2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Complexity bounds for the maximal admissible set of LTI systems with disturbances

Hamid R. Ossareh, *Senior Member, IEEE,* Ilya Kolmanovsky *Fellow, IEEE*

2024 IEEE 63rd Conference on Decision and Control (CDC) | 979-8-3503-1633-9/24/$31.00 ©2024 IEEE | DOI: 10.1109/CDC56724.2024.10886858

***Abstract*— The maximal admissible set (MAS) of a dynamical system characterizes the set of all initial conditions and constant inputs for which the ensuing response satisfies the specified state/output constraints for all time. For a discrete-time, linear time-invariant (LTI) system subject to polytopic constraints and unknown bounded disturbances, the MAS is known to be a polytope, which may not be finitely determined (i.e., it may not be defined by a finite number of inequalities). Thus, the steady-state constraint is usually tightened, which results in a finitely-determined inner approximation of the MAS. However, the complexity of this approximation is not known *a priori* from problem data. This paper presents and compares two computationally efficient methods, based on matrix power series and on quadratic ISS-Lyapunov functions, respectively, to upper bound the complexity of the MAS. The bounds may facilitate the online computation of the MAS and the implementation of robust reference governors and model predictive controllers.**

I. Introduction

Given a dynamical system perturbed by unmeasured bounded disturbances and subject to state/output constraints, the maximal admissible set (MAS) characterizes the set of all initial conditions and constant control inputs (in applications, the inputs may represent reference commands/set-points) the response to which satisfies the constraints for all time. The characterizations of the MAS as defined here (i.e., including constant inputs) were studied in [1] for the case of disturbance-free LTI systems and in [2] for the case of LTI systems with set-bounded disturbance inputs. The MAS is used in the construction of reference/command governors for systems with constraints (see, e.g., [3], [4]). It is also used in Model Predictive Control (MPC) (see, e.g., [5], [6]) and in set-theoretic control (see, e.g., [7], [8]). Various simplifications of the MAS, its applications, and its extensions to other classes of systems have been studied, e.g., in [9]–[20].

The MAS may not be finitely determined (see [1] and [2]), i.e., it may not be described by a finite number of inequalities; however, by tightening the steady-state constraint, a finitely-determined inner approximation of the MAS, which satisfies similar invariance properties as the original MAS, can be computed. It is known that the complexity of this approximation, i.e., the number of inequalities in the set description, is not known *a priori* from the problem data. To find it, one would need to construct the MAS iteratively by adding inequalities one time-step at a time and check for redundancy of the newly added inequalities. Once all the newly added inequalities are redundant, MAS has been fully characterized. In the case of linear systems with linear constraints, the redundancy check can be performed using linear programming (LP) which may be computationally demanding for high-dimensional systems, those with slow dynamics, and those with constraint sets of high complexity. For many applications, solving these LPs is computationally prohibitive in situations where the MAS must be computed online in real-time to accommodate changing models or constraints.

To address the above issues, this paper proposes two computationally-efficient methods to obtain *upper bounds* on the complexity of the MAS for linear systems with linear constraints and set-bounded disturbance inputs. These upper bounds can aid in constructing the MAS without the redundancy checking step, thereby greatly speeding up the computation of MAS (at the expense of having potentially redundant inequalities in its description). This capability is particularly useful in a setting where the MAS must be computed online.

The first method for finding the upper bound leverages matrix power series and the Cayley Hamilton expansion to express the output at a time t as a linear combination of outputs at previous times, which helps determine the time-step after which the constraints become redundant. The second method relies on the decay rate of a quadratic ISS-Lyapunov function towards its constraint-admissible and robustly invariant level set. This method is inspired by the existing literature (see e.g., [4], [21]); however, it is presented here in complete details with explicit bounds and serves as a benchmark against which the efficacy of the first method is evaluated. Both methods were proposed for the disturbance-free case in [22], [23]; the treatment of the systems with disturbances entail several distinct and intricate details such as the existence and construction of MAS. We thus present the theoretical justification for the two methods and provide corresponding algorithms for their computations. We compare the upper bounds obtained from both methods using a numerical study involving systems chosen at random. Our findings show that the second method is not universally applicable, as a constraint admissible, robustly invariant level set may not exist, while the first method finds the upper bound in all cases in which MAS exists. In instances where both methods yield an upper bound, the first method's upper bound is tighter and hence preferred to that of the second

H. R. Ossareh is with the Department of Electrical and Biomedical Engineering, University of Vermont, Burlington, VT USA 05405. I. Kolmanovsky is with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI USA 48105. e-mails: hossareh@uvm.edu, ilya@umich.edu.

This work was partially supported by the National Science Foundation awards CMMI-2238424 and ECCS-1931738.

method.

This paper is organized as follows. In Section II, we review the MAS and state the assumptions. Methods 1 and 2 are presented in Sections III and IV, respectively. The two methods are compared in Section V. Section VI contains concluding remarks.

The paper uses the following notations: $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, and $\mathbb{C}$ denote the sets of non-negative integers, real numbers, n-dimensional vectors of real numbers, $n \times n$ matrices with real entries, and complex numbers, respectively. For a symmetric matrix $P = P^\mathsf{T}$, we say it is positive definite and write $P \succ 0$ if all the eigenvalues of P are strictly positive. We use the variables $t \in \mathbb{Z}^+$, $t^* \in \mathbb{Z}^+$, and $m \in \mathbb{Z}^+$ to denote the discrete time index, the finite determination index (which determines the complexity) of the MAS which is defined in Section II, and the upper bound on the finite determination index, respectively. Given two sets $U, V \subset \mathbb{R}^n$, their Minkowski sum and Pontryagin difference (P-subtraction) are defined, respectively, as follows: $U \oplus V = \{w : w = u + v, u \in U, v \in V\}$ and $U \sim V = \{u : u + v \in U, v \in V\}$. It follows [2] that if $0 \in V$, then $U \sim V \subset U$. Furthermore, if U is compact then so is $U \sim V$.

II. Preliminaries and Problem Statement

In this section, we introduce the maximal admissible set (MAS). We then review its computation and complexity, and state the main assumptions that we use in the paper.

Consider a discrete-time linear time-invariant system described by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_w w(t) \\ y(t) &= Cx(t) + Du(t) + D_w w(t) \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}^+$ is the discrete time index, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^m$, and $w(t) \in \mathbb{R}^s$ are, respectively, the state, output, input, and disturbance vectors.

When computing the MAS for use with reference/command governors, the input $u(t)$ is assumed to be constant (i.e., $u(t) = u, \forall t \in \mathbb{Z}^+$), while the assumption $u = 0$ (or equivalently $B = D = 0$) is made when computing the MAS for use as a terminal set in MPC. In both such scenarios, matrix A is stable as it corresponds to a pre-stabilized system with nominal/terminal control law. This assumption will be made throughout this paper.

Furthermore, we assume that the disturbance belongs to a compact polytope $\mathbb{W}$, i.e.,

$$w(t) \in \mathbb{W}$$

but is otherwise unknown, and the output constraint is defined as

$$y(t) \in \mathbb{Y} \tag{2}$$

where $\mathbb{Y}$ is a compact polytope. We assume that the origin is in the interiors of both $\mathbb{W}$ and $\mathbb{Y}$.

The MAS is defined as the set of all initial conditions, x_0, and constant inputs, u, for which (2) is satisfied for all time, that is:

$$O_\infty = \{(x_0, u) : y(t) \in \mathbb{Y}, \quad \forall w(t) \in \mathbb{W}, \forall t \geq 0\} \tag{3}$$

Under the assumptions of stability and observability of (1), this set is a compact polytope, but it may not be finitely-determined (i.e., may not be described by a finite number of inequalities). However, a finitely-determined inner approximation of it with similar invariance properties as O_∞ can be readily obtained as follows. Let

$$z(t) = x(t) - (I - A)^{-1}Bu$$

where the inverse exists due to the stability of (1). In the new coordinate system, u is eliminated from the state evolution and the dynamics are described by

$$\begin{aligned} z(t+1) &= Az(t) + B_w w(t) \\ y(t) &= Cz(t) + H_0 u + D_w w(t) \end{aligned} \tag{4}$$

where $H_0 = C(I - A)^{-1}B + D$ is the DC gain from u to y. The evolution of the output $y(t), t \geq 1$, is then given by

$$y(t) = CA^t z(0) + H_0 u + D_w w(t) + \sum_{i=0}^{t-1} CA^i B_w w(t-1-i) \tag{5}$$

We can now use the P-subtraction to simplify the description of the MAS in (3). To this end, define the sets $\mathbb{Y}_t$ using the following recursion:

$$\mathbb{Y}_{t+1} = \mathbb{Y}_t \sim CA^t B_w \mathbb{W} \tag{6}$$

$$\mathbb{Y}_0 = \mathbb{Y} \sim D_w \mathbb{W} \tag{7}$$

Using these sets and Equation (5), the MAS can be stated in the z-coordinates as follows:

$$\overline{O}_\infty = \{(z_0, u) : CA^t z_0 + H_0 u \in \mathbb{Y}_t, \quad \forall t \geq 0\}$$

As mentioned previously, this set is generally not finitely determined. However, a finitely-determined, robustly invariant inner approximation, denoted by $\widetilde{O}_\infty$, can be obtained [2] by imposing a tightened-version of the constraint on the steady-state output of the system. Let

$$\mathbb{Y}_\infty = \bigcap \mathbb{Y}_t$$

and consider its compact inner approximation, defined by

$$\widetilde{\mathbb{Y}}_\infty \subset (1 - \epsilon)\mathbb{Y}_\infty \tag{8}$$

where $\epsilon \in (0, 1)$. To obtain $\widetilde{\mathbb{Y}}_\infty$, one can employ an algorithm similar to Algorithm 1 of [10] to obtain an inner approximation of $\mathbb{Y}_\infty$ and then tighten it by $(1 - \epsilon)$. Note that ϵ quantifies a lower bound on the "gap" between $\mathbb{Y}_\infty$ and $\widetilde{\mathbb{Y}}_\infty$. This gap is needed to ensures that $\widetilde{O}_\infty$ is finitely-determined. Using $\widetilde{\mathbb{Y}}_\infty$, the following finitely-determined inner approximation of MAS can be obtained:

$$\begin{aligned} \widetilde{O}_\infty = \{&(z_0, u) : H_0 u \in \widetilde{\mathbb{Y}}_\infty, \\ &CA^t z_0 + H_0 u \in \mathbb{Y}_t,\ t = 0, \ldots, t^*\} \end{aligned}$$

where t^*, referred to as the *finite determination index*, is the last "prediction time-step" required to fully characterize $\widetilde{O}_\infty$. As mentioned in Section I, t^* is not known *a priori*. To find it, one would need to construct $\widetilde{O}_\infty$ iteratively by adding inequalities one time-step at a time and checking for redundancy of the newly added inequality. Thus, the problem addressed in this paper can be formally stated as follows: given the system matrices and the sets $\mathbb{Y}$, $\mathbb{W}$, and $\widetilde{\mathbb{Y}}_\infty$, find an integer $m \in \mathbb{Z}^+$, such that $m \geq t^*$. We present two computationally efficient methods in the next two sections to solve this problem.

To simplify the presentation, we assume in this paper that the output is a scalar (i.e., $q = 1$). The MAS for multi-output systems can be viewed as the intersection of the MAS's for each of the individual outputs. As such, m can be calculated for each output and the largest one may be selected.

To conclude this section, we summarize our assumptions:

Assumption 1: System (1) is asymptotically stable, the pair (A, C) is observable, the input u is constant for all time, the output y is a scalar ($q = 1$), $0 \in \text{int}\widetilde{\mathbb{Y}}_\infty$, and $\mathbb{W}$ is a compact polytope that satisfies $0 \in \text{int}\mathbb{W}$. Furthermore, the constraint set is given by the interval:

$$\mathbb{Y} := \{y : -y^l \leq y \leq y^u\} \tag{9}$$

where $y^l > 0$ and $y^u > 0$ define the lower and upper limits, respectively.

With the form (9), we can express each $\mathbb{Y}_t$ in (6)-(7) as:

$$\mathbb{Y}_t = \{y_t : -y_t^l \leq y_t \leq y_t^u\}$$

where the bounds satisfy the recursion:

$$y_t^u = y_{t-1}^u - \max_{w\in\mathbb{W}} CA^{t-1}B_w w, \quad y_0^u = y^u - \max_{w\in\mathbb{W}} D_w w$$

$$y_t^l = y_{t-1}^l + \min_{w\in\mathbb{W}} CA^{t-1}B_w w, \quad y_0^l = y^l + \min_{w\in\mathbb{W}} D_w w$$

Clearly, we have that $y_t^u \leq y_{t-1}^u$ and $y_t^l \leq y_{t-1}^l$ for any t. Furthermore, we can express $\mathbb{Y}_\infty$ and $\widetilde{\mathbb{Y}}_\infty$ as:

$$\mathbb{Y}_\infty = \{y_\infty : -y_\infty^l \leq y_\infty \leq y_\infty^u\}$$

$$\widetilde{\mathbb{Y}}_\infty = \{\widetilde{y}_\infty : -\widetilde{y}_\infty^l \leq \widetilde{y}_\infty \leq \widetilde{y}_\infty^u\}$$

where $y_\infty^u := \lim_{t\to\infty} y_t^u$ and $y_\infty^l := \lim_{t\to\infty} y_t^l$. For the rest of this paper, we assume that y_t^u, y_t^l, $\widetilde{y}_\infty^l$, and $\widetilde{y}_\infty^u$ are all positive and available. Availability of these parameters is a reasonable assumption as they are also required for the computation of the MAS.

III. Upper bound using Method 1

The general idea behind our first method is to expand A^t in (5) in terms of lower powers of A. As we show, if there exists an integer m such that the sum of the coefficients in the expansion of A^{m+1} is "sufficiently small," then m is an upper bound on t^*. We then show that such an expansion always exists thanks to the Cayley Hamilton Theorem. We begin by stating the main result of this section.

Theorem 1: Consider system (1) and suppose Assumption 1 holds. Suppose there exists an integer m, $m \geq 0$, such that A^{m+1} can be expanded as

$$A^{m+1} = \sum_{i=0}^{m} \alpha_i A^i \tag{10}$$

where α_i satisfy:

$$\begin{aligned} &\sum_i \alpha_i < 1 \\ &\sum_{\alpha_i>0} \alpha_i \left(\frac{y_i^u}{\widetilde{y}_\infty^u} - 1\right) - \sum_{\alpha_i<0} \alpha_i \left(\frac{y_i^l}{\widetilde{y}_\infty^u} + 1\right) \leq \frac{\epsilon}{1-\epsilon} \\ &\sum_{\alpha_i>0} \alpha_i \left(\frac{y_i^l}{\widetilde{y}_\infty^l} - 1\right) - \sum_{\alpha_i<0} \alpha_i \left(\frac{y_i^u}{\widetilde{y}_\infty^l} + 1\right) \leq \frac{\epsilon}{1-\epsilon} \end{aligned} \tag{11}$$

Then, m is an upper bound on the finite determination index, t^*; that is, $t^* \leq m$.

Proof: We use mathematical induction to prove that, for any given initial condition x_0 (or z_0 in the transformed coordinates) and constant input u satisfying $H_0 u \in \widetilde{\mathbb{Y}}_\infty$, we have that: $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $y(t) \in \mathbb{Y}$ for $t \geq m+1$, which means that m is an upper bound on t^*.

For the induction base case, we assume that $y(t) \in \mathbb{Y}$ (i.e., $CA^t z_0 + H_0 u \in \mathbb{Y}_t$) for $t \leq m$ and show that $y(m+1) \in \mathbb{Y}$ (i.e., $CA^{m+1}z_0 + H_0 u \in \mathbb{Y}_{m+1}$). To show this, write:

$$\begin{aligned} CA^{m+1}z_0 + H_0 u &= \sum_{i=0}^{m} \alpha_i (CA^i z_0) + H_0 u \\ &= \sum_{i=0}^{m} \alpha_i (CA^i z_0 + H_0 u) + \left(1 - \sum_{i=0}^{m} \alpha_i\right) H_0 u \end{aligned} \tag{12}$$

where we have added and subtracted $\sum_{i=0}^m \alpha_i H_0 u$ in the last equality. The assumption that $\sum_i \alpha_i < 1$ and $H_0 u \in \widetilde{\mathbb{Y}}_\infty$ imply that the rightmost term satisfies:

$$-\left(1 - \sum_{i=0}^{m} \alpha_i\right) \widetilde{y}_\infty^l \leq \left(1 - \sum_{i=0}^{m} \alpha_i\right) H_0 u \leq \left(1 - \sum_{i=0}^{m} \alpha_i\right) \widetilde{y}_\infty^u$$

Furthermore, the assumption that $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $-y_i^l \leq CA^i x_0 + H_0 u \leq y_i^u$. Thus, breaking up the sum in (12) into positive and negative values of α_i, we obtain the following bounds for $CA^{m+1}z_0 + H_0 u$:

$$-\sum_{\alpha_i>0} \alpha_i y_i^l + \sum_{\alpha_i<0} \alpha_i y_i^u - \left(1 - \sum_{i=0}^{m} \alpha_i\right) \widetilde{y}_\infty^l \leq CA^{m+1}z_0 + H_0 u$$

$$\leq \sum_{\alpha_i>0} \alpha_i y_i^u - \sum_{\alpha_i<0} \alpha_i y_i^l + \left(1 - \sum_{i=0}^{m} \alpha_i\right) \widetilde{y}_\infty^u$$

The above, together with the bottom two conditions in (11), results in:

$$-\frac{1}{1-\epsilon}\widetilde{y}_\infty^l \leq CA^{m+1}z_0 + H_0 u \leq \frac{1}{1-\epsilon}\widetilde{y}_\infty^u$$

Finally, (8) implies that $-y_\infty^l \leq CA^{m+1}z_0 + H_0 u \leq y_\infty^u$, and since $y_\infty^u \leq y_m^u$ and $y_\infty^l \leq y_m^l$, the result follows.

To prove the induction main step, we assume $y(t) \in \mathbb{Y}$ for $t \leq k$, where $k \geq m+1$, and show that $y(k+1) \in \mathbb{Y}$. We proceed as we did in the base case, but this time decompose $A^{k+1} = A^{m+1}A^{k-m}$:

$$CA^{k+1}z_0 + H_0 u = CA^{m+1}A^{k-m}z_0 + H_0 u$$

$$= \sum_{i=0}^{m} \alpha_i (CA^{i+k-m} z_0 + H_0 u) + \left(1 - \sum_{i=0}^{m} \alpha_i\right) H_0 u$$

The assumption $y(t) \in \mathbb{Y}$ for $t \le k$ together with $0 \le i + k - m \le k$ imply that $CA^{i+k-m}z_0 + H_0 u$ in the above sum satisfies: $-y^l_{i+k-m} \le CA^{i+k-m}z_0 + H_0 u \le y^u_{i+k-m}$. Furthermore, $y^u_{i+k-m} \le y^u_i$ and $y^l_{i+k-m} \le y^l_i$, which implies that $-y^l_i \le CA^{i+k-m}z_0 + H_0 u \le y^u_i$. The rest of the proof from this point on follows the same arguments as in the induction base case. This concludes the proof. ■

Remark 1: For the case of systems without control inputs (i.e., $u = 0$), the tightening of steady-state constraint is not required in the definition of MAS. Thus, the first condition in (11) is not necessary (this condition is only used to handle the steady-state constraint in the proof of the theorem). For such a case, condition (11) can be restated as:

$$\begin{aligned} \sum_{\alpha_i>0} \alpha_i \frac{y^u_i}{\tilde{y}^u_\infty} - \sum_{\alpha_i<0} \alpha_i \frac{y^l_i}{\tilde{y}^u_\infty} \le 1 \\ \sum_{\alpha_i>0} \alpha_i \frac{y^l_i}{\tilde{y}^l_\infty} - \sum_{\alpha_i<0} \alpha_i \frac{y^u_i}{\tilde{y}^l_\infty} \le 1 \end{aligned} \tag{13}$$

Remark 2: For the case of systems without disturbances (i.e., $w = 0$), we have that $y^l_i = y^l_\infty = y^l$ and $y^u_i = y^u_\infty = y^u$. Furthermore, $\tilde{y}^u_\infty$ and $\tilde{y}^l_\infty$ can be simply chosen as $\tilde{y}^u_\infty = (1-\epsilon)y^u$ and $\tilde{y}^l_\infty = (1-\epsilon)y^l$. Thus, condition (11) can be restated as:

$$\begin{aligned} &\sum_i \alpha_i < 1 \\ &\sum_{\alpha_i<0} |\alpha_i| \left(\frac{y^l}{y^u(1-\epsilon)} + 1\right) \le \frac{\epsilon}{1-\epsilon} \\ &\sum_{\alpha_i<0} |\alpha_i| \left(\frac{y_u}{y^l(1-\epsilon)} + 1\right) \le \frac{\epsilon}{1-\epsilon} \end{aligned} \tag{14}$$

We now prove the existence of, and a develop a method to construct, the expansion in (10) satisfying condition (11). Recall that the characteristic polynomial of any square matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\Delta(s) := \det(sI - A)$, which can be written as:

$$\Delta(s) = s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 \tag{15}$$

The Cayley Hamilton theorem (see [24]) states that any square matrix satisfies its own characteristic polynomial, i.e., $\Delta(A) = 0$. This result allows us to express A^t, for any $t \ge n$, as a finite power series in lower powers of A. Specifically, A^n can be expanded as:

$$A^n = -c_0 I - c_1 A - \ldots - c_{n-1}A^{n-1} \tag{16}$$

where c_i are the coefficients in (15) and are uniquely defined. Similarly, A^{n+1} can be expanded in the same powers of A:

$$\begin{aligned} A^{n+1} = A(A^n) &= -c_0 A - \ldots - c_{n-2}A^{n-1} - c_{n-1}A^n \\ &= (c_0 c_{n-1})I + (-c_0 + c_1 c_{n-1})A + \ldots + \\ &\quad (-c_{n-2} + c_{n-1}c_{n-1})A^{n-1} \end{aligned}$$

Generalizing the above to any $t \ge n$:

$$A^t = \sum_{i=0}^{n-1} \beta_i(t) A^i \tag{17}$$

where $\beta_i(t)$ denotes the i-th coefficient in the expansion of the t-th power of A. Note that expansion of A^t in lower powers of A is generally not unique, but $\beta_i(t)$ in (17) are, by construction, uniquely defined.

To simplify the presentation, we stack the coefficients of the t-th power into a vector and denote it by $\beta(t)$:

$$\beta(t) = [\beta_0(t) \cdots \beta_{n-1}(t)]^T \in \mathbb{R}^n$$

The following fact characterizes $\beta(t)$ and its convergence properties as $t \to \infty$.

Theorem 2: ([23]) Let $A \in \mathbb{R}^{n\times n}$ be any square matrix and let $\beta(t), t \ge n$, be the vector of coefficients in the expansion of A^t, as defined above. Then, $\beta(t)$ satisfies the recursion:

$$\beta(t+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix} \beta(t) \tag{18}$$

with initial condition $\beta(n) = [-c_0 \ \cdots \ -c_{n-1}]^T$. Also, if A is asymptotically stable, then $\lim_{t\to\infty} \beta(t) = 0$.

The above theorem guarantees the existence of an integer m such that the coefficients of the expansion of A^{m+1} as defined in (10) satisfy condition (11). To see this, compute $\beta(t)$ using the recursion in (18) for increasing t starting from $t = n$, and stop when

$$\begin{aligned} &\sum_i \beta_i < 1 \\ &\sum_{\beta_i>0} \beta_i \left(\frac{y^u_i}{\tilde{y}^u_\infty} - 1\right) - \sum_{\beta_i<0} \beta_i \left(\frac{y^l_i}{\tilde{y}^u_\infty} + 1\right) \le \frac{\epsilon}{1-\epsilon} \\ &\sum_{\beta_i>0} \beta_i \left(\frac{y^l_i}{\tilde{y}^l_\infty} - 1\right) - \sum_{\beta_i<0} \beta_i \left(\frac{y^u_i}{\tilde{y}^l_\infty} + 1\right) \le \frac{\epsilon}{1-\epsilon} \end{aligned} \tag{19}$$

Note that such t always exists, because according to Theorem 2, $\beta(t) \to 0$ as $t \to \infty$ and thus the left hand side of (19) can be made arbitrarily small. Such t corresponds to $m+1$ in Theorem 1, where the α_i in (11) are related to $\beta_i(t)$ in (19) as follows: $\alpha_i = \beta_i(t)$ for $i = 0, \ldots, n-1$ and $\alpha_i = 0$ for $i = n, \ldots, m$. This leads to Algorithm 1 for finding an upper bound for t^*. Note that the algorithm does not explicitly use B, B_w, C, D, or D_w. However, these matrices are required for the computation of $y^l_i, y^u_i, \tilde{y}^l_\infty$, and $\tilde{y}^u_\infty$.

Algorithm 1 Compute upper bound on t^* using Method 1

Input: $A, y^l_i, y^u_i, \tilde{y}^l_\infty, \tilde{y}^u_\infty, \epsilon$
Output: m **such that** $t^* \le m$

1: Compute the Cayley Hamilton coefficients, c_i, using (15). Set $t = n$ and initialize $\beta(n)$ as in Theorem 2.
2: If $\beta(t)$ satisfies (19), then: $m = t - 1$, STOP.
3: Increment t by 1. Compute $\beta(t)$ using (18). Go to step 2.

Remark 3: The Cayley Hamilton-based expansion in (17) provides only one possible expansion for A^{m+1} in Theorem 1. There may be other expansions that lead to smaller upper bounds for t^*.

IV. Upper bound using Method 2

The second method to find an upper bound on t^* relies on the level sets of quadratic (ISS-) Lyapunov functions.

Consider the Lyapunov function

$$V(z) = z^T P z \tag{20}$$

where $P = P^\mathsf{T} \succ 0$. For each real number $r > 0$, we define the r-th level set of $V(z)$ by

$$\Omega_r = \{z \in \mathbb{R}^n : V(z) \leq r\}, \tag{21}$$

which is geometrically the area contained by an ellipsoid in $\mathbb{R}^n$. To proceed, define the following two sets:

$$\widetilde{O}_{n-1} = \big\{(z_0, u) : H_0 u \in \widetilde{\mathbb{Y}}_\infty, CA^t z_0 + H_0 u \in \mathbb{Y}_t, \quad t = 0, \dots, n-1\big\} \tag{22}$$

$$\mathbb{Z} = \{z : Cz \in \mathbb{Y} \sim \widetilde{\mathbb{Y}}_\infty \sim D_w \mathbb{W}\} \tag{23}$$

The first set is the set of all initial conditions and inputs such that the constraints are satisfied for the first n time-steps. This set is compact and satisfies $\widetilde{O}_\infty \subset \widetilde{O}_{n-1}$. The second set is the set of all states that satisfy the constraints for all realizations of the steady-state admissible inputs and disturbances. The set $\mathbb{Z}$ is not generally compact, but $\widetilde{O}_\infty$ and $\widetilde{O}_{n-1}$ are, thanks to the observability assumption. The compactness of $\widetilde{O}_{n-1}$ is the main reason why it is employed in the analysis that follows. If $\mathbb{Z}$ itself is compact, then $\widetilde{O}_{n-1}$ may be replaced by $\mathbb{Z}$ in the subsequent presentation.

Consider now two level sets of $V(z)$, Ω_{r_1} and Ω_{r_2}, which are, respectively, the largest level set inscribed in $\mathbb{Z}$ and the smallest level set circumscribing $\widetilde{O}_{n-1}$ along the z-coordinates. Mathematically, $r_1, r_2 \in \mathbb{R}$ are defined by:

$$r_1 = \max\big\{r : \ \Omega_r \subset \mathbb{Z}\big\} \tag{24}$$

$$r_2 = \min\big\{r : \ \mathtt{Proj}_z \widetilde{O}_{n-1} \subset \Omega_r\big\}, \tag{25}$$

where $\mathtt{Proj}_z$ denotes the projection onto z-coordinates. To find an upper bound on t^*, we determine the conditions under which Ω_{r_1} is robustly positively invariant, and then quantify the longest time it takes for any initial state in Ω_{r_2} to enter Ω_{r_1}. Indeed, if the state enters Ω_{r_1}, it will stay in Ω_{r_1} and the constraints will be satisfied. Thus, this time would be an upper bound on t^*. See Fig. 1 for an illustration of these sets.

We state the main results in the following theorem.

Theorem 3: Consider system (4) with Lyapunov function (20), and suppose Assumption 1 holds. Suppose that both (22) and (23) are non-empty and have the origin in their interiors. Let $\zeta > 0$ be a number such that $Q = -(\zeta + 1)A^T PA + P \succ 0$ and $r_1 > \frac{M}{1-\sigma}$, where

$$\sigma = 1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$$
$$M = \lambda_{max}(P)\left(\frac{1}{\zeta} + 1\right) \max_{w \in \mathbb{W}} w^T B_w^T B_w w \tag{26}$$

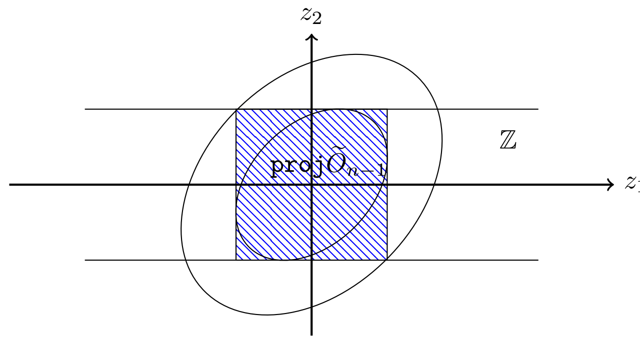


Fig. 1. Illustration of the key idea behind the second method. The set $\mathbb{Z}$ is illustrated as the strip between the two horizontal lines. The set $\mathtt{proj}\widetilde{O}_{n-1}$ is the hatched box. The smaller ellipse is the largest level set of $V(z)$ inscribed in $\mathbb{Z}$. The larger ellipse is the smallest level set of $V(z)$ circumscribing $\mathtt{proj}\widetilde{O}_{n-1}$.

Then, we have that $r_2 > r_1$, and that an upper bound on t^* is given by

$$m = \mathrm{floor}\left(\frac{\log\left(\frac{r_1(1-\sigma)-M}{r_2(1-\sigma)-M}\right)}{\log(\sigma)}\right) \tag{27}$$

where the floor operator returns the previous largest integer.

Proof: Note that r_1 exists because $\mathbb{Z}$ is convex and non-empty and has the origin in its interior, and r_2 exists because $\widetilde{O}_{n-1}$ is compact. The rest of the proof leverages three facts from linear algebra. First, the eigenvalues of a symmetric, positive-definite matrix are all real and positive. Second, for any $P = P^\mathsf{T} \succ 0$, we have that $\lambda_{min}(P)x^T x \leq x^T P x \leq \lambda_{max}(P) x^T x$, where λ_{min} and λ_{max} are well-defined thanks to the first fact. Given $V(x) = x^T P x$, the second fact allows us to write $-x^T x \leq -\frac{V(x)}{\lambda_{max}(P)}$, which we use below. Third, for any vectors x and y, and any $\zeta > 0$, we have that $2x^T y \leq \zeta x^T x + \frac{1}{\zeta} y^T y$. This follows by simplifying the inequality, $(\sqrt{\zeta}x - \frac{1}{\sqrt{\zeta}}y)^T(\sqrt{\zeta}x - \frac{1}{\sqrt{\zeta}}y) \geq 0$. Note that, for a fixed ζ, this bound is tight for $y = \zeta x$.

We now write the change in the Lyapunov function along the trajectories as:

$$\begin{aligned} & V(z(t+1)) - V(z(t)) \\ &= z^T(A^T PA - P)z + w^T B_w^T P B_w w + 2z^T A^T P B_w w \\ &= z^T((1+\zeta)A^T PA - P)z + (\frac{1}{\zeta} + 1) w^T B_w^T P B_w w \\ &\leq z^T((1+\zeta)A^T PA - P)z + (\frac{1}{\zeta} + 1) \max_{w \in \mathbb{W}} w^T B_w^T P B_w w \\ &\leq -\lambda_{min}(Q) z^T z + M \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(z) + M \end{aligned}$$

where, starting from the second line, we have dropped the argument (t) to simplify notation and, on the third line, we have employed the third fact above with $x^T = z^T A^T P^{\frac{1}{2}}$ and $y = P^{\frac{1}{2}} B_w w$. We now simplify the above expression as follows: $V(z(t+1)) \leq \sigma V(z(t)) + M$, which implies that:

$$V(z(t)) \leq \sigma^t V(z(0)) + \frac{1-\sigma^t}{1-\sigma} M \tag{28}$$

where σ is as defined in the Theorem. Note that any Lyapunov level set satisfying $r > \frac{M}{1-\sigma}$ is a robustly positively invariant set because the Lyapunov difference is negative. Thus, by the assumption stated in the theorem, Ω_{r_1} is invariant and also constraint admissible. The goal is now to quantify the longest time it takes for any initial state in Ω_{r_2} to enter Ω_{r_1}, which implies that the constraints will be satisfied for all time-steps afterwards. To this end, note that any $z(0) \in \Omega_{r_2}$ satisfies $V(z(0)) \leq r_2$. Therefore, $V(z(t)) \leq \sigma^t r_2 + \frac{1-\sigma^t}{1-\sigma}M$. Furthermore, to ensure $z(t) \in \Omega_{r_1}$, we must have $V(z(t)) \leq r_1$. Therefore, we set $V(z(t)) \leq \sigma^t r_2 + \frac{1-\sigma^t}{1-\sigma}M \leq r_1$, which implies that

$$t > \frac{\log\left(\frac{r_1(1-\sigma)-M}{r_2(1-\sigma)-M}\right)}{\log(\sigma)}$$

Any t satisfying the above will be an upper bound on t^*. Since we are interested in the smallest integer time-step after which the constraints are redundant, we take the floor of the right hand side of the above.

Finally, $\widetilde{O}_\infty$ contains all positive invariant, constraint-admissible sets, which implies that $\Omega_{r_1} \subset \widetilde{O}_\infty$. Thus, we have the following inclusions: $\Omega_{r_1} \subset \widetilde{O}_\infty \subset \widetilde{O}_{n-1} \subset \Omega_{r_2}$, which means that $r_2 \geq r_1$, as required. ■

Note that, for a fixed ζ, the choice of P in Theorem 3 affects the upper bound, m, in a complicated manner through r_1, r_2, σ, and M. Similarly, for a fixed P, the choice of ζ affects m through σ and M. It thus remains to find a suitable P and ζ in Theorem 3. It is generally desirable to minimize σ to ensure fast decay rate of the Lyapunov function along the trajectories. It is also desirable for $\frac{M}{1-\sigma}$ to be as small as possible so that r_1 will satisfy $r_1 > \frac{M}{1-\sigma}$ as required by the theorem. Therefore, we propose to select P such that σ is as small as possible, and select ζ to ensure $\frac{M}{1-\sigma}$ is as small as possible. Of course, these choices will not necessarily result in the globally minimal value for the upper bound on t^*. However, our numerical studies showed that this was the case in most instances. The next theorem describes the choices for P and ζ. In this theorem, $\rho := \max_i |\lambda_i(A)|$ denotes the spectral radius of A.

Theorem 4: Suppose $\zeta < (\frac{1}{\rho(A)^2} - 1)$. Then, the scalar σ in Theorem 3 satisfies $0 \leq \sigma < 1$. Furthermore, the matrix P that results in the smallest σ is obtained by solving the Lyapunov equation

$$\bar{A}^T P \bar{A} - P = Q \tag{29}$$

where $\bar{A} = \sqrt{1+\zeta}A$ and $Q = I$. The corresponding value of σ is $\sigma = \rho(A)^2$. Finally, the value of $\frac{M}{1-\sigma}$ in Theorem 2 is minimized by

$$\zeta = -\frac{3}{4} + \sqrt{1 + \frac{8}{\rho(A)^2}} \tag{30}$$

Proof: The proofs for $0 \leq \sigma < 1$, the choice of P and Q, and $\sigma = \rho(A)^2$ are similar to the proof of Theorem 6 of [23]. To prove the choice of ζ, we note that with $Q = I$, $\lambda_{max}(P) = \frac{1}{1-\rho(A)^2}$ and $\sigma = \rho(A)^2$. Thus, $\frac{M}{1-\sigma}$ can be written as:

$$\frac{M}{1-\sigma} = \left(\frac{\frac{1}{\zeta}+1}{(1-(\zeta+1)\rho(A)^2)^2}\right) \max_{w\in\mathbb{W}} w^T B_w^T B_w w$$

The expression in the large parentheses is the only term that depends on ζ. Taking the derivative of this expression, setting it equal to 0, and solving for ζ, we obtain the formula in the theorem. ■

Procedures for computing r_1 and r_2 in the theorem are well-established, see, e.g., [25]. Specifically, under the assumptions of Theorem 3, the set $\mathbb{Z}$ is the interval $[-y_0^l + \widetilde{y}_\infty^l, y_0^u - \widetilde{y}_\infty^u]$ of the real line. Thus, r_1 can be found by

$$r_1 = \frac{(\min\{y_0^l - \widetilde{y}_\infty^l, y_0^u - \widetilde{y}_\infty^u\})^2}{CP^{-1}C^T} \tag{31}$$

To find r_2, we first compute $\widetilde{O}_{n-1}$ and convert it from H-representation into the V-representation. Let the vertices of $\widetilde{O}_{n-1}$ in the V-representation be denoted by $v_j \in \mathbb{R}^{n+m}$, where the first n components correspond to the z-coordinates and the next m components correspond to the u-coordinates. Then, r_2 can be found by

$$r_2 = \max_j\{\bar{v}_j^T P \bar{v}_j\} \tag{32}$$

where $\bar{v}_j \in \mathbb{R}^n$ is a vector consisting of the first n components of v_j.

Remark 4: Polynomial time algorithms exist that can convert a polytope from the H-representation to V-representation, see e.g., [26]. However, these algorithms may be computationally intensive in higher dimensions. This may hamper the use of Method 2, e.g., in situations where the upper bound on t^* must be computed online. To remedy this, one can replace $\widetilde{O}_{n-1}$ by any compact superset with known vertices (if such superset is available), or apply the algorithms in [27] to directly find the bounding ellipsoid Ω_{r_2}. These algorithms are applicable in situations where information about the size or aspect ratio of $\widetilde{O}_{n-1}$ is available.

The above results lead to Algorithm 2 for finding an upper bound for t^*.

Algorithm 2 Compute upper bound on t^* using Method 2

Input: $A, y_i^l, y_i^u, \widetilde{y}_\infty^l, \widetilde{y}_\infty^u, \epsilon$
Output: m **such that** $t^* \leq m$

1: Compute: ζ using (30), P using (29) with $Q = I$, $\sigma = \rho(A)^2$, and M using (26).
2: Compute r_1 using (31). If $r_1 \leq \frac{M}{1-\sigma}$, $m = \infty$. STOP.
3: Construct $\widetilde{O}_{n-1}$ as in (22), convert to V-representation, and compute r_2 using (32).
4: Compute m using expression (27).

Remark 5: The condition $r_1 > \frac{M}{1-\sigma}$ in Theorem 3 may not be met, even with the P and ζ computed using Theorem 4. This would mean that there is no robustly invariant level set of the Lyapunov function that is also constraint admissible. In such situations, Method 2 cannot produce an upper bound on t^*. We will explore this further in the next section, where we provide a numerical comparison between the two methods.

V. Numerical Comparison

This section presents a comparative analysis of the upper bounds provided by Algorithm 1 for Method 1 (i.e., the power series-based method) and Algorithm 2 for Method 2 (i.e., the Lyapunov-based method). Since this comparison cannot be carried out analytically, we conduct a Monte Carlo study of randomly-generated systems using Matlab 2020b.

To generate each random system, we first randomly generate n, the order of the system, by sampling the uniform distribution between 1 and 5. We then generate a state-space model with that order by using Matlab's `drss` command, which returns a set of Lyapunov-stable systems with possibly repeated poles. To ensure Assumption 1 is robustly satisfied, we reject systems for which the spectral radius is greater than 0.999 and the smallest singular value of the observability matrix is less than 0.0001. For simplicity, we assume that the constraint is symmetric ($y^u = y^l = 1$), the disturbance w is a scalar bounded to the interval $[-0.01, 0.01]$, and that D_w is 0. The value of ϵ was chosen to be 0.01.

Using the above methodology, we generate a total of 16,000 random systems. For each randomly generated system, we compute $\widetilde{O}_\infty$, and the systems for which $\widetilde{O}_\infty$ was empty are thrown out. A total of 15,810 systems remain. For each of the remaining systems, we compute t^*, as well as the upper bounds on t^* using Algorithms 1 and 2. We denote these upper bounds by m_1 and m_2 respectively, where the subscript refers to the respective method.

Notably, only 5229 systems (about one third of the total) result in viable upper bounds using Method 2. The remaining two-thirds of the systems do not satisfy $r_1 > \frac{M}{1-\sigma}$ so a bound cannot be found. Method 1, however, results in a bound for all the systems considered. To compare the upper bounds against the true value of t^*, we construct the histograms of $m_i - t^*$, $i = 1, 2$, as seen in Fig. 2 for Method 1 and Fig. 3 for Method 2. The statistics of these distributions are shown in the respective figure legend. These statistics should be compared with caution: Method 2 did not yield a bound for many of the systems in which t^* was large, so the statistics appear smaller.

To directly compare the two methods, Fig. 4 shows the histogram of $m_2 - m_1$. Interestingly, Method 1 outperforms Method 2 in all cases. This is consistent with our earlier results reported in [23] for the disturbance-free case. Investigation of this observation is an interesting topic for future research. In conclusion, not only is Method 1 more broadly applicable, it results in tighter bounds as compared to Method 2.

From these figures, it may appear that the upper bounds are too conservative for some systems. To investigate, we plot, in Fig. 5a, $m_1 - t^*$ as a function of the spectral radius of A, ρ. As can be seen, large $m_1 - t^*$ (i.e., loose bound) can be attributed to large ρ (i.e., slow systems). To investigate further, we normalize both t^* and its upper bound

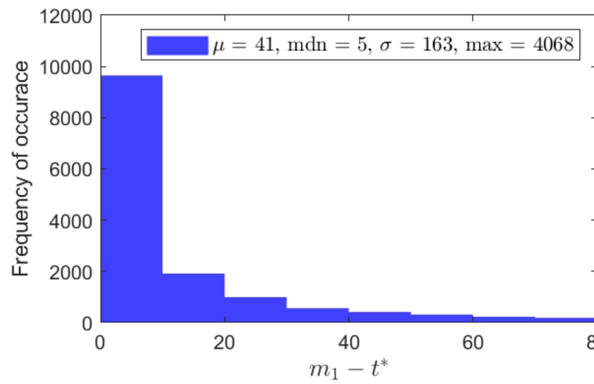


Fig. 2. Histogram of $m_1 - t^*$ (i.e., the tightness of the upper bound obtained via Method 1). In the legend, μ, σ, and mdn refer to the mean, standard deviation, and median, respectively.

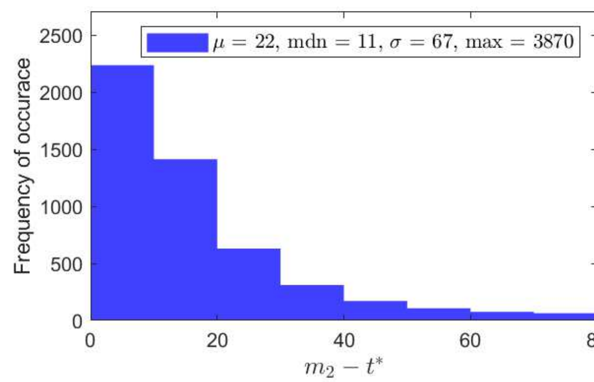


Fig. 3. Histogram of $m_2 - t^*$ (i.e., the tightness of the upper bound obtained via Method 2).

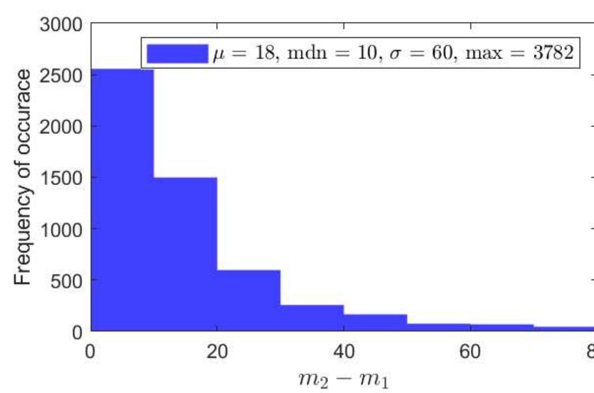
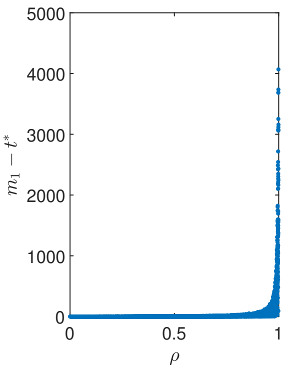
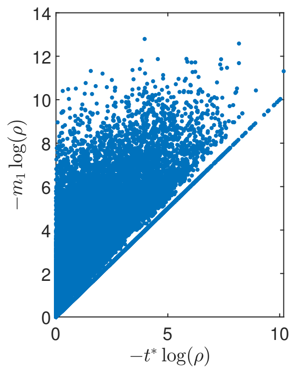


Fig. 4. Comparison between the upper bound provided by Methods 1, m_1, and by Method 2, m_2. Interestingly, $m_1 \le m_2$ in all cases.



(a) $m_1 - t^*$ vs. ρ.



(b) t^* vs. m_1, each scaled by $\log(\rho)$.

Fig. 5. Analysis of m_1 as a function of the spectral radius, $\rho(A)$.

m_1 to allow for a fair comparison between the different systems. The normalization is achieved by scaling t^* and m_1 by $\log(\rho)$. Taking logarithms is inspired by the fact that continuous-time poles and discrete-time poles are related through $z = e^{sT_s}$, where T_s is the sample time. Assuming $T_s = 1$ to allow for direct comparison between the systems, we obtain $s = \log(z)$. Thus, scaling by $\log(\rho)$ normalizes each t^* or m by the "continuous-time time constant" of the system. The results are reported in Fig. 5b. As can be seen, in the normalized coordinates, the spread is narrow and the upper bound is not as conservative as it appeared before. Similar plots can be generated for Method 2.

VI. Conclusion

This paper introduced two computationally efficient methods for obtaining upper bounds on the finite determination index of the (inner approximation) of the Maximal Admissible Set for discrete-time LTI systems subject to bounded disturbances. The first method is based on matrix power series while the second method is based on Lyapunov analysis. We provided a rigorous introduction to both methods, along with a detailed numerical comparison. Our results show that Method 1 outperforms Method 2 and is applicable to a broader range of systems.

In future work, we plan to investigate the reasons behind the superior performance of Method 1 in our numerical study. We also aim to explore other power series expansions beyond those provided by the Cayley-Hamilton method, in order to improve the upper bounds in Method 1.

References

[1] E. G. Gilbert and K. T. Tan, "Linear systems with state and control constraints: The theory and application of maximal output admissible sets," *IEEE Transactions on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 1991.

[2] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, no. 4, pp. 317–367, 1998.

[3] E. G. Gilbert, I. Kolmanovsky, and K. T. Tan, "Discrete-time reference governors and the nonlinear control of systems with state and control constraints," *International Journal of Robust and Nonlinear Control*, vol. 5, no. 5, pp. 487–504, 1995.

[4] E. Garone, S. Di Cairano, and I. Kolmanovsky, "Reference and command governors for systems with constraints: A survey on theory and applications," *Automatica*, vol. 75, pp. 306–328, 2017.

[5] E. F. Camacho and C. B. Alba, *Model Predictive Control*. Springer Science & Business Media, 2013.

[6] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing Madison, WI, 2017, vol. 2.

[7] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[8] F. Blanchini and S. Miani, *Set-theoretic Methods in Control*. Springer, 2008, vol. 78.

[9] C. E. Dórea and J.-C. Hennet, "Computation of maximal admissible sets of constrained linear systems," in *Proc. of 4th IEEE Med. Symposium*, 1996, pp. 286–291.

[10] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.

[11] B. Pluymers, J. A. Rossiter, J. A. K. Suykens, and B. De Moor, "The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty," in *Proceedings of the 2005 American Control Conference*, June 2005, pp. 804–809 vol. 2.

[12] S. V. Raković and M. Fiacchini, "Invariant approximations of the maximal invariant set or "encircling the square"," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 6377–6382, 2008.

[13] K. Hirata and Y. Ohta, "Exact determinations of the maximal output admissible set for a class of nonlinear systems," *Automatica*, vol. 44, no. 2, pp. 526–533, 2008.

[14] C. Danielson, A. Weiss, K. Berntorp, and S. Di Cairano, "Path planning using positive invariant sets," in *55th IEEE Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 5986–5991.

[15] H. R. Ossareh, "Reference governors and maximal output admissible sets for linear periodic systems," *International Journal of Control*, vol. 93, no. 1, pp. 113–125, 2020.

[16] S. M. Vaselnia, S. Aghaei, and V. Puig, "An inclusion-based approach for determination of a safe maximal output admissible set," in *2021 European Control Conference (ECC)*. IEEE, 2021, pp. 646–650.

[17] E. Pérez, C. Arino, F. X. Blasco, and M. A. Martínez, "Maximal closed loop admissible set for linear systems with non-convex polyhedral constraints," *Journal of Process Control*, vol. 21, no. 4, pp. 529–537, 2011.

[18] J. Osorio and H. R. Ossareh, "A stochastic approach to maximal output admissible sets and reference governors," in *2018 IEEE Conference on Control Technology and Applications (CCTA)*. IEEE, 2018, pp. 704–709.

[19] Y. Benfatah, A. El Bhih, M. Rachik, and A. Tridane, "On the maximal output admissible set for a class of bilinear discrete-time systems," *International Journal of Control, Automation and Systems*, vol. 19, pp. 3551–3568, 2021.

[20] H. R. Ossareh, "A data-driven formulation of the maximal admissible set and the data-enabled reference governor," *IEEE Control Systems Letters*, vol. 7, pp. 3465–3470, 2023.

[21] E. Garone, I. Kolmanovsky, and S. Di Cairano, "Full-day workshop on reference supervision for constraint enforcement: theory and applications," *Workshop at 2014 IEEE Conference on Decision and Control, url: https://saas.ulb.ac.be/CDC2014/*, 2014.

[22] H. R. Ossareh and I. Kolmanovsky, "On complexity bounds for the maximal admissible set of linear time-invariant systems," *IEEE Transactions on Automatic Control*, pp. 1–8, 2024.

[23] ——, "On complexity bounds for the maximal admissible set of linear time-invariant systems," 2023. [Online]. Available: https://arxiv.org/abs/2302.02246

[24] C.-T. Chen, *Linear System Theory and Design, Third Edition*. Oxford University Press, Inc., 1998.

[25] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex Optimization*. Cambridge university press, 2004.

[26] D. Avis, K. Fukuda, and S. Picozzi, "On canonical representations of convex polyhedra," in *Mathematical Software*. World Scientific, 2002, pp. 350–360.

[27] Y. Makarychev, N. S. Manoj, and M. Ovsiankin, "Streaming algorithms for ellipsoidal approximation of convex polytopes," in *Conference on Learning Theory*. PMLR, 2022, pp. 3070–3093.