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# EXISTENCE OF MAXIMAL AND MINIMAL WEAK SOLUTIONS AND FINITE DIFFERENCE APPROXIMATIONS FOR ELLIPTIC SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We establish the existence of maximal and minimal weak solutions between ordered pairs of weak sub- and super-solutions for a coupled system of elliptic equations with quasimonotone nonlinearities on the boundary. We also formulate a finite difference method to approximate the solutions and establish the existence of maximal and minimal approximations between ordered pairs of discrete sub- and super-solutions. Monotone iterations are formulated for constructing the maximal and minimal solutions when the nonlinearity is monotone. Numerical simulations are used to explore existence, nonexistence, uniqueness and non-uniqueness properties of positive solutions. When the nonlinearities do not satisfy the monotonicity condition, we prove the existence of weak maximal and minimal solutions using Zorn's lemma and a version of Kato's inequality up to the boundary.

## 1. Introduction

We consider the coupled system of elliptic equations with nonlinear boundary conditions

$$\begin{aligned} -\Delta u_i + u_i &= 0 \quad \text{in } \Omega; \\ \frac{\partial u_i}{\partial \eta} &= f_i(x, u_1, u_2) \quad \text{on } \partial\Omega,\ i = 1, 2, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz ($C^{0,1}$) boundary $\partial\Omega$, $N \geq 2$, and $\partial/\partial\eta := \eta(x)\cdot\nabla$ denotes the outer normal derivative on the boundary $\partial\Omega$. For each $i = 1, 2$, $f_i : \partial\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function; that is, $f_i(\cdot, u_1, u_2)$ is measurable for all $(u_1, u_2) \in \mathbb{R}^2$ and $f_i(x, \cdot, \cdot)$ is continuous for a.e. $x \in \partial\Omega$. Throughout this article we assume that each $f_i$ satisfies the quasimonotonicity condition

- (A1) the functions $f_i$ are quasimonotone nondecreasing in the sense that $f_1(x, u_1, u_2)$ is nondecreasing in $u_2$ for all fixed $x \in \partial\Omega, u_1 \in \mathbb{R}$, and $f_2(x, u_1, u_2)$ is nondecreasing in $u_1$ for all fixed $x \in \partial\Omega, u_2 \in \mathbb{R}$.

In this article, we establish the existence of maximal and minimal weak solutions for (1.1). In particular, we use a monotone iteration method when the $f_i$'s are monotone nondecreasing in both variables $u_1$ and $u_2$, and for the $f_i$'s that are nonmonotone in one of the variables, we utilize the surjectivity of a bounded, pseudomonotone and coercive operator, Zorn's lemma and a version of Kato's inequality up to the boundary to obtain the existence of maximal and minimal solutions.

To visualize solutions, we utilize numerical methods. Some common numerical approximation techniques for reaction diffusion equations can be found in [19, 20, 23, 24, 25]. We use the finite difference method to approximate solutions inspired by the results for approximating semilinear elliptic problems with Dirichlet boundary conditions in [15]. We formulate a finite difference method for (1.1) and prove admissibility and stability results. We also formulate a methodology

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for finding maximal and minimal solutions in the monotone case. This methodology is used to generate bifurcation diagrams for several one-dimensional examples. The numerical study complements the analytical results.

Elliptic equations are important for studying mathematical models in problems such as chemical reactions, ecology, population dynamics and combustion theory. Extensive studies have been done when the elliptic equation has linear boundary conditions which includes Dirichlet, Neumann, and Robin boundary conditions. However, there are certain scenarios where chemical reactions, the biological bonding, or species interactions may occur in a narrow layer or region near the boundary. In such cases, linear boundary conditions are deficient to describe the mathematical model (see e.g. [5, 10, 13, 16, 22, 21] and the references therein). Therefore, studying elliptic problems with nonlinear boundary conditions has been of much interest over the past few decades (see e.g. [2, 3, 4, 5, 9, 11, 16, 17, 26] and the references therein).

In [4] the authors studied the existence of a maximal and a minimal weak solution between ordered pairs of sub and supersolutions for elliptic (scalar) equations with nonlinear boundary conditions for both monotone and nonmonotone nonlinearities. The aim of this paper is to extend the results to the case of elliptic systems with quasimonotone nonlinearities on the boundary, and approximate these solutions using the finite difference method. We shall point out that quasimonotone elliptic systems have been studied by several authors, we refer to [7, 18, 22]. In [18], the authors proved the existence of a maximal weak solution between ordered pairs of sub and supersolutions for quasimonotone elliptic systems with linear boundary conditions, namely, Dirichlet boundary conditions. Furthermore, they assumed that the nonlinearities are not necessarily differentiable or even continuous. In such a case the monotone iteration procedure is not applicable, and the main ingredient is the use of pseudomonotone operators theory. As for [7] the author extended the results in [18] to the $p$-Laplacian case but still with Dirichlet boundary conditions. The author in [22] considered quasimonotone elliptic systems with smooth nonlinear boundary conditions, and they proved the existence of maximal and minimal classical solutions using monotone iteration methods. Here, we address the existence of maximal and minimal weak solutions for (1.1) assuming that the nonlinearities are Carathéodory.

Throughout this paper $H^1(\Omega)$ denotes the usual real Sobolev space of functions on $\Omega$; the product Sobolev space $H^1(\Omega)\times H^1(\Omega)$ will be denoted by $(H^1(\Omega))^2$ and is endowed with the norm $\|(u_1,u_2)\|_{(H^1(\Omega))^2} := \|u_1\|_{H^1(\Omega)} + \|u_2\|_{H^1(\Omega)}$. Moreover, the product space $(H^1(\Omega))^2$ is reflexive as $H^1(\Omega)$ is reflexive (see e.g. [12, p.15]). Besides the Sobolev spaces, we use the real Lebesgue space $L^q(\partial\Omega)$, and the compactness of the trace operator $\Gamma : H^1(\Omega)\to L^q(\partial\Omega)$ with $\Gamma u = u|_{\partial\Omega}$ (see e.g. [1, 8], [6, Thm 2.79], and [12, Chapter 6]), that is, $\Gamma$ is continuous (compact) if

$$\begin{gathered} 1\le q\le \frac{2(N-1)}{N-2} \quad (1\le q<\frac{2(N-1)}{N-2}), \quad \text{if } N>2,\\ 1\le q \quad (1\le q) \quad \text{if } N=2. \end{gathered} \tag{1.2}$$

To keep the notation simple, we will use the following: $U := (u_1,u_2)$, $\overline{U} := (\overline{u}_1,\overline{u}_2)$ and $\underline{U} := (\underline{u}_1,\underline{u}_2)$. The inequality $U\le V$ means $u_i(x)\le v_i(x)$ a.e. $x\in\Omega$ and a.e. $x\in\partial\Omega$ for each $i=1,2$.

**Definition 1.1.** We say that a function $(u_1,u_2)\in (H^1(\Omega))^2$ is a *weak solution* to (1.1) if

(i) For every $i=1,2$, $f_i(\cdot,u_1(\cdot),u_2(\cdot))\in L^r(\partial\Omega)$ for some $r>1$ if $N=2$ and $f_i(\cdot,u_1(\cdot),u_2(\cdot))\in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ if $N>2$ and
(ii) $\int_\Omega (\nabla u_i\nabla\psi + u_i\psi) = \int_{\partial\Omega} f_i(x,u_1,u_2)\psi$ for all $\psi\in H^1(\Omega)$.

**Definition 1.2.** We say that a function $(\overline{u}_1,\overline{u}_2)\in (H^1(\Omega))^2$ is a *weak supersolution* to (1.1) if

(i) For every $i=1,2$, $f_i(\cdot,\overline{u}_1(\cdot),\overline{u}_2(\cdot))\in L^r(\partial\Omega)$ for some $r>1$ if $N=2$ and $f_i(\cdot,\overline{u}_1(\cdot),\overline{u}_2(\cdot))\in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ if $N>2$ and
(ii) $\int_\Omega (\nabla \overline{u}_i\nabla\psi + \overline{u}_i\psi) \ge \int_{\partial\Omega} f_i(x,\overline{u}_1,\overline{u}_2)\psi$ for all $0\le\psi\in H^1(\Omega)$.

A *weak subsolution* $(\underline{u}_1,\underline{u}_2)$ is defined by reversing the inequality in (ii) above. Observe that the integrals on the right hand side of (ii) of Definition 1.1 and Definition 1.2 make sense since (i) holds and $r=\frac{2(N-1)}{N}$ is the conjugate of $\frac{2(N-1)}{N-2}$ when $N>2$.

In what follows, we state our main results which include the case where the $f_i$'s are monotone nondecreasing in both variables $u_1$ and $u_2$ (see Theorem 1.3) and the case where the $f_i$'s are nonmonotone in one of the variables (see Theorem 1.4).

**Theorem 1.3.** *Assume that there exist a weak subsolution $(\underline{u}_1, \underline{u}_2)$ and a weak supersolution $(\overline{u}_1, \overline{u}_2)$ of (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (\overline{u}_1, \overline{u}_2)$ a.e. on $\overline{\Omega}$, and*

- (A2) *there exists $k_1 > 0$ such that for all $x \in \partial\Omega$ and $(s_1, s_2) \in \mathbb{R}^2$ with $(\underline{u}_1(x), \underline{u}_2(x)) \le (s_1, s_2) \le (\overline{u}_1(x), \overline{u}_2(x))$, $f_1(x, s_1, s_2) + k_1 s_1$ is nondecreasing in $s_1$.*
- (A3) *there exists $k_2 > 0$ such that for all $x \in \partial\Omega$ and $(s_1, s_2) \in \mathbb{R}^2$ with $(\underline{u}_1(x), \underline{u}_2(x)) \le (s_1, s_2) \le (\overline{u}_1(x), \overline{u}_2(x))$, $f_2(x, s_1, s_2) + k_2 s_2$ is nondecreasing in $s_2$.*

*Then, there exist a minimal weak solution $(u_{1,*}, u_{2,*})$ and a maximal weak solution $(u_1^*, u_2^*)$ to (1.1), that is, if $(u_1, u_2)$ is any weak solution to (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (u_1, u_2) \le (\overline{u}_1, \overline{u}_2)$, then $(u_{1,*}, u_{2,*}) \le (u_1, u_2) \le (u_1^*, u_2^*)$.*

In the next theorem, we remove the monotonicity conditions (A2) and (A3) and obtain the following result.

**Theorem 1.4.** *Assume that there exists a pair of ordered weak subsolution $(\underline{u}_1, \underline{u}_2)$ and supersolution $(\overline{u}_1, \overline{u}_2)$ of (1.1), and that the following condition holds:*

- (A4) *there exist $K_1, K_2 \in L^r(\partial\Omega)$, $r > \frac{2(N-1)}{N}$, such that $|f_i(x, s_1, s_2)| \le K_i(x)$ a.e. $x \in \partial\Omega$, whenever $\underline{u}_i(x) \le s_i \le \overline{u}_i(x)$, $i = 1, 2$.*

*Then there exists a weak solution $(u_1, u_2)$ of (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (u_1, u_2) \le (\overline{u}_1, \overline{u}_2)$. Moreover, there exist a minimal weak solution $(u_{1,*}, u_{2,*})$ and a maximal weak solution $(u_1^*, u_2^*)$ to (1.1); that is, for any weak solution $(u_1, u_2)$ to (1.1) with $(\underline{u}_1, \underline{u}_2) \le (u_1, u_2) \le (\overline{u}_1, \overline{u}_2)$, we have $(u_{1,*}, u_{2,*}) \le (u_1, u_2) \le (u_1^*, u_2^*)$.*

In the course of the proofs, we will need the following result on the existence of weak solutions for the single equations case.

**Proposition 1.5.** *Consider the nonlinear problem*

$$\begin{aligned} -\Delta u + u &= 0 \quad \text{in } \Omega; \\ \frac{\partial u}{\partial \eta} &= f(x, u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

*where $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Suppose that there exists a pair of a weak subsolution $\underline{u}$ and a weak supersolution $\overline{u}$ such that $\underline{u} \le \overline{u}$ in $\overline{\Omega}$, and that there exists $K \in L^r(\partial\Omega)$, $r > \frac{2(N-1)}{N}$, such that $|f(x, s)| \le K(x)$ a.e. $x \in \partial\Omega$ and for all $s \in \mathbb{R}$ satisfying $\underline{u}(x) \le s \le \overline{u}(x)$. Then (1.3) has at least one weak solution $u$ such that $\underline{u} \le u \le \overline{u}$.*

The proof of Proposition 1.5 can be found in [4], which relies on the surjectivity of bounded, coercive pseudomonotone operators, Zorn's lemma and a version of Kato's inequality up to the boundary for single equations.

This article is organized as follows. In section 2, we prove Theorem 1.3. Section 3 is devoted to the proof of Theorem 1.4. In Section 4, we formulate and analyze a finite difference method to approximate solutions and generate bifurcation diagrams for several one-dimensional cases. Finally, in Section 5, the appendix, we state a version of Kato's inequality up to the boundary for single equations and employ it to prove that the componentwise maximum of two solutions of (1.1) is a subsolution of (1.1) (see Proposition 5.4) and componentwise minimum of two solutions of (1.1) is a supersolution of (1.1) (see Proposition 5.6), results necessary to prove Theorem 1.4.

## 2. Proof of Theorem 1.3

We will first construct a monotone operator and then use a corresponding iterative scheme to show the existence of a minimal (maximal) solution using the convergence of a sequence of weak subsolutions (supersolutions).

**(I)** Construction of the monotone operator. We define the map $T : J \to (H^1(\Omega))^2$ by $T(U) = W$, where $J := \{U = (u_1, u_2) \in (H^1(\Omega))^2 : \underline{U} \le U \le \overline{U}\}$ and $W = (w_1, w_2)$ is the unique weak solution of the decoupled system

$$\begin{gathered} -\Delta w_i + w_i = 0 \quad \text{in } \Omega; \\ \frac{\partial w_i}{\partial \eta} + k w_i = f_i(x, u_1, u_2) + k u_i \quad \text{on } \partial\Omega,\ i = 1, 2, \end{gathered} \tag{2.1}$$

where $k = k_1 + k_2 \ge 0$. Observe that, for each $i$, (2.1) is a linear equation for $w_i$. Also notice that $f_i(x, u_1, u_2) + k u_i \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Indeed, $f_i(\cdot, u_1(\cdot), u_2(\cdot)) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ by Definition 1.1 and (1.2) implies $u_i \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ since $\frac{2(N-1)}{N-2} > \frac{2(N-1)}{N}$. Hence, by the existence results for linear elliptic equations (see e.g. [14, P.160-162], there exists a unique solution of (2.1). Thus, $T$ is well-defined.

Now, we prove that $T$ is monotonically nondecreasing and maps $J$ into itself. Indeed, let $U, V \in J$ with $U \le V$. We have that $T(U) = W = (w_1, w_2)$ and $T(V) = Z = (z_1, z_2)$ satisfy the following:

$$\begin{gathered} -\Delta w_i + w_i = 0 \quad \text{in } \Omega; \\ \frac{\partial w_i}{\partial \eta} + k w_i = f_i(x, u_1, u_2) + k u_i \quad \text{on } \partial\Omega,\ i = 1, 2, \end{gathered} \tag{2.2}$$

and

$$\begin{gathered} -\Delta z_i + z_i = 0 \quad \text{in } \Omega, \\ \frac{\partial z_i}{\partial \eta} + k z_i = f_i(x, v_1, v_2) + k v_i \quad \text{on } \partial\Omega,\ i = 1, 2. \end{gathered} \tag{2.3}$$

By (A2) and the fact that $(u_1, u_2) \le (v_1, v_2)$ and $k > 0$, we have $f_1(x, u_1, u_2) + k u_1 \le f_1(x, v_1, u_2) + k v_1$. Applying the quasimonotonicity condition (A1), we conclude $f_1(x, u_1, u_2) + k u_1 \le f_1(x, v_1, v_2) + k v_1$. Similarly, by (A1) and (A3), we have $f_2(x, u_1, u_2) + k u_2 \le f_2(x, v_1, v_2) + k v_2$.

Substracting (2.2) and (2.3), we obtain that $w_i - z_i$ satisfies

$$\begin{gathered} -\Delta(w_i - z_i) + (w_i - z_i) = 0 \quad \text{in } \Omega; \\ \frac{\partial(w_i - z_i)}{\partial \eta} + k(w_i - z_i) \le 0 \quad \text{on } \partial\Omega. \end{gathered}$$

It follows from the comparison principle (see e.g. [8]) that $w_i \le z_i$. So, $T(U) \le T(V)$.

Now, we show that $T(J) \subseteq J$. With the monotonicity of $T$, it is sufficient to show that $\underline{U} \le T(\underline{U})$ and $T(\overline{U}) \le \overline{U}$. Take the subsolution $\underline{U} = (\underline{u}_1, \underline{u}_2)$. Then $T(\underline{U}) = (\underline{w}_1, \underline{w}_2)$ satisfies the system

$$\begin{gathered} -\Delta \underline{w}_i + \underline{w}_i = 0 \quad \text{in } \Omega; \\ \frac{\partial \underline{w}_i}{\partial \eta} + k \underline{w}_i = f_i(x, \underline{u}_1, \underline{u}_2) + k \underline{u}_i \quad \text{on } \partial\Omega,\ i = 1, 2. \end{gathered}$$

Using the fact that $\underline{U}$ is a subsolution to (1.1), we obtain that

$$\begin{gathered} -\Delta(\underline{u}_i - \underline{w}_i) + (\underline{u}_i - \underline{w}_i) = 0 \quad \text{in } \Omega; \\ \frac{\partial(\underline{u}_i - \underline{w}_i)}{\partial \eta} + k(\underline{u}_i - \underline{w}_i) \le 0 \quad \text{on } \partial\Omega,\ i = 1, 2. \end{gathered}$$

By the comparison principle, we have $\underline{u}_i - \underline{w}_i \le 0$ in $\overline{\Omega}$. Hence, $\underline{U} \le T(\underline{U})$. Similarly, we can show $T(\overline{U}) \le \overline{U}$. Hence,

$$\underline{U} \le T(\underline{U}) \le T(\overline{U}) \le \overline{U}. \tag{2.4}$$

Thus, $T$ maps $J$ into itself.

**(II)** Construction of minimal and maximal weak solutions. We construct a monotone sequence $\{U_n\} = \{(u_{1,n}, u_{2,n})\}$ and a monotone sequence $\{W_n\} = \{(w_{1,n}, w_{2,n})\}$ using the linear iteration process as follows:

$$U_n = T(U_{n-1}) \text{ with } U_0 = \underline{U}, \quad \text{and} \quad W_n = T(W_{n-1}) \text{ with } W_0 = \overline{U},$$

where $U_n$ and $W_n$ are weak solutions of

$$\begin{gathered} -\Delta u_{i,n} + u_{i,n} = 0 \quad \text{in } \Omega; \\ \frac{\partial u_{i,n}}{\partial \eta} + k u_{i,n} = f_i(x, u_{1,n-1}, u_{2,n-1}) + k u_{i,n-1} \quad \text{on } \partial\Omega, \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} -\Delta w_{i,n} + w_{i,n} = 0 \quad \text{in } \Omega; \\ \frac{\partial w_{i,n}}{\partial \eta} + k w_{i,n} = f_i(x, w_{1,n-1}, w_{2,n-1}) + k w_{i,n-1} \quad \text{on } \partial\Omega \end{gathered}$$

for each $i = 1, 2$. The monotonicity of $T$ and (2.4) imply that

$$\underline{U} = U_0 \le U_1 \le U_2 \le \dots \le W_n \le W_{n-1} \le \dots \le W_0 = \overline{U}.$$

We now proceed to show that the sequences $U_n$ and $W_n$ are weakly convergent. Since $U_n = T(U_{n-1})$ is a weak solution of (2.5), we have that

$$\int_\Omega (\nabla u_{i,n} \nabla \psi + u_{i,n}\psi) + k\int_{\partial\Omega} u_{i,n}\psi = \int_{\partial\Omega} \big(f_i(x, u_{1,n-1}, u_{1,n-1}) + k u_{i,n-1}\big)\psi \tag{2.6}$$

for all $\psi \in H^1(\Omega)$. Taking $\psi = u_{i,n}$ for each $i = 1, 2$, we obtain

$$\int_\Omega \big(|\nabla u_{i,n}|^2 + u_{i,n}^2\big) + k\int_{\partial\Omega} u_{i,n}^2 = \int_{\partial\Omega} \big(f_i(x, u_{1,n-1}, u_{1,n-1}) + k u_{i,n-1}\big)u_{i,n}. \tag{2.7}$$

Observe that

$$\begin{aligned} &\|f_i(x, u_{1,n-1}, u_{2,n-1}) + k u_{i,n-1}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \\ &\le \|f_i(x, \underline{u}_1, \underline{u}_2) + k\underline{u}_i\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} + \|f_i(x, \overline{u}_1, \overline{u}_2) + k\overline{u}_i\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \le \widetilde{C}\,, \end{aligned} \tag{2.8}$$

where $\widetilde{C}$ is a constant independent of $n$.

Hence, for each $i = 1, 2$, $f_i(\cdot, u_{1,n-1}(\cdot), u_{2,n-1}(\cdot)) + k u_{i,n-1}(\cdot) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Then, the right-hand side of (2.7) can be estimated using Hölder's inequality and the bound (2.8), and so (2.7) can be estimated as follows:

$$\begin{aligned} \|u_{i,n}\|^2_{H^1(\Omega)} &\le \|u_{i,n}\|^2_{H^1(\Omega)} + k\|u_{i,n}\|^2_{L^2(\partial\Omega)} \\ &\le \|f_i(x, u_{1,n-1}, u_{2,n-1}) + k u_{i,n-1}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \|u_{i,n}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} \\ &\le \widetilde{C}\Big(\|\overline{U}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)} + \|\underline{U}\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)}\Big) \le C', \end{aligned}$$

where $C'$ is a constant independent of $n$. Hence,

$$\|U_n\|_{(H^1(\Omega))^2} = \|u_{1,n}\|_{H^1(\Omega)} + \|u_{2,n}\|_{H^1(\Omega)} \le C, \tag{2.9}$$

where $C$ is a constant independent of $n$. Since $U_n$ is uniformly bounded in $(H^1(\Omega))^2$ and $(H^1(\Omega))^2$ is reflexive, it follows that there exists a subsequence (relabeled) $U_n$ which converges weakly to $U_* = (u_{1,*}, u_{2,*}) \in (H^1(\Omega))^2$.

Now, let's show that $f_i(x, U_n) + k u_{i,n}$ converges weakly to $f_i(x, U_*) + k u_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$, where we have adopted the notation $f_i(x, V) = f_i(x, v_1, v_2)$ for $V = (v_1, v_2)$. From (2.9), we have that the sequence $U_n$ is monotone increasing and bounded. Therefore, $U_n$ converges pointwise to $U_*$, that is, $U_*(x) = \lim_{n\to\infty} U_n(x)$ and $\underline{U}(x) \le U_*(x) \le \overline{U}(x)$ a.e. $x \in \Omega$ and a.e. $x \in \partial\Omega$. Since $f_i$ is continuous with respect to the second and third variables, it follows that

$$\lim_{n\to\infty} [f_i(x, U_n(x)) + k u_{i,n}(x)] = f_i(x, U_*(x)) + k u_{i,*}(x).$$

By the Lebesgue Dominated Convergence Theorem, $f_i(x, U_n(x)) + k u_{i,n}(x)$ converges strongly to $f_i(x, U_*(x)) + k u_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. Hence, we have $f_i(x, U_n(x)) + k u_{i,n}(x)$ converges weakly to $f_i(x, U_*(x)) + k u_{i,*}$, that is, for all $\psi \in H^1(\Omega)$,

$$\lim_{n\to\infty} \int_{\partial\Omega} [f_i(x, U_n) + k u_{i,n}]\psi = \int_{\partial\Omega} [f_i(x, U_*(x)) + k u_{i,*}]\psi. \tag{2.10}$$

We will show that $U_* = (u_{1,*}, u_{2,*})$ is a weak solution of (1.1). By the continuity of the trace operator (1.2) and the embedding $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ into $L^{\frac{2(N-1)}{N}}(\partial\Omega)$, it follows that $u_{i,*} \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. Using (2.8), we have

$$\begin{aligned}\|f_i(x, U_*)\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} &= \|f_i(x, U_*) + ku_{i,*} - ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \\ &\le \|f_i(x, U_*) + ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} + \|ku_{i,*}\|_{L^{\frac{2(N-1)}{N}}(\partial\Omega)} \le C,\end{aligned}$$

where $C$ is a positive constant. Hence, $f_i(x, U_*) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$.

Since $u_{i,n}$ converges weakly to $u_{i,*}$ in $H^1(\Omega)$, $u_{i,n}$ converges strongly in $L^2(\partial\Omega)$, and $f_i(x, U_n) + ku_{i,n}$ converges weakly to $f_i(x, U_*) + ku_{i,*}$ in $L^{\frac{2(N-1)}{N}}(\partial\Omega)$. By taking the limit in (2.6) as $n$ goes to $\infty$ and using (2.10), we obtain

$$\begin{aligned}\int_\Omega (\nabla u_{i,*}\nabla\psi + u_{i,*}\psi) + \int_{\partial\Omega} ku_{i,*}\psi &= \lim_{n\to\infty}\Big(\int_\Omega (\nabla u_{i,n}\nabla\psi + u_{i,n}\psi) + \int_{\partial\Omega} ku_n\psi\Big) \\ &= \lim_{n\to\infty}\Big(\int_{\partial\Omega} (f_i(x, U_{n-1}) + ku_{i,n-1})\psi\Big) \\ &= \int_{\partial\Omega} (f_i(x, U_*) + ku_{i,*})\psi\end{aligned}$$

for any $\psi \in H^1(\Omega)$. Hence,

$$\int_\Omega (\nabla u_{i,*}\nabla\psi + u_{i,*}\psi) = \int_{\partial\Omega} f_i(x, U_*)\psi \quad \text{for all } \psi \in H^1(\Omega).$$

Thus $U_*$ is a weak solution to (1.1).

Finally, let us show that $U_*$ is the minimal weak solution to (1.1) in the interval $[\underline{U}, \overline{U}]$. Let $V$ be a weak solution to (1.1) such that $\underline{U} \le V \le \overline{U}$. Then $V$ is a weak supersolution of (1.1). Repeating the above iteration procedure with $U_0 = \underline{U}$, we obtain $\underline{U} \le U_* \le V$. Thus $U_*$ is a weak minimal solution.

In a similar way, we can construct the maximal weak solution $U^*$ to (1.1) for which $\lim_{n\to\infty} W_n = U^*$ with $W_0 = \overline{U}$. This completes the proof of Theorem 1.3.

## 3. Proof of Theorem 1.4

The proof is based on an application of Zorn's lemma where we construct a nonempty set of subsolutions and show that the set has a maximal element which will turn out to be a solution to (1.1). We then use a version of Kato's Inequality up to the boundary to prove that the maximal element of the set of subsolutions is in fact a maximal solution to (1.1). Similarly, we can show the existence of a minimal solution to (1.1) by applying Zorn's lemma on a set of supersolutions.

We proceed to show the existence of a maximal solution. This proof involves several steps described below.

**Step 1.** Existence of a uniformly bounded subsolution of (1.1). Let $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2) \in (H(\Omega))^2$ be a subsolution to (1.1) such that $\underline{U} \le \widetilde{U} \le \overline{U}$. We will show the existence of a subsolution $W = (w_1, w_2)$ of (1.1) such that $\widetilde{U} \le W \le \overline{U}$ and $\|W\|_{(H^1(\Omega))^2} \le C$, where $C$ is a constant depending on $\overline{u}_i, \underline{u}_i, \Omega, K_i$.

Consider the equations

$$\begin{aligned}-\Delta u_1 + u_1 = 0 \quad &\text{in } \Omega, \\ \frac{\partial u_1}{\partial\eta} = f_1(x, u_1, \widetilde{u}_2) \quad &\text{on } \partial\Omega,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}-\Delta u_2 + u_2 = 0 \quad &\text{in } \Omega, \\ \frac{\partial u_2}{\partial\eta} = f_2(x, \widetilde{u}_1, u_2) \quad &\text{on } \partial\Omega.\end{aligned} \tag{3.2}$$

We set $f_1(x, s_1) := f_1(x, s_1, \widetilde{u}_2(x))$. It follows from condition (A4) that whenever $\widetilde{u}_1 \le s_1 \le \overline{u}_1$, $\widetilde{u}_2 \le \overline{u}_2$ and $r > \frac{2(N-1)}{N}$, it holds $|f_1(x, s_1)| = |f_1(x, s_1, \widetilde{u}_2(x))| \le K_1(x) \in L^r(\partial\Omega)$. Then, applying Proposition 1.5 to (3.1), we obtain a solution $w_1 \in H^1(\Omega)$ such that $\widetilde{u}_1 \le w_1 \le \overline{u}_1$. Similarly, setting $f_2(x, s_2) := f_2(x, \widetilde{u}_1, s_2)$ and applying Proposition 1.5 to (3.2), we obtain a solution $w_2$ such that $\widetilde{u}_2 \le w_2 \le \overline{u}_2$. Since $f_i$ is quasimonotone non-decreasing, it follows that $f_1(x, w_1, \widetilde{u}_2) \le f_1(x, w_1, w_2)$ and $f_2(x, \widetilde{u}_1, w_2) \le f_2(x, w_1, w_2)$. Hence,

$$\int_\Omega (\nabla w_i \nabla \psi + w_i \psi) \le \int_{\partial\Omega} f_i(x, w_1, w_2)\psi \quad \text{for all } \psi \in H^1(\Omega).$$

Using (A4) and $\underline{u}_i \le \widetilde{u}_i \le w_i \le \overline{u}_i$, we have that $f_i(x, w_1, w_2) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N}$. Hence, $f_i(x, w_1, w_2) \in L^{\frac{2(N-1)}{N}}(\partial\Omega)$ for $i = 1, 2$. So, $W = (w_1, w_2)$ is a subsolution of (1.1).

Furthermore, $\|W\|_{(H_1(\Omega))^2} \le M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$, where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is a constant. Indeed, by using Hölder's inequality, (A4) and the continuity of the trace operator, we obtain

$$\begin{aligned} \|w_1\|^2_{H^1(\Omega)} &= \int_\Omega |\nabla w_1|^2 + w_1^2 \\ &= \int_{\partial\Omega} f_1(x, w_1, \widetilde{u}_2) w_1 \\ &\le \int_{\partial\Omega} K_1(x) w_1 \le \int_{\partial\Omega} K_1(x) \overline{u}_1 \\ &\le \|K_1\|_{L^r(\partial\Omega)} \|\overline{u}_1\|_{L^{r'}(\partial\Omega)} \le C_1, \end{aligned}$$

where $r'$ is the conjugate of $r$ and $r' < \frac{2(N-1)}{N-2}$. Similarly $\|w_2\|^2_{H^1(\Omega)} \le C_2$. Hence, $\|W\|_{(H^1(\Omega))^2} \le M$, where $M$ is constant independent of $W$.

**Step 2.** Zorn's Lemma. Consider the set $\mathcal{A}$ consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a subsolution $(\widetilde{u}_1, \widetilde{u}_2)$ of (1.1) satisfying

$$(\underline{u}_1, \underline{u}_2) \le (\widetilde{u}_1, \widetilde{u}_2) \le (w_1, w_2) \le (\overline{u}_1, \overline{u}_2),$$

where $w_1$ and $w_2$ are solutions of (3.1) and (3.2), respectively, for the pair $\widetilde{u}_1, \widetilde{u}_2$.

We will first check the hypothesis of Zorn's Lemma, and then we derive the existence of a maximal element of $\mathcal{A}$. From Step 1, we observe that $\mathcal{A} \neq \emptyset$. Let $Y = \{W_n = (w_{1,n}, w_{2,n})\}_{n \ge 1}$ be a chain in $(\mathcal{A}, \le)$ where the countability of the indexing set is guaranteed by the separability of the product space $(H^1(\Omega))^2$. By reordering we have without loss of generality $Y = \{W_n\}_{n \ge 1}$ is an (componentwise) increasing sequence in $\mathcal{A}$. Now, let us show that $Y$ has an upper bound in $\mathcal{A}$. Since $W_n = (w_{1,n}, w_{2,n})$ belongs to $\mathcal{A}$ for every $n$, there exists a subsolution $(\widetilde{u}_{1,n}, \widetilde{u}_{2,n})$ of (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (\widetilde{u}_{1,n}, \widetilde{u}_{2,n}) \le (w_{1,n}, w_{2,n}) \le (\overline{u}_1, \overline{u}_2)$ and $w_{1,n}$ and $w_{2,n}$ are solutions of (3.1) and (3.2), respectively, for the pair $\widetilde{u}_{1,n}, \widetilde{u}_{2,n}$. From Step 1, it follows that $\|W_n\|_{(H^1(\Omega))^2} \le M$, where $M(\overline{u}_i, \underline{u}_i, \Omega, K_i)$ is a constant independent of $n$. By the reflexivity of $(H^1(\Omega))^2$, there is a subsequence (relabeled) $W_n$ which converges weakly to $W_* = (w_{1,*}, w_{2,*})$. Since the sequence $\{w_{1,n}\}$ is monotonically increasing and bounded above, $\{w_{1,n}\}$ converges pointwise to $w_{1,*}$. Similarly, $\{w_{2,n}\}$ converges pointwise to $w_{2,*}$. By the continuity of $f_i(x, \cdot, \cdot)$, it follows that

$$f_i(x, w_{1,n}(x), w_{2,n}(x)) \to f_i(x, w_{1,*}(x), w_{2,*}(x)); \ i = 1, 2$$

as $n \to \infty$. Using (A4), $|f_1(x, w_{1,n}, w_{2,n})| \le K_1(x)$, where $K_1(x) \in L^r(\partial\Omega)$ with $r > \frac{2(N-1)}{N-2} > 1$. Therefore, by the dominated convergence theorem, we have

$$\|f_1(x, w_{1,n}(x), w_{2,n}(x)) - f_1(x, w_{1,*}(x), w_{2,*}(x))\|_{L^r(\partial\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Using Holder's inequality and (3.3), we have that for any test function $\psi \in H^1(\Omega)$,

$$\begin{aligned} &\Big|\int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi - \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi\Big| \\ &\le \|f_1(x, w_{1,n}(x), w_{2,n}(x)) - f_1(x, w_{1,*}(x), w_{2,*}(x))\|_{L^r(\partial\Omega)} . \|\psi\|_{L^{r'}(\partial\Omega)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi = \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi \tag{3.4}$$

for all $\psi \in H^1(\Omega)$. Utilizing (3.4) and the quasimonotonicity condition of $f_1$, we have

$$\begin{aligned}\int_\Omega (\nabla w_{1,*}\nabla\psi + w_{1,*}\psi) &= \lim_{n\to\infty}\int_\Omega (\nabla w_{1,n}\nabla\psi + w_{1,n}\psi)\\ &= \lim_{n\to\infty}\int_{\partial\Omega} f_1(x, w_{1,n}(x), \widetilde{u}_{2,n}(x))\psi\\ &\le \lim_{n\to\infty}\int_{\partial\Omega} f_1(x, w_{1,n}(x), w_{2,n}(x))\psi\\ &= \int_{\partial\Omega} f_1(x, w_{1,*}(x), w_{2,*}(x))\psi\end{aligned}$$

and, similarly,

$$\int_\Omega (\nabla w_{2,*}\nabla\psi + w_{2,*}\psi) \le \int_{\partial\Omega} f_2(x, w_{1,*}(x), w_{2,*}(x))\psi$$

for all $0 \le \psi \in H^1(\Omega)$. Hence, $W_* = (w_{1,*}, w_{2,*})$ is a subsolution of (1.1).

Taking $\widetilde{U} = W_* = (w_{1,*}, w_{2,*})$ and applying Step 1, there exists a subsolution $V = (v_1, v_2)$ of (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (w_{1,*}, w_{2,*}) \le (v_1, v_2) \le (\overline{u}_1, \overline{u}_2)$ with $v_1$ and $v_2$ solutions of (3.1), (3.2), respectively, for the pair $(w_{1,*}, w_{2,*})$. This implies $V \in \mathcal{A}$ and is an upper bound of $Y$. By Zorn's Lemma (see e.g. [6]), $\mathcal{A}$ has a maximal element $Z = (z_1, z_2) \in \mathcal{A}$. We claim that $Z = (z_1, z_2)$ is a subsolution of (1.1). Indeed, $Z = (z_1, z_2) \in \mathcal{A}$ implies that there exists a subsolution to (1.1), $(\widetilde{z}_1, \widetilde{z}_2)$, such that

$$(\underline{u}_1, \underline{u}_2) \le (\widetilde{z}_1, \widetilde{z}_2) \le (z_1, z_2) \le (\overline{u}_1, \overline{u}_2)$$

where $z_1$ and $z_2$ are solutions of (3.1) and (3.2), respectively, for the pair $\widetilde{z}_1, \widetilde{z}_2$.

By the quasimonotonicity of $f_1$ and $f_2$, it follows that

$$\int_\Omega (\nabla z_1\nabla\psi + z_1\psi) = \int_{\partial\Omega} f_1(x, z_1, \widetilde{z}_2)\psi \le \int_{\partial\Omega} f_1(x, z_1, z_2)\psi$$

and

$$\int_\Omega (\nabla z_2\nabla\psi + z_2\psi) = \int_{\partial\Omega} f_2(x, \widetilde{z}_1, z_2)\psi \le \int_{\partial\Omega} f_2(x, z_1, z_2)\psi$$

for all $0 \le \psi \in H^1(\Omega)$. Therefore, $Z = (z_1, z_2)$ is a subsolution of (1.1).

**Step 3.** $Z = (z_1, z_2)$ is a solution of (1.1). Taking $\widetilde{U} = Z$ and applying Step 1, we have that there exists $Z^* = (z_1^*, z_2^*)$ that is a subsolution of (1.1) with $z_i \le z_i^* \le \overline{u}_i$, $i = 1, 2$, and $z_1^*$ and $z_2^*$ solutions of (3.1) and (3.2), respectively, for the pair $(z_1, z_2)$. From the definition of the set $\mathcal{A}$, we have that $Z^* = (z_1^*, z_2^*) \in \mathcal{A}$. Using the fact that $Z$ is a maximal element of $\mathcal{A}$, we obtain that $Z^* \le Z$. Hence, $Z = Z^*$, and so, for every $\psi \in H^1(\Omega)$,

$$\int_\Omega (\nabla z_1\nabla\psi + z_1\psi) = \int_\Omega (\nabla z_1^*\nabla\psi + z_1^*\psi)\int_{\partial\Omega} f_1(x, z_1, z_2)\psi$$

and

$$\int_\Omega (\nabla z_2\nabla\psi + z_2\psi) = \int_\Omega (\nabla z_2^*\nabla\psi + z_2^*\psi) = \int_{\partial\Omega} f_2(x, z_1, z_2)\psi.$$

Therefore, $(z_1, z_2)$ is a solution of (1.1) and $(\underline{u}_1, \underline{u}_2) \le (z_1, z_2) \le (\overline{u}_1, \overline{u}_2)$.

**Step 4.** $Z = (z_1, z_2)$ is a maximal solution of (1.1). Let $U = (u_1, u_2)$ be any solution of (1.1) such that $(\underline{u}_1, \underline{u}_2) \le (u_1, u_2) \le (\overline{u}_1, \overline{u}_2)$. Therefore, by Proposition 5.4 in the Appendix (see Section 5), $V = (v_1, v_2)$ is a subsolution of (1.1) with $v_1 = \max\{u_1, z_1\}$ and $v_2 = \max\{u_2, z_2\}$. Then it follows from **Step 1** that there exists $\widehat{Z} = (\widehat{z}_1, \widehat{z}_2) \in \mathcal{A}$ such that $\underline{U} \le V \le \widehat{Z} \le \overline{U}$, which implies $V \in \mathcal{A}$. As $Z$ is a maximal element of $\mathcal{A}$, hence, $V \le Z$. On the other hand, since $V = (\max\{u_1, z_1\}, \max\{u_2, z_2\})$, we have, $Z \le V \le \widehat{Z}$. Consequently, $Z = V$ implying $U \le Z$, and it follows that $Z = (z_1, z_2)$ is the unique maximal solution of (1.1).

By a similar approach, we can show the existence of a minimal solution of (1.1).

## 4. Finite difference approximations

In this section, we use a finite difference (FD) method for approximating solutions of the problem

$$\begin{aligned} -\Delta u_i + u_i &= 0 \quad \text{in } \Omega; \\ \frac{\partial u_i}{\partial \eta} &= \lambda f_i(u_1, u_2), \quad \text{on } \partial\Omega,\ i = 1, 2, \end{aligned} \tag{4.1}$$

where $\lambda$ is parameter and $f_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, in addition to (A1), satisfies the following conditions:

(i) $f_1(s_1, s_2)$ is locally Lipschitz continuous in $s_1$ and $f_2(s_1, s_2)$ is locally Lipschitz continuous in $s_2$.
(ii) $f_i(0, 0) \geq 0$.
(iii) $f_i$'s are sublinear, i.e.
$$\lim_{\|(s_1,s_2)\|_1 \to \infty} \frac{f_i(s_1, s_2)}{\|(s_1, s_2)\|_1} = 0,$$
where $\|(s_1, s_2)\|_1 = |s_1| + |s_2|$.

We prove the existence of nonnegative solutions for the discrete problem generated by the FD method (see Section 4.1) in between an ordered pair of discrete sub and supersolutions which turn out to be uniformly bounded independent of the discretization parameter $h$. This result and the corresponding sub and supersolution technique is a discrete analogue of Theorem 1.3. In fact, we find exact sub and supersolutions for the discrete problem. We formulate a monotone iteration to find the maximal nonnegative solution bounded above by the supersolution. Several bifurcation diagrams generated using MATLAB are provided.

4.1. **Formulation.** The main idea in our FD formulation is to approximate all differential operators by discrete operators using difference quotients. We first discretize the domain and, at each of the grid points, we approximate the value of the solution by solving the algebraic system of equations that results from replacing the differential operators with discrete difference operators.

Assume the domain $\Omega$ is an $N$-rectangle, where $N \geq 1$ is the dimension. In other words, $\Omega = (a_1, b_1) \times (a_2, b_2) \times \dots (a_N, b_N)$. Let $M_i \geq 2$ be a positive integer and $h_i = \frac{b_i - a_i}{M_i - 1}$ for $i = 1, 2, \dots N$. Define $h = (h_1, h_2, \dots h_N) \in \mathbb{R}^N$, $M = \prod_{i=1}^N (M_i)$, and $\mathbb{N}_M^N = \{\alpha = (\alpha_1, \alpha_2, \dots \alpha_N) \mid 1 \leq \alpha_i \leq M_i, i = 1, 2, \dots N\}$. Next we partition $\Omega$ into $\prod_{i=1}^N (M_i - 1)$ sub-$N$ rectangles with grid points $x_\alpha = (a_1 + (\alpha_1 - 1)h_1, a_2 + (\alpha_2 - 1)h_2, \dots, a_N + (\alpha_N - 1)h_N)$ for each multi-index $\alpha \in \mathbb{N}_M^N$. We call $\mathcal{T}_h = \{x_\alpha\}_{\alpha \in \mathbb{N}_M^N}$ a grid for $\overline{\Omega}$.

Let $\{e_i\}_{i=1}^N$ denote the canonical basis vectors for $\mathbb{R}^N$. We define the discrete operators for approximating first order partial derivatives $\frac{\partial}{\partial x_i} u(x)$ by

$$\begin{aligned} \delta^+_{x_i,h_i} u(x) &:= \frac{u(x + h_i e_i) - u(x)}{h_i}, \\ \delta^-_{x_i,h_i} u(x) &:= \frac{u(x) - u(x - h_i e_i)}{h_i}, \\ \delta_{x_i,h_i} u(x) &:= \frac{1}{2}\delta^+_{x_i,h_i} u(x) + \frac{1}{2}\delta^-_{x_i,h_i} u(x) = \frac{u(x + h_i e_i) - u(x - h_i e_i)}{2h_i} \end{aligned} \tag{4.2}$$

for the function $u : \mathbb{R}^N \to \mathbb{R}$ and

$$\begin{aligned} \delta^+_{x_i,h_i} u_h(x_\alpha) &:= \frac{u_h(x_{\alpha+e_i}) - u_h(x_\alpha)}{h_i}, \\ \delta^-_{x_i,h_i} u_h(x_\alpha) &:= \frac{u_h(x_\alpha) - u_h(x_{\alpha-e_i})}{h_i}, \\ \delta_{x_i,h_i} u_h(x_\alpha) &:= \frac{1}{2}\delta^+_{x_i,h_i} u_h(x_\alpha) + \frac{1}{2}\delta^-_{x_i,h_i} u_h(x_\alpha) = \frac{u_h(x_{\alpha+e_i}) - u_h(x_{\alpha-e_i})}{2h_i} \end{aligned} \tag{4.3}$$

for all $x_\alpha \in \mathcal{T}_h \cap \Omega$ for the grid function $u_h : \mathcal{T}_h \to \mathbb{R}$. Note that the discrete operators $\delta^\pm_{x_i,h_i}$ are first-order accurate whereas $\delta_{x_i,h_i}$ is second-order accurate. We also define the corresponding discrete gradient operators

$$[\nabla_h^{\pm}]_i := \delta^{\pm}_{x_i,h_i}\,, \quad [\nabla_h]_i := \delta_{x_i,h_i}.$$

Let $\widetilde{\partial\Omega} \subset \partial\Omega$ be such that $\widetilde{\partial\Omega} := \partial\Omega \setminus \{\text{the points where } \partial\Omega \text{ is not smooth}\}$. For $x \in \mathcal{T}_h \cap \widetilde{\partial\Omega}$, we define the discrete outer normal derivative using the discrete gradient operator $\nabla_h^*$ by

$$[\nabla_h^* u(x)]_i := \begin{cases} \delta^{+}_{x_i,h_i} u(x) & \text{if } \eta_i(x) < 0, \\ \delta^{-}_{x_i,h_i} u(x) & \text{if } \eta_i(x) > 0, \\ \delta_{x_i,h_i} u(x) & \text{if } \eta_i(x) = 0 \end{cases}$$

to ensure that $\nabla_h^* u(x) \cdot \eta$ does not require points outside of the domain $\overline{\Omega}$. Note that the discrete outward normal derivative approximation is only first order accurate.

Next, we define the second order central difference operators for approximating second order nonmixed partial derivatives $\frac{\partial^2}{\partial x_i^2} u(x)$ by

$$\delta^2_{x_i,h_i} u(x) := \delta^{\pm}_{x_i,h_i}(\delta^{\mp}_{x_i,h_i}(u(x))) = \frac{u(x+h_ie_i) - 2u(x) + u(x-h_ie_i)}{h_i^2} \tag{4.4}$$

for the function $u : \mathbb{R}^N \to \mathbb{R}$ and

$$\delta^2_{x_i,h_i} u_h(x_\alpha) := \frac{u_h(x_{\alpha+e_i}) - 2u_h(x_\alpha) + u_h(x_{\alpha-e_i})}{h_i^2} \tag{4.5}$$

for all $x_\alpha \in \mathcal{T}_h \cap \Omega$ for the grid function $u_h : \mathcal{T}_h \to \mathbb{R}$. Finally, we define the second order discrete Laplacian operator $\Delta_h$ by

$$\Delta_h := \sum_{i=1}^N \delta^2_{x_i,h_i}.$$

In this section we use the following discrete problem to approximate the solutions to (4.1). Here the grid functions $u_{i,h}$ are an approximation for $u_i$ over the grid $\mathcal{T}_h$ for $i = 1, 2$:

$$\begin{gathered} -\Delta_h u_{1,h} + u_{1,h} = 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ -\Delta_h u_{2,h} + u_{2,h} = 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ \nabla_h^* u_{1,h} \cdot \eta - \lambda f_1(u_{1,h}, u_{2,h}) = 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \\ \nabla_h^* u_{2,h} \cdot \eta - \lambda f_2(u_{1,h}, u_{2,h}) = 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}. \end{gathered} \tag{4.6}$$

**Remark 4.1.** Note that we are eliminating the set of points (a set with measure 0) where the outward normal derivative is not defined. Once $U_h$ is defined over $\mathcal{T}_h \cap (\Omega \cup \widetilde{\partial\Omega})$, it can be extended to $\mathcal{T}_h$ in post-processing.

4.2. **Existence and stability.** We use sub and supersolution theory in the discrete setting to prove existence and stability results for solutions to (4.6). First, we define discrete sub and supersolutions of (4.6). We say $\underline{U}_h = (\underline{u}_{1,h}, \underline{u}_{2,h})$ is a subsolution of (4.6) if it satisfies the following conditions:

$$\begin{gathered} -\Delta_h \underline{u}_{1,h} + \underline{u}_{1,h} \le 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ -\Delta_h \underline{u}_{2,h} + \underline{u}_{2,h} \le 0 \quad \text{in } \mathcal{T}_h \cap \Omega, \\ \nabla_h^* \underline{u}_{1,h} \cdot \eta - \lambda f_1(\underline{u}_{1,h}, \underline{u}_{2,h}) \le 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \\ \nabla_h^* \underline{u}_{2,h} \cdot \eta - \lambda f_2(\underline{u}_{1,h}, \underline{u}_{2,h}) \le 0 \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}. \end{gathered} \tag{4.7}$$

We can define the supersolution $\overline{U}_h = (\overline{u}_{1,h}, \overline{u}_{2,h})$ by reversing the inequalities in (4.7). We focus on proving the existence and stability of nonnegative solutions to (4.6).

**Step 1.** Constructing discrete sub and supersolutions of (4.6). It is clear that $\varphi := (0, 0)$ is a subsolution of (4.6) since $f_i(0) \ge 0$ for $i = 1, 2$. To construct a supersolution of (4.6), let us define

$m_i := \frac{a_i+b_i}{2}$ for $i = 1, 2, \dots, N$ to be the midpoint of the domain $\Omega$ along the $x_i$ direction, and, for some $c \gg 1$ specified later, define the quadratic function $\phi : \overline{\Omega} \to \mathbb{R}$ as follows:

$$\phi(x) := c\sum_{i=1}^{N}[(x_i - m_i)^2 + 4]. \tag{4.8}$$

Clearly, $\phi(x) \geq 4cN$ on $\overline{\Omega}$. Observe that $\frac{\partial^2}{\partial x_i^2}\phi(x) = 2c$. Since the operators $\delta^2_{x_i,h_i}$ are exact for quadratic functions, we have

$$\begin{aligned} -\Delta_h\phi(x) + \phi(x) &= -\Delta\phi(x) + \phi(x) \\ &= -2cN + c\sum_{i=1}^{N}(x_i - m_i)^2 + 4cN \\ &= 2cN + c\sum_{i=1}^{N}(x_i - m_i)^2 > 0 \end{aligned} \tag{4.9}$$

for all $x \in \mathcal{T}_h \cap \Omega$. Choose $x \in \mathcal{T}_h \cap \widetilde{\partial\Omega}$, and define $H_i = m_i - a_i = \frac{b_i - a_i}{2}$. Suppose $x_i = a_i$. Then $\eta = -e_i$, and by the convexity of $\phi$, we have

$$\begin{aligned} \nabla_h^*\phi(x)\cdot\eta &= -\delta^+_{x_i,h_i}\phi(x) \\ &\geq -\delta^+_{x_i,H_i}\phi(x) \\ &= -\frac{\phi(x + H_ie_i) - \phi(x)}{H_i} \\ &= \frac{-c(a_i + H_i - m_i)^2 + 4 + c(a_i - m_i)^2 - 4}{H_i} \\ &= c\frac{b_i - a_i}{2}. \end{aligned}$$

Similarly, if $x_i = b_i$, then, $\eta = e_i$, and by the convexity of $\phi(x)$ it holds

$$\begin{aligned} \nabla_{h_i}^*\phi(x)\cdot\eta &= \delta^-_{x_i,h_i}\phi(x) \\ &\geq \delta^-_{x_i,H_i}\phi(x) \\ &= \frac{\phi(x) - \phi(x - H_ie_i)}{H_i} \\ &= \frac{c(b_i - m_i)^2 + 4 - c(b_i - H_i - m_i)^2 - 4}{H_i} \\ &= c\frac{b_i - a_i}{2}. \end{aligned}$$

Observe that $\|(\phi(x), \phi(x))\|_1 \geq 8cN \to \infty$ as $c \to \infty$. Furthermore,

$$\|(\phi(x), \phi(x))\|_1 \leq cM,$$

where $M = 2(\sum_{i=1}^{N} 4H_i^2 + 4N)$. Hence, by the sublinearity of $f_i$,

$$0 \leq \frac{f_i(\phi(x), \phi(x))}{\|(\phi(x), \phi(x))\|_1} \leq \frac{f_i(cM, cM)}{8cN} = \frac{M}{8N}\frac{f_i(cM, cM)}{cM} \to 0$$

as $c \to \infty$ since $\frac{M}{8N}$ is a constant independent of $c$. Therefore, there exists $c \gg 1$ such that

$$\frac{\nabla_h^*\phi(x)\cdot\eta}{(\|(\phi(x), \phi(x))\|_1)} \geq \frac{c(b_i - a_i)}{2(\|(\phi(x), \phi(x))\|_1)} \geq \frac{b_i - a_i}{(\sum_{k=1}^{N} 4H_k^2 + 4N)} > \frac{\lambda f_j(\phi(x), \phi(x))}{\|(\phi(x), \phi(x))\|_1} \tag{4.10}$$

for $j = 1, 2$. Thus, $\nabla_h^*\phi(x)\cdot\eta \geq \lambda f_j(\phi(x), \phi(x))$ for all $x \in \widetilde{\partial\Omega}$ and for $j = 1, 2$. Finally, combining (4.9) and (4.10), we conclude that $(\phi(x), \phi(x))$ is a supersolution of (4.6).

**Step 2.** Forming a monotone iteration. Let us assume the reaction terms $f_1, f_2$ are Lipschitz continuous and let $U_h^{(0)}$ be a discrete supersolution of (4.6). Consider the fixed point iteration

$$U_h^{(n+1)} = \mathcal{M}_K U_h^{(n)} \tag{4.11}$$

for all $n \geq 0$, where $K$ is the maximum of the Lipschitz constants for $f_1, f_2$ in (4.1) and $\mathcal{M}_K$ is defined such that

$$\begin{aligned} -\Delta_h u_{1,h}^{(n+1)} + u_{1,h}^{(n+1)} + \lambda K u_{1,h}^{(n+1)} &= \lambda K u_{1,h}^{(n)} \quad \text{in } \mathcal{T}_h \cap \Omega, \\ -\Delta_h u_{2,h}^{(n+1)} + u_{2,h}^{(n+1)} + \lambda K u_{2,h}^{(n+1)} &= \lambda K u_{2,h}^{(n)} \quad \text{in } \mathcal{T}_h \cap \Omega, \\ \nabla_h^* u_{1,h}^{(n+1)} \cdot \eta + \lambda K u_{1,h}^{(n+1)} &= \lambda f_1(u_{1,h}^{(n)}, u_{2,h}^{(n)}) + \lambda K u_{1,h}^{(n)} \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}, \\ \nabla_h^* u_{2,h}^{(n+1)} \cdot \eta + \lambda K u_{2,h}^{(n+1)} &= \lambda f_2(u_{1,h}^{(n)}, u_{2,h}^{(n)}) + \lambda K u_{2,h}^{(n)} \quad \text{on } \mathcal{T}_h \cap \widetilde{\partial\Omega}. \end{aligned} \tag{4.12}$$

Before we proceed with the following theorem, let us write the above mapping on grid functions as an equivalent transformation for vectors. Let $J_0 = |\mathcal{T}_h \cap (\Omega \cup (\widetilde{\partial\Omega}))|$ and $\mathbf{U} \in \mathbb{R}^{2J_0}$ denote the vectorization of the grid function $U_h$. Notationally, the $I$ subscript will correspond to grid function values in $\mathcal{T}_h \cap \Omega$ and the $B$ subscript will correspond to grid function values in $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Then, (4.12) is equivalent to

$$M\mathbf{U}^{(n+1)} = \lambda \mathbf{F}(\mathbf{U}^{(n)}), \tag{4.13}$$

where

$$M = \begin{bmatrix} L_I & \underline{\underline{0}} & L_B & \underline{\underline{0}} \\ \underline{\underline{0}} & L_I & \underline{\underline{0}} & L_B \\ B_I & \underline{\underline{0}} & B_B & \underline{\underline{0}} \\ \underline{\underline{0}} & B_I & \underline{\underline{0}} & B_B \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{u_{1,I}} \\ \mathbf{u_{2,I}} \\ \mathbf{u_{1,B}} \\ \mathbf{u_{2,B}} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F_{1,I}} \\ \mathbf{F_{2,I}} \\ \mathbf{F_{1,B}} \\ \mathbf{F_{2,B}} \end{bmatrix}$$

for $L_I$ and $L_B$ matrices corresponding to $-\Delta_h + (1+\lambda K)I$; $B_I$ and $B_B$ matrices corresponding to $\nabla_h^* \cdot \eta + \lambda K I$; $\mathbf{F_{i,I}}$ corresponding to $\lambda K$ for $i = 1,2$; and $\mathbf{F_{i,B}}$ corresponding to $\lambda f_i(\mathbf{u_1}, \mathbf{u_2}) + \lambda K$ for $i = 1,2$. Clearly, $M$ is diagonally dominant since it is positive on the diagonal, non-positive for all off-diagonal terms, and the row sum is positive. Hence, $M$ is a Z-matrix, and by Gershgorin's Circle Theorem, $M$ is non-singular since the real part of all of its eigenvalues are always positive. Therefore, it follows that $M$ is a monotone matrix.

**Remark 4.2.** Notice that the iteration (4.12) is well defined and $f_i(s_1, s_2) + K s_i$ is increasing in $s_1$ and $s_2$ for $i = 1,2$.

**Theorem 4.3.** *Let $U_h = (u_{1,h}, u_{2,h})$ be a nonnegative subsolution of (4.6) and $U_h^{(0)} = (u_{1,h}^{(0)}, u_{2,h}^{(0)})$ be a supersolution of (4.6) such that $U_h^{(0)} \geq U_h$. Then, $U_h^{(1)} = (u_{1,h}^{(1)}, u_{2,h}^{(1)}) = \mathcal{M}_K U_h^{(0)}$ is a supersolution of (4.6) with $U_h \leq U_h^{(1)} \leq U_h^{(0)}$.*

*Proof.* Observe that, by the definition of $\mathcal{M}_K$ described in (4.12), for $i = 1,2$ in $\mathcal{T}_h \cap \Omega$, we have

$$-\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} + \lambda K u_{i,h}^{(1)} = \lambda K u_{i,h}^{(0)} \geq \lambda K u_{i,h} \geq -\Delta_h u_{i,h} + u_{i,h} + \lambda K u_{i,h}. \tag{4.14}$$

Also, on $\mathcal{T}_h \cup \widetilde{\partial\Omega}$, we have

$$\begin{aligned} \nabla_h^* u_{1,h}^{(1)} \cdot \eta + \lambda K u_{1,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(0)} \\ &\geq \lambda f_1(u_{1,h}^{(0)}, u_{2,h}) + \lambda K u_{1,h}^{(0)} \\ &\geq \lambda f_1(u_{1,h}, u_{2,h}) + \lambda K u_{1,h} \\ &\geq \nabla_h^* u_{1,h} \cdot \eta + \lambda K u_{1,h} \end{aligned}$$

by the quasimonotonicity of $f_i$ and Remark 4.2. In a similar fashion there holds

$$\begin{aligned} \nabla_h^* u_{2,h}^{(1)} \cdot \eta + \lambda K u_{2,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\ &\geq \lambda f_1(u_{1,h}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\ &\geq \lambda f_1(u_{1,h}, u_{2,h}) + \lambda K u_{2,h} \\ &\geq \nabla_h^* u_{2,h} \cdot \eta + \lambda K u_{2,h}. \end{aligned}$$

Hence $M\mathbf{U}^{(1)} \geq M\mathbf{U}$, and it follows that $\mathbf{U}^{(1)} \geq \mathbf{U}$. Thus $U_h^{(1)} \geq U_h$.

Next, for $i = 1, 2$,

$$\begin{aligned}
-\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} + \lambda K u_{i,h}^{(1)} &= \lambda K u_{i,h}^{(0)} \\
&= \lambda K u_{i,h}^{(0)} + 0 \\
&\leq -\Delta_h u_{i,h}^{(0)} + u_{i,h}^{(0)} + \lambda K u_{i,h}^{(0)}
\end{aligned}$$

in $\mathcal{T}_h \cap \Omega$ since $U_h^{(0)}$ is a nonnegative supersolution of (4.6). Furthermore,

$$\begin{aligned}
\nabla_h^* u_{i,h}^{(1)} \cdot \eta + \lambda K u_{i,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{i,h}^{(0)} \\
&\leq \nabla_h^* u_{i,h}^{(0)} \cdot \eta + \lambda K u_{i,h}^{(0)}
\end{aligned}$$

on $\mathcal{T}_h \cap \widetilde{\partial\Omega}$ since $U_h^{(0)}$ is a nonnegative supersolution of (4.6). Hence, $M\mathbf{U}^{(1)} \leq M\mathbf{U}^{(0)}$ which implies $\mathbf{U}^{(1)} \leq \mathbf{U}^{(0)}$. Thus, $U_h^{(1)} \leq U_h^{(0)}$.

Since, $U_h^{(1)} \leq U_h^{(0)}$, (4.14) implies

$$-\Delta_h u_{i,h}^{(1)} + u_{i,h}^{(1)} = \lambda K u_{i,h}^{(0)} - \lambda K u_{i,h}^{(1)} \geq 0$$

in $\mathcal{T}_h \cap \Omega$ for $i = 1, 2$. Also, from the definition of $\mathcal{M}_K$ combined with the quasimonotonicity of $f_i$ and Remark 4.2, we obtain

$$\begin{aligned}
\nabla_h^* u_{1,h}^{(1)} \cdot \eta + \lambda K u_{1,h}^{(1)} &= \lambda f_1(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(0)} \\
&\geq \lambda f_1(u_{1,h}^{(1)}, u_{2,h}^{(0)}) + \lambda K u_{1,h}^{(1)} \\
&\geq \lambda f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)}) + \lambda K u_{1,h}^{(1)}
\end{aligned}$$

on $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Thus

$$\nabla_h^* u_{1,h}^{(1)} \cdot \eta \geq f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)})$$

on $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Similarly,

$$\begin{aligned}
\nabla_h^* u_{2,h}^{(1)} \cdot \eta + \lambda K u_{2,h}^{(1)} &= \lambda f_2(u_{1,h}^{(0)}, u_{2,h}^{(0)}) + \lambda K u_{2,h}^{(0)} \\
&\geq \lambda f_2(u_{1,h}^{(0)}, u_{2,h}^{(1)}) + \lambda K u_{2,h}^{(1)} \\
&\geq \lambda f_1(u_{1,h}^{(1)}, u_{2,h}^{(1)}) + \lambda K u_{2,h}^{(1)}
\end{aligned}$$

on $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Thus

$$\nabla_h^* u_{2,h}^{(1)} \cdot \eta \geq \lambda f_2(u_{1,h}^{(1)}, u_{2,h}^{(1)})$$

on $\mathcal{T}_h \cap \widetilde{\partial\Omega}$. Therefore, $U_h^{(1)}$ is a supersolution of (4.6). ☐

**Step 3.** Existence of a fixed point.

**Theorem 4.4.** *Let $0 \leq U_h$ be a subsolution to (4.6) and $U_h^{(0)} \geq U_h$ be a supersolution of (4.6). Then the sequence $U_h^{(n)}$ defined by (4.12) converges to a solution of (4.6). Furthermore, if $\|U_h^{(0)}\|_{l^\infty([\mathcal{T}_h]^2)}$ is bounded independent of $h$, then the solution is $\ell^\infty$ norm stable.*

*Proof.* Observe that, by Theorem 4.3, we have

$$0 \leq U_h \leq U_h^{(n+1)} \leq U_h^{(n)} \leq U_h^{(n-1)} \leq \cdots \leq U_h^{(1)} \leq U_h^{(0)}$$

for all $n \geq 1$. Thus, the sequence $\{U_h^{(n)}\}_{n=0}^\infty$ is convergent since it is monotone and bounded. Let $V_h : [\widetilde{\mathcal{T}_h}]^2 \to \mathbb{R}^2$ such that $U_h^{(n)} \to V_h$ in $l^\infty([\widetilde{\mathcal{T}_h}]^2)$ for $\widetilde{\mathcal{T}_h} = \mathcal{T}_h \cap (\Omega \cup \widetilde{\partial\Omega})$. Clearly, $V_h$ is a fixed point of (4.11). Thus $V_h$ is a solution of (4.6) with

$$0 \leq U_h \leq V_h \leq U_h^{(n+1)} \leq U_h^{(n)} \leq U_h^{(n-1)} \leq \cdots \leq U_h^{(1)} \leq U_h^{(0)}$$

from which the stability result follows. ☐

**Remark 4.5.** Let $\phi$ be defined by (4.8) so that it is bounded by a constant depending only on $\Omega$, $\lambda$, and $f$ for $f$ satisfying the hypotheses for problem (4.1). Then $\phi$ is bounded independent of $h$, and it follows that the grid functions $U_h$ defined by $U_h(x) := (0,0)$ for all $x \in \mathcal{T}_h$ and $U_h^{(0)}$ defined by $U_h^{(0)}(x) := (\phi(x), \phi(x))$ for all $x \in \mathcal{T}_h$ for $c$ sufficiently large satisfy the assumptions in Theorem 4.4 to ensure the finite difference approximation exists and is $\ell^\infty$ norm stable.

4.3. **Example 1.** In this example, we consider the nonlinearities

$$\begin{aligned} f_1(u_1,u_2) &= 100\sqrt{u_2} - 2\sqrt{u_1}, \\ f_2(u_1,u_2) &= 4\sqrt{u_1} + \sqrt{u_2}. \end{aligned} \tag{4.15}$$

Clearly the $f_i$'s are Carateódory, quasimonotone non-decreasing and sublinear functions satisfying the assumption (A2) and (A3) in Theorem 1.3. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2}(0,0) = \infty$ and $\frac{\partial f_2}{\partial u_1}(0,0) = \infty$. Solutions can be found in Figure 1 for $\lambda = 0.5$ and $\lambda = 3$, and computed bifurcation diagrams can be found in Figure 2. When choosing the supersolution $U_h^{(0)}$ according to Remark 4.5, the solution for $\lambda = 0.5$ was found using $c = 16,384$ and the solution for $\lambda = 3$ was found using $c = 524,288$.

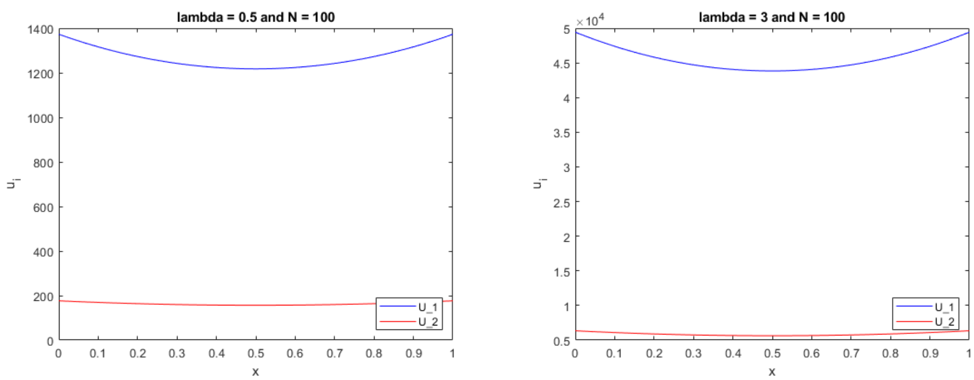



FIGURE 1. Graphs of $U_1$ and $U_2$ for $\lambda = 0.5$ and $\lambda = 3$

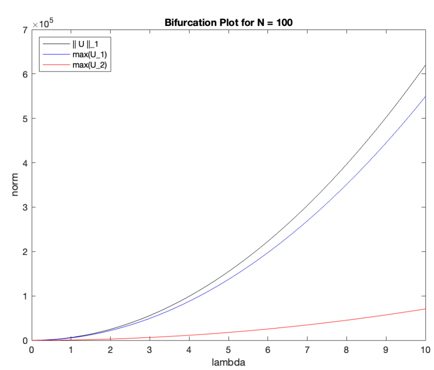



FIGURE 2. Bifurcation diagrams of (4.1) for $\lambda \in (0,10)$ when $f_1(u_1,u_2) = 100\sqrt{u_2} - 2\sqrt{u_1}$, $f_2(u_1,u_2) = 4\sqrt{u_1} + \sqrt{u_2}$.

4.4. **Example 2.** In this example, we consider the nonlinearities

$$\begin{aligned} f_1(u_1,u_2) &= \arctan(u_2), \\ f_2(u_1,u_2) &= 3\arctan(u_1). \end{aligned} \tag{4.16}$$

Clearly the $f_i$'s are Caratheodory, quasimonotone, non-decreasing, and sublinear functions satisfying the assumption (A2) and (A3) in Theorem 1.3. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2}(0,0) > 0$ and $\frac{\partial f_2}{\partial u_1}(0,0) > 0$. Solutions can be found in Figure 3 for $\lambda = 0.5$ and $\lambda = 3$, and computed bifurcation diagrams can be found in Figure 4. We see nonexistence of a positive solution for $\lambda$ small. When choosing the supersolution $U_h^{(0)}$ according to Remark 4.5, the solution for $\lambda = 0.5$ was found using $c = 4$ and the solution for $\lambda = 3$ was found using $c = 16$.

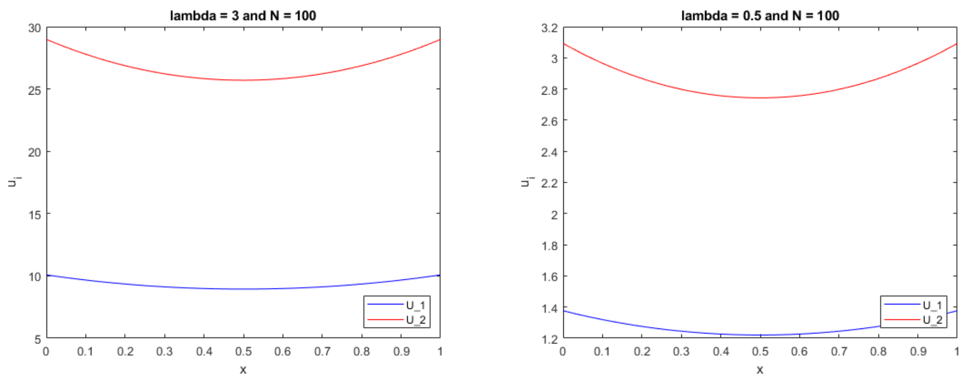
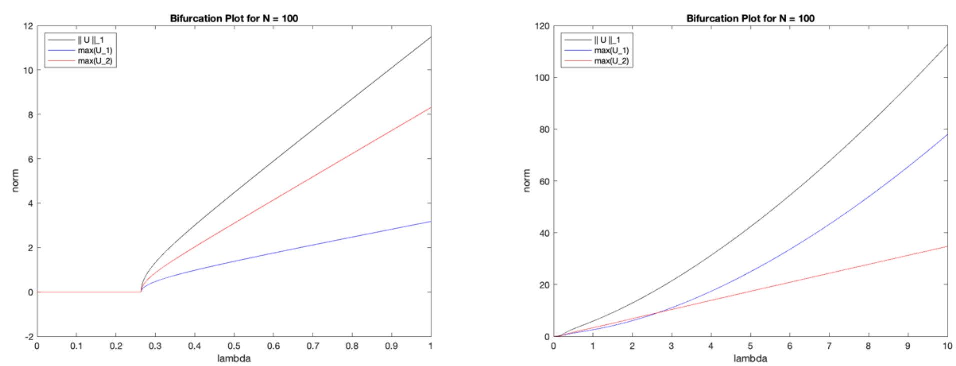



FIGURE 3. Graphs of $U_1$ and $U_2$ for $\lambda = 0.5$ and $\lambda = 3$.




Bifurcation diagram when $\lambda \in (0,1)$. Bifurcation diagram when $\lambda \in (0,10)$.

FIGURE 4. Bifurcation diagrams of (4.1) for different ranges of $\lambda$ when $f_1(u_1,u_2) = \arctan(u_2)$, $f_2(u_1,u_2) = 3\arctan(u_1)$.

4.5. **Example 3.** In this example, we consider the nonlinearities

$$f_1(u_1,u_2) = e^{\frac{10u_1}{1+u_1^2}} - 1 + 5\sqrt[3]{u_2^2+1},$$

$$f_2(u_1,u_2) = e^{\frac{u_2}{1+u_2^2}} - 1.$$

Clearly the $f_i$'s are Caratheodory, quasimonotone, non-decreasing, and sublinear functions satisfying the assumption (A2) and (A3) in Theorem 1.3. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2}(0,0) = \infty$ and $\frac{\partial f_2}{\partial u_1}(0,0) = 0$. Solutions can be found in Figure 5 for $\lambda = 0.5$ and $\lambda = 3$, and computed bifurcation diagrams can be found in Figure 6. Also in Figure 6, notice that there exists a range of $\lambda$ for which we see non coexistence of positive solutions. Our method and monotone solver found the maximal solution based on our supersolution. Observe that, in Subfigure 6(A), the vertical lines are jump discontinuities corresponding to finding maximal solutions. We used the method of continuation to find the exact branches of the bifurcation curves in Subfigure 6(B). When choosing the supersolution $U_h^{(0)}$ according to Remark 4.5, the solution for $\lambda = 0.5$ was found using $c = 512$ and the solution for $\lambda = 3$ was found using $c = 65536$.

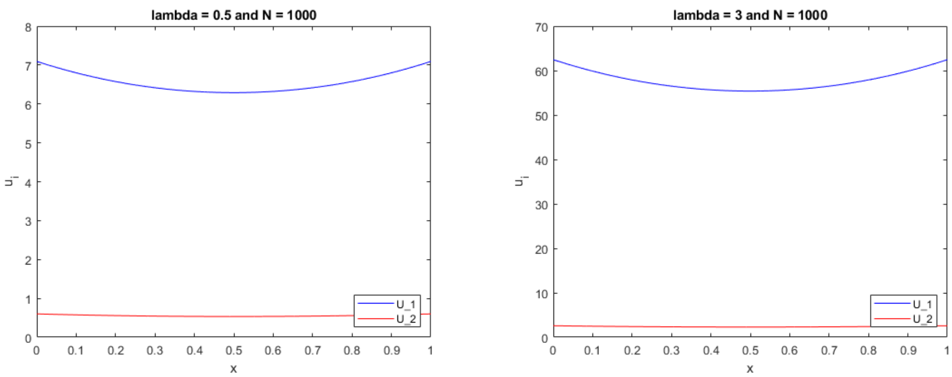
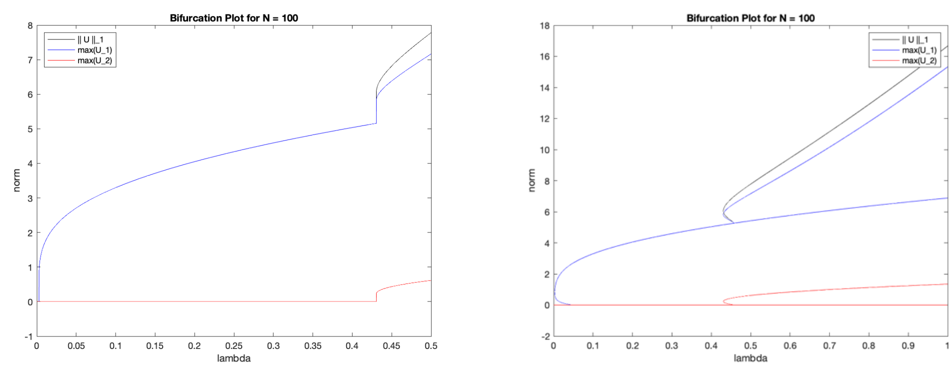



FIGURE 5. Graphs of $U_1$ and $U_2$ for $\lambda = 0.5$ and $\lambda = 3$.




Bifurcation diagram when $\lambda \in (0, 0.5)$. Bifurcation diagram when $\lambda \in (0, 1)$.

FIGURE 6. Bifurcation diagrams of (4.1) for different ranges of $\lambda$ when $f_1(u_1, u_2) = e^{\frac{10u_1}{1+u_1^2}} - 1 + 5\sqrt[3]{u_2^2 + 1}$, $f_2(u_1, u_2) = e^{\frac{u_2}{1+u_2^2}} - 1$.

4.6. **Example 4.** In this example, we consider the nonlinearities

$$f_1(u_1, u_2) = e^{\frac{u_1}{1+u_1^2}} - 1 + \frac{1}{3}\sqrt[3]{u_2^2 + 1},$$
$$f_2(u_1, u_2) = \arctan(u_2) + e^{\frac{u_2}{1+u_2^2}} - 1.$$

Clearly the $f_i$'s are Caratheodory, quasimonotone, non-decreasing, and sublinear functions satisfying the assumption (A2) and (A3) in Theorem 1.3. Moreover, $f_i(0,0) = 0$, $\frac{\partial f_1}{\partial u_2}(0,0) = \infty$ and $\frac{\partial f_2}{\partial u_1}(0,0) > 0$. Solutions can be found in Figure 7 for $\lambda = 0.5$ and $\lambda = 3$, and computed bifurcation diagrams can be found in Figure 8. In Figure 7, notice that for $\lambda = 0.5$, $u_2 > u_1$, whereas, for $\lambda = 3$, $u_1 > u_2$. When choosing the supersolution $U_h^{(0)}$ according to Remark 4.5, the solution for $\lambda = 0.5$ was found using $c = 1$ and the solution for $\lambda = 3$ was found using $c = 32$.

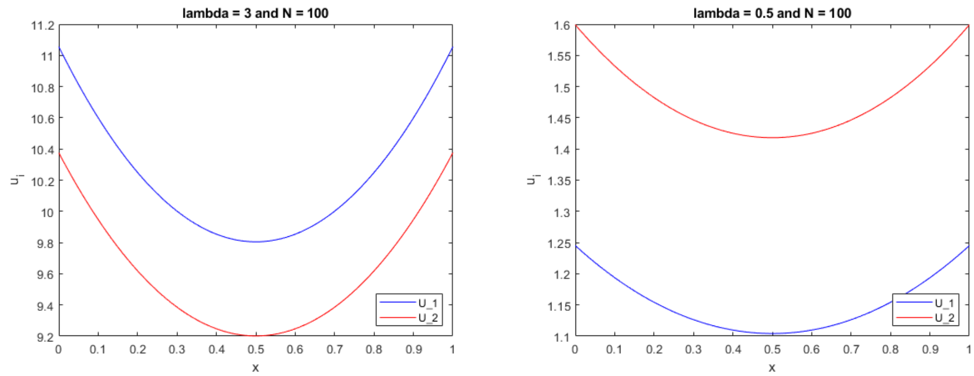
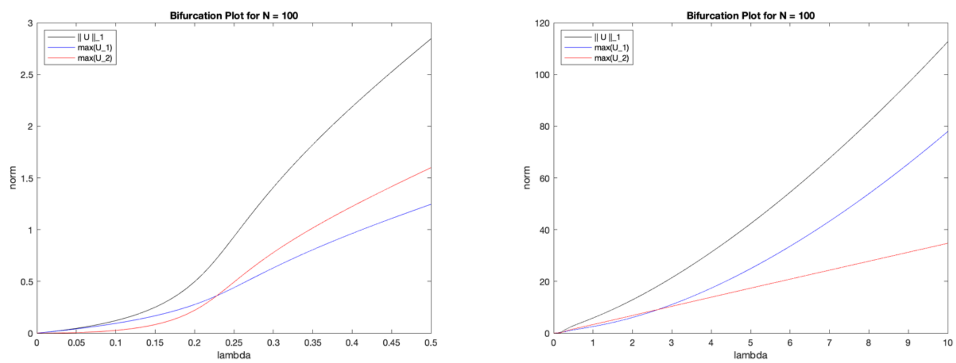



FIGURE 7. Graphs of $U_1$ and $U_2$ for $\lambda = 0.5$ and $\lambda = 3$.




Bifurcation diagram when $\lambda \in (0, 0.5)$. Bifurcation diagram when $\lambda \in (0, 10)$.

FIGURE 8. Bifurcation diagrams of (4.1) for different ranges of $\lambda$ when $f_1(u_1,u_2) = e^{\frac{u_1}{1+u_1^2}} - 1 + \frac{1}{3}\sqrt[3]{u_2^2+1}$, $f_2(u_1,u_2) = \arctan(u_2) + e^{\frac{u_2}{1+u_2^2}} - 1$.

## 5. Appendix

In this section we use Kato's inequality up to the boundary for single equations to prove that the componentwise maximum of two solutions of (1.1) is a subsolution of (1.1) (see Proposition 5.4) and componentwise minimum of two solutions of (1.1) is a supersolution of (1.1) (see Proposition 5.6), which we used in Step 4 of the proof of Theorem 1.4 (see Section 3). Now we have Kato's inequality up to the boundary for single equations.

**Proposition 5.1.** *Let $u_1$ and $u_2$ be functions in $H^1(\Omega)$ such that there exist $f_1$ and $f_2$ in $L^r(\partial\Omega)$, for $r \geq \frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i \nabla \psi + u_i \psi) \leq \int_{\partial\Omega} f_i \psi \quad \text{for all } 0 \leq \psi \in H^1(\Omega)\,, \tag{5.1}$$

*for $i = 1, 2$. Then, $u := \max\{u_1, u_2\}$ satisfies*

$$\int_\Omega (\nabla u \nabla \psi + u\psi) \leq \int_{\partial\Omega} f\psi \quad \text{for all } \ 0 \leq \psi \in H^1(\Omega)\,,$$

*where*

$$f(x) := \begin{cases} f_1(x) & \text{if } u_1(x) > u_2(x) \text{ a.e. } x \in \partial\Omega, \\ f_2(x) & \text{if } u_1(x) \leq u_2(x) \text{ a.e. } x \in \partial\Omega \end{cases}$$

For a proof of the above proposition, see [4, Theorem 2.4].

**Corollary 5.2.** *Let $u_1$ and $u_2$ be functions in $H^1(\Omega)$ such that there exist $f_1$ and $f_2$ in $L^r(\partial\Omega)$, for $r \geq \frac{2(N-1)}{N}$, satisfying*

$$\int_\Omega (\nabla u_i \nabla \psi + u_i \psi) \geq \int_{\partial\Omega} f_i \psi \quad \text{for all } 0 \leq \psi \in H^1(\Omega)\,,$$

*for $i = 1, 2$. Then, $u := \min\{u_1, u_2\}$ satisfies*

$$\int_\Omega (\nabla u \nabla \psi + u \psi) \geq \int_{\partial\Omega} f \psi, \quad \text{for all } 0 \leq \psi \in H^1(\Omega)\,,$$

*where*

$$f(x) := \begin{cases} f_1(x) & \text{if } u_1(x) < u_2(x) \text{ a.e. } x \in \partial\Omega \\ f_2(x) & \text{if } u_1(x) \geq u_2(x) \text{ a.e. } x \in \partial\Omega. \end{cases}$$

For a proof of the above corollary, see [4, Corollary 2.5]. Let $\mathcal{A}$ be a set consisting of $(w_1, w_2) \in (H^1(\Omega))^2$ such that there exists a subsolution $(\widetilde{u}_1, \widetilde{u}_2)$ of (1.1) satisfying

$$(\underline{u}_1, \underline{u}_2) \leq (\widetilde{u}_1, \widetilde{u}_2) \leq (w_1, w_2) \leq (\overline{u}_1, \overline{u}_2),$$

where $w_1$ and $w_2$ are solutions of (3.1) and (3.2), respectively, for the pair $(\widetilde{u}_1, \widetilde{u}_2)$.

**Lemma 5.3.** *Let $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2) \in \mathcal{A}$ be any two subsolutions of (1.1). Then, the pair $(\max\{\alpha_1, \beta_1\}, \max\{\alpha_2, \beta_2\})$ is a subsolution of (1.1).*

*Proof.* Since $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ belong to $\mathcal{A}$, there exist $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ and $(\widetilde{\beta}_1, \widetilde{\beta}_2)$ such that $\underline{u}_i \leq \widetilde{\alpha}_i \leq \alpha_i \leq \overline{u}_i$, $i = 1, 2$, and $\underline{u}_i \leq \widetilde{\beta}_i \leq \beta_i \leq \overline{u}_i$, $i = 1, 2$, which satisfy

$$\begin{gathered} -\Delta\alpha_1 + \alpha_1 = 0 \quad \text{in } \Omega, \\ \frac{\partial \alpha_1}{\partial \eta} = f_1(x, \alpha_1, \widetilde{\alpha}_2) \quad \text{on } \partial\Omega, \end{gathered} \tag{5.2}$$

$$\begin{gathered} -\Delta\alpha_2 + \alpha_2 = 0 \quad \text{in } \Omega, \\ \frac{\partial \alpha_2}{\partial \eta} = f_2(x, \widetilde{\alpha}_1, \alpha_2) \quad \text{on } \partial\Omega, \end{gathered} \tag{5.3}$$

$$\begin{gathered} -\Delta\beta_1 + \beta_1 = 0 \quad \text{in } \Omega, \\ \frac{\partial \beta_1}{\partial \eta} = f_1(x, \beta_1, \widetilde{\beta}_2) \quad \text{on } \partial\Omega, \end{gathered} \tag{5.4}$$

$$\begin{gathered} -\Delta\beta_2 + \beta_2 = 0 \quad \text{in } \Omega, \\ \frac{\partial \beta_2}{\partial \eta} = f_2(x, \widetilde{\beta}_1, \beta_2) \quad \text{on } \partial\Omega. \end{gathered} \tag{5.5}$$

We define $\gamma_1 := \max\{\alpha_1, \beta_1\}$ and $\gamma_2 := \max\{\alpha_2, \beta_2\}$. By the quasimonotonicity of $f_i$, the following inequalities hold:

$$\begin{gathered} f_1(x, \alpha_1, \widetilde{\alpha}_2) \leq f_1(x, \alpha_1, \gamma_2), \\ f_1(x, \beta_1, \widetilde{\beta}_2) \leq f_1(x, \beta_1, \gamma_2), \\ f_2(x, \widetilde{\alpha}_1, \alpha_2) \leq f_2(x, \gamma_1, \alpha_2), \\ f_2(x, \widetilde{\beta}_1, \beta_2) \leq f_2(x, \gamma_1, \beta_2). \end{gathered}$$

Let

$$g_1(x) := \begin{cases} f_1(x, \alpha_1, \gamma_2) & \text{if } \alpha_1(x) > \beta_1(x) \text{ a.e. } x \in \partial\Omega, \\ f_1(x, \beta_1, \gamma_2) & \text{if } \alpha_1(x) \leq \beta_1(x) \text{ a.e. } x \in \partial\Omega \end{cases}$$

and

$$g_2(x) := \begin{cases} f_2(x, \gamma_1, \alpha_2) & \text{if } \alpha_2(x) > \beta_2(x) \text{ a.e. } x \in \partial\Omega, \\ f_2(x, \gamma_1, \beta_2) & \text{if } \alpha_2(x) \leq \beta_2(x) \text{ a.e. } x \in \partial\Omega. \end{cases}$$

Notice that $\alpha_1$ and $\beta_1$ are subsolutions of

$$\begin{gathered}-\Delta v+v=0 \quad \text{in } \Omega,\\ \frac{\partial v}{\partial \eta}=g_1(x) \quad \text{on } \partial\Omega,\end{gathered} \tag{5.6}$$

and $\alpha_2$ and $\beta_2$ are subsolutions of

$$\begin{gathered}-\Delta v+v=0 \quad \text{in } \Omega,\\ \frac{\partial v}{\partial \eta}=g_2(x) \quad \text{on } \partial\Omega.\end{gathered} \tag{5.7}$$

Then, by Proposition 5.1, along with equations (5.6) and (5.7), we have that $(\gamma_1,\gamma_2)$ is a subsolution of (1.1). Thus, the pair $(\max\{\alpha_1,\beta_1\},\max\{\alpha_2,\beta_2\})$ is a subsolution of (1.1) which completes the proof. □

**Proposition 5.4.** *Suppose that $(u_1,u_2)$ and $(v_1,v_2)$ are two solutions of (1.1) such that $(\underline{u}_1,\underline{u}_2)\le(u_1,u_2)$, $(v_1,v_2)\le(\overline{u}_1,\overline{u}_2)$, where $(\underline{u}_1,\underline{u}_2)$ and $(\overline{u}_1,\overline{u}_2)$ are sub- and supersolutions of (1.1), respectively. Then $(\max\{u_1,v_1\},\max\{u_2,v_2\})$ is a subsolution of (1.1).*

*Proof.* Observe that any solution $(u_1,u_2)$ of (1.1) such that $(\overline{u}_1,\overline{u}_2)\le(u_1,u_2)\le(\underline{u}_1,\underline{u}_2)$ belongs to $\mathcal{A}$ as $(u_1,u_2)$ is a solution of (3.1) and (3.2) for the pair $(u_1,u_2)$. The rest of the proof follows from Lemma 5.3. □

Next, we show that if $(u_1,u_2)$ and $(v_1,v_2)$ are two solutions of (1.1), then the pair of functions $(\min\{u_1,v_1\},\min\{u_2,v_2\})$ is a supersolution of (1.1). For that purpose, consider the equations

$$\begin{gathered}-\Delta u_1+u_1=0 \quad \text{in } \Omega,\\ \frac{\partial u_1}{\partial \eta}=f_1(x,u_1,\widehat{u}_2) \quad \text{on } \partial\Omega,\end{gathered} \tag{5.8}$$

and

$$\begin{gathered}-\Delta u_2+u_2=0 \quad \text{in } \Omega,\\ \frac{\partial u_2}{\partial \eta}=f_2(x,\widehat{u}_1,u_2) \quad \text{on } \partial\Omega.\end{gathered} \tag{5.9}$$

Let $\mathcal{B}$ consisting of $(w_1,w_2)\in(H^1(\Omega))^2$ such that there exists a supersolution $(\widehat{u}_1,\widehat{u}_2)$ of (1.1) satisfying

$$(\underline{u}_1,\underline{u}_2)\le(w_1,w_2)\le(\widehat{u}_1,\widehat{u}_2)\le(\overline{u}_1,\overline{u}_2), \tag{5.10}$$

where $w_1$ and $w_2$ are solutions of (5.8) and (5.9), respectively, for the pair $\widehat{u}_1,\widehat{u}_2$.

**Lemma 5.5.** *Let $(\alpha_1,\alpha_2)$ and $(\beta_1,\beta_2)\in\mathcal{B}$ be any two supersolutions of (1.1). Then the pair $(\min\{\alpha_1,\beta_1\},\min\{\alpha_2,\beta_2\})$ is a supersolution of (1.1).*

*Proof.* Since $(\alpha_1,\alpha_2)$ and $(\beta_1,\beta_2)$ belong to $\mathcal{B}$, there exist $(\widehat{\alpha}_1,\widehat{\alpha}_2)$ and $(\widehat{\beta}_1,\widehat{\beta}_2)$ such that $\underline{u}_i\le\alpha_i\le\widehat{\alpha}_i\le\overline{u}_i$, $i=1,2$ and $\underline{u}_i\le\beta_i\le\widehat{\beta}_i\le\overline{u}_i$, $i=1,2$, which satisfy

$$\begin{gathered}-\Delta\alpha_1+\alpha_1=0 \quad \text{in } \Omega,\\ \frac{\partial \alpha_1}{\partial \eta}=f_1(x,\alpha_1,\widehat{\alpha}_2) \quad \text{on } \partial\Omega,\end{gathered} \tag{5.11}$$

$$\begin{gathered}-\Delta\alpha_2+\alpha_2=0 \quad \text{in } \Omega,\\ \frac{\partial \alpha_2}{\partial \eta}=f_2(x,\widehat{\alpha}_1,\alpha_2) \quad \text{on } \partial\Omega,\end{gathered} \tag{5.12}$$

$$\begin{gathered}-\Delta\beta_1+\beta_1=0 \quad \text{in } \Omega\\ \frac{\partial \beta_1}{\partial \eta}=f_1(x,\beta_1,\widehat{\beta_2}) \quad \text{on } \partial\Omega,\end{gathered} \tag{5.13}$$

$$\begin{gathered}-\Delta\beta_2+\beta_2=0 \quad \text{in } \Omega,\\ \frac{\partial \beta_2}{\partial \eta}=f_2(x,\widehat{\beta_1},\beta_2) \quad \text{on } \partial\Omega.\end{gathered} \tag{5.14}$$

Now, define $\gamma_1 := \min\{\alpha_1, \beta_1\}$ and $\gamma_2 := \min\{\alpha_2, \beta_2\}$. Observe that the quasimonotonicity of $f_i$ leads to the inequalities

$$\begin{gathered} f_1(x, \alpha_1, \widehat{\alpha}_2) \geq f_1(x, \alpha_1, \gamma_2), \\ f_1(x, \beta_1, \widehat{\beta}_2) \geq f_1(x, \beta_1, \gamma_2) \end{gathered} \tag{5.15}$$

$$\begin{gathered} f_2(x, \widehat{\alpha}_1, \alpha_2) \geq f_2(x, \gamma_1, \alpha_2), \\ f_2(x, \widehat{\beta}_1, \beta_2) \geq f_2(x, \gamma_1, \beta_2). \end{gathered} \tag{5.16}$$

Let

$$g_1(x) := \begin{cases} f_1(x, \alpha_1, \gamma_2) & \text{if } \alpha_1(x) < \beta_1(x) \text{ a.e. } x \in \partial\Omega, \\ f_1(x, \beta_1, \gamma_2) & \text{if } \alpha_1(x) \geq \beta_1(x) \text{ a.e. } x \in \partial\Omega, \end{cases}$$

and

$$g_2(x) := \begin{cases} f_2(x, \gamma_1, \alpha_2) & \text{if } \alpha_2(x) < \beta_2(x) \text{ a.e. } x \in \partial\Omega, \\ f_2(x, \gamma_1, \beta_2) & \text{if } \alpha_2(x) \geq \beta_2(x) \text{ a.e. } x \in \partial\Omega. \end{cases}$$

Notice that $\alpha_1$ and $\beta_1$ are supersolutions of

$$\begin{gathered} -\Delta v + v = 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = g_1(x) \quad \text{on } \partial\Omega, \end{gathered} \tag{5.17}$$

and $\alpha_2$ and $\beta_2$ are supersolutions of the following:

$$\begin{gathered} -\Delta v + v = 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = g_2(x) \quad \text{on } \partial\Omega. \end{gathered} \tag{5.18}$$

Then, by Corollary 5.2, along with equations (5.17) and (5.18), we have that $(\gamma_1, \gamma_2)$ is a supersolution of (1.1). This completes the proof. □

**Proposition 5.6.** *Suppose that $(u_1, u_2)$ and $(v_1, v_2)$ are two solutions of (1.1) such that $(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2)$, $(v_1, v_2) \leq (\overline{u}_1, \overline{u}_2)$, where $(\underline{u}_1, \underline{u}_2)$ and $(\overline{u}_1, \overline{u}_2)$ are sub- and supersolutions of (1.1), respectively. Then $(\min\{u_1, v_1\}, \min\{u_2, v_2\})$ is a supersolution of (1.1).*

*Proof.* Observe that any solution $(u_1, u_2)$ of (1.1) such that $(\overline{u}_1, \overline{u}_2) \leq (u_1, u_2) \leq (\underline{u}_1, \underline{u}_2)$ belongs to $\mathcal{B}$. The rest of the proof follows from Lemma 5.5. □

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