STABILIZATION DISTANCE BOUNDS FROM LINK FLOER HOMOLOGY

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Abstract. We consider the set of connected surfaces in the 4-ball with boundary a fixed knot in the 3-sphere. We define the stabilization distance between two surfaces as the minimal g such that we can get from one to the other using stabilizations and destabilizations through surfaces of genus at most g. Similarly, we consider a double point distance between two surfaces of the same genus which is the minimum over all regular homotopies connecting the two surfaces of the maximal number of double points appearing in the homotopy.

To many of the concordance invariants defined using Heegaard Floer homology, we construct an analogous invariant for a pair of surfaces. We show that these give lower bounds on the stabilization distance and the double point distance. We compute our invariants for some pairs of deform-spun slice disks by proving a trace formula on the full infinity knot Floer complex, and by determining the action on knot Floer homology of an automorphism of the connected sum of a knot with itself that swaps the two summands. We use our invariants to find pairs of slice disks with arbitrarily large distance with respect to many of the metrics we consider in this paper. We also answer a slice-disk analogue of Problem 1.105 (B) from Kirby's problem list by showing the existence of non-0-cobordant slice disks.

Contents

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1. Introduction

There is a natural stabilization operation on smooth, oriented surfaces in 4-manifolds where one attaches an embedded 1-handle to the surface. This operation was recently considered by Baykur and Sunukjian [3]. When the 1-handle is unknotted, this does not change the fundamental group of the surface complement. They asked the following question: If two surfaces are topologically isotopic, then do they become smoothly isotopic after a single unknotted 1-handle stabilization? They verified this for most known constructions of pairs of exotic surfaces, such as rim surgery. A related question is how many stabilizations are required to make a given pair of surfaces isotopic.

There is a parallel notion of stabilization for 4-manifolds. A classical result of Wall [51] states that if two smooth, simply-connected 4-manifolds are homeomorphic, then they become diffeomorphic after taking connected sums with some number of copies of $\mathbb{S}^2 \times \mathbb{S}^2$. It is an open conjecture whether a single copy of $\mathbb{S}^2 \times \mathbb{S}^2$ always suffices. Recently, Lin and Mukherjee [25] constructed a pair of surfaces-with-boundary in a punctured K3 that are topologically isotopic but not smoothly isotopic, and remain so after stabilizing their complements once with $\mathbb{S}^2 \times \mathbb{S}^2$. They showed this using family Bauer–Furuta invariants.

In this paper, we construct invariants that provide lower bounds on the number of 1-handle stabilizations required to make two surfaces isotopic. Using a generalization of 1-handle stabilization, we endow the set of smooth, connected, oriented, and properly embedded surfaces in B^4 with boundary a knot K with a type of metric that we call the *stabilization distance*. The stabilization distance between two surfaces bounds from below the number of 1-handle stabilizations required to make them isotopic. Using the link Floer TQFT, we define several invariants of pairs of surfaces bounding K that give lower bounds on the stabilization distance. We compute these invariants for certain pairs of slice disks arising from deform-spinning, and observe they often give non-trivial lower bounds. Furthermore, we give examples in Section 10 where these lower bounds can be arbitrarily large.

Throughout this paper, we work in the smooth category, and all manifolds are assumed to be oriented, unless otherwise stated.

1.1. **Metric filtrations on the set of surfaces bounding a knot.** In Definition 2.9, we introduce a very general type of stabilization operation for surfaces in 4-manifolds that extends the 1-handle stabilization considered by Baykur and Sunukjian. Let S be a properly embedded surface in a 4-manifold W. To obtain the stabilization of S, we choose a 4-ball $B \subseteq \operatorname{int}(W)$ such that $B \cap S$ is a collection of disks that can be isotoped into ∂B^4 relative to their boundaries. In particular, $\partial B \cap S$ is an unlink. We then replace $S \cap B$ with a properly embedded surface $S_0 \subseteq B$ such that $\partial S_0 = \partial B \cap S$. Any two surfaces in the same relative homology class can be related by a finite sequence of such stabilizations and destabilizations (in fact, 1-handle stabilizations suffice).

Let K be a knot in $\mathbb{S}^3$. We denote by $\operatorname{Surf}(K)$ the set of isotopy classes of connected, properly embedded surfaces in B^4 with boundary K. In Definition 2.14, we introduce the *stabilization distance* $\mu_{\rm st}(S, S')$ of a pair of surfaces S, $S' \in \operatorname{Surf}(K)$ to be the minimum of

$$\max\{g(S_1), \dots, g(S_k)\}$$

over sequences of connected, properly embedded surfaces $S_1, \dots, S_k$ in $\operatorname{Surf}(K)$ connecting S and S' such that consecutive surfaces are related by a stabilization or a destabilization. Furthermore, if K is slice and $S \in \operatorname{Surf}(K)$, then we define the *destabilizing genus* $g_{\rm dest}(S)$ to be the stabilization distance of S from the subset of slice disks.

We also define the *M-distance* function $M_{(S,S')}\colon [0,2] \to \mathbb{R}^{\ge 0}$ of a pair of surfaces S, $S' \in \operatorname{Surf}(K)$, which is similar to the stabilization distance, but where the stabilization operation is allowed to change the ambient 4-manifold. Instead of changing the surface in a 4-ball, one can glue in a pair (X_0, S_0), where $\partial X_0 = \mathbb{S}^3$ and ∂S_0 is an unlink, and $b_2^+(X_0) = b_1(X_0) = 0$. The *M-degree* of a pair (W, S), where W is a compact 4-manifold and S is a properly embedded surface, is defined in Section 7. It is a function on $[0, 2]$ that measures not only the genus of S, but also a homological quantity depending on $[S] \in H_2(W)$ and the intersection form Q_W of W. The M-distance of a pair of surfaces S, $S' \in \operatorname{Surf}(K)$ minimizes the maximal M-degree along sequences $(W_1, S_1) \dots, (W_n, S_n)$ connecting (B^4, S) and (B^4, S') such that (W_i, S_i) and (W_{i+1}, S_{i+1}) are related by the above stabilization operation.

Another notion of distance we consider in this paper is the *cobordism distance*, which we denote $\mu_{\rm Cob}(S, S')$. If S, $S' \in \operatorname{Surf}(K)$, we set $\mu_{\rm Cob}(S, S')$ to be the minimal g such that there is a 5-dimensional cobordism $(I \times B^4, Y)$, where Y is a smoothly and properly embedded, oriented 3-manifold-with-corners such that

$$\partial Y = -(\{0\} \times S) \cup (\{1\} \times S') \cup (I \times K),$$

projection of Y to I is Morse, and such that each regular level set of Y is a surface such that the sum of the genera of its components is at most g. We say that S and S' are *strictly g-cobordant* if $\mu_{\rm Cob}(S, S') = g$. Compare this to the notion of *g-cobordism* of 2-knots defined by Melvin [30], where one requires every component of each level set to have genus at most g, and *g-concordance*, where, in addition, $Y \approx I \times S^2$. Two surfaces are strictly 0-cobordant if and only if they are 0-cobordant. In particular, μ_{Cob} defines an ultrametric on the set of 0-cobordism classes of surfaces. It is straightforward to see that

$$\mu_{\text{Cob}}(S,S') \leq \mu_{\text{st}}(S,S').$$

Note that Sunukjian showed that there are infinitely many distinct 0-concordance classes of 2-knots in S^4 [50]. Dai and Miller [6] improved this result to show that the 0-concordance monoid of 2-knots was infinitely generated.

For $g \in \mathbb{N}$, let $\text{Surf}_g(K)$ denote the subset of $\text{Surf}(K)$ consisting of genus g surfaces. If S, $S' \in \text{Surf}_g(K)$, then they are regularly homotopic relative to K. We define the *double point distance* $\mu_{\text{Sing}}(S,S')$ as $\widetilde{\mu}_{\text{Sing}}(S,S')+g$, where $\widetilde{\mu}_{\text{Sing}}(S,S')$ is obtained by minimizing half the maximal number of double points that appear during a regular homotopy from S to S'; see Definition 6.2. When $g(S) \neq g(S')$, we set $\mu_{\text{Sing}}(S,S') = \infty$. Motivated by an earlier version of this paper, Singh [48] showed that

$$\mu_{\text{st}}(S,S') \leq \mu_{\text{Sing}}(S,S') + 1$$

using techniques from Gabai's proof [10] of the 4-dimensional light bulb theorem. It is an open problem whether the $+1$ is necessary in the above formula.

Both μ_{st} and μ_{Sing} are *metric filtrations* on $\text{Surf}(K)$; i.e., they are nonnegative, symmetric, and satisfy the ultrametric inequality

$$\mu(S,S'') \leq \max\{\, \mu(S,S'), \mu(S',S'')\,\}$$

for any S, S', $S'' \in \text{Surf}(K)$; see Section 2.2. Furthermore, $M_{(S,S')}(t)$ is also a metric filtration for every $t \in [0,2]$. If S, $S' \in \text{Surf}_0(K)$, then $\mu_{\text{st}}(S,S') = 0$ if and only if S and S' are related by genus zero stabilizations, hence $\text{Surf}_0(K)/\{\text{2-knots}\}$ is an ultrametric space. However, in general, $\mu_{\text{st}}(S,S) = g(S)$, and if $\mu_{\text{st}}(S,S') = g(S) = g(S')$, it is possible that S and S' are not related by genus zero stabilizations.

1.2. **Lower bounds from Heegaard Floer homology.** Our aim is to provide computable lower bounds on the stabilization distance, the double point distance, the destabilizing genus, and the cobordism distance using the link Floer TQFT of the second author [58], which extends the TQFT of the first author [18] from $\widehat{HFL}$ to the full infinity complex $\mathcal{CFL}^\infty$. If $\mathbb{K} = (K,w,z)$ is a doubly-based knot in $\mathbb{S}^3$, then $\mathcal{CFL}^\infty(\mathbb{K})$ is a $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex over the two-variable Laurent polynomial ring

$$\mathcal{R}^\infty := \mathbb{F}_2[U,V,U^{-1},V^{-1}].$$

We give an overview of $\mathcal{CFL}^\infty(\mathbb{K})$ in Section 3.1. There is a variation, denoted $\mathcal{CFL}^-(\mathbb{K})$, corresponding to the subspace of $\mathcal{CFL}^\infty(\mathbb{K})$ in nonnegative bi-filtration, which is a module over the ring

$$\mathcal{R}^- := \mathbb{F}_2[U,V].$$

Knot Floer homology has been used by many authors to provide numerical concordance invariants that provide deep geometric information. Important examples are the knot invariants τ, ν, V_k for $k \in \mathbb{N}$, and Υ, introduced by Ozsváth–Szabó [36, 42], Hom–Wu [17], Rasmussen [44], and Ozsváth–Stipsicz–Szabó [34], respectively. Since τ, ν, V_k, and Υ are concordance invariants, they all vanish when the knot is slice.

It is well known that the knot invariants τ, ν, and V_k give lower bounds on the 4-ball genus $g_4(K)$:

$$\tau(K) \leq \nu(K) \leq g_4(K) \text{ and } V_k(K) \leq \left\lceil \frac{g_4(K)-k}{2} \right\rceil$$

whenever $k \leq g_4(K)$; see [36, Theorem 1.1], [17, Section 2], and [45, Theorem 2.3]. The invariant Υ also gives lower bounds on the 4-ball genus; see [34, Theorem 1.11]. The invariants τ and Υ satisfy more general versions of the genus bound involving surfaces in negative definite 4-manifolds W with $b_1(W) = b_2^+(W) = 0$; see [36, Theorem 1.1] and [57, Theorem 1.1].

In Section 4, we show that, by mirroring their constructions, we can define secondary versions of all of the above knot invariants for a pair of surfaces S, $S' \in \text{Surf}(K)$ in B^4 with boundary a knot

K in $\mathbb{S}^3$. We show that these give formally analogous lower bounds on the metric filtrations μ_{st}, μ_{Sing}, and M:

Theorem 1.1. *Let S, $S' \in \mathrm{Surf}(K)$. Then*

$$\tau(S,S') \le \nu(S,S') \le \min\{\, \mu_{\mathrm{st}}(S,S'),\ \mu_{\mathrm{Sing}}(S,S')\,\}.$$

Furthermore,

$$\Upsilon_{(S,S')}(t) \le M_{(S,S')}(t)$$

for every $t \in [0,2]$. Finally, for every $k \in \mathbb{N}$,

$$V_k(S,S') \le \left\lceil \frac{\mu_{\mathrm{Sing}}(S,S') - k}{2} \right\rceil. \tag{1.1}$$

If S and S' are disks, then Equation (1.1) *also holds with μ_{st} in place of μ_{Sing}.*

The bounds stated in Theorem 1.1 are proven separately throughout the paper; see Theorems 5.13, 5.14, 6.7, 6.14, and 7.5, as well as Proposition 6.8. In Section 4.7, we introduce a novel invariant $\kappa(S,S')$ that does not have an analogue for knots, and which only gives a lower bound on $\mu_{\mathrm{Sing}}(S,S')$; see Theorem 6.9.

Using the link Floer TQFT, we also introduce another integer invariant $\mathcal{I}(S)$ of a surface S which has $g(S) > 0$ and has boundary a slice knot K; see Definition 5.6. In Theorem 5.10, we prove that $\mathcal{I}(S)$ bounds the stabilization distance between S and the subset of slice disks:

$$\mathcal{I}(S) \le g_{\mathrm{dest}}(S).$$

However, $\mathcal{I}(S)$ is in general difficult to compute, as it involves determining the link cobordism maps for infinitely many decorations on the surface S. Giving a lower bound on $\mathcal{I}(S)$ is theoretically feasible, since it only involves finding two decorations on S satisfying a simple condition, whose induced cobordism maps disagree. We define an analogue of $\kappa(S,S')$ for a single surface, $\kappa_0(S)$, which gives a computable lower bound on $g_{\mathrm{dest}}(S)$; see Theorem 5.11.[1]

The invariants τ, κ, and κ_0 can all be derived from Υ. As opposed to the case of knots, $\Upsilon_{(S,S')}(t)$ is not a symmetric function. We will see in Theorem 4.20 that, for all $t \in [0,2]$ sufficiently close to 0, we have

$$\Upsilon_{(S,S')}(t) = \tau(S,S') \cdot t.$$

However, for t sufficiently close to 2, we have

$$\Upsilon_{(S,S')}(t) = \begin{cases} (\kappa_0(S) - g(S)) \cdot (2-t) + g(S) \cdot t & \text{if } g(S) > g(S'), \\ (\kappa(S,S') - g(S)) \cdot (2-t) + g(S) \cdot t & \text{if } g(S) = g(S'). \end{cases}$$

We now review the construction of the invariants. If $S \in \mathrm{Surf}(K)$, then it can be viewed as a link cobordism from $\emptyset$ to $(\mathbb{S}^3, K)$. If we decorate it such that the type-$\mathbf{w}$ region is a bigon, then it induces a filtered chain map

$$\mathbf{t}^{\infty}_{S,\mathbf{z}} \colon \mathcal{R}^{\infty} \to \mathcal{CFL}^{\infty}(\mathbb{K}),$$

well-defined up to filtered chain homotopy; see Section 4.2. If instead the type-$\mathbf{z}$ region is a bigon, we obtain a map $\mathbf{t}^{\infty}_{S,\mathbf{w}}$. We call $\mathbf{t}^{\infty}_{S,\mathbf{z}}$ and $\mathbf{t}^{\infty}_{S,\mathbf{w}}$ the *extremal principal invariants* of the surface S.

Given surfaces S, $S' \in \mathrm{Surf}(K)$, the invariants

$$\tau(S,S'),\ \nu(S,S'),\ V_k(S,S'),\ \text{and}\ \Upsilon_{S,S'}(t)$$

are all extracted from the pair of maps $(\mathbf{t}^{\infty}_{S,\mathbf{z}}, \mathbf{t}^{\infty}_{S',\mathbf{z}})$ by algebraically mirroring the construction of their knot invariant counterparts. Hence, we think of our invariants as *secondary* versions of the knot invariants. The invariant $\kappa_0(S)$ is derived from $\mathbf{t}^{\infty}_{S,\mathbf{w}}$, and we obtain $\kappa(S,S')$ from the pair $(\mathbf{t}^{\infty}_{S,\mathbf{w}}, \mathbf{t}^{\infty}_{S',\mathbf{w}})$.

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[1] In subsequent work [21], we show that $\mathcal{I}(S), \kappa_0(S) \in \{g(S), g(S)+1\}$, and hence $\mathcal{I}(S)$ and $\kappa_0(S)$ give potential obstructions to surfaces being stabilized (cf. Proposition 5.5 therein). By analyzing the proof of [21, Theorem 1.7] and by considering cases where the bound is sharp, one may find surfaces where $\mathcal{I}(S) = g(S) + 1$. For example, any genus 2 slice surface for $T_{2,3} \# T_{2,3}$ has $\mathcal{I}(S) = 3$.

For example, to obtain τ, we set $U = 0$ and $V = 1$ in the $\mathcal{R}$-module $\mathcal{CFL}^-(\mathbb{K})$, and obtain the $\mathbb{Z}$-filtered complex $\widehat{CFK}^{fil,z}(\mathbb{K})$ whose homology is $\widehat{HF}(\mathbb{S}^3) \cong \mathbb{F}_2$. Given a pair of surfaces S, $S' \in \mathrm{Surf}(K)$, the elements $\mathbf{t}^-_{S,\mathbf{z}}(1)$ and $\mathbf{t}^-_{S',\mathbf{z}}(1)$ of $\mathcal{CFL}^-(\mathbb{K})$ give rise to elements $\widehat{t}_{S,\mathbf{z}}(1)$ and $\widehat{t}_{S',\mathbf{z}}(1)$ of $\widehat{CFK}^{fil,z}(\mathbb{K})$. We define the invariant

$$\tau(S,S') := \min\{n \geq \max\{g,g'\} : \big[\widehat{t}_{S,\mathbf{z}}(1)\big] = \big[\widehat{t}_{S',\mathbf{z}}(1)\big] \text{ as elements of } H_*(\widehat{CFK}^{fil,z}_n(\mathbb{K}))\},$$

where $\widehat{CFK}^{fil,z}_n(\mathbb{K})$ is the part of $\widehat{CFK}^{fil,z}(\mathbb{K})$ in filtration at most n.

1.3. **Computing the invariants for deform-spun disks using the trace formula.** In Section 10, we compute the secondary invariants for several pairs of deform-spun slice disks, using the computer algebra software SageMath [46]. We exhibit several examples where the lower bound on the stabilization distance is 2 or 3, and a family with arbitrarily large distance. We note that, in our examples, the pairs of slice disks are *not* topologically isotopic relative to their boundaries.

Let K be a knot in $\mathbb{S}^3$, and suppose that the 3-ball B intersects K in an unknotted arc. Then $(\mathbb{S}^3 \setminus \mathrm{int}(B), K \setminus \mathrm{int}(B))$ is a ball-arc pair (B^3, a). Suppose that we are given an isotopy $\phi\colon I \times \mathbb{S}^3 \to \mathbb{S}^3$ of $\mathbb{S}^3$ that is the identity on B, and such that $\phi_0 = \mathrm{Id}_{\mathbb{S}^3}$ and $\phi_1(K) = K$, where $\phi_t(x) := \phi(t,x)$ for every $t \in I$ and $x \in \mathbb{S}^3$. Then the *deform-spun* slice disk $D_{K,\phi} \subseteq B^4$ is defined by taking

$$\bigcup_{t\in I} \{t\} \times \phi_t(a) \subseteq I \times B^3,$$

and rounding the corners along $\{0,1\} \times \partial B^3$. This is a slice disk of $-K\#K$. The isotopy class of $D_{K,\phi}$, relative to $-K\#K$, only depends on the diffeomorphism ϕ_1, so we will write D_{K,ϕ_1} instead of $D_{K,\phi}$.

In this paper, we consider three automorphisms of a ball-arc pair. The first is the *roll-spinning* automorphism r, which corresponds to a positive Dehn twist in the longitudinal direction of the knot K. The automorphism r is supported in a neighborhood of K. The second is the *twist-spinning* automorphism t, which is similar to roll-spinning, but instead twists in the meridional direction. The third automorphism $\mathbf{R}^\pi$ that we consider is specific to knots of the form $K\#K$. The *summand-swapping* automorphism $\mathbf{R}^\pi$ of $K\#K$ is the composition of an isometry of $\mathbb{R}^3$ that swaps the two copies of K, and a half Dehn twist that ensures $\mathbf{R}^\pi$ fixes $K\#K$ pointwise.

We show that, if K is a knot in $\mathbb{S}^3$, then the roll-spun and twist-spun slice disks $D_{K,r}$, $D_{K,t} \in \mathrm{Surf}_0(-K\#K)$ satisfy

$$\mu_{st}(D_{K,r}, D_{K,t}) \leq 2;$$

see Proposition 2.21. We conjecture that the upper bound can be improved to 1.

We obtain lower bounds on the stabilization distance and the double point distance using our secondary invariants. For this end, we first compute the extremal principal invariant $\mathbf{t}^\infty_{D_{K,\phi_1}}(1) \in \mathcal{CFL}^\infty(-K\#K)$ of a deform-spun slice disk. We define the canonical cotrace map

$$\mathrm{cotr}\colon \mathcal{R}^\infty \to \mathcal{CFL}^\infty(Y,\mathbb{L},\mathfrak{s}) \otimes_{\mathcal{R}^\infty} \mathcal{CFL}^\infty(-Y,-\mathbb{L},\mathfrak{s})$$

as the dual of the trace pairing.

Theorem 1.2. *Let $D_{K,\varphi}$ be a slice disk of the doubly-based knot $-\mathbb{K}\#\mathbb{K} = (-K\#K, w, z)$, obtained by deform-spinning a doubly-based knot $\mathbb{K} = (K, w, z)$ in $\mathbb{S}^3$ using an automorphism φ of $(\mathbb{S}^3, K)$. If we write $C := \mathcal{CFL}^\infty(\mathbb{K})$, then*

$$E \circ \mathbf{t}^\infty_{D_{K,\varphi}} \simeq (\mathrm{id} \otimes \varphi_*) \circ \mathrm{cotr} \in \mathrm{Hom}_{\mathcal{R}^\infty}(\mathcal{R}^\infty, C^\vee \otimes C),$$

where $E\colon \mathcal{CFL}^\infty(-\mathbb{K}\#\mathbb{K}) \to C^\vee \otimes C$ is the chain homotopy equivalence induced by a pair-of-pants link cobordism.

Our proof of Theorem 1.2 uses a much more general trace formula for the full link Floer TQFT, Theorem 9.3, extending a result from our earlier work [22, Theorem 1.1] for sutured Floer homology, as well as a result of the second author for the graph TQFT [55, Theorem 1.6].

Hence, to compute the secondary invariants of deform-spun slice disks, it remains to determine the induced diffeomorphism map φ_* in specific examples. The map r_* induced by the basepoint moving diffeomorphism is well-known, and is given by the formula

$$r_* \simeq \mathrm{id} + \Phi \circ \Psi,$$

which was proven on $HFK^-(\mathbb{K})$ by Sarkar [47], and extended to $\mathcal{CFL}^\infty(\mathbb{K})$ by the second author [54]. The maps Φ and Ψ are two special endomorphisms of knot Floer homology, which we describe in Section 3.2. It turns out that $\tau(D_{K,\mathrm{id}}, D_{K,r}) \leq 1$ for the canonical deform-spun slice disks $D_{K,\mathrm{id}}$ and the roll-spun slice disks $D_{K,r}$; see Proposition 10.1 .

To provide more interesting examples beyond roll-spinning, we compute a formula for the map induced by the summand-swapping automorphism $\mathbf{R}^\pi$ on $\mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K})$ in Theorem 8.2. We prove that there is a chain homotopy equivalence identifying $\mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K})$ and $\mathcal{CFL}^\infty(\mathbb{K})\otimes\mathcal{CFL}^\infty(\mathbb{K})$ under which

$$\mathbf{R}^\pi_* \simeq \mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + (\Phi \circ \Psi) \otimes \mathrm{id} + \Phi \otimes \Psi),$$

where Sw is the map that swaps the two tensor factors. To our knowledge, after Sarkar's formula, this is the only other known formula for a mapping class group action on the knot Floer complexes of a family of knots in $\mathbb{S}^3$.

1.4. **Families with large distance.** Using the trace formula and our invariants, we prove the following:

Theorem 1.3. *Given $n \geq 0$, there is a knot K_n and a pair of slice disks D_1 and D_2 for K_n, such that $\tau(D_1, D_2) \geq n$. In particular $\omega(D_1, D_2) \geq n$ for $\omega \in \{\mu_{\mathrm{st}}, \mu_{\mathrm{Sing}}, \mu_{\mathrm{Cob}}\}$.*

The slice disks appearing in our proof of Theorem 1.3 are deform spun slice disks of

$$T_{p,q}\#T_{p,q}\#\bar{T}_{p,q}\#\bar{T}_{p,q}$$

for various p and q.

We note that the above results appeared after the work of Miller and Powell [31], who constructed for each n a pair of slice disks such that any stabilization sequence between D_1 and D_2 required at least n stabilizations. Their result does not imply ours, since it focuses on the total number of stabilizations, as opposed to the maximal genus. Hence it does not give a lower bound on the cobordism distance. See also work of Miyazaki [33].

Theorem 1.3 gives an answer to a slice disk analogue of Problem 1.105 (B) of Kirby's Problem List [1] that asks whether every 2-knot is 0-null-cobordant:

Corollary 1.4. *For every g, there is a knot K_g and a pair of slice disks that are not strictly g-cobordant. In particular, there are slice disks that are not 0-cobordant in the sense of Melvin* [30].

Remark 1.5. We note that Dai, Mallick and Stoffregen have independently found examples of slice disks with large stabilization distance [5]. Additionally, they use some of the techniques of this paper in their work to study equivariant knots.

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2. The stabilization distance of a pair of surfaces

In this section, we first review deform-spun slice disks. We then introduce the notion of a metric filtration, which is a generalization of an ultrametric, but where the distance of a point from itself can be nonzero. We proceed to define the stabilization distance of a pair of surfaces with boundary a given knot, which is an instance of a metric filtration. Finally, we show that the stabilization distance of a 1-roll-spun and a 1-twist-spun slice disk is always at most two.

2.1. **Deform-spun slice disks.** We review the definitions of deform-spun slice disks [23, Section 3]. Originally, Litherland [27] introduced deform-spinning to construct 2-knots in $\mathbb{R}^4$, generalizing twist-spinning, due to Zeeman [52], and roll-spinning, due to Fox [7]. An analogous construction can be used to obtain slice disks in B^4. The following is [23, Definition 3.1]:

Definition 2.1. Let K be a knot in $\mathbb{S}^3 = -B^3 \cup B^3$ such that K intersects $-B^3$ in an unknotted arc, and write $a = K \cap B^3$. Furthermore, let $\phi\colon I \times \mathbb{S}^3 \to \mathbb{S}^3$ be an isotopy of $\mathbb{S}^3$ such that $\phi_0 = \mathrm{Id}_{\mathbb{S}^3}$, $\phi_t|_{-B^3} = \mathrm{Id}_{-B^3}$ for every $t \in I$, and $\phi_1(K) = K$. The *deform-spun* slice disk $D_{K,\phi} \subseteq B^4$ is defined by taking

$$\bigcup_{t \in I} \{t\} \times \phi_t(a) \subseteq I \times B^3,$$

and rounding the corners along $\{0,1\} \times \partial B^3$. The surface $D_{K,\phi}$ is a slice disk for $-K \# K$, where $-K$ stands for $(-\mathbb{S}^3, -K)$.

Note that $\partial \mathbb{R}^4_+ = \mathbb{R}^3 = \mathbb{R}^3_+ \cup \mathbb{R}^3_-$. Intuitively, we consider the arc a in $\mathbb{R}^3_-$, which we rotate about the plane $\mathbb{R}^2 = \mathbb{R}^3_+ \cap \mathbb{R}^3_-$ in $\mathbb{R}^4_+$, while applying the isotopy ϕ, until we reach $\mathbb{R}^3_+$. The following result is [23, Lemma 3.3], which immediately follows from the work of Hatcher [12].

Lemma 2.2. *Let φ be an automorphism of $(\mathbb{S}^3, K)$ such that $\varphi|_{-B^3} = \mathrm{Id}_{-B^3}$. Then there is an isotopy $\phi\colon I \times \mathbb{S}^3 \to \mathbb{S}^3$ as in Definition 2.1, such that $\phi_1 = \varphi$. Furthermore, the isotopy class of the deform-spun disk $D_{K,\phi}$ only depends on φ, which we denote by $D_{K,\varphi}$.*

We now recall the definition of roll-spinning, based on the description of Litherland [27, Example 2.2]. The following is [23, Definition 3.5]:

Definition 2.3. Let K be a knot in $\mathbb{S}^3$. Choose a tubular neighborhood $N(K)$ of K as well as an identification $N(K) \approx K \times B^2$ which induces the Seifert framing of K. Let $X = \mathbb{S}^3 \setminus \mathrm{int}(N(K))$ be the knot exterior. Furthermore, let $\partial X \times I$ be a collar of ∂X in X, with $\partial X \times \{0\}$ be identified with $\partial X \subseteq X$. We identify K with $\mathbb{R}/\mathbb{Z}$. Choose a smooth monotonic function $\varphi\colon \mathbb{R} \to I$ such that $\varphi(s) = 0$ for $s \le 0$ and $\varphi(s) = 1$ for $s \ge 1$. We define the *rolling diffeomorphism* $r\colon (\mathbb{S}^3, K) \to (\mathbb{S}^3, K)$ by the formula

$$r(\bar{x}, \bar{\theta}, s) = \left(\overline{x + \varphi(s)}, \bar{\theta}, s\right) \text{ for } (\bar{x}, \bar{\theta}, s) \in K \times \partial B^2 \times I \approx \partial X \times I,$$

and let $r(p) = p$ for $p \in \mathbb{S}^3 \setminus (\partial X \times I)$.

Similarly, we define the *twisting diffeomorphism* $t\colon (\mathbb{S}^3, K) \to (\mathbb{S}^3, K)$ by the formula

$$t(\bar{x}, \bar{\theta}, s) = \left(\bar{x}, \overline{\theta + \varphi(s)}, s\right) \text{ for } (\bar{x}, \bar{\theta}, s) \in K \times \partial B^2 \times I \approx \partial X \times I,$$

and let $t(p) = p$ for $p \in \mathbb{S}^3 \setminus (\partial X \times I)$.

Let $B \subseteq N(K)$ be an open ball that intersects K in an arc. Then $(\mathbb{S}^3 \setminus B, K \setminus B)$ is diffeomorphic to a ball-arc pair (B^3, a), and $r|_B = \mathrm{id}_B$. We define the (k,l)-*twist-roll-spin* of K to be $D_{K, t^k r^l}$.

Note that $D_{K,\mathrm{id}}$ is simply the spun slice disk of K, obtained using the identity deformation. We will call this the *canonical* slice disk of $-K \# K$.

2.2. **Metric filtrations.**

Definition 2.4. Let X be a set. We say that a function $\mu\colon X \times X \to \mathbb{R}^{\ge 0}$ is a *metric filtration* on X if it is symmetric, and satisfies the *ultrametric inequality*

$$\mu(x, x'') \le \max\{\mu(x, x'), \mu(x', x'')\}$$

for every x, x', $x'' \in X$.

The metric filtrations appearing in the paper will all be instances of the following construction.

Example 2.5. Let X be a path-connected topological space, and $f\colon X \to \mathbb{R}^{\geq 0}$ a continuous function. Given points x, $x' \in X$, we define

$$\mu(x,x') := \inf_{\substack{\gamma\colon I \to X \\ \gamma(0)=x,\ \gamma(1)=x'}} \max_{t \in I} (f \circ \gamma)(t),$$

where the infimum is taken over continuous paths γ. This is clearly a metric filtration. Note that $\mu(x,x) = f(x)$.

Similarly, let $G = (V,E)$ be a connected graph, and $f\colon V \to \mathbb{R}^{\geq 0}$ a function. For x, $x' \in V$, we define $\mu(x,x')$ to be the infimum of

$$\max\{f(x_1),\dots,f(x_n)\}$$

over paths $x_1,\dots,x_n$ in V such that $x_1 = x$ and $x_n = x'$. This is a special case of the above construction: Set X to be the 1-complex associated to G, and extend f over the 1-cells of X linearly. Typically we will be interested in graphs where f is integrally valued on V. For these graphs, we can replace the infimum with a minimum.

Definition 2.6. Let μ be a metric filtration on the set X. Then we define its *normalization* as

$$\widetilde{\mu}(x,x') := \mu(x,x') - \min\{\mu(x,x),\mu(x',x')\}.$$

Recall that $g\colon X \times X \to \mathbb{R}^{\geq 0}$ is a pseudometric on X if it is symmetric, satisfies the triangle inequality, and $g(x,x) = 0$ for every $x \in X$ (but $g(x,y) = 0$ does not necessarily imply that $x = y$).

Lemma 2.7. *Let μ be a metric filtration on the set X. Then its normalization $\widetilde{\mu}$ is a pseudometric.*

Proof. Choose points x, x', $x'' \in X$, and write $a = \mu(x,x)$, $a' = \mu(x',x')$, and $a'' = \mu(x'',x'')$. It is clear that $\widetilde{\mu}(x,x) = 0$. If we apply the ultrametric inequality to the triple x, x', x we obtain that $\mu(x,x') \geq a$. Similarly, by considering the triple x', x, x', we get that $\mu(x,x') \geq a'$, and hence $\mu(x,x') \geq \max\{a,a'\}$. In particular, $\widetilde{\mu}(x,x') \geq 0$.

It remains to prove the triangle inequality

$$\widetilde{\mu}(x,x'') \leq \widetilde{\mu}(x,x') + \widetilde{\mu}(x',x'').$$

By definition,

$$\widetilde{\mu}(x,x') = \mu(x,x') - \min\{a,a'\} \quad \text{and} \quad \widetilde{\mu}(x',x'') = \mu(x',x'') - \min\{a',a''\}.$$

Without loss of generality, we can assume that

$$\max\{\mu(x,x'),\mu(x',x'')\} = \mu(x',x'').$$

Hence, by the ultrametric inequality, $\mu(x,x'') \leq \mu(x',x'')$. So it suffices to show that

$$-\min\{a,a''\} \leq \mu(x,x') - \min\{a,a'\} - \min\{a',a''\}.$$

We saw that $\max\{a,a'\} \leq \mu(x,x')$, hence we only need to prove that

$$\min\{a,a'\} + \min\{a',a''\} \leq \max\{a,a'\} + \min\{a,a''\}.$$

This holds because $\max\{a,a'\}$ bounds both terms on the left-hand side from above, while $\min\{a,a''\}$ bounds at least one of them from above. □

2.3. **The stabilization distance.** In this section, we describe a collection of topological numerical invariants associated to pairs of surfaces bounding a knot. Given a properly embedded surface S in a 4-manifold W, we describe several ways of increasing the genus of S within W. Our description is inspired by Baykur and Sunukjian [3]. The most general notion, and the one we will focus on in this paper, is the following:

Definition 2.8. Let S be a connected and properly embedded surface in a 4-manifold W. Suppose that $B^4 \subseteq \operatorname{int}(W)$ is an embedded 4-ball such that $\partial B^4 \cap S$ is an unlink of m components, written as $U_1 \cup \dots \cup U_m$. Furthermore, suppose that $S \cap B^4$ is a collection of m pairwise disjoint and properly embedded disks $D_1 \cup \dots \cup D_m$ that can simultaneously be smoothly isotoped into ∂B^4 relative to their boundaries. Let S_0 be an oriented, connected, properly embedded genus n surface in B^4 such that $\partial S_0 = U_1 \cup \dots \cup U_m$. The surface

$$S' := (S \setminus \mathrm{int}(B^4)) \cup S_0$$

is the (m,n)-*stabilization* of S along (B^4, S_0). We say the *genus* of an (m,n)-stabilization is $m+n-1$, which we note is $g(S') - g(S)$.

If S' is the (m,n)-stabilization of S, then we say that S is the (m,n)-*destabilization* of S'. In this paper, a *stabilization* refers to an (m,n)-stabilization and a *destabilization* to an (m,n)-destabilization for some m and n.

A schematic of an (m,n)-stabilization is shown in Figure 2.1.

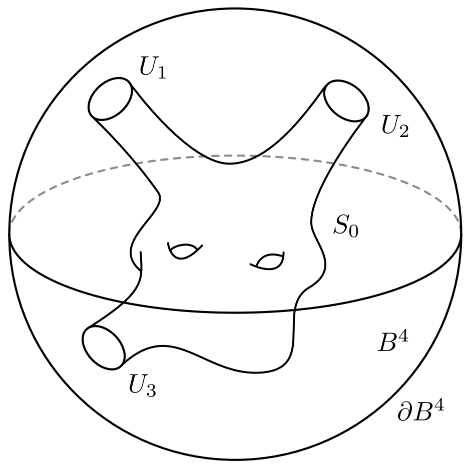

FIGURE 2.1. A $(3,2)$-stabilization.

Note that performing a $(1,0)$-stabilization is the same as taking the connected sum of a surface S with a 2-knot contained in a small 4-ball, disjoint from S. We additionally define the following special cases of stabilization:

Definition 2.9. Let S be a properly embedded surface in the 4-manifold W, and suppose that S' is obtained from S by an (m,n)-stabilization along (B^4, S_0).

(1) We say S' is an *unknotted surface stabilization* of S if $m = 1$ and S_0 is smoothly isotopic into ∂B^4 relative to ∂S_0. If, furthermore, $n = 1$, then we call this a *trivial stabilization*.
(2) We say S' is a *1-handle stabilization* of S if $(m,n) = (2,0)$ and $S_0 \cup D_1 \cup D_2$ is the boundary of a 3-dimensional 1-handle that is embedded in W.

Unknotted surface stabilization and 1-handle stabilization have been introduced by Boyle [4], and studied further by Baykur and Sunukjian [3].

Lemma 2.10. *The surface S' can be obtained from S by a genus g unknotted surface stabilization if and only if it can be obtained by g disjoint trivial stabilizations.*

Proof. Suppose that S' is obtained from S by an unknotted surface stabilization along (B^4, S_0). After isotoping S_0 into $\mathbb{S}^3$, it becomes a Seifert surface of the unknot ∂S_0. If $g > 0$, the map

$$\pi_1(S_0 \setminus N(\partial S_0)) \to \pi_1(\mathbb{S}^3 \setminus N(\partial S_0)) \cong \mathbb{Z}$$

is not injective, hence S_0 is compressible in $\mathbb{S}^3$ by the loop theorem. Compressing corresponds to reversing a 1-handle stabilization in $\mathbb{S}^3$. If we push the interior of the 1-handle into B^4, it becomes unknotted. By induction on the genus of S_0, we see that S' can be obtained from S by g consecutive trivial stabilizations. However, in dimension 4, we can always isotope consecutive trivial stabilizations to be disjoint from each other. The opposite implication is straightforward. $\square$

As in [23, Definition 3.8], we can define the *peripheral map*

$$h_S \colon \pi_1(\partial W \setminus \partial S) \to \pi_1(W \setminus S)$$

for a properly embedded surface S in W. Given surfaces S and S' in W with $\partial S = \partial S'$, we say that their peripheral maps are *equivalent* if there is an isomorphism

$$g \colon \pi_1(W \setminus S) \to \pi_1(W \setminus S')$$

such that $h_{S'} = g \circ h_S$. The equivalence class of the peripheral map is clearly an invariant of S up to ambient isotopy in W fixing ∂W pointwise.

Lemma 2.11. *Suppose that S' is obtained from S by a trivial stabilization. Then their peripheral maps are equivalent.*

Proof. Boyle [4, Lemma 11] showed that $\pi_1(W \setminus S) \cong \pi_1(W \setminus S')$. Indeed, a trivial stabilization corresponds to taking the connected sum of (W, S) and $(\mathbb{S}^4, T^2)$, where T^2 is an unknotted torus. Since $\mathbb{S}^4 \setminus T^2$ is homotopy equivalent to the suspension of $\mathbb{S}^3 \setminus T^2$, we have $\pi_1(\mathbb{S}^4 \setminus T^2) \cong \mathbb{Z}$, generated by the meridian of T^2. Hence, the claim follows from the Seifert–van Kampen theorem. As the connected sum is taken in the interior of W, it follows that the peripheral maps are equivalent. □

As shown by Boyle [4, Lemma 9], a nontrivial 1-handle stabilization might change the fundamental group of the surface complement, though it is always a quotient of the original group. Based on his work, Baykur and Sunukjian [3, Lemma 3] determined when two 1-handle stabilizations give isotopic surfaces. If the 1-handle h is attached along the points a, $b \in S$, then we can act on the homotopy class of the core of h by either adding the class of the meridian of S, or pre- or post-composing with the push-off of a loop in $\pi_1(S, a)$ or $\pi_1(S, b)$. The equivalence class of the homotopy class of the core of h determines the resulting surface up to isotopy, also in the case when Σ has boundary.

Corollary 2.12. *Let D_φ and $D_{\varphi'}$ be deform-spun slice disks of a knot $-K \# K$, where φ and φ' are non-isotopic automorphisms of $(\mathbb{S}^3, K)$ that are fixed in a neighborhood of a point of K. Then one cannot obtain $D_{\varphi'}$ from D_φ by a sequence of trivial stabilizations and destabilizations (or, equivalently, by unknotted surface stabilizations and destabilizations).*

Proof. According to the proof of [23, Proposition 3.9], the peripheral maps of D_φ and $D_{\varphi'}$ are inequivalent. Hence, the result follows from Lemma 2.11. □

Proposition 2.13. *Let S and S' be compact, properly embedded surfaces in the compact 4-manifold W such that $\partial S = \partial S'$ and $[S] = [S'] \in H_2(W, \partial S)$. Then Σ and Σ' become ambient isotopic relative to ∂W after finitely many 1-handle stabilizations.*

Proof. This is a relative version of [3, Theorem 5], and the proof is analogous. The idea is that one removes a neighborhood of $S \cap S'$, and chooses a relative Seifert manifold M for

$$(S \cup -S') \setminus N(S \cap S').$$

Then a self-indexing Morse function on M with only index 1 and 2 critical points that is minimal along S and maximal along S' gives the required handles. □

Definition 2.14. Suppose that S and S' are connected, properly embedded surfaces in the compact 4-manifold W such that $\partial S = \partial S'$ is a knot K in ∂W. We define the *stabilization distance* of the pair (S, S'), for which we write $\mu_{\text{st}}(S, S')$, to be the minimum of

$$\max\{g(S_1), \dots, g(S_k)\}$$

over sequences of connected, properly embedded surfaces $S_1, \dots, S_k$ in W such that

(1) $S_1 = S$ and $S_k = S'$,
(2) $\partial S_i = K$ for $i \in \{1, \dots, k\}$, and
(3) S_i and S_{i+1} are related by a stabilization or destabilization, up to proper isotopy, for $i \in \{1, \dots, k-1\}$.

We define $\mu_{\text{st}}(S,S')$ to be ∞ if no such sequence exists. Analogously, we define $\overline{\mu}_{\text{st}}(S,S')$ by minimizing

$$\max\{g(S_1),\dots,g(S_k)\}-\min\{g(S_1),\dots,g(S_k)\}$$

over the same set of sequences. Finally, we let

$$\widetilde{\mu}_{\text{st}}(S,S')=\mu_{\text{st}}(S,S')-\min\{g(S),g(S')\},$$

which we call the *normalized stabilization distance.*

The *(trivial) 1-handle distance* of S and S' is defined similarly to μ_{st}, except that S_i and S_{i+1} are related by adding or removing a (trivial) 1-handle. We denote the 1-handle distance by $\mu_1(S,S')$, and the trivial 1-handle distance by $\mu_1^0(S,S')$.

We observe that

$$\widetilde{\mu}_{\text{st}}(S,S')\le\overline{\mu}_{\text{st}}(S,S')\le\mu_{\text{st}}(S,S')\le\mu_1(S,S')\le\mu_1^0(S,S').$$

Furthermore, if $[S]=[S']$ in $H_2(W,K)$, then $\mu_1(S,S')$ is finite by Proposition 2.13, and hence so are $\mu_{\text{st}}(S,S')$ and $\overline{\mu}_{\text{st}}(S,S')$. On the other hand, $\mu_1^0(S,S')$ might be infinite by Corollary 2.12. Note that $\overline{\mu}_{\text{st}}(S,S')=0$ if and only if S and S' become isotopic after taking connected sums with 2-knots. Since $\mu_{\text{st}}(S,S)=g(S)$, the normalized distance satisfies $\widetilde{\mu}_{\text{st}}(S,S)=0$.

Consider the graph whose vertices are isotopy classes (rel. boundary) of connected surfaces in W with boundary the knot K in a fixed relative homology class in $H_2(W,K)$, and whose edges correspond to (m,n)-stabilization for some m and n. If we apply the procedure outlined in Example 2.5 to the genus function, we obtain μ_{st}, which is hence a metric filtration in the sense of Definition 2.4. Its normalization in the sense of Definition 2.6 is $\widetilde{\mu}_{\text{st}}$. So, as a special case of Lemma 2.7, we obtain the following:

Lemma 2.15. *Let W be a compact 4-manifold, and K a knot in ∂W. The normalized stabilization distance $\widetilde{\mu}_{\text{st}}$ is a pseudometric when restricted to surfaces in a given class in $H_2(W,K)$.*

For a knot K in $\mathbb{S}^3$, let us write $\text{Surf}(K)$ for the set of isotopy classes of connected, oriented, properly embedded surfaces in B^4 with boundary K, and $\text{Surf}(K)/\{\text{2-knots}\}$ for $\text{Surf}(K)$ modulo genus 0 stabilizations. We denote by $\text{Surf}_g(K)$ the subset of genus g surfaces in $\text{Surf}(K)$. If K is slice, we will write

$$\mathcal{D}(K):=\text{Surf}_0(K)$$

for the set of isotopy classes of slice disks of K in B^4, and $\mathcal{D}(K)/\{\text{2-knots}\}$ for $\mathcal{D}(K)$ modulo genus 0 stabilizations.

Remark 2.16. Note that $(\text{Surf}(K)/\{\text{2-knots}\},\widetilde{\mu}_{\text{st}})$ is a *pseudometric space* (i.e., $\widetilde{\mu}_{\text{st}}(S,S')=0$ can hold for $S\neq S'$ in $\text{Surf}(K)/\{\text{2-knots}\}$), while $(\text{Surf}(K)/\{\text{2-knots}\},\overline{\mu}_{\text{st}})$ is a *metric space.* Furthermore, μ_{st} is a metric filtration, so it satisfies the ultrametric inequality

$$\mu_{\text{st}}(S,S'')\le\max\{\mu_{\text{st}}(S,S'),\mu_{\text{st}}(S',S'')\}$$

for any S, S', $S''\in\text{Surf}(K)$, and so does $\widetilde{\mu}_{\text{st}}$ when restricted to $\text{Surf}_g(K)$.

If one of S, $S'\in\text{Surf}(K)$ is a disk, and $S_1,\dots,S_k$ is a sequence of surfaces connecting S and S', as in Definition 2.14, then

$$\min\{g(S_1),\dots,g(S_k)\}=0,$$

so $\mu_{\text{st}}(S,S')$ and $\overline{\mu}_{\text{st}}(S,S')$ are both obtained by minimizing $\max\{g(S_1),\dots,g(S_k)\}$. Hence μ_{st} and $\overline{\mu}_{\text{st}}$ agree on $\mathcal{D}(K)/\{\text{2-knots}\}$, and $(\mathcal{D}(K)/\{\text{2-knots}\},\mu_{\text{st}})$ is an *ultrametric space*; i.e., a metric space that satisfies the ultrametric inequality. Our invariants from Heegaard Floer homology naturally give bounds on μ_{st}, hence we will not study $\overline{\mu}_{\text{st}}$ in the rest of this paper.

Definition 2.17. If K is a slice knot and $S\in\text{Surf}(K)$, we define the *destabilizing genus* $g_{\text{dest}}(S)$ of S to be the minimum of

$$\max\{g(S_1),\dots,g(S_n)\}$$

over sequences of properly embedded surfaces $S_1,\dots,S_n$ in B^4 such that

(1) S_{i+1} is obtained from S_i via stabilization or destabilization for $i\in\{1,\dots,n-1\}$,

(2) $S_1 = S$, and S_n is a slice disk of K.

By definition, $g_{\text{dest}}(S) \geq g(S)$. Furthermore, if D is a slice disk of K, then $g_{\text{dest}}(S) \leq \widetilde{\mu}_{\text{st}}(S, D)$, and hence $g_{\text{dest}}(S)$ is finite. In fact, $g_{\text{dest}}(S)$ is the distance of S from $\mathcal{D}(K)/\{2\text{-knots}\}$ in the pseudometric space $(\text{Surf}(K)/\{2\text{-knots}\}, \widetilde{\mu}_{\text{st}})$.

Proposition 2.18. *Let S_1, $S_2 \in \text{Surf}(K)$. Then*

$$\mu_{\text{st}}(S_1, S_2) \leq 2g(K) + \max\{g(S_1), g(S_2)\},$$

where $g(K)$ is the Seifert genus of K.

Proof. Let S be a minimal genus Seifert surface for K, and choose an open ball $B \subseteq \text{int}(S)$. Consider the product $S \times I \subseteq \mathbb{S}^3$, where we identify S with $S \times \{0\}$. For $i \in \{1, 2\}$, isotope S_i near ∂S_i such that it becomes a surface S_i' with boundary $K \times \{1\}$. We let

$$\Sigma_i := ((S \setminus B) \times \{0\}) \cup (\partial B \times I) \cup ((S \setminus B) \times \{1\}) \cup S_i'.$$

Then Σ_i is a surface of genus $2g(K) + g(S_i)$ that can be obtained from S_i by $2g(K)$ 1-handle stabilizations in $\mathbb{S}^3$. Indeed, let $a_1, \dots, a_{2g}$ be pairwise disjoint arcs in $S \setminus B$ with boundary on ∂B that span $H_1(S, B)$. If we compress Σ_i along the curves

$$(a_i \times \{0, 1\}) \cup (\partial a_i \times I)$$

using the compressing disks $a_i \times I \subseteq S \times I$, we obtain S_i, up to proper isotopy.

If we push $\Sigma_i \setminus (S \times \{0\})$ into $\text{int}(B^4)$ relative to $\partial B \times \{0\}$, we obtain a surface Σ_i' that is a stabilization of S with $m = 1$. The sequence of surfaces S_1, Σ_1', S, Σ_2', S_2 satisfies the requirements of Definition 2.14, and has maximal genus $2g(K) + \max\{g(S_1), g(S_2)\}$. □

Corollary 2.19. *If K is a slice knot and $S \in \text{Surf}_g(K)$, then*

$$g_{\text{dest}}(S) \leq 2g(K) + g.$$

Proposition 2.20. *Let K be a slice knot, and S, $S' \in \text{Surf}(K)$. Then*

$$\widetilde{\mu}_{\text{st}}(S, S') \geq |g_{\text{dest}}(S) - g_{\text{dest}}(S')|.$$

Proof. Since the claim is symmetric in S and S', we can assume that $g_{\text{dest}}(S) \leq g_{\text{dest}}(S')$. Let D be a slice disk for K such that $\widetilde{\mu}_{\text{st}}(D, S) = g_{\text{dest}}(S)$. Then

$$g_{\text{dest}}(S) + \widetilde{\mu}_{\text{st}}(S, S') = \widetilde{\mu}_{\text{st}}(D, S) + \widetilde{\mu}_{\text{st}}(S, S') \geq \widetilde{\mu}_{\text{st}}(D, S') \geq g_{\text{dest}}(S')$$

by the triangle inequality, and the result follows. □

2.4. **An upper bound on the distance between 1-roll-spun and 1-twist-spun slice disks.** Let $t^n r^m$ denote the n-twist-m-roll-spinning diffeomorphism of K. We will show the following:

Proposition 2.21. *If K is a knot in $\mathbb{S}^3$, then the deform-spun slice disks $D_{K,r}$, $D_{K,t} \in \text{Surf}_0(-K\#K)$ satisfy*

$$\mu_{st}(D_{K,r}, D_{K,t}) \leq 2.$$

Proof. Let $B_0 \subseteq \mathbb{S}^3$ denote a 3-ball that intersects K in an unknotted arc. We consider the knotted ball-arc pair (B, a), where $B := \mathbb{S}^3 \setminus \text{int}(B_0)$ and $a = K \cap B$. We present both slice disks $D_{K,r}$ and $D_{K,t}$ as movies of ball-arc pairs which start and end at (B, a).

We begin with describing a movie for $D_{K,t^n r}$ for any $n \in \mathbb{N}$. We pick a diagram $\mathcal{D}$ for K with $\text{wr}(\mathcal{D}) = n$. We view the diagram $\mathcal{D}$ as nearly being embedded in a plane P. The movie for $D_{K,t^n r}$ consists of rotating the diagram $\mathcal{D}$ about an axis perpendicular to P (and shifting along the axis perpendicular to P slightly), while translating $\mathcal{D}$ in the plane so that the image of K intersects B in an unknotted arc. This is a movie for $D_{K,t^n r}$ for some n. The exponent n is equal to the difference between the blackboard framing and the Seifert framing. Since the difference between the blackboard framing and the Seifert framing is $\text{wr}(\mathcal{D})$, it follows that this movie represents $D_{K,t^{\text{wr}(\mathcal{D})} r}$.

Next, we describe a movie for the slice disk $D_{K,t}$. We pick a line ℓ in $\mathbb{R}^3$ which coincides with K inside B_0, and is disjoint from K outside a small neighborhood of B_0. The movie for $D_{K,t}$ is

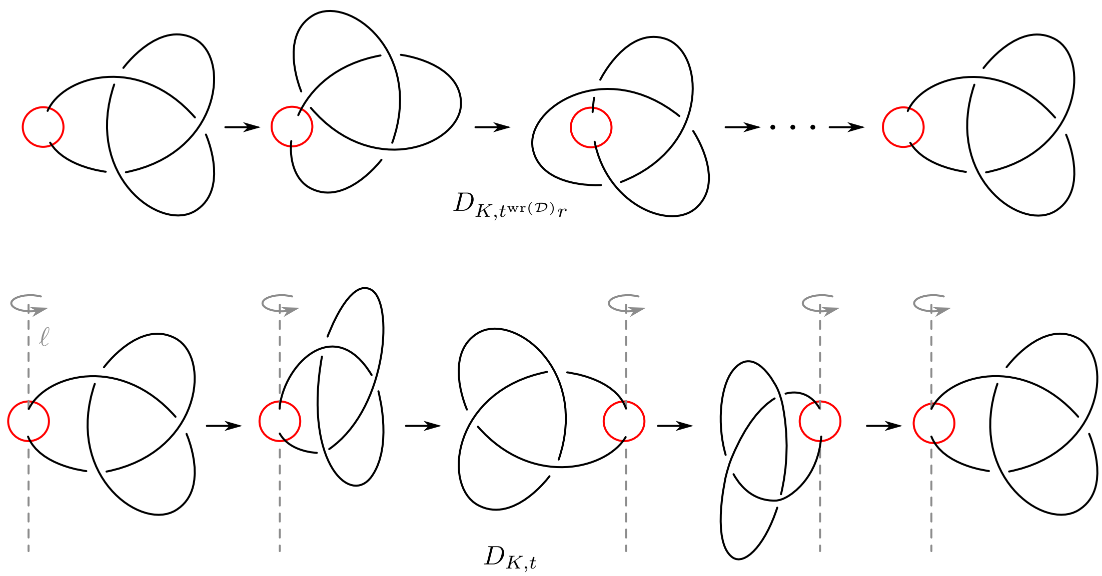

FIGURE 2.2. The slice disks $D_{K,t^{\mathrm{wr}(\mathcal{D})}r}$ and $D_{K,t}$ of $-K\#K$. In the top row, we rotate the diagram counterclockwise a full turn in the plane, and consecutive frames differ by a small rotation.

obtained by rotating a in a full twist about ℓ. Schematics of the movies for $D_{K,t^{\mathrm{wr}(\mathcal{D})}r}$ and $D_{K,t}$ are shown in Figure 2.2.

We now present a stabilization sequence from $D_{K,t^{\mathrm{wr}(\mathcal{D})}r}$ to $D_{K,t}$ that has maximal genus two. Let us write $\{a_s : s \in I\}$ for the movie of arcs corresponding to $D_{K,t^{\mathrm{wr}(\mathcal{D})}r}$. Suppose that $a_s = \phi_s(K) \cap B$ for a 1-parameter family of rigid motions $\phi_s \colon \mathbb{S}^3 \to \mathbb{S}^3$ for $s \in I$ that nearly preserve the plane P. We give K a parametrization $\gamma(s)$ such that $\phi_s(\gamma(s))$ is the center point of the ball $B_0 = \mathbb{S}^3 \setminus \mathrm{int}(B)$ (note that B_0 is the region inside the red ball in Figure 2.2).

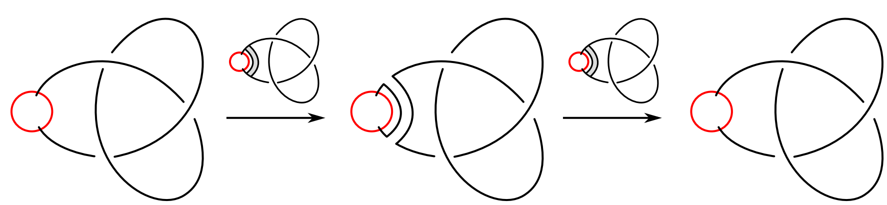

FIGURE 2.3. Attaching a 1-handle to the disk $D_{K,t}$ by adding a pair of bands to the beginning of the movie for $D_{K,t}$.

Let $0 < s_1 < \cdots < s_n < 1$ be the times such that $\gamma(s_i)$ is the lower point of a crossing of $\mathcal{D}$, which we denote c_i. Let $\mathrm{Cr}(\mathcal{D})$ be the set $\{c_1, \dots, c_n\}$ of crossings of $\mathcal{D}$. As s passes s_i for some $i \in \{1, \dots, n\}$, the overstrand of the crossing c_i passes over ∂B in the movie a_s.

As a first step, we attach a 1-handle to $D_{K,t^{\mathrm{wr}(\mathcal{D})}r}$, by adding two bands to the beginning of the movie a_s, as in Figure 2.3. We can move the second band to the end of the movie. This breaks each arc a_s into a knot K_s disjoint from ∂B, which we can view as a copy of our original knot K, as well as a small, boundary-parallel arc attached to ∂B. We now wish to continuously pull the family of knots K_s downward, such that they do not pass over ∂B for any s (or, phrased another way, such that there is a path from ∂B to $\infty \in \mathbb{S}^3$ disjoint from K_s for all $s \in I$).

Given $\mathbf{c} \subseteq \mathrm{Cr}(\mathcal{D})$, we let $S_{\mathbf{c}} \in \mathrm{Surf}_1(-K\#K)$ denote the genus one surface obtained by modifying the movie $\{a_s : s \in I\}$ such that the upper strand of c_i passes over ∂B if $c_i \in \mathbf{c}$, and the upper strand of c_i passes under ∂B if $c_i \notin \mathbf{c}$; see Figure 2.4.

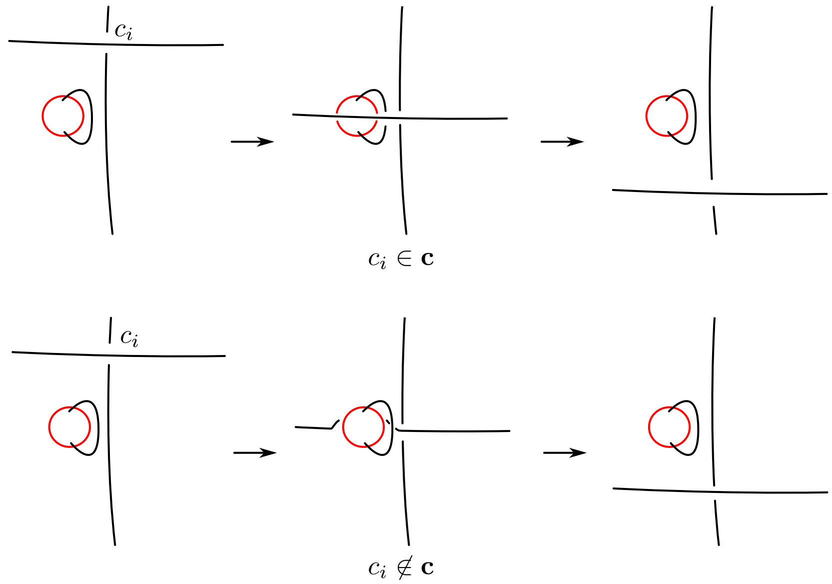

FIGURE 2.4. On the top row, we show a portion of the surface $S_{\mathbf{c}}$ when $c_i \in \mathbf{c}$. The upper strand of a crossing c_i passes over ∂B. On the bottom row, we show a portion of the movie for $S_{\mathbf{c}}$ when $c_i \notin \mathbf{c}$. The upper strand of the crossing c_i passes underneath ∂B.

Let c_i and c_j be consecutive crossings in $\mathbf{c}$ of opposite sign. We claim that $S_{\mathbf{c}}$ and $S_{\mathbf{c}\setminus\{c_i,c_j\}}$ become isotopic after a single stabilization. The stabilization is obtained by attaching a band connecting the upper strands of the crossings c_i and c_j, followed by attaching the dual band; see Figure 2.5.

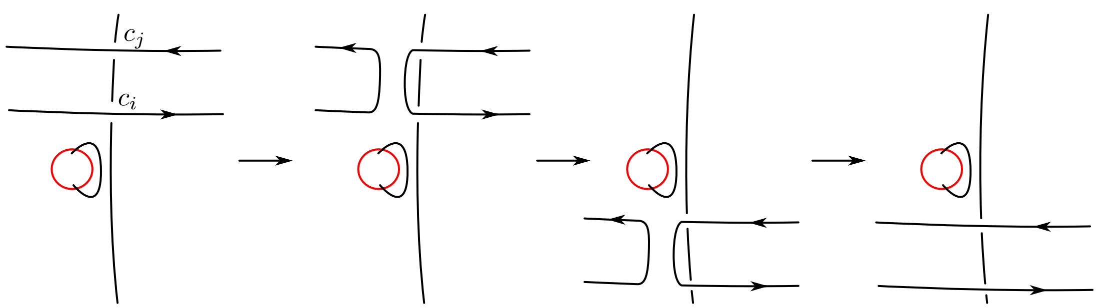

FIGURE 2.5. A movie for a common stabilization of $S_{\mathbf{c}}$ and $S_{\mathbf{c}\setminus\{c_i,c_j\}}$, when c_i and c_j are consecutive crossings in $\mathbf{c}$ of opposite sign. In the movie, a band is added between the first and the second frames. An isotopy connects the second and third frames. The third and fourth frames are related by attaching the dual band.

In the case that $\mathrm{wr}(\mathcal{D}) = 0$, the number of positive crossings is equal to the number of negative crossings, so we can eliminate all crossings from $\mathbf{c}$ pairwise via the stabilization sequence described above. Hence

$$\mu_{st}(S_{\mathrm{Cr}(\mathcal{D})}, S_\emptyset) \leq 2.$$

We note that $S_{\mathrm{Cr}(\mathcal{D})}$ is a genus one stabilization of $D_{K,r}$, while the surface $S_\emptyset$ is described by a movie that starts at (B, a), then has a copy of K break off and move away from ∂B. This copy of K rotates $\mathrm{tw}(\mathcal{D})$-many times near a plane far away from ∂B, where $\mathrm{tw}(\mathcal{D})$ denotes the *twisting number* of $\mathcal{D}$, and then K is reattached to the arc on ∂B via a band.

It is a general fact that $\mathrm{tw}(\mathcal{D})+\mathrm{wr}(\mathcal{D})$ is always odd for any diagram of a knot (as can be verified by noting that Reidemeister moves and crossing changes do not change the quantity $\mathrm{tw}(\mathcal{D})+\mathrm{wr}(\mathcal{D})$ modulo 2, and that $\mathrm{tw}+\mathrm{wr}=1$ for a trivial diagram of an unknot). Since we picked $\mathcal{D}$ to satisfy $\mathrm{wr}(\mathcal{D})=0$, we conclude that

$$\mathrm{tw}(\mathcal{D})\equiv 1 \pmod 2.$$

Since a path of rotations about a line in $\mathbb{R}^3$ induces the generator of $\pi_1(\mathrm{SO}(3))\cong\mathbb{Z}_2$, we can assume that the copy of K which breaks off rotates in a plane exactly once.

Hence, our movie for $S_\emptyset$ becomes a copy of K which breaks off of (B,a) as in Figure 2.3, then makes one full rotation in a plane (away from ∂B), and is finally reconnected to the arc attached to ∂B. A continuous deformation transforms this final movie into a stabilization of the movie for $D_{K,t}$ shown in Figure 2.2. □

We conclude this section with the following conjecture (compare Proposition 10.1, below).

Conjecture 2.22. *If K is a knot in $\mathbb{S}^3$, then the deform-spun slice disks $D_{K,r}$, $D_{K,t}\in\mathrm{Surf}_0(-K\#K)$ satisfy*

$$\mu_{st}(D_{K,r},D_{K,t})\le 1.$$

3. Background on the link Floer TQFT

In this section, we recall some previous results about the link Floer TQFT which we will need to compute the effect of stabilization on the link cobordism maps in Section 5, to determine the map induced by the summand-swapping diffeomorphism in Section 8, and to prove the trace formula in Section 9.

3.1. The full link Floer TQFT.

We first recall the category whose objects are multi-based links, and whose morphisms are decorated link cobordisms, following the notation of the second author [58]; see also the equivalent construction of the first author [18].

Definition 3.1. A *multi-based link* $\mathbb{L}=(L,\mathbf{w},\mathbf{z})$ in a closed, oriented (not necessarily connected) 3-manifold Y is an oriented link $L\subseteq Y$, together with two disjoint collections of basepoints $\mathbf{w}$, $\mathbf{z}\subseteq L$ such that

1. each component of L has at least two basepoints;
2. the basepoints along a link component of L alternate between $\mathbf{w}$ and $\mathbf{z}$, as one traverses the link;
3. each component of Y has at least one component of L, and each component of L has at least two basepoints.

Definition 3.2. Let Y_1 and Y_2 be 3-manifolds containing multi-based links $\mathbb{L}_1=(L_1,\mathbf{w}_1,\mathbf{z}_1)$ and $\mathbb{L}_2=(L_2,\mathbf{w}_2,\mathbf{z}_2)$, respectively. A *decorated link cobordism* from $(Y_1,\mathbb{L}_1)$ to $(Y_2,\mathbb{L}_2)$ is a pair $(W,\mathcal{F})=(W,(S,\mathcal{A}))$, where

1. W is an oriented cobordism from Y_1 to Y_2,
2. S is a properly embedded, oriented surface in W with $\partial S=-L_1\cup L_2$, and
3. $\mathcal{A}$ is a properly embedded 1-manifold in S that divides S into two subsurfaces, $S_{\mathbf{w}}$ and $S_{\mathbf{z}}$, that meet along $\mathcal{A}$, such that $\mathbf{w}_1$, $\mathbf{w}_2\subseteq S_{\mathbf{w}}$ and $\mathbf{z}_1$, $\mathbf{z}_2\subseteq S_{\mathbf{z}}$.

Multi-based links and equivalence classes of decorated link cobordisms form a category.

The first author [18] showed that decorated link cobordisms induce functorial maps on the hat version of link Floer homology. The second author [58] extended this to the *full infinity complex*, denoted $\mathcal{CFL}^\infty$, which is a minor variation of the infinity complexes of Ozsváth–Szabó [37] and Rasmussen [44]. We now review the construction of $\mathcal{CFL}^\infty$.

Let $\mathcal{R}^-$ denote the ring $\mathbb{F}_2[U,V]$, and let $\mathcal{R}^\infty$ denote the ring $\mathbb{F}_2[U,V,U^{-1},V^{-1}]$, obtained by inverting U and V in $\mathcal{R}^-$. Suppose that $\mathbb{L}=(L,\mathbf{w},\mathbf{z})$ is a multi-based link in Y. Given a multi-pointed diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ for $(Y,\mathbb{L})$ (see [41, Definition 3.1]), the complex $\mathcal{CFL}^\infty(Y,\mathbb{L},\mathfrak{s})$ is the free module over $\mathcal{R}^\infty$ generated by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. Over $\mathbb{F}_2$, the generators are the monomials

$$U^i V^j \cdot \mathbf{x},$$

where $i,\, j \in \mathbb{Z}$.

The module $\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s})$ has a filtration $\mathcal{G}$ over $\mathbb{Z} \oplus \mathbb{Z}$, where the subset $\mathcal{G}_{(n,m)} \subseteq \mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s})$ is generated over $\mathbb{F}_2$ by monomials $U^i V^j \cdot \mathbf{x}$ with $i \geq n$ and $j \geq m$. We denote the $\mathcal{R}^-$-submodule $\mathcal{G}_{(0,0)}$ by $\mathcal{CFL}^-(Y, \mathbb{L}, \mathfrak{s})$, and call it the *full minus complex*. It is generated over $\mathbb{F}_2$ by monomials $U^i V^j \cdot \mathbf{x}$ with $i,\, j \geq 0$.

There is a filtered, $\mathcal{R}^\infty$-equivariant endomorphism ∂ of $\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s})$, defined by the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) \cdot U^{n_{\mathbf{w}}(\phi)} V^{n_{\mathbf{z}}(\phi)} \cdot \mathbf{y},$$

which satisfies $\partial \circ \partial = 0$.

When $[L] = 0$ in $H_1(Y)$ and $\mathfrak{s}$ is torsion, the chain complex $(\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}), \partial)$ has several gradings. Ozsváth and Szabó defined a homological grading and an Alexander grading. It is convenient for our purposes to repackage their two gradings into three gradings that satisfy a linear dependency. Namely, there are two homological gradings, $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$, and an Alexander grading A, which together satisfy

$$A = \frac{1}{2}(\mathrm{gr}_{\mathbf{w}} - \mathrm{gr}_{\mathbf{z}}).$$

Note that V is $+1$ graded with respect to A, and U is -1 graded.

When $\mathbb{K} = (K, w, z)$ is a doubly-based knot in $\mathbb{S}^3$, we will write $\mathcal{CFL}^\infty(\mathbb{K})$ for $\mathcal{CFL}^\infty(\mathbb{S}^3, \mathbb{K}, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the unique Spin^c structure on $\mathbb{S}^3$.

The second author [58] constructed cobordism maps for the full knot and link Floer complexes. Given a decorated link cobordism $(W, \mathcal{F})$ from $(Y_1, \mathbb{L}_1)$ to $(Y_2, \mathbb{L}_2)$ and a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, there is an induced $\mathcal{R}^\infty$-equivariant, filtered chain map

$$F_{W, \mathcal{F}, \mathfrak{s}} \colon \mathcal{CFL}^\infty(Y_1, \mathbb{L}_1, \mathfrak{s}|_{Y_1}) \to \mathcal{CFL}^\infty(Y_2, \mathbb{L}_2, \mathfrak{s}|_{Y_2}),$$

well-defined up to filtered, $\mathcal{R}^\infty$-equivariant chain homotopy.

3.2. **Basepoint actions on link Floer homology.** Let $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ be a multi-based link in the 3-manifold Y, and fix $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. We recall that, for each $w \in \mathbf{w}$ and $z \in \mathbf{z}$, there are distinguished endomorphisms Φ_w and Ψ_z of $\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s})$. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a diagram for $(Y, \mathbb{L})$, then Φ_w and Ψ_z can be defined via the formulas

$$\Phi_w(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} n_w(\phi) \# \widehat{\mathcal{M}}(\phi) \cdot U^{n_{\mathbf{w}}(\phi) - 1} V^{n_{\mathbf{z}}(\phi)} \cdot \mathbf{y}, \tag{3.1}$$

and

$$\Psi_z(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} n_z(\phi) \# \widehat{\mathcal{M}}(\phi) \cdot U^{n_{\mathbf{w}}(\phi)} V^{n_{\mathbf{z}}(\phi) - 1} \cdot \mathbf{y} \tag{3.2}$$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. According to [56, Lemma 4.1], the endomorphisms Φ_w and Ψ_z are the link cobordism maps induced by the two decorations of $(I \times Y, I \times L)$ shown in Figure 3.1. When we are working with doubly-based knots, we will often write Φ and Ψ for the maps Φ_w and Ψ_z, respectively.

According to [58, Lemma 4.9], the basepoint actions satisfy

$$\Phi_w^2 \simeq \Psi_z^2 \simeq 0. \tag{3.3}$$

Note that the dividing sets on the decorated link cobordisms corresponding to Φ_w^2 and Ψ_z^2 both contain a closed curve that bounds a disk in either $S_{\mathbf{w}}$ or $S_{\mathbf{z}}$.

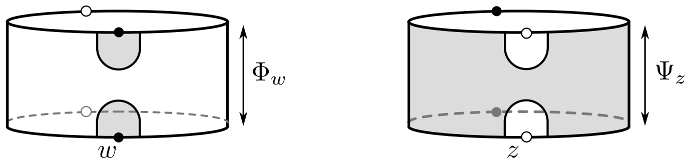

FIGURE 3.1. The two decorated link cobordisms for Φ_w and Ψ_z. The diagrams indicate the decorations on the surface $I \times L$. Here we denote $\mathbf{w}$ by solid basepoints, and $\mathbf{z}$ by open basepoints. The shaded regions denote $\Sigma_{\mathbf{w}}$ and the unshaded regions denote $\Sigma_{\mathbf{z}}$.

3.3. **Quasi-stabilizations and basepoint moving maps.** We now review the quasi-stabilization maps. Suppose $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a multi-based link in Y, and suppose that w and z are two new basepoints contained in a single component of $L \setminus (\mathbf{w} \cup \mathbf{z})$. Let us assume that w immediately follows z with respect to the orientation of L, and write

$$\mathbb{L}^+_{w,z} := (L, \mathbf{w} \cup \{w\}, \mathbf{z} \cup \{z\}).$$

There are two *quasi-stabilization* maps

$$S^+_{w,z}, T^+_{w,z} \colon \mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}) \to \mathcal{CFL}^\infty(Y, \mathbb{L}^+_{w,z}, \mathfrak{s}),$$

as well as two *quasi-destabilization* maps $S^-_{w,z}$ and $T^-_{w,z}$, defined in the opposite direction. If instead z follows w with respect to the orientation of L, then we obtain similar maps $S^\pm_{z,w}$ and $T^\pm_{z,w}$.

We briefly summarize the construction of the quasi-stabilization maps. See [29, Section 6] and [58, Section 4] for further details. Suppose $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a diagram for $(Y, \mathbb{L})$. Let w' and z' denote the basepoints adjacent to w and z on $\mathbb{L}$. There is a component A of $\Sigma \setminus \boldsymbol{\alpha}$ which contains w' and z'. We pick a simple closed curve $\alpha_s \subseteq A$ that cuts A into two components, one of which contains w' and the other z'. We add another curve, β_0, that bounds a small disk on Σ, which is cut into two bigons by α_s, and is disjoint from the other $\boldsymbol{\alpha}$ curves. Inside one bigon, we place w. In the other bigon, we place z. See Figure 3.2.

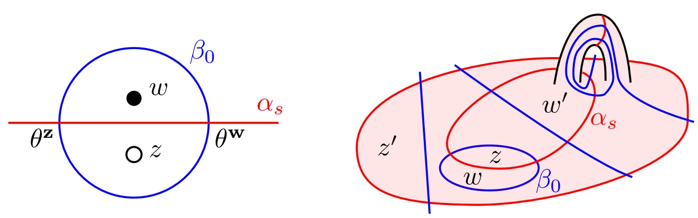

FIGURE 3.2. A quasi-stabilization of Heegaard diagrams. On the left, a local picture of the quasi-stabilization near w and z is shown. On the right, we have a complete example of a quasi-stabilization. The shaded region denotes A. Also shown are the points w' and z'.

We write $\theta^{\mathbf{w}}$ and $\theta^{\mathbf{z}}$ for the higher $\mathrm{gr}_{\mathbf{w}}$- and $\mathrm{gr}_{\mathbf{z}}$-graded intersection points of $\alpha_s \cap \beta_0$, respectively. The maps $S^+_{w,z}$ and $T^+_{w,z}$ are defined via the formulas

$$S^+_{w,z}(\mathbf{x}) := \mathbf{x} \times \theta^{\mathbf{w}} \quad \text{and} \quad T^+_{w,z}(\mathbf{x}) := \mathbf{x} \times \theta^{\mathbf{z}} \tag{3.4}$$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, extended $\mathcal{R}^\infty$-equivariantly. The quasi-destabilization maps are defined dually, via the equations

$$S^-_{w,z}(\mathbf{x} \times \theta^{\mathbf{w}}) = 0, \quad S^-_{w,z}(\mathbf{x} \times \theta^{\mathbf{z}}) = \mathbf{x}, \quad T^-_{w,z}(\mathbf{x} \times \theta^{\mathbf{w}}) = \mathbf{x}, \quad \text{and} \quad T^-_{w,z}(\mathbf{x} \times \theta^{\mathbf{z}}) = 0, \tag{3.5}$$

extended $\mathcal{R}^\infty$-equivariantly. The decorations on $(I \times Y, I \times L)$ inducing the quasi-stabilization maps $S^+_{w,z}$, $S^-_{w,z}$, $T^+_{w,z}$, and $T^-_{w,z}$ are shown in Figure 3.3.

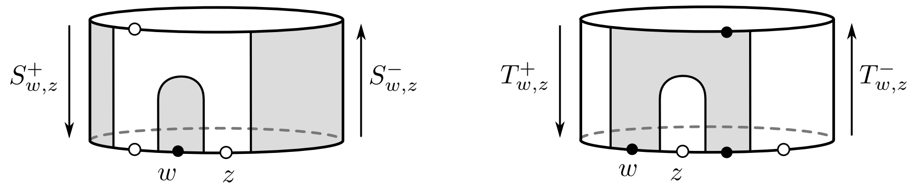

FIGURE 3.3. The decorated link cobordisms for the quasi-stabilization maps.

We will need the following relations between the quasi-stabilization maps and the basepoint actions:

$$T^+_{w,z} \simeq \Psi_z S^+_{w,z}, \tag{3.6}$$

$$T^-_{w,z} \simeq S^-_{w,z}\Psi_z, \tag{3.7}$$

$$\Phi_w \simeq S^+_{w,z}S^-_{w,z}, \tag{3.8}$$

$$\Psi_z \simeq T^+_{w,z}T^-_{w,z}. \tag{3.9}$$

Proofs of equations (3.6)–(3.9) can be found in [58, Lemmas 4.10 and 4.11]. Examples of the corresponding dividing sets for the relations in equations (3.6)–(3.9) appear in Figure 3.4.

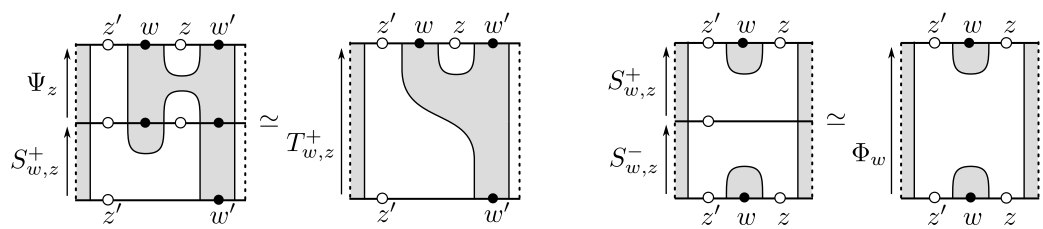

FIGURE 3.4. Dividing set manipulations corresponding to the relations $T^+_{w,z} \simeq \Psi_z S^+_{w,z}$ and $\Phi_w \simeq S^+_{w,z}S^-_{w,z}$.

Next, we review the connection between the basepoint moving maps and the quasi-stabilization maps. We first focus on using the quasi-stabilization maps to move a single basepoint, while fixing all other basepoints. Suppose that $(L, \mathbf{w}_0 \cup \{w'\}, \mathbf{z})$ is a multi-based link in Y. Suppose that (w, z) are a new pair of basepoints in a single component of $L \setminus (\mathbf{w}_0 \cup \{w'\} \cup \mathbf{z})$, such that z is adjacent to w'. Suppose further that, according to the orientation of L, the three basepoints appear in the order w', z, w. We can construct a diffeomorphism

$$\tau^{w \leftarrow w'} \colon (Y, L, \mathbf{w}_0 \cup \{w'\}, \mathbf{z}) \to (Y, L, \mathbf{w}_0 \cup \{w\}, \mathbf{z}),$$

by moving w' to w along the arc connecting them, but fixing all of Y outside a neighborhood of this arc. According to [58, Lemma 4.24],

$$T^+_{w,z} \simeq T^+_{z,w'}\tau^{w\leftarrow w'}_*. \tag{3.10}$$

Equation (3.10) has a simple description in terms of dividing sets and link cobordisms, shown in Figure 3.5.

Using the quasi-stabilization maps, we can also move a pair of adjacent basepoints at the same time. Suppose that $\mathbb{L} = (L, \mathbf{w}_0, \mathbf{z}_0)$ is a multi-based link (though we allow the case where $\mathbf{w}_0$ and $\mathbf{z}_0$ intersect a single component of L trivially). Suppose that (w', z', w, z) are four new consecutive basepoints on a single component of $L \setminus (\mathbf{w}_0 \cup \mathbf{z}_0)$. We assume that these four basepoints appear in the order w', z', w and z on the link, with z being the first and w' being the last with respect to the orientation of L. There is a diffeomorphism

$$\rho^{(w',z')\leftarrow(w,z)} \colon (Y, L, \mathbf{w}_0 \cup \{w\}, \mathbf{z}_0 \cup \{z\}) \to (Y, L, \mathbf{w}_0 \cup \{w'\}, \mathbf{z}_0 \cup \{z'\}),$$

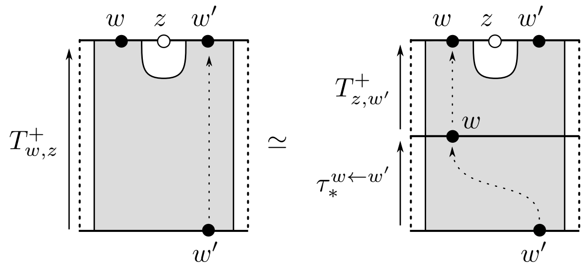

Figure 3.5. The interpretation of equation (3.10) in terms of decorated link cobordisms.

obtained by moving the pair (w,z) to (w',z'), but fixing everything outside a neighborhood of an interval containing the four basepoints (w',z',w,z). According to [58, Lemma 4.27], there is a chain homotopy

$$\rho_*^{(w',z')\leftarrow(w,z)} \simeq S^-_{w,z} T^+_{w',z'}. \tag{3.11}$$

Equation (3.11) can be interpreted in terms of the manipulation of dividing sets shown in Figure 3.6.

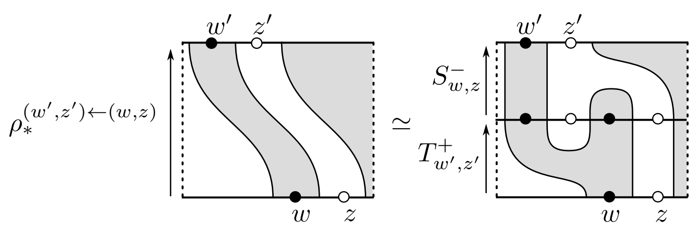

Figure 3.6. The interpretation of the relation $\rho_*^{(w',z')\leftarrow(w,z)} \simeq S^-_{w,z} T^+_{w',z'}$ in terms of dividing sets.

Another useful relation is the following:

$$T^+_{w,z} S^-_{w,z} + S^+_{w,z} T^-_{w,z} + \mathrm{id} \simeq 0. \tag{3.12}$$

See [58, Lemma 4.13]. Note that Equation (3.12) follows immediately from the formulas in Equations (3.4) and (3.5). For a pictorial description, see Figure 3.7. Equation (3.12) is an example of the *bypass relation*, which is often helpful when doing computations in the link Floer TQFT.

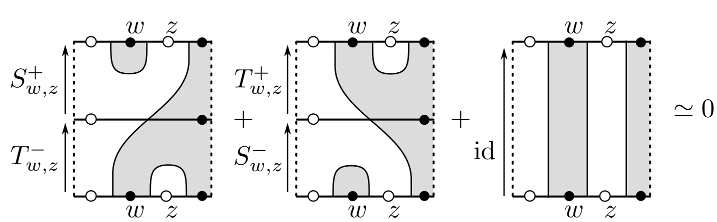

Figure 3.7. A pictorial description of the bypass relation, equation (3.12).

3.4. **Cobordism maps for saddles.** Next, we discuss the maps for saddle cobordisms. Suppose that $\mathbb{L}=(L,\mathbf{w},\mathbf{z})$ is a multi-based link in Y, and that B is an embedded band for L that has both ends in subarcs of $L\setminus(\mathbf{w}\cup\mathbf{z})$ that run from $\mathbf{w}$ to $\mathbf{z}$, or has both ends in subarcs that run from $\mathbf{z}$ to $\mathbf{w}$. Assuming that $\mathbb{L}(B)$, the result of band surgery, is also a valid multi-based link, there is a map

$$F_B^{\mathbf{z}}\colon \mathcal{CFL}^\infty(Y,\mathbb{L},\mathfrak{s}) \to \mathcal{CFL}^\infty(Y,\mathbb{L}(B),\mathfrak{s}),$$

described in [58, Section 6]. The map $F_B^{\mathbf{z}}$ corresponds to a saddle link cobordism, with an index 1 critical point occurring in the type-$\mathbf{z}$ subregion. There is another map $F_B^{\mathbf{w}}$, with the same domain and codomain as $F_B^{\mathbf{z}}$, that corresponds to adding a band to the type-$\mathbf{w}$ subregion.

The relation between the band maps and the basepoint maps is studied in [58, Section 9.1]. According to [58, Lemma 9.1],

$$F_B^{\mathbf{z}}\Phi_w + \Phi_w F_B^{\mathbf{z}} \simeq 0. \tag{3.13}$$

In contrast, the map $F_B^{\mathbf{z}}$ does not always commute with Ψ_z. Instead, if z is either of the two $\mathbf{z}$-basepoints adjacent to the ends of B, then

$$F_B^{\mathbf{z}}\Psi_z + \Psi_z F_B^{\mathbf{z}} \simeq F_B^{\mathbf{w}}, \tag{3.14}$$

according to [58, Proposition 9.3]. In fact, the three dividing sets corresponding to the maps in equation (3.14) can be interpreted as a bypass relation on a saddle cobordism; see [58, Figure 9.1].

Note that, if z and z' are the two $\mathbf{z}$-basepoints adjacent to the ends of B, then equation (3.14) holds for both z and z'. As a consequence, if we sum both relations, we obtain

$$F_B^{\mathbf{z}}(\Psi_z + \Psi_{z'}) + (\Psi_z + \Psi_{z'})F_B^{\mathbf{z}} \simeq 0. \tag{3.15}$$

3.5. **Birth cobordisms and quasi-stabilizations.** We recall from [58, Section 7.1] the birth cobordism map. Suppose $\mathbb{U} = (U, w, z)$ is a doubly-based unknot, which is unlinked from the multi-based link in Y, and is given a distinguished Seifert disk D. In this situation, there is a well defined birth map

$$\mathcal{B}^+_{\mathbb{U},D} \colon \mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}) \to \mathcal{CFL}^\infty(Y, \mathbb{L} \cup \mathbb{U}, \mathfrak{s}).$$

The map $\mathcal{B}^+_{\mathbb{U},D}$ corresponds to a birth cobordism in $I \times Y$, where the disk portion of the link cobordism surface is decorated with a single dividing arc.

The map $\mathcal{B}^+_{\mathbb{U},D}$ can be computed as follows. We pick a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $(Y, \mathbb{L})$ such that $w, z \in \Sigma$, and $D \cap \Sigma$ consists of an embedded arc in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ that connects w and z. We add two new curves, α_0 and β_0, that bound a small disk containing $D \cap \Sigma$, and intersect in a pair of points. Let $\theta^+_{\alpha_0,\beta_0} \in \alpha_0 \cap \beta_0$ denote the higher Maslov graded intersection point (the designation is the same for both $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$). The map $\mathcal{B}^+_{\mathbb{U},D}$ is defined via the formula

$$\mathcal{B}^+_{\mathbb{U},D}(\mathbf{x}) = \mathbf{x} \times \theta^+_{\alpha_0,\beta_0}$$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and extended $\mathcal{R}^\infty$-equivariantly. There is also a death map $\mathcal{D}^-_{\mathbb{U},D}$, defined in the opposite direction, though it will not make an appearance in this paper.

Suppose L is a link in Y, and U is an unknot that bounds a Seifert disk D, disjoint from L. Suppose further that B is a band connecting U and L, which is disjoint from the interior of D. Let

$$\phi \colon (Y, L) \to (Y, (L \cup U)(B))$$

denote a diffeomorphism which is the identity outside a neighborhood of $B \cup D$.

Since a birth cobordism adds two basepoints, the composition of a birth cobordism map and a band map is not simply the diffeomorphism map ϕ_*. Instead the composition is a quasi-stabilization. More precisely, if $\mathbb{U} = (U, w, z)$, and B is an α-band (i.e., has both ends on strands of L that lie in the α-handlebody), then

$$F_B^{\mathbf{w}}\mathcal{B}^+_{\mathbb{U},D} \simeq T^+_{w,z}\phi_*. \tag{3.16}$$

A proof of equation (3.16) can be found in [58, Proposition 8.5]. Note that there are other variations of equation (3.16). For example, if B is instead a β-band, then

$$F_B^{\mathbf{w}}\mathcal{B}^+_{\mathbb{U},D} \simeq T^+_{z,w}\phi_*. \tag{3.17}$$

A schematic illustrating equation (3.16) is shown in Figure 3.8.

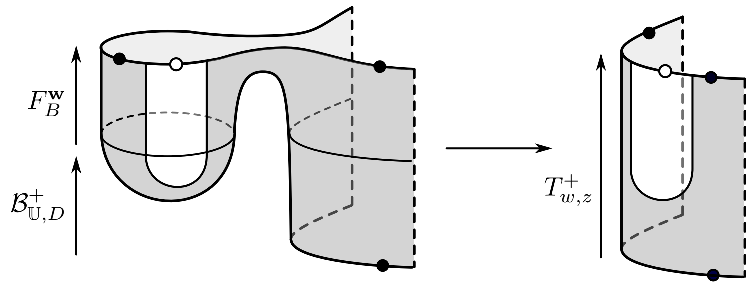

FIGURE 3.8. A manipulation demonstrating equation (3.16). The composition of a birth cobordism followed by a band is (up to diffeomorphism) a quasi-stabilization.

3.6. **4-dimensional connected sums of link cobordisms.** Suppose $(W_1,\mathcal{F}_1)$ and $(W_2,\mathcal{F}_2)$ are two link cobordisms, and pick two embedded 4-balls $D_1 \subseteq W_1$ and $D_2 \subseteq W_2$ such that $D_i \cap \mathcal{F}_i$ consists of a 2-dimensional disk which intersects the dividing set of $\mathcal{F}_i$ in a single arc. We glue $W_1 \setminus \operatorname{int}(D_1)$ to $W_2 \setminus \operatorname{int}(D_2)$ using an orientation-reversing diffeomorphism which restricts to an orientation-reversing diffeomorphism of $\mathcal{F}_1\cap \partial D_1$ and $\mathcal{F}_2 \cap \partial D_2$, and is compatible with the dividing sets. We write $(W_1\# W_2, \mathcal{F}_1\# \mathcal{F}_2)$ for the resulting link cobordism. Using a handle cancellation argument in the connected sum region, one can prove that

$$F_{W_1\# W_2,\mathcal{F}_1\#\mathcal{F}_2,\mathfrak{s}_1\#\mathfrak{s}_2} \simeq F_{W_1,\mathcal{F}_1,\mathfrak{s}_1}\otimes F_{W_2,\mathcal{F}_2,\mathfrak{s}_2}. \tag{3.18}$$

See [56, Proposition 5.2] for a detailed proof. To make use of equation (3.18), it will be convenient to have a more explicit description of the cobordism map for $(W_1\# W_2, \mathcal{F}_1\#\mathcal{F}_2)$. To this end, suppose the following:

(1) $(Y_1,\mathbb{L}_1)$ and $(Y_2,\mathbb{L}_2)$ are two 3-manifolds with multi-based links.
(2) $\mathbb{S}_0\subseteq Y_1\sqcup Y_2$ is a framed 0-sphere with one foot in Y_1 and the other in Y_2.
(3) $\mathbb{S}_2\subseteq Y_1\# Y_2$ is the dual framed 2-sphere.
(4) B is a band in $Y_1\# Y_2$ that connects $\mathbb{L}_1$ and $\mathbb{L}_2$, and intersects $\mathbb{S}_2$ in a single arc.
(5) B is adjacent to the basepoints w_1 and z_1 on $\mathbb{L}_1$, as well as w_2 and z_2 on $\mathbb{L}_2$.
(6) B' is the band in $Y_1\# Y_2$ attached to $\mathbb{L}_1\#\mathbb{L}_2$ dual to B.

When $(W_i,\mathcal{F}_i)$ is the identity cobordism of $(Y_i,\mathbb{L}_i)$ for $i\in\{1,2\}$, equation (3.18) can be rewritten as

$$F_{\mathbb{S}_2}F_{B'}^{\mathbf{w}}\Phi_w F_B^{\mathbf{w}} F_{\mathbb{S}_0}\simeq \mathrm{id}, \tag{3.19}$$

where $w\in\{w_1,w_2\}$. Figure 3.9 shows equation (3.19) in terms of dividing sets.

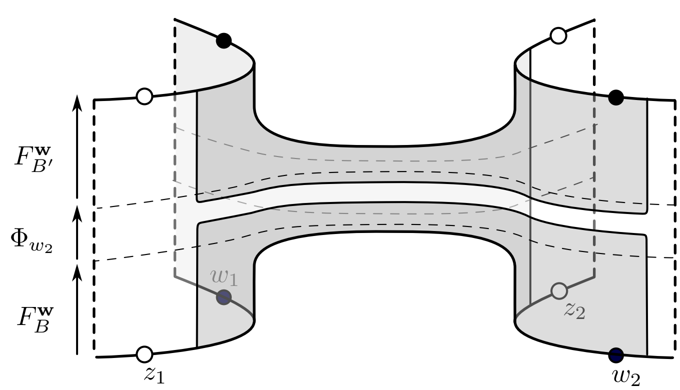

FIGURE 3.9. A schematic of equation (3.19), the effect of taking the connected sum of two link cobordisms. There is additionally a 4-dimensional 1-handle and 3-handle, which are not shown.

4. Heegaard Floer invariants of surfaces

4.1. **Variations on the knot Floer complex.** In this section, we describe several variations on the full infinity complex $\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s})$, which we will use to define our invariants. We focus on the case that $Y = \mathbb{S}^3$ and $\mathbb{L} = \mathbb{K} = (K, w, z)$ is a doubly-based knot.

The first variation we consider is the *standard infinity complex*, denoted $CFK^\infty(\mathbb{K})$. It is defined as the homogeneous subset of $\mathcal{CFL}^\infty(\mathbb{K})$ in Alexander grading zero; i.e., the $\mathbb{F}_2$-vector space generated by monomials

$$U^i V^j \cdot \mathbf{x}$$

with $A(\mathbf{x}) + j - i = 0$. Since the actions of U and V are -1 and $+1$ graded with respect to the Alexander grading, the chain complex $CFK^\infty(\mathbb{K})$ is not an $\mathbb{F}_2[U,V]$-module. However the product $\widehat{U} := UV$ is 0-graded with respect to A, and we view $CFK^\infty(\mathbb{K})$ as an $\mathbb{F}_2[\widehat{U}, \widehat{U}^{-1}]$-module. The complex $CFK^\infty(\mathbb{K})$ contains essentially the same information as $\mathcal{CFL}^\infty(\mathbb{K})$. We will use $CFK^\infty(\mathbb{K})$ to define our invariants $V_k(S, S')$ and $\mathcal{I}(S)$.

There are two *small minus complexes*,

$$CFK^-_{U=0}(\mathbb{K}) := \mathcal{CFL}^-(\mathbb{K}) \otimes_{\mathcal{R}^-} \mathcal{R}^-/(U) \quad \text{and} \quad CFK^-_{V=0}(\mathbb{K}) := \mathcal{CFL}^-(\mathbb{K}) \otimes_{\mathcal{R}^-} \mathcal{R}^-/(V),$$

which are modules over $\mathbb{F}_2[V]$ and $\mathbb{F}_2[U]$, respectively. By inverting V or U, respectively, we obtain the *small infinity complexes*

$$CFK^\infty_{U=0}(\mathbb{K}) \quad \text{and} \quad CFK^\infty_{V=0}(\mathbb{K}),$$

whose homologies are canonically isomorphic to $\mathbb{F}_2[V, V^{-1}]$ and $\mathbb{F}_2[U, U^{-1}]$, respectively.

By setting $V = 1$ in the complex $CFK^-_{U=0}(\mathbb{K})$, we obtain the *Alexander filtered complex*

$$\widehat{CFK}^{fil,z}(\mathbb{K}),$$

described by Ozsváth and Szabó [36]. It has an increasing filtration over $\mathbb{Z}$; i.e., we have an increasing sequence of subcomplexes

$$\widehat{CFK}^{fil,z}_i(\mathbb{K}) \subseteq \widehat{CFK}^{fil,z}_{i+1}(\mathbb{K})$$

for $i \in \mathbb{Z}$. The subspace $\widehat{CFK}^{fil,z}_i(\mathbb{K})$ is generated by the set of intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $A(\mathbf{x}) \le i$. The differential counts holomorphic disks which are allowed to pass over z, but not w. Note that

$$\widehat{CFK}^{fil,z}_i(\mathbb{K}) = \widehat{CFK}^{fil,z}(\mathbb{K}) \cong \widehat{CF}(\mathbb{S}^3)$$

for sufficiently large i, while $\widehat{CFK}^{fil,z}_i(\mathbb{K}) = \{0\}$ for sufficiently negative i. In particular, the total homology of $\widehat{CFK}^{fil,z}(\mathbb{K})$ is isomorphic to $\widehat{HF}(\mathbb{S}^3) \cong \mathbb{F}_2$.

Symmetrically, one can filter using the w basepoint to get a $\mathbb{Z}$-filtered chain complex $\widehat{CFK}^{fil,w}(\mathbb{K})$. The filtration $\widehat{CFK}^{fil,w}(\mathbb{K})$ is *decreasing*; i.e.,

$$\widehat{CFK}^{fil,w}_{i+1}(\mathbb{K}) \subseteq \widehat{CFK}^{fil,w}_i(\mathbb{K}),$$

where $\widehat{CFK}^{fil,w}_i(\mathbb{K})$ is generated over $\mathbb{F}_2$ by intersection points $\mathbf{x}$ with $A(\mathbf{x}) \ge i$.

Remark 4.1. The chain complex $CFK^-_{U=0}(\mathbb{K})$ and the filtered chain complex $\widehat{CFK}^{fil,z}(\mathbb{K})$ contain equivalent information. To see this, note that $\widehat{CFK}^{fil,z}(\mathbb{K})$ is obtained from $CFK^-_{U=0}(\mathbb{K})$ by setting $V = 1$, and using the filtration induced by the Alexander grading. In the other direction, $CFK^-_{U=0}(\mathbb{K})$ is obtained by taking a basis of intersection points of $\widehat{CFK}^{fil,z}(\mathbb{K})$, and weighting an intersection point $\mathbf{y}$ appearing in $\partial \mathbf{x}$ by $V^{A(\mathbf{x})-A(\mathbf{y})}$.

Furthermore, there is a conjugation symmetry of knot Floer homology that allows one to recover $CFK^-_{V=0}(\mathbb{K})$ from $CFK^-_{U=0}(\mathbb{K})$, and vice versa, and similarly recover $\widehat{CFK}^{fil,w}(\mathbb{K})$ from $\widehat{CFK}^{fil,z}(\mathbb{K})$. Our invariant $\tau(S, S')$ will be defined in Section 4.3 using $\widehat{CFK}^{fil,z}(\mathbb{K})$, while $\kappa(S, S')$ and $\kappa_0(S)$ in Section 4.7 in terms of $CFK^-_{U=0}(\mathbb{K})$.

Another variation we use to construct our invariants is the *t-modified complex*, denoted $tCFK^-(\mathbb{K})$, due to Ozsváth–Stipsicz–Szabó [34]. If $t = \frac{m}{n} \in [0, 2]$ is a rational number with m and n relatively

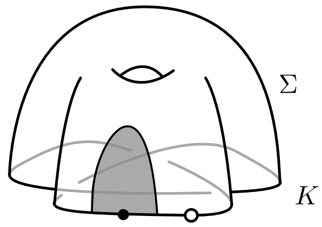

FIGURE 4.1. The dividing set $\mathcal{A}_{\mathbf{z}}$ used to define the map $\mathbf{t}^{\infty}_{\Sigma,\mathbf{z}}$.

prime integers and $n \neq 0$, then we consider the ring $\mathbb{F}_2[v^{1/n}]$, where v has grading -1. The ring $\mathbb{F}_2[v^{1/n}]$ can be given an action of $\mathbb{F}_2[U,V]$ by having U act by v^{2-t} and V act by v^t. The chain complex $tCFK^-(\mathbb{K})$ is defined as the tensor product

$$tCFK^-(\mathbb{K}) := \mathcal{CFL}^-(\mathbb{K}) \otimes_{\mathbb{F}_2[U,V]} \mathbb{F}_2[v^{1/n}].$$

The invariant $\Upsilon_{(S,S')}(t)$ will be defined in Section 4.6 using $tCFK^-(\mathbb{K})$.

A final variation is the *hat complex*

$$\widehat{CFK}(\mathbb{K}) := \mathcal{CFL}^-(\mathbb{K}) \otimes_{\mathcal{R}^-} \mathcal{R}^-/(U,V).$$

We will not discuss $\widehat{CFK}(\mathbb{K})$ extensively in this paper, since it does not contain enough information to compute most of our invariants.

4.2. **The principal invariants of a surface bounding a knot.** We now describe two generalizations of the slice disk invariant $t_D \in \widehat{HFK}(\mathbb{K})$, defined by Marengon and the first author [19], to higher genus surfaces in the full infinity complex.

Definition 4.2. Let $\mathbb{K} = (K, w, z)$ be a multi-based knot in $\mathbb{S}^3$, and let $S \in \mathrm{Surf}_g(K)$ be a surface in B^4 bounding K. If $\mathcal{A}$ is a decoration on S consisting of a single arc which divides S into two connected subsurfaces, then we say that the map $F_{B^4,(S,\mathcal{A})}$ is a *principal invariant* of the surface S.

Let $\mathcal{A}_{\mathbf{z}}$ denote the decoration on S consisting of a single arc such that $g(S_{\mathbf{z}}) = g(S)$ and $g(S_{\mathbf{w}}) = 0$; see Figure 4.1. We define

$$\mathbf{t}^{\infty}_{S,\mathbf{z}} := F_{B^4,(S,\mathcal{A}_{\mathbf{z}})} \colon \mathcal{R}^{\infty} \to \mathcal{CFL}^{\infty}(\mathbb{K}).$$

Similarly, if $\mathcal{A}_{\mathbf{w}}$ denotes the decoration on S with $g(S_{\mathbf{w}}) = g(S)$ and $g(S_{\mathbf{z}}) = 0$, we define

$$\mathbf{t}^{\infty}_{S,\mathbf{w}} := F_{B^4,(S,\mathcal{A}_{\mathbf{w}})} \colon \mathcal{R}^{\infty} \to \mathcal{CFL}^{\infty}(\mathbb{K}).$$

We call $\mathbf{t}^{\infty}_{S,\mathbf{w}}$ and $\mathbf{t}^{\infty}_{S,\mathbf{z}}$ the *extremal principal invariants* of the surface S.

Both $\mathbf{t}^{\infty}_{S,\mathbf{w}}$ and $\mathbf{t}^{\infty}_{S,\mathbf{z}}$ are filtered, $\mathcal{R}^{\infty}$-equivariant chain map that are well-defined up to filtered, $\mathcal{R}^{\infty}$-equivariant chain homotopy. Furthermore, both of them induce isomorphisms on homology by [57, Theorem 9.9]. By [57, Theorem 1.4], the map $\mathbf{t}^{\infty}_{S,\mathbf{w}}$ decreases the Alexander grading by $g(S)$, while $\mathbf{t}^{\infty}_{S,\mathbf{z}}$ increases it by $g(S)$. When $D \in \mathcal{D}(K)$ is a slice disk for K, then $\mathcal{A}_{\mathbf{z}} = \mathcal{A}_{\mathbf{w}}$, and we denote $\mathbf{t}^{\infty}_{D,\mathbf{z}} = \mathbf{t}^{\infty}_{D,\mathbf{w}}$ by $\mathbf{t}^{\infty}_{D}$.

Although the chain complexes $\mathcal{CFL}^{\infty}(\mathbb{K})$ and $\mathcal{CFL}^-(\mathbb{K})$ contain equivalent information, it is convenient to also define maps

$$\mathbf{t}^{-}_{S,\mathbf{w}},\, \mathbf{t}^{-}_{S,\mathbf{z}} \colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K})$$

as the cobordism maps on the full minus complexes. For a slice disk D, we again have $\mathbf{t}^{-}_{D,\mathbf{w}} = \mathbf{t}^{-}_{D,\mathbf{z}}$, which we denote by $\mathbf{t}^{-}_{D}$. By [22, Theorem 1.4], when we set $U = V = 0$, the map $\mathbf{t}^{-}_{D}$, defined using the maps from [58], becomes t_D, defined in [19].

We note that the elements $[\mathbf{t}^{-}_{S,\mathbf{w}}(1)]$ and $[\mathbf{t}^{-}_{S,\mathbf{z}}(1)]$ in the homology group $\mathcal{HFL}^-(\mathbb{K}) := H_*(\mathcal{CFL}^-(\mathbb{K}))$ contain exactly the same information as the $\mathcal{R}^-$-equivariant, $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy types of the maps

$$\mathbf{t}^{-}_{S,\mathbf{w}},\, \mathbf{t}^{-}_{S,\mathbf{z}} \colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K}),$$

since two filtered, equivariant maps from $\mathcal{R}^-$ to $\mathcal{CFL}^-(\mathbb{K})$ are filtered, equivariantly chain homotopic if and only if their values on $1 \in \mathcal{R}^-$ differ by $\partial\eta$ for some $\eta \in \mathcal{CFL}^-(\mathbb{K})$. Nonetheless, we will usually not view $\mathbf{t}^-_{S,\mathbf{w}}$ and $\mathbf{t}^-_{S,\mathbf{z}}$ as elements of $\mathcal{HFL}^-(\mathbb{K})$ because, on its own, the group $\mathcal{HFL}^-(\mathbb{K})$ is not sufficient to define our invariants.

We now describe some basic properties of the invariants $\mathbf{t}^\infty_{S,\mathbf{w}}$ and $\mathbf{t}^\infty_{S,\mathbf{z}}$. Let r denote the rolling automorphism of $(\mathbb{S}^3,\mathbb{K})$ that consists of a Dehn twist about K; see Definition 2.3. Then

$$r_* \circ \mathbf{t}^\infty_{S,\mathbf{z}} \simeq \mathbf{t}^\infty_{S,\mathbf{z}}, \tag{4.1}$$

and similarly for $\mathbf{t}^\infty_{S,\mathbf{w}}$, as the corresponding dividing sets differ by a Dehn twist about ∂S, and are hence isotopic.

Definition 4.3. Let C and C' be free, $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complexes over $\mathcal{R}^\infty$. We say that a map $f\colon C \to C'$ is *skew-equivariant* and *skew-filtered* if

$$f\circ U = V\circ f, \quad f\circ V = U \circ f, \quad \text{and} \quad f(\mathcal{G}_{i,j}(C)) \subseteq \mathcal{G}_{j,i}(C').$$

Given skew-equivariant and skew-filtered chain maps $f, g\colon C \to C'$, we say that they are *skew-equivariant and skew-filtered chain homotopic*, and write $f \approx g$, if they are chain homotopic through a skew-equivariant and skew-filtered chain homotopy.

There is a skew-equivariant and skew-filtered homotopy automorphism

$$\iota_K\colon \mathcal{CFL}^\infty(\mathbb{K}) \to \mathcal{CFL}^\infty(\mathbb{K}),$$

defined as the composition of a tautological conjugation automorphism of $\mathcal{CFL}^\infty(\mathbb{K})$, and the map induced by a half Dehn twist about K that switches w and z; see [13, Section 6.1]. The map ι_K satisfies $\iota_K^2 \simeq r_*$.

Using the conjugation formula for the link cobordism maps [56, Theorem 1.3], as well as the same manipulation of dividing sets as in equation (4.1), one obtains

$$\mathbf{t}^\infty_{S,\mathbf{w}} \circ \iota_{\mathcal{R}^\infty} \approx \iota_K \circ \mathbf{t}^\infty_{S,\mathbf{z}}, \tag{4.2}$$

where $\iota_{\mathcal{R}^\infty}\colon \mathcal{R}^\infty \to \mathcal{R}^\infty$ denotes the unique involution that switches U and V. Using equation (4.1), we can rewrite equation (4.2) as

$$\iota_K \circ \mathbf{t}^\infty_{S,\mathbf{w}} \circ \iota_{\mathcal{R}^\infty} \simeq \mathbf{t}^\infty_{S,\mathbf{z}}. \tag{4.3}$$

Hence, together with the conjugation automorphism, $\mathbf{t}^\infty_{S,\mathbf{w}}$ and $\mathbf{t}^\infty_{S,\mathbf{z}}$ provide essentially equivalent information.

In the following subsections, we introduce several invariants of a pair of surfaces S, $S' \in \mathrm{Surf}(K)$, in order to give lower bounds on their stabilization distance. These are all derived from the principal invariants for S and S'. Furthermore, our invariants τ, ν, V_k, and Υ are constructed as algebraic analogues of the homonymous knot invariants from knot Floer homology. Hence, we shall sometimes call these *secondary versions* of their knot invariant counterparts.

4.3. **The tau invariant.** Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$. We now describe a map

$$\tau\colon\ \mathrm{Surf}(K) \times \mathrm{Surf}(K) \to \mathbb{Z}^{\geq 0}.$$

It is a secondary version of the concordance invariant defined by Ozsváth and Szabó [36]. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a doubly-pointed diagram representing $\mathbb{K}$. We will define the invariant $\tau(S,S')$ for S, $S' \in \mathrm{Surf}(K)$ in terms of the Alexander filtered complex $\widehat{CFK}^{fil,z}(\mathbb{K})$ described in Section 4.1.

Following Section 4.2, a surface $S \in \mathrm{Surf}_g(K)$ induces a chain map $\widehat{t}_{S,\mathbf{z}}\colon \mathbb{F}_2 \to \widehat{CFK}^{fil,z}(\mathbb{K})$ whose image is contained in $\widehat{CFK}^{fil,z}_g(\mathbb{K})$. We recall that for sufficiently large n, we have

$$H_*(\widehat{CFK}^{fil,z}_n(\mathbb{K})) \cong \widehat{HF}(S^3) = \mathbb{F}. \tag{4.4}$$

Additionally, it follows from the reduction theorem [58, Theorem C] that with respect to the isomorphism in equation (4.4), the element $\widehat{t}_{S,\mathbf{z}}(1)$ represents $1 \in \mathbb{F}$. We make the following definition:

Definition 4.4. Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$. Given surfaces $S \in \mathrm{Surf}_g(K)$ and $S' \in \mathrm{Surf}_{g'}(K)$, we define the invariant

$$\tau(S, S') = \min\left\{ n \geq \max\{g, g'\} : \left[\widehat{t}_{S,\mathbf{z}}(1)\right] = \left[\widehat{t}_{S',\mathbf{z}}(1)\right] \text{ in } H_*\left(\widehat{CFK}^{\mathit{fil},z}_n(\mathbb{K})\right)\right\}.$$

It is straightforward to see that $\tau(S, S')$ is independent of the choice of basepoints on K, and is furthermore a finite integer.

Lemma 4.5. *Let S_1, S_1', S_2, $S_2' \in \mathrm{Surf}(K)$ be surfaces such that $[S_1] = [S_2]$ and $[S_1'] = [S_2']$ in $\mathrm{Surf}(K)/\{$2-knots$\}$. Then*

$$\tau(S_1, S_1') = \tau(S_2, S_2').$$

Proof. This follows from the observation that $\widehat{t}_{S,\mathbf{z}}$ is unchanged, up to filtered chain homotopy, if we take the connected sum of S with a 2-knot, which can be shown by adapting the proof of [23, Lemma 4.2]. ☐

We recall that the concordance invariant $\tau(K)$ may be computed from the $\mathbb{F}_2[V]$-module

$$CFK^-_{U=0}(\mathbb{K}) := \mathcal{CFL}^-(\mathbb{K}) \otimes_{\mathcal{R}^-} \mathcal{R}^-/(U),$$

obtained from $\mathcal{CFL}^-(\mathbb{K})$ by setting $U = 0$; see Ozsváth–Szabó–Thurston [43, Lemma A.2]. (Note that Ozsváth, Szabó, and Thurston use the $V = 0$ version of knot Floer homology. Using the conjugation symmetry, these are equivalent perspectives).

Analogously, we can reformulate $\tau(S, S')$ in terms of $CFK^-_{U=0}(\mathbb{K})$. Let us write $t^-_{S,\mathbf{z}}$ and $t^-_{S',\mathbf{z}}$ for the maps from $\mathbb{F}_2[V]$ to $CFK^-_{U=0}(\mathbb{K})$ induced by $\mathbf{t}^-_{S,\mathbf{z}}$ and $\mathbf{t}^-_{S',\mathbf{z}}$, respectively.

Lemma 4.6. *Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$. If $S \in \mathrm{Surf}_g(K)$ and $S' \in \mathrm{Surf}_{g'}(K)$, then*

$$\tau(S, S') = \min\{\, n \geq \max\{g, g'\} : V^{n-g} \cdot [t^-_{S,\mathbf{z}}(1)] = V^{n-g'} \cdot [t^-_{S',\mathbf{z}}(1)] \text{ in } HFK^-_{U=0}(\mathbb{K}) \,\}.$$

Proof. Let us write $\zeta(S, S')$ for the right-hand side. If $n = \zeta(S, S')$, then

$$V^{n-g} \cdot t^-_{S,\mathbf{z}}(1) + V^{n-g'} \cdot t^-_{S',\mathbf{z}}(1) = \partial x$$

for some $x \in CFK^-_{U=0}(\mathbb{K})$. Note that ∂ preserves the Alexander grading, V increases it by one, $A(t^-_{S,\mathbf{z}}(1)) = g$, and $A(t^-_{S',\mathbf{z}}(1)) = g'$. Hence, we can assume that $A(x) = n$. Consequently, using the identification

$$\widehat{CFK}^{\mathit{fil},z}(\mathbb{K}) \cong CFK^-_{U=0}(\mathbb{K}) \otimes_{\mathbb{F}_2[V]} \mathbb{F}_2[V]/(V-1),$$

we have $x \otimes 1 \in \widehat{CFK}^{\mathit{fil},z}_n(\mathbb{K})$, and

$$\widehat{t}_{S,\mathbf{z}}(1) + \widehat{t}_{S',\mathbf{z}}(1) = \partial(x \otimes 1) \in \widehat{CFK}^{\mathit{fil},z}_n(\mathbb{K}).$$

It follows that

$$\tau(S, S') \leq \zeta(S, S').$$

Conversely, suppose $n = \tau(S, S')$. Then $\widehat{t}_{S,\mathbf{z}}(1) + \widehat{t}_{S',\mathbf{z}}(1) = \partial x$ for some $x \in \widehat{CFK}^{\mathit{fil},z}_n(\mathbb{K})$. We write x as a sum of intersection points $x = \sum_{i=1}^n \mathbf{x}_i$. We define an element $\widetilde{x} \in CFK^-_{U=0}(\mathbb{K})$ in Alexander grading n via the formula $\widetilde{x} = \sum_{i=1}^n V^{n-A(\mathbf{x}_i)}\mathbf{x}_i$. The element $\widetilde{x}$ satisfies

$$V^{n-g} \cdot t^-_{S,\mathbf{z}}(1) + V^{n-g'} \cdot t^-_{S',\mathbf{z}}(1) = \partial\widetilde{x}.$$

Compare Remark 4.1. It follows that

$$\zeta(S, S') \leq \tau(S, S'),$$

which concludes the proof. ☐

Let K_1 and K_2 be knots in $\mathbb{S}^3$. Given surfaces $S_1 \in \mathrm{Surf}(K_1)$ and $S_2 \in \mathrm{Surf}(K_2)$, their boundary connected sum $S_1 \natural S_2$ is an element of $\mathrm{Surf}(K_1 \# K_2)$.

Proposition 4.7. *Let K_1 and K_2 be knots in $\mathbb{S}^3$. If S_1, $S_1' \in \mathrm{Surf}(K_1)$ and S_2, $S_2' \in \mathrm{Surf}(K_2)$ are surfaces of genera g_1, g_1', g_2, and g_2', respectively, then*

$$\tau(S_1 \natural S_2, S_1' \natural S_2') \leq \max\left\{\tau(S_1, S_1') + \max\{g_2, g_2'\}, \tau(S_2, S_2') + \max\{g_1, g_1'\}\right\}.$$

Furthermore, when $g_1 = g_1' = g_2 = g_2' = 0$, then equality holds.

Proof. Using the connected sum formula [56, Proposition 5.1], there is chain homotopy equivalence

$$F\colon \mathit{CFK}^-_{U=0}(\mathbb{K}_1) \otimes_{\mathbb{F}_2[V]} \mathit{CFK}^-_{U=0}(\mathbb{K}_2) \to \mathit{CFK}^-_{U=0}(\mathbb{K}_1 \# \mathbb{K}_2),$$

where the decoration on $\mathbb{K}_1 \# \mathbb{K}_2$ is the decoration of $\mathbb{K}_1$. Furthermore, the map F can be taken to be the link cobordism map for a 1-handle cobordism containing a band. By the functoriality of the link cobordism maps,

$$F \circ (t^-_{S_1,\mathbf{z}} \otimes t^-_{S_2,\mathbf{z}}) \simeq t^-_{S_1 \natural S_2, \mathbf{z}},$$

and similarly for S_1' and S_2'.

By the Künneth theorem for tensor products over a PID, there is a short exact sequence

$$0 \to \mathit{HFK}^-_{U=0}(\mathbb{K}_1) \otimes_{\mathbb{F}_2[V]} \mathit{HFK}^-_{U=0}(\mathbb{K}_2) \xrightarrow{G} \mathit{HFK}^-_{U=0}(\mathbb{K}_1 \# \mathbb{K}_2) \to \mathrm{Tor}^1_{\mathbb{F}_2[V]}(\mathit{HFK}^-(\mathbb{K}_1), \mathit{HFK}^-_{U=0}(\mathbb{K}_2)) \to 0,$$

where G is the composition of the natural map

$$\mathit{HFK}^-_{U=0}(\mathbb{K}_1) \otimes_{\mathbb{F}_2[V]} \mathit{HFK}^-_{U=0}(\mathbb{K}_2) \to H_*(\mathit{CFK}^-_{U=0}(\mathbb{K}_1) \otimes_{\mathbb{F}_2[V]} \mathit{CFK}^-_{U=0}(\mathbb{K}_2))$$

and F_*. The map G sends $[t^-_{S_1,\mathbf{z}}(1)] \otimes [t^-_{S_2,\mathbf{z}}(1)]$ to $[t^-_{S_1 \natural S_2,\mathbf{z}}(1)]$.

For $i \in \{1,2\}$, the $\mathbb{F}_2[V]$-module $\mathit{HFK}^-_{U=0}(\mathbb{K}_i)$ splits (non-canonically) as $\mathbb{F}_2[V] \oplus T_i$, where T_i is a torsion $\mathbb{F}_2[V]$-module, and $t^-_{S_i,\mathbf{z}}(1) = V^{g_i} \oplus s_i$ and $t^-_{S_i',\mathbf{z}}(1) = V^{g_i'} \oplus s_i'$ for some s_i, $s_i' \in T_i$. Let

$$n = \max\left\{\tau(S_1, S_1') + \max\{g_2, g_2'\}, \tau(S_2, S_2') + \max\{g_1, g_1'\}\right\}. \tag{4.5}$$

Then we claim that

$$V^{n-g_1-g_2} \cdot ([t^-_{S_1,\mathbf{z}}(1)] \otimes [t^-_{S_2,\mathbf{z}}(1)]) = V^{n-g_1'-g_2'} \cdot ([t^-_{S_1',\mathbf{z}}(1)] \otimes [t^-_{S_2',\mathbf{z}}(1)]) \tag{4.6}$$

as elements of $\mathit{HFK}^-(\mathbb{K}_1) \otimes \mathit{HFK}^-(\mathbb{K}_2)$. This is equivalent to

$$V^{n-g_1-g_2}((V^{g_1} \otimes V^{g_2}) \oplus (s_1 \otimes V^{g_2}) \oplus (V^{g_1} \otimes s_2) \oplus (s_1 \otimes s_2)) =$$
$$V^{n-g_1'-g_2'}((V^{g_1'} \otimes V^{g_2'}) \oplus (s_1' \otimes V^{g_2'}) \oplus (V^{g_1'} \otimes s_2') \oplus (s_1' \otimes s_2')).$$

For $i \in \{1,2\}$ and $k \geq \tau(S_i, S_i')$, by Lemma 4.6, we have

$$V^k \oplus (V^{k-g_i} s_i) = V^{k-g_i} \cdot [t^-_{S_i,\mathbf{z}}(1)] = V^{k-g_i'} \cdot [t^-_{S_i',\mathbf{z}}(1)] = V^k \oplus (V^{k-g_i'} s_i').$$

Together with equation (4.5), this implies that $V^{n-g_i} s_i = V^{n-g_i'} s_i'$ and

$$V^{n-g_1-g_2}(s_1 \otimes s_2) = V^{n-g_1'-g_2'}(s_1' \otimes s_2'),$$

so equation (4.6) holds. The result follows by applying G to equation (4.6), and invoking Lemma 4.6.

When $g_1 = g_1' = g_2 = g_2' = 0$, then we choose

$$n = \tau(S_1 \natural S_2, S_1' \natural S_2').$$

By Lemma 4.6, n satisfies equation (4.6), which implies that $V^n s_i = V^n s_i'$ for $i \in \{1,2\}$, and hence

$$V^n \cdot [t^-_{S_i,\mathbf{z}}] = V^n \cdot [t^-_{S_i',\mathbf{z}}].$$

So $n \geq \max\{\tau(S_1, S_1'), \tau(S_2, S_2')\}$, and equality holds, as claimed. ∎

4.4. **An infinitesimal refinement of tau.** In this section, we describe a refinement of $\tau(S,S')$, inspired by work of Ozsváth–Szabó [42], Hom–Wu [17], and Hom [16],

Let $\bar{\mathbb{Z}}$ denote $\mathbb{Z}\cup\{-\infty,\infty\}$, and write $\bar{\mathbb{Z}}^{\le 0}=[-\infty,0]\cap\bar{\mathbb{Z}}$. Given a knot K in $\mathbb{S}^3$, we will define a symmetric map

$$\tau^+\colon \operatorname{Surf}(K)\times\operatorname{Surf}(K)\to\mathbb{N}\times\bar{\mathbb{Z}}^{\le 0}.$$

The invariant $\tau^+(S,S')$ takes the form

$$\tau^+(S,S'):=(\tau(S,S'),\tau'(S,S')),$$

where $\tau(S,S')$ is the integer defined in Section 4.3, and $\tau'(S,S')$ is an element of $\bar{\mathbb{Z}}^{\le 0}$ that we define shortly. We will think of τ' as a second-order version of τ, or an infinitesimal refinement.

To define τ', we introduce some notation. If $(i,j)\in\mathbb{Z}\oplus\mathbb{Z}$, let

$$R_{i,j}:=\{\,(m,n)\in\mathbb{Z}\oplus\mathbb{Z}: m\ge i,\, n\ge j\,\}.$$

If $\mathcal{S}\subseteq\mathbb{Z}\oplus\mathbb{Z}$, let $H(\mathcal{S})$ denote the *filtered hull* of $\mathcal{S}$; i.e.,

$$H(\mathcal{S})=\bigcup_{(i,j)\in\mathcal{S}}R_{i,j}.$$

We say that $\mathcal{S}$ is a *filtered* shape if

$$\mathcal{S}=H(\mathcal{S}).$$

Let w and z be basepoints on K, and write $\mathbb{K}=(K,w,z)$. If $\mathcal{S}\subseteq\mathbb{Z}\oplus\mathbb{Z}$, let $C(\mathbb{K},\mathcal{S})$ denote the subspace of $\mathit{CFK}^\infty(\mathbb{K})$ generated over $\mathbb{F}_2$ by monomials $U^iV^j\cdot\mathbf{x}$ with $A(\mathbf{x})+j-i=0$ and $(i,j)\in\mathcal{S}$. If $\mathcal{S}$ is a filtered shape, then $C(\mathbb{K},\mathcal{S})$ is a subcomplex of $\mathit{CFK}^\infty(\mathbb{K})$.

More generally, we say $\mathcal{S}\subseteq\mathbb{Z}\times\mathbb{Z}$ is a *sub-quotient* shape if, whenever (i,j), $(m,n)\in\mathcal{S}$ with $i\le m$ and $j\le n$, then the entire rectangle spanned by the points (i,j) and (m,n) is contained in $\mathcal{S}$. If $\mathcal{S}$ is a sub-quotient shape, then $C(\mathbb{K},\mathcal{S})$ is in fact a sub-quotient complex of $\mathit{CFK}^\infty(\mathbb{K})$; i.e., there are subcomplexes $C_{\text{in}}(\mathbb{K},\mathcal{S})\subseteq C_{\text{out}}(\mathbb{K},\mathcal{S})\subseteq\mathit{CFK}^\infty(\mathbb{K})$ such that $C_{\text{out}}(\mathbb{K},\mathcal{S})/C_{\text{in}}(\mathbb{K},\mathcal{S})$ is chain isomorphic to $C(\mathbb{K},\mathcal{S})$. Indeed, the two sub-complexes of $\mathit{CFK}^\infty(\mathbb{K})$ are

$$C_{\text{out}}(\mathbb{K},\mathcal{S}):=C\left(\mathbb{K},H(\mathcal{S})\right),\text{ and}$$
$$C_{\text{in}}(\mathbb{K},\mathcal{S}):=C\left(\mathbb{K},H(\mathcal{S})\setminus\mathcal{S}\right).$$

We note that, if $\mathcal{S}\subseteq\mathbb{Z}\oplus\mathbb{Z}$ is an arbitrary subset, then its filtered hull $H(\mathcal{S})$ is automatically a filtered shape. It is an easy exercise to show that, if $\mathcal{S}$ is a sub-quotient shape, then $H(\mathcal{S})\setminus\mathcal{S}$ is filtered. Hence $C_{\text{in}}(\mathbb{K},\mathcal{S})$ and $C_{\text{out}}(\mathbb{K},\mathcal{S})$ are both subcomplexes of $\mathit{CFK}^\infty(\mathbb{K})$.

We note that the map $\mathbf{t}^-_{S,\mathbf{z}}$ naturally has image in the $g(S)$ Alexander graded subspace of $\mathcal{CFL}^-(\mathbb{K})$. Hence, there is a well-defined map

$$V^{-g(S)}\cdot\mathbf{t}^-_{S,\mathbf{z}}\colon\mathbb{F}_2[\widehat{U}]\to C(\mathbb{K},R_{0,-g(S)}).$$

Furthermore, if $\mathcal{S}$ is a sub-quotient shape of $\mathbb{Z}\oplus\mathbb{Z}$ such that $R_{0,-g(S)}\subseteq H(\mathcal{S})$, then there is a natural map

$$q\colon C(\mathbb{K},R_{0,-g(S)})\to C(\mathbb{K},\mathcal{S}),$$

which is the composition of the inclusion map $C(\mathbb{K},R_{0,-g(S)})\to C_{\text{out}}(\mathbb{K},\mathcal{S})$, followed by the quotient map $C_{\text{out}}(\mathbb{K},\mathcal{S})\to C_{\text{out}}(\mathbb{K},\mathcal{S})/C_{\text{in}}(\mathbb{K},\mathcal{S})\cong C(\mathbb{K},\mathcal{S})$. In particular, $[V^{-g(S)}\cdot\mathbf{t}^-_{S,\mathbf{z}}(1)]$ determines a well-defined element of $H_*(C(\mathbb{K},\mathcal{S}))$.

Define

$${}_nI:=\{0\}\times([-n,\infty)\cap\mathbb{Z}),$$

which is a sub-quotient shape. Noting that $C(\mathbb{K},{}_nI)$ is chain isomorphic to $\widehat{\mathit{CFK}}{}^{fil,z}_n(\mathbb{K})$, we obtain the following:

Lemma 4.8. *Given a doubly-based knot $\mathbb{K}=(K,w,z)$ and surfaces S, $S'\in\operatorname{Surf}(K)$, we have*

$$\tau(S,S')=\min\left\{n\ge\max\{g(S),g(S')\}:[V^{-g(S)}\cdot\mathbf{t}^-_{S,\mathbf{z}}(1)]=[V^{-g(S')}\cdot\mathbf{t}^-_{S',\mathbf{z}}(1)]\text{ in }H_*(C(\mathbb{K},{}_nI))\right\}.$$

We are now ready to define the refinement $\tau'(S,S')$.

Definition 4.9. For n, $m \in \mathbb{Z}$, let

$${}_nL_m := {}_nI \cup ([0,m] \cap \mathbb{Z}) \times \{-n\}.$$

This is an L-shaped subset of $\mathbb{Z} \times \mathbb{Z}$, and hence a sub-quotient shape.

Let $\mathbb{K} = (K, w, z)$ be a doubly-pointed knot in $\mathbb{S}^3$, let S, $S' \in \mathrm{Surf}(K)$, and write $\tau = \tau(S, S')$. Then we define

$$\tau'(S,S') = -\sup\{\, m \in \mathbb{Z} : [V^{-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1) - V^{-g(S')} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] = 0 \in H_*\left(C(\mathbb{K}, {}_\tau L_m\right)\,\}.$$

Note that ${}_\tau L_0 = {}_\tau I$, and $[V^{-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1) - V^{-g(S')} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] = 0$ in $H_*(C(\mathbb{K}, {}_\tau I))$ by the definition of τ. It follows that $\tau'(S,S') \leq 0$. However, if $x \in C(\mathbb{K}, {}_\tau I)$ satisfies

$$\partial x = V^{-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1) - V^{-g(S')} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)$$

in $C(\mathbb{K}, {}_\tau I)$, then ∂x might have some nonzero terms in $([0,m] \cap \mathbb{Z}) \times \{-\tau\}$ for $m > 0$. Hence $[V^{-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1) - V^{-g(S')} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)]$ might not be zero in $H_*\left(C(\mathbb{K}, {}_\tau L_m)\right)$ for $m > 0$, and this is what $\tau'(S, S')$ measures.

The invariant $\tau^+(S, S')$ was inspired by the concordance invariant $\nu(K)$, due to Ozsváth and Szabó [42, Definition 9.1], which gives an improved 4-ball genus bound over $\tau(K)$ by at most 1. We now extract an analogue of ν from τ^+ for pairs of surfaces in $\mathrm{Surf}(K)$, though there is some information lost when doing this as τ' can take any value in $\bar{\mathbb{Z}}^{\leq 0}$.

Definition 4.10. Let K be a knot in $\mathbb{S}^3$ and S, $S' \in \mathrm{Surf}(K)$. Then let

$$\nu(S,S') = \begin{cases} \tau(S,S') & \text{if } \tau'(S,S') = -\infty, \\ \tau(S,S') + 1 & \text{otherwise.} \end{cases}$$

We will see that ν gives a lower bound on the stabilization and double point distances in Proposition 6.8.

4.5. **A sequence of local h-invariants.** Let K be a knot in $\mathbb{S}^3$ and S, $S' \in \mathrm{Surf}_g(K)$. Modeled on the invariants $V_k(K)$ of large surgeries from knot Floer homology, also referred to as Rasmussen's *local h-invariants* [44], we describe a sequence of integer invariant $V_k(S, S')$ for $k \geq g$, such that

$$V_g(S,S') \geq V_{g+1}(S,S') \geq \cdots \geq 0,$$

and such that $V_k(S, S') = 0$ for k sufficiently large.

Definition 4.11. Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$, and let S, $S' \in \mathrm{Surf}_g(K)$. To define $V_k(S, S')$, we consider the subcomplex

$$A^-_k(\mathbb{K}) := C(\mathbb{K}, R_{0,-k})$$

of $CFK^\infty(\mathbb{K})$. We think of the complexes $A^-_k(\mathbb{K})$ as modules over the ring $\mathbb{F}_2[\widehat{U}]$, where $\widehat{U} = UV$.

The map $\mathbf{t}^-_{S,\mathbf{z}}$ increases the Alexander grading by $g(S)$. If $k \geq g(S)$, then $V^{-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)$ has Alexander grading 0, and determines a well-defined element of $H_*(A^-_k(\mathbb{K}))$. We define the invariant

$$V_k(S,S') := \min\left\{\, n \in \mathbb{N} : \widehat{U}^n \cdot [V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = \widehat{U}^n \cdot [V^{-g} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(A^-_k(\mathbb{K}))\,\right\}.$$

Remark 4.12. The above definition of V_k can be easily adapted to surfaces of different genera; however, we specialize to the case when $g(S) = g(S')$ since our topological applications for V_k only hold when this is the case.

We now show that the invariants V_k of pairs of surfaces in $\mathrm{Surf}_g(K)$ satisfy many of the same properties as Rasmussen's local h-invariants. The reader should compare the following to [44, Proposition 7.6]:

Lemma 4.13. *If $k \geq g$, then*

$$V_k(S,S') \geq V_{k+1}(S,S') \geq V_k(S,S') - 1.$$

Proof. There is a natural, grading-preserving inclusion of chain complexes
$$i_k\colon A_k^-(\mathbb{K}) \hookrightarrow A_{k+1}^-(\mathbb{K}),$$
which becomes an isomorphism on homology after we invert $\widehat{U}$, and satisfies
$$(i_k)_*([V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)]) = [V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)].$$
Hence
$$V_{k+1}(S,S') \le V_k(S,S').$$
Multiplication by $\widehat{U}$ induces a -2-graded inclusion $A^-_{k+1}(\mathbb{K}) \hookrightarrow A^-_k(\mathbb{K})$ of chain complexes, which becomes an isomorphism on homology after we invert $\widehat{U}$. The map sends $[V^{-g}\cdot \mathbf{t}_{S,\mathbf{z}}(1)] \in H_*(A^-_{k+1}(\mathbb{K}))$ to $\widehat{U}\cdot[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] \in H_*(A^-_k(\mathbb{K}))$, and similarly for S'. Hence, if
$$\widehat{U}^n\cdot[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = \widehat{U}^n\cdot[V^{-g}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(A^-_{k+1}(\mathbb{K})), \text{ then}$$
$$\widehat{U}^{n+1}\cdot[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = \widehat{U}^{n+1}\cdot[V^{-g}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(A^-_{k}(\mathbb{K})).$$
We conclude that $V_k(S,S') \le V_{k+1}(S,S')+1$. □

The reader should compare the following to [44, Proposition 7.7]:

Lemma 4.14. *If S, $S' \in \mathrm{Surf}_g(K)$ and $g \le k < \tau(S,S')$, then $0 < V_k(S,S')$.*

Proof. By Lemma 4.6, we can describe $\tau(S,S')$ as the minimal n such that $V^{n-g}\cdot[t^-_{S,\mathbf{z}}(1)] = V^{n-g}\cdot[t^-_{S',\mathbf{z}}(1)]$ in
$$\mathit{HFK}^-_{U=0}(\mathbb{K}) = H_*\left(\mathcal{CFL}^-(\mathbb{K})\otimes \mathbb{F}_2[U,V]/(U)\right).$$
Note that multiplication by V^k determines an inclusion of chain complexes
$$A^-_k(\mathbb{K}) \hookrightarrow \mathcal{CFL}^-(\mathbb{K}),$$
which we compose with the natural map $\mathcal{CFL}^-(\mathbb{K}) \to \mathit{CFK}^-_{U=0}(\mathbb{K})$ given by $\mathbf{x}\mapsto \mathbf{x}\otimes 1$, corresponding to setting $U=0$. The induced map on homology sends $[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)]$, $[V^{-g}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \in H_*(A^-_k(\mathbb{K}))$ to $V^{k-g}\cdot[t^-_{S,\mathbf{z}}(1)]$, $V^{k-g}\cdot[t^-_{S',\mathbf{z}}(1)] \in \mathit{HFK}^-_{U=0}(\mathbb{K})$, respectively. If $g\le k<\tau(S,S')$, then
$$V^{k-g}\cdot[t^-_{S,\mathbf{z}}(1)] \neq V^{k-g}\cdot[t^-_{S',\mathbf{z}}(1)] \text{ in } \mathit{HFK}^-_{U=0}(\mathbb{K}),$$
and consequently,
$$[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] \neq [V^{-g}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(A^-_k(\mathbb{K})).$$
Hence $V_k(S,S')>0$, as claimed. □

4.6. **The upsilon invariant.** Let K be a knot in $\mathbb{S}^3$, and let S, $S'\in \mathrm{Surf}(K)$. We now describe our invariant
$$\Upsilon_{(S,S')}\colon [0,2]\to \mathbb{R}^{\ge 0}.$$
It is a secondary version of Ozsváth, Stipsicz, and Szabó's [34] invariant $\Upsilon_K(t)$.

We recall the *t-modified* version of knot Floer homology, described by Ozsváth, Stipsicz, and Szabó. Suppose that $t=\frac{m}{n}\in[0,2]$ is a rational number with $m\in\mathbb{N}$ and $n\in\mathbb{Z}_+$ relatively prime. We define $t\mathit{CFK}^-(\mathbb{K})$ to be the free $\mathbb{F}_2[v^{1/n}]$-module generated by $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, where v is a formal variable. Similarly, let $t\mathit{CFK}^\infty(\mathbb{K})$ be the free $\mathbb{F}_2[v^{1/n},v^{-1/n}]$-module generated by $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. The modules $t\mathit{CFK}^-(\mathbb{K})$ and $t\mathit{CFK}^\infty(\mathbb{K})$ are equipped with a differential ∂ that satisfies
$$\partial \mathbf{x} := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\cdot v^{tn_z(\phi)+(2-t)n_w(\phi)}\cdot \mathbf{y}$$
for $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$.

We note that $t\mathit{CFK}^-(\mathbb{K})$ and $t\mathit{CFK}^\infty(\mathbb{K})$ can easily be expressed in terms of $\mathcal{CFL}^-(\mathbb{K})$ and $\mathcal{CFL}^\infty(\mathbb{K})$, respectively, as we now describe. We give $\mathbb{F}_2[v^{1/n}]$ the structure of an $\mathbb{F}_2[U,V]$-module, where U acts by v^{2-t} and V acts by v^t. With this action, we have a canonical isomorphism
$$t\mathit{CFK}^-(\mathbb{K})\cong \mathcal{CFL}^-(\mathbb{K})\otimes_{\mathbb{F}_2[U,V]}\mathbb{F}_2[v^{1/n}],$$

as well as a similar isomorphism involving $tCFK^\infty(\mathbb{K})$ and $\mathcal{CFL}^\infty(\mathbb{K})$. Note that, in particular, if $(W,\mathcal{F})\colon (\mathbb{S}^3,\mathbb{K}_1)\to(\mathbb{S}^3,\mathbb{K}_2)$ is a decorated link cobordism, then the map $F_{W,\mathcal{F},\mathfrak{s}}$ determines a t-modified version

$$tF_{W,\mathcal{F},\mathfrak{s}} := F_{W,\mathcal{F},\mathfrak{s}}\otimes \mathrm{id}_{\mathbb{F}_2[v^{1/n}]}\colon tCFK^-(\mathbb{K}_1)\to tCFK^-(\mathbb{K}_2).$$

Finally, we note that there is a t-grading on $\mathcal{CFL}^-(\mathbb{K})$, defined via the formula

$$\mathrm{gr}_t(\mathbf{x}) := \frac{t}{2}\cdot \mathrm{gr}_{\mathbf{z}}(\mathbf{x}) + \left(1-\frac{t}{2}\right)\cdot \mathrm{gr}_{\mathbf{w}}(\mathbf{x}).$$

This induces a well-defined grading on $tCFK^-(\mathbb{K})$, for which we also write gr_t. With respect to gr_t, the variable v is -1 graded.

If $S\in \mathrm{Surf}_g(K)$ and $S'\in\mathrm{Surf}_{g'}(K)$, the invariants $\mathbf{t}^-_{S,\mathbf{z}}$ and $\mathbf{t}^-_{S',\mathbf{z}}$ admit t-modified versions $t\mathbf{t}^-_{S,\mathbf{z}}$ and $t\mathbf{t}^-_{S',\mathbf{z}}$, respectively. Furthermore, the elements $t\mathbf{t}^-_{S,\mathbf{z}}(1)$ and $t\mathbf{t}^-_{S',\mathbf{z}}(1)$ for $1\in\mathbb{F}_2[v^{1/n}]$ have gr_t-grading $-t\cdot g$ and $-t\cdot g'$, respectively.

Definition 4.15. For $t=\frac{m}{n}\in[0,2]$, we define

$$\Upsilon_{(S,S')}(t) := \min\{\, s=k/n\ge \max\{t\cdot g, t\cdot g'\} : v^{s-t\cdot g}\cdot[t\mathbf{t}^-_{S,\mathbf{z}}(1)] = v^{s-t\cdot g'}\cdot [t\mathbf{t}^-_{S',\mathbf{z}}(1)]\in tHFK^-(\mathbb{K})\,\}.$$

There is an alternate definition of the invariant $\Upsilon_{(S,S')}(t)$, which is more amenable to computations, and is based on Livingston's description of the corresponding knot invariant [28]. If $t\in[0,2]$, there is a filtration $\mathcal{G}^t_s(\mathbb{K})$ of $CFK^\infty(\mathbb{K})$ which is indexed by a parameter $s\in\mathbb{R}$. The set $\mathcal{G}^t_s(\mathbb{K})$ is the $\mathbb{F}_2$-module generated by monomials $U^iV^j\cdot\mathbf{x}$ with $A(\mathbf{x})+j-i=0$ and

$$t\cdot j+(2-t)\cdot i\ge -s.$$

If $s\ge t\cdot g(S)$, then it is straightforward to see that $[V^{-g(S)}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)]$ is a well-defined element of $H_*(\mathcal{G}^t_s(\mathbb{K}))$. It is not hard to adapt [28, Section 14.1] to establish the following:

Lemma 4.16. *If $S\in\mathrm{Surf}_g(K)$ and $S'\in\mathrm{Surf}_{g'}(K)$, then*

$$\Upsilon_{(S,S')}(t)=\min\left\{s\ge t\cdot\max\{g,g'\} : [V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = [V^{-g'}\cdot\mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(\mathcal{G}^t_s(\mathbb{K}))\right\}.$$

4.7. **The kappa and kappa-nought invariants.** If $\mathbb{K}=(K,w,z)$ is a doubly-based knot in $\mathbb{S}^3$, let $CFK^-_{U=0}(\mathbb{K})$ and $CFK^\infty_{U=0}(\mathbb{K})$ denote the small minus and infinity knot Floer complexes described in Section 4.1.

Lemma 4.17. *If $g(S)>0$, then*

$$[t^\infty_{S,\mathbf{w}}(1)]=0\in HFK^\infty_{U=0}(\mathbb{K}).$$

Proof. For $n\in\mathbb{Z}$, let $CFK^\infty_{U=0}(\mathbb{K})_n$ denote the subspace of $CFK^\infty_{U=0}(\mathbb{K})$ in Alexander grading n. Explicitly, the subspace $CFK^\infty_{U=0}(\mathbb{K})_n$ is generated by monomials of the form $V^i\cdot\mathbf{x}$, where

$$A(\mathbf{x})+i=n.$$

We define the reduction map

$$R^n_w\colon CFK^\infty_{U=0}(\mathbb{K})_n\to \widehat{CF}(\mathbb{S}^3,w),$$

by the formula $R^n_w(V^i\cdot\mathbf{x})=\mathbf{x}$. It is straightforward to see that R^n_w is a chain map. Furthermore, since the differential on $CFK^\infty_{U=0}(\mathbb{K})$ preserves the Alexander grading, the map R^n_w is a chain isomorphism. Consider the chain isomorphism

$$R_w := \bigoplus_{n\in\mathbb{Z}} R^n_w\colon CFK^\infty_{U=0}(\mathbb{K})\to\bigoplus_{n\in\mathbb{Z}}\widehat{CF}(\mathbb{S}^3,w).$$

In particular, since $\widehat{HF}(\mathbb{S}^3,w)$ is supported in $\mathrm{gr}_{\mathbf{w}}$-grading 0, it follow that $HFK^\infty_{U=0}(\mathbb{K})$ is as well. The map $\mathbf{t}^\infty_{S,\mathbf{w}}(1)$ has $\mathrm{gr}_{\mathbf{w}}$-grading $-2g(S)$ by the grading formula in [57, Theorem 1.4]. It follows that

$$[\mathbf{t}^\infty_{S,\mathbf{w}}(1)]=0\in HFK^\infty_{U=0}(\mathbb{K}),$$

completing the proof. □

Definition 4.18. Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$, and let $S \in \mathrm{Surf}_g(K)$ for $g > 0$. Then we let

$$\kappa_0(S) := \min\left\{ n \geq g : V^{n-g} \cdot [t^-_{S,\mathbf{w}}(1)] = 0 \text{ in } \mathit{HFK}^-_{U=0}(\mathbb{K}) \right\}.$$

If $S \in \mathrm{Surf}_0(K)$, we set $\kappa_0(S) = 0$.

We note that the element $\mathbf{t}^-_{S,\mathbf{w}}(1)$ lives in Alexander grading $-g$. Hence

$$\kappa_0(S) = g + \min\left\{ k \in \mathbb{N} : [\widehat{t}_{S,\mathbf{w}}(1)] = 0 \text{ in } H_*\left(\widehat{\mathit{CFK}}^{fil,z}_{k-g}(\mathbb{K})\right) \right\}.$$

Definition 4.19. If $g > 0$ and S, $S' \in \mathrm{Surf}_g(K)$, we define the invariant

$$\kappa(S, S') := \min\left\{ n \geq g : V^{n-g} \cdot [t^-_{S,\mathbf{w}}(1)] = V^{n-g} \cdot [t^-_{S',\mathbf{w}}(1)] \text{ in } \mathit{HFK}^-_{U=0}(\mathbb{K}) \right\}.$$

Note that

$$\kappa(S, S') \leq \max\{\kappa_0(S), \kappa_0(S')\}.$$

We emphasize that the invariant $\tau(S, S')$ is defined in terms of the maps $t^-_{S,\mathbf{z}}$ and $t^-_{S',\mathbf{z}}$, while $\kappa(S, S')$ is defined in terms of the maps $t^-_{S,\mathbf{w}}$ and $t^-_{S',\mathbf{w}}$. Also, unlike the invariant $\tau(S, S')$, the definition of $\kappa(S, S')$ only makes sense if $g(S) = g(S') > 0$.

4.8. **Upsilon near** 0 **and** 2. Ozsváth, Stipsicz, and Szabó [34, Proposition 1.6] proved that the knot invariant $\Upsilon_K(t) = -\tau(K) \cdot t$ near $t = 0$. In this section, we prove a similar result for $\Upsilon_{(S,S')}(t)$.

Theorem 4.20. *Suppose that S, $S' \in \mathrm{Surf}(K)$. For all $t \in [0, 2]$ sufficiently close to 0, we have*

$$\Upsilon_{(S,S')}(t) = \tau(S, S') \cdot t.$$

For t sufficiently close to 2, we have

$$\Upsilon_{(S,S')}(t) = \begin{cases} (\kappa_0(S) - g(S)) \cdot (2 - t) + g(S) \cdot t & \text{if } g(S) > g(S'), \\ (\kappa(S, S') - g(S)) \cdot (2 - t) + g(S) \cdot t & \text{if } g(S) = g(S'). \end{cases}$$

Proof. The argument we present is an adaptation of Livingston's proof of the analogous fact [28, Theorem 13.1] for the knot invariants $\Upsilon_K(t)$, and we use the reformulation of $\Upsilon_{(S,S')}(t)$ in terms of filtrations on CFK^∞ from Lemma 4.16. We focus on $\Upsilon_{(S,S')}(t)$ near $t = 2$ when $g(S) > g(S')$, since the other cases are straightforward adaptations of this. Let us write $g = g(S)$ and $g' = g(S')$.

Define the following sub-quotient shapes of $\mathbb{Z} \oplus \mathbb{Z}$:

$$H_{-g+1} := \{ (i, j) : j \geq -g + 1 \},$$
$$T_{-g,k} := \{ (i, j) : i \geq k,\ j = -g \}$$
$$Z_{-g,k} := H_{-g+1} \cup T_{-g,k}.$$

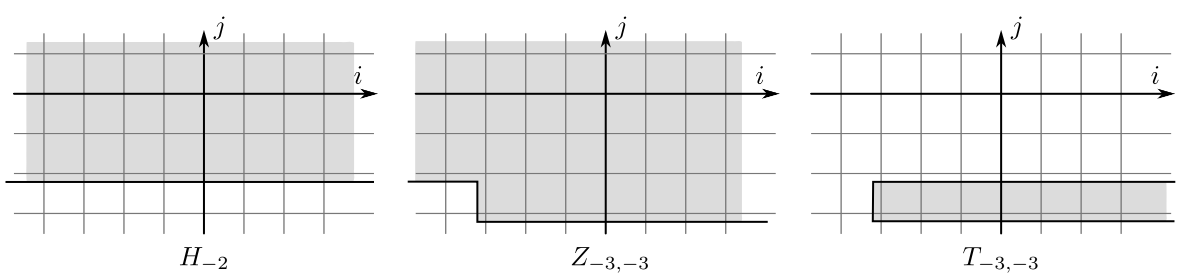

FIGURE 4.2. Examples of the sub-quotient shapes H_{-g+1}, $Z_{-g,k}$, and $T_{-g,k}$ of $\mathbb{Z} \oplus \mathbb{Z}$.

It is straightforward to see that, for t sufficient close to 2, the complex $\mathcal{G}^t_s(\mathbb{K})$ is always equal to $C(\mathbb{K}, Z_{i,j})$ for some i, j. Hence $[V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = [V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}] \in H_*(\mathcal{G}^t_s(\mathbb{K}))$ if and only if

$$C(\mathbb{K}, Z_{-g,m}) \subseteq \mathcal{G}^t_s(\mathbb{K}),$$

where

$$m := \max \left\{ k \le 0 : [V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = [V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(C(\mathbb{K}, Z_{-g,k})) \right\}.$$

Consequently, an easy computation shows that

$$\Upsilon_{(S,S')}(t) = -m(2-t) + gt, \tag{4.7}$$

for t near 2.

There is a short exact sequence of chain complexes

$$0 \to C(\mathbb{K}, H_{-g+1}) \xrightarrow{i} C(\mathbb{K}, Z_{-g,k}) \xrightarrow{q} C(\mathbb{K}, T_{-g,k}) \to 0. \tag{4.8}$$

Since $H_*(C(\mathbb{K}, T_{-g,k}))$ is a torsion $\mathbb{F}_2[\widehat{U}]$-module (in fact, $\widehat{U}$ has vanishing action), while $H_*(C(\mathbb{K}, H_{-g+1}))$ is torsion-free (in fact, isomorphic to $\mathbb{F}_2[\widehat{U}]$), the connecting homomorphism of the long exact sequence associated to equation (4.8) vanishes. Consequently, there is a short exact sequence

$$0 \to H_*(C(\mathbb{K}, H_{-g+1})) \xrightarrow{i} H_*(C(\mathbb{K}, Z_{-g,k})) \xrightarrow{q} H_*(C(\mathbb{K}, T_{-g,k})) \to 0. \tag{4.9}$$

Furthermore, since $H_*(C(\mathbb{K}, H_{-g+1})) \cong \mathbb{F}_2[\widehat{U}]$, it follows that q is injective on the torsion submodule of $H_*(C(\mathbb{K}, Z_{-g,k}))$. Since $[V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] + [V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)]$ is a torsion element of $H_*(C(\mathbb{K}, Z_{-g,k}))$, it follows that $[V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1) + V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)]$ is zero in $H_*(C(\mathbb{K}, Z_{-g,k}))$ if and only if its image under q is zero in $H_*(C(\mathbb{K}, T_{-g,k}))$. Consequently,

$$m = \max \left\{ k \le 0 : [V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = [V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(C(\mathbb{K}, T_{-g,k})) \right\}. \tag{4.10}$$

Note that if $g > g'$, then $[V^{-g'} \cdot \mathbf{t}^-_{S',\mathbf{z}}] = 0$ as an element of $H_*(C(\mathbb{K}, T_{-g,k}))$, so equation (4.10) implies that

$$m = \max \left\{ k \le 0 : [V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = 0 \text{ in } H_*(C(\mathbb{K}, T_{-g,k})) \right\}. \tag{4.11}$$

Next, we note that multiplication by $\widehat{U}^g$ induces a chain isomorphism between $C(\mathbb{K}, T_{-g,k})$ and $C(\mathbb{K}, T_{0,k+g})$. The group $C(\mathbb{K}, T_{0,k+g})$ is the $\mathbb{F}_2$-module generated by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $A(\mathbf{x}) \ge k+g$. The differential on $C(\mathbb{K}, T_{0,k+g})$ counts holomorphic disks which are allowed to go over w, but not z. This is simply the subcomplex $\widehat{CFK}^{fil,w}_{k+g}(\mathbb{K}) \subseteq \widehat{CFK}^{fil,w}(\mathbb{K})$. Applying the conjugation symmetry of knot Floer homology to equation (4.11) implies that

$$\begin{aligned} m &= \max \left\{ k \le 0 : [U^g \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = 0 \text{ in } H_*(C(\mathbb{K}, T_{0,k+g})) \right\} \\ &= -\min \left\{ k \ge 0 : [V^g \cdot \mathbf{t}^-_{S,\mathbf{w}}(1)] = 0 \text{ in } H_*(\widehat{CFK}^{fil,z}_{k-g}(\mathbb{K})) \right\} \\ &= g - \kappa_0(S). \end{aligned} \tag{4.12}$$

Equations (4.7) and (4.12) together imply that, for t near 2, we have

$$\Upsilon_{(S,S')}(t) = (\kappa_0(S) - g) \cdot (2-t) + g \cdot t.$$

Similar arguments apply when $g = g'$, and for t close to 0. □

4.9. **Further properties of the secondary invariants.** Our secondary invariants satisfy a monotonicity condition with respect to stacking link cobordisms.

Proposition 4.21. *Suppose that $(I \times \mathbb{S}^3, S)$ is a link cobordism from $(\mathbb{S}^3, K_0)$ to $(\mathbb{S}^3, K_1)$, and S_0, $S_0' \in \operatorname{Surf}(K_0)$. Let S_1, $S_1' \in \operatorname{Surf}(K_1)$ be the surfaces obtained by stacking S_0 or S_0' and S, respectively. Then*

$$\tau(S_0, S_0') + g(S) \ge \tau(S_1, S_1').$$

When $g(S_0) = g(S_0')$, the invariant κ satisfies an analogous inequality; furthermore,

$$V_k(S_0, S_0') \ge V_{k+g(S)}(S_1, S_1')$$

for $k \ge g(S_0)$. Finally,

$$\Upsilon_{(S_0,S_0')}(t) + (1 - |1-t|) \cdot g(S) \ge \Upsilon_{(S_1,S_1')}(t),$$

for any S_0, $S_0' \in \mathrm{Surf}(K)$ *and* $t \in [0,2]$.

Proof. Choose basepoints w_i and z_i on K_i for $i \in \{0,1\}$, and a decoration $\mathcal{A}$ on S such that $S_{\mathbf{w}}$ is a strip containing w_0 and w_1. Using the functoriality of the link cobordism maps, it sends $\mathbf{t}^{\infty}_{S_0,\mathbf{z}}(1)$ to $\mathbf{t}^{\infty}_{S_1,\mathbf{z}}(1)$ and $\mathbf{t}^{\infty}_{S_0',\mathbf{z}}(1)$ to $\mathbf{t}^{\infty}_{S_1',\mathbf{z}}(1)$. Furthermore, by [57, Theorem 1.4], the map $F_{I\times \mathbb{S}^3,(S,\mathcal{A})}$ increases the Alexander grading by $g(S)$. From these two facts, all claims can be proven quickly. □

A concordance $\mathcal{C}$ from K_0 to K_1 is called *invertible* if there is a concordance $\mathcal{C}'$ from K_1 to K_0 such that $\mathcal{C}' \circ \mathcal{C}$ is the identity cobordism from K_0 to itself; see Sumners [49].

Corollary 4.22. *Suppose that* K_0 *and* K_1 *are knots in* $\mathbb{S}^3$ *and* S_0, $S_0' \in \mathrm{Surf}(K_0)$. *If* $\mathcal{C}$ *is an invertible concordance from* K_0 *to* K_1, *let* S_1 *and* S_1' *denote the surfaces in* $\mathrm{Surf}(K_1)$ *obtained by stacking* S_0 *or* S_0' *and* $\mathcal{C}$, *respectively. Then*

$$\omega(S_0,S_0') = \omega(S_1,S_1')$$

for $\omega \in \{\tau,\Upsilon\}$. *If* $g(S_0) = g(S_0')$, *then the same equality holds for* $\omega \in \{V_k,\kappa\}$, *provided* $k \geq g$.

Proof. Let $\mathcal{C}'$ be a left inverse of $\mathcal{C}$. We first apply Proposition 4.21 to the surfaces S_0, S_0', and to the concordance $\mathcal{C}$, to obtain that $\omega(S_0,S_0') \geq \omega(S_1,S_1')$. Note that we recover S_0 if we stack S_1 and $\mathcal{C}'$, and S_0' if we stack S_1' and $\mathcal{C}'$. Hence, if we apply Proposition 4.21 to the surfaces S_1, S_1', and to the concordance $\mathcal{C}'$, we obtain that $\omega(S_1,S_1') \geq \omega(S_0,S_0')$. □

Like μ_{st} and μ_{Sing}, our secondary invariants satisfy the following ultrametric inequality:

Proposition 4.23. *If* K *is a knot in* $\mathbb{S}^3$ *and* S_1, S_2, $S_3 \in \mathrm{Surf}(K)$, *then*

$$\omega(S_1,S_3) \leq \max\{\,\omega(S_1,S_2),\omega(S_2,S_3)\,\}$$

for $\omega \in \{\tau,\Upsilon\}$, *where the inequality is to be taken pointwise when* $\omega = \Upsilon$. *If* $g(S_1) = g(S_2) = g(S_3)$, *then the inequality holds with* $\omega \in \{V_k,\kappa\}$, *as well.*

Proof. All of the invariants are described in terms of when two distinguished elements become equal in homology, after multiplying by some power of V, $\widehat{U}$, or v. □

Lemma 4.24. *Let* K *be a knot in* $\mathbb{S}^3$, *and suppose* S_1, S_2, $S_3 \in \mathrm{Surf}(K)$. *If we endow* $\mathbb{N} \times \bar{\mathbb{Z}}^{\leq 0}$ *with the lexicographic ordering, then the map* τ^+ *satisfies the ultrametric inequality*

$$\tau^+(S_1,S_3) \leq \max\{\tau^+(S_1,S_2),\tau^+(S_2,S_3)\}.$$

Proof. For i, $j \in \{1,2,3\}$, let us write $\tau_{ij} = \tau(S_i,S_j)$ and $\tau_{ij}' = \tau'(S_i,S_j)$, and let

$$\mathbf{t}_i := V^{-g(S_i)} \cdot \mathbf{t}^-_{S_i,\mathbf{z}}(1).$$

Without loss of generality, we can suppose that $\tau_{12} \leq \tau_{23}$. By Proposition 4.23, we have $\tau_{13} \leq \max\{\tau_{12},\tau_{23}\} = \tau_{23}$. As we have endowed $\mathbb{N} \times \bar{\mathbb{Z}}^{\leq 0}$ with the lexicographic ordering, it suffices to consider the case when $\tau_{13} = \tau_{23}$, which we denote by τ, as otherwise $\tau^+(S_1,S_3) < \tau^+(S_2,S_3)$.

Choose basepoints w and z on K, and write $\mathbb{K} = (K,w,z)$. First, assume that $\tau_{12} < \tau$. By the definition of τ_{12}, there exists $x \in C(\mathbb{K}, {}_{\tau_{12}}I)$ such that $\partial x = \mathbf{t}_1 - \mathbf{t}_2$. If we write $m := -\tau_{23}'$, by definition, there is an $y \in C(\mathbb{K}, {}_{\tau}L_m)$ such that $\partial y = \mathbf{t}_2 - \mathbf{t}_3$. Since $\tau_{12} < \tau$ and ∂ respects the Alexander filtration, there is an inclusion of complexes

$$\iota\colon C(\mathbb{K}, {}_{\tau_{12}}I) \to C(\mathbb{K}, {}_{\tau}L_m).$$

It follows that $\iota(x) + y \in C(\mathbb{K}, {}_{\tau}L_m)$ satisfies $\partial(\iota(x)+y) = \mathbf{t}_1 - \mathbf{t}_3$. As $\tau_{13} = \tau$, it follows that $-\tau_{13}' \geq m$, hence $\tau_{13}' \leq \tau_{23}'$ and $\tau^+(S_1,S_3) \leq \tau^+(S_2,S_3)$.

Now suppose that $\tau_{12} = \tau$. If we write $m' := -\tau_{12}'$ and $m := -\tau_{23}'$, then there exist $x \in C(\mathbb{K}, {}_{\tau}L_{m'})$ and $y \in C(\mathbb{K}, {}_{\tau}L_m)$ such that $\partial x = \mathbf{t}_1 - \mathbf{t}_2$ and $\partial y = \mathbf{t}_2 - \mathbf{t}_3$. Without loss of generality, we can assume that $m' \geq m$. Then there is a natural projection

$$\pi\colon C(\mathbb{K}, {}_{\tau}L_{m'}) \to C(\mathbb{K}, {}_{\tau}L_m)$$

that is a chain map, and preserves the elements $\mathbf{t}_i$ for $i \in \{1,2,3\}$. Since $x + \pi(y) \in C({}_{\tau}L_m)$ and $\partial(x+\pi(y)) = \mathbf{t}_1 - \mathbf{t}_3$, we have $-\tau_{13}' \geq m$, hence $\tau_{13}' \leq \tau_{23}'$ and $\tau^+(S_1,S_3) \leq \tau^+(S_2,S_3)$. □

5. Link Floer homology and the stabilization distance

In this section we prove our main technical results about stabilizations and the link Floer TQFT, and show that our invariants τ and V_k give lower bounds on μ_{st}, while κ_0 and $\mathcal{I}$ give lower bounds on g_{dest}.

5.1. **Algebraic reduction.** In this section, we consider the relation between the Heegaard Floer homology of multi-pointed 3-manifolds and the link Floer homology of unlinks.

There are two natural ways to reduce $\mathcal{CFL}^-$ to CF^- via a tensor product. Let $M_{V=1}$ denote the $(\mathbb{F}[U,V],\mathbb{F}[\widehat{U}])$-bimodule with underlying vector space $\mathbb{F}[\widehat{U}]$, where U acts on the left by $\widehat{U}$, and V acts by 1. We have $\widehat{U}$ act on the right by ordinary multiplication. There is also an $(\mathbb{F}[U,V],\mathbb{F}[\widehat{U}])$-bimodule $M_{U=1}$ with underlying vector space $\mathbb{F}[\widehat{U}]$, defined similarly, except that we have V act on the left by $\widehat{U}$ and we have U act by 1.

There are canonical isomorphisms

$$\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})\otimes_{\mathbb{F}[U,V]} M_{V=1}\cong CF^-(Y,\mathbf{w},\mathfrak{s})$$
$$\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})\otimes_{\mathbb{F}[U,V]} M_{U=1}\cong CF^-(Y,\mathbf{z},\mathfrak{s}-PD[L]).$$

These isomorphisms are obtained by taking any Heegaard diagram for $(Y,\mathbb{L})$, and ignoring the $\mathbf{z}$ basepoints, or ignoring the $\mathbf{w}$ basepoints.

In particular, for any $\mathbb{F}[U,V]$-equivariant map F from $\mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}_1)$ to $\mathcal{CFL}^-(Y_2,\mathbb{L}_2,\mathfrak{s}_2)$, we obtain a map $F|_{V=1}$ from $CF^-(Y_1,\mathbf{w}_1,\mathfrak{s}_1)$ to $CF^-(Y_2,\mathbf{w}_2,\mathfrak{s}_2)$. There is also a map $F|_{U=1}$ from $CF^-(Y_1,\mathbf{z}_1,\mathfrak{s}_1-PD[L_1])$ to $CF^-(Y_2,\mathbf{z}_2,\mathfrak{s}_2-PD[L_2])$.

An important special case is when $\mathbb{L}$ is an unlink, and each link component has exactly two basepoints. We say that a diagram for $(Y,\mathbb{L})$ is a *minimal unlink diagram* if each $\mathbf{w}$ basepoint occurs in the same component of $\Sigma\setminus(\boldsymbol{\alpha}\cup\boldsymbol{\beta})$ as a $\mathbf{z}$ basepoint. In this case, a Seifert disk is canonically specified by picking a collection of arcs in $\Sigma\setminus(\boldsymbol{\alpha}\cup\boldsymbol{\beta})$ which connects each $\mathbf{w}$ basepoint to a $\mathbf{z}$ basepoint. By pushing the interiors of these arcs off of Σ, in both directions, a collection of Seifert disks for $\mathbb{L}$ is spanned. In particular, there is a canonical Seifert surface S of $\mathbb{L}$ which is determined by the diagram, and $A_S(U^iV^j\mathbf{x})=j-i$ for all intersection points $\mathbf{x}$.

Additionally, in the case of a minimal unlink diagram $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$, there is a canonical isomorphism

$$\mathcal{CFL}^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z},\mathfrak{s})\cong CF^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathfrak{s})\otimes_{\mathbb{F}[\widehat{U}]}\mathbb{F}[U,V], \tag{5.1}$$

where we view $\widehat{U}$ as acting on $\mathbb{F}[U,V]$ via the product UV.

In particular, if we are given minimal unlink diagrams for $(Y_1,\mathbb{L}_1)$ and $(Y_2,\mathbb{L}_2)$ as well as an $\mathbb{F}[U,V]$-equivariant map F from $\mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}_1)$ to $\mathcal{CFL}^-(Y_2,\mathbb{L}_2,\mathfrak{s}_2)$, we may view $F|_{U=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}$ and $F|_{V=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}$ as also being maps from $\mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}_1)$ to $\mathcal{CFL}^-(Y_2,\mathbb{L}_2,\mathfrak{s}_2)$. For our purposes, it is useful to compare these maps to the original map F:

Lemma 5.1. *Suppose that $\mathbb{L}_1\subseteq Y_1$ and $\mathbb{L}_2\subseteq Y_2$ are unlinks, and pick minimal unlink diagrams for $(Y_1,\mathbb{L}_1)$ and $(Y_2,\mathbb{L}_2)$, respectively. Suppose that $F\colon\mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}_1)\to\mathcal{CFL}^-(Y_2,\mathbb{L}_2,\mathfrak{s}_2)$ is an $\mathbb{F}[U,V]$-equivariant map, which is homogeneously graded with respect to the Alexander grading, and shifts the Alexander grading by Δ.*

(1) If $\Delta\ge 0$, then

$$F=V^{\Delta}\cdot(F|_{V=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]})\quad\textit{and}\quad U^{\Delta}\cdot F=(F|_{U=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}).$$

(2) If $\Delta\le 0$, then

$$V^{-\Delta}\cdot F=(F|_{V=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]})\quad\textit{and}\quad F=U^{-\Delta}\cdot(F|_{U=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}).$$

Proof. Consider the claim for $\Delta\ge 0$. In this case, the maps F and $F|_{V=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}$ agree up to an overall power of V. Since $F|_{V=1}\otimes\mathrm{id}_{\mathbb{F}[U,V]}$ preserves the Alexander grading, the overall power is V^{Δ}. The same argument works for the other claims. ☐

Remark 5.2. Lemma 5.1 is stated using two fixed minimal unlink diagrams for $(Y_1,\mathbb{L}_1)$ and $(Y_2,\mathbb{L}_2)$, so we do not claim that the map $F|_{V=1}\otimes \mathrm{id}_{\mathbb{F}[U,V]}$ and $F|_{U=1}\otimes \mathrm{id}_{\mathbb{F}[U,V]}$ are natural maps. We may view these maps as being natural if we fix a set of Seifert disks for $\mathbb{L}_1$ and $\mathbb{L}_2$.

5.2. **Stabilizations and link Floer homology.** In this section, we prove our main computational results about stabilizations and link Floer homology. Before we state our computational results, we recall that the link cobordism maps admit extensions

$$F_{W,\mathcal{F},\mathfrak{s}}\colon \Lambda^*(H_1(W)/\operatorname{Tors})\otimes \mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}|_{Y_1})\to \mathcal{CFL}^-(Y_2,\mathbb{L}_2,\mathfrak{s}|_{Y_2})$$

that incorporate the action of $\Lambda^*(H_1(W)/\operatorname{Tors})$, similar to the cobordism maps of Ozsváth and Szabó [40]; see [57, Section 12.2] for a description.

If F and G are two maps from $\mathbb{F}_2[U,V]$ to $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$, we say that $F\simeq G$ *modulo the action of* $H_1(Y)$, and write

$$F\simeq G \mod H_1(Y),$$

if there are classes $[\gamma_1],\dots,[\gamma_k]\in H_1(Y)$, as well as $\mathbb{F}_2[U,V]$-equivariant maps $J_1,\dots,J_k$ from $\mathbb{F}_2[U,V]$ to $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$, such that

$$F+G\simeq \sum_{i=1}^k A_{\gamma_i}\circ J_i.$$

(Note that if γ and γ' are homologous 1-cycles in Y, then $A_\gamma\simeq A_{\gamma'}$.)

Lemma 5.3. *Suppose that* $(W,\mathcal{F})\colon \emptyset\to (Y,\mathbb{U})$ *is a decorated link cobordism from the empty set to a doubly-based unknot* $\mathbb{U}$ *in* Y*, equipped with a Seifert disk* D*, and let* $\mathfrak{s}\in \mathrm{Spin}^c(W)$*. Pick a Heegaard diagram for* $(Y,\mathbb{U})$ *where the* w *and* z *basepoints are immediately adjacent. Write* $\mathcal{F}=(S,\mathcal{A})$*, and suppose that* $H_1(Y)\to H_1(W)$ *is a surjection.*

(1) Suppose $\mathcal{A}$ *has a single component (necessarily a non-closed arc), and write*

$$h(S\cup D,\mathfrak{s}):=\frac{\langle c_1(\mathfrak{s}),[S\cup D]\rangle-[S\cup D]\cdot[S\cup D]}{2}.$$

If $h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})\ge 0$*, then*

$$F_{W,\mathcal{F},\mathfrak{s}}\simeq U^{g(S_{\mathbf{w}})}V^{g(S_{\mathbf{z}})+h(S\cup D,\mathfrak{s})}\cdot F_{W,\mathfrak{s}}\otimes \mathrm{id}_{\mathbb{F}[U,V]} \mod H_1(Y),$$

with respect to the isomorphism from Equation (5.1)*. If* $h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})\le 0$*, then*

$$F_{W,\mathcal{F},\mathfrak{s}}\simeq U^{g(S_{\mathbf{w}})-h(S\cup D,\mathfrak{s})}V^{g(S_{\mathbf{z}})}\cdot F_{W,\mathfrak{s}-PD[S]}\otimes \mathrm{id}_{\mathbb{F}[U,V]} \mod H_1(Y).$$

(2) If $\mathcal{A}$ *has a closed component* γ*, then*

$$F_{W,\mathcal{F},\mathfrak{s}}\simeq 0 \mod H_1(Y).$$

Proof. The proof is a modification of [57, Proposition 9.7]. Consider first Claim (1), in the case when $h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})\ge 0$. Note that the latter quantity is the Alexander grading change of the map $F_{W,\mathcal{F},\mathfrak{s}}$. The reduction $F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}$ is computed explicitly in [58, Theorem C], and depends only on W, $\mathfrak{s}$, and the embedding of the subsurface $S_{\mathbf{w}}$ in W. When $S_{\mathbf{w}}$ is a connected surface with a single boundary component, according to [57, Lemma 9.6], the $V=1$ reduction satisfies

$$F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}\simeq F_{W,\mathfrak{s}}(\xi_{\mathbf{w}}\otimes -), \tag{5.2}$$

where $\xi_{\mathbf{w}}\in \Lambda^*(H_1(W)/\operatorname{Tors})\otimes \mathbb{F}_2[\widehat{U}]$ is an element equal to $\widehat{U}^{g(S_{\mathbf{w}})}$ modulo $H_1(W)$. Since $H_1(Y)$ surjects onto $H_1(W)$, we can commute $\xi_{\mathbf{w}}$ with $F_{W,\mathfrak{s}}$ to obtain the relation

$$F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}\simeq \widehat{U}^{g(S_{\mathbf{w}})}\cdot F_{W,\mathfrak{s}} \mod H_1(Y).$$

Applying Lemma 5.1, we conclude that

$$F_{W,\mathcal{F},\mathfrak{s}}\simeq V^{h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})}\widehat{U}^{g(S_{\mathbf{w}})}\cdot F_{W,\mathfrak{s}}\otimes \mathrm{id}_{\mathbb{F}[U,V]} \mod H_1(Y). \tag{5.3}$$

Claim (1) in this case follows by rearranging equation (5.3) using the fact that $\widehat{U}=UV$.

The argument fails when the Alexander grading $h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})$ is negative, since multiplication by $V^{h(S\cup D,\mathfrak{s})+g(S_{\mathbf{z}})-g(S_{\mathbf{w}})}$ is not a filtered map. Instead we must consider the $U=1$ reduction of $F_{W,\mathcal{F},\mathfrak{s}}$. According to [57, Lemma 9.6], the $U=1$ reduction satisfies

$$F_{W,\mathcal{F},\mathfrak{s}}|_{U=1}\simeq F_{W,\mathfrak{s}-PD[S]}(\xi_{\mathbf{z}}\otimes -),$$

for an element $\xi_{\mathbf{z}}\in \Lambda^*(H_1(W)/\operatorname{Tors})\otimes \mathbb{F}_2[\widehat{U}]$ equal to $\widehat{U}^{g(S_{\mathbf{z}})}$ modulo $H_1(W)$. Using this fact, the formula

$$F_{W,\mathcal{F},\mathfrak{s}}\simeq U^{g(S_{\mathbf{w}})-h(S\cup D,\mathfrak{s})}V^{g(S_{\mathbf{z}})}\cdot (F_{W,\mathfrak{s}-PD[S]}\otimes \mathrm{id}_{\mathbb{F}[U,V]}) \mod H_1(Y)$$

can be established by the same strategy as before.

Next, we consider Claim (2), where $\mathcal{A}$ contains a closed component. As in the proof of Claim (1), the key will be to consider the maps $F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}$ and $F_{W,\mathcal{F},\mathfrak{s}}|_{U=1}$. Let Δ denote the quantity

$$\Delta := h(S\cup D,\mathfrak{s})+\frac{\chi(S_{\mathbf{w}})-\chi(S_{\mathbf{z}})}{2},$$

which we note is the Alexander grading of the map $F_{W,\mathcal{F},\mathfrak{s}}$.

We first consider the case when $\Delta\ge 0$. Let us write $C_{\mathbf{w},0}$ and $C_{\mathbf{z},0}$ for the components of $S_{\mathbf{w}}$ and $S_{\mathbf{z}}$ that intersect ∂S. We will reduce to the case when $\partial C_{\mathbf{w},0}$ contains a closed component disjoint from ∂S. If $C_{\mathbf{w},0}=S_{\mathbf{w}}$, then $\partial C_{\mathbf{w},0}$ trivially contains a closed component disjoint from ∂S. If $C_{\mathbf{w},0}$ is not the only component of $S_{\mathbf{w}}$, then, since S is connected, we can find a properly embedded path $\gamma_{\mathbf{z}}\colon I\to C_{\mathbf{z},0}$, with both endpoints on $\mathcal{A}$, such that $\gamma_{\mathbf{z}}(0)\in C_{\mathbf{w},0}$, and $\gamma_{\mathbf{z}}(1)$ is a point in the boundary of another component, $C_{\mathbf{w},1}$ of $S_{\mathbf{w}}$. There are four cases we consider:

(1) $C_{\mathbf{w},1}$ is planar, and $|\partial C_{\mathbf{w},1}|=1$.
(2) $C_{\mathbf{w},1}$ is planar, and $|\partial C_{\mathbf{w},1}|=2$.
(3) $C_{\mathbf{w},1}$ is planar, and $|\partial C_{\mathbf{w},1}|>2$.
(4) $g(C_{\mathbf{w},1})>0$.

In Case (1), the surface $C_{\mathbf{w},1}$ is topologically a disk, which is necessarily disjoint from ∂S, since $|\mathcal{A}\cap \partial S|=2$. We claim that the map $F_{W,\mathcal{F},\mathfrak{s}}\simeq 0$. Indeed, the cobordism map $F_{W,\mathcal{F},\mathfrak{s}}$ can be factored through the composition of a quasi-stabilization, followed by a quasi-destabilization, and such a composition clearly vanishes.

We next consider Case (2), when $C_{\mathbf{w},1}$ is an annulus, which is disjoint from ∂S. In this case, we also have $F_{W,\mathcal{F},\mathfrak{s}}\simeq 0$. To see this, pick a properly embedded path $\gamma_{\mathbf{w}}\colon I\to C_{\mathbf{w},1}$ that connects the two boundary components of $C_{\mathbf{w},1}$, and such that $\gamma_{\mathbf{w}}(0)=\gamma_{\mathbf{z}}(1)$; see the top of Figure 5.1. We concatenate $\gamma_{\mathbf{z}}$ and $\gamma_{\mathbf{w}}$ to get a path γ. A neighborhood of γ is the domain of a bypass. Let $\mathcal{A}'$ and $\mathcal{A}''$ denote the other two dividing sets in the bypass triple; see the bottom row of Figure 5.1. The bypass relation (relation (3.12) above and its interpretation in terms of decorated cobordisms from Figure 3.7) implies that

$$F_{W,(S,\mathcal{A}),\mathfrak{s}}\simeq F_{W,(S,\mathcal{A}'),\mathfrak{s}}+F_{W,(S,\mathcal{A}''),\mathfrak{s}}. \tag{5.4}$$

The key observation is that $\mathcal{A}'$ and $\mathcal{A}''$ are actually isotopic, so equation (5.4) implies that $F_{W,(S,\mathcal{A}),\mathfrak{s}}\simeq 0$. The isotopy between $\mathcal{A}'$ and $\mathcal{A}''$ is shown in Figure 5.1.

We now consider Cases (3) and (4), when $C_{\mathbf{w},1}$ is planar and $|\partial C_{\mathbf{w},1}|>2$, or when $g(C_{\mathbf{w},1})>0$, respectively. In both cases, we let $\gamma_{\mathbf{w}}\colon I\to C_{\mathbf{w},1}$ be a properly embedded curve which is non-separating and satisfies $\gamma_{\mathbf{z}}(1)=\gamma_{\mathbf{w}}(0)$; see the top of Figure 5.2. If $g(C_{\mathbf{w},1})>0$, we require that both ends of $\gamma_{\mathbf{w}}$ are on the same component of $\partial C_{\mathbf{w},1}$. We let γ denote the concatenation of $\gamma_{\mathbf{w}}$ and $\gamma_{\mathbf{z}}$. As in Case (2), we consider the bypass triple obtained by taking a regular neighborhood of the image of γ. Let $\mathcal{A}'$ and $\mathcal{A}''$ denote the other two dividing sets in the bypass triple, shown on the bottom of Figure 5.2.

Let $C'_{\mathbf{w},0}$ and $C''_{\mathbf{w},0}$ denote the type-$\mathbf{w}$ subregions of $S\setminus \mathcal{A}'$ and $S\setminus\mathcal{A}''$ that intersect ∂S. In Cases (3) and (4), it is easy to check that $\partial C'_{\mathbf{w},0}$ and $\partial C''_{\mathbf{w},0}$ both contain a closed curve disjoint from ∂S, so it is sufficient to show the main claim for each of $F_{W,(S,\mathcal{A}'),\mathfrak{s}}$ and $F_{W,(S,\mathcal{A}''),\mathfrak{s}}$ separately. Case (4) is illustrated in Figure 5.2.

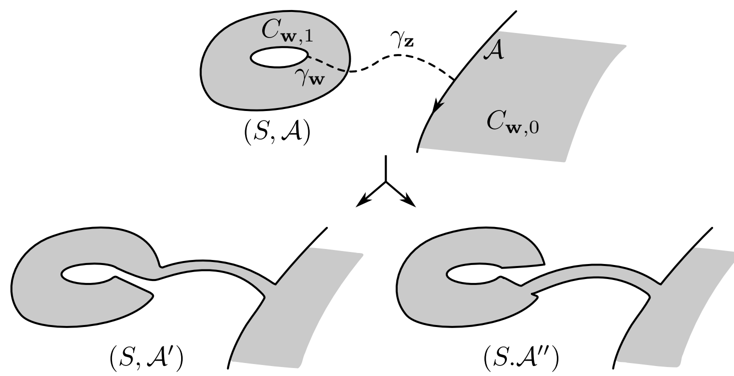

FIGURE 5.1. When $C_{\mathbf{w},1}$ is an annulus, the two other dividing sets in a bypass are isotopic.

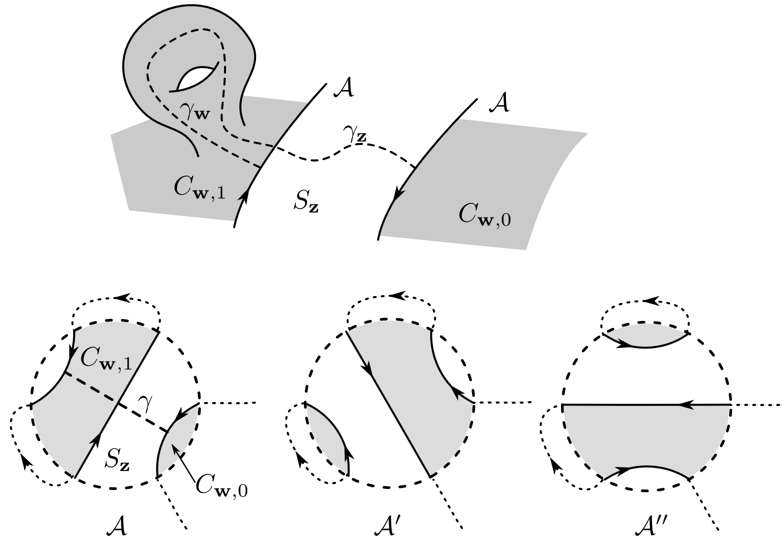

FIGURE 5.2. The bypass relation in Case (4), when $g(C_{\mathbf{w},1}) > 0$. On the bottom row, the domain of the bypass is shown. The dotted lines outside of the domain of the bypass indicate the configuration of the dividing set outside the bypass region.

We now proceed to show that if $\mathcal{A}$ is a dividing set on S such that $\partial C_{\mathbf{w},0}$ contains a closed component γ disjoint from ∂S, then

$$F_{W,\mathcal{F},\mathfrak{s}}|_{V=1} = A_\gamma \circ G \tag{5.5}$$

for some map G, where A_γ denotes the homology action of the curve γ. To establish equation (5.5), we must recall some additional facts about the functor $F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}$. According to [58, Theorem C], the chain homotopy type of the map $F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}$ depends only on W, the embedded surface $S_{\mathbf{w}}$ (which is not properly embedded) and $\mathfrak{s} \in \text{Spin}^c(W)$. To describe the reduction in more detail, we recall that a *ribbon 1-skeleton* of $S_{\mathbf{w}}$ is a choice of embedded graph $\Gamma_{\mathbf{w}} \subseteq S_{\mathbf{w}}$ such that $\Gamma_{\mathbf{w}} \cap \partial S_{\mathbf{w}} = \{w\}$, and $S_{\mathbf{w}}$ is a regular neighborhood of $\Gamma_{\mathbf{w}}$ in S; see [58, Definition 14.5].

There is a simple way to construct the ribbon 1-skeleton of the subsurface $S_{\mathbf{w}}$. One starts with a collection of arcs $a \subseteq S_{\mathbf{w}}$ such that $a \cap \partial S_{\mathbf{w}} = \{w\}$, and such that each component of $S_{\mathbf{w}}$ contains exactly one arc. One then takes inward translates $C_1, \dots, C_n$ of the boundary components of $\partial S_{\mathbf{w}}$ which do not contain a basepoint of $\mathbf{w}$, which one connects to a by adjoining an embedded arc (disjoint from the other arcs). The complement of this graph in $S_{\mathbf{w}}$ consists of a collection of $|\partial S_{\mathbf{w}}|$

connected surfaces, each with a single boundary component. The total genus of these surfaces is $g(S_{\mathbf{w}})$. We then pick a geometric symplectic basis of H_1 of the complement of this graph (i.e., a collection of simple closed curves $A_1, \dots, A_g, B_1, \dots, B_g$ that form a basis of H_1 and satisfy $|A_i \cap B_j| = \delta_{ij}$). By connecting a with one of the curves in each pair in the symplectic basis by an arc, we obtain a ribbon 1-skeleton of $S_{\mathbf{w}}$. An example is shown in Figure 5.3.

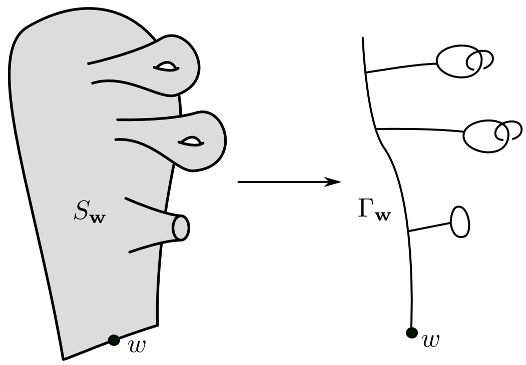

FIGURE 5.3. A ribbon 1-skeleton $\Gamma_{\mathbf{w}}$ for a genus 2 component of $S_{\mathbf{w}}$ with 2 boundary components.

The second author [53] constructed maps on CF^- induced by cobordisms with embedded ribbon graphs. By [58, Theorem C], the reduction $F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}$ is chain homotopic to the graph cobordism map $F_{W,\Gamma_{\mathbf{w}},\mathfrak{s}}$ for a ribbon 1-skeleton $\Gamma_{\mathbf{w}}$ of $S_{\mathbf{w}}$. Note that $\Gamma_{\mathbf{w}}$ inherits a ribbon structure; i.e., a cyclic ordering of the edges adjacent to each vertex, from the orientation of $S_{\mathbf{w}}$.

By picking an appropriate ribbon 1-skeleton $\Gamma_{\mathbf{w}}$ of $\Sigma_{\mathbf{w}}$ (see Figure 5.4), we can decompose the graph cobordism $(W, \Gamma_{\mathbf{w}})$ such that it is a sequence of graph cobordisms $(W_3, \gamma_3) \circ (W_2, \Gamma_2) \circ (W_1, \Gamma_1)$, satisfying the following conditions:

(1) W_1 is a 4-dimensional 1-handlebody, and the graph Γ_1 intersects ∂W_1 in a single point.
(2) W_2 is a cylinder $I \times Y$, and Γ_2 is a graph of the form $(I \times \{p\}) \cup \gamma$, as shown in Figure 5.4, where γ is a loop induced by one of the boundary components of $S_{\mathbf{w}}$ which is disjoint from ∂S.
(3) W_3 is a cobordism between two connected 3-manifolds, and γ_3 is a path connecting the two components.

This can be achieved as follows. We pick $\Gamma_{\mathbf{w}}$ by first taking an arc a in $\Sigma_{\mathbf{w}}$ such that $a \cap \partial \Sigma_{\mathbf{w}} = \{w\}$. We then join closed loops (as described above) for components of $\partial S_{\mathbf{w}}$ not containing w, and also for a symplectic basis $A_1, \dots, A_g, B_1, \dots, B_g$ as above. We assume that of these loops, γ is joined the closest to w along the arc a. We pick an ordered handle decomposition for W into 0-, 1-, 2- and 3-handles. We let W_1 be the the union of the 0- and 1-handles. We let W_2 be a regular neighborhood of ∂W_1, and we let W_3 be the 2-handles and 3-handles. By flowing using a gradient like Morse function for this handle decomposition, we may isotope all of the closed loops of $\Gamma_{\mathbf{w}}$ so that they lie below W_3. Therefore we may assume that $\Gamma_{\mathbf{w}} \cap W_2$ consists of a subarc of a with γ spliced in, and that $W_3 \cap \Gamma_{\mathbf{w}}$ consists only of a single arc, as claimed above.

The composition law for graph cobordism maps implies that

$$F_{W,\Gamma_{\mathbf{w}},\mathfrak{s}} \simeq F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}} \circ F_{W_2,\Gamma_2,\mathfrak{s}|_{W_2}} \circ F_{W_1,\Gamma_1,\mathfrak{s}|_{W_1}}. \tag{5.6}$$

Since γ_3 is a path, the map $F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}}$ agrees with Ozsváth and Szabó's cobordism map. By [55, Proposition 4.6], we have

$$F_{W_2,\Gamma_2,\mathfrak{s}|_{W_2}} \simeq A_\gamma. \tag{5.7}$$

Since $H_1(Y)$ surjects onto $H_1(W)$, we have

$$F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}} \circ A_\gamma \simeq A_\gamma \circ F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}}. \tag{5.8}$$

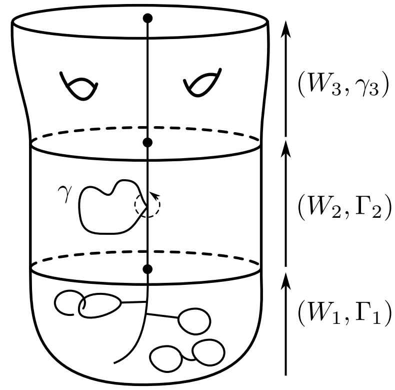

FIGURE 5.4. A decomposition of the graph cobordism $(W, \Gamma_{\mathbf{w}})$. The loop γ in Γ_2 corresponds to a closed curve in $\partial C_{\mathbf{w},0}$ disjoint from ∂S.

Combining equations (5.6), (5.7), and (5.8), we obtain the relation

$$F_{W,\mathcal{F},\mathfrak{s}}|_{V=1} \simeq F_{W,\Gamma_{\mathbf{w}},\mathfrak{s}} \simeq A_\gamma \circ F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}} \circ F_{W_1,\Gamma_1,\mathfrak{s}|_{W_1}}. \tag{5.9}$$

Since we assumed that the Alexander grading shift Δ was nonnegative, by using Lemma 5.1 we obtain

$$\begin{aligned} F_{W,\mathcal{F},\mathfrak{s}} &\simeq V^{\Delta} \cdot (F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}) \otimes \mathrm{id}_{\mathbb{F}[U,V]} \\ &\simeq V^{\Delta} \cdot (A_\gamma \circ G) \otimes \mathrm{id}_{\mathbb{F}[U,V]} \\ &\simeq (A_\gamma \otimes \mathrm{id}_{\mathbb{F}[U,V]}) \circ (V^{\Delta} \cdot G \otimes \mathrm{id}_{\mathbb{F}[U,V]}) \\ &\simeq A_\gamma \circ (V^{\Delta} \cdot G \otimes \mathrm{id}_{\mathbb{F}[U,V]}), \end{aligned}$$

where $G \simeq F_{W_3,\gamma_3,\mathfrak{s}|_{W_3}} \circ F_{W_1,\Gamma_1,\mathfrak{s}|_{W_1}}$. In the last line, we are using the fact that A_γ preserves the Alexander grading, so $(A_\gamma|_{V=1}) \otimes \mathrm{id}_{\mathbb{F}[U,V]}$ coincides with the ordinary action of A_γ on $\mathcal{CFL}^-(Y, \mathbb{U})$ by Lemma 5.1. This proves the claim.

The case when the Alexander grading change Δ is negative is handled similarly, using the $U = 1$ reductions instead. $\square$

Next, we compute the effect of a stabilization, for a simple dividing set:

Lemma 5.4. *Suppose that $(W, \mathcal{F})\colon (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2)$ is a decorated link cobordism with $b_1(W) = 0$. Write $\mathcal{F} = (S, \mathcal{A})$, and suppose that S is connected. Let S' be a (n, g)-stabilization of S along (B^4, S_0). Let $D_1, \dots, D_n$ denote the components of $S \cap B^4$, and let $\widehat{D} \subseteq S$ be a disk that contains $D_1, \dots, D_n$ and intersects $\mathcal{A}$ in a single arc. Consider the subsurface*

$$S_0' := (\widehat{D} \setminus (D_1 \cup \dots \cup D_n)) \cup S_0 \subseteq S'.$$

Let $\mathcal{A}'$ be a dividing set on S' that agrees with $\mathcal{A}$ outside $\widehat{D}$, and write $\mathcal{F}' = (S', \mathcal{A}')$.

(1) Suppose that $\mathcal{A}'$ intersects S_0' in a single arc that divides S_0' into two connected components. Let $S_{\mathbf{w}}$ and $S_{\mathbf{w}}'$ denote the type-$\mathbf{w}$ regions, and $S_{\mathbf{z}}$ and $S_{\mathbf{z}}'$ the type-$\mathbf{z}$ regions of $\mathcal{F}$ and $\mathcal{F}'$, respectively. Then

$$F_{W,\mathcal{F}',\mathfrak{s}} \simeq U^{g(S_{\mathbf{w}}') - g(S_{\mathbf{w}})} V^{g(S_{\mathbf{z}}') - g(S_{\mathbf{z}})} \cdot F_{W,\mathcal{F},\mathfrak{s}}.$$

(2) If $\mathcal{A} \cap S_0'$ contains a closed component, then

$$F_{W,\mathcal{F}',\mathfrak{s}} \simeq 0.$$

Proof. We first show Claim (1). Consider the punctured disk

$$\widehat{D}_0 := \widehat{D} \setminus (D_1 \cup \dots \cup D_n).$$

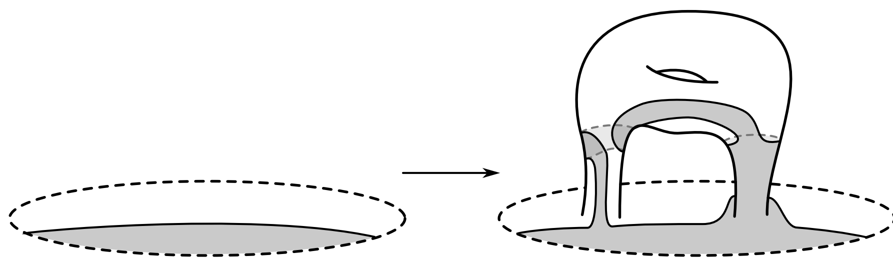

FIGURE 5.5. An example of a stabilization considered in Lemma 5.4. The dividing set $\mathcal{A}$ in the region $\widehat{D} \subseteq S$ is shown on the left, and the dividing set $\mathcal{A}'$ on the stabilization S' is shown on the right.

Let N denote the total space of the unit normal disk bundle of $\widehat{D}_0$ in $W \setminus B^4$. Note that N is diffeomorphic to $\widehat{D}_0 \times B^2$. Define

$$W_0 := B^4 \cup N,$$

which, after rounding corners, we can view as a codimension 0 submanifold of W with smooth boundary. In fact, W_0 is a 4-dimensional genus $n-1$ handlebody $(\mathbb{S}^1 \times B^3)^{\natural(n-1)}$. Let Y denote ∂W_0. We observe that S_0', as defined above, is equal to $W_0 \cap S'$. We can view (W_0, S_0') as a link cobordism from the empty link to the pair (Y, K) where $K = \partial \widehat{D} \times \{0\}$.

Let us write $C_1, \dots, C_n$ for the components of $\partial \widehat{D}_0 \setminus \partial \widehat{D}$, and $U_1, \dots, U_n$ for the components of the unlink $S_0 \cap \partial B^4$. We can view

$$\partial N = (\widehat{D}_0 \times \partial B^2) \cup (\partial \widehat{D} \times B^2) \cup \bigcup_{i=1}^{n} (C_i \times B^2).$$

Hence, we can write

$$Y = \left(\partial N \setminus \bigcup_{i=1}^{n} (C_i \times B^2) \right) \cup \left(\partial B^4 \setminus \bigcup_{i=1}^{n} N(U_i) \right),$$

where the two manifolds are glued along their n torus boundary components.

We now claim that K is an unknot in Y. It is at this step that we use the fact that $S \cap \partial B^4$ is an unlink. To see that K is an unknot, we will construct a Seifert disk D_K for K in Y. Let r denote a radial arc from $0 \in B^2$ to a point $p \in \partial B^2$. Let A denote the annulus

$$A := \partial \widehat{D} \times r \subseteq Y.$$

We then attach the punctured disk $\widehat{D}_0 \times \{p\}$ to the annulus A. The resulting surface has boundary

$$\partial(A \cup (\widehat{D}_0 \times \{p\})) = K \cup \bigcup_{i=1}^{n} (C_i \times \{p\}).$$

Next, we note that the image of $C_i \times \{p\}$ in $\partial N(U_i) \subseteq \partial B^4$ is the Seifert longitude, since the disks $D_i \subseteq S \cap B^4$ can be pushed into ∂B^4 to give Seifert disks of U_i that intersect $\partial N(U_i)$ along $C_i \times \{p\}$. By capping $C_i \times \{p\}$ with Seifert disks of the U_i, we obtain the Seifert disk D_K of K in Y.

Let us write $\mathcal{F}_0$ for the decorated surface $(\widehat{D}, \mathcal{A} \cap \widehat{D})$, $\mathcal{F}_0'$ for the decorated surface $(S_0', \mathcal{A}' \cap S_0')$, and $\mathfrak{s}_0$ for $\mathfrak{s}|_{W_0}$. Since $H_2(W_0) = 0$, by applying Lemma 5.3 to both $F_{W_0, \mathcal{F}_0, \mathfrak{s}_0}$ and $F_{W_0, \mathcal{F}_0', \mathfrak{s}_0}$, we compute that

$$F_{W_0, \mathcal{F}_0', \mathfrak{s}_0} \simeq U^{g(S'_{0,\mathbf{w}})} V^{g(S'_{0,\mathbf{z}})} \cdot F_{W_0, \mathcal{F}_0, \mathfrak{s}_0} \mod H_1(Y). \tag{5.10}$$

Write $W_1 := W \setminus \text{int}(W_0)$, $\mathcal{F}_1 := \mathcal{F}' \cap W_1$, and $\mathfrak{s}_1 := \mathfrak{s}|_{W_1}$. Since W_0 is a 4-dimensional handlebody, we conclude that $b_1(W_1) = 0$. Noting that the map $\delta \colon H^1(Y) \to H^2(W)$ is trivial, and using the

Spin^c composition law, we conclude from equation (5.10) that

$$F_{W,\mathcal{F}',\mathfrak{s}} \simeq U^{g(S'_{0,\mathbf{w}})} V^{g(S'_{0,\mathbf{z}})} \cdot F_{W,\mathcal{F},\mathfrak{s}}. \tag{5.11}$$

Noting that $g(S'_{\mathbf{w}})-g(S_{\mathbf{w}}) = g(S'_{0,\mathbf{w}})$ and $g(S'_{\mathbf{z}})-g(S_{\mathbf{z}}) = g(S'_{0,\mathbf{z}})$, the proof of Claim (1) is complete.

We now consider Claim (2). In this case, Lemma 5.3 implies that $F_{W_0,\mathcal{F}'_0,\mathfrak{s}} \simeq \sum_{i=1}^k A_{\gamma_i} \circ J_i$ for some filtered, equivariant maps $J_1, \dots, J_k$. The map $H_1(Y) \to H_1(W_1)/\operatorname{Tors}$ is trivial since $b_1(W) = 0$ and W_0 is a 4-dimensional 1-handlebody. Hence, using the composition law,

$$F_{W,\mathcal{F}',\mathfrak{s}} \simeq F_{W_1,\mathcal{F}_1,\mathfrak{s}_1} \circ F_{W_0,\mathcal{F}'_0,\mathfrak{s}_0} \simeq F_{W_1,\mathcal{F}_1,\mathfrak{s}_1} \circ \left(\sum_{i=1}^k A_{\gamma_i} \circ J_i \right) \simeq 0,$$

concluding the proof of Claim (2). □

Lemma 5.4 computes the result of a stabilization of a link cobordism, when the dividing set is nicely arranged on the stabilization. However, to prove geometric bounds on the secondary versions of V_k, we will need to consider more general dividing sets on stabilizations.

Definition 5.5. Suppose that (B^4, S) is an undecorated knot cobordism from $\emptyset$ to an arbitrary knot K in $\mathbb{S}^3$. Let $\mathbb{K}$ denote K decorated with two basepoints, and let $\mathfrak{s}_0$ be the unique Spin^c structure on B^4. We say that S satisfies the *decoration-independence condition (DI)* if the following holds:

(DI) For any decoration $\mathcal{F} = (S, \mathcal{A})$ whose dividing set intersects K in exactly two points,
  1. the filtered, equivariant chain homotopy type of the map
$$F_{B^4,\mathcal{F},\mathfrak{s}_0} \colon \mathbb{F}_2[U,V] \to \mathcal{CFL}^-(\mathbb{K})$$
depends only on $g(S_{\mathbf{w}})$ and $g(S_{\mathbf{z}})$ when $|\mathcal{A}| = 1$, and
  2. $F_{B^4,\mathcal{F},\mathfrak{s}_0} \simeq 0$ when $|\mathcal{A}| > 1$.

Note that, if S is a stabilization of a slice disk (B^4, D), then, by Lemma 5.4, the link cobordism (B^4, S) satisfies the decoration-independence condition (DI).

Definition 5.6. Let S and $\mathbb{K}$ be as in Definition 5.5. Suppose $d \geq g(S)$ is an integer. We say that S satisfies the decoration-independence condition (DI) *above degree* d if for any decoration $\mathcal{F} = (S, \mathcal{A})$ compatible with $\mathbb{K}$, and for any i, $j \in \mathbb{N}$ satisfying $i + j + g(S) \geq d$,

1. the chain homotopy type of the map $U^i V^j \cdot F_{B^4,\mathcal{F},\mathfrak{s}_0}$ depends only on $i+g(S_{\mathbf{w}})$ and $j+g(S_{\mathbf{z}})$ when $|\mathcal{A}| = 1$, and
2. $U^i V^j \cdot F_{B^4,\mathcal{F},\mathfrak{s}} \simeq 0$ when $|\mathcal{A}| > 1$.

We define the invariant $\mathcal{I}(S) \in \mathbb{N}$ to be the minimal $d \geq g(S)$ such that S satisfies condition (DI) above degree d.

Remark 5.7. The quantity $\mathcal{I}(S)$ is finite for every surface S. This can be seen as follows. Two $\mathbb{F}[U,V]$-equivariant chain maps f, $g\colon \mathbb{F}[U,V] \to \mathcal{CFL}^-(S^3, \mathbb{K})$ are $\mathbb{F}[U,V]$-equivariantly chain homotopic if and only if $[f(1)] = [g(1)]$, as elements of $\mathcal{HFL}^-(S^3, \mathbb{K})$. However, the rank of $\mathcal{HFL}^-(S^3, \mathbb{K})$ in $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading $(-2n, -2m)$ is 1 whenever n, $m \geq 0$ and $n + m$ is sufficiently large.

Note that, to compute $\mathcal{I}(S)$, one would need to determine the cobordism maps for infinitely many dividing sets on S, which is a formidable task. However, to obtain a lower bound on $\mathcal{I}(S)$, it suffices to find two dividing sets $\mathcal{A}_1$ and $\mathcal{A}_2$ on S, both consisting of a single arc, and integers i_1, i_2, j_1, and j_2, such that

$$i_1 + g(S_{1,\mathbf{w}}) = i_2 + g(S_{2,\mathbf{w}}) \text{ and } j_1 + g(S_{1,\mathbf{z}}) = j_2 + g(S_{2,\mathbf{z}}), \text{ but}$$
$$U^{i_1} V^{j_1} \cdot F_{B^4,(S,\mathcal{A}_1),\mathfrak{s}_0} \not\simeq U^{i_2} V^{j_2} \cdot F_{B^4,(S,\mathcal{A}_2),\mathfrak{s}_0},$$

where $S_{k,\mathbf{w}}$ and $S_{k,\mathbf{z}}$ denote the type-$\mathbf{w}$ and type-$\mathbf{z}$ subsurfaces of S with respect to the decoration $\mathcal{A}_k$ for $k \in \{1, 2\}$. In this case, $\mathcal{I}(S) > i_1 + i_2 + g(S)$.

Proposition 5.8. *Suppose that* (B^4, S) *satisfies the decoration-independence condition* (DI) *above degree* d*, and let* S' *be a stabilization of* S*. Then* (B^4, S') *satisfies condition* (DI) *above degree* $\max\{d, g(S')\}$.

Our proof of Proposition 5.8 uses the following combinatorial lemma about dividing sets on surfaces:

Lemma 5.9. *Suppose that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are three dividing sets that fit into a bypass triple on a surface S with $|\partial S| = 1$, and $|\mathcal{A}_i \cap \partial S| = 2$ for $i \in \{1, 2, 3\}$. Then the number of $\mathcal{A}_i$ that have no closed loops is even.*

Proof. If $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ all have a closed loop, then the statement is true since 0 is even, so instead assume that $\mathcal{A}_1$ has no closed loops. Note that this implies that $|\mathcal{A}_1| = 1$, since $|\partial S \cap \mathcal{A}_1| = 2$. The dividing sets $\mathcal{A}_i$ can be consistently oriented, by declaring their orientation to be the boundary orientation of $S_{\mathbf{w}}$. Let D denote the bypass region. The set $\mathcal{A}_1 \cap D$ consists of three arcs, which we label as a_1, a_2, and a_3; see Figure 5.6. The main claim can be proven by considering separately six cases, corresponding to the possible relative orderings of the arcs a_1, a_2, and a_3, as they appear on $\mathcal{A}_1$. Let us first consider the case when the arcs appear ordered (a_1, a_2, a_3), read left-to-right. In this case, $\mathcal{A}_1$ has no closed loops by assumption, and by inspecting Figure 5.6, we see that exactly one of $\mathcal{A}_2$ and $\mathcal{A}_3$ also has no closed loops. The arguments when the arcs appear along $\mathcal{A}_1$ with ordering (a_1, a_3, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2), or (a_3, a_2, a_1) are easy modifications of the above argument. □

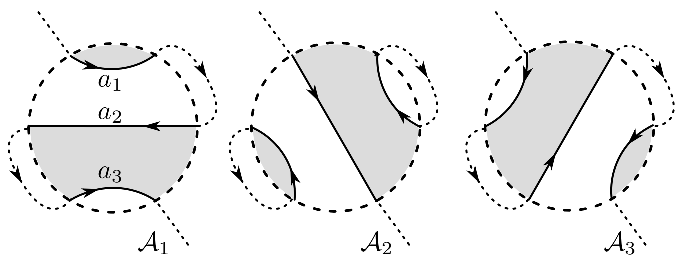

FIGURE 5.6. The proof of Lemma 5.9, when the arcs a_1, a_2, and a_3 appear on $\mathcal{A}_1$ with order (a_1, a_2, a_3), read left-to-right. The bypass region is the disk shown. The dashed lines outside the bypass regions represent the configuration of dividing arcs outside the bypass region. In the case at hand, $\mathcal{A}_1$ and $\mathcal{A}_3$ have no closed components, while $\mathcal{A}_2$ has two.

Proof of Proposition 5.8. Fix integers i, $j \geq 0$ such that $i + j + g(S') \geq d$. Analyzing the proof of Lemma 5.4, we can find a 4-dimensional 1-handlebody W_0 whose boundary we denote Y, such that

(1) $S' \cap Y = S \cap Y$ is an unknot in Y;
(2) $S \cap W_0$ is a disk, and $S' \cap W_0$ is a connected, genus $g(S') - g(S)$ surface with only one boundary component.

Let J denote $S' \cap Y$. Note that Lemma 5.4 immediately implies the statement for any dividing set $\mathcal{A}' \subseteq S'$ (connected or disconnected) that intersects J in exactly two points.

We now show the main claim by induction on $|\mathcal{A}' \cap J|$. We have established the base case, $|\mathcal{A}' \cap J| = 2$. If $\mathcal{A}'$ is a dividing set on S' with $|\mathcal{A}' \cap J| \geq 4$, then, using the bypass relation as shown in Figure 5.7, we can write

$$F_{B^4,(S',\mathcal{A}'),\mathfrak{s}_0} \simeq F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0} + F_{B^4,(S',\mathcal{A}'''),\mathfrak{s}_0}, \tag{5.12}$$

where $\mathcal{A}''$ and $\mathcal{A}''$ are dividing sets satisfying

$$|\mathcal{A}'' \cap J| = |\mathcal{A}''' \cap J| = |\mathcal{A}' \cap J| - 2.$$

Let us write $S''_{\mathbf{w}}$, $S''_{\mathbf{z}}$, $S'''_{\mathbf{w}}$, and $S'''_{\mathbf{z}}$ for the type-$\mathbf{w}$ and type-$\mathbf{z}$ subregions of $S' \setminus \mathcal{A}''$ and $S' \setminus \mathcal{A}'''$. There are two cases to consider: when $\mathcal{A}'$ has no closed components, or when $\mathcal{A}'$ has at least one closed component.

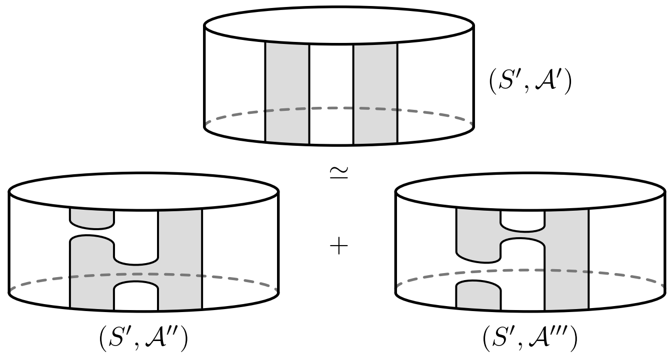

FIGURE 5.7. Reducing $|J\cap\mathcal{A}'|$ by 2, using the bypass relation. The annulus shown is a neighborhood of J in the surface S'. Using a small isotopy, we may push the two bigons in each of the bottom two annuli out of the neighborhood of J which is shown.

Let us consider the case when $\mathcal{A}'$ has no closed components. In this case, by Lemma 5.9, we know that exactly one of $\mathcal{A}''$ and $\mathcal{A}'''$ has no closed components, while the other has a closed component. For definiteness, let us say that $\mathcal{A}''$ has no closed components. Note that, in this case, $g(S'_{\mathbf{w}})=g(S''_{\mathbf{w}})$ and $g(S'_{\mathbf{z}})=g(S''_{\mathbf{z}})$.

By our inductive hypothesis, we know that $U^iV^j\cdot F_{B^4,(S',\mathcal{A}'''),\mathfrak{s}_0}\simeq 0$. Combining this with equation (5.12), we conclude that

$$U^iV^j\cdot F_{B^4,(S',\mathcal{A}'),\mathfrak{s}_0}\simeq U^iV^j\cdot F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0}.$$

By the inductive hypothesis, $U^iV^j\cdot F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0}$ depends only on the integers $i+g(S''_{\mathbf{w}})$ and $j+g(S''_{\mathbf{z}})$, and hence the same holds for $U^iV^j\cdot F_{B^4,(S',\mathcal{A}'),\mathfrak{s}_0}$.

Next, we consider the case when $\mathcal{A}'$ has a closed component. We wish to show that

$$U^iV^j\cdot F_{B^4,(S',\mathcal{A}'),\mathfrak{s}_0}\simeq 0.$$

By Lemma 5.9, one of the following two cases holds: Either $\mathcal{A}''$ and $\mathcal{A}'''$ both have a closed component, or neither $\mathcal{A}''$ nor $\mathcal{A}'''$ has a closed component. If $\mathcal{A}''$ and $\mathcal{A}'''$ both have a closed component, then $U^iV^j\cdot F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0}$ and $U^iV^j\cdot F_{B^4,(S',\mathcal{A}'''),\mathfrak{s}_0}$ are both chain homotopic to zero, by induction. If neither $\mathcal{A}''$ and $\mathcal{A}'''$ have a closed component, we note that $g(S''_{\mathbf{w}})=g(S'''_{\mathbf{w}})$ and $g(S''_{\mathbf{z}})=g(S'''_{\mathbf{z}})$, so

$$U^iV^j\cdot F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0}\simeq U^iV^j\cdot F_{B^4,(S',\mathcal{A}'''),\mathfrak{s}_0}$$

by induction. In both cases, the sum

$$U^iV^j\cdot F_{B^4,(S',\mathcal{A}''),\mathfrak{s}_0}+U^iV^j\cdot F_{B^4,(S',\mathcal{A}'''),\mathfrak{s}_0}\simeq 0.$$

Hence, by equation (5.12), $U^iV^j\cdot F_{B^4,(S',\mathcal{A}'),\mathfrak{s}_0}\simeq 0$, completing the proof. □

5.3. **Destabilizing genus bounds from $\mathcal{I}$ and κ_0.** In this section, we show that the invariants $\kappa_0(S)$ and $\mathcal{I}(S)$ give lower bounds on the quantity $g_{\text{dest}}(S)$, introduced in Definition 2.17. We begin with the invariant $\mathcal{I}(S)$ from Definition 5.6.

Theorem 5.10. *If K is slice a knot in $\mathbb{S}^3$ and $S\in\operatorname{Surf}(K)$, then*

$$\mathcal{I}(S)\le g_{\text{dest}}(S).$$

Proof. Let $S_1,\dots,S_n\in\operatorname{Surf}(K)$ be a sequence of surfaces as in Definition 2.17 connecting $S_1=S$ with a slice disk $S_n=D$, such that $g_{\text{dest}}(S)=\max\{g(S_1),\dots,g(S_n)\}$.

The result follows immediately from Proposition 5.8, with the following explanation. The disk S_n trivially satisfies the decoration-independence condition (DI) above degree 0. By Proposition 5.8, if S_k for $k\in\{2,\dots,n\}$ satisfies condition (DI) above degree d and S_{k-1} is a stabilization of S_k, then S_{k-1} satisfies condition (DI) above degree $\max\{d,g(S_{k-1})\}$. Using the stabilization formula, Lemma 5.4, the converse is also true: If S_k satisfies condition (DI) above degree d and S_{k-1} is a destabilization of S_k, then S_{k-1} also satisfies condition (DI) above degree d. Hence, by induction, we see that $S = S_1$ satisfies condition (DI) above degree $d = \max\{g(S_1), \dots, g(S_n)\}$, and hence $\mathcal{I}(S) \le g_{\text{dest}}(S)$. □

We note that the invariant $\mathcal{I}(S)$ is not easy to determine, since it involves computing the cobordism maps for infinitely many decorations on S. The invariant $\kappa_0(S)$ defined in Section 4.7 is easier to compute because it involves calculating just a single cobordism map on $\mathit{HFK}^-_{U=0}$, as opposed to infinitely many on $\mathcal{CFL}^-$. We now prove that $\kappa_0(S)$ also bounds $g_{\text{dest}}(S)$:

Theorem 5.11. *If K is a slice knot in $\mathbb{S}^3$ and $S \in \operatorname{Surf}(K)$, then*

$$\kappa_0(S) \le g_{\text{dest}}(S).$$

Proof. Suppose that $g(S) > 0$. Recall that $\mathbf{t}^-_{S,\mathbf{w}}$ is defined by decorating S with a dividing set consisting of a single arc such that $g(S_{\mathbf{w}}) = g(S)$ and $g(S_{\mathbf{z}}) = 0$. Suppose that $S_1, \dots, S_n$ is a stabilization sequence of surfaces in $\operatorname{Surf}(K)$ such that $S_1 = S$ and S_n is a slice disk. Let

$$d := \max\{g(S_1), \dots g(S_n)\}.$$

By Theorem 5.10, the surface S satisfies the decoration-independence condition (DI) above degree d. There are two cases: $d = g(S)$ or $d > g(S)$. If $d = g(S)$, then the stabilization formula implies that

$$\mathbf{t}^-_{S,\mathbf{w}} \simeq U^{g(S)} \cdot \mathbf{t}^-_{S_n},$$

so $\mathbf{t}^-_{S,\mathbf{w}}$ vanishes on $\mathit{HFK}^-_{U=0}$, implying that

$$\kappa_0(S) = g(S) = g_{\text{dest}}(S).$$

We now consider the second case, where $d > g(S)$. We note that

$$V^{d-g(S)} \cdot \mathbf{t}^-_{S,\mathbf{w}} \simeq F_{B^4,(S',\mathcal{A}'_{\mathbf{w}})},$$

where $(S', \mathcal{A}'_{\mathbf{w}})$ is obtained from $(S, \mathcal{A}_{\mathbf{w}})$ by performing $d - g(S)$ trivial 1-handle stabilizations along $S_{\mathbf{z}}$. Since $(S', \mathcal{A}'_{\mathbf{w}})$ satisfies condition (DI), by definition the map $F_{B^4,(S',\mathcal{A}'_{\mathbf{w}})}$ depends only on the dividing set through the genera of the type-$\mathbf{w}$ and type-$\mathbf{z}$ subregions. Hence, if $\mathcal{A}' \subseteq S'$ is any other dividing set on S' consisting of a single arc, such that the genera of the type-$\mathbf{w}$ and type-$\mathbf{z}$ subregions are the same as those of $(S', \mathcal{A}'_{\mathbf{w}})$, then

$$F_{B^4,(S',\mathcal{A}'_{\mathbf{w}})} \simeq F_{B^4,(S',\mathcal{A}')}.$$

We pick a dividing set $\mathcal{A}' \subseteq S'$ such that one of the trivial stabilizations of S' occurs in the type-$\mathbf{w}$ subregion. See Figure 5.8.

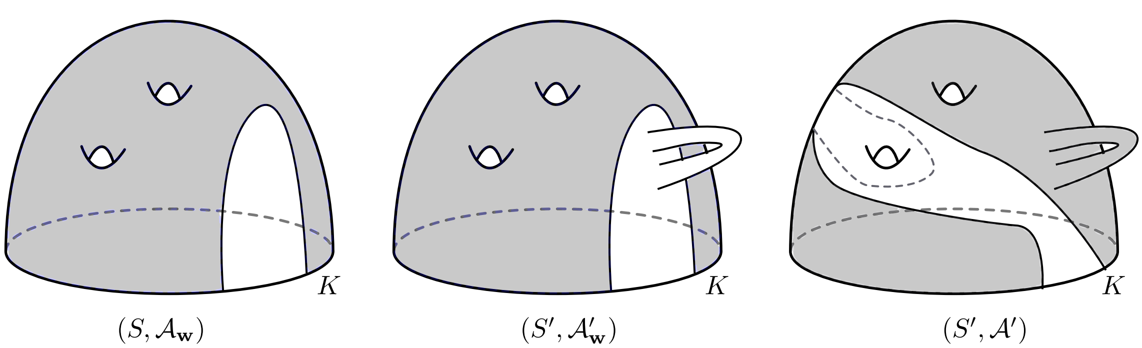

FIGURE 5.8. The surfaces $(S, \mathcal{A}_{\mathbf{w}})$, $(S', \mathcal{A}'_{\mathbf{w}})$, and $(S', \mathcal{A}')$ from the proof of Theorem 5.11.

Using the stabilization formula, we conclude that there is a decorated surface $\mathcal{F}$ such that

$$V^{d-g(S)} \cdot F_{B^4,(S,\mathcal{A}_{\mathbf{w}})} \simeq F_{B^4,(S',\mathcal{A}')} \simeq U \cdot F_{B^4,\mathcal{F}},$$

from which we conclude that $V^{d-g(S)} \cdot t^-_{S,\mathbf{w}} \simeq 0$ since the action of U is trivial on $\mathit{CFK}_{U=0}$. □

Remark 5.12. If S is a genus $g>0$ stabilization of a surface, then $\kappa_0(S)=g(S)$, since $\mathbf{t}^-_{S,\mathbf{w}} \simeq U^g \cdot G$ for some map G, so $[t^-_{S,\mathbf{w}}(1)]=0$ in $\mathit{HFK}^-_{U=0}(\mathbb{K})$. Also, we note that if S satisfies the decoration independence condition (DI) at degree $d>g(S)$, then the map $V^{d-g(S)}\cdot \mathbf{t}^-_{S,\mathbf{w}}$ vanishes on $\mathit{HFK}^-_{U=0}$. This follows by adapting the argument from the proof of Theorem 5.11. Hence

$$\kappa_0(S) \le \max\{g(S)+1, \mathcal{I}(S)\}.$$

5.4. **Stabilization distance bounds from τ and V_k.**

Theorem 5.13. *Let K be a knot in $\mathbb{S}^3$, and let S, $S' \in \mathrm{Surf}(K)$. Then*

$$\tau(S,S') \le \mu_{\mathrm{st}}(S,S').$$

Proof. Let us write $m=\mu_{\mathrm{st}}(S,S')$. Suppose that $S_1,\dots,S_k$ is a stabilization sequence of surfaces in B^4 connecting S and S', as in Definition 2.14, such that

$$\max\{\, g(S_1),\dots,g(S_k)\,\} = m.$$

Let $\mathbb{K}$ denote K decorated with two basepoints. By Lemma 5.4, if S_{i+1} is obtained from S_i by a stabilization, then the map $\mathbf{t}^-_{S_{i+1},\mathbf{z}}$ is filtered chain homotopic to $V^{g(S_{i+1})-g(S_i)}\cdot \mathbf{t}^-_{S_i,\mathbf{z}}$. Similarly, if S_{i+1} is obtained from S_i by a destabilization, then the map $\mathbf{t}^-_{S_{i+1},\mathbf{z}}$ is filtered chain homotopic to $V^{g(S_i)-g(S_{i+1})}\cdot \mathbf{t}^-_{S_{i+1},\mathbf{z}}$. It follows that all of the maps $V^{m-g(S_i)}\cdot \mathbf{t}^-_{S_i,\mathbf{z}}$ coincide for $i\in\{1,\dots,n\}$. In particular,

$$V^{m-g(S)}\cdot \mathbf{t}^-_{S,\mathbf{z}} \simeq V^{m-g(S')}\cdot \mathbf{t}^-_{S',\mathbf{z}}. \tag{5.13}$$

The map $\mathbf{t}^-_{S_i,\mathbf{z}}$ on $\mathcal{CFL}^-$ increases the Alexander grading by $g(S_i)$, so $V^{-g(S_i)}\cdot \mathbf{t}^-_{S_i,\mathbf{z}}$ determines a well-defined map from $\mathbb{F}_2[\widehat{U}]$ into $C(\mathbb{K},R_{0,-g(S_i)}) \subseteq \mathit{CFK}^\infty(\mathbb{K})$. Hence, from equation (5.13), we conclude that the induced elements $[V^{-g(S)}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)]$ and $[V^{-g(S')}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)]$ coincide in $H_*(C(\mathbb{K},{}_mI))$. By Lemma 4.8, this implies that $\tau(S,S')\le m$, completing the proof. □

A different algebraic perspective on the previous proof can be given using the formulation of $\tau(S,S')$ in terms of $\mathit{HFK}^-_{U=0}(\mathbb{K})$ described in Lemma 4.6, and the computation of the effect of stabilizations from Lemma 5.4.

We now show that the local h-invariants give a lower bound on the stabilization distance between two slice disks:

Theorem 5.14. *If D and D' are slice disks of K and $k \le \mu_{\mathrm{st}}(D,D')$, then*

$$V_k(D,D') \le \left\lceil \frac{\mu_{\mathrm{st}}(D,D')-k}{2} \right\rceil.$$

If $k \ge \mu_{\mathrm{st}}(D,D')$, then $V_k(D,D')=0$.

Proof of Theorem 5.14. Suppose first that $k \le \mu_{\mathrm{st}}(D,D')$, and that $S_1,\dots,S_n$ is a sequence of embedded surfaces in $\mathrm{Surf}(K)$ such that S_{i+1} is either obtained from S_i by a stabilization or destabilization. Further, we assume that $S_1=D$ and $S_n=D'$. Let d denote $\max\{g(S_1),\dots,g(S_n)\}$. Since S_n is a slice disk, $g_{\mathrm{dest}}(S_i)\le d$ for $i\in\{1,\dots,n\}$. Furthermore, $\mathcal{I}(S_i)\le g_{\mathrm{dest}}(S_i)$ by Theorem 5.10, hence S_i satisfies the decoration-independence condition (DI) above degree d.

Next, we fix an integer k such that $0\le k\le d$. By increasing d by 1, if necessary, we may assume that $(d-k)/2$ is an integer. Let $\mathbb{K}$ denote K decorated with two basepoints. We decorate each surface S_i with a single dividing arc $\mathcal{A}_i$, and we pick nonnegative integers n_i and m_i such that $g(S_i)+n_i+m_i=d$ and

$$g(S_{i,\mathbf{w}})+n_i=\frac{d-k}{2} \text{ and } g(S_{i,\mathbf{z}})+m_i=\frac{d+k}{2}.$$

We note that, since each S_i satisfies condition (DI) above degree d, it follows that the map $U^{n_i}V^{m_i}\cdot F_{B^4,(S_i,\mathcal{A}_i),\mathfrak{s}_0}$ depends on the dividing set $\mathcal{A}_i$ only up to the quantities $n_i+g(S_{i,\mathbf{w}})$ and $m_i+g(S_{i,\mathbf{z}})$, and is independent of the choice of $\mathcal{A}_i$. Using the stabilization formula, Lemma 5.4, it thus follows that all of the maps $U^{n_i}V^{m_i} \cdot F_{W,(S_i,\mathcal{A}_i),\mathfrak{s}_0}$ are chain homotopic. In particular,

$$U^{(d-k)/2}V^{(d+k)/2} \cdot \mathbf{t}^\infty_D \simeq U^{(d-k)/2}V^{(d+k)/2} \cdot \mathbf{t}^\infty_{D'}. \tag{5.14}$$

Note that $U^{(d-k)/2}V^{(d+k)/2} \cdot \mathbf{t}^\infty_{D_i}(1)$ is not an element of $CFK^\infty(\mathbb{K})$, since it lives in Alexander grading k. In fact, $U^{(d-k)/2}V^{(d+k)/2} \cdot \mathbf{t}^\infty_{D_i}(1)$ is an element of the subcomplex of $\mathcal{CFL}^-(\mathbb{K})$ of Alexander grading k. Multiplication by V^{-k} gives a chain isomorphism between the subset of $\mathcal{CFL}^-(\mathbb{K})$ in Alexander grading k and the subcomplex of $\mathcal{CFL}^\infty(\mathbb{K})$ generated over $\mathbb{F}_2$ by elements $U^iV^j \cdot \mathbf{x}$ with $A(\mathbf{x}) + (i-j) = 0$, $i \geq 0$, and $j \geq -k$. The latter is $A^-_k(\mathbb{K})$, by definition. Hence, from equation (5.14), it follows that

$$\widehat{U}^{(d-k)/2} \cdot [\mathbf{t}^\infty_D(1)] = \widehat{U}^{(d-k)/2} \cdot [\mathbf{t}^\infty_{D'}(1)] \in H_*(A^-_k(\mathbb{K})),$$

where $\widehat{U} = UV$. It follows that

$$V_k(D, D') \leq \frac{d-k}{2},$$

completing the proof when $k \leq \mu_{\text{st}}(D, D')$.

The statement for $k \geq \mu_{\text{st}}(D, D')$ follows from the statement for $k = \mu_{\text{st}}(D, D')$, together with the monotonicity result from Lemma 4.13. □

6. Regular homotopies and the double point distance

6.1. **The double point distance.** If K is a knot in $\mathbb{S}^3$, we denote by $\text{Imm}(K)$ the set of *immersed* connected surfaces in B^4 with boundary K. Furthermore, for $g \in \mathbb{N}$, we write $\text{Imm}_g(K)$ for the subset of $\text{Imm}(K)$ consisting of genus g surfaces. If S, $S' \in \text{Imm}(K)$, then a *regular homotopy* from S to S' is a 1-parameter family $\{\, S_t : t \in I \,\}$ in $\text{Imm}(K)$ that is continuous in the C^∞-topology, and such that $S_0 = S$ and $S_1 = S'$. If S and S' are regularly homotopic, then $g(S) = g(S')$. For a *generic* regular homotopy, at all but finitely many t, the surface S_t is embedded away from finitely many transverse double points. At finitely many t, the immersion S_t has a single non-transverse double point, where a pair of double points of opposite signs is created or canceled. In particular, the algebraic number of double points is constant along a generic regular homotopy.

Lemma 6.1. *Let K be a knot in $\mathbb{S}^3$, and let $g \in \mathbb{N}$. Then any two surfaces S, $S' \in \text{Surf}_g(K)$ are regularly homotopic relative to K.*

Proof. By extending the proof of Hirsch [15, Theorem 8.2] using the relative version of his h-principle [15, Theorem 5.9], we obtain that the regular homotopy class of S relative to K is determined by the relative normal Euler class of S. Since S is embedded and $[S] = 0$ in $H_2(B^4, \partial B^4)$, there is a 3-manifold-with-boundary M embedded in B^4 such that $S \subseteq \partial M$ and $\partial M \setminus S \subseteq \mathbb{S}^3$ is a Seifert surface of K. In particular, M induces a normal framing of S that restricts to the Seifert framing along K. Hence, the normal Euler class of S relative to the Seifert framing vanishes. Since the same holds for S', we obtain that S and S' are regularly homotopic relative to K. □

The regular homotopy class of a generic immersed surface $S \in \text{Imm}(K)$ is determined by the algebraic number of its double points. If $\{\, S_t : t \in I \,\}$ is a regular homotopy such that S_0 is embedded, then the algebraic number of double points of S_t is zero for every $t \in I$ where S_t is generic.

Definition 6.2. Given an immersed surface $S \in \text{Imm}(K)$, let $\text{Sing}(S)$ be the set of its double points (this might be infinite when S is not generic). If S, $S' \in \text{Surf}_g(K)$, then we define

$$\widetilde{\mu}_{\text{Sing}}(S, S') := \frac{1}{2} \min_{\{\, S_t : t \in I \,\}} \max\{\, |\text{Sing}(S_t)| : t \in I \,\},$$

where the minimum is taken over all generic regular homotopies $\{\, S_t : t \in I \,\}$ such that $S_0 = S$ and $S_1 = S'$. Furthermore, we set

$$\mu_{\text{Sing}}(S, S') = \widetilde{\mu}_{\text{Sing}}(S, S') + g.$$

When $S,\, S' \in \operatorname{Surf}(K)$ and $g(S) \neq g(S')$, we set $\mu_{\mathrm{Sing}}(S,S') = \infty$. We call $\mu_{\mathrm{Sing}}(S,S')$ the *double point distance* between S and S'.

Since $\widetilde{\mu}_{\mathrm{Sing}}(S,S') = 0$ if and only if S and S' are isotopic, the function $\widetilde{\mu}_{\mathrm{Sing}}$ is an ultrametric on $\operatorname{Surf}_g(K)$ for every $g \in \mathbb{N}$. Furthermore, μ_{Sing} is a metric filtration whose normalization is $\widetilde{\mu}_{\mathrm{Sing}}$. The goal of this section is to prove that, if $S,\, S' \in \operatorname{Surf}_g(K)$, then

$$\tau(S,S') \leq \mu_{\mathrm{Sing}}(S,S'). \tag{6.1}$$

If $g > 0$, we will also show that

$$\kappa(S,S') \leq \mu_{\mathrm{Sing}}(S,S'). \tag{6.2}$$

Equations (6.1) and (6.2) are proven in Theorems 6.7 and 6.9. Finally, in Theorem 6.14, we will show that the local h-invariants also give lower bounds on $\mu_{\mathrm{Sing}}(S,S')$.

6.2. **Movies of immersions and regular homotopies.** Suppose that $S \in \operatorname{Imm}(K)$ is the image of a proper immersion

$$f\colon \bar{S} \to B^4.$$

Let $B' \subseteq \operatorname{int}(B^4)$ be a ball disjoint from S. After a suitable identification between $B^4 \setminus \operatorname{int}(B')$ and $I \times S^3$, we can view S as an immersed surface in $I \times \mathbb{S}^3$, satisfying $S \cap (\{0\} \times \mathbb{S}^3) = \emptyset$ and $S \cap (\{1\} \times \mathbb{S}^3) = K$. We can visualize S by considering the movie $\{\, S^s : s \in I \,\}$, where S^s is obtained by projecting $(\{s\} \times \mathbb{S}^3) \cap S$ into $\mathbb{S}^3$. We orient S^s as the boundary of $([0,s] \times \mathbb{S}^3) \cap S$ using the outward-normal-first convention.

If S is generic, then $\pi_I \circ f$ is a Morse function on $\bar{S}$ and the double points are on regular level sets, where $\pi_I\colon I \times \mathbb{S}^3 \to I$ is the projection onto the I-factor. Hence S^s is an immersed link whenever s is a regular value. If s is a critical value of index zero, then an unknotted component is born. If s has index one, the link undergoes a saddle move, and if it has index two, an unknotted component dies. Generically, passing a double point of S locally corresponds to a crossing change of S^s; see Gompf–Stipsicz [11, Figure 6.25]. We now explain why this is true, and how to read off the intersection sign.

Lemma 6.3. *Generically, as we pass a positive (negative) double point p of S, a negative (positive) crossing of S^s changes to a positive (negative) crossing; see Figure 6.1.*

Proof. Suppose that S is the image of an immersion $f\colon \bar{S} \looparrowright B^4$. The set of points $x \in \bar{S}$ such that S is not transverse to the sets $\{s\} \times \mathbb{S}^3$ at $f(x)$ is generically 0-dimensional, and hence disjoint from the two preimages of p. Hence, generically, passing the double point p corresponds to two strands of S^s passing through each other.

Write the double point $p \in S \subseteq I \times \mathbb{S}^3$ as $p = (s_0, p_0)$, where $s_0 \in I$ and $p_0 \in \mathbb{S}^3$. Suppose that, at $s = s_0$, a negative crossing of S^s turns into a positive crossing. Let $v_+,\, v_- \in T_{p_0}\mathbb{S}^3$ denote oriented tangent vectors for the upper and lower strands of the crossing, respectively. Let $\gamma\colon (s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{S}^3$ denote the trajectory of a point on the upper strand, chosen to pass through a point on the lower strand at s_0. By inspection of the crossing change, the triple $(v_+, v_-, \gamma'(s_0))$ is a positive basis of $T_{p_0}\mathbb{S}^3$. Using the product orientation on $I \times \mathbb{S}^3$, the 4-tuple $(\partial/\partial s, v_+, v_-, \gamma'(s_0))$ is an oriented basis for $I \times \mathbb{S}^3$. It is easy to see that oriented bases for the tangent spaces of the two sheets of S at p are

$$(\partial/\partial s + \gamma'(s_0), v_+) \quad \text{and} \quad (\partial/\partial s - \gamma'(s_0), v_-),$$

respectively, which concatenate to form a positive basis of $I \times \mathbb{S}^3$.

A similar argument applies when a positive crossing turns into a negative one at p. ☐

Now suppose that $\{\, S_t : t \in [-1,1] \,\}$ is a generic regular homotopy in $\operatorname{Imm}(K)$, and that a pair of double points p_+ and p_- appear as t passes $0 \in [-1,1]$. The immersed surface S_0 has a non-transverse double point $p \in B^4$. Write $p = (s_0, p_0)$, where $s_0 \in I$ and $p_0 \in \mathbb{S}^3$.

A local model for a double point creation can be visualized via a 2-parameter family

$$\{\, S_t^s : (s,t) \in [s_0 - \epsilon, s_0 + \epsilon] \times [-\epsilon, \epsilon] \,\}$$

of immersed links in $\mathbb{S}^3$ that is constant outside a neighborhood $N(p_0)$ containing the crossing. The families $\{ S_t^{s_0-\epsilon} : t \in [-\epsilon, \epsilon] \}$ and $\{ S_t^{s_0+\epsilon} : t \in [-\epsilon, \epsilon] \}$ are constant and have a positive crossing in $N(p_0)$; we denote this link by L_+. For $t < 0$, the intersection $S_t^s \cap N(p_0)$ is a positive crossing and the family of links S_t^s is embedded (and hence isotopic to L_+) for all $s \in [s_0 - \epsilon, s_0 + \epsilon]$. For $t > 0$, the positive crossing $L_+ \cap N(p_0)$ changes to a negative crossing, and then back to a positive crossing. Let L_- be the link obtained by changing $L_+ \cap N(p_0)$ to a negative crossing. If we fix t, self-intersections in the 1-parameter family $\{ S_t^s \colon s \in [s_0 - \epsilon, s_0 + \epsilon] \}$ correspond to double points of the surface the family traces out in $[s_0 - \epsilon, s_0 + \epsilon] \times \mathbb{S}^3$. The movie $\{ S_t^s : s \in [s_0 - \epsilon, s_0 + \epsilon] \}$ for $t > 0$ is shown in the top of Figure 6.1. We prove the above in the following lemma.

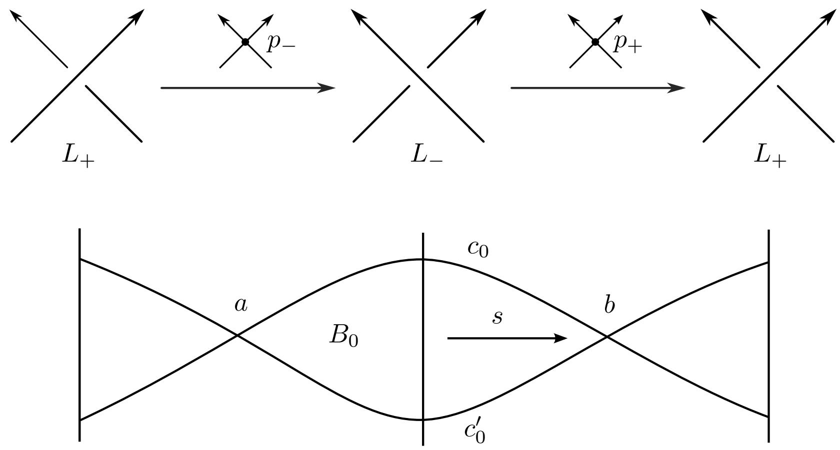

FIGURE 6.1. The top row shows a movie of a pair of double points after they have been born during a regular homotopy of an immersed surface. The bottom row shows the standard model of a Whitney disk used in the proof, which gives a canonical neighborhood of the pair of canceling double points.

Lemma 6.4. *Let $\{ S_t : t \in [-1, 1] \}$ be a generic regular homotopy of immersed surfaces such that a pair of double points is born at time 0 at a point $p \in B^4$. Furthermore, let $B' \subseteq \text{int}(B^4)$ be a ball disjoint from S_t for every $t \in [-1, 1]$. Then there is an identification of $B^4 \setminus \text{int}(B')$ with $I \times \mathbb{S}^3$ and an $\epsilon > 0$ such that, if p corresponds to (s_0, p_0), the 1-parameter family of immersed links*

$$\{ S_t^s : s \in [s_0 - \epsilon, s_0 + \epsilon] \}$$

is diffeomorphic to the constant 1-parameter family $L_+ := S_{-\epsilon}^{s_0-\epsilon}$ for $t = -\epsilon$, where L_+ has a positive crossing in $N(p_0)$, and to the 1-parameter family shown on the top row of Figure 6.1 for $t = \epsilon$, where the positive crossing of L_+ in $N(p_0)$ changes to negative, and then back to positive.

Proof. Choose an identification between $B^4 \setminus \text{int}(B')$ and $I \times S^3$ such that $s(p_-) < s(p_+)$, where s is the I-coordinate. We write S_t as the image of a 1-parameter family of immersions $f_t \colon \bar{S} \looparrowright I \times \mathbb{S}^3$. Up to isotopy, we can express any regular homotopy of a surface in a 4-manifold as a composition of finger moves and Whitney moves; see Gabai [9, Proposition 4.3] and Freedman–Quinn [8, Section 1.5]. In particular, p_+ and p_- admit a Whitney disk B. A neighborhood of the Whitney disk can be put in the standard form of Milnor [32, Lemma 6.7]; see the bottom row of Figure 6.1. This is given by an embedding $\varphi \colon U \times \mathbb{R} \times \mathbb{R} \to B^4$, where U is a neighborhood of the disk B_0 in $\mathbb{R}^2$ enclosed by arcs c_0 and c_0' that transversely intersect at points a and b. We have $\varphi(B_0) = B$, $\varphi(a) = p_+$ and $\varphi(b) = p_-$, and let us write $c = \varphi(c_0)$ and $c' = \varphi(c_0')$. The preimages of the two branches of S_1 meeting at p_+ and p_- are $(U \cap c_0) \times \mathbb{R} \times \{0\}$ and $(U \cap c_0') \times \{0\} \times \mathbb{R}$, respectively. The isotopy S_t at the finger move is modeled on the isotopy of c_0' shown in [32, Figure 6.3] that creates the intersection points a and b with c_0. This isotopy is constant in the normal $\mathbb{R} \times \mathbb{R}$ direction.

The immersed surface S_0 has a non-transverse double point. If x, $y \in \bar{S}$ are the two preimages of the double point, let v denote a generator of the 1-dimensional vector space $(f_0)_*(T_x\bar{S}) \cap (f_0)_*(T_y\bar{S})$.

For $t > 0$, the movie S_t^s has an extra pair of double points. By the choice of the coordinate function s, we have $ds(v) \neq 0$ and $s(p_-) < s(p_+)$. Both p_+ and p_- correspond to a crossing change in the movie $\{ S_t^s : s \in [s_0 - \epsilon, s_0 + \epsilon] \}$ for $t > 0$ by Lemma 6.3. By arranging for the Whitney disk to be symmetric about $s = s_0$ in a small neighborhood of s_0, the movie for the second double point is obtained by reversing the movie for the first double point. When $t < 0$, the curves c_0 and c_0' become disjoint, and so the movie $\{ S_t^s : s \in [s_0 - \epsilon, s_0 + \epsilon] \}$ is just an isotopy of the link L_+, completing the proof. □

6.3. **The desingularization of an immersed surface.**

Definition 6.5. Suppose $S \in \operatorname{Imm}(K)$ is a generic immersed surface in B^4; i.e., an immersion with only transverse double points. The *desingularization* of (B^4, S) is the link cobordism $(B^4(S), \widehat{S})$ obtained as follows:

1. The 4-manifold $B^4(S)$ is constructed by blowing up the 4-manifold at each *negative* double point of S. Topologically, this corresponds to connected summing with $\overline{\mathbb{CP}}^2$.
2. The surface $\widehat{S}$ is constructed from the proper transform of S in $B^4(S)$ by resolving each *positive* double point (increasing the genus of S by 1 at each point).

Definition 6.5 makes sense for any immersed oriented link cobordism as well. For a movie presentation of the resolution of a positive double point, see Figure 6.2, taken from the book of Gompf and Stipsicz [11, Figure 6.30]. For a movie of the blowup of a negative double point, see Figure 6.3. The 4-dimensional 2-handle of $\overline{\mathbb{CP}}^2$ is attached along a (-1)-framed unknot that links the negative crossing of L_+.

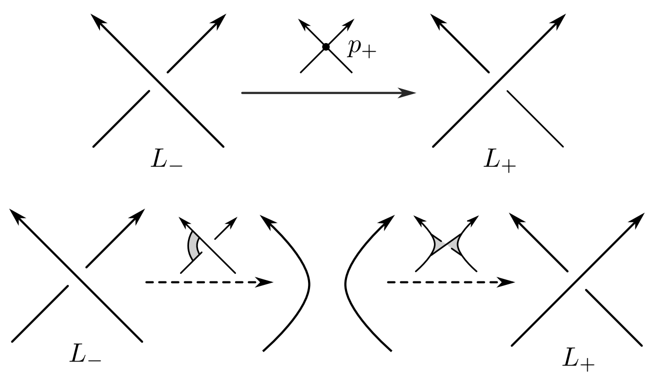

FIGURE 6.2. Resolving a positive double point. The top row is the singular knot cobordism. The bottom is our choice of resolution.

Let $(W, \mathcal{F})$ with $\mathcal{F} = (S, \mathcal{A})$ be an immersed, decorated link cobordism, such that the double points are disjoint from $\mathcal{A}$. Furthermore, suppose that the two branches meeting at a double point either both lie in $S_{\mathbf{w}}$, or they both lie in $S_{\mathbf{z}}$. We write $(W(S), \widehat{\mathcal{F}})$ for the decorated link cobordism with double points resolved as described above.

Lemma 6.6. *Suppose $(W_0, \mathcal{F}_0)$ is a non-singular link cobordism, and $(W, \mathcal{F})$ with $\mathcal{F} = (S, \mathcal{A})$ is obtained from $(W_0, \mathcal{F}_0)$ by a double point birth, corresponding to a tangency between either two branches of $S_{\mathbf{w}}$, or two branches of $S_{\mathbf{z}}$. Let $(W(S), \widehat{\mathcal{F}})$ denote the resolved link cobordism, as described above. Then $W(S) = W_0 \# \overline{\mathbb{CP}}^2$, and $\widehat{\mathcal{F}}$ is obtained from $\mathcal{F}_0$ by a 1-handle stabilization along $S_{\mathbf{w}}$ or $S_{\mathbf{z}}$, and disjoint from $\overline{\mathbb{CP}}^2$.*

Let $\widehat{\mathfrak{s}}$ be a Spin^c *structure on $\widehat{W}$ such that $\langle c_1(\widehat{\mathfrak{s}}), E \rangle = \pm 1$, where E denotes the exceptional divisor in $\widehat{W}$, and agrees with $\mathfrak{s}$ on W. Then*

$$F_{\widehat{W}, \widehat{\mathcal{F}}, \widehat{\mathfrak{s}}} \simeq U \cdot F_{W, \mathcal{F}, \mathfrak{s}}$$

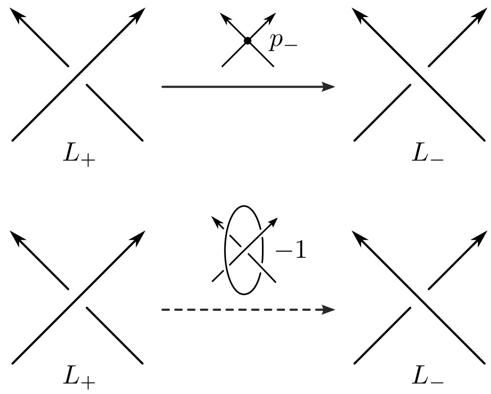

FIGURE 6.3. Resolving a negative double point. The top is the singular knot cobordism. The bottom is the resolution, obtained by blowing up the surface at the double point.

if the double points both occur in $S_{\mathbf{w}}$*, and*

$$F_{\widehat{W},\widehat{\mathcal{F}},\widehat{\mathfrak{s}}} \simeq V \cdot F_{W,\mathcal{F},\mathfrak{s}},$$

if the double points both occur in $S_{\mathbf{z}}$*.*

Proof. By Lemma 6.4, there is a movie presentation of $\mathcal{F}$ as in the top of Figure 6.1, where a positive crossing changes to negative, and then back to positive. Furthermore, the movie of the decorated surface $\mathcal{F}_0$ only differs from that of $\mathcal{F}$ by locally changing the above movie to one where the positive crossing stays positive.

We consider the composition of the resolutions of the positive and negative double points shown in Figures 6.2 and 6.3; see the top row of Figure 6.4. We can arrange that the resolution of the negative double point occurs immediately before the resolution of the positive double point. The resolution of the positive double point is a pair of saddles, corresponding to attaching bands B_1 and B_2. The composition of the two resolutions can be rearranged such that we first attach the band B_1, then attach a 4-dimensional 2-handle along the -1 framed unknot $\mathcal{U}$, and finally attach the band B_2; see the second row of Figure 6.4.

We can slide the band B_2 over $\mathcal{U}$, though it gains a full right-handed twist when we do this; see the third row of Figure 6.4. The unknot $\mathcal{U}$ is now totally unlinked from the knot and bands, and B_1 and B_2 are simply dual bands, corresponding to a 1-handle stabilization of the surface $\mathcal{F}_0$ disjoint from the -1 framed unknot giving the $\overline{\mathbb{CP}}^2$ summand of $W(S)$.

Applying Lemma 5.4 for the effect of the 1-handle stabilization, and using the standard blow-up formula for -1 surgery on an unknot contained in a ball disjoint from the link, we see that the composition is multiplication by either U or V, depending on whether the 1-handle is added to $\Sigma_{\mathbf{w}}$ or $\Sigma_{\mathbf{z}}$. Using the composition law, the proof is complete. □

6.4. **Tau, nu, and the double point distance.** We now prove that τ gives a lower bound on μ_{Sing}:

Theorem 6.7. *If* S*,* $S' \in \text{Surf}_g(K)$*, then*

$$\tau(S, S') \le \mu_{\text{Sing}}(S, S').$$

Proof. Suppose that $\{ S_t : t \in I \}$ is a generic regular homotopy between embedded surfaces S, $S' \in \text{Surf}(K)$. The immersion S_t fails to be self-transverse at times $s_1, \dots, s_{n-1} \in (0,1)$. Pick a point $t_i \in (s_{i-1}, s_i)$ for every $i \in \{2, \dots, n-1\}$, and let $S_i = S_{t_i}$. We write $S_1 = S$ and $S_n = S'$. Let $(B^4(S_i), \widehat{\mathcal{F}}_i)$ denote the desingularization of (B^4, S_i), as described in Definition 6.5. Let $\widehat{\mathfrak{s}}_i$ denote any maximal Spin^c structure on $B^4(S_i)$ (i.e., $c_1(\mathfrak{s})^2 + b_2(B^4(S_i)) = 0$), such that $\widehat{\mathfrak{s}}_{i+1}$ is obtained by blowing up or blowing down $\widehat{\mathfrak{s}}_i$.

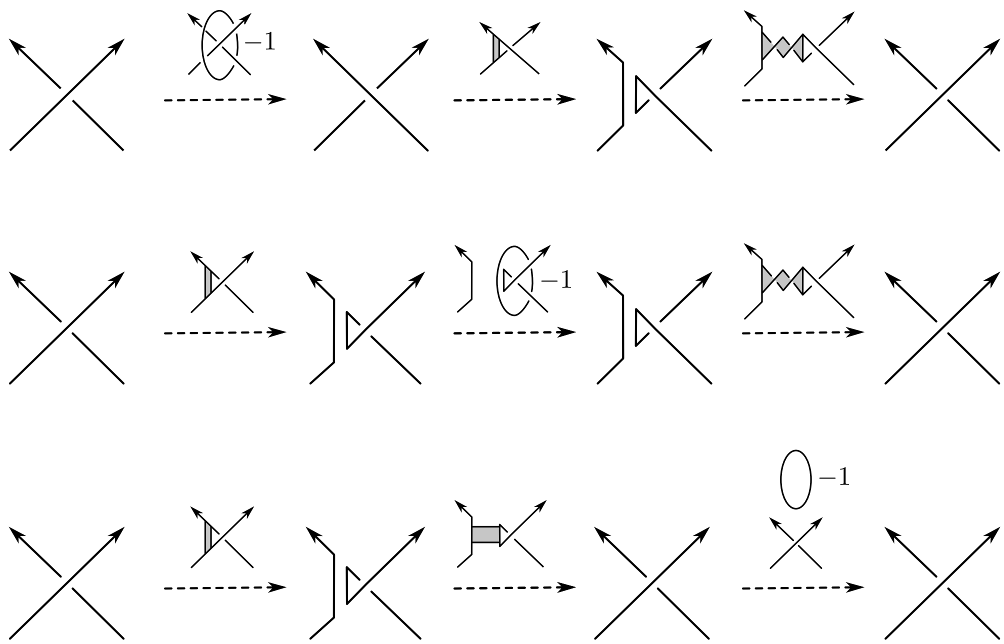

FIGURE 6.4. In the top row, we show the movie of the resolution of a canceling pair of double points, which consists of a blowup followed by two band moves. We then commute the blowup and the first band move, giving rise to the movie in the second row. Finally, we slide the second band over the (-1)-framed 2-handle, giving the third row. The two bands form a tube, and the blowup now happens away from the surface.

We decorate each $\widehat{\mathcal{F}}_i$ such that the type-$\mathbf{w}$ region is a bigon along K, and the rest of $\widehat{\mathcal{F}}_i$ is of type-$\mathbf{z}$. In particular, all double points occur in regions of type-$\mathbf{z}$. If S_i is obtained from S_{i-1} via a double point birth, then Lemma 6.6 implies that

$$F_{B^4(S_{i+1}),\widehat{\mathcal{F}}_{i+1},\widehat{\mathfrak{s}}_{i+1}} \simeq V \cdot F_{B^4(S_i),\widehat{\mathcal{F}}_i,\widehat{\mathfrak{s}}_i}.$$

Similarly, if S_{i+1} is obtained from S_i via a double point cancellation, then

$$V \cdot F_{B^4(S_{i+1}),\widehat{\mathcal{F}}_{i+1},\widehat{\mathfrak{s}}_{i+1}} \simeq F_{B^4(S_i),\widehat{\mathcal{F}}_i,\widehat{\mathfrak{s}}_i}.$$

It follows that

$$V^n \cdot \mathbf{t}^\infty_{S,\mathbf{z}} \simeq V^n \cdot \mathbf{t}^\infty_{S',\mathbf{z}},$$

where n is the maximal number of positive double points of any S_i. Since the algebraic count of double points of each S_i is zero, we have $n = \frac{1}{2}\max\{\,|\operatorname{Sing}(S_t)| : t \in I\,\}$. It now follows from Lemma 4.6 that

$$\tau(S,S') \le \frac{1}{2}\max\{\,|\operatorname{Sing}(S_t)| : t \in I\,\} + g.$$

Hence $\tau(S,S') \le \mu_{\mathrm{Sing}}(S,S')$, as claimed. □

We now show that ν, introduced in Section 4.4, gives a slightly better lower bound on the stabilization distance and the double point distance than τ.

Proposition 6.8. *If S, $S' \in \operatorname{Surf}(K)$, then*

$$\nu(S,S') \le \min\{\mu_{\mathrm{st}}(S,S'), \mu_{\mathrm{Sing}}(S,S')\}. \tag{6.3}$$

Proof. Let $\mathbb{K}$ be K decorated with two basepoints, and write $g = g(S)$ and $g' = g(S')$. By Theorems 5.13 and 6.7, if n is either $\mu_{\mathrm{st}}(S,S')$ or $\mu_{\mathrm{Sing}}(S,S')$, then $\tau(S,S') \le n$ (recall that

$\mu_{\mathrm{Sing}}(S,S')=\infty$ when $g\neq g'$, so the inequality obviously holds in this case). Furthermore, their proofs imply that

$$V^{n-g}\cdot \mathbf{t}^\infty_{S,\mathbf{z}}\simeq V^{n-g'}\cdot \mathbf{t}^\infty_{S',\mathbf{z}}. \tag{6.4}$$

If $\tau(S,S')<n$, then equation (6.3) automatically holds, since $\nu(S,S')\le \tau(S,S')+1$, so it is sufficient to consider the case when $\tau(S,S')=n$. It follows from equation (6.4) that

$$V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)-V^{-g'}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)=\partial x \tag{6.5}$$

for some $x\in C(\mathbb{K},i\ge 0,j\ge -n)$. It follows that the elements $\mathbf{t}^-_{S,\mathbf{z}}(1)$ and $\mathbf{t}^-_{S',\mathbf{z}}(1)$ agree in the homology of any quotient of $C(\mathbb{K},i\ge 0,j\ge -n)$ by a filtered subcomplex. Since $C(\mathbb{K},{}_nL_m)$ is the quotient of $C(\mathbb{K},i\ge 0,j\ge -n)$ by a filtered subcomplex for any $m\in\mathbb{N}$, it follows that

$$[V^{-g}\cdot \mathbf{t}^-_{S,\mathbf{z}}(1)-V^{-g'}\cdot \mathbf{t}^-_{S',\mathbf{z}}(1)]=0\in H_*(C(\mathbb{K},{}_nL_m))$$

for all $m\in\mathbb{N}$. Hence $\tau'(S,S')=-\infty$ and $\nu(S,S')=\tau(S,S')=n$. □

6.5. **Kappa and the double point distance.** In this section, we show that the kappa invariant also gives a lower bound on the double point distance.

Theorem 6.9. *Let K be a knot in $\mathbb{S}^3$. If S, $S'\in \mathrm{Surf}_g(K)$ and $g>0$, then*

$$\kappa(S,S')\le \mu_{\mathrm{Sing}}(S,S').$$

The proof requires several steps. Suppose that $S\in \mathrm{Imm}(K)$ is a generic, properly immersed surface in B^4 (by generic, we mean S has a discrete collection of transverse double points, disjoint from ∂S). The surface S is the image of an immersion $f\colon \bar{S}\looparrowright B^4$, and write $\widehat{S}\subseteq B^4(S)$ for the desingularization of S. Let

$$\mathbf{p}^+\subseteq \bar{S}$$

denote the preimages of the positive double points of S. Note that each positive double point of S contributes two points to $\mathbf{p}^+$. Let P be a subset of the positive double points of S.

Definition 6.10. Suppose that $S\in\mathrm{Imm}(K)$ is a generic immersed surface. Let $T\subseteq \bar{S}$ denote an embedded tree such that the following hold:

- (*T1*) $T\cap \partial\bar{S}$ consists of a single point.
- (*T2*) Each point of $f^{-1}(P)$ is a leaf of T, and T is disjoint from $\mathbf{p}^+\setminus f^{-1}(P)$.

Given such a tree T, we define an induced decoration $\mathcal{A}_{\mathbf{w}}(T)$ on $\bar{S}$, as follows. The underlying dividing set of $\mathcal{A}_{\mathbf{w}}(T)$ is $\partial N(T)\setminus \partial\bar{S}$. We declare $N(T)$ to be the type-$\mathbf{z}$ subregion, and the complement of $N(T)$ to be the type-$\mathbf{w}$ subregion. We note that the decoration $\mathcal{A}_{\mathbf{w}}(T)$ on $\bar{S}$ induces a decoration on the desingularized surface $\widehat{S}\subseteq B^4(S)$, for which we also write $\mathcal{A}_{\mathbf{w}}(T)$. There is an analogous decoration $\mathcal{A}_{\mathbf{z}}(T)$, obtained by reversing the roles of $\mathbf{w}$ and $\mathbf{z}$.

Note that $g(\widehat{S}_{\mathbf{w}})=g(\bar{S})+|\mathbf{p}^+|/2-|P|$ and $g(\widehat{S}_{\mathbf{z}})=|P|$. We now prove the following, somewhat surprising fact:

Proposition 6.11. *Suppose that $S\in\mathrm{Imm}(K)$ is a generic immersed surface in B^4 with boundary K, and that P is a subset of the positive double points of S. Let $T\subseteq\bar{S}$ be a tree satisfying conditions (T1) and (T2). If $\mathfrak{s}\in \mathrm{Spin}^c(B^4(S))$, the chain homotopy type of the map*

$$F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T)),\mathfrak{s}}\colon \mathcal{R}^\infty\to \mathcal{CFL}^\infty(\mathbb{K})$$

is independent of the choice of tree T.

To prove Proposition 6.11, we need a set of moves that can be used to connect two trees satisfying conditions (*T1*) and (*T2*). We introduce the following *tree-moves*:

- (*TM1*) T is replaced by another tree T' satisfying (*T1*) and (*T2*) such that $\partial N(T)$ and $\partial N(T')$ are isotopic through dividing sets which are fixed on $\partial\bar{S}$ and never intersect $\mathbf{p}^+$.

(*TM2*) Suppose that e is an edge of T which contains a point p of $\mathbf{p}^+$, and that e' is an embedded path in $\bar{S}$ such that $e' \cap T = \partial e'$, and such that $e \cap e'$ consists of a single point t. The tree T is replaced with the tree T' formed by adding e', and removing a segment of e that is not between p and t; see the top row of Figure 6.5.

Lemma 6.12. *If $\bar{S}$ is a surface, $\mathbf{p}^+ \subseteq \bar{S}$ is a collection of points, and $f^{-1}(P) \subseteq \mathbf{p}^+$ is a chosen subset, then any two trees satisfying (T1) and (T2) can be connected by moves (TM1) and (TM2).*

Proof. We pick a subset $A \subseteq \bar{S}$ consisting of $g(\bar{S})$ compressing curves on $\bar{S}$ such that we are left with a disk after surgering $\bar{S}$ on A. If T is a tree satisfying (*T1*) and (*T2*), we give T the partial order determined by setting $T \cap \partial S$ to be the maximal point. If $t \in T$, define

$$L(t) = \{x \in T : x \le t\}.$$

As a first step, we show that any tree T satisfying conditions (*T1*) and (*T2*) can be connected by moves (*TM1*) and (*TM2*) to a tree which is disjoint from A. To establish this, we show that if $|A \cap T| > 0$, we can always reduce $|A \cap T|$ by 1, using moves (*TM1*) and (*TM2*). To do this, we pick any point $t \in A \cap T$ such $L(t) \cap A = \emptyset$. There are two cases: either $L(t) \cap \mathbf{p}^+ = \emptyset$ or $L(t) \cap \mathbf{p}^+ \neq \emptyset$.

If $L(t) \cap \mathbf{p}^+ = \emptyset$, then we can just isotope $L(t)$ (an instance of move (*TM1*)) so that it no longer intersects A, thus reducing $|A \cap T|$.

Next, we consider the case that $L(t) \cap \mathbf{p}^+ \neq \emptyset$. In this case, we pick a point $t' \in L(t)$ such that $L(t')$ is a subset of a single edge of T, and $L(t')$ contains a point $p \in \mathbf{p}^+$. We let e' be any embedded path in $S \setminus A$ such that $e' \cap T = \partial e'$, and $\partial e'$ consists of t' and another point of T which is not contained in $L(t)$. Let e denote a subinterval of the edge of T containing t', such that t' is the smaller endpoint of e. We can then use move (*TM2*) to replace e with e'. This reduces $L(t) \cap \mathbf{p}^+$ by 1, and does not increase $|A \cap T|$. Repeating this procedure, we may reduce to the case that $L(t) \cap \mathbf{p}^+ = \emptyset$. Arguing as before, an isotopy of $L(t)$ can then be used to reduce $|A \cap T|$ by 1.

Hence, if T and T' are two trees satisfying conditions (*T1*) and (*T2*), by applying moves (*TM1*) and (*TM2*), we may assume that T and T' are both disjoint from A. We may compress $\bar{S}$ along A to get a disk D, containing T and T', as well as a collection of $2g(\bar{S})$ points $\mathbf{p}$ corresponding to the curves in A. We note that T and T' are disjoint from $\mathbf{p}$. Furthermore, isotoping an edge of T or T' across a point in $\mathbf{p}$ may be achieved by move (*TM2*). In this manner, we can reduce the claim to the case that S is a disk, and it is straightforward to see that in this situation that T and T' can be related by applying move (*TM1*). □

Proof of Proposition 6.11. By Lemma 6.12, it is sufficient to show invariance of $F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T)),\mathfrak{s}}$ under moves (*TM1*) and (*TM2*). First note that, up to isotopy, the decoration $\mathcal{A}_{\mathbf{w}}(T)$ depends only on a regular neighborhood of $T \subseteq \bar{S}$, so move (*TM1*) does not change the cobordism map $F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T)),\mathfrak{s}}$.

We now address move (*TM2*). Suppose that e is an edge of T which has exactly one endpoint at a point $p \in \mathbf{p}^+$ and another at a vertex $v \in T \setminus \mathbf{p}^+$. Suppose that e' is an embedded path on $\bar{S}$ such that $e' \cap T = \partial e'$ and $e \cap e' = \{t\}$. Let T' denote the tree obtained by removing a segment of e not between t and p, and inserting e'. There is a bypass relation

$$F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T)),\mathfrak{s}} + F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T')),\mathfrak{s}} + F_{B^4(S),(\widehat{S},\mathcal{A}''),\mathfrak{s}} \simeq 0 \tag{6.6}$$

for a third decoration $\mathcal{A}'' \subseteq \widehat{S}$, which is shown in Figure 6.5.

We now claim that

$$F_{B^4(S),(\widehat{S},\mathcal{A}''),\mathfrak{s}} \simeq 0 \tag{6.7}$$

for any $\mathfrak{s} \in \mathrm{Spin}^c(B^4(S))$, which will complete the proof when combined with equation (6.6). The key observation is that the dividing set $\mathcal{A}''$ has a closed loop, which is homologically essential in the tube that is added to form the desingularized surface $\widehat{S}$. Let B be a 4-ball in $B^4(S)$ containing the point $f(p)$. We note that $\widehat{S}$ intersects ∂B in a negative Hopf link H. Let A_0 denote the annulus that is inserted into B to form the desingularized surface $\widehat{S}$. By isotoping $\mathcal{A}''$ in $\widehat{S}$, we may assume that $\mathcal{A}''$ intersects A_0 in the dividing set $\mathcal{A}_0''$ shown in Figure 6.6.

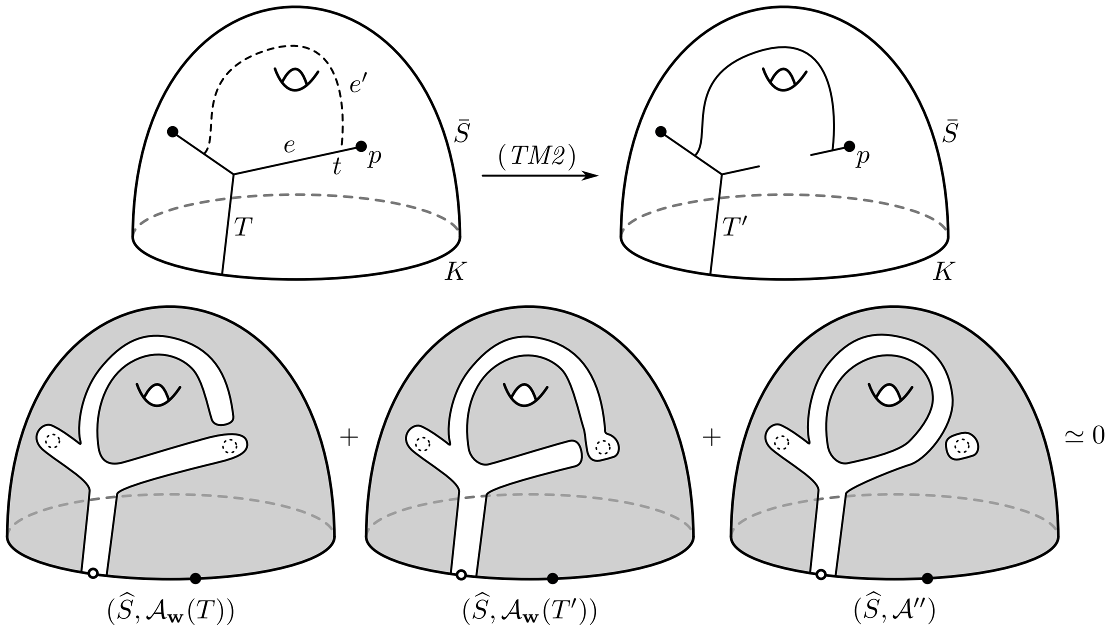

FIGURE 6.5. The graphs T and T' on the top row are related by move ($TM2$). The point p is in $\mathbf{p}^+$. On the bottom row, the associated decorations $\mathcal{A}_{\mathbf{w}}(T)$ and $\mathcal{A}_{\mathbf{w}}(T')$ on the desingularized surface $\widehat{S}$ are shown, as well as a third decoration $\mathcal{A}''$, which fits into a bypass triple with $\mathcal{A}_{\mathbf{w}}(T)$ and $\mathcal{A}_{\mathbf{w}}(T')$. The dashed circles denote where a tube is added on the desingularized surface.

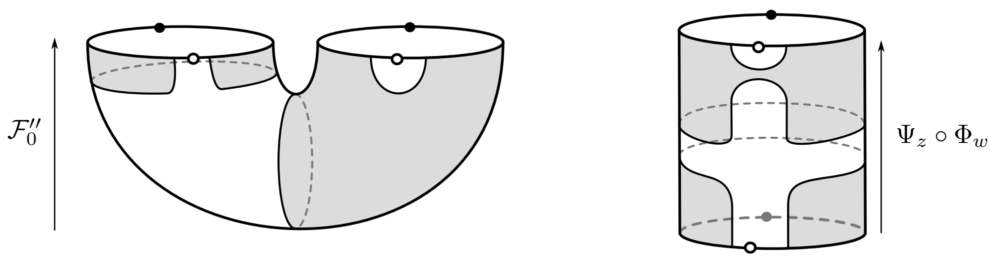

FIGURE 6.6. On the left is the annulus $A_0 \subseteq \widehat{S}$, which is added to desingularize the immersed surface S at the negative double point $f(p)$. The decoration $\mathcal{A}_0''$ on A_0 is shown. On the right is the dividing set corresponding to the map $\Phi_w \circ \Psi_z$, on a cylindrical link cobordism.

Let us write $\mathcal{F}''$ for $(\widehat{S}, \mathcal{A}'')$ and $\mathcal{F}_0'' = (A_0, \mathcal{A}_0'')$. Using the composition law, it is sufficient to show that

$$F_{B,\mathcal{F}_0'',\mathfrak{s}_0} \simeq 0,$$

where $\mathfrak{s}_0$ is the unique torsion Spin^c structure on B.

Let $\mathbb{H}^+$ denote the positive Hopf link, decorated with basepoints w_1 and z_1 on one component, and w_2 and z_2 on the other. By isotoping the dividing set on $\mathcal{F}_0''$, we may factor the map $F_{B,\mathcal{F}_0'',\mathfrak{s}_0}$ through

$$\Phi_{w_1}\Psi_{z_1} \colon \mathcal{CFL}^-(\mathbb{H}^+) \to \mathcal{CFL}^-(\mathbb{H}^+).$$

A diagram for $\mathbb{H}^+$ is shown in Figure 6.7. The differential can be computed by simply counting bigons. It is not hard to see that the complex $\mathcal{CFL}^-(\mathbb{H}^+)$ is chain homotopy equivalent to the complex shown in Figure 6.8. The chain homotopy type of the maps Φ_{w_1} and Ψ_{z_1} are also shown

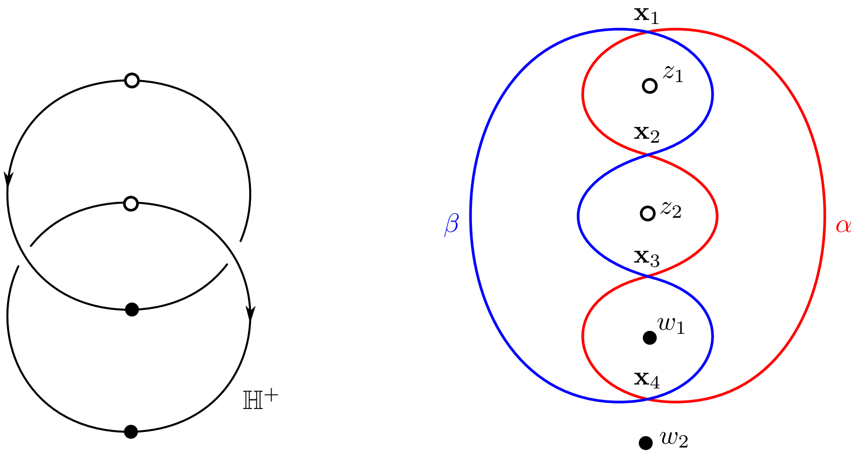

FIGURE 6.7. The positive Hopf link (left), and a genus zero Heegaard diagram (right).

$$\mathcal{CFL}^-(\mathbb{H}^+)=\begin{pmatrix} \mathbf{x}_1 & \xrightarrow{V} & \mathbf{x}_2\\ \big\downarrow U & & V\big\uparrow\\ \mathbf{x}_4 & \xleftarrow[U]{} & \mathbf{x}_3\end{pmatrix},\quad \Phi_{w_1}=\begin{pmatrix} \mathbf{x}_1 & & \mathbf{x}_2\\ & & \\ \mathbf{x}_4 & \longleftarrow & \mathbf{x}_3\end{pmatrix},\quad \Psi_{z_1}=\begin{pmatrix} \mathbf{x}_1 & \longrightarrow & \mathbf{x}_2\\ & & \\ \mathbf{x}_4 & & \mathbf{x}_3\end{pmatrix}$$

FIGURE 6.8. The chain complex $\mathcal{CFL}^-(\mathbb{H}^+)$, as well as the maps Φ_{w_1} and Ψ_{z_1}.

in Figure 6.8. Examining the maps Φ_{w_1} and Ψ_{z_1} shown in Figure 6.8, we see that $\Phi_{w_1}\Psi_{z_1}$ vanishes, completing the proof. $\square$

Definition 6.13. Given a generic immersed surface $S \in \operatorname{Imm}(K)$ that is the image of an immersion $f\colon \bar{S} \looparrowright B^4$, as well as a subset P of the positive double points of S, pick a tree $T \subseteq \bar{S}$ satisfying conditions (*T1*) and (*T2*). We form the decoration $\mathcal{A}_{\mathbf{w}}(T)$ of the desingularization $(B^4(S), \widehat{S})$ as in Definition 6.10. For $\mathfrak{s} \in \operatorname{Spin}^c(B^4(S))$, we define the map

$$\mathbf{t}^-_{S,P,\mathbf{w},\mathfrak{s}}\colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K})$$

to be the decorated link cobordism map $F_{B^4(S),(\widehat{S},\mathcal{A}_{\mathbf{w}}(T)),\mathfrak{s}}$, which is independent of the choice of T by Proposition 6.11. Analogously, we can define the map

$$\mathbf{t}^-_{S,P,\mathbf{z},\mathfrak{s}}\colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K}).$$

We can now prove that $\kappa(S,S')$ is a lower bound for $\mu_{\operatorname{Sing}}(S,S')$:

Proof of Theorem 6.9. Suppose that $S_1,\dots,S_n$ is a sequence of immersed surfaces, each of which is obtained from the previous via creating or canceling a pair of double points, up to diffeomorphism fixing ∂B^4 pointwise, and $S_1 = S$ and $S_n = S'$. Furthermore, let P_k be the set of all positive double points of S_k for $k \in \{1,\dots,n\}$.

Using Lemma 6.6, we conclude that, if

$$m := \frac{1}{2}\max\{|\operatorname{Sing}(S_1)|,\dots,|\operatorname{Sing}(S_n)|\},$$

and $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ is a stabilization sequence of Spin^c structures on $B^4(S_1),\dots,B^4(S_n)$ that are all maximal, then the filtered chain homotopy types of the maps

$$V^{m-\frac{1}{2}|\operatorname{Sing}(S_i)|}\cdot \mathbf{t}^-_{S_k,P_k,\mathbf{w},\mathfrak{s}}$$

coincide for $k \in \{1, \dots, n\}$. In particular, the maps on $\mathit{CFK}^-_{U=0}$ coincide, so

$$V^m \cdot [t^-_{S,\mathbf{w}}(1)] = V^m \cdot [t^-_{S',\mathbf{w}}(1)] \text{ in } \mathit{HFK}^-_{U=0}(\mathbb{K}),$$

and hence $\kappa(S, S') \le m + g$. □

6.6. **The local h-invariants and the double point distance.** In this section, we show the following:

Theorem 6.14. *Suppose that S, $S' \in \mathrm{Surf}_g(K)$ and $g \le k \le \mu_{\mathrm{Sing}}(S, S')$. Then*

$$V_k(S, S') \le \left\lceil \frac{\mu_{\mathrm{Sing}}(S, S') - k}{2} \right\rceil.$$

If $k \ge \mu_{\mathrm{Sing}}(S, S')$, then $V_k(S, S') = 0$.

Definition 6.15. Suppose that $S \in \mathrm{Imm}(K)$ is a generic immersed surface. We denote by P^+ the set of positive double points of S. Furthermore, let d be an integer satisfying $d \ge g(S) + |P^+|$. We say that S satisfies the *singular, decoration-independence condition (SDI) above degree* d if the following holds:

(*SDI*) For all maximal $\mathfrak{s} \in \mathrm{Spin}^c(B^4(S))$, for every i, $j \in \mathbb{N}$ satisfying

$$g(S) + |P^+| + i + j \ge d,$$

and for all $P \subseteq P^+$, the chain homotopy type of the map

$$U^i V^j \cdot \mathbf{t}^-_{S,P,\mathbf{z},\mathfrak{s}} \colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K})$$

depends only on $\mathfrak{s}$ and the quantities

$$i + |P| \quad \text{and} \quad j + |P^+| - |P|.$$

Lemma 6.16. *Let $S \in \mathrm{Imm}(K)$ be a generic immersion with positive double points P^+, and suppose that S' is obtained from S by the birth of a pair of double points. Write p^+ and p^- for the new positive and negative double points of S', respectively. If S satisfies the singular decoration-independence condition* (*SDI*) *above degree d, then S' satisfies condition* (*SDI*) *above degree*

$$\max\{d, g(S') + |P^+ \cup \{p^+\}|\}.$$

Similarly, if S' satisfies (*SDI*) *above degree d, then so does S.*

Proof. Consider first the case that S satisfies condition (*SDI*) above degree d. Let $\mathfrak{s} \in \mathrm{Spin}^c(B^4(S))$ and $\mathfrak{s}' \in \mathrm{Spin}^c(B^4(S'))$ be such that $\mathfrak{s}'$ agrees with $\mathfrak{s}$ on $B^4(S) \setminus N(p^-)$ and $\langle c_1(\mathfrak{s}'), E \rangle = \pm 1$, where E is the exceptional divisor that appears after blowing up $B^4(S)$ at p^-.

Let $P \subseteq P^+$ and $P' \subseteq P^+ \cup \{p^+\}$. By Lemma 6.6, we have that

$$U^i V^j \cdot \mathbf{t}^-_{S',P',\mathbf{z},\mathfrak{s}'} \simeq \begin{cases} U^{i+1} V^j \cdot \mathbf{t}^-_{S,P' \setminus \{p^+\},\mathbf{z},\mathfrak{s}} & \text{if } p^+ \in P' \\ U^i V^{j+1} \cdot \mathbf{t}^-_{S,P',\mathbf{z},\mathfrak{s}} & \text{if } p^+ \notin P'. \end{cases} \tag{6.8}$$

Since S satisfies condition (*SDI*) above degree d, the expression on the right side of equation (6.8) depends only on the quantities $i + |P'|$ and $j + 1 + |P^+| - |P'|$, as long as

$$i + j + 1 + |P^+| + g(S) \ge d.$$

Hence S' satisfies condition (*SDI*) above degree d if $d > g(S) + |P^+|$, or above degree $d + 1$ if $d = g(S) + |P^+|$.

Next, we suppose that S' satisfies condition (*SDI*) above degree

$$d \ge g(S') + |P^+ \cup \{p^+\}| = g(S') + |P^+| + 1.$$

The stabilization formula from equation (6.8) shows that S satisfies condition (*SDI*) above degree d, as well. □

We can now prove that $V_k(S, S')$ gives a lower bound on $\mu_{\mathrm{Sing}}(S, S')$:

Proof of Theorem 6.14. Suppose first that $g \le k \le \mu_{\mathrm{Sing}}(S,S')$. Pick a sequence of generic immersed surfaces $S_1,\dots,S_n \in \mathrm{Imm}(K)$ such that consecutive surfaces differ by the birth or death of a pair of double points. Furthermore, assume that $S_1 = S$ and $S_n = S'$. Note that, trivially, S and S' satisfy the singular decoration-independence condition (SDI) above degree $g := g(S) = g(S')$. We pick a stabilization sequence of maximal Spin^c structures $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ on $B^4(S_1),\dots,B^4(S_n)$, respectively.

Write $m = \frac{1}{2}\max\{|\mathrm{Sing}(S_1)|,\dots,|\mathrm{Sing}(S_n)|\}$. By Lemma 6.16, each of the immersed surfaces S_i satisfies condition (SDI) above degree $g+m$. By adding one additional birth-death pair of double points, if necessary, we may assume that $\frac{m+g-k}{2}$ and $\frac{m-g+k}{2}$ are both integers. Note also that both expressions are nonnegative, since $g \le k \le m+g$ by assumption. Let P_l^+ be the set of positive double points of S_l for $l \in \{1,\dots,n\}$. Since

$$\frac{m+g-k}{2} + \frac{m-g+k}{2} = m$$

and $|P_l^+| \le m$ for all l, we can pick subsets $P_l \subseteq P_l^+$ such that

$$|P_l| \le \frac{m+g-k}{2} \qquad \text{and} \qquad |P_l^+| - |P_l| \le \frac{m-g+k}{2}.$$

We then pick sequences of nonnegative integers i_l and j_l such that

$$i_l + |P_l| = \frac{m+g-k}{2} \qquad \text{and} \qquad j_l + |P_l^+| - |P_l| = \frac{m-g+k}{2}$$

for all $1 \le l \le n$.

Using our computation of the effect of a double point birth from Lemma 6.6, as well as Lemma 6.16 to change the decorations as needed, we conclude that the maps

$$U^{i_l} V^{j_l} \cdot \mathbf{t}^-_{S_l,P_l,\mathbf{z},\mathfrak{s}_l} \colon \mathcal{R}^- \to \mathcal{CFL}^-(\mathbb{K})$$

are all filtered chain homotopic to one another. In particular,

$$U^{\frac{g+m-k}{2}} V^{\frac{m-g+k}{2}} \cdot \mathbf{t}^-_{S,\mathbf{z}} \simeq U^{\frac{g+m-k}{2}} V^{\frac{m-g+k}{2}} \cdot \mathbf{t}^-_{S',\mathbf{z}}. \tag{6.9}$$

The maps in equation (6.9) increase the Alexander grading by k. Multiplying equation (6.9) by V^{-k} and rearranging, we conclude that

$$\widehat{U}^{\frac{g+m-k}{2}} \cdot [V^{-g} \cdot \mathbf{t}^-_{S,\mathbf{z}}(1)] = \widehat{U}^{\frac{g+m-k}{2}} \cdot [V^{-g} \cdot \mathbf{t}^-_{S',\mathbf{z}}(1)] \text{ in } H_*(A_k^-(\mathbb{K})),$$

completing the proof for $k \le \mu_{\mathrm{Sing}}(S,S')$.

To verify that $V_k(S,S') = 0$ when $k \ge \mu_{\mathrm{Sing}}(S,S')$, we note that the above result implies that $V_{\mu_{\mathrm{Sing}}(S,S')}(S,S') = 0$. The monotonicity result from Lemma 4.13 then implies the claim for $k > \mu_{\mathrm{Sing}}(S,S')$. ☐

7. Upsilon and an infinite family of topological metric filtrations

Let K be a knot in $\mathbb{S}^3$, and let $S, S' \in \mathrm{Surf}(K)$. The invariant $\Upsilon_{(S,S')}$ gives a family of algebraically defined functions

$$\Upsilon_{(S,S')}(t) \colon \mathrm{Surf}(K) \times \mathrm{Surf}(K) \to \mathbb{R}^{\ge 0}$$

parametrized by $t \in [0,2]$. In this section, we describe a topologically defined family $M_{(S,S')}(t)$ of functions that are bounded below by $\Upsilon_{(S,S')}(t)$.

7.1. The topological M-metric on $\mathrm{Surf}(K)$.

The topological M-metric will be defined using the following generalized stabilization operation:

Definition 7.1. Suppose that (W,S) bounds $(\mathbb{S}^3,K)$, and that $B^4 \subseteq \mathrm{int}(W)$ is an embedded 4-ball such that $\partial B^4 \cap S$ is an n-component unlink U_n. Further, suppose that $B^4 \cap S$ consists of disks $D_1,\dots,D_n$ that can be smoothly isotoped into ∂B^4 relative to U_n. We say that (W',S') is a *generalized stabilization* of (W,S) if it is formed by removing $(B^4, S\cap B^4)$ from (W,S), and gluing in a link cobordism (X_0,S_0) such that the following hold:

1. (X_0,S_0) is a cobordism from $\emptyset$ to $(\partial B^4, U_n)$,
2. $b_2^+(X_0) = b_1(X_0) = 0$,

(3) S_0 is connected and oriented.

We remark that, although Definition 7.1 clearly generalizes the stabilization operation from Section 2.8, it may still seem somewhat unmotivated. We note that, after a double point creation, the desingularization of an immersed surface changes by a generalized stabilization:

Example 7.2. Suppose that S is an immersed surface in B^4 which bounds K, and S' is obtained from S by a creation of a pair of canceling double points. If $(W, \widehat{S})$ and $(W', \widehat{S}')$ denote their desingularizations, as defined in Definition 6.5, then $(W', \widehat{S}')$ can be obtained from $(W, \widehat{S})$ by a generalized stabilization. Indeed, since a double point creation can be achieved by a finger move supported in a neighborhood of a path λ connecting two points on S, we can take B^4 to be a neighborhood of λ. Clearly, ∂B^4 intersects S along two disks, and $S \cap \partial B^4$ is a 2-component unlink. Since S' differs from S only inside B^4, the desingularization $(W, \widehat{S})$ can be obtained by cutting out B^4 and gluing in $(\overline{\mathbb{CP}}^2 \setminus B^4, S_0)$ for an annulus S_0 in $\overline{\mathbb{CP}}^2 \setminus B^4$.

If W is a compact, oriented 4-manifold with boundary $\mathbb{S}^3$, we let $\operatorname{Char}(W)$ denote the set of *characteristic vectors* of the intersection form Q_W; i.e., the set of elements $C \in H^2(W) \cong H^2(W, \partial W)$ such that

$$\langle\, x \cup x, [W, \partial W]\, \rangle \equiv \langle\, C \cup x, [W, \partial W]\, \rangle \mod 2$$

for every $x \in H^2(W, \partial W)$. It is well known that $\operatorname{Char}(W) = \{\, c_1(\mathfrak{s}) : \mathfrak{s} \in \operatorname{Spin}^c(W)\, \}$.

Suppose (W, S) is a link cobordism from $\emptyset$ to $(\mathbb{S}^3, K)$, with $b_1(W) = b_2^+(W) = 0$. For $C \in \operatorname{Char}(W)$, let

$$H_t(W, [S], C) := \frac{C^2 + b_2(W) - 2t\langle\, C, [S]\, \rangle + 2t[S] \cdot [S]}{4},$$

where $[S] \in H_2(W) \cong H_2(W, \partial W)$. Furthermore, for $\mathfrak{s} \in \operatorname{Spin}^c(W)$, we write

$$H_t(W, [S], \mathfrak{s}) := H_t(W, [S], c_1(\mathfrak{s})).$$

If $t \in [0, 2]$, we define the *M-degree* of (W, S) to be the function

$$M_{(W,S)}(t) := \min_{C \in \operatorname{Char}(W)} -H_t(W, [S], C) + t \cdot g(S).$$

Definition 7.3. Suppose that K is a knot in $\mathbb{S}^3$, and S, $S' \in \operatorname{Surf}(K)$. We say that $\vec{\mathcal{S}} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$ is a *generalized stabilization sequence* from S to S' if the following hold:

1. Each $\mathcal{S}_i = (W_i, S_i)$ is a link cobordism bounding $(\mathbb{S}^3, K)$, such that S_i is connected and $b_1(W_i) = b_2^+(W_i) = 0$.
2. $\mathcal{S}_1 = (B^4, S)$ and $\mathcal{S}_n = (B^4, S')$.
3. Up to diffeomorphism fixing $\mathbb{S}^3$ pointwise, $\mathcal{S}_{i+1}$ is obtained from $\mathcal{S}_i$ via a generalized stabilization or destabilization.

We write $P_{\mathrm{st}}(S, S')$ for the set of stabilization sequences connecting S and S'.

Definition 7.4. Let K be a knot in $\mathbb{S}^3$ and suppose S, $S' \in \operatorname{Surf}(K)$. Given a stabilization sequence $\vec{\mathcal{S}} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$ from S to S', we define the *M-degree* of the sequence $\vec{\mathcal{S}}$ to be the function

$$M_{\vec{\mathcal{S}}}(t) := \max_{1 \le i \le n} M_{(W_i, S_i)}(t).$$

Furthermore, the *M-distance* of S and S' is the function $M_{(S,S')} \colon [0, 2] \to \mathbb{R}^{\ge 0}$ defined by

$$M_{(S,S')}(t) := \min_{\vec{\mathcal{S}} \in P_{\mathrm{st}}(S, S')} M_{\vec{\mathcal{S}}}(t).$$

For each $t \in [0, 2]$, the quantity $M_{(S,S')}(t)$ is a metric filtration on $\operatorname{Surf}(K)$.

7.2. **The M-metric and Υ.** In this section, we prove the following:

Theorem 7.5. *If K is a knot in $\mathbb{S}^3$ and S, $S' \in \mathrm{Surf}(K)$, then*

$$\Upsilon_{(S,S')}(t) \le M_{(S,S')}(t).$$

The proof of Theorem 7.5 is similar to the proof of Theorem 5.13, our bound on τ. It is convenient to introduce the following notation. Suppose $(W,\mathcal{F})\colon (Y_1,\mathbb{L}_1) \to (Y_2,\mathbb{L}_2)$ is a decorated link cobordism, and $\mathfrak{s} \in \mathrm{Spin}^c(W)$. Write $\mathcal{F} = (S,\mathcal{A})$. If $\mathfrak{s}|_{Y_i}$ is torsion and $\mathbb{L}_i = (L_i, \mathbf{w}_i, \mathbf{z}_i)$ is null-homologous for $i \in \{1,2\}$, we define the quantities

$$G_{\mathbf{w}}(W,\mathcal{F},\mathfrak{s}) := \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \chi(S_{\mathbf{w}}) - \frac{1}{2}(|\mathbf{w}_1| + |\mathbf{w}_2|)$$

and

$$G_{\mathbf{z}}(W,\mathcal{F},\mathfrak{s}) := \frac{c_1(\mathfrak{s} - PD[S])^2 - 2\chi(W) - 3\sigma(W)}{4} + \chi(S_{\mathbf{z}}) - \frac{1}{2}(|\mathbf{z}_1| + |\mathbf{z}_2|).$$

For $t \in [0,2]$, we define

$$G_t(W,\mathcal{F},\mathfrak{s}) := \left(1 - \frac{t}{2}\right) \cdot G_{\mathbf{w}}(W,\mathcal{F},\mathfrak{s}) + \frac{t}{2} \cdot G_{\mathbf{z}}(W,\mathcal{F},\mathfrak{s}).$$

In the case when $(W,\mathcal{F})$ is a cobordism from $\emptyset$ to $(\mathbb{S}^3,\mathbb{K})$ with $b_2^+(W) = b_1(W) = 0$, and the dividing set $\mathcal{A}$ consists of a single arc that divides S into two components, we have

$$G_{\mathbf{w}}(W,\mathcal{F},\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 + b_2(W)}{4} - 2g(S_{\mathbf{w}}) \text{ and } G_{\mathbf{z}}(W,\mathcal{F},\mathfrak{s}) = \frac{c_1(\mathfrak{s} - PD[S])^2 + b_2(W)}{4} - 2g(S_{\mathbf{z}}).$$

In this situation, we also have

$$G_t(W,\mathcal{F},\mathfrak{s}) = H_t(W,[S],\mathfrak{s}) - (2-t)\cdot g(S_{\mathbf{w}}) - t \cdot g(S_{\mathbf{z}}). \tag{7.1}$$

We now compute the effect of a generalized stabilization; cf. Lemma 5.4:

Lemma 7.6. *Suppose that $(W,\mathcal{F})\colon (Y_1,\mathbb{L}_1) \to (Y_2,\mathbb{L}_2)$ is a decorated link cobordism with $b_1(W) = 0$. Write $\mathcal{F} = (S,\mathcal{A})$. Suppose that (W',S') is a generalized stabilization of (W,S), obtained by cutting out $B^4 \subseteq W$ and gluing in a link cobordism (X_0,S_0) with $b_1(X_0) = b_2^+(X_0) = 0$, as in Definition 7.1. Let $D_1,\dots,D_n$ denote the components of $S \cap B^4$, and suppose that $\widehat{D} \subseteq S$ is a disk that contains each of $D_1,\dots,D_n$, and intersects $\mathcal{A}$ in a single arc. Define*

$$S_0' := (\widehat{D} \setminus (D_1 \cup \dots \cup D_n)) \cup S_0,$$

and suppose that $\mathcal{A}'$ is a dividing set on S' that intersects S_0' in a single arc, and agrees with $\mathcal{A}$ outside $\widehat{D}$. Write $\mathcal{F}' = (S',\mathcal{A}')$. If $\mathfrak{s}' \in \mathrm{Spin}^c(W')$ agrees with $\mathfrak{s} \in \mathrm{Spin}^c(W)$ on $W \setminus B^4$, then

$$F_{W',\mathcal{F}',\mathfrak{s}'} \simeq U^{-d_1/2} V^{-d_2/2} \cdot F_{W,\mathcal{F},\mathfrak{s}},$$

where $d_1 := G_{\mathbf{w}}(W',\mathcal{F}',\mathfrak{s}') - G_{\mathbf{w}}(W,\mathcal{F},\mathfrak{s})$ and $d_2 := G_{\mathbf{z}}(W',\mathcal{F}',\mathfrak{s}') - G_{\mathbf{z}}(W,\mathcal{F},\mathfrak{s})$.

Proof. Let $\widehat{D}_0$ denote the punctured disk $\widehat{D} \setminus B^4$. We write $N(\widehat{D}_0)$ for the total space of the unit normal disk bundle of $\widehat{D}_0$ in $W \setminus B^4$, and let

$$W_0 := N(\widehat{D}_0) \cup B^4 \text{ and } W_0' := N(\widehat{D}_0) \cup X_0.$$

Note that W_0 and W_0' are topologically obtained from B^4 and X_0, respectively, by attaching a collection of 4-dimensional 1-handles. Write $Y := \partial W_0 = \partial W_0'$. We can view $(W_0,\widehat{D})$ and (W_0',S_0') as undecorated link cobordisms from the empty set to the knot

$$K := \partial \widehat{D} \times \{0\} = S' \cap Y \subseteq N(\widehat{D}_0)$$

in Y. As in Lemma 5.4, the knot K is an unknot in Y, since we can explicitly construct a Seifert disk D_K. Let us write $\mathcal{F}_0 = (\widehat{D},\mathcal{A})$ and $\mathcal{F}_0' = (S_0',\mathcal{A}')$.

Suppose $\mathfrak{s}' = \mathfrak{s} \# \mathfrak{t}$, for $\mathfrak{t} \in \mathrm{Spin}^c(X_0)$. Consider the quantity

$$h(S_0' \cup D_K, \mathfrak{s}') := \frac{\langle c_1(\mathfrak{s}'), [S_0' \cup D_K] \rangle - [S_0' \cup D_K] \cdot [S_0' \cup D_K]}{2}.$$

If $h(S_0'\cup D_K,\mathfrak{s}')+g(S'_{0,\mathbf{z}})-g(S'_{0,\mathbf{w}})\ge 0$, then, according to Lemma 5.3,

$$F_{W_0',\mathcal{F}_0',\mathfrak{s}'|_{W_0'}}\simeq U^{g(S'_{0,\mathbf{w}})}V^{g(S'_{0,\mathbf{z}})+h(S_0'\cup D_K,\mathfrak{s}')}\cdot (F_{W_0',\mathfrak{s}'|_{W_0'}}\otimes \mathbb{F}[U,V]) \mod H_1(Y), \tag{7.2}$$

while

$$F_{W_0,\mathcal{F}_0,\mathfrak{s}|_{W_0}}\simeq (F_{W_0,\mathfrak{s}|_{W_0}}\otimes \mathbb{F}[U,V]) \mod H_1(Y). \tag{7.3}$$

Up to diffeomorphism, we can write W_0 and W_0' as the compositions

$$W_0\cong W_1\circ B^4 \text{ and } W_0'\cong W_1\circ X_0,$$

where $W_1\cong (I\times \mathbb{S}^3)\cup N(\widehat{D}_0)$. We note W_1 is a 1-handle cobordism.

The map

$$F_{X_0,\mathfrak{t}}\colon HF^-(\mathbb{S}^3)\to HF^-(\mathbb{S}^3)$$

is an injection since $b_1(X_0)=b_2^+(X_0)=0$, by the proof of [35, Theorem 9.1]. The map $F_{X_0,\mathfrak{t}}$ has grading

$$d:=\frac{c_1(\mathfrak{t})^2+b_2(X_0)}{4},$$

and hence must be chain homotopic to multiplication by $\widehat{U}^{-d/2}$. Hence

$$F_{W_0',\mathfrak{s}'|_{W_0'}}\simeq F_{W_1,\mathfrak{s}|_{W_1}}\circ F_{X_0,\mathfrak{t}}\simeq \widehat{U}^{-d/2}\cdot F_{W_1,\mathfrak{s}|_{W_1}}\circ F_{B^4,\mathfrak{s}|_{B^4}}\simeq \widehat{U}^{-d/2}\cdot F_{W_0,\mathfrak{s}|_{W_0}}. \tag{7.4}$$

Write $W_2:=W\setminus W_1$. The inclusion $H_1(Y)\to H_1(W_2)/\operatorname{Tors}$ is trivial, since Y is the boundary of a 4-dimensional 1-handlebody in W. Similarly, the coboundary maps $H^1(Y)\to H^2(W)$ and $H^1(Y)\to H^2(W')$ are both trivial. Hence, combining equations (7.2), (7.3), and (7.4), and using the Spin^c composition law, we conclude that

$$F_{W',\mathcal{F}',\mathfrak{s}'}\simeq U^{-d/2+g(S'_{0,\mathbf{w}})}V^{-d/2+h(S_0'\cup D_K,\mathfrak{s}')+g(S'_{0,\mathbf{z}})}\cdot F_{W,\mathcal{F},\mathfrak{s}}.$$

It is easy to see that

$$-\frac{d}{2}+g(S'_{0,\mathbf{w}})=-\frac{1}{2}\left(G_{\mathbf{w}}(W',\mathcal{F}',\mathfrak{s}')-G_{\mathbf{w}}(W,\mathcal{F},\mathfrak{s})\right)$$

and

$$-\frac{d}{2}+h(S_0'\cup D_K,\mathfrak{s}')+g(S'_{0,\mathbf{z}})=-\frac{1}{2}\left(G_{\mathbf{z}}(W',\mathcal{F}',\mathfrak{s}')-G_{\mathbf{z}}(W,\mathcal{F},\mathfrak{s})\right),$$

which completes the proof in the case when $h(S_0'\cup D_K,\mathfrak{s}')+g(S'_{0,\mathbf{z}})-g(S'_{0,\mathbf{w}})\ge 0$.

The proof when $h(S_0'\cup D_K,\mathfrak{s}')+g(S'_{0,\mathbf{z}})-g(S'_{0,\mathbf{w}})\le 0$ is an easy modification, using the corresponding subcase of Lemma 5.3. ☐

Lemma 7.6 also immediately computes the effect of a generalized stabilization on the t-modified versions of the link cobordism maps:

Corollary 7.7. *If $(W',\mathcal{F}')$ is a generalized stabilization of $(W,\mathcal{F})$ and $\mathfrak{s}'\in \operatorname{Spin}^c(W')$ restricts to $\mathfrak{s}$ on $W\setminus B^4$, then*

$$tF_{W',\mathcal{F}',\mathfrak{s}'}\simeq v^{-G_t(W',\mathcal{F}',\mathfrak{s}')+G_t(W,\mathcal{F},\mathfrak{s})}\cdot tF_{W,\mathcal{F},\mathfrak{s}}.$$

We can now prove Theorem 7.5:

Proof of Theorem 7.5. Fix $t\in[0,2]$. Suppose $\vec{\mathcal{S}}=\{\mathcal{S}_1,\dots,\mathcal{S}_n\}$ is a stabilization sequence connecting (B^4,S) and (B^4,S'), and write $\mathcal{S}_i=(W_i,S_i)$. Decorate each S_i with a dividing set $\mathcal{A}_i$ consisting of a single arc, such that the type-$\mathbf{w}$ subregion has genus 0, and the type-$\mathbf{z}$ subregion has genus $g(S_i)$. We can assume the dividing arc is very near to the knot K, and the the type-$\mathbf{w}$ subregion is unaffected by any of the stabilizations. Write $\mathcal{F}_i=(S_i,\mathcal{A}_i)$.

Let us call a sequence $\vec{\mathfrak{s}}=\{\mathfrak{s}_1,\dots,\mathfrak{s}_n\}$ of Spin^c structures on $W_1,\dots,W_n$, respectively, a *stabilization sequence* if, whenever (W_{i+1},S_{i+1}) is obtained by stabilizing (W_i,S_i) with the negative definite link cobordism (X_0,S_0), the Spin^c structure $\mathfrak{s}_{i+1}$ can be written as $\mathfrak{s}_i\#\mathfrak{t}_i$ for $\mathfrak{t}_i\in\operatorname{Spin}^c(X_0)$. We require an analogous condition whenever W_{i+1} is a generalized destabilization of W_i. We define

$$M_{\vec{\mathcal{S}},\vec{\mathfrak{s}}}(t):=\max_{1\le i\le n}-G_t(W_i,\mathcal{F}_i,\mathfrak{s}_i)=\max_{1\le i\le n}-H_t(W_i,[S_i],\mathfrak{s}_i)+t\cdot g(S_i),$$

where the second equality follows from equation (7.1) and the fact that $g(S_{i,\mathbf{w}}) = 0$ and $g(S_{i,\mathbf{z}}) = g(S_i)$.

By Corollary 7.7, if $\vec{\mathfrak{s}} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_n\}$ is a stabilization sequence of Spin^c structures on $W_1, \dots, W_n$, then all of the elements

$$v^{M_{\vec{\mathcal{S}},\vec{\mathfrak{s}}}(t)+G_t(W_i,\mathcal{F}_i,\mathfrak{s}_i)} \cdot [tF_{W_i,\mathcal{F}_i,\mathfrak{s}_i}(1)]$$

coincide in $tHFK^-(\mathbb{K})$. In particular,

$$v^{M_{\vec{\mathcal{S}},\vec{\mathfrak{s}}}(t)-t\cdot g(S)} \cdot [tF_{S,\mathbf{z}}(1)] = v^{M_{\vec{\mathcal{S}},\vec{\mathfrak{s}}}(t)-t\cdot g(S')} \cdot [tF_{S',\mathbf{z}}(1)], \tag{7.5}$$

as $G_t(W_1, \mathcal{F}_1, \mathfrak{s}_1) = -t \cdot g(S)$ and $G_t(W_n, \mathcal{F}_n, \mathfrak{s}_n) = -t \cdot g(S')$.

Suppose that $(W_{i\pm1}, S_{i\pm1})$ is obtained by stabilizing (W_i, S_i) using the negative definite link cobordism (X_0, S_0), and $\mathfrak{s}_i \in \text{Spin}^c(W_i)$ and $\mathfrak{t}_i \in \text{Spin}^c(X_0)$. Then

$$H_t(W_i\# X_0, [S_i] + [S_0], \mathfrak{s}_i\#\mathfrak{t}_i) = H_t(W_i, [S_i], \mathfrak{s}_i) + H_t(X_0, [S_0], \mathfrak{t}_i).$$

Hence, the Spin^c structure $\mathfrak{s}_i\#\mathfrak{t}_i$ minimizes $-H_t(W_{i\pm1}, [S_{i\pm1}], \mathfrak{s}_i\#\mathfrak{t}_i)$ for a fixed t if and only if $\mathfrak{s}_i$ minimizes $-H_t(W_i, [S_i], \mathfrak{s}_i)$ and $\mathfrak{t}_i$ minimizes $-H_t(X_0, [S_0], \mathfrak{t}_i)$. It follows that we can always construct a stabilization sequence of Spin^c structures $\vec{\mathfrak{s}} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_n\}$ such that

$$-H_t(W_i, [S_i], \mathfrak{s}_i) = \min_{\mathfrak{s}\in\text{Spin}^c(W_i)} -H_t(W_i, [S_i], \mathfrak{s}) = M_{(W_i,S_i)}(t) - t \cdot g(S_i).$$

Then

$$M_{\vec{\mathcal{S}},\vec{\mathfrak{s}}}(t) = \max_{1\le i\le n} -H_t(W_i, [S_i], \mathfrak{s}_i) + t \cdot g(S_i) = \max_{1\le i\le n} M_{(W_i,S_i)}(t) = M_{\vec{\mathcal{S}}}(t). \tag{7.6}$$

Combining equations (7.5) and (7.6), we conclude that

$$\Upsilon_{(S,S')}(t) \le M_{\vec{\mathcal{S}}}(t) \tag{7.7}$$

for any $t \in [0, 2]$. Minimizing equation (7.7) over all $\vec{\mathcal{S}} \in P_{\text{st}}(S, S')$ yields the result. □

8. The summand-swapping diffeomorphism

If K is a knot in $\mathbb{S}^3$, one can construct an order n automorphism of the knot $K^{\#n}$, corresponding to cyclically permuting the summands. In this section, we investigate the case when $n = 2$, and compute the induced map on knot Floer homology. We will use this in Section 10 to construct pairs of slice disks for which we can explicitly compute the secondary invariants defined in Section 4.

8.1. **Construction of the diffeomorphism map.** If $K \subseteq \mathbb{S}^3$ is a knot, there is a diffeomorphism

$$R^\pi\colon (\mathbb{S}^3, K\#K) \to (\mathbb{S}^3, K\#K)$$

that switches the two summands of $K\#K$. In fact, for an appropriate embedding of $K\#K$ into $\mathbb{S}^3$, the diffeomorphism R^π can be realized as an order 2 rigid motion of $\mathbb{S}^3$: Isotope K into the $y \ge 1$ half-space of $\mathbb{R}^3 \subseteq \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ such that the line segment $I := [-1, 1] \times \{(1, 0)\} \subseteq K$. For $\varphi \in \mathbb{R}$, let R^φ be φ-rotation about the z-axis. If we let $K\#K$ be the equivariant smoothing of

$$(K \setminus I) \cup R^\pi(K \setminus I) \cup R^{\pi/2}(I) \cup R^{-\pi/2}(I),$$

then R^π is an orientation-preserving automorphism of $(\mathbb{S}^3, K\#K)$.

In particular, the knot $K\#K$ is 2-periodic, and has no fixed points. We pick two basepoints, w, $z \in K \setminus I$, such that w follows z with respect to the orientation of K. We let w' and z' denote their images on $R^\pi(K \setminus I)$ under the map R^π. We write $\mathbb{K} = (K, w, z)$ and $\mathbb{K}\#\mathbb{K} = (K\#K, w, z)$.

We define the element

$$\mathbf{R}^\pi := \rho \circ R^\pi \in \text{MCG}(\mathbb{S}^3, K\#K, w, z),$$

where

$$\rho\colon (\mathbb{S}^3, K\#K, w', z') \to (\mathbb{S}^3, K\#K, w, z)$$

is a half-twist diffeomorphism in the direction of the knot's orientation that swaps the pairs (w, z) and (w', z'). We note that the diffeomorphism $(\mathbf{R}^\pi)^2$ is isotopic to a full Dehn twist along $K\#K$. Hence, by [54, Theorem B],

$$(\mathbf{R}^\pi_*)^2 \simeq \text{id} + \Phi^{\mathbb{K}\#\mathbb{K}}_w \circ \Psi^{\mathbb{K}\#\mathbb{K}}_z, \tag{8.1}$$

where $\Phi_w^{\mathbb{K}\#\mathbb{K}}$ and $\Psi_z^{\mathbb{K}\#\mathbb{K}}$ are the basepoint actions on $\mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K})$ described in Section 3.2; see also the work of Sarkar [47].

Remark 8.1. Note that any $\mathbb{F}[U,V]$-equivariant homotopy equivalence

$$\mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K}) \simeq \mathcal{CFL}^\infty(\mathbb{K}) \otimes \mathcal{CFL}^\infty(\mathbb{K})$$

will intertwine $\Phi_w^{\mathbb{K}\#\mathbb{K}}$ (resp. $\Psi_z^{\mathbb{K}\#\mathbb{K}}$) with $\Phi_w \otimes \mathrm{id} + \mathrm{id} \otimes \Phi_w$ (resp. $\Psi_z \otimes \mathrm{id} + \mathrm{id} \otimes \Psi_z$), up to chain homotopy. This is because if C and C' are free chain complexes over $\mathbb{F}[U,V]$ and $F\colon C \to C'$ is a chain map, one easily shows that $F \circ \Phi \simeq \Phi' \circ F$, where Φ and Φ' are the algebraic analogs of the map Φ_w on C and C'.

We now wish to compute a formula for the chain homotopy type of the induced map $\mathbf{R}^\pi_*$. Note that there is a filtered chain map

$$\mathrm{Sw}\colon \mathcal{CFL}^\infty(\mathbb{K}) \otimes \mathcal{CFL}^\infty(\mathbb{K}) \to \mathcal{CFL}^\infty(\mathbb{K}) \otimes \mathcal{CFL}^\infty(\mathbb{K}),$$

obtained by switching the two factors. Note that Sw cannot be chain homotopic to $\mathbf{R}^\pi$, since $\mathrm{Sw} \circ \mathrm{Sw} = \mathrm{id}$, which would violate equation (8.1). In this section, we prove the following:

Theorem 8.2. *Let $\mathbb{K} = (K, w, z)$ be a doubly-based knot in $\mathbb{S}^3$, and consider the doubly-based knot $\mathbb{K}\#\mathbb{K} = (K\#K, w, z)$ defined above. Then there is a filtered chain homotopy equivalence between $\mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K})$ and $\mathcal{CFL}^\infty(\mathbb{K}) \otimes_{\mathcal{R}^\infty} \mathcal{CFL}^\infty(\mathbb{K})$ that intertwines $\mathbf{R}^\pi_*$ with*

$$\mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes (\Phi_w \circ \Psi_z) + \Psi_z \otimes \Phi_w)\,.$$

Remark 8.3. We say the endomorphisms F and G of a chain complex C are *homotopy conjugate* if there is a homotopy automorphism $A\colon C \to C$ such that $F \circ A \simeq A \circ G$. It is not hard to see that the four maps

$$\begin{aligned}
&\mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes (\Phi_w \circ \Psi_z) + \Psi_z \otimes \Phi_w)\,,\\
&\mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes (\Phi_w \circ \Psi_z) + \Phi_w \otimes \Psi_z)\,,\\
&\mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + (\Phi_w \circ \Psi_z) \otimes \mathrm{id} + \Psi_z \otimes \Phi_w)\,,\\
&\mathrm{Sw} \circ (\mathrm{id} \otimes \mathrm{id} + (\Phi_w \circ \Psi_z) \otimes \mathrm{id} + \Phi_w \otimes \Psi_z)
\end{aligned}$$

are all homotopy conjugate endomorphisms of $\mathcal{CFL}^\infty(\mathbb{K}) \otimes \mathcal{CFL}^\infty(\mathbb{K})$. Indeed, the map A can be taken to be one of the maps Sw, $\mathrm{id} \otimes \mathrm{id} + \Phi_w \otimes \Psi_z$, or $\mathrm{id} \otimes \mathrm{id} + \Psi_z \otimes \Phi_w$, since $\Phi_w^2 \simeq 0$, $\Psi_z^2 \simeq 0$, and $\Phi_w \circ \Psi_z \simeq \Psi_z \circ \Phi_w$.

8.2. Proof of the formula for the summand-swapping diffeomorphism map.

Proof of Theorem 8.2. Let us write $\mathbb{K}\#\mathbb{K} = (K\#K, w, z)$, where w and z appear on the right copy of K. Let w' and z' denote their images under R^π, on the left copy of K. We define our connected sum map

$$E\colon \mathcal{CFL}^\infty(\mathbb{K}\#\mathbb{K}) \to \mathcal{CFL}^\infty(K, w', z') \otimes_{\mathcal{R}^\infty} \mathcal{CFL}^\infty(\mathbb{K})$$

as the composition

$$E := F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'},$$

where $F_{\mathbb{S}_2}$ denotes the 3-handle map induced by the framed 2-sphere $\mathbb{S}_2$ that separates the two copies of K after the band surgery along B. A schematic of the link cobordism corresponding to E is shown in Figure 8.1.

The map E is a chain homotopy equivalence. Indeed, E is the map induced by a pair-of-pants link cobordism in a 3-handle cobordism that is diffeomorphic to the reverse of one of the two connected sum cobordisms constructed in [56, Section 5] (in fact, it is diffeomorphic to the link cobordism inducing the map E_1, described therein). According to [58, Proposition 5.1], the map E is a chain homotopy equivalence, and a homotopy inverse is given by turning around and reversing the orientation of the cobordism.

Expanding the definitions of E and $\mathbf{R}^\pi$, and observing that $\rho \circ R^\pi$ and $R^\pi \circ \rho$ are equal as automorphisms of $(S^3, \mathbb{K}\#\mathbb{K})$, we have

$$E\mathbf{R}^\pi_* \simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'} R^\pi_* \rho_*. \tag{8.2}$$

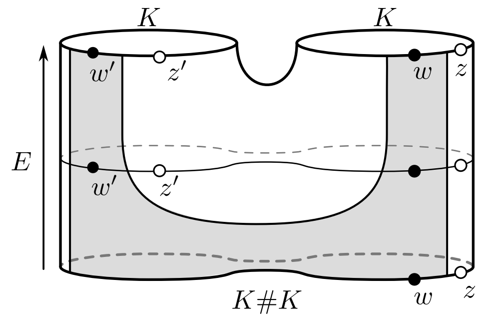

FIGURE 8.1. A schematic of the map $E := F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'}$. A decomposition of the surface is shown, corresponding to the factors of $F_B^{\mathbf{z}}$ and $T^+_{w',z'}$ in E. The 3-handle map $F_{\mathbb{S}_2}$ is not shown.

We note that, for the doubly-based knot $\rho(\mathbb{K}\#\mathbb{K}) = (K\#K, w', z')$, we have

$$T^+_{w',z'} R^\pi_* \simeq R^\pi_* T^+_{w,z} \tag{8.3}$$

by the functoriality of the quasi-stabilization operation. Similarly,

$$F_B^{\mathbf{z}} R^\pi_* \simeq R^\pi_* F_B^{\mathbf{z}}, \tag{8.4}$$

since the diffeomorphism R^π preserves the connected sum band B setwise. Finally, we note that

$$F_{\mathbb{S}_2} R^\pi_* \simeq \operatorname{Sw} F_{\mathbb{S}_2}, \tag{8.5}$$

since the framed sphere $\mathbb{S}_2$ is fixed setwise by R^π. We remark that, in equation (8.5), R^π reverses the orientation of the framed 2-sphere $\mathbb{S}_2$, though this has no effect on the cobordism map.

Applying the relations from equations (8.3), (8.4), and (8.5) to equation (8.2) yields

$$E\mathbf{R}^\pi_* \simeq \operatorname{Sw} F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w,z} \rho_*. \tag{8.6}$$

Next, we examine the expression $F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w,z} \rho_*$ appearing in equation (8.6). We perform the following manipulations:

$$\begin{aligned} F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w,z} \rho_* &\simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w,z} S^-_{w,z} T^+_{w',z'} && \text{(equation (3.11))} \\ &\simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'} + F_{\mathbb{S}_2} F_B^{\mathbf{z}} S^+_{w,z} T^-_{w,z} T^+_{w',z'} && \text{(equation (3.12)).} \end{aligned} \tag{8.7}$$

Next, we compute

$$\begin{aligned} F_{\mathbb{S}_2} F_B^{\mathbf{z}} S^+_{w,z} T^-_{w,z} T^+_{w',z'} &\simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} S^+_{w,z} S^-_{w,z} \Psi_z T^+_{w',z'} && \text{(equation (3.7))} \\ &\simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} \Phi_w \Psi_z T^+_{w',z'} && \text{(equation (3.8)).} \end{aligned} \tag{8.8}$$

We note that $\Psi_{z'} T^+_{w',z'} \simeq 0$, since $T^+_{w',z'} \simeq \Psi_{z'} S^+_{w',z'}$ and $\Psi^2_{z'} \simeq 0$ by equations (3.6) and (3.3). Hence

$$F_{\mathbb{S}_2} F_B^{\mathbf{z}} \Phi_w \Psi_z T^+_{w',z'} \simeq F_{\mathbb{S}_2} F_B^{\mathbf{z}} \Phi_w (\Psi_z + \Psi_{z'}) T^+_{w',z'}. \tag{8.9}$$

From equations (3.13) and (3.15), we conclude that

$$F_{\mathbb{S}_2} F_B^{\mathbf{z}} \Phi_w (\Psi_z + \Psi_{z'}) T^+_{w',z'} \simeq F_{\mathbb{S}_2} \Phi_w (\Psi_z + \Psi_{z'}) F_B^{\mathbf{z}} T^+_{w',z'}. \tag{8.10}$$

Finally, we note that Φ_w, Ψ_z, and $\Psi_{z'}$ commute with $F_{\mathbb{S}_2}$ by [58, Lemma 8.3], hence

$$F_{\mathbb{S}_2} \Phi_w (\Psi_z + \Psi_{z'}) F_B^{\mathbf{z}} T^+_{w',z'} \simeq (\operatorname{id} \otimes (\Phi_w \Psi_z) + \Psi_{z'} \otimes \Phi_w) F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'}. \tag{8.11}$$

Combining, equations (8.7)–(8.11), and using the fact that $E := F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w',z'}$, we see that

$$F_{\mathbb{S}_2} F_B^{\mathbf{z}} T^+_{w,z} \rho_* \simeq (\operatorname{id} \otimes \operatorname{id} + \operatorname{id} \otimes \Phi_w \Psi_z + \Psi_{z'} \otimes \Phi_w) E. \tag{8.12}$$

By applying Sw to equation (8.12), and combining it with equation (8.6), we obtain that

$$E\mathbf{R}^\pi_* \simeq \operatorname{Sw}(\operatorname{id} \otimes \operatorname{id} + \operatorname{id} \otimes \Phi_w \Psi_z + \Psi_{z'} \otimes \Phi_w) E.$$

Upon relabeling w' and z' as w and z, we obtain the formula in the statement. □

9. The trace formula

If $(Y, \mathbb{L})$ is a multi-based link, the identity decorated link cobordism

$$(W_{\mathrm{id}}, \mathcal{F}_{\mathrm{id}})\colon (Y, \mathbb{L}) \to (Y, \mathbb{L})$$

is constructed by decorating $(I \times Y, I \times L)$ with a dividing set $\mathcal{A} = I \times \mathbf{p}$, where $\mathbf{p} \subseteq L$ consists of one point in each component of $L \setminus (\mathbf{w} \cup \mathbf{z})$.

By changing which ends of $(W_{\mathrm{id}}, \mathcal{F}_{\mathrm{id}})$ are designated as incoming or outgoing, we get two other decorated link cobordisms, which we denote by

$$(W_{\mathrm{tr}}, \mathcal{F}_{\mathrm{tr}})\colon (-Y \sqcup Y, -\mathbb{L} \sqcup \mathbb{L}) \to \emptyset \quad \text{and} \quad (W_{\mathrm{cotr}}, \mathcal{F}_{\mathrm{cotr}})\colon \emptyset \to (Y \sqcup -Y, \mathbb{L} \sqcup -\mathbb{L}).$$

The $\mathcal{R}$-module $\mathcal{CFL}^\infty(-Y, -\mathbb{L}, \mathfrak{s})$ is canonically isomorphic to $\operatorname{Hom}_{\mathcal{R}^\infty}(\mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}), \mathcal{R}^\infty)$, and hence there is a *canonical trace pairing*

$$\mathrm{tr}\colon \mathcal{CFL}^\infty(-Y, -\mathbb{L}, \mathfrak{s}) \otimes_{\mathcal{R}^\infty} \mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}) \to \mathcal{R}^\infty.$$

Similarly there is a canonical cotrace map

$$\mathrm{cotr}\colon \mathcal{R}^\infty \to \mathcal{CFL}^\infty(Y, \mathbb{L}, \mathfrak{s}) \otimes_{\mathcal{R}^\infty} \mathcal{CFL}^\infty(-Y, -\mathbb{L}, \mathfrak{s}),$$

obtained by dualizing the trace pairing. In this section, we prove the following:

Theorem 9.1. *The trace and cotrace cobordisms induce the canonical trace and cotrace maps:*

$$F_{W_{\mathrm{tr}}, \mathcal{F}_{\mathrm{tr}}, \mathfrak{s}} \simeq \mathrm{tr} \quad \textit{and} \quad F_{W_{\mathrm{cotr}}, \mathcal{F}_{\mathrm{cotr}}, \mathfrak{s}} \simeq \mathrm{cotr}\,.$$

Our proof of Theorem 9.1 is similar to the proofs of [55, Theorem 1.6] and [22, Theorem 1.2].

9.1. Heegaard triples and link cobordisms.

Definition 9.2. We say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is a *Heegaard link triple* if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a Heegaard triple diagram decorated with two disjoint collections of basepoints, $\mathbf{w}$ and $\mathbf{z}$. Furthermore, for each $\boldsymbol{\sigma} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, each component of $\Sigma \setminus \boldsymbol{\sigma}$ is planar, and contains exactly one $\mathbf{w}$ basepoint, and exactly one $\mathbf{z}$ basepoint.

We remark that such a Heegaard triple is called a *doubly multi-pointed Heegaard triple* in [58]. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is a Heegaard link triple and $\boldsymbol{\sigma}, \boldsymbol{\eta} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, then we write $(Y_{\sigma,\eta}, \mathbb{L}_{\sigma,\eta})$ for the multi-based link defined by the diagram $(\Sigma, \boldsymbol{\sigma}, \boldsymbol{\eta}, \mathbf{w}, \mathbf{z})$. There is a decorated link cobordism

$$(X_{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})\colon (Y_{\alpha,\beta} \sqcup Y_{\beta,\gamma}, \mathbb{L}_{\alpha,\beta} \sqcup \mathbb{L}_{\beta,\gamma}) \to (Y_{\alpha,\gamma}, \mathbb{L}_{\alpha,\gamma}),$$

described in [22, Section 9.4], which is a refinement of the construction from [39, Section 8.1]. The 4-manifold $X_{\alpha,\beta,\gamma}$ is constructed as the union

$$X_{\alpha,\beta,\gamma} := (\Delta \times \Sigma) \cup (e_\alpha \times U_\alpha) \cup (e_\beta \times U_\beta) \cup (e_\gamma \times U_\gamma),$$

where U_α, U_β, and U_γ denote handlebodies with boundary Σ, with compressing curves $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, respectively.

The decorated surface $\mathcal{F}_{\alpha,\beta,\gamma} = (S_{\alpha,\beta,\gamma}, \mathcal{A}_{\alpha,\beta,\gamma})$ is constructed as follows. We pick embedded paths in $\Sigma \setminus \boldsymbol{\alpha}$, $\Sigma \setminus \boldsymbol{\beta}$, and $\Sigma \setminus \boldsymbol{\gamma}$ connecting the $\mathbf{z}$-basepoints to the $\mathbf{w}$-basepoints, and then push the interiors of these arcs into the interior of U_α, U_β, or U_γ, respectively. We obtain three sets of properly embedded arcs $\ell_\alpha \subseteq U_\alpha$, $\ell_\beta \subseteq U_\beta$, and $\ell_\gamma \subseteq U_\gamma$. The surface $S_{\alpha,\beta,\gamma}$ is defined as

$$S_{\alpha,\beta,\gamma} := (\Delta \times (\mathbf{w} \cup \mathbf{z})) \cup (e_\alpha \times \ell_\alpha) \cup (e_\beta \times \ell_\beta) \cup (e_\gamma \times \ell_\gamma).$$

To obtain $\mathcal{A}_{\alpha,\beta,\gamma}$, choose subsets $\mathbf{p}_\alpha \subseteq \ell_\alpha$, $\mathbf{p}_\beta \subseteq \ell_\beta$, and $\mathbf{p}_\gamma \subseteq \ell_\gamma$ consisting of one point in the interior of each component of ℓ_α, ℓ_β, and ℓ_γ, and set

$$\mathcal{A}_{\alpha,\beta,\gamma} := (e_\alpha \times \mathbf{p}_\alpha) \cup (e_\beta \times \mathbf{p}_\beta) \cup (e_\gamma \times \mathbf{p}_\gamma).$$

Theorem 9.3. *If $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{w},\mathbf{z})$ is a Heegaard link triple, then the cobordism map $F_{W_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma},\mathfrak{s}}$ is chain homotopic to the holomorphic triangle map*

$$F_{\alpha,\beta,\gamma,\mathfrak{s}}\colon \mathcal{CFL}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z},\mathfrak{s}|_{Y_{\alpha,\beta}})\otimes \mathcal{CFL}^\infty(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{w},\mathbf{z},\mathfrak{s}|_{Y_{\beta,\gamma}})\to \mathcal{CFL}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma},\mathbf{w},\mathbf{z},\mathfrak{s}|_{Y_{\alpha,\gamma}}),$$

defined by the formula

$$F_{\alpha,\beta,\gamma,\mathfrak{s}}(\mathbf{x}\otimes\mathbf{y}):=\sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma}\sum_{\substack{\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{z})\\ \mu(\psi)=0\\ \mathfrak{s}_{\mathbf{w}}(\psi)=\mathfrak{s}}}\#\mathcal{M}(\psi)\cdot U^{n_{\mathbf{w}}(\psi)}V^{n_{\mathbf{z}}(\psi)}\cdot\mathbf{z}$$

for $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $\mathbf{y}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$.

We now demonstrate that the trace formula follows quickly from Theorem 9.3:

Proof of Theorem 9.1. We will focus on the claim that $F_{W_{\text{tr}},\mathcal{F}_{\text{tr}},\mathfrak{s}}\simeq \text{tr}$. The claim about the cotrace cobordism follows from the formula for the trace cobordism. Indeed, if $(W,\mathcal{F})$ is a decorated link cobordism from $(Y_1,\mathbb{L}_1)$ to $(Y_2,\mathbb{L}_2)$, and

$$(W^\vee,\mathcal{F}^\vee)\colon(-Y_2,-\mathbb{L}_2)\to(-Y_1,-\mathbb{L}_1)$$

is the cobordism obtained by turning around $(W,\mathcal{F})$, then it is straightforward to adapt the proof of [40, Theorem 3.5] to see that $F_{W^\vee,\mathcal{F}^\vee,\mathfrak{s}}$ is equal to the dual map

$$(F_{W,\mathcal{F},\mathfrak{s}})^\vee\colon\ \text{Hom}_{\mathcal{R}^\infty}(\mathcal{CFL}^\infty(Y_2,\mathbb{L}_2,\mathfrak{s}|_{Y_2}),\mathcal{R}^\infty)\to \text{Hom}_{\mathcal{R}^\infty}(\mathcal{CFL}^\infty(Y_1,\mathbb{L}_1,\mathfrak{s}|_{Y_1}),\mathcal{R}^\infty).$$

To establish the formula for $F_{W_{\text{tr}},\mathcal{F}_{\text{tr}},\mathfrak{s}}$, we pick a diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ for $(Y,\mathbb{L})$, as well as a small Hamiltonian translate $\boldsymbol{\beta}'$ of $\boldsymbol{\beta}$, and we consider the Heegaard triple $(\Sigma,\boldsymbol{\beta}',\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$. The decorated link cobordism $(X_{\beta',\alpha,\beta},\mathcal{F}_{\beta',\alpha,\beta})$ is in fact equal to $(I\times Y,I\times L)$ with a neighborhood of $\{\frac{1}{2}\}\times U_\beta$ removed. Hence, using Theorem 9.3 and the composition law, we can write

$$F_{W_{\text{tr}},\mathcal{F}_{\text{tr}},\mathfrak{s}}(\mathbf{x}\otimes\mathbf{y})=(G\circ F_{\beta',\alpha,\beta,\mathfrak{s}})(\mathbf{x}\otimes\mathbf{y}),\tag{9.1}$$

where $\mathbf{x}\in\mathbb{T}_{\beta'}\cap\mathbb{T}_\alpha$ and $\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, and G is a composition of 3-handle and 4-handle maps. The map G takes the form

$$G(\Theta)=\begin{cases}1 & \text{if } \Theta=\Theta^-_{\beta',\beta},\\ 0 & \text{otherwise}\end{cases}$$

for $\Theta\in\mathbb{T}_{\beta'}\cap\mathbb{T}_\beta$, and extended $\mathcal{R}^\infty$-equivariantly. On the other hand equation (9.1) says that $F_{W_{\text{tr}},\mathcal{F}_{\text{tr}},\mathfrak{s}}(\mathbf{x}\otimes\mathbf{y})$ is exactly equal to the count of Maslov index 0 holomorphic triangles with vertices $\mathbf{x}$, $\mathbf{y}$, and $\Theta^-_{\beta',\beta}$. Note that $\Theta^-_{\beta',\beta}=\Theta^+_{\beta,\beta'}$, and that the transition map

$$\Phi^\alpha_{\beta\to\beta'}\colon\mathcal{CFL}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z},\mathfrak{s})\to\mathcal{CFL}^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}',\mathbf{w},\mathbf{z},\mathfrak{s})$$

can be computed via the triangle map $F_{\alpha,\beta,\beta',\mathfrak{s}}(-\otimes\Theta^+_{\beta,\beta'})$. Observing that $F_{\alpha,\beta,\beta',\mathfrak{s}}$ and $F_{\beta',\alpha,\beta,\mathfrak{s}}$ count the exact same holomorphic triangles, we conclude that

$$F_{W_{\text{tr}},\mathcal{F}_{\text{tr}},\mathfrak{s}}(\mathbf{x}\otimes\mathbf{y})=\text{tr}(\mathbf{x}\otimes\Phi^\alpha_{\beta\to\beta'}(\mathbf{y})),$$

completing the proof. □

9.2. Compound 1- and 3-handle maps and some related counts of holomorphic curves.

Suppose that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ and $(\Sigma_0,\boldsymbol{\xi},\boldsymbol{\zeta},\mathbf{w}_0,\mathbf{z}_0)$ are two multi-pointed Heegaard diagrams, and that we have a choice of injection

$$i\colon\mathbf{w}_0\to\mathbf{z}.$$

Suppose further that $(\Sigma_0,\boldsymbol{\xi},\boldsymbol{\zeta},\mathbf{w}_0,\mathbf{z}_0)$ satisfies the following:

- (*D1*) The curves $\boldsymbol{\xi}$ can be related to the curves $\boldsymbol{\zeta}$ by a sequence of isotopies and handleslides in the complement of $\mathbf{w}_0$ and $\mathbf{z}_0$.
- (*D2*) Each $\mathbf{w}_0$-basepoint is contained in the same region of $\Sigma_0\setminus(\boldsymbol{\xi}\cup\boldsymbol{\zeta})$ as a $\mathbf{z}_0$-basepoint.
- (*D3*) $|\xi_i\cap\zeta_j|=2\delta_{ij}$, and $\xi_i\cap\zeta_i$ consists of two points that have relative Maslov grading 1.

Note that condition (*D1*) implies that $(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{w}_0, \mathbf{z}_0)$ is a diagram for an unlink $\mathbb{U}$ in $(\mathbb{S}^1 \times \mathbb{S}^2)^{\# g(\Sigma_0)}$, each of whose components contains exactly two basepoints. Condition (*D2*) implies that $\mathrm{gr}_{\mathbf{w}}(\mathbf{x}) = \mathrm{gr}_{\mathbf{z}}(\mathbf{x})$ for any intersection point $\mathbf{x} \in \mathbb{T}_\xi \cap \mathbb{T}_\zeta$. Finally, condition (*D3*) implies that there is a top-graded intersection point $\Theta^+_{\xi,\zeta} \in \mathbb{T}_\xi \cap \mathbb{T}_\zeta$, and a bottom-graded intersection point $\Theta^-_{\xi,\zeta} \in \mathbb{T}_\xi \cap \mathbb{T}_\zeta$.

We form the surface $\Sigma \#_i \Sigma_0$ by joining Σ and Σ_0 together with a connected sum tube for each point $w_0 \in \mathbf{w}_0$, which is attached near the points w_0 and $i(w_0)$. Let us write

$$\mathbf{z}' := (\mathbf{z} \setminus i(\mathbf{w}_0)) \cup \mathbf{z}_0.$$

There is an induced Heegaard diagram $(\Sigma \#_i \Sigma_0, \boldsymbol{\alpha} \cup \boldsymbol{\xi}, \boldsymbol{\beta} \cup \boldsymbol{\zeta}, \mathbf{w}, \mathbf{z}')$.

We define the *compound 1-handle map*

$$F_1^{\xi,\zeta} \colon \mathcal{CFL}^\infty_{J_s}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}) \to \mathcal{CFL}^\infty_{J_s(\mathbf{T})}(\Sigma \#_i \Sigma_0, \boldsymbol{\alpha} \cup \boldsymbol{\xi}, \boldsymbol{\beta} \cup \boldsymbol{\zeta}, \mathbf{w}, \mathbf{z}')$$

via the formula

$$F_1^{\xi,\zeta}(\mathbf{x}) := \mathbf{x} \times \Theta^+_{\xi,\zeta}$$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and extended $\mathcal{R}^\infty$-equivariantly. Here J_s and $J_s(\mathbf{T})$ are 1-parameter families of almost complex structures on $\Sigma \times [0,1] \times \mathbb{R}$ and $(\Sigma \#_i \Sigma_0) \times [0,1] \times \mathbb{R}$, respectively, that we will describe shortly.

Similarly there is a *compound 3-handle map*

$$F_3^{\xi,\zeta} \colon \mathcal{CFL}^\infty_{J_s(\mathbf{T})}(\Sigma \#_i \Sigma_0, \boldsymbol{\alpha} \cup \boldsymbol{\xi}, \boldsymbol{\beta} \cup \boldsymbol{\zeta}, \mathbf{w}, \mathbf{z}') \to \mathcal{CFL}^\infty_{J_s}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}),$$

defined via the formula

$$F_3^{\xi,\zeta}(\mathbf{x} \times \Theta) := \begin{cases} \mathbf{x} & \text{if } \Theta = \Theta^-_{\xi,\zeta}, \\ 0 & \text{if } \Theta \neq \Theta^-_{\xi,\zeta} \end{cases}$$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\Theta \in \mathbb{T}_\xi \cap \mathbb{T}_\zeta$, and extended $\mathcal{R}^\infty$-equivariantly.

We now wish to show that the compound 1-handle and 3-handle maps are chain maps. This involves an argument involving analyzing how holomorphic curves behave as one degenerates the almost complex structure. Lipshitz's cylindrical reformation of Heegaard Floer homology [26] provides the technical framework necessary to perform the analysis. Let us write $n = |\mathbf{w}_0|$, the number of connected sum tubes we add. Given almost complex structures J_s and J_s' on $\Sigma \times [0,1] \times \mathbb{R}$ and $\Sigma_0 \times [0,1] \times \mathbb{R}$ that are split in a neighborhood of the connected sum points, as well as an n-tuple of positive numbers $\mathbf{T} = (T_1, \dots, T_n)$, we can form an almost complex structure $J_s(\mathbf{T})$ on $(\Sigma \#_i \Sigma_0) \times [0,1] \times \mathbb{R}$ by inserting necks of length $T_1, \dots, T_n$ along the connected sum tubes.

Proposition 9.4. *Suppose that* $(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{w}_0, \mathbf{z}_0)$ *is a Heegaard diagram satisfying conditions (D1), (D2), and (D3). If* $\mathbf{T}$ *is an* n*-tuple of neck lengths, all of whose components are sufficiently large, the compound 1-handle map* $F_1^{\xi,\zeta}$ *and the compound 3-handle map* $F_3^{\xi,\zeta}$ *are chain maps.*

Proposition 9.4 follows from a careful analysis of how holomorphic curves in $(\Sigma \#_i \Sigma_0) \times [0,1] \times \mathbb{R}$ degenerate as one sends all components of $\mathbf{T}$ to $+\infty$, simultaneously. The technical details of the proof can be found in [55, Proposition 6.1].

There is an analogue of Proposition 9.4 for the holomorphic triangle maps, which we will need for our proof of the trace formula. Suppose that $(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0)$ is a Heegaard link triple satisfying the following:

(*T*) All three sub-diagrams of $(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0)$ satisfy conditions (*D1*), (*D2*), and (*D3*).

Note that condition (*T*) implies that there are top-graded intersection points $\Theta^+_{\xi,\zeta}$, $\Theta^+_{\xi,\tau}$, and $\Theta^+_{\zeta,\tau}$, as well as bottom-graded intersection points $\Theta^-_{\xi,\zeta}$, $\Theta^-_{\xi,\tau}$, and $\Theta^-_{\zeta,\tau}$.

Suppose $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ and $\mathcal{T}_0 = (\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0)$ are Heegaard link triples such that the latter satisfies condition (*T*) above, and that we have a fixed injection

$$i \colon \mathbf{w}_0 \to \mathbf{z}.$$

We can construct a surface $\Sigma \#_i \Sigma_0$ as we did before, as well as a Heegaard link triple

$$\mathcal{T} \# \mathcal{T}_0 := (\Sigma \#_i \Sigma_0, \boldsymbol{\alpha} \cup \boldsymbol{\xi}, \boldsymbol{\beta} \cup \boldsymbol{\zeta}, \boldsymbol{\gamma} \cup \boldsymbol{\tau}, \mathbf{w}, \mathbf{z}'),$$

where $\mathbf{z}' := (\mathbf{z} \setminus i(\mathbf{w}_0)) \cup \mathbf{z}_0$.

Using a Mayer–Vietoris argument, it is not hard to see that there is an isomorphism

$$\operatorname{Spin}^c(X_{\alpha\cup\xi,\beta\cup\zeta,\gamma\cup\tau}) \cong \operatorname{Spin}^c(X_{\alpha,\beta,\gamma}) \times \operatorname{Spin}^c(X_{\xi,\zeta,\tau}),$$

under which $\mathfrak{s}_{\mathbf{w}\cup\mathbf{w}_0}(\psi\#\psi_0)$ is identified with $(\mathfrak{s}_{\mathbf{w}}(\psi), \mathfrak{s}_{\mathbf{w}_0}(\psi_0))$. For triples $(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0)$ satisfying condition (T), the 4-manifold $X_{\xi,\zeta,\tau}$ becomes $\#^{g(\Sigma_0)}(\mathbb{S}^1 \times \mathbb{S}^3)$ once we glue in 3- and 4-handles along the boundary. In particular, there is a unique Spin^c structure $\mathfrak{s}_0 \in \operatorname{Spin}^c(X_{\xi,\zeta,\tau})$ which restricts to the torsion Spin^c structure on all three boundary components. If $\mathfrak{s} \in \operatorname{Spin}^c(X_{\alpha,\beta,\gamma})$, there is thus a well-defined Spin^c structure $\mathfrak{s}\#\mathfrak{s}_0 \in \operatorname{Spin}^c(X_{\alpha\cup\xi,\beta\cup\zeta,\gamma\cup\tau})$.

The holomorphic triangle counts from [55, Proposition 6.3] carry over to our present situation without change to imply the following:

Proposition 9.5. *Suppose that $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ and $\mathcal{T}_0 = (\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0)$ are Heegaard link triples such that the latter satisfies condition (T), and $i\colon \mathbf{w}_0 \to \mathbf{z}$ is a chosen injection. Let $\mathcal{T}\#\mathcal{T}_0$ denote the Heegaard link triple described above. Then, for a tuple of sufficiently large neck-lengths $\mathbf{T}$, the following hold:*

$$F_{\mathcal{T}\#\mathcal{T}_0,J(\mathbf{T}),\mathfrak{s}\#\mathfrak{s}_0}(F_1^{\xi,\zeta}(-)\otimes F_1^{\zeta,\tau}(-)) = F_1^{\xi,\tau}F_{\mathcal{T},J,\mathfrak{s}}(-\otimes -),$$
$$F_3^{\xi,\tau}F_{\mathcal{T}\#\mathcal{T}_0,J(\mathbf{T}),\mathfrak{s}\#\mathfrak{s}_0}(F_1^{\xi,\zeta}(-)\otimes -) = F_{\mathcal{T},J,\mathfrak{s}}(-\otimes F_3^{\zeta,\tau}(-)),$$
$$F_3^{\xi,\tau}F_{\mathcal{T}\#\mathcal{T}_0,J(\mathbf{T}),\mathfrak{s}\#\mathfrak{s}_0}(-\otimes F_1^{\zeta,\tau}(-)) = F_{\mathcal{T},J,\mathfrak{s}}(F_3^{\xi,\zeta}(-)\otimes -).$$

Remark 9.6. Consider the special case when

$$(\Sigma_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\tau}, \mathbf{w}_0, \mathbf{z}_0) = (\mathbb{S}^2, \xi, \zeta, \tau, \{w_0, w_0'\}, \{z_0, z_0'\})$$

is a Heegaard triple where w_0 and z_0, and also w_0' and z_0' are adjacent, and where any two of ξ, ζ, and τ intersect in two points. Then Propositions 9.4 and 9.5 imply more standard relations between the holomorphic disk and triangle counts, and the 1-handle and 3-handle maps, for 1-handles with feet attached near the $\mathbf{z}$ basepoints on the original Heegaard diagram. Compare [40, Theorem 2.14] and [53, Lemma 8.5 and Theorem 8.8].

Finally, we need an additional holomorphic triangle count, due to Manolescu and Ozsváth [29, Proposition 6.2], which is useful in the proof that the quasi-stabilization maps are well defined. Suppose that $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is a Heegaard triple, and write A for a distinguished component of $\Sigma \setminus \boldsymbol{\alpha}$. Let w and z be the two basepoints in A. Suppose α_s is a simple closed curve in A that divides A into two components, one of which contains w, and the other contains z. Let $p \in \alpha_s \setminus (\boldsymbol{\beta} \cup \boldsymbol{\gamma})$ be an arbitrary choice of point. We form the quasi-stabilized Heegaard triple $\mathcal{T}^+$ via the formula

$$\mathcal{T}^+ := (\Sigma, \boldsymbol{\alpha} \cup \{\alpha_s\}, \boldsymbol{\beta} \cup \{\beta_0\}, \boldsymbol{\gamma} \cup \{\gamma_0\}, \mathbf{w} \cup \{w_0\}, \mathbf{z} \cup \{z_0\}),$$

where α_s, β_0, γ_0, w_0, and z_0 are as shown in Figure 9.1. (Compare Figure 3.2). The curves β_0 and γ_0 are contained in a small disk centered at the point p. The basepoints w_0 and z_0 are both contained in this disk, and are in the regions bounded by β_0 and γ_0.

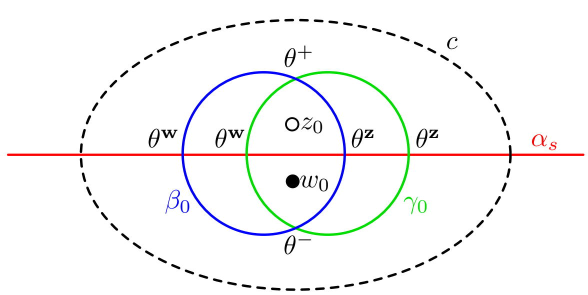

FIGURE 9.1. The quasi-stabilized Heegaard triple $\mathcal{T}^+ = (\Sigma, \boldsymbol{\alpha} \cup \{\alpha_s\}, \boldsymbol{\beta} \cup \{\beta_0\}, \boldsymbol{\gamma} \cup \{\gamma_0\}, \mathbf{w} \cup \{w_0\}, \mathbf{z} \cup \{z_0\})$ considered in Proposition 9.7.

Abusing notation slightly, write

$$\alpha_s \cap \beta_0 = \{\theta^{\mathbf{w}}, \theta^{\mathbf{z}}\}, \qquad \alpha_s \cap \gamma_0 = \{\theta^{\mathbf{w}}, \theta^{\mathbf{z}}\}, \qquad \text{and} \qquad \beta_0 \cap \gamma_0 = \{\theta^+, \theta^-\},$$

where $\theta^{\mathbf{w}}$ denotes the top $\mathrm{gr}_{\mathbf{w}}$-graded intersection point, and $\theta^{\mathbf{z}}$ denotes the top $\mathrm{gr}_{\mathbf{z}}$-graded intersection point. When the relative gradings coincide, we write θ^+ for the top-graded intersection point, and θ^- for the bottom.

We note that, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is a Heegaard triple and $(\Sigma, \boldsymbol{\alpha} \cup \{\alpha_s\}, \boldsymbol{\beta} \cup \{\beta_0\}, \boldsymbol{\gamma} \cup \{\gamma_0\}, \mathbf{w}, \mathbf{z})$ is a quasi-stabilization, then the 4-manifolds $X_{\alpha,\beta,\gamma}$ and $X_{\alpha\cup\{\alpha_0\},\beta\cup\{\beta_0\},\gamma\cup\{\gamma_0\}}$ are canonically diffeomorphic, since the handlebodies U_α, U_β, and U_γ are unchanged after we quasi-stabilize the triple. In particular,

$$\mathrm{Spin}^c(X_{\alpha,\beta,\gamma}) \cong \mathrm{Spin}^c(X_{\alpha\cup\{\alpha_0\},\beta\cup\{\beta_0\},\gamma\cup\{\gamma_0\}}).$$

We will need the following holomorphic curve count:

Proposition 9.7. *Suppose that $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ is a Heegaard link triple, and $\mathcal{T}^+$ is its quasi-stabilization. Write $\mathbb{U}$ for a doubly-based unknot containing the basepoints w_0 and z_0, with Seifert disk D, intersecting Σ in an arc connecting w_0 and z_0 disjoint from β_0 and γ_0. Then*

$$T^+_{w_0,z_0} F_{\mathcal{T},\mathfrak{s}}(\mathbf{x} \otimes \mathbf{y}) = F_{\mathcal{T}^+}(T^+_{w_0,z_0}(\mathbf{x}) \otimes \mathcal{B}^+_{\mathbb{U},D}(\mathbf{y})) \text{ and } S^+_{w_0,z_0} F_{\mathcal{T}}(\mathbf{x} \otimes \mathbf{y}) = F_{\mathcal{T}^+,\mathfrak{s}}(S^+_{w_0,z_0}(\mathbf{x}) \otimes \mathcal{B}^+_{\mathbb{U},D}(\mathbf{y})).$$

Proof. From the definitions of the maps $S^+_{w_0,z_0}$, $T^+_{w_0,z_0}$, and $\mathcal{B}^+_{\mathbb{U},D}$ (see Sections 3.3 and 3.5), the main claim is equivalent to the claim that

$$F_{\mathcal{T},\mathfrak{s}}(\mathbf{x} \otimes \mathbf{y}) \times \theta^{\mathbf{w}} = F_{\mathcal{T}^+,\mathfrak{s}}\left((\mathbf{x} \times \theta^{\mathbf{w}}) \otimes (\mathbf{y} \times \theta^+)\right) \text{ and } F_{\mathcal{T},\mathfrak{s}}(\mathbf{x} \otimes \mathbf{y}) \times \theta^{\mathbf{z}} = F_{\mathcal{T}^+,\mathfrak{s}}\left((\mathbf{x} \times \theta^{\mathbf{z}}) \otimes (\mathbf{y} \times \theta^+)\right),$$

which is exactly the statement of [29, Proposition 6.2]. □

9.3. **Twisted conjugate Heegaard diagrams for links.** Analogous to the proofs of the trace formulas in [55] and [22], to prove Theorem 9.3, we define a special kind of operation on a Heegaard diagram for a link, whose result we will call the *twisted conjugate* of the original. We describe the construction presently.

If $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a multi-based link, we write $\overline{\mathbb{L}}$ for the multi-based link $(L, \mathbf{z}, \mathbf{w})$, obtained by switching the roles of the basepoints. Given a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $(Y, \mathbb{L})$, we can obtain a diagram for $(Y, \overline{\mathbb{L}})$ by reversing the orientation of Σ, and switching the roles of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The resulting diagram $\bar{\mathcal{H}} := (\bar{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z}, \mathbf{w})$ is referred to as the *conjugate* of $\mathcal{H}$; see [38, Section 2.2] and [13, Section 6.1].

To obtain a diagram for $(Y, \mathbb{L})$, we can modify the embedding of $\bar{\Sigma}$ in a neighborhood of L. By isotoping $\bar{\Sigma}$ along L in the positive direction according to the orientation of L, we obtain the *positive twisted conjugate* diagram $Tw^+(\bar{\mathcal{H}})$. Analogously, if we twist in the negative direction, we obtain the *negative twisted conjugate* $Tw^-(\bar{\mathcal{H}})$. The diagrams $Tw^+(\bar{\mathcal{H}})$ and $Tw^-(\bar{\mathcal{H}})$ are illustrated in Figure 9.2. We write $Tw^+(\bar{\Sigma})$ and $Tw^-(\bar{\Sigma})$ for the underlying Heegaard surfaces of the twisted conjugate diagrams.

9.4. **Doubling Heegaard diagrams for links.** An additional type of operation on Heegaard diagrams we will encounter in the proof of Theorem 9.3 is *doubling*. In the case of links, if $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a diagram for $(Y, \mathbb{L})$, then there are four natural variations of the doubling procedure, producing four diagrams

$$D^{\mathbf{z}}_\alpha(\mathcal{H}), \quad D^{\mathbf{z}}_\beta(\mathcal{H}), \quad D^{\mathbf{w}}_\alpha(\mathcal{H}), \quad \text{and} \quad D^{\mathbf{w}}_\beta(\mathcal{H}). \tag{9.2}$$

For our purposes, it will be sufficient to consider any one of the four diagrams in equation (9.2). We focus on $D^{\mathbf{z}}_\alpha(\mathcal{H})$, whose construction we describe presently.

We first construct the underlying Heegaard surface $D^{\mathbf{z}}_\alpha(\Sigma)$. Let $N(\Sigma) = [-1,1] \times \Sigma$ denote a regular neighborhood of Σ in Y. Let $N(\mathbf{z})$ denote a regular neighborhood of the basepoints in Σ, which is a collection of disks, and let us write

$$\Sigma_0 := \Sigma \setminus N(\mathbf{z}).$$

We define

$$D^{\mathbf{z}}_{\alpha,0}(\Sigma) := \partial([-1,0] \times \Sigma_0).$$

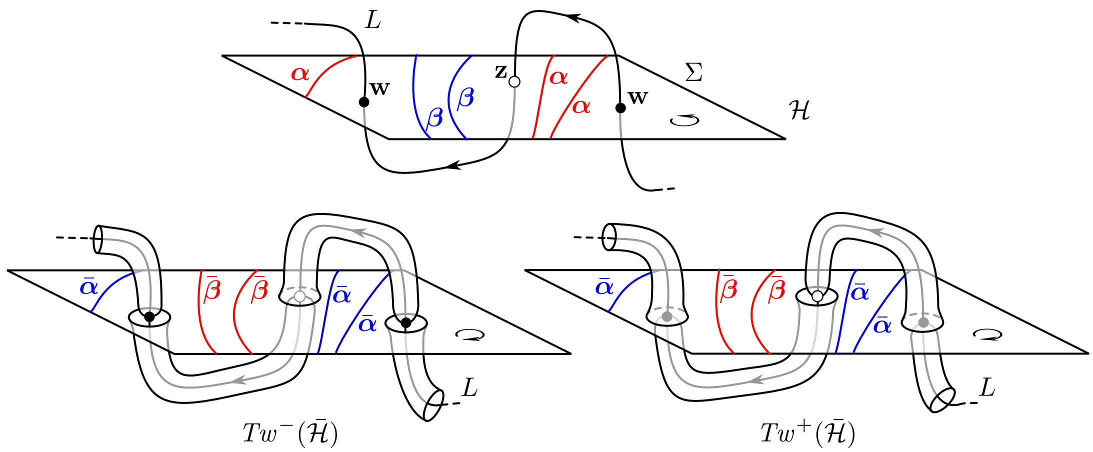

FIGURE 9.2. On top, we show a link diagram $\mathcal{H}$, and its twisted conjugates $Tw^-(\bar{\mathcal{H}})$ bottom left and $Tw^+(\bar{\mathcal{H}})$ bottom right.

We can view $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$ as being formed by gluing a copy of $\Sigma \setminus N(\mathbf{z})$ to $\bar{\Sigma} \setminus N(\mathbf{z})$ along their boundaries. Note that $D^{\mathbf{z}}_{\alpha,0}(\Sigma) \cap L$ consists of the original basepoints $\mathbf{w} \subseteq \Sigma$, as well as another collection of basepoints $\mathbf{z}'$, which are the images of $\mathbf{w}$ on $\bar{\Sigma}$.

We now describe the attaching curves on $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$. Let $m = |\mathbf{w}| = |\mathbf{z}|$. We pick embedded and pairwise disjoint arcs $\lambda_1, \dots, \lambda_m$ on Σ, each traveling from a $\mathbf{z}$-basepoint to a $\mathbf{w}$-basepoint. We assume further that each basepoint in $\mathbf{w} \cup \mathbf{z}$ is an endpoint of exactly one λ_i. We assume that the interiors of the λ_i are disjoint from $\mathbf{w} \cup \mathbf{z}$.

Next, we pick a collection A of subarcs of $\partial \Sigma_0$, such that each component of $\partial \Sigma_0$ contains exactly one subarc. We further require that A be disjoint from each λ_i. Pick a collection $d_1, \dots, d_n$ of properly embedded and pairwise disjoint arcs on Σ_0 that have both boundary components on A, are disjoint from the λ_i, and such that they form a basis of $H_1(\Sigma_0, A)$.

By doubling the arcs $d_1, \dots, d_n$ across the connected sum tubes onto all of $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$, we obtain n pairwise disjoint simple closed curves $\delta_1, \dots, \delta_n$ on $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$ that do not intersect the arcs λ_i. Let us write

$$\boldsymbol{\Delta} := \{\delta_1, \dots, \delta_n\}.$$

We can now define an initial version of the doubled diagram as

$$D^{\mathbf{z}}_{\alpha,0}(\mathcal{H}) := (D^{\mathbf{z}}_{\alpha,0}(\Sigma), \boldsymbol{\alpha} \cup \bar{\boldsymbol{\beta}}, \boldsymbol{\Delta}, \mathbf{w}, \mathbf{z}'),$$

where $\bar{\boldsymbol{\beta}}$ is the copy of $\boldsymbol{\beta}$ on $\bar{\Sigma}$.

Via an isotopy of Y supported in a neighborhood of L that fixes $\mathbf{w}$, we can move $\mathbf{z}'$ to $\mathbf{z}$. We let $D^{\mathbf{z}}_{\alpha}(\Sigma)$ denote the Heegaard surface obtained by isotoping $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$ in such a manner. The diagram $D^{\mathbf{z}}_{\alpha}(\mathcal{H})$ is similarly obtained by pushing forward the attaching curves on $D^{\mathbf{z}}_{\alpha}(\mathcal{H})$ under such an isotopy; see Figure 9.3.

A diagram $D^{\mathbf{z}}_{\beta}(\mathcal{H})$ can be constructed using a variation of the above construction, by having $[0,1] \times (\Sigma \setminus N(\mathbf{z}))$ play the role of the $\boldsymbol{\beta}$-handlebody. Diagrams $D^{\mathbf{w}}_{\alpha}(\mathcal{H})$ and $D^{\mathbf{w}}_{\beta}(\mathcal{H})$ can be defined similarly, by instead adding tubes near the $\mathbf{w}$-basepoints.

We now proceed to show that $D^{\mathbf{z}}_{\alpha}(\mathcal{H})$ is a valid Heegaard diagram for $(L, \mathbf{w}, \mathbf{z})$. Note that it is clearly sufficient to show that $D^{\mathbf{z}}_{\alpha,0}(\mathcal{H})$ is a valid Heegaard diagram for $(L, \mathbf{w}, \mathbf{z}')$. To this end, we prove the following fact about the $\boldsymbol{\Delta}$ curves:

Lemma 9.8. *The curves $\delta_1, \dots, \delta_n$ are homologically independent in both $H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma) \setminus N(\mathbf{w}))$ and $H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma) \setminus N(\mathbf{z}'))$.*

Proof. Let p_i denote the point $\lambda_i \cap \partial \Sigma_0$, and write $\mathbf{p} = \{p_1, \dots, p_m\}$. Since the λ_i, as well as their images on $\bar{\Sigma}$, are disjoint from the δ_j, if follows that $\delta_1, \dots, \delta_n$ are homologically independent in

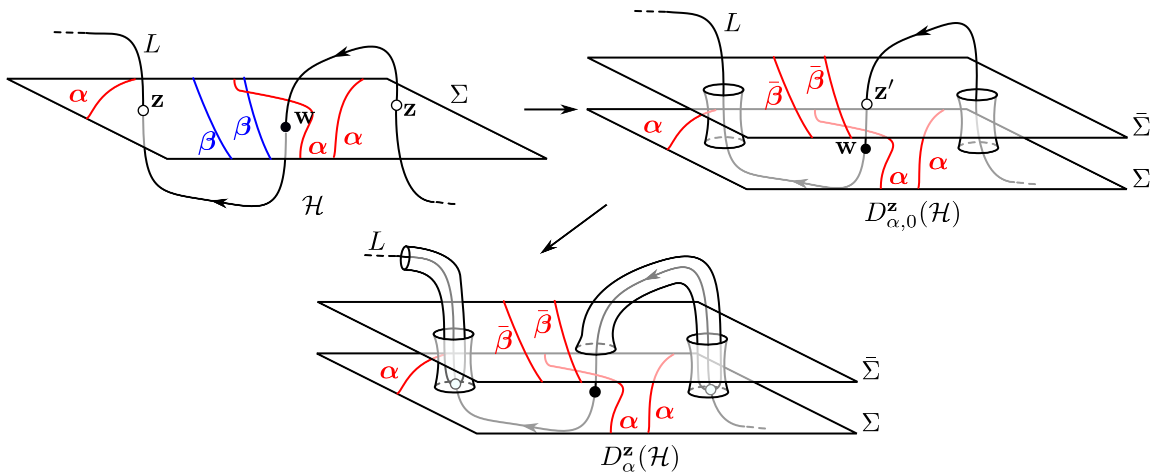

FIGURE 9.3. The link diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is shown top left, the preliminary double $D^{\mathbf{z}}_{\alpha,0}(\mathcal{H})$ top right, and the double $D^{\mathbf{z}}_{\alpha}(\mathcal{H})$ at the bottom. The curves $\boldsymbol{\Delta}$ bounding disks in the $\boldsymbol{\beta}$-handlebody of $D^{\mathbf{z}}_{\alpha}(\mathcal{H})$ (i.e., the region between the two copies of Σ) are not shown.

$H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{w}))$ if and only if they are independent in $H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{z}'))$, which in turn occurs if an only if they are homologically independent in $H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{p}))$. Noting that $D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{p})$ can be viewed as Σ_0 glued to $\bar{\Sigma}_0$ along the arcs A, we consider the sequence

$$H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{p})) \to H_1(D^{\mathbf{z}}_{\alpha,0}(\Sigma)\setminus N(\mathbf{p}), \bar{\Sigma}_0) \to H_1(\Sigma_0, A). \tag{9.3}$$

Here, the first map is induced by inclusion, and the second is the inverse of the excision isomorphism. Since the curves $\delta_1, \dots, \delta_n$ are mapped to $d_1, \dots, d_n$ by the composition of the two maps in equation (9.3), which are homologically independent in $H_1(\Sigma_0, A)$, we conclude that the curves $\delta_1, \dots, \delta_n$ are also homologically independent. □

Lemma 9.8 implies that $\boldsymbol{\Delta}$ is a valid set of attaching curves on $D^{\mathbf{z}}_{\alpha,0}(\Sigma)$, as the following basic lemma demonstrates:

Lemma 9.9. *Suppose that Σ_0 is a connected surface-with-boundary, and $\delta_1, \dots, \delta_n \subseteq \Sigma_0$ is a collection of pairwise disjoint simple closed curves, with $n = g(\Sigma_0) + |\partial\Sigma_0| - 1$. Then each component of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ is planar and contains exactly one component of $\partial\Sigma_0$ if and only if $\delta_1, \dots, \delta_n$ are homologically independent $H_1(\Sigma_0)$.*

Proof. Assume first that each component of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ is planar and contains exactly one component of $\partial\Sigma_0$. Each curve δ_i determines two boundary components of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$. A simple Mayer–Vietoris argument for gluing along these two boundary components shows that the curves $\delta_1, \dots, \delta_n$ are homologically independent in $H_1(\Sigma_0)$.

Conversely, suppose that $\delta_1, \dots, \delta_n$ are homologically independent in $H_1(\Sigma_0)$. We note that, if any component of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ does not contain a component of $\partial\Sigma_0$, then we obtain a non-trivial relation in $H_1(\Sigma_0)$ amongst the δ_i. Hence each component of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ contains at least one component of $\partial\Sigma_0$. If any component C of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ is non-planar or contains more than one component of $\partial\Sigma_0$, then we can pick a simple closed curve δ' in C such that $[\delta']$ is not in the span of the classes $[\delta_i]$, for $\delta_i \subseteq \partial C$. Using a Mayer–Vietoris argument, such a class $[\delta']$ remains homologically independent from the classes $[\delta_i]$ in $H_1(\Sigma_0)$, and is clearly disjoint from the curves $\delta_1, \dots, \delta_n$. However, it is easily verified that the maximal rank of a subspace of $H_1(\Sigma_0)$ on which the intersection form Q_{Σ_0} vanishes is $n = g(\Sigma_0) + |\partial\Sigma_0| - 1$, so such a curve δ' cannot exist, since we would obtain a subspace of rank $n+1$ on which Q_{Σ_0} vanished. Hence each component of $\Sigma_0 \setminus (\delta_1 \cup \cdots \cup \delta_n)$ must be planar and contain exactly one component of $\partial\Sigma_0$. □

9.5. **Transition maps and doubled Heegaard diagrams.** Suppose that $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ is a Heegaard diagram for $(Y,\mathbb{L})$. Let $D_\alpha^{\mathbf{z}}(\mathcal{H})=(D_\alpha^{\mathbf{z}}(\Sigma),\boldsymbol{\alpha}\cup\bar{\boldsymbol{\beta}},\boldsymbol{\Delta},\mathbf{w},\mathbf{z})$ denote the doubled diagram from Section 9.4, and let $Tw^-(\bar{\mathcal{H}})=(Tw^-(\bar{\Sigma}),\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\alpha}},\mathbf{w},\mathbf{z})$ be the negative twisted conjugate from Section 9.3. In this section, we describe compact formulas for the transition maps between the link Floer complexes for $\mathcal{H}$, $D_\alpha^{\mathbf{z}}(\mathcal{H})$, and $Tw^-(\bar{\mathcal{H}})$.

As a first observation, we note that $(D_\alpha^{\mathbf{z}}(\Sigma),\boldsymbol{\beta}\cup\bar{\boldsymbol{\beta}},\boldsymbol{\Delta},\mathbf{w},\mathbf{z})$ is a multi-pointed diagram for an unlink in $(\mathbb{S}^1\times\mathbb{S}^2)^{\#g(\Sigma)}$, where each component has exactly two basepoints. Hence there is a well-defined top-graded generator

$$[\Theta^+_{\beta\cup\bar{\beta},\Delta}]\in\mathcal{HFL}^-(D_\alpha^{\mathbf{z}}(\mathcal{H}),\boldsymbol{\beta}\cup\bar{\boldsymbol{\beta}},\boldsymbol{\Delta},\mathbf{w},\mathbf{z},\mathfrak{s}_0),$$

where $\mathfrak{s}_0$ is the torsion Spin^c structure on $(\mathbb{S}^1\times\mathbb{S}^2)^{\#g(\Sigma)}$.

Lemma 9.10. *If $\mathcal{H}$ is a Heegaard diagram and $D_\alpha^{\mathbf{z}}(\mathcal{H})$ is its double, then*

$$\Phi_{\mathcal{H}\to D_\alpha^{\mathbf{z}}(\mathcal{H})}(-)\simeq F_{\alpha\cup\bar{\beta},\beta\cup\bar{\beta},\Delta}(F_1^{\bar{\beta},\bar{\beta}}(-)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}).$$

Proof. The key observation is that the map $F_1^{\bar{\beta},\bar{\beta}}$ is equal to a composition of 1-handle maps, while

$$F_{\alpha\cup\bar{\beta},\beta\cup\bar{\beta},\Delta}(-\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta})$$

is chain homotopic to the 2-handle map for a collection of 2-handles that cancel the 1-handles which were added by $F_1^{\bar{\beta},\bar{\beta}}$. See [55, Proposition 7.2] for a detailed proof of a closely related result. ☐

Next, we need a simple formula for the transition map from $Tw^-(\bar{\mathcal{H}})$ to $D_\alpha^{\mathbf{z}}(\mathcal{H})$. A handle cancellation argument yields the following:

Lemma 9.11. *There is a chain homotopy*

$$\Phi_{Tw^-(\bar{\mathcal{H}})\to D_\alpha^{\mathbf{z}}(\mathcal{H})}(-)\simeq F_{\alpha\cup\bar{\beta},\alpha\cup\bar{\alpha},\Delta}(F_1^{\alpha,\alpha}(-)\otimes\Theta^+_{\alpha\cup\bar{\alpha},\Delta}).$$

Finally, we describe a formula for the transition map from $D_\alpha^{\mathbf{z}}(\mathcal{H})$ to $\mathcal{H}$, which is essentially just the dual of Lemma 9.10. Suppose $D_1,\dots,D_n$ are compressing disks attached to the $\bar{\Sigma}$ portion of $\Sigma\#_i\bar{\Sigma}$ that bound the curves in $\bar{\boldsymbol{\beta}}$. If we surger $\Sigma\#_i\bar{\Sigma}$ along the $\bar{\boldsymbol{\beta}}$ curves using the compressing disks $D_1,\dots,D_n$, we simply obtain the original Heegaard surface Σ (up to isotopy, relative to $L\cap\Sigma$). With this in mind, the handle cancellation argument used to prove Lemma 9.10 implies the following:

Lemma 9.12. *There is a chain homotopy*

$$\Phi_{D_\alpha^{\mathbf{z}}(\mathcal{H})\to\mathcal{H}}(-)\simeq F_3^{\bar{\beta},\bar{\beta}}F_{\alpha\cup\bar{\beta},\Delta,\beta\cup\bar{\beta}}(-\otimes\Theta^+_{\Delta,\beta\cup\bar{\beta}}).$$

9.6. **Intertwining maps and connected sums.** Suppose that $(Y_j,\mathbb{L}_j)$ for $j\in\{1,2\}$ is a 3-manifolds with a multi-based link, and $\mathcal{H}_j=(\Sigma_j,\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j,\mathbf{w}_j,\mathbf{z}_j)$ is a Heegaard diagram for $(Y_j,\mathbb{L}_j)$. Suppose also we have chosen a bijection i from $\mathbf{z}_1$ to $\mathbf{w}_2$. We can form the *generalized connected sum* of $(Y_1,\mathbb{L}_1)$ and $(Y_2,\mathbb{L}_2)$, for which we write $(Y_1\#_iY_2,\mathbb{L}_1\#_i\mathbb{L}_2)$, by deleting 3-balls centered at each point in $\mathbf{z}_1$ and $\mathbf{w}_2$, and gluing the boundary components according to our chosen bijection between $\mathbf{z}_1$ and $\mathbf{w}_2$. The link $\mathbb{L}_1\#_i\mathbb{L}_2$ is decorated with the basepoints $\mathbf{w}_1$ and $\mathbf{z}_2$.

We can construct a Heegaard surface $\Sigma_1\#_i\Sigma_2$ for $(Y_1\#_iY_2,\mathbb{L}_1\#_i\mathbb{L}_2)$ by adding a connected sum tube between Σ_1 and Σ_2 near each basepoint in $\mathbf{z}_1$ and the corresponding basepoint in $\mathbf{w}_2$. We define

$$\mathcal{H}_1\#_i\mathcal{H}_2:=(\Sigma_1\#_i\Sigma_2,\boldsymbol{\alpha}_1\cup\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1\cup\boldsymbol{\beta}_2,\mathbf{w}_1,\mathbf{z}_2)$$

for the resulting diagram.

Adapting the construction from [38, Section 6.2], we can define an *intertwining map*

$$\mathcal{G}\colon\mathcal{CFL}^\infty(\mathcal{H}_1,\mathfrak{s}_1)\otimes_{\mathcal{R}^\infty}\mathcal{CFL}^\infty(\mathcal{H}_2,\mathfrak{s}_2)\to\mathcal{CFL}^\infty(\mathcal{H}_1\#_i\mathcal{H}_2,\mathfrak{s}_1\#\mathfrak{s}_2),$$

via the formula

$$\mathcal{G}(-,-):=F_{\alpha_1\cup\alpha_2,\beta_1\cup\alpha_2,\beta_1\cup\beta_2}(F_1^{\alpha_2,\alpha_2}(-)\otimes F_1^{\beta_1,\beta_1}(-)).\tag{9.4}$$

We now show that the map $\mathcal{G}$ is chain homotopic to a link cobordism map. We define the decorated link cobordism $(W,\mathcal{F})$, as follows. Write $\mathbf{z}_1=\{z_{1,1},\dots,z_{1,n}\}$ and $\mathbf{w}_2=\{w_{2,1},\dots,w_{2,n}\}$, ordered such that $z_{1,i}$ and $w_{2,i}$ are paired. The 4-manifold W is obtained by attaching n 1-handles, such that the i^{th} 1-handle has one foot at $z_{1,i}$, and the other foot at $w_{2,i}$. We construct a surface S inside the 1-handle cobordism by attaching a band inside each 1-handle. We construct a dividing set $\mathcal{A} \subseteq S$ as follows. For each pair $(z_{1,i}, w_{2,i})$ we add a dividing arc a_i which has one end on Y_1, and the other on Y_2, and travels through the 1-handle connecting $z_{1,i}$ to $w_{2,i}$. One end of a_i occurs immediately after $z_{1,i}$, and the other end occurs immediately after $w_{2,i}$. The remaining arcs of $\mathcal{A}$ are of the form $I \times \{p\}$, for points $p \in L_1 \cup L_2$. We define $\mathcal{F} := (S, \mathcal{A})$; see Figure 9.4.

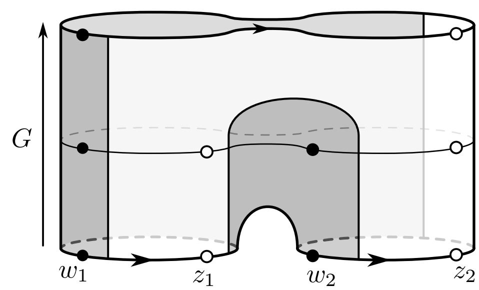

FIGURE 9.4. The decorated link cobordism used to define the map G, when $\mathbb{L}_1$ and $\mathbb{L}_2$ are both doubly-based knots. The orientation of $\mathbb{L}_1$ and $\mathbb{L}_2$ is shown.

We define the cobordism map

$$G := F_{W,\mathcal{F}}.$$

In the case when $\mathbb{L}_1$ and $\mathbb{L}_2$ are doubly-based knots, using the decomposition shown in Figure 9.4, we see

$$G \simeq S^-_{w_2,z_1} F^{\mathbf{w}}_B F_1, \tag{9.5}$$

where F_1 denotes the 1-handle map. More generally, if $\mathbb{L}_1$ and $\mathbb{L}_2$ have many basepoints, the cobordism map G is a composition of n terms which each have the form shown in equation (9.5).

Note that there is an asymmetry between Y_1 and Y_2 in the definition of G. At each pair of basepoints we delete, we could instead do a type-$\mathbf{z}$ band map, followed by the $T^-_{w_2,z_1}$ quasi-stabilization map. The corresponding decorated link cobordism is not diffeomorphic to $(W, \mathcal{F})$ (they can be distinguished by looking at the order in which the boundary components appear on the subsurface $S_{\mathbf{w}}$, with respect to the boundary orientation). There is a similar asymmetry also in the definition of $\mathcal{G}$ since the formula defining $\mathcal{G}$ is not invariant under switching the roles of Y_1 and Y_2.

Proposition 9.13. *The intertwining map $\mathcal{G}$ is chain homotopic to the link cobordism map G.*

Proof. The proof is similar to the proof of [55, Proposition 8.1]. The idea is that we exhibit a chain homotopy inverse of G, which we denote E, and show that

$$E \circ \mathcal{G} \simeq \mathrm{id}\,. \tag{9.6}$$

For notational simplicity, we restrict to the case when $\mathbb{L}_1$ and $\mathbb{L}_2$ are both doubly-based knots. The proof we present extends to the more general case by an elaboration of notation.

We define the map E via the formula

$$E := F_3 F^{\mathbf{w}}_B S^+_{w_2,z_1}. \tag{9.7}$$

We note E is the cobordism map for the decorated link cobordism obtained by turning around and reversing the orientation of the link cobordism used to define G. The fact that E and G are chain homotopy inverses of each other follows from [58, Proposition 5.1]. Using equation (9.7), we see equation (9.6) is equivalent to

$$F_3 F^{\mathbf{w}}_B S^+_{w_2,z_1} F_{\alpha_1\cup\alpha_2,\beta_1\cup\alpha_2,\beta_1\cup\beta_2}(F_1^{\alpha_2,\alpha_2} \otimes F_1^{\beta_1,\beta_1}) \simeq \mathrm{id}\,. \tag{9.8}$$

We pick two curves in the connected sum region of $\Sigma_1\#\Sigma_2$, which we label as ξ_s and ζ_0. The curve ζ_0 bounds a small disk containing the basepoints w_2 and z_1, while ξ_s wraps all the way around the connected sum tube; see Figure 9.5. We write

$$\xi_s\cap\zeta_0=\{\theta^{\mathbf{w}}_{\xi_s,\zeta_0},\theta^{\mathbf{z}}_{\xi_s,\zeta_0}\},$$

where $\theta^{\mathbf{w}}_{\xi_s,\zeta_0}$ is the top $\mathrm{gr}_{\mathbf{w}}$-graded intersection point and $\theta^{\mathbf{z}}_{\xi_s,\zeta_0}$ denotes the top $\mathrm{gr}_{\mathbf{z}}$-graded intersection point.

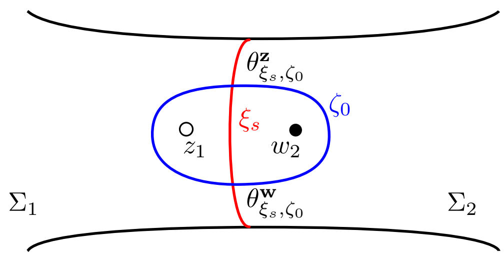

FIGURE 9.5. The connected sum region of $\Sigma_1\#\Sigma_2$.

The map $S^+_{w_2,z_1}$ appearing in equation (9.8) is defined by the equation

$$S^+_{w_2,z_1}(\mathbf{x}):=\mathbf{x}\times\theta^{\mathbf{w}}_{\xi_s,\zeta_0},$$

extended $\mathcal{R}^\infty$-equivariantly. Similarly, there is a birth map (corresponding to the cobordism map for a doubly-based unknot being born), given by the formula

$$\mathcal{B}^+(\mathbf{x}):=\mathbf{x}\times\theta^+_{\zeta_0,\zeta_0},$$

where $\theta^+_{\zeta_0,\zeta_0}$ denotes the top-graded intersection point of ζ_0 and a small Hamiltonian translate of ζ_0. Note that, to ease the notational burden, we will henceforth not distinguish between a curve and its Hamiltonian translate (though, implicitly, when we are counting holomorphic triangles, we must translate some of the curves using Hamiltonians).

We introduce the following shorthand notation for sets of attaching curves on $\Sigma_1\#\Sigma_2$:

$$L:=\boldsymbol{\alpha}_1\cup\boldsymbol{\alpha}_2,\qquad M:=\boldsymbol{\beta}_1\cup\boldsymbol{\alpha}_2,\qquad\text{and}\qquad R:=\boldsymbol{\beta}_1\cup\boldsymbol{\beta}_2.$$

We define

$$L_0:=\boldsymbol{\alpha}_1\cup\{\zeta_0\}\cup\boldsymbol{\alpha}_2\qquad\text{and}\qquad L_s:=\boldsymbol{\alpha}_1\cup\{\xi_s\}\cup\boldsymbol{\alpha}_2,$$

and we define M_0, M_s, R_0, and R_s similarly.

By Proposition 9.7,

$$\begin{aligned}&S^+_{w_2,z_1}F_{L,M,R}(F_1^{\alpha_2,\alpha_2}(-)\otimes F_1^{\beta_1,\beta_1}(-))\simeq\\&F_{L_s,M_0,R_0}(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes\mathcal{B}^+F_1^{\beta_1,\beta_1}(-)).\end{aligned}\tag{9.9}$$

The band map $F_B^{\mathbf{w}}$ is defined via the triangle count

$$F_B^{\mathbf{w}}(-):=F_{L_s,R_0,R_s}(-\otimes(\Theta^+_{R,R}\times\theta^{\mathbf{z}}_{\zeta_0,\xi_s})).\tag{9.10}$$

We note that

$$\Theta^+_{R,R}\times\theta^{\mathbf{z}}_{\zeta_0,\xi_s}=T^+_{w_2,z_1}(\Theta^+_{R,R}),$$

by definition, so equation (9.10) reads

$$F_B^{\mathbf{w}}(-)=F_{L_s,R_0,R_s}(-\otimes T^+_{w_2,z_1}(\Theta^+_{R,R})).\tag{9.11}$$

Combining equations (9.9) and (9.11), and using associativity, we see that

$$\begin{aligned}&F_B^{\mathbf{w}}S^+_{w_2,z_1}\mathcal{G}(-,-)\simeq\\&F_{L_s,R_0,R_s}\left(F_{L_s,M_0,R_0}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes\mathcal{B}^+F_1^{\beta_1,\beta_1}(-)\right)\otimes T^+_{w_2,z_1}(\Theta^+_{R,R})\right)\simeq\\&F_{L_s,M_0,R_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes F_{M_0,R_0,R_s}\left(\mathcal{B}^+F_1^{\beta_1,\beta_1}(-)\otimes T^+_{w_2,z_1}(\Theta^+_{R,R})\right)\right).\end{aligned}\tag{9.12}$$

Using Proposition 9.7, equation (9.12) becomes

$$F_{L_s,M_0,R_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes T^+_{w_2,z_1}F_{M,R,R}\left(F_1^{\beta_1,\beta_1}(-)\otimes \Theta^+_{R,R}\right)\right). \tag{9.13}$$

The expression $F_{M,R,R}(-\otimes \Theta^+_{R,R})$ is the change of diagrams map for shifting the curves R slightly, which we can safely delete, since we are already precomposing with a change of diagrams map on $\mathcal{CFL}^\infty(Y_1,\mathbb{L}_1,\mathfrak{s}_1)\otimes \mathcal{CFL}^\infty(Y_2,\mathbb{L}_2,\mathfrak{s}_2)$. Hence equation (9.13) becomes

$$F_{L_s,M_0,R_s}(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes T^+_{w_2,z_1}F_1^{\beta_1,\beta_1}(-)). \tag{9.14}$$

Define the map

$$\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}\colon \mathcal{CFL}^\infty(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2)\to \mathcal{CFL}^\infty(\Sigma_1,\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)\otimes \mathcal{CFL}^\infty(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2)$$

via the formula

$$\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(\mathbf{x}) := \Theta^+_{\beta_1,\beta_1}\otimes \mathbf{x},$$

extended $\mathcal{R}^\infty$-equivariantly. Next, we claim

$$T^+_{z_1,w_2}F_1^{\beta_1,\beta_1}(-)\simeq F^{\mathbf{w}}_{B'}F_1^{\xi_s,\xi_s}\,\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(-), \tag{9.15}$$

where

$$F^{\mathbf{w}}_{B'}\colon \mathcal{CFL}^\infty(\Sigma_1\#\Sigma_2,M_s,R_s)\to \mathcal{CFL}^\infty(\Sigma_1\#\Sigma_2,M_0,R_s)$$

is the band map

$$F^{\mathbf{w}}_{B'}(-) := F_{M_0,M_s,R_s}(T^+_{z_1,w_2}(\Theta^+_{M,M})\otimes -). \tag{9.16}$$

Note that $F^{\mathbf{w}}_{B'}$ is an α-band map, because the handlebody U_{M_0} is playing the role of the α-handlebody. Furthermore, the quasi-stabilization map $T^+_{z_1,w_2}$ appears in equation (9.16) instead of $T^+_{w_2,z_1}$ because U_{M_0} is now playing the role of the α-handlebody instead of the β-handlebody, so the induced orientation of the strands it contains are reversed, and hence, in this handlebody, z_1 now immediately follows w_2.

It is possible to establish equation (9.15) via a direct holomorphic triangle count. Indeed, by using Proposition 9.5, one could delete the portion added via the compound 1-handle map, and reduce the computation to a holomorphic triangle count supported in a disk, involving three isotopic attaching curves. A holomorphic triangle count could then be performed by using a neck-stretching argument, as in [58, Lemma 8.6].

A more conceptually enlightening approach for proving equation (9.15), and the approach we take, is to interpret the maps appearing as cobordism maps and use properties of the link Floer TQFT. We note that the map $\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}$ can be written as the composition of a single 0-handle map, followed by $|\boldsymbol{\beta}_1|$ 1-handle maps. After rearranging handles and canceling the 0-handle added by $\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}$ with the 1-handle added by $F_1^{\xi_s,\xi_s}$, the composition $F_1^{\xi_s,\xi_s}\,\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}$ can be rewritten as the composition of the cobordism map F_1 induced by attaching $|\boldsymbol{\beta}_1|$ 1-handles to Y_2, followed by a birth cobordism map, which adds the doubly-based knot $\mathbb{U}=(U,w_1,z_1)$. Hence, we can write

$$F_1^{\xi_s,\xi_s}\,\mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}\simeq \mathcal{B}^+_{\mathbb{U}}F_1. \tag{9.17}$$

Similarly,

$$F_1^{\beta_1,\beta_1}\simeq \phi_*F_1, \tag{9.18}$$

where ϕ is an isotopy of $Y_2\#(\mathbb{S}^1\times\mathbb{S}^2)^{\#|\boldsymbol{\beta}_1|}$ that moves the knot in Y_2 into the 1-handle region formed when we attach $(\Sigma_1,\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)$.

We can decompose the isotopy ϕ as the composition of an isotopy ϕ_0, which fixes w_2 and z_2 but moves the link, followed by an isotopy $\tau^{w_1\leftarrow w_2}$ that fixes the link setwise, is supported in a neighborhood of the link, fixes z_2, but moves w_2 to w_1. Using equations (9.17) and (9.18), we see that equation (9.15) is equivalent to

$$T^+_{z_1,w_2}\tau_*^{w_1\leftarrow w_2}(\phi_0)_*F_1\simeq F^{\mathbf{w}}_{B'}\mathcal{B}^+_{\mathbb{U}}F_1. \tag{9.19}$$

We now simply note that equation (3.10) implies $T^+_{z_1,w_2}\tau_*^{w_1\leftarrow w_2} \simeq T^+_{w_1,z_1}$, while equation (3.16) implies $F^{\mathbf{w}}_{B'}\mathcal{B}^+_{\mathbb{U}} \simeq T^+_{w_1,z_1}(\phi_0)_*$. Together, these establish equation (9.19), and hence also equation (9.15).

If we substitute the formula (9.16) for $F^{\mathbf{w}}_{B'}$ into equation (9.15), equation (9.14) becomes

$$F_{L_s,M_0,R_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes F_{M_0,M_s,R_s}\left(T^+_{z_1,w_2}(\Theta^+_{M,M})\otimes F_1^{\xi_s,\xi_s}\operatorname{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(-)\right)\right). \tag{9.20}$$

By associativity, we see that equation (9.20) is chain homotopic to

$$F_{L_s,M_s,R_s}\left(F_{L_s,M_0,M_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes T^+_{z_1,w_2}(\Theta^+_{M,M})\right)\otimes F_1^{\xi_s,\xi_s}\operatorname{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(-)\right) \tag{9.21}$$

After post-composing equation (9.21) with the 3-handle map $F_3^{\xi_s,\xi_s}$, and pulling the 3-handle map inside the outer triangle map using Proposition 9.5, we obtain that the composition $E\circ\mathcal{G}$ is chain homotopic to

$$F_{\mathcal{T}_1\sqcup\mathcal{T}_2}\left(F_3^{\xi_s,\xi_s}F_{L_s,M_0,M_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes T^+_{z_1,w_2}(\Theta^+_{M,M})\right)\otimes\operatorname{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(-)\right). \tag{9.22}$$

where $\mathcal{T}_1\sqcup\mathcal{T}_2$ denotes the disjoint union of the Heegaard triples $(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)$ and $(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2)$. Since the outer triangle map is on the disjoint union of Σ_1 and Σ_2, we direct our attention to the inner triangle map. We claim that

$$F_3^{\xi_s,\xi_s}F_{L_s,M_0,M_s}\left(S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}(-)\otimes T^+_{z_1,w_2}(\Theta^+_{M,M})\right)\simeq\operatorname{Top}^+_{(\Sigma_2,\alpha_2,\alpha_2)}. \tag{9.23}$$

Note that, by the definition of $F^{\mathbf{w}}_B$, the left-hand side of equation (9.23) is

$$F_3^{\xi_s,\xi_s}F^{\mathbf{w}}_B S^+_{w_2,z_1}F_1^{\alpha_2,\alpha_2}. \tag{9.24}$$

Our strategy for proving equation (9.23) will be to manipulate equation (9.24) using algebraic properties of the TQFT until it becomes the cobordism map for the disjoint union of the identity cobordism $I\times Y_1$, and a 4-dimensional handlebody bounding Y_{α_2,α_2}.

We can write $F_1^{\alpha_2,\alpha_2}$ as $(\phi_*)F_1$, where F_1 is a 1-handle cobordism, and ϕ is a diffeomorphism that moves a small portion of the link near z_1 into Y_{α_2,α_2}, and sends z_1 to z_2. Note that we can write ϕ as a composition $\rho^{z_1\to z_2}\circ\phi_0$, where ϕ_0 moves a small portion of K_1 near z_1, but fixes z_1, and $\rho^{z_1\to z_2}$ is a diffeomorphism that is fixed outside a neighborhood of the subarc of $\phi_0(K_1)$ containing z_1 and z_2, but sends z_1 to z_2. We note that the map $\rho_*^{z_1\to z_2}$ satisfies the relation

$$\rho_*^{z_1\to z_2}\simeq S^-_{w_2,z_1}T^+_{z_2,w_2}, \tag{9.25}$$

by [58, Lemma 4.25]; cf. equation (3.11) and Figure 3.6.

We perform the following manipulation:

$$\begin{aligned}\phi_* &\simeq \rho_*^{z_1\to z_2}(\phi_0)_*\\ &\simeq S^-_{w_2,z_1}T^+_{z_2,w_2}(\phi_0)_* &&\text{(equation (9.25))}\\ &\simeq S^-_{w_2,z_1}F^{\mathbf{w}}_B\mathcal{B}^+_{\mathbb{U}} &&\text{(equation (3.17))},\end{aligned} \tag{9.26}$$

where $\mathbb{U}$ denotes (U,w_2,z_2). Hence equation (9.24) becomes

$$F_3^{\xi_s,\xi_s}F^{\mathbf{w}}_B S^+_{w_2,z_1}S^-_{w_2,z_1}F^{\mathbf{w}}_B\mathcal{B}^+_{\mathbb{U}}F_1. \tag{9.27}$$

We note that the 1-handles of F_1 can be moved to the left of all the other maps. After moving F_1 to the left, the birth cobordism map $\mathcal{B}^+_{\mathbb{U}}$ becomes the composition of a 0-handle map F_0 (which adds a 4-ball containing $\mathbb{U}$), and the 1-handle map $F_1^{\xi_s,\xi_s}$. We also note that $S^+_{w_2,z_1}S^-_{w_2,z_1}\simeq\Phi_{w_2}$ by equation (3.8). Hence equation (9.27) becomes

$$F_1'F_3^{\xi_s,\xi_s}F^{\mathbf{w}}_B\Phi_{w_2}F^{\mathbf{w}}_B F_1^{\xi_s,\xi_s}F_0, \tag{9.28}$$

where F_1' denotes the cobordism map for attaching $|\boldsymbol{\alpha}_2|$ 1-handles to the 3-sphere added by F_0. Using equation (3.19), we can reduce equation (9.28) to the expression

$$F_1'F_0,$$

which is clearly just $\operatorname{Top}^+_{(\Sigma_2,\alpha_2,\alpha_2)}$, establishing equation (9.23).

Applying the relation from equation (9.23) to equation (9.22), it follows that $E \circ \mathcal{G}$ is chain homotopic to

$$F_{\mathcal{T}_1 \sqcup \mathcal{T}_2}\left(\mathrm{Top}^+_{(\Sigma_2,\alpha_2,\alpha_2)}(-) \otimes \mathrm{Top}^+_{(\Sigma_1,\beta_1,\beta_1)}(-)\right).$$

This holomorphic triangle count appears on the disjoint union of Σ_1 and Σ_2, and is clearly just the tensor product $\Phi^{\alpha_1}_{\beta_1 \to \beta_1} \otimes \Phi^{\alpha_2 \to \alpha_2}_{\beta_2}$, completing the proof. ☐

Remark 9.14. There is another chain homotopy equivalence E' from the connected sum to the disjoint union, defined via the formula $E' := F_3 F_B^{\mathbf{z}} T^+_{w_2,z_1}$. The map E' corresponds to a pair-of-pants link cobordism where the type-$\mathbf{w}$ and type-$\mathbf{z}$ regions have been switched from the cobordism corresponding to E. One might expect the above argument to also go through using E' to try to cancel $\mathcal{G}$, by just replacing each type-T quasi-stabilization map with a type-S quasi-stabilization map, and replacing each type-$\mathbf{w}$ band map with a type-$\mathbf{z}$ band map. However, the careful reader will discover that such a strategy fails at equation (9.15).

9.7. **Proof of the triangle cobordism formula.** We now prove that the cobordism map induced by $(X_{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$ is chain homotopic to the holomorphic triangle map:

Proof of Theorem 9.3. A handlebody description of the 4-manifold $X_{\alpha,\beta,\gamma}$ is given in the proof of [55, Theorem 9.1]. Let f_β denote a Morse function on the handlebody U_β compatible with the attaching curves $\boldsymbol{\beta} \subseteq \Sigma$, which has Σ as a maximal level set. The 4-manifold $X_{\alpha,\beta,\gamma}$ has the following handlebody description:

- A 1-handle for each index 0 critical point of f_β, with one foot at the critical point in $\bar{U}_\beta \subseteq Y_{\alpha,\beta}$, and the other foot at its image in $U_\beta \subseteq Y_{\beta,\gamma}$.
- A 2-handle for each index 1 critical point of f_β. The attaching sphere is equal to the union of the corresponding descending manifold in U_β, concatenated across the connected sum tubes with its mirror image in $\bar{U}_\beta$. There is a canonical framing specified by taking an arbitrary framing in the portion in U_β, and mirroring it in the portion in $\bar{U}_\beta$.

Let W_1 denote the 1-handle cobordism, and let W_2 denote the 2-handle cobordism. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the intersection of the decorated surface $\mathcal{F}_{\alpha,\beta,\gamma}$ with W_1 and W_2, respectively. Note that $\mathcal{F}_1$ is obtained by attaching a collection of bands, one for each 1-handle, each containing a single dividing arc that meets both $Y_{\alpha,\beta}$ and $Y_{\beta,\gamma}$. Hence, we can write

$$F_{X_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma}} \simeq F_{W_2,\mathcal{F}_2} \circ F_{W_1,\mathcal{F}_1}. \tag{9.29}$$

Let $\mathcal{H}_{\alpha,\beta} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, and let $Tw^-(\bar{\mathcal{H}}_{\beta,\gamma}) = (Tw^-(\bar{\Sigma}), \bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\beta}}, \mathbf{w}, \mathbf{z})$ denote the negative twisted conjugate of the diagram $\mathcal{H}_{\beta,\gamma} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ described in Section 9.3. To show the main claim, we need to compute the cobordism map starting at the diagram $\mathcal{H}_{\alpha,\beta} \sqcup \mathcal{H}_{\beta,\gamma}$. However, for the computation, it is more convenient to start at the diagram $\mathcal{H}_{\alpha,\beta} \sqcup Tw^-(\bar{\mathcal{H}}_{\beta,\gamma})$. Hence, we precompose with the change of diagrams map $\mathrm{id} \otimes \Phi_{\mathcal{H}_{\beta,\gamma} \to Tw^-(\bar{\mathcal{H}}_{\beta,\gamma})}$. To simplify notation, we will omit writing this change of diagrams map for most of the proof, though it will reappear at the end.

We note that, by Proposition 9.13,

$$F_{W_1,\mathcal{F}_1} \simeq \mathcal{G}, \tag{9.30}$$

where

$$\mathcal{G}\colon \mathcal{CFL}^\infty(\mathcal{H}_{\alpha,\beta}) \otimes \mathcal{CFL}^\infty(Tw^-(\bar{\mathcal{H}}_{\beta,\gamma})) \to \mathcal{CFL}^\infty(\Sigma \#_i Tw^-(\bar{\Sigma}), \boldsymbol{\alpha} \cup \bar{\boldsymbol{\gamma}}, \boldsymbol{\beta} \cup \bar{\boldsymbol{\beta}}, \mathbf{w}, \mathbf{z})$$

is the intertwining map defined by the equation

$$\mathcal{G}(-,-) := F_{\alpha \cup \bar{\gamma}, \beta \cup \bar{\gamma}, \beta \cup \bar{\beta}}(F_1^{\bar{\gamma},\bar{\gamma}}(-) \otimes F_1^{\beta,\beta}(-)).$$

Hence

$$F_{X_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma}} \simeq F_{W_2,\mathcal{F}_2} \circ \mathcal{G}. \tag{9.31}$$

Next, it is not hard to see that the Heegaard triple $(\Sigma\#_i Tw^-(\bar{\Sigma}),\boldsymbol{\alpha}\cup\bar{\boldsymbol{\gamma}},\boldsymbol{\beta}\cup\bar{\boldsymbol{\beta}},\boldsymbol{\Delta},\mathbf{w},\mathbf{z})$ can be used to compute the 2-handle cobordism map $F_{W_2,\mathcal{F}_2}$, where $\boldsymbol{\Delta}$ was defined in Section 9.4; see [55, Lemma 7.7] for a detailed argument in a very closely related context. It follows that

$$F_{W_2,\mathcal{F}_2}\simeq F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}(-\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}).$$

Hence, omitting the initial factor of $\mathrm{id}\otimes\Phi_{\mathcal{H}_{\beta,\gamma}\to Tw^-(\bar{\mathcal{H}}_{\beta,\gamma})}$, we have, by associativity,

$$\begin{aligned}F_{X_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma}}(-,-)&\simeq(F_{W_2,\mathcal{F}_2}\circ\mathcal{G})(-,-)\\ &\simeq F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}\left(F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\gamma},\beta\cup\bar{\beta}}\left(F_1^{\bar{\gamma},\bar{\gamma}}(-)\otimes F_1^{\beta,\beta}(-)\right)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}\right)\\ &\simeq F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\gamma},\Delta}\left(F_1^{\bar{\gamma},\bar{\gamma}}(-)\otimes F_{\beta\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}\left(F_1^{\beta,\beta}(-)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}\right)\right).\end{aligned}\tag{9.32}$$

It is not hard to see that the Heegaard diagram corresponding to the codomain of the map in equation (9.32) is the double $D^{\mathbf{z}}_\alpha(\mathcal{H}_{\alpha,\gamma})$ of the diagram $\mathcal{H}_{\alpha,\gamma}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma},\mathbf{w},\mathbf{z})$, so we must postcompose with the transition map $\Phi_{D^{\mathbf{z}}_\alpha(\mathcal{H}_{\alpha,\gamma})\to\mathcal{H}_{\alpha,\gamma}}$, which we computed in Lemma 9.12. Accordingly, our expression from equation (9.32) for $F_{X_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma}}$ becomes

$$F_3^{\bar{\gamma},\bar{\gamma}}F_{\alpha\cup\bar{\gamma},\Delta,\gamma\cup\bar{\gamma}}\left(F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\gamma},\Delta}\left(F_1^{\bar{\gamma},\bar{\gamma}}(-)\otimes F_{\beta\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}\left(F_1^{\beta,\beta}(-)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}\right)\right)\otimes\Theta^+_{\Delta,\gamma\cup\bar{\gamma}}\right).\tag{9.33}$$

Associativity implies that equation (9.33) is chain homotopic to

$$F_3^{\bar{\gamma},\bar{\gamma}}F_{\alpha\cup\bar{\gamma},\beta\cup\bar{\gamma},\gamma\cup\bar{\gamma}}\left(F_1^{\bar{\gamma},\bar{\gamma}}(-)\otimes F_{\beta\cup\bar{\gamma},\Delta,\gamma\cup\bar{\gamma}}\left(F_{\beta\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}\left(F_1^{\beta,\beta}(-)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}\right)\otimes\Theta^+_{\Delta,\gamma\cup\bar{\gamma}}\right)\right).\tag{9.34}$$

Using Proposition 9.5, we see equation (9.34) is chain homotopic to

$$F_{\alpha,\beta,\gamma}\left(-\otimes F_3^{\bar{\gamma},\bar{\gamma}}\left(F_{\beta\cup\bar{\gamma},\Delta,\gamma\cup\bar{\gamma}}\left(F_{\beta\cup\bar{\gamma},\beta\cup\bar{\beta},\Delta}\left(F_1^{\beta,\beta}(-)\otimes\Theta^+_{\beta\cup\bar{\beta},\Delta}\right)\otimes\Theta^+_{\Delta,\gamma\cup\bar{\gamma}}\right)\right)\right).\tag{9.35}$$

Lemmas 9.11 and 9.12 imply that equation (9.35) is chain homotopic to

$$F_{\alpha,\beta,\gamma}(-\otimes\Phi_{Tw^-(\bar{\mathcal{H}}_{\beta,\gamma})\to\mathcal{H}_{\beta,\gamma}}).$$

The transition map inside the triangle map cancels the initial factor of $\mathrm{id}\otimes\Phi_{\mathcal{H}_{\beta,\gamma}\to Tw^-(\mathcal{H}_{\beta,\gamma})}$, which we have been omitting writing up until now. Hence $F_{X_{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma}}\simeq F_{\alpha,\beta,\gamma}$, concluding the proof. ☐

10. Examples

In this section, we perform some model computations to illustrate our invariants defined in Section 4 for pairs of slice disks. Our two main examples will be slice disks constructed by roll-spinning, and deform-spinning using the rigid motion deformation from Section 8. See Section 2.1 for the definitions of roll-spinning and deform-spinning.

10.1. Invariants of deform-spun slice disks.

As stated in the introduction, our main computational results rely on a formula for the fundamental principal invariants of deform-spun slice disks, generalizing [23, Theorem 5.1] from the hat to the full infinity version of knot Floer homology:

Theorem 1.2. *Let $D_{K,\varphi}$ be a slice disk of the knot $-K\#K$, obtained by deform-spinning a knot K in $\mathbb{S}^3$ using an automorphism φ of $(\mathbb{S}^3,K)$. Let w and z be basepoints on K, and write $C:=\mathcal{CFL}^\infty(K,w,z)$. Then*

$$E\circ\mathbf{t}^\infty_{D_{K,\varphi}}\simeq(\mathrm{id}\otimes\varphi_*)\circ\mathrm{cotr}\in\mathrm{Hom}_{\mathcal{R}^\infty}(\mathcal{R}^\infty,C^\vee\otimes C),$$

where the chain homotopy equivalence $E\colon\mathcal{CFL}^\infty(-K\#K,w,z)\to C^\vee\otimes C$ is described in [56, Section 5].

Proof. This follows from the trace formula in Theorem 9.1, using the same argument as the proof of [23, Theorem 5.1]. ☐

We now prove the following:

Proposition 10.1. *Let $D_{K,\mathrm{id}}$ and $D_{K,r}$ be the canonical and the 1-roll-spun slice disks of $-K\#K$, respectively. Then*

$$\tau(D_{K,\mathrm{id}}, D_{K,r}) \le 1.$$

Proof. Let w and z be basepoints on K, and write $\mathbb{K} = (K, w, z)$ and $-\mathbb{K}\#\mathbb{K} = (-K\#K, w, z)$. By Lemma 4.6, we can calculate $\tau(D_{K,\mathrm{id}}, D_{K,r})$ using $\mathit{HFK}^-_{V=0}(-\mathbb{K}\#\mathbb{K})$. By Theorem 1.2,

$$E \circ t^-_{D_{K,\mathrm{id}}} \simeq \mathrm{cotr} \in \mathrm{Hom}\left(\mathbb{F}_2[U], \mathit{CFK}^-_{V=0}(\mathbb{K})^\vee \otimes_{\mathbb{F}_2[U]} \mathit{CFK}^-_{V=0}(\mathbb{K})\right). \tag{10.1}$$

Furthermore, $r_* \simeq \mathrm{id} + \Phi_w \circ \Psi_z$ by [54, Theorem B], so

$$E \circ t^-_{D_{K,r}} \simeq (\mathrm{id} \otimes (\mathrm{id} + \Phi_w \circ \Psi_z)) \circ \mathrm{cotr}. \tag{10.2}$$

Hence, if we can show that $U \cdot \Phi_w \circ \Psi_z$ is U-equivariantly chain homotopic to zero, then

$$U \cdot t^-_{D_{K,\mathrm{id}}} \simeq U \cdot t^-_{D_{K,r}},$$

so $\tau(D_{K,\mathrm{id}}, D_{K,r}) \le 1$. We note that Φ_w has a simple algebraic interpretation on $\mathit{CFK}^-_{V=0}(\mathbb{K})$. It is the map obtained by writing the differential as a matrix with entries in $\mathbb{F}_2[U]$, and then differentiating each entry; cf. equation (3.1). According to [14, Proposition 6.3], since $\mathit{CFK}^-_{V=0}(\mathbb{K})$ is a finitely generated, free, $\mathbb{Z}$-graded chain complex over $\mathbb{F}_2[U]$, the map $U \cdot \Phi_w$ is U-equivariantly chain homotopic to zero, and hence so is $U \cdot \Phi_w \circ \Psi_z$. The claim follows. ☐

Question 10.2. In light of Proposition 10.1, it is natural to ask whether $\mu_{\mathrm{st}}(D_{K,\mathrm{id}}, D_{K,r}) \le 1$ for any roll-spun slice disks $D_{K,\mathrm{id}}$ and $D_{K,r}$; cf. Conjecture 2.22. This would give a topological proof of Proposition 10.1 by Theorem 5.13.

10.2. **Computational examples.** In this section, we compute the invariants τ, V_k, and Υ for several pairs of deform-spun slice disks. We begin by considering the complex $\mathcal{CFL}^\infty(4_1)$ for the figure-eight knot 4_1, which is shown in Figure 10.1.

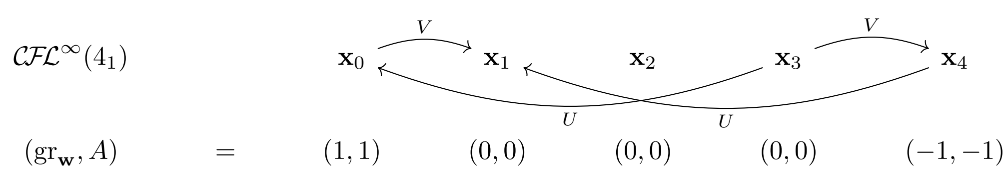

FIGURE 10.1. The complex $\mathcal{CFL}^\infty(\mathbb{S}^3, 4_1)$.

Lemma 10.3. *Let K be the figure-eight knot, and let $D_{K,\mathrm{id}}$ and $D_{K,r}$ denote the canonical and the 1-roll-spun slice disk slice disks of $-4_1\#4_1$. Then*

$$\tau(D_{K,\mathrm{id}}, D_{K,r}) = 1,\ V_0(D_{K,\mathrm{id}}, D_{K,r}) = 1,\ \textit{and}\ V_1(D_{K,\mathrm{id}}, D_{K,r}) = 0.$$

Proof. The Alexander filtered chain complex $\widehat{\mathit{CFK}}^{\mathit{fil}}(4_1)$, obtained by setting $U = 0$ and $V = 1$, has the form

$$\widehat{\mathit{CFK}}^{\mathit{fil}}(4_1) = \big((\mathbf{x}_0)_1 \longrightarrow (\mathbf{x}_1)_0 \qquad (\mathbf{x}_2)_0 \qquad (\mathbf{x}_3)_0 \longrightarrow (\mathbf{x}_4)_{-1}\big).$$

The notation $(\mathbf{x}_i)_j$ means the intersection points $\mathbf{x}_i$, which has Alexander grading j. Using equations (3.1) and (3.2),

$$(\Phi_w \circ \Psi_z)(\mathbf{x}_3) = \mathbf{x}_1,$$

and $\Phi_w \circ \Psi_z$ vanishes on all other generators. The complex $\widehat{\mathit{CFK}}^{\mathit{fil}}(-4_1)$ is obtained by dualizing $\widehat{\mathit{CFK}}^{\mathit{fil}}(4_1)$ (note that, although 4_1 is amphichiral, to compute the trace formula, it is better to ignore this fact). Hence

$$\widehat{\mathit{CFK}}^{\mathit{fil}}(-4_1) = \big((\mathbf{x}_0^\vee)_{-1} \longleftarrow (\mathbf{x}_1^\vee)_0 \qquad (\mathbf{x}_2^\vee)_0 \qquad (\mathbf{x}_3^\vee)_0 \longleftarrow (\mathbf{x}_4^\vee)_1\big).$$

Using equations (10.1) and (10.2),

$$E(\widehat{t}_{D_{K,r}}(1)-\widehat{t}_{D_{K,\mathrm{id}}}(1))=(\mathrm{id}\otimes(\Phi_w\circ\Psi_z))\circ \mathrm{cotr}(1)=\mathbf{x}_3^\vee\otimes(\Phi_w\circ\Psi_z)(\mathbf{x}_3)=\mathbf{x}_3^\vee\otimes\mathbf{x}_1.$$

We now observe that $\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ is nonzero in the homology of the 0-filtration level

$$\mathcal{G}_0\left(\widehat{CFK}^{fil}(-4_1)\otimes_{\mathbb{F}_2}\widehat{CFK}^{fil}(4_1)\right),$$

where $\mathcal{G}_i$ denotes Alexander filtration level i. Indeed, the only elements mapped to $\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ by the differential are $\mathbf{x}_4^\vee\otimes\mathbf{x}_1$ and $\mathbf{x}_3^\vee\otimes\mathbf{x}_0$. However, neither of these are in $\mathcal{G}_0$; instead, they are in $\mathcal{G}_1$. Hence $\tau(D_{K,\mathrm{id}},D_{K,r})=1$.

We now consider the invariants V_0 and V_1. The complex $A_0^-(-4_1\#4_1)$ has 25 generators over $\mathbb{F}_2[\widehat{U}]$. The generators are the monomials $U^nV^m\cdot\mathbf{x}_i^\vee\otimes\mathbf{x}_j$, where n, $m\geq 0$, and

$$A(\mathbf{x}_i^\vee)+A(\mathbf{x}_j)+m-n=0. \tag{10.3}$$

Similarly, $A_1^-(-4_1\#4_1)$ is generated by the monomials $U^nV^m\cdot\mathbf{x}_i^\vee\otimes\mathbf{x}_j$, where $n\geq 0$, $m\geq -1$, and satisfy equation (10.3).

As above, we can identify $t^-_{D_{K,r}}(1)-t^-_{D_{K,\mathrm{id}}}(1)$ with $U^0V^0\cdot\mathbf{x}_3^\vee\otimes\mathbf{x}_1$. It is straightforward to see that $U^0V^0\cdot\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ is not a boundary in $A_0^-(-4_1\#4_1)$, so $V_0\geq 1$. However,

$$\partial(U^1V^0\cdot\mathbf{x}_4^\vee\otimes\mathbf{x}_1)=U^1V^1\cdot\mathbf{x}_3^\vee\otimes\mathbf{x}_1:=\widehat{U}\cdot\mathbf{x}_3^\vee\otimes\mathbf{x}_1 \text{ and}$$
$$\partial(U^0V^{-1}\cdot\mathbf{x}_4^\vee\otimes\mathbf{x}_1)=\mathbf{x}_3^\vee\otimes\mathbf{x}_1,$$

implying that $V_0\leq 1$ and $V_1=0$. □

Lemma 10.4. *Let K denote the figure-eight knot, and let $D_{K,\mathrm{id}}$ and $D_{K,r}$ be as in Lemma 10.3. Then $\Upsilon_{(D_{K,\mathrm{id}},D_{K,r})}(t)$ takes the form shown in Figure 10.2.*

Proof. The proof is similar to Lemma 10.3. Therein, we computed $t^-_{D_{K,r}}(1)-t^-_{D_K,\mathrm{id}}(1)$ to be $\mathbf{x}_3^\vee\otimes\mathbf{x}_1\in CFK^\infty(4_1)^\vee\otimes CFK^\infty(4_1)$. The two elements $y_1=V^{-1}\mathbf{x}_4^\vee\otimes\mathbf{x}_1$ and $y_2=U^{-1}\mathbf{x}_3^\vee\otimes\mathbf{x}_0$ lie in homogeneous $(\mathrm{gr}_{\mathbf{w}},A)$-grading $(-1,0)$. It is straightforward to check that for $t\in[0,1]$, y_1 satisfies $\partial y_1=\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ and $y_1\in\mathcal{G}_t^t(\bar{4}_1\#4_1)$. Furthermore, if $s<t$, then there are no elements $z\in\mathcal{G}_s^t(\bar{4}_1\#4_1)$ such that $\partial z=\mathbf{x}_3^\vee\otimes\mathbf{x}_1$. Similarly, if $t\in[1,2]$, then $\partial y_2=\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ and $y_2\in\mathcal{G}_{2-t}^t(\bar{4}_1\#4_1)$, and there are no elements $z\in\mathcal{G}_s^t(\bar{4}_1\#4_1)$ such that $\partial z=\mathbf{x}_3^\vee\otimes\mathbf{x}_1$ for $s<2-t$. □

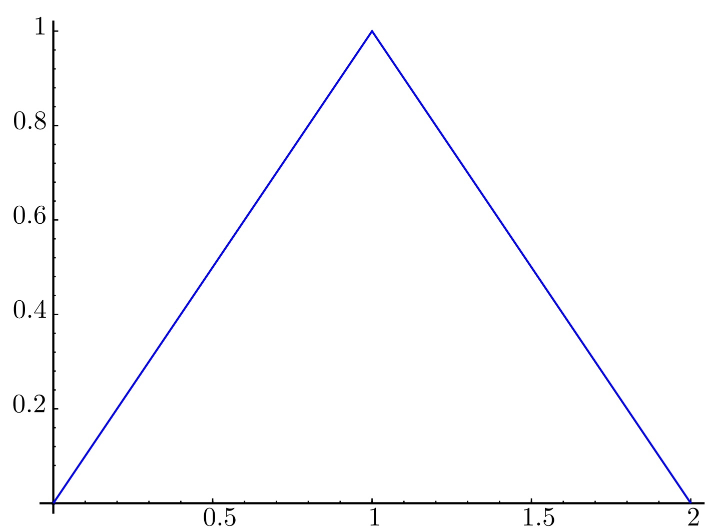

FIGURE 10.2. $\Upsilon_{D_{4_1,\mathrm{id}},D_{4_1,r}}(t)$.

Our remaining examples were computed with the help of SageMath [46]. The program that computed these invariants can be found at [24].

The next examples we consider are built from the knots $T_{3,4}$ and $T_{4,5}$. Their associated full infinity complexes are shown in Figure 10.3. If K is a knot, we will write $D_{K\#K,\mathbf{R}^\pi}$ for the deform-spun slice disk induced by the summand-swapping diffeomorphism $\mathbf{R}_\pi$ of $(\mathbb{S}^3, K\#K)$ described in Section 8.

$$\mathcal{CFL}^\infty(T_{3,4}) \qquad \mathbf{x}_0 \xleftarrow{\;V\;} \mathbf{x}_1 \xrightarrow{\;U^2\;} \mathbf{x}_2 \xleftarrow{\;V^2\;} \mathbf{x}_3 \xrightarrow{\;U\;} \mathbf{x}_4$$

$$(\mathrm{gr}_{\mathbf{w}}, A) \quad = \quad (-6,-3) \quad (-5,-2) \quad (-2,0) \quad (-1,2) \quad (0,3)$$

$$\mathcal{CFL}^\infty(T_{4,5}) \qquad \mathbf{x}_0 \xleftarrow{\;V\;} \mathbf{x}_1 \xrightarrow{\;U^3\;} \mathbf{x}_2 \xleftarrow{\;V^2\;} \mathbf{x}_3 \xrightarrow{\;U^2\;} \mathbf{x}_4 \xleftarrow{\;V^3\;} \mathbf{x}_5 \xrightarrow{\;U\;} \mathbf{x}_6$$

$$(\mathrm{gr}_{\mathbf{w}}, A) \quad = \quad (-12,-6) \quad (-11,-5) \quad (-6,-2) \quad (-5,0) \quad (-2,2) \quad (-1,5) \quad (0,6)$$

FIGURE 10.3. The complexes $\mathcal{CFL}^\infty(T_{3,4})$ and $\mathcal{CFL}^\infty(T_{4,5})$.

The following has been computed using SageMath:

Lemma 10.5. *(1) For the pair $(D_{T_{3,4}\#T_{3,4},\mathbf{R}^\pi}, D_{T_{3,4}\#T_{3,4},\mathrm{id}})$, we have $\tau = 2$, $V_0 = 1$, $V_1 = 1$, and $V_2 = 0$. A plot of Υ is shown on the left-hand side of Figure 10.4.*
(2) For the pair $(D_{T_{4,5}\#T_{4,5},\mathbf{R}^\pi}, D_{T_{4,5}\#T_{4,5},\mathrm{id}})$, we have $\tau = 3$, $V_0 = 2$, $V_1 = 1$, $V_2 = 1$, and $V_3 = 0$. A plot of Υ is shown on the right-hand side of Figure 10.4.

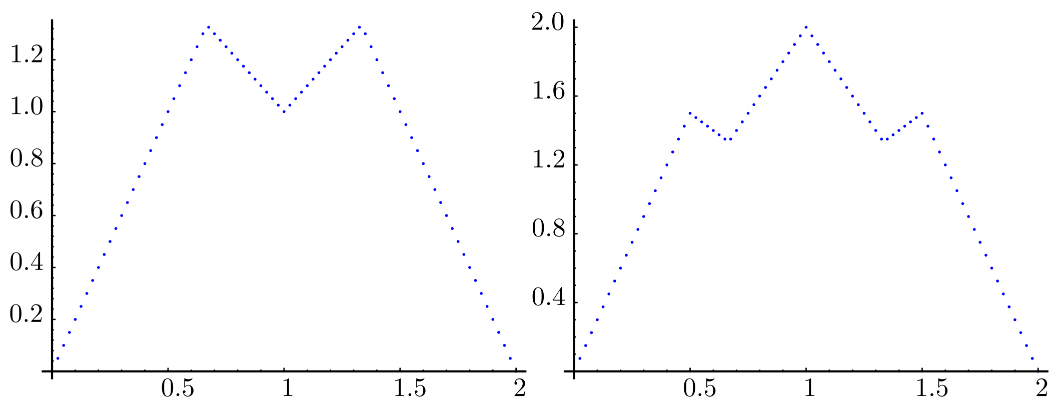

FIGURE 10.4. The $\Upsilon(t)$ functions for the pairs $(D_{T_{3,4}\#T_{3,4},\mathbf{R}^\pi}, D_{T_{3,4}\#T_{3,4},0}, \mathrm{id})$ (left) and $(D_{T_{4,5}\#T_{4,5},\mathbf{R}^\pi}, D_{T_{4,5}\#T_{4,5},\mathrm{id}})$ (right), computed using SageMath.

An immediate corollary of Lemma 10.5 and Theorems 5.13 and 6.7 is the following:

Corollary 10.6. *Let $\omega \in \{\mu_{\mathrm{st}}, \mu_{\mathrm{Sing}}\}$. Then*

$$\omega(D_{T_{3,4}\#T_{3,4},\mathbf{R}^\pi}, D_{T_{3,4}\#T_{3,4},\mathrm{id}}) \geq 2, \text{ and}$$
$$\omega(D_{T_{4,5}\#T_{4,5},\mathbf{R}^\pi}, D_{T_{4,5}\#T_{4,5},\mathrm{id}}) \geq 3.$$

10.3. **Slice disks with large stabilization distance.** We now prove Theorem 1.3 of the introduction.

Theorem 10.7. *Given $n \geq 0$, there is a knot K_n and a pair of slice disks D_1 and D_2 for K_n such that $\tau(D_1, D_2) \geq n$.*

If K is a knot in S^3, consider the *V-torsion order* of $\mathit{HFK}^-_{U=0}(K)$, for which we write $\mathrm{Tor}_V(K)$. This is the minimal $n \in \mathbb{N} \cup \{\infty\}$ such that

$$V^n \cdot \mathrm{Tor}(\mathit{HFK}^-_{U=0}(K)) = \{0\}.$$

See [2] and [20] for examples of applications of the torsion order in knot Floer homology.

Let K be a knot in S^3, and consider the slice knot $J = K_1 \# K_2 \# \bar{K}_3 \# \bar{K}_4$, where each K_i denotes a copy of K. We define two slice disks for J, which are boundary connected sums of slice disks for pairs of summands, as follows. Let D_1 be the spun slice disk obtained by viewing J as $(K_1\#\bar{K}_3)\#(K_2\#\bar{K}_4)$ and taking the boundary connected sum of the Artin spun slice disks for $K_1\#\bar{K}_3$ and $K_2\#\bar{K}_4$, and let D_2 be the spun slice disks obtained by viewing J as $(K_1\#\bar{K}_4)\#(K_2\#\bar{K}_3)$, and taking a similar boundary connected sum.

Lemma 10.8. *If K is a knot and D_1 and D_2 are the slice disks for $J = K\#K\#\bar{K}\#\bar{K}$ described above, then*

$$\tau(D_1, D_2) = \mathrm{Tor}_V(K).$$

Proof. Firstly, note that the connected sum formula and the duality formula for mirroring knots implies that $\mathrm{Tor}_V(L) = \mathrm{Tor}_V(-L)$, and also $\mathrm{Tor}_V(L\#M) = \max(\mathrm{Tor}_V(L), \mathrm{Tor}_V(M))$ for any knots L and M. In particular $\mathrm{Tor}_V(J) = \mathrm{Tor}_V(K)$.

We claim firstly that

$$\tau(D_1, D_2) \le \mathrm{Tor}_V(K). \tag{10.4}$$

This follows from algebraic considerations. Indeed, $[t^-_{D_1}(1)] = [t^-_{D_2}(1)] + \sigma$, where $\sigma \in \mathit{HFK}^-_{U=0}(J)$ is V-torsion. Hence, if $n = \mathrm{Tor}_V(K)$, then $V^n \cdot \sigma = 0$, so

$$V^n \cdot [t^-_{D_1}] = V^n \cdot [t_{D_2}].$$

This establishes equation (10.4).

To establish the reverse inequality of equation (10.4), we argue as follows. Consider the connected sum decomposition of J as $(K_1\#\bar{K}_3)\#(K_2\#\bar{K}_4)$. The corresponding 2-sphere gives a pair-of-pants cobordism from (S^3, J) to $(S^3, K\#\bar{K}) \sqcup (S^3, K\#\bar{K})$. Denote the cobordism map by F. By composing $t^-_{D_1}$ and $t^-_{D_2}$ with F, we may view the induced elements as chain maps

$$T_1, T_2 \in \mathrm{Hom}_{\mathbb{F}[V]}(\mathit{CFK}^-_{U=0}(K\#\bar{K}), \mathit{CFK}^-_{U=0}(K\#\bar{K})).$$

Since F is a homotopy equivalence,

$$V^n \cdot [t^-_{D_1}] = V^n \cdot [t^-_{D_1}]$$

if and only if $V^n \cdot [F(t^-_{D_1})] = V^n \cdot [F(t^-_{D_1})]$, which in turn occurs if and only if $V^n \cdot T_1 \simeq V^n \cdot T_2$, where $\simeq$ denotes $\mathbb{F}[V]$-equivariant chain homotopy.

The maps T_1 and T_2 may be identified with concordance maps for concordances from $K\#\bar{K}$ to $K\#\bar{K}$. The map T_1 is identified with the identity map id. On the other hand, the map T_2 is the concordance map for a concordance which factors through the unknot. In particular, T_2 must annihilate all torsion, as $\mathrm{Tor}_V(\text{Unknot}) = 0$. In particular, if $V^n \cdot \mathrm{id} \simeq V^n \cdot T_2$, then n must be larger than the torsion order of $K\#\bar{K}$. ☐

We now prove Theorem 10.7:

Proof of Theorem 10.7. It suffices to construct knots where $\mathrm{Tor}_V(K) \ge n$. This is straightforward. For example, $\mathrm{Tor}_V(T_{p,q}) = \min(p, q) - 1$. (This fact is well known, but a proof may be found in [20, Lemma 5.3]). ☐

11. The cobordism distance

In this section, we consider the following notion of distance between two surfaces:

Definition 11.1. Suppose that $g \in \mathbb{N}$, and S, S' are two slice surfaces of a knot $K \subseteq S^3$. We say that S and S' are *strictly g-cobordant* if there is a smoothly embedded, orientable 3-manifold $Y \subseteq I \times B^4$, such that the following are satisfied:

1. $\partial Y = (I \times K) \cup -(\{0\} \times S) \cup (\{1\} \times S')$.
2. Projection of Y onto I is Morse.
3. The sum of the genera of the components of each regular level set of Y at most g.

We write $\mu_{\mathrm{Cob}}(S,S')$ for the minimal g such that S and S' are strictly g-cobordant. We call this quantity the *cobordism distance* of S and S'. In the case of a 0-cobordism, this coincides with the notion of a 0-*cobordism* introduced by Melvin [30] for 2-knots. He defined a *g-cobordism* to be one where each component of every level set has genus at most g.

The main result of this section is that the invariant $\tau(D,D')$ gives a lower bound on $\mu_{\mathrm{Cob}}(D,D')$ for slice disks D and D'.

Theorem 11.2. *Suppose D and D' are two slice disks of a knot K in S^3. Then*

$$\tau(D,D')\le \mu_{\mathrm{Cob}}(D,D').$$

The main additional subtlety in the proof of Theorem 11.2 is that the level sets of a strict g-cobordism Y need not be connected, whereas the link cobordism maps vanish when there is a closed component. Hence, some care is required in the proof.

11.1. **Tubing disconnected surfaces.** In this section, we describe a way of meaningfully assigning cobordism maps to disconnected surfaces by tubing the components together.

Definition 11.3. Suppose that W is a compact 4-manifold with boundary Y, and S is a properly embedded, orientable surface in W. Suppose further that ∂S is equal to a knot $K\subseteq Y$. A *tubing* of S is a properly embedded surface $\widehat{S}\subseteq W$ with $\partial\widehat{S}=K$, obtained by attaching tubes to S which are the boundaries of 3-dimensional 1-handles in W. Furthermore, we assume $g(\widehat{S})=g(S)$ and $\widehat{S}$ is connected.

We now prove a local relation for the graph cobordism maps (cf. [59, Lemma 6.2]):

Lemma 11.4. *The graph cobordism maps satisfy the relation shown on the bottom of Figure 11.1.*

Proof. We begin with the bypass relation for the knot Floer cobordism maps, which is shown on the top of Figure 11.1. We may take the underlying link cobordism to be $I\times K$, where K is an unknot. The bypass relation for the link cobordism maps is proven in [56, Lemma 1.4]. We then obtain the graph relation by considering the $V=1$ reductions of the graph cobordism maps, following [58, Theorem C]. $\square$

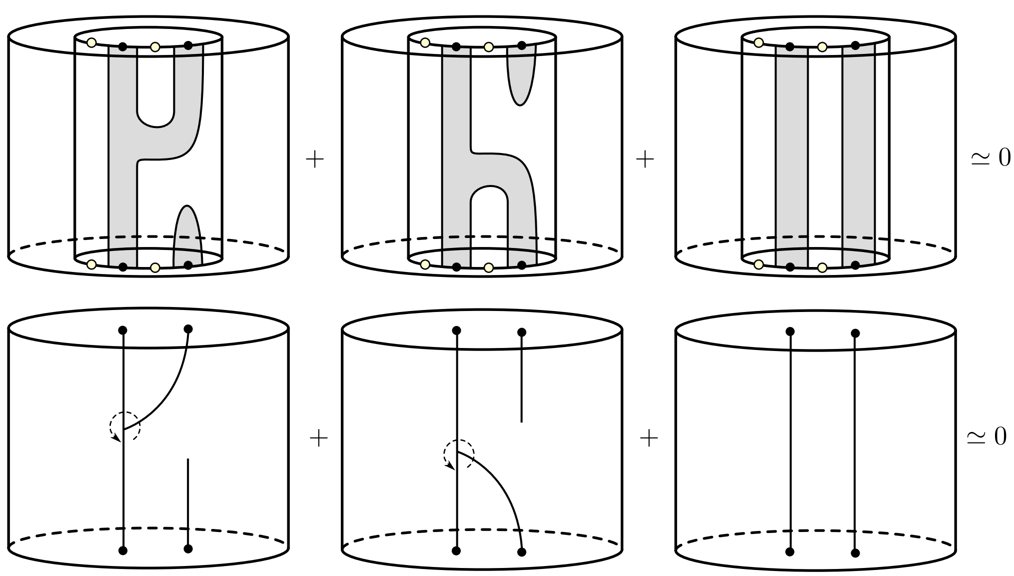

FIGURE 11.1. The bypass relation, as well as an induced relation obtained by setting $V=1$, in terms of graph cobordisms.

Proposition 11.5. *Suppose that S is a properly embedded, oriented surface in B^4 with boundary equal to a knot K. Suppose that $\widehat{S}_1$ and $\widehat{S}_2$ are two tubings of S. Then*

$$\mathbf{t}^-_{\widehat{S}_1,\mathbf{z}} \simeq \mathbf{t}^-_{\widehat{S}_2,\mathbf{z}}.$$

Proof. Any tubing of S is isotopic to a tubing where each tube has one foot on the component of S containing K, and one foot on a closed component of S. Since tubes are boundaries of 3-dimensional 1-handles, we may assume that, after an isotopy, any two such tubings have disjoint tubes. In particular, it suffices to change tubes one at a time.

We assume that T and T' are tubes which have their feet on the same components of S. We assume the feet of the tubes are very close, and we pick an open neighborhood of the two tubes which is diffeomorphic to $S^1 \times B^3$. We can factor the two cobordism maps through $(S^1 \times S^2, O_2)$, where O_2 is a two-component unlink in $S^1 \times S^2$.

We will prove the tube relation shown in Figure 11.2. This tube relation may be proven by considering the $V = 1$ reduction of the link cobordism maps, and then applying the graph relation shown in Figure 11.1. Since the link in $S^1 \times S^2$ is an unlink with 2 basepoints per component, the link cobordism maps are determined by the graph cobordism maps, and so it is sufficient to prove the analogous formula for the graph cobordism maps. We do this in Figure 11.3, using the local relation from Lemma 11.4, which is shown in Figure 11.1.

We now claim that the cobordism map for the right-most surface in Figure 11.2 factors through the H_1-action. This is proven as follows. The $V = 1$ reduction factors through the H_1-action by [55, Proposition 4.6]. Since the Alexander grading change of the link cobordism map is zero, Lemma 5.1 implies that the link cobordism itself factors through the H_1-action. In particular, once we compose this cobordism map for $S^1 \times B^3$ with the remainder of the cobordism map for the surface in B^4, we obtain the 0-map since $b_1(B^4) = 0$.

□

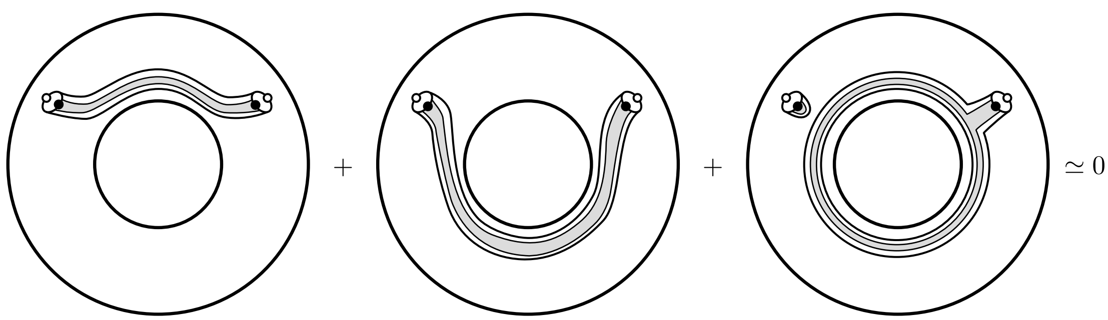

FIGURE 11.2. A relation involving tubes.

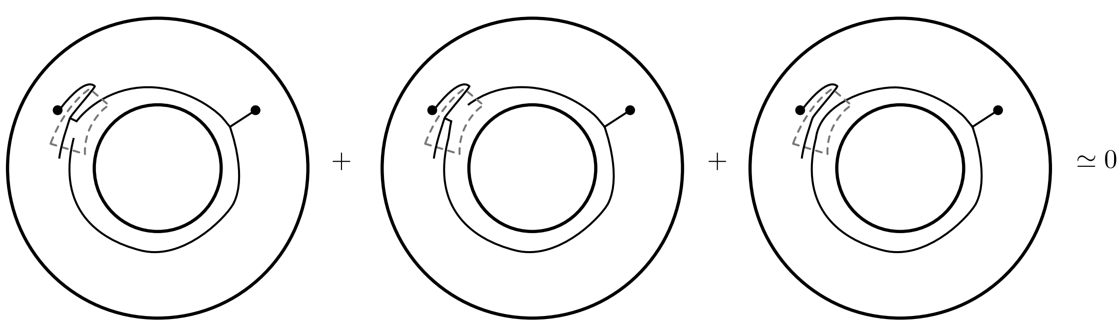

FIGURE 11.3. A relation involving graphs.

11.2. **Proof of Theorem 11.2.**

Proof of Theorem 11.2. Suppose that Y is a strict g-cobordism between slice disks D and D' of $K \subseteq S^3$. The projection from Y onto the factor I of $I \times B^4$ is a Morse function, by assumption. We may understand this Morse function as determining a sequence of 3-dimensional handles attached to $\{0\} \times D$, which build the 3-manifold Y. These handles may be of any index in $\{0, 1, 2, 3\}$. A 0-handle or 3-handle corresponds to adding or deleting an unknotted 2-sphere that is unlinked from the rest of the surface. Attaching a 1-handle corresponds to a 1-handle stabilization, while a 2-handle corresponds to 1-handle destabilization. Let us write $S_1, \dots, S_n$ for a sequence of surfaces induced by a strict g-cobordism. By definition

$$k = \max_{1 \le i \le n} g(S_i),$$

where $g(S_i)$ is the sum of the genera of the components of S_i.

The surfaces $S_1, \dots, S_n$ will in general not be connected. Let $\widehat{S}_i$ be any tubing of S_i for $i \in \{1, \dots, n\}$. Note that, by definition, $g(\widehat{S}_i) = g(S_i)$. We decorate each $\widehat{S}_i$ with a dividing set such that $(\widehat{S}_i)_{\mathbf{w}}$ is a bigon. We write $\widehat{\mathcal{S}}_i$ for this decorated surface. Proposition 11.5 implies that the map $\mathbf{t}^-_{\widehat{\mathcal{S}}_i,\mathbf{z}}$ is independent of the choice of tubing, and hence depends only on S_i.

If S_i is obtained from S_{i-1} by a 0-handle, then we can pick tubes so that $\widehat{S}_i$ is isotopic to $\widehat{S}_{i-1}$. If S_i is obtained by a 3-handle, then, after changing tubes if necessary, the same is true. In particular, $\mathbf{t}^-_{\widehat{\mathcal{S}}_i,\mathbf{z}} = \mathbf{t}^-_{\widehat{\mathcal{S}}_{i-1},\mathbf{z}}$ if S_i is obtained from S_{i-1} by attaching a 0-handle or a 3-handle.

If S_i is obtained by attaching a 1-handle to S_{i-1}, then either $g(S_i) = g(S_{i-1})$ or $g(S_i) = g(S_{i-1}) + 1$. In the first case, the 1-handle connects two different components of S_{i-1}, and consequently, after changing tubes if necessary, $\widehat{S}_i$ and $\widehat{S}_{i-1}$ are isotopic. In the second case, $\widehat{S}_i$ is obtained by stabilizing $\widehat{S}_{i-1}$. We have the same conclusions for a 2-handle attachment, with the roles of S_{i-1} and S_i reversed. Consequently, using the formula in Lemma 5.4 for stabilization, the maps $V^k \cdot \mathbf{t}^-_{\widehat{S}_i,\mathbf{z}}$ coincide for all i. So $\tau(D, D') \le k$ by Lemma 4.6, which completes the proof. ☐

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