$. Parts (1) and (3) of Lemma 3.1 imply that a λ-path $(\beta_1, \dots, \beta_d)$ of maximal length can only contain simple roots $\beta_i \in \Delta$, and that $d = |D(\lambda)|$. For a pre-dominant integral weight λ, we denote by $\mathrm{MPath}(\lambda) \subset \mathrm{Path}(\lambda)$ the subset of longest λ-paths.

Now we can state Nakada's colored hook formula.

Theorem 3.2 (Nakada's colored hook formula [34, Theorem 7.1]). *Let λ be a pre-dominant integral weight. Then*

$$\sum \frac{1}{\beta_1} \cdot \frac{1}{\beta_1 + \beta_2} \cdot \dots \cdot \frac{1}{\beta_1 + \beta_2 + \dots + \beta_r} = \prod_{\beta \in D(\lambda)} \left(1 + \frac{1}{\beta}\right),$$

where the sum is over all $r \geqslant 0$ and λ-paths $(\beta_1, \beta_2, \dots, \beta_r)$.

Taking the lowest degree terms of the formula in theorem 3.2, we get the following corollary.

Corollary 3.3 ([34, Corollary 7.2]). *Let λ be a pre-dominant integral weight and $d = |D(\lambda)|$. Then*

$$\sum_{(\beta_1, \beta_2, \dots, \beta_d) \in \mathrm{MPath}(\lambda)} \frac{1}{\beta_1} \cdot \frac{1}{\beta_1 + \beta_2} \cdot \dots \cdot \frac{1}{\beta_1 + \beta_2 + \dots + \beta_d} = \prod_{\beta \in D(\lambda)} \frac{1}{\beta}.$$

This formula generalizes the classical hook formula (1) and the Peterson–Proctor formula (5); see also §9 below. It is also related to an equality of rational functions involving root partitions for cluster variables cf. [12, 13].

3.2. Peterson minuscule elements.

For any integral weight π, D. Peterson defined the notion of a π-minuscule Weyl group element, see below and [40, 10, 42, 48, 49, 19]. Examples of π-minuscule elements are the minimal length representatives in W^P, where P is the maximal parabolic associated to a minuscule fundamental weight. Minuscule elements are fully commutative [48]; in particular, in type A they are 321-avoiding. We will use this notion to rewrite Nakada's formula and its generalizations considered in this paper in terms of Weyl group elements.

Definition 3.4 (π-minuscule elements). *Let π be an integral weight. An element $w \in W$ is called* **π-minuscule** *if there is a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ such that*

$$\langle s_{i_{k+1}} s_{i_{k+2}} \cdots s_{i_\ell}(\pi), \alpha_{i_k}^\vee \rangle = 1 \qquad (1 \leqslant k \leqslant \ell) \tag{8}$$

Equivalently,

$$s_{i_k}s_{i_{k+1}}s_{i_{k+2}}\cdots s_{i_\ell}(\pi)=\pi-\alpha_{i_\ell}-\ldots-\alpha_{i_k}. \tag{9}$$

From the definition it follows that if w is π-minuscule, then its length is $\ell(w)=\mathrm{ht}(\pi-w(\pi))$. The next result follows immediately from analyzing the inversion set of w.

Lemma 3.5 ([49, Proposition 5.1]). *If $w\in W$ then*

$$w \text{ is } \pi\text{-minuscule} \iff \langle\pi,\gamma^\vee\rangle=1 \text{ for all } \gamma\in R^+ \text{ such that } ws_\gamma<w.$$

From now on we restrict to the case when π is a a dominant integral weight. Let $W_\pi=\mathrm{Stab}_W(\pi)$ denote the stabilizer subgroup of π inside W. This determines the parabolic subgroup P such that $W_P=W_\pi$, containing simple roots $\Sigma_P:=\{\alpha_i\in\Sigma\mid\langle\pi,\alpha_i^\vee\rangle=0\}$. Let W^π denote the set of minimal length representatives for the cosets W/W_π.

Remark 3.6. It follows from Lemma 3.5 that if w is π-minuscule, then $w\in W^\pi$, and the property (8) holds for any reduced expression of $w=s_{i_1}s_{i_2}\cdots s_{i_\ell}$.

We also need the following definition.

Definition 3.7. *For $u,w\in W^\pi$ and $\beta\in R^+$, define*

$$u\xrightarrow{\beta}w \quad \text{if} \quad s_\beta wW_\pi=uW_\pi, \text{ and } u<w.$$

Lemma 3.8. *Let π be dominant integral and $u,w\in W^\pi$ such that $u\xrightarrow{\beta}w$. Then the following hold:*

(a) The root β is unique.

(b) There exists a unique positive root γ such that $s_\beta wW_P=ws_\gamma W_P$. Furthermore,

$$\gamma=-w^{-1}(\beta) \quad \text{and} \quad ws_\gamma<w.$$

(c) If in addition w is π-minuscule then $\langle w(\pi),\beta^\vee\rangle=-1$.

Proof. The uniqueness of β follows from [16, Lemma 4.1]. Since $s_\beta w<w$ it follows that $w^{-1}(\beta)<0$, thus $\gamma:=-w^{-1}(\beta)$ is a positive root, and $s_\gamma W_P=w^{-1}s_\beta wW_P=w^{-1}uW_P$. Since $w^{-1}u\notin W_P$ by hypothesis, the uniqueness of γ follows from [9, Lemma 2.2]. Finally, since $ws_\gamma W_P=uW_P$ and $u<w$, then necessarily $ws_\gamma<w$. This finishes the proof of (b). Part (c) follows from (b) and Lemma 3.5. ☐

The relation between pre-dominant integral weights and π-minuscule elements is given by the following proposition.

Proposition 3.9 ([34, Propositions 10.1 and 10.3]). *There is a bijection between the following two sets*

$$\{\text{pre-dominant integral weights } \lambda\} \longleftrightarrow \{(\pi,w)\mid \pi \text{ is dominant integral, and } w \text{ is } \pi\text{-minuscule}\}.$$

Here, λ is determined by (π,w) by the formula $\lambda=w(\pi)$. Conversely, for any pre-dominant integral weight λ, take a maximal λ-path $(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_d})\in\mathrm{MPath}(\lambda)$ and set $w=s_{i_1}s_{i_2}\cdots s_{i_d}$, $\pi=w^{-1}(\lambda)$. Then π is a dominant integral weight and w is π-minuscule. Moreover $D(\lambda)=S(w)=\{\beta\in R^+\mid s_\beta w<w\}$, and the correspondence $(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_d})\mapsto s_{i_1}s_{i_2}\cdots s_{i_d}$ from $\mathrm{MPath}(\lambda)$ to $\mathrm{Red}(w)$ is bijective.

Corollary 3.10. *Let π be a dominant integral weight, w be π-minuscule and $\lambda=w(\pi)$. For $\beta\in R^+$, let $u\in W^\pi$ denote the minimal length representative in $s_\beta wW_\pi$. Then:*

(a) $\lambda\xrightarrow{\beta}s_\beta(\lambda)$ is equivalent to $u\xrightarrow{\beta}w$.

(b) If $u \xrightarrow{\beta} w$, then u is also π-minuscule.

Proof. Part (a) follows from:

$$\lambda \xrightarrow{\beta} s_\beta(\lambda) \Leftrightarrow \langle \lambda, \beta^\vee \rangle = -1 \Leftrightarrow \langle \pi, w^{-1}(\beta^\vee) \rangle = -1 \Leftrightarrow w^{-1}\beta < 0 \Leftrightarrow s_\beta w < w \Leftrightarrow u \xrightarrow{\beta} w.$$

Here in the third equivalence, the $\Rightarrow$ direction follows from the fact that π is dominant, while the $\Leftarrow$ direction follows from Lemma 3.8.

To prove (b), observe that if $u \xrightarrow{\beta} w$, then $s_\beta \lambda$ is also a pre-dominant integral weight by Lemma 3.1(2). Because of Proposition 3.9,

$$s_\beta \lambda = w'(\pi')$$

for some dominant integral weight π', $w' \in W$ such that w' is π'-minuscule. Therefore,

$$s_\beta w(\pi) = w'(\pi'),$$

and because the dominant weights π, π' are in the fundamental domain for the W-action, it follows that $\pi = \pi'$, and that $s_\beta w W_\pi = w' W_\pi$. By Remark 3.6, $w' \in W^\pi$. Thus, $w' = u$, and u is π-minuscule. ☐

With Corollary 3.10, Nakada's formula (Theorem 3.2) can be reformulated as follows.

Theorem 3.11 (Nakada). *Let π be a dominant integral weight, and $w \in W$ be a π-minuscule element. Then:*

$$\sum \frac{1}{\beta_1} \cdot \frac{1}{\beta_1 + \beta_2} \cdot \dots \cdot \frac{1}{\beta_1 + \beta_2 + \dots + \beta_r} = \prod_{\beta \in S(w)} \left(1 + \frac{1}{\beta}\right),$$

where the summation is over $r \geqslant 0$ and paths $x_r \xrightarrow{\beta_r} x_{r-1} \to \dots \to x_1 \xrightarrow{\beta_1} x_0 = w$. In particular,

$$\sum_{x_d \xrightarrow{\beta_d} x_{d-1} \to \dots \to x_1 \xrightarrow{\beta_1} x_0 = w} \frac{1}{\beta_1} \cdot \frac{1}{\beta_1 + \beta_2} \cdot \dots \cdot \frac{1}{\beta_1 + \beta_2 + \dots + \beta_d} = \prod_{\beta \in S(w)} \frac{1}{\beta},$$

where $d = |S(w)| = \ell(w)$.

This statement will generalized in Corollary 9.3 below.

3.3. Bruhat intervals of π-minuscule elements. The main goal of this section is to prove Proposition 3.13, stating that the intervals in the weak and ordinary (or strong) Bruhat order determined by two π-minuscule elements coincide. Special cases appeared in the works by Stembridge [48, Thm. 7.1] (for π a minuscule weight) and Proctor [42, §10] (for G simply laced), but we have not seen Proposition 3.13 in the generality we need.

For $u, w \in W$ the (left) weak Bruhat order is defined by $u <_L w$ iff $\ell(wu^{-1}) = \ell(w) - \ell(u)$. Equivalently, $w = vu$ and $\ell(vu) = \ell(v) + \ell(u)$. Observe that if $u <_L w$ then $u < w$ in the strong Bruhat order, but in general the two orders are different.

Example 3.12. Consider the example of type A_2. The Weyl group has two generators s_1, s_2. Let $w = s_1 s_2$ and $u = s_1$. Then $u < w$, but $u \not<_L w$ as $\ell(wu^{-1}) \neq \ell(w) - \ell(u)$.

For $v < w \in W^P$, set $[v, w]^P \coloneqq \{x \in W^P \mid v \leqslant x \leqslant w\}$.

Proposition 3.13. *Let π is a dominant integral weight with $\mathrm{Stab}_W(\pi) = W_P$ and let $u < w$ be π-minuscule elements in W^P. Then the interval $[u, w]^P \subset W^P$ is the same in the weak and strong Bruhat orders, i.e.,*

$$\{v \in W^P : u \leqslant v \leqslant w\} = \{v' \in W^P : u \leqslant_L v' \leqslant_L w\}.$$

Proof. Let $v \in W^P$ such that $v < w$. Then it suffices to show that $v \leqslant_L w$ when w covers v in the strong Bruhat order in W^P, i.e., $\ell(w) - \ell(v) = 1$. Write $vW_P = s_\beta w W_P = w s_\gamma W_P$ for positive roots β, γ as in Lemma 3.8. Since w is π-minuscule, v is also π-minuscule by Corollary 3.10 (b). Using that $\ell(w) = \mathrm{ht}(\pi - w(\pi))$ we obtain

$$1 = \ell(w) - \ell(v) = \mathrm{ht}(s_\beta w(\pi) - w(\pi)) = \mathrm{ht}(w(\pi) - \langle w(\pi), \beta^\vee \rangle \beta - w(\pi)) = \mathrm{ht}(\beta).$$

The last equality follows because the multiplicity $\langle w(\pi), \beta^\vee \rangle = -1$ by Lemma 3.8(c). Therefore β must be a simple root, and, furthermore, $w = s_\beta v$, thus $v \leqslant_L w$. ☐

The special case $u = id$ of the following corollary has been proved by Nakada [34, Proposition 10.3].

Corollary 3.14. *Assume the hypotheses from Proposition 3.13. Let $d \coloneqq \ell(w) - \ell(u)$. There is a bijection between the set of reduced words of wu^{-1} and the set of maximal length paths from u to w, sending a reduced word $wu^{-1} = s_{\beta_1} \cdot \ldots \cdot s_{\beta_d}$ to the path*

$$u = x_d \xrightarrow{\beta_d} x_{d-1} \to \ldots \to x_1 \xrightarrow{\beta_1} x_0 = w.$$

Proof. Consider any path $u = x_d \xrightarrow{\beta_d} x_{d-1} \to \ldots \to x_1 \xrightarrow{\beta_1} x_0 = w$ in the (strong) Bruhat interval $[u,w]^P$. Then $\ell(x_{i-1}) - \ell(x_i) = 1$ for all i, and since w is π-minuscule all x_i's must be also. Because weak and strong Bruhat orders coincide in $[u,w]^P$ by Proposition 3.13, $x_i = s_{\beta_i} x_{i-1}$ and each of the roots β_i must be simple. Then $s_{\beta_d} s_{\beta_{d-1}} \cdot \ldots \cdot s_{\beta_1} = uw^{-1}$ and this decomposition is reduced. This associates a reduced word of wu^{-1} to the given path (by reading in reverse), and it easily follows that this correspondence is a bijection. ☐

4. Chern–Schwartz–MacPherson classes of Schubert cells

In this section, we recall the definition of the equivariant Chern–Schwartz–MacPherson (CSM) and Segre–MacPherson (SM) classes, then we recall the Chevalley formula for the CSM and SM classes of Schubert cells in partial flag manifolds [1, 50].

4.1. Definition. Let X be a complex algebraic variety. The group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_W c_W 1\!\!1_W$, where the sum is over a finite set of constructible subsets $W \subset X$, $c_W \in \mathbb{Z}$ are integers, and $1\!\!1_W$ is the characteristic function of W. For a proper morphism $f : Y \to X$, there is a linear map $f_* : \mathcal{F}(Y) \to \mathcal{F}(X)$, such that for any constructible subset $W \subset Y$, we have $f_*(1\!\!1_W)(x) = \chi_{top}(f^{-1}(x) \cap W)$, where $x \in X$ and χ_{top} denotes the topological Euler characteristic. Thus $\mathcal{F}$ can be considered as a (covariant) functor from the category of complex algebraic varieties and proper morphisms to the category of abelian groups.

According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor of constructible functions on a complex algebraic variety X to the homology functor, where all morphisms are proper, such that if X is smooth then $c_*(1\!\!1_X) = c(TX) \cap [X]$, where $c(TX)$ denotes the total Chern class of the tangent bundle TX and $[X]$ denotes the fundamental class. This conjecture was proved by MacPherson [27]; the class $c_*(1\!\!1_X)$ for possibly singular X was shown to coincide with a class defined earlier by M.-H. Schwartz [45, 46, 5].

The theory of CSM classes was later extended to the equivariant setting by Ohmoto [38]. If X has an action of a torus T, Ohmoto defined the group $\mathcal{F}^T(X)$ of *equivariant* constructible functions. We will need the following properties of this group:

1. If $W \subseteq X$ is a constructible set which is invariant under the T-action, its characteristic function $1\!\!1_W$ is an element of $\mathcal{F}^T(X)$. We will denote by $\mathcal{F}^T_{inv}(X)$ the subgroup of $\mathcal{F}^T(X)$ consisting of T-invariant constructible functions on X.

(The group $\mathcal{F}^T(X)$ also contains other elements, but this will be immaterial for us.)

(2) Every proper T-equivariant morphism $f : Y \to X$ of algebraic varieties induces a homomorphism $f_*^T : \mathcal{F}^T(X) \to \mathcal{F}^T(Y)$. The restriction of f_*^T to $\mathcal{F}_{inv}^T(X)$ coincides with the ordinary push-forward f_* of constructible functions. See [38, §2.6].

Ohmoto proves [38, Theorem 1.1] that there is an equivariant version of MacPherson transformation $c_*^T : \mathcal{F}^T(X) \to H_*^T(X)$ that satisfies $c_*^T(\mathbb{1}_X) = c^T(TX) \cap [X]_T$ if X is a smooth variety, and that is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ {\scriptstyle f_*^T}\downarrow & & \downarrow{\scriptstyle f_*^T} \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X). \end{array}$$

If X is smooth, we will identify the (equivariant) homology and cohomology groups, by Poincaré duality: $H_*^T(X) \simeq H_T^*(X)$.

Definition 4.1. *Let Z be a T-invariant constructible subvariety of X.*

(1) *We denote by* $c_{\mathrm{SM}}(Z) \coloneqq c_*^T(\mathbb{1}_Z) \in H_*^T(X)$ *the* equivariant Chern–Schwartz–MacPherson (CSM) class *of Z.*

(2) *If X is smooth, we denote by* $s_{\mathrm{M}}(Z) \coloneqq \frac{c_*^T(\mathbf{1}_Z)}{c^T(TX)} \in \widehat{H}_T^*(X)$ *the* equivariant Segre–MacPherson (SM) class *of Z, where $\widehat{H}_T^*(X)$ is an appropriate completion of $H_T^*(X)$.*

If X is smooth, we identify $H_T^*(X)$ with $H_*^T(X)$ via the Poincaré duality sending $\kappa \mapsto \kappa \cap [X]_T$. Thus, $c_{\mathrm{SM}}(Z)$ is viewed as a cohomology class. In particular, in the above definition, the SM classes may also be seen as cohomology classes.

4.2. Chevalley formulae for CSM and SM classes of Schubert cells. In this section, we recall the Chevalley formula for the CSM/SM classes of the Schubert cells in the partial flag variety G/P, proved in [1, 50].

Let $H_T^*(X)_{\mathrm{loc}} \coloneqq H_T^*(X) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$ denote the localization of $H_T^*(X)$, where $\operatorname{Frac} H_T^*(pt)$ is the fraction field of $H_T^*(pt)$. Since the transition matrix between the CSM classes $\{c_{\mathrm{SM}}(X(w)^\circ) \mid w \in W^P\}$ (and also the SM classes $\{s_{\mathrm{M}}(Y(w)^\circ) \mid w \in W^P\}$) and the Schubert classes is a triangular matrix with non-zero diagonal terms in $H_T^*(X)_{\mathrm{loc}}$, the CSM classes and SM classes are bases for $H_T^*(pt)_{\mathrm{loc}}$. Moreover, they are dual under the Poincaré pairing $\langle -,- \rangle_{G/P}$ on $H_T^*(G/P)_{\mathrm{loc}}$ (see [1, Theorem 7.1]):

$$\langle s_{\mathrm{M}}(Y(u)^\circ), c_{\mathrm{SM}}(X(w)^\circ) \rangle_{G/P} = \delta_{u,w} \textit{ for any } w, u \in W^P.$$

If φ_{w_0} denotes the automorphism of $H_T^*(G/P)$ induced by the left multiplication by w_0 on G/P, then

$$\varphi_{w_0}(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(Y(w_0 w)^\circ). \tag{10}$$

Theorem 4.2 ([1, 50]). *For any $w \in W^P$ and $\lambda \in X^*(T)_P$, the following holds in $H_T^*(G/P)$:*

$$c_1(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(w)^\circ) = w(\lambda) c_{\mathrm{SM}}(X(w)^\circ) - \sum_{\alpha>0, ws_\alpha<w} \langle \lambda, \alpha^\vee \rangle c_{\mathrm{SM}}(X(ws_\alpha)^\circ),$$

and

$$c_1(\mathcal{L}_\lambda) \cup s_{\mathrm{M}}(Y(w)^\circ) = w(\lambda) s_{\mathrm{M}}(Y(w)^\circ) - \sum_{\alpha>0, ws_\alpha>w} \langle \lambda, \alpha^\vee \rangle s_{\mathrm{M}}(Y(ws_\alpha)^\circ).$$

Proof. The first equality follows from [50, Theorem 3.7] and [1]. We prove the second one. Applying φ_{w_0} to the first equation, we get

$$c_1(\mathcal{L}_\lambda) \cup c_{\mathrm{SM}}(X(w_0 w)^\circ) = w_0 w(\lambda) c_{\mathrm{SM}}(X(w_0 w)^\circ) - \sum_{\substack{\alpha>0,\\ ws_\alpha<w}} \langle \lambda, \alpha^\vee \rangle c_{\mathrm{SM}}(X(w_0 w s_\alpha)^\circ).$$

Here we have used Equation (10) and the fact that $c_1(\mathcal{L}_\lambda)$ is W-invariant. Assume $w_0 w = zu$ for some $z \in W^P$ and $u \in W_P$. Then $w_0 w s_\alpha W_P = zus_\alpha W_P = zs_{u\alpha} W_P$, and we have the following equivalences

$$\begin{aligned} &\alpha > 0, ws_\alpha < w \\ \Leftrightarrow &\alpha > 0, w\alpha < 0 \\ \Leftrightarrow &\alpha \in R^+ \smallsetminus R^+_P, w\alpha < 0 \\ \Leftrightarrow &\beta \coloneqq u\alpha \in R^+ \smallsetminus R^+_P, z\beta > 0 \\ \Leftrightarrow &\beta \in R^+ \smallsetminus R^+_P, zs_\beta > z. \end{aligned}$$

Moreover, $\langle \lambda, \alpha^\vee \rangle = \langle \lambda, \beta^\vee \rangle$ since $\lambda \in X^*(T)_P$. Finally, the second equation holds by the above equivalences and the fact that $\langle \lambda, \gamma^\vee \rangle = 0$ for any $\gamma \in R^+_P$. ☐

5. Segre–MacPherson Littlewood–Richardson (SMLR) coefficients

For any $u, v, w \in W^P$, define the Segre–MacPherson Littlewood–Richardson (SMLR) coefficients $d^v_{u,w} \in H^*_T(pt)_{\mathrm{loc}}$ for the SM classes of Schubert cells by

$$s_{\mathrm{M}}(Y(u)^\circ) \cup s_{\mathrm{M}}(Y(v)^\circ) = \sum_{w\in W^P} d^w_{u,v} s_{\mathrm{M}}(Y(w)^\circ) \in H^*_T(G/P)_{\mathrm{loc}}. \tag{11}$$

A formula for the structure constants $d^w_{u,v}$, in terms of multiplications in the cohomology of Bott-Samelson varieties, was recently obtained by the third-named author in [51, Theorem 5.2]. Furthermore, in the non-equivariant case, it was proved in [44] that $(-1)^{\ell(u)+\ell(v)-\ell(w)} d^w_{u,v} \geqslant 0$. In what follows we will obtain a recursive procedure to calculate the coefficients, based on the equivariant Chevalley formula for the SM classes. Instances of this recursion in various equivariant (quantum) cohomology and K theory rings, appeared in [32, 23, 28, 31, 36, 8, 37].

We start with the following simple lemma.

Lemma 5.1. *The following properties hold for the SMLR coefficients $d^w_{u,v}$:*

(a) *If $d^w_{u,v} \neq 0$, then $u \leqslant w$ and $v \leqslant w$.*
(b) $d^v_{u,v} = s_{\mathrm{M}}(Y(u)^\circ)|_v$.

Proof. To prove (a), localize both sides at the fixed point wP, and observe that

$$s_{\mathrm{M}}(Y(u)^\circ)|_w = 0$$

unless $u \leqslant w$. Part (b) follows again by localization, after restricting both sides of Equation (11) to the fixed point vP. ☐

We record the following explicit localization formula for $s_{\mathrm{M}}(Y(u)^\circ)|_v$, generalizing the one in the complete flag variety case [1, Corollary 9.8].

Proposition 5.2. *Let $u \leqslant v \in W^P$. Fix a reduced expression $v = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ and set $\beta_j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j})$ $(j = 1, 2, \dots, \ell)$. Then*

$$s_{\mathrm{M}}(Y(u)^\circ)|_v = \frac{1}{\prod_{1\leqslant j\leqslant \ell}(1+\beta_j)} \sum \beta_{j_1}\beta_{j_2}\cdots\beta_{j_k},$$

where the summation is over $1 \leqslant j_1 < j_2 < \cdots j_k \leqslant \ell$ with $s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_k}}W_P = uW_P$.

Proof. Let $p : G/B \to G/P$ be the natural projection. In this proof, we use $Y(w)^\circ_B := B^- wB/B \subset G/B$ to denote the Schubert cells in G/B. Since p is a smooth morphism, we may apply the (equivariant) Verdier-Riemann-Roch formula [38, Theorem 4.1] to obtain

$$p^*(s_{\mathrm{M}}(Y(u)^\circ)) = \sum_{z\in W_P} s_{\mathrm{M}}(Y(uz)^\circ_B).$$

Restricting both sides to the fixed point $vB \in G/B$, we get

$$\begin{aligned} s_{\mathrm{M}}(Y(u)^\circ)|_v &= p^*(s_{\mathrm{M}}(Y(u)^\circ))|_{vB} \\ &= \sum_{z\in W_P} s_{\mathrm{M}}(Y(uz)^\circ_B)|_{vB} \\ &= \frac{\prod_{\alpha>0, v\alpha>0}(1-v\alpha)}{\prod_{\alpha>0}(1-v\alpha)} \sum \beta_{j_1}\beta_{j_2}\cdots\beta_{j_k} \\ &= \frac{1}{\prod_{1\leqslant j\leqslant \ell}(1+\beta_j)} \sum \beta_{j_1}\beta_{j_2}\cdots\beta_{j_k}. \end{aligned}$$

Here the third equality follows from [1, Corollary 6.7], and the summations in the last two lines are over $1 \leqslant j_1 < j_2 < \cdots j_k \leqslant \ell$ such that $s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_k}}W_P = uW_P$. □

For any $\lambda \in X^*(T)_P$ and $u, w \in W^P$, define the Chevalley coefficients $c^u_{\lambda,w}$ by the equation

$$c_1(\mathcal{L}_\lambda) \cup s_{\mathrm{M}}(Y(w)^\circ) = \sum_{u\in W^P} c^u_{\lambda,w} s_{\mathrm{M}}(Y(u)^\circ). \tag{12}$$

These coefficients are calculated in Theorem 4.2. The following is our main recursion formula for $d^w_{u,v}$.

Proposition 5.3. *For any $u, v, w \in W^P$ and any weight $\lambda \in X^*(T)_P$, the following holds:*

$$(c^w_{\lambda,w} - c^u_{\lambda,u})d^w_{u,v} = \sum_{u<x} c^x_{\lambda,u} d^w_{x,v} - \sum_{y<w} c^w_{\lambda,y} d^y_{u,v}.$$

Proof. By the associativity of the cup product, we have

$$\Big(c_1(\mathcal{L}_\lambda) \cup s_{\mathrm{M}}(Y(u)^\circ)\Big) \cup s_{\mathrm{M}}(Y(v)^\circ) = c_1(\mathcal{L}_\lambda) \cup \Big(s_{\mathrm{M}}(Y(u)^\circ) \cup s_{\mathrm{M}}(Y(v)^\circ)\Big).$$

By Equation (11), Equation (12) and Lemma 5.1, taking the coefficients of $s_{\mathrm{M}}(Y(w)^\circ)$ in both sides of the above equation gives

$$c^u_{\lambda,u} d^w_{u,v} + \sum_{x>u} c^x_{\lambda,u} d^w_{x,v} = c^w_{\lambda,w} d^w_{u,v} + \sum_{y<w} c^w_{\lambda,y} d^y_{u,v}.$$

This proves the desired equality. □

Note that by Theorem 4.2, the Chevalley coefficient $c^y_{\lambda,x}$ is not equal to 0 only if $x \leqslant y$. Furthermore, if $u \neq w$, it was proved in [31, Proposition A.3] that one can find λ such that $c^w_{\lambda,w} \neq c^u_{\lambda,u}$. Thus, the identity in Proposition 5.3 may be rewritten as:

$$d^w_{u,v} = \frac{1}{c^w_{\lambda,w} - c^u_{\lambda,u}} \left(\sum_{u<x} c^x_{\lambda,u} d^w_{x,v} - \sum_{y<w} c^w_{\lambda,y} d^y_{u,v} \right). \tag{13}$$

Then the same argument as the one from [31] shows that Equation (13) gives a recursive relation for the coefficients $d^w_{u,v}$. We briefly recall the salient points. If $u = w$, then the coefficient $d^u_{u,v}$ is known from Lemma 5.1 and Proposition 5.2. If $u \neq w$, we may choose λ (depending on u, w) such that $c^w_{\lambda,w} - c^u_{\lambda,u} \neq 0$. The coefficients $d^w_{x,v}$ and $d^y_{u,v}$ on the right hand side of Equation (13) satisfy $\ell(w)-\ell(x), \ell(y)-\ell(u) < \ell(w)-\ell(u)$. To conclude, an inductive argument on $u \leqslant w$ and $\ell(w) - \ell(u)$ shows that:

Corollary 5.4. *The coefficients $d^w_{u,v}$ are algorithmically determined by the following:*

- $d^w_{u,v} = 0$ *for* $u \nleq w$*;*
- $d^u_{u,v}$ *(known from Lemma 5.1 and Proposition 5.2);*
- *For $u < w$, the coefficients $d^w_{u,v}$ are determined recursively from Equation (13), in terms of coefficients $d^{w'}_{u',v}$ where $u \leqslant u'$, $v' \leqslant w$ and $\ell(v') - \ell(u') < \ell(w) - \ell(u)$.*

An algorithm to calculate $d^u_{u,v}$, in terms of paths on a decorated version of the Bruhat graph of G/P, will be discussed in §8 below.

6. Equivariant multiplicities

6.1. Definition and a smoothness criterion. Let X be a (finite-type) pure-dimensional scheme with a torus action T, and $p \in X$ a torus fixed point. In this section we recall Brion's definition of the equivariant multiplicity of X at p. The main goal is to prove Theorem 6.7 which provides a smoothness criterion using equivariant multiplicities and CSM classes; it generalizes a similar criterion by Brion, in terms of fundamental classes. This will be needed to identify the right hand side of the Equation (3) with an equivariant multiplicity.

To be consistent with Brion's hypotheses, in this section we consider classes in the equivariant Chow group $A^T_*(X)$ of X defined by Edidin and Graham [14]. There is a degree doubling cycle map $cl_X : A^T_k(X) \to H^T_{2k}(X)$ ($k \geqslant 0$) from the equivariant Chow group to the equivariant Borel–Moore group. We will abuse notation and we will identify classes in the Chow group with their images in the Borel–Moore group under this map. In our main application later in this paper, $X = G/P$ is a flag manifold. In this case the cycle map is an isomorphism; see, e.g., [7, §3.2, Cor. 2].

Let us fix notation. For a T-module $V = \bigoplus_{i=1}^n \mathbb{C}_{\chi_i}$ we let $\Phi(V) \coloneqq \{\chi_i : 1 \leqslant i \leqslant n\}$ denote the set of its weights. The *Chern class* of V is defined by $c^T(V) \coloneqq \prod_{i=1}^n (1+\chi_i)$. The *Euler class* of V is defined by $e^T(V) \coloneqq \prod_{i=1}^n \chi_i$; these are elements in $H^*_T(pt)$. Recall that $H^*_T(pt)_{\mathrm{loc}}$ denotes the fraction field of $H^*_T(pt)$; it is equal to the fraction field of the polynomial ring $\mathbb{Z}[\lambda_1, \dots, \lambda_r]$ in a basis of the characters of T, written additively. We set $\deg \lambda_i = 1$.

Following [7, §4], we say that the fixed point $p \in X$ is non-degenerate if 0 is not a weight of the (Zariski) tangent space T_pX. The set of weights of T_pX will be called the weights of p. For $p \in X$ a non-degenerate point, Brion defined the equivariant multiplicity $e_p(\kappa)$, an element in the fraction field of $H^*_T(pt)$ associated to a class $\kappa \in A^T_*(X)$ in the equivariant Chow group of X. The definition of the equivariant multiplicity is given in [7, §4.2]:

Theorem 6.1 (Brion). *Let $p \in X$ be a non-degenerate fixed point and let $\chi_1, \dots, \chi_n$ be its weights.*

(a) There exists a unique $H_T^*(pt)$-linear map $e_{p,X} : H_*^T(X) \to H_T^*(pt)_{\mathrm{loc}}$ such that $e_{p,X}[p]_T = 1$ and that $e_{p,X}[Y]_T = 0$ for any T-invariant subvariety $Y \subset X$ with $p \notin Y$. Furthermore, the image $Im(e_{p,X})$ is contained in $\frac{1}{\chi_1 \cdot \ldots \cdot \chi_n} H_T^*(pt)$.

(b) For any T-invariant subvariety $Y \subset X$, the rational function $e_{p,X}[Y]_T$ is homogeneous of degree $-\dim Y$ and it coincides with $e_{p,Y}[Y]_T$.

(c) The point p is smooth in X if and only if

$$e_{p,X}[X]_T = \frac{1}{\chi_1 \cdot \ldots \cdot \chi_n}.$$

For $p \in X$ we denote by $\iota_p : \{p\} \to X$ denote the embedding. If p is smooth, then, using the terminology from [17, §B.7], the morphism ι_p is a regular embedding. Therefore it admits a pullback $\iota_p^* : H^T_{2(\dim X - k)}(X) \to H^T_{-2k}(pt) \simeq H_T^{2k}(pt)$. For $\kappa \in H_*^T(X)$, we denote by $\kappa|_p$ the pull back $\iota_p^*(\kappa)$. The following corollary is included in Brion's proof of the theorem above, but for convenience we recall briefly the salient points.

Corollary 6.2. *Let X be a scheme with a torus action T, and let $p \in X$ be a smooth non-degenerate, torus fixed point. Then the equivariant multiplicity is equal to*

$$e_{p,X}(\kappa) = \frac{\kappa|_p}{e^T(T_pX)} \quad \forall \kappa \in H_*^T(X).$$

Proof. We may may find a T-invariant neighborhood V of p such that V is smooth. (For instance, $V = X \smallsetminus X_{sing}$, the complement of the singular locus of X.) By the self intersection formula $([p]_T)|_p = \iota_p^*(\iota_p)_*[p]_T = e^T(T_pX)[p]_T$ in $H_*^T(pt)$. After the identification $H_*^T(pt) \simeq H_T^*(pt)$ which sends $[p]_T \mapsto 1$, the assignment $\kappa \mapsto \frac{\kappa|_p}{e^T(T_pX)}$ satisfies the conditions in part (a) of Theorem 6.1. □

Example 6.3. Let X be a smooth projective algebraic variety with a torus T action and finitely many T-fixed points, which we assume to be non-degenerate. Let $Y \subset X$ be closed and T-stable. By the localization theorem [15, Thm. 2], the fundamental class $[Y]_T$ expands as a sum of classes of fixed points, and by Corollary 6.2 the coefficients are the equivariant multiplicities:

$$[Y]_T = \sum_{p \in X^T} e_{p,X}([Y]_T)[p]_T.$$

We will need the following homological analogue of [2, Lemma 9.3].

Lemma 6.4. *Let $X \subset M$ be a T-equivariant closed embedding of irreducible T-varieties. Let $p \in X$ be a T-fixed point, non-degenerate and smooth in X and M. Denote by $N_{p,X}M$ the normal space of X in M at p. Then:*

$$\iota_p^* c_{\mathrm{SM}}(X) = c^T(T_pX) \cdot e^T(N_{p,X}M) \ .$$

Proof. Since the arguments essentially mimic those from [2], we will be brief. Again we may find a T-invariant neighborhood $p \in V \subset M$ such that $V \cap X$ is smooth. Let $j : V \to M$ be the inclusion. This is an open embedding, thus smooth, with trivial relative tangent bundle. Factor the inclusion ι_p as $\{p\} \xrightarrow{\iota_p'} V \xrightarrow{j} M$. Note that the restriction $i : X \cap V \subset V$ is proper, by base-change. We apply the Verdier–Riemann–Roch [38, Theorem 4.1] to the open embedding (hence smooth morphism) j, to obtain

$$\iota_p^* c_{\mathrm{SM}}(X) = (\iota_p')^* j^* c_{\mathrm{SM}}(X) = (\iota_p')^* i_* c_{\mathrm{SM}}(X \cap V).$$

Since $X \cap V$ is smooth, it follows that $i_* c_{\mathrm{SM}}(X \cap V) = i_*(c^T(T(X \cap V)) \cap [X \cap V]_T) \in H_*^T(V)$. By the self-intersection formula,

$$(\iota'_p)^* i_* c_{\mathrm{SM}}(X \cap V) = c^T(T_p X) \cdot e^T(N_{p, X\cap V} V) = c^T(T_p X) \cdot e^T(N_{p,X} M),$$

as claimed. □

Example 6.5. Consider $\mathbb{C}^3$ with standard basis f_1, f_2, f_3 and the torus action given by $diag(t_1, t_2, t_3).f_i = t_i f_i$. Let $M = \mathbb{P}(\mathbb{C}^3)$ be the projective plane, and denote by $p_i = \mathbb{P}(\langle f_i \rangle)$ the T-fixed points. Consider two invariant lines $L_1 = \mathbb{P}(\langle f_1, f_2 \rangle)$, $L_2 = \mathbb{P}(\langle f_2, f_3 \rangle)$, and their union $X = L_1 \cup L_2$, which is singular at p_2. Using that $\mathbb{1}_{L_1 \cup L_2} = \mathbb{1}_{L_1} + \mathbb{1}_{L_2} - \mathbb{1}_{L_1 \cap L_2}$, one calculates that:

$$c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(L_1) + c_{\mathrm{SM}}(L_2) - c_{\mathrm{SM}}(L_1 \cap L_2) = [L_1]_T + [L_2]_T + [p_1]_T + [p_2]_T + [p_3]_T.$$

Then

$$\begin{aligned} \iota_{p_1}^* c_{\mathrm{SM}}(X) &= \iota_{p_1}^*([L_1]_T) + \iota_{p_1}^*([p_1]_T) \\ &= t_3 - t_1 + (t_3 - t_1)(t_2 - t_1) \\ &= (1 + t_2 - t_1)(t_3 - t_1) \\ &= c^T(T_{p_1} X) e^T(N_{p_1, X} M). \end{aligned}$$

The following is immediate from the previous Lemma.

Corollary 6.6. *Let M be a smooth T-variety, and assume all the other hypotheses from Lemma 6.4. Then:*

$$\iota_p^*(s_{\mathrm{M}}(X)) = \frac{e^T(N_{p,X} M)}{c^T(N_{p,X} M)}.$$

Theorem 6.7. *Let X be an irreducible variety with a T-action. Let $p \in X$ be a non-degenerate T-fixed point with weights $\chi_1, \dots, \chi_n$. Then X is smooth at p if and only if*

$$e_{p,X}(c_{\mathrm{SM}}(X)) = \prod_{i=1}^n \big(1 + \frac{1}{\chi_i}\big).$$

Proof. If X is smooth at p, the claim follows from Corollary 6.2 and Lemma 6.4 (for $X = M$). We now prove the converse. From the definition of the CSM classes it follows that in $H_*^T(X)$,

$$c_{\mathrm{SM}}(X) = [X]_T + \sum a_i [V_i]_T, \tag{14}$$

where $a_i \in H_T^*(pt)$ and $V_i \subset X$ are closed irreducible subvarieties such that the terms $a_i [V_i]_T \in H_{2j}^T(X)$ for $j < \dim X$. Indeed, take a Whitney stratification $X = \bigcup X_i$. We may find equivariant desingularizations $\pi_i : \widetilde{X}_i \to \overline{X_i}$ of the closures of X_i such that π_i is an isomorphism over X_i and $\widetilde{X}_i \smallsetminus \pi_i^{-1}(X_i)$ is a simple normal crossing divisor. Then by additivity and functoriality,

$$c_{\mathrm{SM}}(X) = \sum (\pi_i)_*(c^T(T\widetilde{X}_i)) - (\pi_i)_*(c_{\mathrm{SM}}(\widetilde{X}_i \smallsetminus \pi_i^{-1}(X_i))).$$

It is not difficult to check that in this expression, the only term in $H_{2\dim X}^T(X)$ is $[X]_T$.

From Equation (14) it follows that the leading term (i.e. the term of lowest degree) in the localization $\iota_p^*(c_{\mathrm{SM}}(X))$ is equal to $\iota_p^*[X]_T$. Since the equivariant multiplicity is $H_T^*(pt)$-linear, we deduce that

$$e_{p,X}(c_{\mathrm{SM}}(X)) = e_{p,X}([X]_T) + \sum a_i e_{p,X}([V_i]_T).$$

By part (b) of Theorem 6.1, $\deg e_{p,X}([X]_T) = -\dim X$ and $\deg e_{p,V_i}([V_i]_T) = -\dim V_i$. Then the leading term of $e_{p,X}(c_{\mathrm{SM}}(X))$ has degree $-\dim X$, and it must be equal to $e_{p,X}([X]_T)$. The hypothesis implies that $e_{p,X}([X]_T) = \frac{1}{\chi_1 \cdot \dots \cdot \chi_n}$, and by Brion's criterion from Theorem 6.1(c), X is smooth at p. □

7. Equivariant multiplicities of CSM classes and smoothness of Richardson varieties

Next we apply the results in §6 to prove a smoothness criterion for Richardson varieties in terms of the equivariant multiplicities of their CSM class; cf. Theorem 7.5. This will be used to interpret certain terms in the hook formula from §3.

7.1. Weights of tangent and normal spaces of Schubert varieties.

We recall, and also define, various tangent and normal spaces of Schubert varieties which we use below.

Fix P a parabolic subgroup and $w \in W^P$. The fixed point wP is isolated and the tangent space T_wG/P has weights $\Phi(T_wG/P) := \{-w(\alpha) : \alpha \in R^+ \smallsetminus R_P^+\}$. We consider the following T-submodules of T_wG/P.

- The tangent spaces $T_w(X(w))$, respectively $T_w(Y(w))$, of $X(w)$ and $Y(w)$, at the smooth point wP. They satisfy $T_wX(w) \oplus T_wY(w) = T_wG/P$ and have weights

$$\Phi(T_wY(w)) = \{-w(\alpha) \in \Phi(T_wG/P) : w(\alpha) > 0\};$$

$$\Phi(T_wX(w)) = \Phi(T_wG/P) \smallsetminus \Phi(T_wY(w)) = \{-w(\alpha) \in \Phi(T_wG/P) : w(\alpha) < 0\}.$$

  Note that the condition $w(\alpha) > 0$ (respectively $w(\alpha) < 0$) is equivalent to $ws_\alpha > w$ (respectively $ws_\alpha < w$). Equivalently, $\Phi(T_wX(w)) = S(w)$ from Equation (6).

- The normal spaces $N_{w,X(w)}G/P$, respectively $N_{w,Y(w)}G/P$, of $X(w)$ and $Y(w)$, at the smooth point wP. Note that:

$$\Phi(N_{w,X(w)}G/P) = \Phi(T_wY(w)); \quad \Phi(N_{w,Y(w)}G/P) = \Phi(T_wX(w)).$$

- More generally, let $v \leqslant w$ be two elements in W^P. Define the T-submodule $\tilde{T}_wY(v) \subset T_wG/P$ by the requirement that its weights are:

$$\Phi(\tilde{T}_wY(v)) = \{-w(\alpha) \in \Phi(T_wG/P) : v \leqslant ws_\alpha\}.$$

  The space $\tilde{T}_wY(v)$ is in general not equal to the Zariski tangent space $T_wY(v)$ of $Y(v)$ at wP; see Example 7.1 below. However, if wP is smooth in $Y(v)$ then $\tilde{T}_wY(v)$ is the actual tangent space. The latter will be the case for most of our applications. We will not need it, but observe that in general $\tilde{T}_wY(v)$ is always included in $T_wY(v)$, see e.g. [11] or [25, Prop. 12.1.7].

- As before, let $v \leqslant w$ be two elements in W^P. Define the T-submodule $\tilde{N}_{w,Y(v)}G/P \subset T_wG/P$ by the requirement that its weights are:

$$\Phi(\tilde{N}_{w,Y(v)}G/P) = \{-w(\alpha) \in \Phi(T_wG/P) : v \nleqslant ws_\alpha\} = \{\beta \in \Phi(T_wG/P) : v \nleqslant s_\beta w\},$$

  where the second equality follows from the change of variable $\beta = -w(\alpha)$. Again we observe that if wP is smooth in $Y(v)$ then this is the genuine normal space $N_{w,Y(v)}G/P$ of $Y(v)$ at wP. Also observe that by definition,

$$\tilde{T}_wY(v) \oplus \tilde{N}_{w,Y(v)}G/P = T_wG/P.$$

  Using again (6), it follows that for any $v, w \in W^P$ such that $v \leqslant w$,

$$\begin{aligned}\Phi(\tilde{N}_{w,Y(v)}G/P) &= \{-w\alpha \mid \alpha \in R^+ \smallsetminus R_P^+, v \nleqslant ws_\alpha < w\}\\ &= \{-w\alpha \mid \alpha \in R^+, v \nleqslant ws_\alpha < w\}\\ &= \{\beta > 0 \mid v \nleqslant s_\beta w < w\}\end{aligned}$$

$$=S(w) \setminus S(w/v).$$

Here the third equality follows from the fact

$$\{-w\alpha \mid \alpha \in R^+, ws_\alpha < w\} = \{\beta > 0 \mid s_\beta w < w\}.$$

Example 7.1. Consider the 3-dimensonal quadric $\mathrm{OG}(1,5) = \mathrm{SO}(5)/P$, where P is the maximal parabolic subgroup such that $W_P = \langle s_{\alpha_2} \rangle$ with α_2 the short root. The divisor $Y(s_1)$ is singular at $w = s_1 s_2 s_1$ (e.g., by using Theorem 7.2 below), therefore the Zariski tangent space at w has dimension 3. However, $\dim \tilde{T}_w Y(v) = 2$.

We also recall a smoothness criterion for Schubert varieties due to S. Kumar, see also [7, page 255 (K)] for another proof using equivariant multiplicities.

Theorem 7.2 ([24]). *Let $v, w \in W^P$ be two Weyl group elements such that $v \leqslant w$. Then the Schubert variety $Y(v) \subset G/P$ is smooth at wP if and only if the localization of the equivariant fundamental class $[Y(v)] \in H_T^*(G/P)$ is given by:*

$$[Y(v)]|_w = \prod_{\beta \in \Phi(\tilde{N}_{w,Y(v)}G/P)} \beta.$$

Proof. In [24], the criterion is stated for G/B, but it immediately implies the result in G/P. For completeness, we indicate the main points. We use the same convention as in the proof of Proposition 5.2. Let $p : G/B \to G/P$ be the natural projection, and let $Y(v)_B := \overline{B^- vB/B} \subset G/B$ be the Schubert variety in G/B. Since p is a smooth morphism, and if $v \in W^P$ then $p^{-1}(Y(v)) = Y(v)_B$. In particular, if $v \leqslant w$ are elements in W^P, then $Y(v)$ is smooth at wP if and only if $Y(v)_B$ is smooth at wB. Furthermore, the localization $[Y(vW_P)]|_w = [Y(v)_B]|_{wB}$. This finishes the proof. □

7.2. Equivariant multiplicities of CSM classes of Richardson varieties. In this section we calculate equivariant multiplicities of Richarsdon varieties and their CSM classes. These will show again as factors in the generalized hook formula.

Let $v \leqslant w$ in W^P and consider the Richardson variety $R_w^v := X(w) \cap Y(v) \subset G/P$. The T-fixed point wP is an isolated T-fixed point in R_w^v,and the torus weights of the tangent space of R_w^v at wP are non-zero and distinct, therefore wP is non-degenerate in R_w^v. We need the following Lemma.

Lemma 7.3. *Let w be an element in W^P. Then*

$$c_{\mathrm{SM}}(X(w))|_w \cdot s_{\mathrm{M}}(Y(w))|_w = e^T(T_w G/P).$$

Proof. From Lemma 6.4 we obtain that

$$\begin{aligned} &c_{\mathrm{SM}}(X(w))|_w \cdot s_{\mathrm{M}}(Y(w))|_w \\ =&\frac{c^T(T_w X(w)) e^T(N_{w,X(w)}G/P) c^T(T_w Y(w)) e^T(N_{w,Y(w)}G/P)}{c^T(T_w G/P)} \\ =&e^T(T_w G/P). \end{aligned}$$

Here we used that $T_w X(w) \oplus T_w Y(w) = T_w G/P$ since the intersection $X(w) \cap Y(w)$ is proper and transversal and it consists of the single point wP. □

Proposition 7.4. *Let $v \leqslant w$ in W^P. Then the following equality holds:*

$$e_{w,G/P}(c_{\mathrm{SM}}(R_w^v)) = \frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w}.$$

Proof. From Corollary 6.2 we deduce that

$$e_{w,G/P}(c_{\mathrm{SM}}(R_w^v)) = \frac{c_{\mathrm{SM}}(R_w^v)|_w}{e^T(T_w G/P)}.$$

By Schürmann's transversality formula [43, 1] it follows that

$$c_{\mathrm{SM}}(R_w^v)|_w = (c_{\mathrm{SM}}(X(w)) s_{\mathrm{M}}(Y(v)))|_w = c_{\mathrm{SM}}(X(w))|_w \cdot s_{\mathrm{M}}(Y(v))|_w.$$

Then the claim follows from Lemma 7.3 after combining the two equations. □

Next we use Kumar's smoothness criterion to relate smoothness of Schubert and Richardson varieties in terms of localizations of CSM and SM classes.

Theorem 7.5. *Let $v, w \in W^P$ be two elements such that $v \leqslant w$. Then the following are equivalent:*

(a) *The Schubert variety $Y(v)$ is smooth at wP;*
(b) *The Richardson variety R_w^v is smooth at wP;*
(c) *The localization of $c_{\mathrm{SM}}(Y(v))$ at wP satisfies:*

$$c_{\mathrm{SM}}(Y(v))|_w = e^T(\tilde{N}_{w,Y(v)} G/P) \cdot c^T(\tilde{T}_w Y(v));$$

(d) *The localization of $s_{\mathrm{M}}(Y(v))$ at wP satisfies:*

$$s_{\mathrm{M}}(Y(v))|_w = \frac{e^T(\tilde{N}_{w,Y(v)} G/P)}{c^T(\tilde{N}_{w,Y(v)} G/P)} = \prod_{\beta > 0, v \not\leq s_\beta w} \frac{\beta}{1+\beta};$$

(e) *The equivariant multiplicity of $c_{\mathrm{SM}}(R_w^v)$ at wP satisfies:*

$$e_{w,G/P}(c_{\mathrm{SM}}(R_w^v)) = \prod_{\beta \in S(w/v)} (1 + \frac{1}{\beta}),$$

where $S(w/v)$ is defined in Equation (6).

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Kleiman transversality theorem; see e.g. [3, Cor. 2.10] for a proof. The second equality in (d), and the equivalence $(c) \Leftrightarrow (d)$, follow from the definition of the spaces involved (cf. §7.1) and the definition of CSM and SM classes. The equivalence $(d) \Leftrightarrow (e)$ follows from Proposition 7.4, taking into account that $Y(w)$ is smooth at wP, and using the formula for $s_{\mathrm{M}}(Y(w))|_w$ from Corollary 6.6.

To finish the proof it suffices to show the equivalence $(a) \Leftrightarrow (c)$. The direction $(a) \Rightarrow (c)$ follows from Lemma 6.4. For the reverse direction, observe that Equation (14) implies that the term of lowest degree of $c_{\mathrm{SM}}(Y(v))|_w$ is the localization $[Y(v)]|_w$. Therefore, the hypothesis implies that

$$[Y(v)]|_w = e^T(\tilde{N}_{w,Y(v)} G/P).$$

Then the claim follows from Kumar's smoothness criterion [24]; see Theorem 7.2. □

8. The Λ-Bruhat graph

In this section we introduce the main combinatorial object in this paper: a directed graph depending on an 'admissible function' Λ assigning weights to vertices, and whose sums over weighted paths give algorithms to calculate localization of SM classes of Schubert cells and varieties; cf. Proposition 8.12 and Corollary 8.13. Similar graphs based on Chevalley-type recursions have been used in [28, 31, 36, 37] to provide algorithms for Schubert multiplication in the equivariant quantum cohomology of flag manifolds. In the next section we will use this graph to formulate and prove a generalization of the Nakada's colored formula.

For a parabolic subgroup P recall that $X^*(T)_P$ denotes the set of integral weights orthogonal to roots in R_P^+. We recall the following characterization of the covering relations in the Bruhat order in G/P; see e.g. [16, Lemma 4.1].

Lemma 8.1. *Let $x \neq y$ be two elements in W^P. Then the following are equivalent:*

(a) *There exists $\gamma \in R^+ \smallsetminus R_P^+$ such that $xs_\gamma W_P = yW_P$.*
(b) *There exists $\beta \in R^+$ such that $s_\beta x W_P = yW_P$.*

Furthermore, β and γ are unique with these properties.

Proof. The equivalence of (a) and (b), and the uniqueness of β are proved in [16, Lemma 4.1]. This lemma also shows that s_γ is unique up to a conjugation by an element in W_P. If $s_\gamma W_P = s_{\gamma'} W_P$ then $\gamma = \gamma'$ by [9, Lemma 2.2]. ☐

Remark 8.2. When we analyze edges of the Bruhat graph below, we need to consider situations where $xW_P = ys_{\gamma'} W_P < yW_P = xs_\gamma W_P$ for $x, y \in W^P$. As observed in the proof above $s_{\gamma'}$ is a W_P conjugate of s_γ. Furthermore, if $xW_P < s_\beta x W_P = xs_\gamma W_P$ then $s_\beta x > x$, $\gamma = x^{-1}(\beta)$, and for any weight λ such that $Stab_W(\lambda) = W_P$,

$$\langle \lambda, \gamma^\vee \rangle = \langle \lambda, (\gamma')^\vee \rangle = \langle x(\lambda), \beta^\vee \rangle \quad .$$

For $v < w \in W^P$, recall that $[v,w]^P \coloneqq \{x \in W^P \mid v \leqslant x \leqslant w\}$ and set $[v,w)^P \coloneqq [v,w]^P \smallsetminus \{w\}$.

Definition 8.3 (Admissible function). *Let $v < w \in W^P$. An admissible function, $\Lambda = \Lambda_{v,w} : [v,w]^P \to X^*(T)_P$ is any assignment $x \mapsto \lambda_x$ such that $x(\lambda_x) \neq w(\lambda_x)$ for all $x \in [v,w)^P$.*

Admissible functions always exist, and below are two examples. In both cases the functions are constant; these are the only situations needed in this paper, but see Example 8.8 for a non-constant admissible function.

Example 8.4. (Standard admissible function.) Fix $v < w$ in W^P. For any $x \in [v,w]^P$ define

$$\Lambda(x) = \varpi_P \coloneqq \sum_{\alpha_i \in \Sigma \smallsetminus \Sigma_P} \varpi_i$$

(the sum of the fundamental weights not in P; see §2). Then Λ is admissible because $w(\varpi_P) = \varpi_P$ if and only if $w \in W_P$; cf. [4, Ch.5, §4.6]. We call this the **standard admissible function**.

To illustrate, consider the case of $G = \mathrm{SL}(3,\mathbb{C})$. The set of simple roots is $\Sigma = \{\alpha_1, \alpha_2\}$. If we take $\Sigma_P = \{\alpha_2\}$, then $W^P = \{id, s_1, s_2 s_1\}$ and $\varpi_P = \varpi_1$. The values of the function Λ are:

$$id(\varpi_P) = \varpi_1, \quad s_1(\varpi_P) = \varpi_1 - \alpha_1, \quad s_2 s_1(\varpi_P) = \varpi_1 - \alpha_1 - \alpha_2.$$

Example 8.5. (Dominant weights.) A more general example is when one considers an integral dominant weight π, and the parabolic subgroup determined by the condition $W_P = W_\pi$. Let $v < w$ in W^P. If $x \in [v,w]^P$ is an element such that $x(\pi) = w(\pi)$, it follows that $x = w$ in W^P. This implies that the constant function $\Lambda \equiv \pi$ is admissible.

If $\pi = \varpi_i$ is a fundamental weight, one recovers the standard admissible function for the maximal parabolic P given by the node i.

Admissible functions appeared in [31, §7] in the study of the equivariant quantum cohomology ring of flag manifolds.

Definition 8.6 (Λ-Bruhat graph). *Let $v, w \in W^P$ such that $v < w$, and let $\Lambda : [v,w]^P \to X^*(T)_P$ be an admissible function. To this data we associate a decorated directed graph $\Gamma = (V, E)$ and two functions, $\mathcal{W}_\Lambda : V \to X^*(T)$ and $m_\Lambda : E \to \mathbb{Z}_+$, as follows:*

(1) *The set of vertices is defined by $V = [v,w]^P$.*
(2) *There is an oriented edge $x \to y$ whenever $xW_P < yW_P$ and $yW_P = xs_\gamma W_P$.*

(3) *Each vertex $x \in V$ is decorated by a weight*

$$\mathcal{W}_\Lambda(x) \coloneqq x(\lambda_x) - w(\lambda_x).$$

By the definition of Λ, if $x \neq w$, then the weight $\mathcal{W}_\Lambda(x)$ is not equal to 0. We will call $\mathcal{W}_\Lambda : V \to X^*(T)$ the **Λ-weight function** *of the graph.*

(4) *Each edge $xW_P \to yW_P = xs_\gamma W_P$ is decorated by the* **Chevalley multiplicity***:*

$$m_\Lambda(x,y) \coloneqq \langle \lambda_x, \gamma^\vee \rangle.$$

We will refer to this graph as the **Λ-Bruhat graph** *determined by the triple (v, w, Λ). If $\Lambda \equiv \pi$ is a constant admissible function, then we set $\mathcal{W}_\pi = \mathcal{W}_\Lambda$, $m_\pi = m_\Lambda$.*

If one ignores the orientation and the admissible function Λ, then the Λ-Bruhat graph is the 1-skeleton of the T-action on G/P. This is the graph used in the GKM theory, and to calculate curve neighborhoods of Schubert varieties [9]; it is also called the Bruhat graph. It contains the (unoriented) quantum Bruhat graph from [6] and [26], and it is related to “Games with hook structure” defined by Kawanaka [21].

We provide examples of Λ-Bruhat graphs below. In these examples we decorate the edges with their Chevalley multiplicities, and we remove those edges of Chevalley multiplicity 0, as they will not contribute to our algorithm for the SM structure constants. We will encode the function $\mathcal{W}_\Lambda$ by representing a vertex x as $\frac{x}{\mathcal{W}_\Lambda(x)}$.

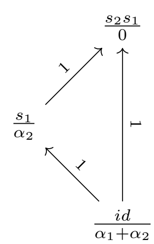
Example 8.7. We continue the Example 8.4 (1), i.e., $G = \mathrm{SL}(3, \mathbb{C})$, $\Sigma_P = \{\alpha_2\}$, $v = id$, $w = s_2 s_1$ and $\Lambda : [id, w]^P \to X^*(T)_P$ is the standard admissible function

$$\Lambda \equiv \varpi_P = \varpi_1.$$

The set of vertices of the Bruhat graph is $V = [id, w]^P = \{id, s_1, s_2 s_1\}$. The weight function is

$$\mathcal{W}_\Lambda(id) = \alpha_1 + \alpha_2, \quad \mathcal{W}_\Lambda(s_1) = \alpha_2, \quad \mathcal{W}_\Lambda(s_2 s_1) = 0.$$

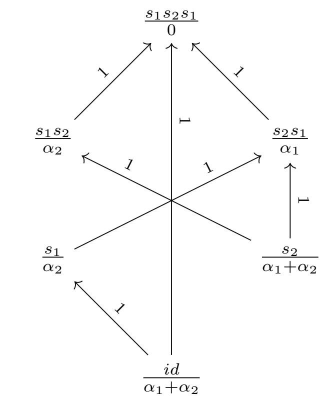
The resulting Λ-Bruhat graph is depicted below.

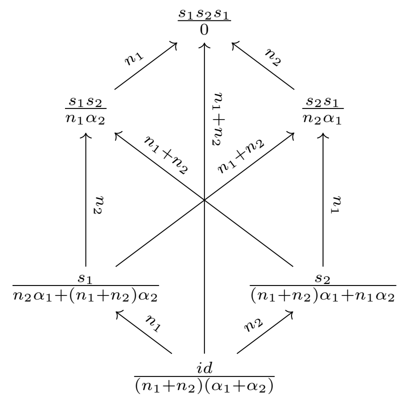


Example 8.8. Consider $G = \mathrm{SL}_3(\mathbb{C})$ and $P = B$. In this case $G/B = \mathrm{Fl}(3)$, the flag variety parametrizing complete flags $(F_1 \subset F_2 \subset \mathbb{C}^3)$. Set $v \coloneqq id$ and $w \coloneqq w_0 = s_1 s_2 s_1$ and consider two functions $\Lambda_i : W \to X^*(T)^+$ defined by:

$$\Lambda_1(x) = \begin{cases} \varpi_1 & \text{if } x \neq s_2 s_1; \\ \varpi_2 & \text{if } x = s_2 s_1. \end{cases} \qquad \Lambda_2(x) \equiv n_1 \varpi_1 + n_2 \varpi_2, \quad (n_1, n_2 > 0).$$

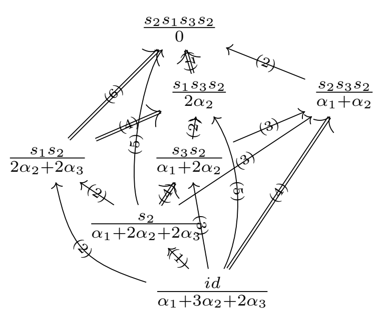
One may easily check that both are admissible functions. The corresponding Λ_i-Bruhat graphs for the triple (id, w_0, Λ_i) are below, from left to right.





Example 8.9. Consider the Lie type B_3, i.e. $G = \mathrm{SO}(7, \mathbb{C})$. The simple roots are $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ with α_3 short. We choose P to be the maximal parabolic determined by the set $\Delta_P = \{\alpha_1, \alpha_3\}$. Geometrically, G/P is the submaximal isotropic Grassmannian $\mathrm{IG}(2, 7)$ parametrizing subspaces of dimension 2 which are isotropic with respect to the non-degenerate symmetric form in $\mathbb{C}^7$. We pick the standard admissible function $\Lambda(x) \equiv \varpi_2$ and $v = id, w = s_2s_1s_3s_2$. All simple edges have Chevalley multiplicity 1, and the double edges have multiplicity 2.

Let us explain in more detail the calculations. Each $u \in [id, w]^P$ has $\ell(u)$ arrows directed to u. For example $w = s_2s_1s_3s_2$ has four arrows directed to w.



We list below the reflections s_γ s.t. $\langle \varpi_2, \gamma^\vee \rangle \neq 0$.

no.	s_γ	γ	$\gamma^\vee$
(1)	s_2	α_2	$\alpha_2^\vee$
(2)	$s_1s_2s_1$	$\alpha_1 + \alpha_2$	$a_1^\vee + a_2^\vee$
(3)	$s_3s_2s_3$	$\alpha_2 + 2\alpha_3$	$\alpha_2^\vee + a_3^\vee$
(4)	$s_2s_3s_2$	$\alpha_2 + \alpha_3$	$2\alpha_2^\vee + a_3^\vee$
(5)	$s_3s_2s_1s_2s_3$	$\alpha_1 + \alpha_2 + 2\alpha_3$	$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$
(6)	$s_1s_2s_3s_2s_1$	$\alpha_1 + \alpha_2 + \alpha_3$	$2\alpha_1 + 2\alpha_2^\vee + \alpha_3^\vee$
(7)	$s_2s_3s_2s_1s_2s_3s_2$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$

$$\langle \varpi_2, \gamma^\vee \rangle = \begin{cases} 1 & \text{for } (1), (2), (3), (5) \\ 2 & \text{for } (4), (6), (7) \end{cases}$$

$y \xleftarrow{(i)} x$ indicates $xs_\gamma W_P = yW_P$, for (i)-th s_γ in the above list.

Next we record the following lemma.

Lemma 8.10. *(a) Let (v, w, Λ) be a Λ-Bruhat graph and let (x, y) be an edge such that $yW_P = s_\beta xW_P = xs_\gamma W_P$. If $\Lambda(x) = \Lambda(y) = \lambda$ then $x(\lambda) - y(\lambda) = m_\Lambda(x, y)\beta$.*

(b) Let $\pi \in X(T)_P$ be a dominant integral weight and assume we are given a constant admissible function $\mathcal{W}_\pi \equiv \pi$. Then for any edge $x \xrightarrow{\beta} y$ as in (a) (with the notation from Definition 3.7):

$$\mathcal{W}_\pi(x) = \mathcal{W}_\pi(y) + m_\pi(x, y)\beta. \tag{15}$$

Proof. By definition,

$$x(\lambda) - y(\lambda) = x(\lambda) - xs_\gamma(\lambda) = \langle \lambda, \gamma^\vee \rangle x(\gamma) = m_\Lambda(x, y)\beta,$$

where the last equality follows from Remark 8.2. This proves part (a). Part (b) follows form (a) and the definition of $\mathcal{W}_\pi$. $\square$

In the next section we will analyze Λ-Bruhat graphs for minuscule elements, and we will need the following result.

Proposition 8.11. *Let $v < w$ be π-minuscule elements, and let P be the parabolic subgroup defined by $W_P = Stab_W(\pi)$. Consider the Λ-Bruhat graph (v, w, Λ) with the constant admissible function $\Lambda \equiv \pi$. Then the following hold:*

(1) Let $x, y \in [v, w]^P$ be two elements and let $x \to y$ be an edge such that $xW_P < yW_P = xs_\gamma W_P$. Then the Chevalley multiplicity satisfies:

$$m_\pi(x, y) = \langle \pi, \gamma^\vee \rangle = 1.$$

(2) The $\Lambda \equiv \pi$-weight of the vertex x (cf. Definition 8.6, parts (3) and (4)) is equal to:

$$\mathcal{W}_\pi(x) = x(\pi) - w(\pi) = \sum_{i=0}^{s} \beta_i \quad ,$$

where $x = x_0 \stackrel{\beta_1}{\to} x_1 \to \ldots \to x_{s-1} \stackrel{\beta_s}{\to} x_s = w$ is any chain in the Λ-Bruhat graph.

(3) The admissible function $\mathcal{W}_\pi$ is injective.

Proof. By Corollary 3.10(2), since w is π-minuscule, each element in $[v, w]^P$ is again π-minuscule. Let γ' be the positive root such that $xW_P = ys_{\gamma'}W_P$ (cf. Lemma 3.8). From Remark 8.2, the multiplicity $m_\pi(x, y)$ is equal to $\langle \pi, (\gamma')^\vee \rangle$. Since $ys_{\gamma'}W_P < yW_P$ implies that $ys_{\gamma'} < y$, and since y is π-minuscule, $\langle \pi, (\gamma')^\vee \rangle = 1$ from Lemma 3.5. This proves part (1).

From Lemma 8.10(b) it follows that for any chain as in the hypothesis,

$$\mathcal{W}_\pi(x) = x(\pi) - w(\pi) = \sum m_\pi(x_{i-1}, x_i)\beta_i.$$

Since by part (1) all multiplicities $m_\pi(x_{i-1}, x_i) = 1$, part (2) follows. Finally, if $\mathcal{W}_\pi(x) = \mathcal{W}_\pi(y)$ then $x(\pi) = y(\pi)$, thus $x = y$ in W^P, proving part (3). ☐

We now give a formula for the coefficients $d_{u,w}^w$ from Equation (11) in terms of summation over weighted paths in the Λ-Bruhat graph.

Proposition 8.12. *For any $v \leqslant w \in W^P$, fix an admissible function $\Lambda : [v, w]^P \to X^*(T)_P$, and let $\Gamma = (v, w, \Lambda)$ be the associated Λ-Bruhat graph. Then for $u \in [v, w]^P$,*

$$d_{u,w}^w = \left(\sum \frac{m_\Lambda(x_r, x_{r-1})}{\mathcal{W}_\Lambda(x_r)} \cdot \frac{m_\Lambda(x_{r-1}, x_{r-2})}{\mathcal{W}_\Lambda(x_{r-1})} \cdot \ldots \cdot \frac{m_\Lambda(x_1, x_0)}{\mathcal{W}_\Lambda(x_1)} \right) d_{w,w}^w. \tag{16}$$

where the sum is over integers $r \geqslant 0$, and over all directed paths $u = x_r \to x_{r-1} \to \ldots \to x_0 = w$ in Γ.

Proof. There is nothing to do when $u = w$. If $u < w$, set $\lambda_u \coloneqq \Lambda(u)$. Then from Equation (13),

$$\begin{aligned} d_{u,w}^w &= \frac{1}{w(\lambda_u) - u(\lambda_u)} \sum_{u<x} c_{\lambda_u,u}^x d_{x,w}^w = \frac{1}{w(\lambda_u) - u(\lambda_u)} \sum_{x=us_\alpha>u;\alpha>0} c_{\lambda_u,u}^x d_{x,w}^w \\ &= \frac{1}{w(\lambda_u) - u(\lambda_u)} \sum_{x=us_\alpha>u;\alpha>0} \langle -\lambda_u, \alpha^\vee \rangle d_{x,w}^w = \frac{1}{\mathcal{W}_\Lambda(u)} \sum_{u\to x} m_\Lambda(u, x) d_{x,w}^w \\ &= \sum_{u\to x} \frac{m_\Lambda(u, x)}{\mathcal{W}_\Lambda(u)} d_{x,w}^w. \end{aligned}$$

Here the second equality follows from Theorem 4.2 and the definition of $c_{\lambda,u}^x$ in Equation (12), and the rest are from the definitions. Then the statement follows by induction descending from w, on those elements x such that $u < x \leqslant w$. ☐

From Lemma 5.1 we deduce another interpretation of the sum in the previous proposition:

Corollary 8.13. *Let $v \leqslant w$ be two elements in W^P. With the hypotheses from Proposition 8.12, the following hold:*

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = \sum \frac{m_\Lambda(x_r, x_{r-1})}{\mathcal{W}_\Lambda(x_r)} \cdot \frac{m_\Lambda(x_{r-1}, x_{r-2})}{\mathcal{W}_\Lambda(x_{r-1})} \cdot \ldots \cdot \frac{m_\Lambda(x_1, x_0)}{\mathcal{W}_\Lambda(x_1)}, \tag{17}$$

where the sum is over integers $r \geqslant 0$, and over all directed paths $v \leqslant x_r \to x_{r-1} \to \ldots \to x_0 = w$ in Γ.

Proof. By Lemma 5.1 and the additivity of CSM classes, the left hand side of (17) is equal to

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = \frac{\sum_{v \leqslant u} s_{\mathrm{M}}(Y(u)^\circ)|_w}{s_{\mathrm{M}}(Y(w)^\circ)|_w} = \sum_{v \leqslant u \leqslant w} \frac{d^w_{u,w}}{d^w_{w,w}}.$$

Here we also used that $s_{\mathrm{M}}(Y(u)^\circ)|_w = 0$ if $w \not\geqslant u$. Then the claim follows from Proposition 8.12. □

Remark 8.14. By Proposition 5.2 and Proposition 7.4, the left hand side of Equation (17) is an equivariant multiplicity, which may be calculated explicitly. Fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and set $\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ $(j = 1, 2, \ldots, \ell)$. Then:

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = e_{w,G/P}(c_{\mathrm{SM}}(R^v_w)) = \frac{\sum \beta_{j_1}\beta_{j_2} \cdot \ldots \cdot \beta_{j_k}}{\beta_1 \cdot \ldots \cdot \beta_\ell}; \tag{18}$$

here the summation is over $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant \ell$ such that $vW_P \leqslant s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_k}} W_P$.

As we observed in Theorem 7.5, if $Y(v)$ is smooth at wP, then both the numerator and the denominator of this expression may be written as products. This is the key observation which leads to a generalization of Nakada's hook formula in the next section.

9. A generalized colored hook formula

In this section we prove the main result of this paper - the generalization of Nakada's colored hook formula, together with several corollaries of it.

9.1. The colored hook formula and consequences.

Recall that for $v < w \in W$, $S(w/v) \coloneqq \{\beta \in R^+ \mid v \leqslant s_\beta w < w\}$.

Theorem 9.1. *Let $v \leqslant w \in W^P$, and fix an admissible function $\Lambda : [v, w]^P \to X^*(T)_P$ with the associated Λ-Bruhat graph $\Gamma = (v, w, \Lambda)$. Then:*

$Y(v) \subset G/P$ is smooth at $wP \in G/P$ if and only if

$$\sum \frac{m_\Lambda(x_r, x_{r-1})}{\mathcal{W}_\Lambda(x_r)} \cdot \frac{m_\Lambda(x_{r-1}, x_{r-2})}{\mathcal{W}_\Lambda(x_{r-1})} \cdot \ldots \cdot \frac{m_\Lambda(x_1, x_0)}{\mathcal{W}_\Lambda(x_1)} = \prod_{\beta \in S(w/v)} \left(1 + \frac{1}{\beta}\right),$$

where the sum is over integers $r \geqslant 0$, and over all directed paths $v \leqslant x_r \to x_{r-1} \to \ldots \to x_0 = w$ in Γ.

Proof. We proved in Proposition 7.4 that

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = e_{w,G/P}(c_{\mathrm{SM}}(R^v_w)).$$

Then by Theorem 7.5, $Y(v) \subset G/P$ is smooth at $wP \in G/P$ if and only if

$$\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w} = \prod_{\beta \in S(w/v)} (1 + \frac{1}{\beta}).$$

Now observe that by Corollary 8.13 the left hand side of this expression is the sum in the statement. □

An important particular case of Theorem 9.1 is to consider a constant admissible function. For example, let $\pi \in X^*(T)$ be any dominant integral weight, and as usual define P by $\mathrm{Stab}_W(\pi) = W_P$. Set $\Lambda(x) \equiv \pi$ for $x \in [v,w]^P$, and recall that $\mathcal{W}_\pi$ denotes the associated weight function. Let $m_i = \langle \pi, \gamma_i^\vee \rangle$ be the multiplicity of the edge $x_i \to x_{i-1}$ and let β_i be the positive root β_i such that $x_{i-1}W_P = s_{\beta_i} x_i W_P$. With this notation, and by Theorem 9.1, we deduce the following.

Corollary 9.2. *Under the above assumptions, we have the following equivalence:*

$Y(v) \subset G/P$ is smooth at $wP \in G/P$ if and only if

$$\sum \frac{m_r}{m_1\beta_1 + m_2\beta_2 + \cdots + m_r\beta_r} \cdot \ldots \cdot \frac{m_1}{m_1\beta_1} = \prod_{\beta \in S(w/v)} \left(1 + \frac{1}{\beta}\right),$$

where the sum is over all integers $r \geqslant 0$, and over all directed paths $v \leqslant x_r \xrightarrow{\beta_r} x_{r-1} \xrightarrow{\beta_{r-1}} \ldots \xrightarrow{\beta_1} x_0 = w$ in $\Gamma = (v, w, \pi)$.

Proof. Immediate from Theorem 9.1, since $\mathcal{W}_\pi(x_k) = \sum_{i=1}^k m_i\beta_i$, by Equation (15). □

Corollary 9.3. *Let $v \leqslant w \in W$ be two π-minuscule elements for a dominant integral weight π, and $P \subset G$ be the parabolic subgroup satisfying* $\mathrm{Stab}_W(\pi) = W_P$. *Then:*

$Y(v) \subset G/P$ is smooth at $wP \in G/P$ if and only if

$$\sum \frac{1}{\beta_1 + \beta_2 + \cdots + \beta_r} \cdot \ldots \cdot \frac{1}{\beta_1} = \prod_{\beta \in S(w/v)} \left(1 + \frac{1}{\beta}\right), \tag{19}$$

where the sum is as in Corollary 9.2.

Proof. First observe that by Corollary 3.10(2), since w is π-minuscule, each representative $x \in W^P$ in a chain to w is also π-minuscule. Then from Corollary 9.2, we only need to show that any edge $x \to y$ has multiplicity $m_\pi(x, y) = 1$. This follows from Proposition 8.11. □

If $v = id$, Corollary 9.3 recovers Nakada's colored hook formula stated in Theorem 3.11.

Corollary 9.3 implies a skew version of the classical Peterson formula [10, 42]; see also [22] for related recent work in this direction.

Recall that $\mathrm{Red}(w)$ denotes the set of all reduced expressions of w and $\mathrm{ht}(\beta)$ is the height of the positive root β (see §2).

Corollary 9.4 (A skew Peterson formula). *Let π be a dominant integral weight, and let P be the parabolic subgroup defined by* $\mathrm{Stab}_W(\pi) = W_P$. *Let $v \leqslant w \in W$ be π-minuscule elements such that $Y(vW_P)$ is smooth at wP. Then*

$$\# \,\mathrm{Red}(wv^{-1}) = \frac{(\ell(w) - \ell(v))!}{\prod_{\beta \in S(w/v)} \mathrm{ht}(\beta)} \;. \tag{20}$$

Proof. Consider the term of lowest degree $-\#S(w/v)$ in the expression in Corollary 9.3. This corresponds to taking the summation over *maximal paths* in Corollary 9.3, and yields

$$\sum \frac{1}{\beta_1+\beta_2+\cdots+\beta_r}\cdot\ldots\cdot\frac{1}{\beta_1}=\prod_{\beta\in S(w/v)}\frac{1}{\beta}. \tag{21}$$

The paths considered are the same as the paths from Corollary 3.14, in particular each β_i is a simple root. Now specialize each simple root $\alpha_i \mapsto 1$. The right hand side gives $\frac{1}{\prod_{\beta\in S(w/v)}\mathrm{ht}(\beta)}$. On the left hand side, each summand specializes to $\frac{1}{(\ell(w)-\ell(v))!}$, and the number of summands is equal to the number of reduced expressions of wv^{-1}, again by Corollary 3.14. □

Remark 9.5. One can show that in the hypotheses from Corollary 9.3, and if W is a simply laced Weyl group, then then $Y(v)$ is smooth at w if and only if wv^{-1} is π'-minuscule, where π' is dominant integral. Furthermore, in this case the identities (19) and (20) for $[v,w]^\pi$ coincide with those corresponding to the interval $[id,wv^{-1}]^{\pi'}$. The proof requires developing techniques different from those employed in this paper and will appear in a separate note.

If W is not simply laced then it is possible that $Y(v)$ is smooth at w, but wv^{-1} is not dominant minuscule. An example is in type F_4 for $v=s_1$, $w=s_1s_3s_2s_4s_3s_2s_1$, with α_1,α_2 short roots. However, in this case one can show that if $Y(v)$ is smooth at w, the 'skew' Nakada's identity is obtained from the 'straight' formula applied to suitable elements in type E_6. The transformation between the two cases is given by the folding of the root system E_6 into F_4. This suggests that folding may lead to a more general statement relating skew and straight Nakada formulae.

Remark 9.6. Let w be dominant minuscule and $v\in[id,w]^P$. If we remove the condition that $Y(vW_P)$ is smooth at wP, by equations (17), (18), and the proof of Corollary 9.4, we obtain

$$\#\,\mathrm{Red}(wv^{-1})=(\ell(w)-\ell(v))!\frac{\sum \mathrm{ht}(\beta_{j_1})\mathrm{ht}(\beta_{j_2})\cdots\mathrm{ht}(\beta_{j_{\ell(v)}})}{\mathrm{ht}(\beta_1)\cdots\mathrm{ht}(\beta_\ell)} \tag{22}$$

where the summation is over $1\leqslant j_1<j_2<\cdots<\ell(v)\leqslant\ell$ such that $v=s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_{\ell(v)}}}$ is a reduced decomposition.

Observe also that the right hand side is the specialization of $(\ell(w)-\ell(v))!\frac{[Y(v)]|_w}{[Y(w)]|_w}$ by taking a root to its height. To relate to the right hand side in Theorem 9.1, observe that the fraction $\frac{[Y(v)]|_w}{[Y(w)]|_w}$ may also be obtained by taking the lowest degree terms in the numerator and denominator of the fraction $\frac{s_{\mathrm{M}}(Y(v))|_w}{s_{\mathrm{M}}(Y(w))|_w}$. This fact follows from Proposition 5.2 above, or, more directly, because the Segre class satisfies:

$$s_{\mathrm{M}}(Y(v))=[Y(v)]+\text{ higher order terms in } H_T^*(G/P).$$

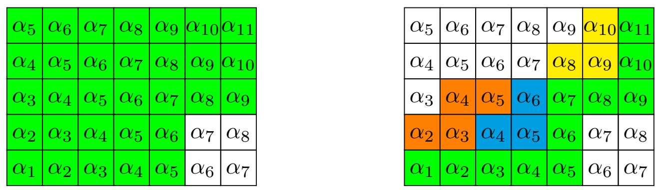
10. Examples

In this section we give some examples illustrating Theorem 9.1 and its corollaries.

10.1. Grassmannians. Let $X=G/P=\mathrm{Gr}(k,n)$ be a Grassmann manifold, where P is the maximal parabolic subgroup with $\Delta_P=\Delta\setminus\{\alpha_k\}$. The permutations in W^P are in bijection to the set of partitions $\mu=(\mu_1,\ldots,\mu_k)$ in the $k\times(n-k)$ rectangle. We denote by w_μ the element in W^P corresponding to μ. This bijection has the property that $\ell(w_\mu)=|\mu|=\mu_1+\ldots+\mu_k$ (the number of boxes of μ) and $w_\mu<w_\nu$ if and only if $\mu\subset\nu$. Furthermore, there exists an edge $w_\mu\to w_\nu$ in the Bruhat graph if and only if the skew shape ν/μ is a *rim hook* in ν. (Recall that the

skew shape ν/μ consists of the boxes in the Young diagram of ν which are not in μ. A rim hook in ν is a non-empty connected collection B of boxes which intersect the boundary of ν in at least one point, ν/B remains a partition, and B does not contain 2×2 square.) One may read the reduced decompositions of w_μ, w_ν, and also the root β such that $w_\nu W_P = s_\beta w_\mu W_P$ directly from the diagrams involved; cf. [20, 8], the latter, for more general cominuscule Grassmannians. We illustrate this in the example below.

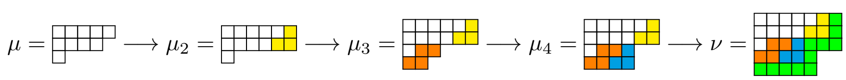
Example 10.1. Take $G/P = \mathrm{Gr}(5,12)$; then $\Delta_P = \{\alpha_i : 1 \leqslant i \leqslant 11\} \smallsetminus \{\alpha_5\}$. Consider $\nu = (7,7,7,5,5)$ included in the 5×7 rectangle. The partition ν corresponds to the green portion in the left diagram below.



One may read a reduced decomposition for w_ν by reading the labels of ν bottom to top, right to left:

$$w_\nu = s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_{10} s_9 s_8 s_7 s_6 s_5 s_4 s_{11} s_{10} s_9 s_8 s_7 s_6 s_5.$$

Consider $\mu = (5,4,1)$. A path in the Bruhat graph from μ to ν is given by removing rim hooks starting from ν. One example is the path below, where the rim hooks removed are, in order, starting from ν: green, cyan, orange, yellow.



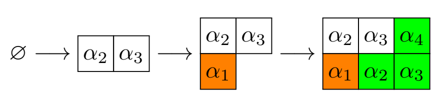
The root β associated to an edge $\mu_i \to \mu_{i+1}$ is the sum of the simple roots in the boxes of the skew shape μ_{i+1}/μ_i. In our case, if $w_\nu W_P = s_\beta w_{\mu_4} W_P$ then $\beta = \sum_{i=1}^{11} \alpha_i$ (the sum of the labels of the green rim hook). Different ways to remove rim hooks, including removing hooks in different order, will result in different paths in Equation (19), and all such paths are considered. Each *maximal* path has length $|\nu/\mu| = |\nu| - |\mu|$.

In order to illustrate Corollary 9.3 we consider the standard admissible function $\Lambda \equiv \omega_P = \omega_k$. Then each element $w \in W^P$ is ω_k-minuscule, and all edge multiplicities are equal to 1. Further, $Y(w_\mu W_P)$ is smooth at $w_\nu W_P$ if and only if the skew partition ν/μ is a union of straight shapes. (This may be deduced e.g. from [20, Corollary 9.2, 9.3] or from [18].) In the example above, $Y(w_\mu)$ is singular at w_ν.

Example 10.2. We now consider $G/P = \mathrm{Gr}(2,5)$ and the standard admissible function $\Lambda \equiv \omega_2$. Take $v \coloneqq w_\varnothing < w \coloneqq w_{(3,3)}$. Then $Y(v) = G/P$ is smooth, and the set $S(w/v)$ is equal to $S(w)$, and it corresponds to the roots of all possible rim hooks which may be removed from w. Then:

$$S(w/v) = S(w) = \{\alpha_3, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4\}$$

Consider the path



As before, we indicated the removed hooks with colors, and also their box labels. In the left hand side of the summation from Corollary 9.3 this corresponds to the product

$$\frac{1}{\beta_1+\beta_2+\beta_3}\times\frac{1}{\beta_1+\beta_2}\times\frac{1}{\beta_1}$$
$$=\frac{1}{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}\times\frac{1}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\times\frac{1}{\alpha_2+\alpha_3+\alpha_4}.$$

Another path is given by

$$\varnothing\longrightarrow\begin{array}{|c|c|}\hline \alpha_2 & \alpha_3 \\ \hline\end{array}\longrightarrow\begin{array}{|c|c|c|}\hline \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline\end{array}$$

giving the product

$$\frac{1}{\beta_1+\beta_2}\times\frac{1}{\beta_1}=\frac{1}{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}\times\frac{1}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.$$

Nakada's formula from Corollary 9.3 states that if one sums over all possible paths, one obtains:

(23)
$$\prod_{\beta\in S(w/v)}\Big(1+\frac{1}{\beta}\Big)=\Big(1+\frac{1}{\alpha_3}\Big)\times\Big(1+\frac{1}{\alpha_2+\alpha_3}\Big)\times\Big(1+\frac{1}{\alpha_3+\alpha_4}\Big)$$
$$\times\Big(1+\frac{1}{\alpha_1+\alpha_2+\alpha_3}\Big)\times\Big(1+\frac{1}{\alpha_2+\alpha_3+\alpha_4}\Big)\times\Big(1+\frac{1}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\Big).$$

Example 10.3. We illustrate Peterson's formula from Corollary 9.4. We continue with u, w defined in the previous example. If one takes the leading term of (23) and specializes each simple root to 1, then one obtains

$$1\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{4}=\frac{1}{4!\times 6}$$

Each maximal path has length $\ell(w)-\ell(v)=6$ and it contributes with $1/6!$ to the left hand side of (19). The number of maximal paths is equal to the number $\mathrm{Red}(w)$ of reduced decompositions of w, and this number must satisfy:

$$\frac{\mathrm{Red}(w)}{6!}=\frac{1}{4!\times 6}$$

therefore $\mathrm{Red}(w)=\frac{6!}{4!\times 6}=5$. One may check directly that the 5 reduced decompositions of w are:

$$s_3s_2s_1s_4s_3s_2,\quad s_3s_2s_4s_1s_3s_2,\quad s_3s_2s_4s_3s_1s_2,\quad s_3s_4s_2s_1s_3s_2,\quad s_3s_4s_2s_3s_1s_2.$$

They correspond respectively to 5 maximal paths from v to w, given by removing boxes from 1 to 6 (in order) in the diagrams below:

6	5	4
3	2	1

6	5	3
4	2	1

6	4	3
5	2	1

6	5	2
4	3	1

6	4	2
5	3	1

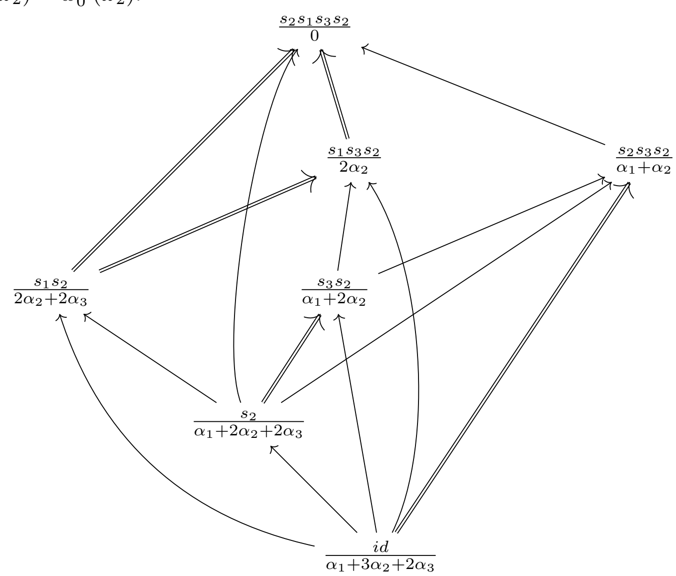
(Of course, these also correspond to the 5 standard Young tableaux on the diagram of w.)

10.2. Submaximal isotropic Grassmannians. In this section we consider G/P to be $\mathrm{IG}(k, 2n+1)$, the isotropic Grassmannian parametrizing k-dimensional subspaces of $\mathbb{C}^{2n+1}$ isotropic with respect to a symmetric non-degenerate bilinear form on $\mathbb{C}^{2n+1}$. Then $\Delta=\{\alpha_i : 1\leqslant i\leqslant n\}$ and by convention α_n is short. In this case, $\Delta_P=\Delta\setminus\{\alpha_k\}$ and the multiplicities may be 2. This is an example of a 'non-minuscule' flag manifold, and the equality from Corollary 9.2 is in general stronger than the Nakada's colored hook formula.

We consider the standard admissible function $\Lambda \equiv \omega_k$. The root β for an edge $u \to u'$ such that $u'W_P = s_\beta u W_P$ may be found from Equation (15):

$$\beta = \frac{\mathcal{W}_\Lambda(u) - \mathcal{W}_\Lambda(u')}{m(u,u')} = \frac{u(\omega_k) - u'(\omega_k)}{m(u,u')}.$$

To illustrate, consider $\mathrm{IG}(2,7)$ and the standard admissible function $\Lambda \equiv \omega_2$. Recall the Λ-Bruhat graph from Example 8.9 above, where the simple edges have multiplicity 1 and the double edges have multiplicity 2. The denominator at u is equal to $\mathcal{W}_\Lambda(u) = u(\omega_2) - w_0^P(\omega_2)$.



Every Schubert variety is smooth at $s_2s_1s_3s_2$ (the Schubert point), except for $Y(s_2)$ (the Schubert divisor). This may be checked directly e.g. by the smoothness criterion from Theorem 7.2; we will also recover it from Theorem 9.1 in an example below.

Example 10.4. Let us consider the case $v = s_3s_2$, $w = s_2s_1s_3s_2$ in the above situation. In this case $S(w) = \{\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_2+\alpha_3, \alpha_1+2\alpha_2+2\alpha_3\}$ and $S(w/v) = \{\alpha_2, \alpha_1+\alpha_2\}$. The localization is $Y(v)|_w$ is equal to $(\alpha_2+\alpha_3)(\alpha_1+2\alpha_2+2\alpha_3)$. By Theorem 7.2, we see $Y(v)$ that is smooth at w. To check Equation (19), we calculate the left hand side. Depending on $u \in [v,w]$, we divide into the following four cases.

(1) $u = v$ There are two paths
  The path $v \to s_1s_3s_2 \to w$ contributes with $\frac{1}{\alpha_1+2\alpha_2} \times \frac{2}{2\alpha_2}$;
  The path $v \to s_2s_3s_2 \to w$ contributes with $\frac{1}{\alpha_1+2\alpha_2} \times \frac{1}{\alpha_1+\alpha_2}$;
(2) $u = s_1s_3s_2$
  The path $s_1s_3s_2 \to w$ contributes with $\frac{2}{2\alpha_2}$;
(3) $u = s_2s_3s_2$
  The path $s_2s_3s_2 \to w$ contributes with $\frac{1}{\alpha_1+\alpha_2}$;
(4) $u = w$
  The trivial path $w = w$ contributes with 1.

In this case Theorem 9.1 (or Corollary 9.2) gives the equality:

$$1 + \frac{1}{\alpha_1+\alpha_2} + \frac{2}{2\alpha_2} + \frac{1}{\alpha_1+2\alpha_2} \times \frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+2\alpha_2} \times \frac{2}{2\alpha_2} = \left(1 + \frac{1}{\alpha_2}\right)\left(1 + \frac{1}{\alpha_1+\alpha_2}\right)$$

(This equality is equivalent to the fact that $Y(v)$ is smooth at wP.)

Example 10.5. Let us consider another case $v = s_2$, $w = s_2 s_3 s_2$ in the above situation. In this case

$$S(w) = \{\alpha_2, \alpha_2+\alpha_3, \alpha_2+2\alpha_3\}, \quad S(w/v) = \{\alpha_2, \alpha_2+2\alpha_3\}, \quad [Y(v)]|_w = 2(\alpha_2+\alpha_3),$$

and by Theorem 7.2 $Y(s_2)$ is singular at w, therefore we do not expect the identity from Theorem 9.1 to hold. Then:

(1) The path $v \to s_3 s_2 \to w$ contributes with $\frac{2}{\alpha_2+2\alpha_3} \times \frac{1}{\alpha_2}$;
(2) The path $s_2 \to w$ contributes with $\frac{1}{\alpha_2+2\alpha_3}$;
(3) The path $s_3 s_2 \to w$ contributes with $\frac{1}{\alpha_2}$;
(4) The trivial path contributes with 1.

Observe that

$$1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2+2\alpha_3} + \frac{1}{\alpha_2} \times \frac{2}{\alpha_2+2\alpha_3} \neq \left(1+\frac{1}{\alpha_2}\right)\left(1+\frac{1}{\alpha_2+2\alpha_3}\right).$$

By Theorem 9.1, this confirms that $Y(s_2)$ is singular at $s_1 s_3 s_2$. (In fact, one may check using [24] that the multiplicity at this singular point is equal to 2. Interestingly, $Y(s_2)$ is rationally smooth at w, since $\ell(w) - \ell(v) < 3$.)

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