# Probabilistic Data-driven Invariance for Constrained Control of Nonlinear Systems

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***Abstract*— We present a novel direct data-driven method for computing constraint-admissible positive invariant sets for general nonlinear systems with compact constraint sets. Our approach employs machine learning techniques to lift the state space and approximate invariant sets using finite data. The invariant sets are parameterized as sub-level-sets of scalar linear functions in the lifted space, which is suitable for control applications. We provide probabilistic guarantees of invariance through scenario optimization, with probability bounds on robustness against the uncertainty inherent in the data-driven framework. As the amount of data increases, these probability bounds approach 1. We use our invariant sets to switch between a collection of controllers to select a controller which enforces constraints. We demonstrate the practicality of our method by applying it to a nonlinear autonomous driving lane-keeping scenario.**

***Index Terms*— Data-driven control, Control of constrained systems, Lyapunov methods, Machine learning.**

## I. Introduction

Constraint enforcement is a crucial control objective for cyber-physical systems. These constraints can arise from safety considerations, physical limitations on the system, or performance specifications. A fundamental tool for constrained control is invariance. A positive invariant (PI) set characterizes the subset of states where, once the state enters, it remains there indefinitely. A controlled invariant (CI) set describes the subset of states where, for each initial state within the set, there exists a control input that ensures the system remains in the set for all future time. Invariant sets are important since they reduce the infinite-horizon control objective of perpetual constraint enforcement into a one-step problem, which is suitable for real-time implementation.

This paper presents a novel data-driven approach for computing PI sets. Standard algorithms [1], [2] synthesize PI sets for linear [3] and nonlinear [4] systems with mathematically accessible models. For unmodeled systems, an indirect data-driven approach is typically employed, wherein a model is identified from data, which is then used in a model-based algorithm. However, this approach can be challenging due to nonlinearity and uncertainty. Although recent advancements have been made toward direct data-driven approaches for linear systems [5], there are few model-free approaches capable of computing true invariant sets for nonlinear systems [6].

Our approach parameterizes invariant sets as sub-level sets of linear functions in a lifted space, enabling us to leverage recent advances in data-driven and learning-based techniques [6]–[8]. However, providing guarantees of invariance is challenging since approximations of an invariant set are not necessarily approximately invariant, e.g. [9]. Providing guarantees often requires inaccessible information about the system, such as the Lipschitz bounds or excessively large data-set. Even with a known Lipschitz constant, providing deterministic guarantees requires assuming an increased contraction for the invariant set, and an excessively large data-set [6].

Additionally, synthesizing constraint admissible CI (CACI) sets from data is challenging because a CACI set only implies the existence of a control input that supports invariance and satisfies the constraints. The dataset may not necessarily contain such an appropriate control input. Furthermore, certifying that a sampled control input leads to perpetual constraint enforcement requires a CACI set. To overcome these issues, we consider a switching between a collection of stabilizing controllers, which can be created using existing data-driven methods [10], based on our data-driven constraint admissible PI (CAPI) sets. For many systems, however, control-orient models of the closed-loop dynamics are unavailable or inaccessible [6], for instance, when the controller is a neural network or includes a human in the loop. Finally, we create a switched controller using our CAPI sets to enforce constraints.

Our **contributions** are summarized as follows. We relax the assumptions from [6] and present a method to compute invariant sets from finite data. We provide probabilistic guarantees of invariance through a scenario optimization approach [11]. Our data-driven approach to computing CAPI sets uses data to determine for which states and with what probability an existing controller enforces constraints. Finally, we present a supervisory controller using our CAPI sets to switch between a collection of existing stabilizing controllers to ensure constraint enforcement and improve performance.

*Notation and definitions:* The set of natural numbers from $N$ to $M$, inclusive, is denoted by $\mathbb{N}_N^M$ when $N \leq M$, and by $\mathbb{N}_N^M = \emptyset$ when $N > M$. We often drop the time index $t$, and denote $x = x_t$, and $x^+ = x_{t+1}$. For an autonomous system $x^+ = f(x)$, a set $\mathcal{O}$ is PI if for every $x \in \mathcal{O}$, we have $x^+ \in \mathcal{O}$. For a controlled system $x^+ = f(x, u)$, a set $\mathcal{C}$ is CI if for every $x \in \mathcal{C}$, there exists an input $u$ such that $x^+ \in \mathcal{C}$. The predecessor of a set $\mathcal{X}$ under the mapping $x^+ = f(x)$ is denoted by $\mathrm{Pre}(\mathcal{X}) = \{x : f(x) \in \mathcal{X}\}$. According to the

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geometric condition for invariance, a set $\mathcal{O}$ is PI if and only if $\mathcal{O} \subseteq \text{Pre}(\mathcal{O})$. A CAPI set (CACI set) is a PI (CI) set that is a subset $\mathcal{O} \subseteq \mathcal{X}$ ($\mathcal{C} \subseteq \mathcal{X}$) of the constraint set $\mathcal{X}$. A max-PI set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the largest CAPI set.

## II. Probabilistic data-driven invariance problem

Consider an unmodeled discrete-time, potentially nonlinear, time-invariant dynamical system $x^+ = f(x,u)$, where the mapping $f : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ is defined on a constraint set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ for the state $x \in \mathcal{X}$ and an input set $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ for inputs $u \in \mathcal{U}$ with a codomain $\mathcal{Y} \subseteq \mathbb{R}^{n_x}$. Accordingly, the transitions $f(x,u)$ for states $x \in \mathcal{Y} \setminus \mathcal{X}$ may be undefined, as they lie outside the constraint set $\mathcal{X}$. We define the closed-loop

$$x^+ = f_j(x), \quad j \in \mathbb{N}_1^{n_\kappa}, \tag{1a}$$

$$x \in \mathcal{X} = \big\{x \in \mathbb{R}^{n_x} : \; g(x) \leq 1\big\}, \tag{1b}$$

where $f_j : \mathcal{X} \to \mathcal{Y}$ is given by $f_j(x) = f(x, \kappa_j(x))$ for the controller $\kappa_j : \mathcal{X} \to \mathcal{U}$ that is selected from a known set of available stabilizing state-feedback controllers $\mathcal{K} = \{\kappa_j : j \in \mathbb{N}_1^{n_\kappa}\}$. We assume that the constraint set $\mathcal{X}$ is compact and defined as a sub-level of a continuous function $g : \mathbb{R}^{n_x} \to \mathbb{R}$.

We consider the case where the closed-loop dynamics (1a) are unmodeled, and our knowledge is limited to data-sets

$$\mathcal{D}_j = \big\{x_j^i, x_j^{i+} : \; i \in \mathbb{N}_1^{n_s}\big\}, \quad j \in \mathbb{N}_1^{n_\kappa} \tag{2}$$

with size $n_s$, collected for each controller $\kappa_j$. Each data-set $\mathcal{D}_j$ is comprised of pairs $(x_j^i, x_j^{i+})$ of states $x_j^i$ and their successors $x_j^{i+}$ for the system (1) under the $j$-th controller $\kappa_j \in \mathcal{K}$. Our assumptions about the system (1) are summarized below.

*Assumption 1 (System assumptions):*

(a) The state $x$ is available.
(b) The constraint set $\mathcal{X}$ is compact.
(c) The existing controllers $\kappa_j \in \mathcal{K}$ ensure Lyapunov stability.
(d) Performance metrics $J_j(x,r)$ are known for $\kappa_j \in \mathcal{K}$.

Assumption 1(a) holds for well-instrumented dynamical systems. For example, the state $x$ of many robots includes measurable physical variables such as position and velocity. Alternatively, for *differentially flat* nonlinear autonomous systems, the state $x$ can be defined as a window of any *flat* measurement $\{y_k : k \in \mathbb{N}_{k=t}^{t-n}\}$ [12]. Assumption 1(c) is necessary but not sufficient for enforcing bounded constraints, since a stabilizing controller may cause constraint violations $x_t \notin \mathcal{X}$ at some time $t > 0$ for some initial conditions. We will use Assumption 1(d) to quantify and optimize the performance of our switched controller based on the known performance metrics $J_j(x,r)$ of each controller $\kappa_j$ at the current state $x$ and command $r$ e.g., energy consumption, tracking error, and settling time.

### A. Probabilistic data-driven invariance problem

In this section, we define the problem of synthesizing a probabilistic CAPI set for the closed-loop system (1) under a fixed controller $\kappa_j$ using the corresponding data-set $\mathcal{D}_j$. We seek to characterize the set of initial states $\mathcal{O}_j$ for which the closed-loop system (1) satisfies the constraint $x_t \in \mathcal{X}$ for all times $t \geq 0$. The solution $\mathcal{O}_j$ is the well-known CAPI set [13] i.e., a feasible solution of the optimization

$$\sup_{\mathcal{O}_j} \text{ volume of } \mathcal{O}_j \tag{3a}$$

$$\text{s.t. } x^+ = f_j(x) \in \mathcal{O}_j, \qquad \forall x \in \mathcal{O}_j, \tag{3b}$$

$$x \in \mathcal{X}, \qquad \forall x \in \mathcal{O}_j. \tag{3c}$$

Solving (3) presents two challenges. First, the conditions (3b) and (3c) should be verified on an unknown set $\mathcal{O}_j$. We will provide a method in Section III to address this challenge. Second, when a control-oriented model of the dynamics is unavailable, synthesizing $\mathcal{O}_j$ requires infinite data, as the conditions (3b) and (3c) can only be verified on data samples. It is unclear how to provide deterministic guarantees that $\mathcal{O}_j$ is CAPI using finite data without making additional assumptions about the system and data, such as Lipschitz continuity and adequate data density. Thus, we instead seek probabilistic guarantees based on the available data (2).

For a candidate CAPI set $\mathcal{O}_j$, we define its inaccuracy as

$$\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \tag{4a}$$
$$:= \mathbb{P}_x\{(x \in \mathcal{O}_j \wedge f_j(x) \notin \mathcal{O}_j) \vee (x \in \mathcal{O}_j \wedge x \notin \mathcal{X}\},$$

the probability of a randomly sampled state $x \in \mathcal{O}_j$ violating the conditions (3b) and (3c). Ideally, we aim to bound $\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \leq \epsilon$ and minimize the bound $\epsilon \in [0,1]$. However, the data-driven set $\mathcal{O}_j$ is itself a random variable since it is computed using the randomly sampled data-set $\mathcal{D}_j$. Consequently, we can only bound the probability $\mathbb{P}_{\mathcal{D}_j}\big\{\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} > \epsilon\big\} < \beta$ of violating the inaccuracy bound with some $\beta \in [0,1]$. In other words, $\mathcal{O}_j$ may indeed fail as a $(1-\epsilon)$-accurate solution of (3), but with a probability of at most $\beta$. This problem is defined below.

*Problem 1 (Probabilistic data-driven* CAPI *set synthesis):* Given a random data-set (2), find a set $\mathcal{O}_j$ that satisfies (3b) and (3c) with accuracy $1-\epsilon$ and confidence $1-\beta$, i.e.,

$$\mathbb{P}_{\mathcal{D}_j}\big\{\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \leq \epsilon\big\} \geq 1-\beta. \tag{4b}$$

### B. Invariance based switched control problem

We will leverage our data-driven CAPI sets to determine the switching between the available controllers in $\mathcal{K}$ to enforce constraints and optimize performance. The switched control problem for constraint enforcement is defined below.

*Problem 2 (Switched control for constraint enforcement):* Given a collection of CAPI sets $\mathcal{O}_j$, $j \in \mathbb{N}_1^{n_\kappa}$ for the closed-loop dynamics (1), compute an optimal switching sequence $j_\tau^\star$ for the controller $\kappa_{j_\tau^\star}$ that enforces the constraint $x_\tau \in \mathcal{X}$ for every $\tau \geq t$, while optimizing the current performance $J_{j_t^\star}(x_t, r_t)$ at each time $t$.

## III. Probabilistic data-driven invariance

Without loss of generality, we can parameterize each set $\mathcal{O}_j$ as a sub-level set,

$$\mathcal{O}_j = \big\{x \in \mathbb{R}^{n_x} : \; V_j(x) \leq 1\big\}, \tag{5}$$

of a scalar function $V_j(x)$. The function $V_j(x)$ must satisfy $V_j(x) \leq 1 \Rightarrow V_j(x^+) \leq 1$ to enforce the PI set condition $x \in \mathcal{O}_j \Rightarrow x^+ \in \mathcal{O}_j$ from (3b), which makes the synthesis of $V_j(x)$ challenging. We instead enforce the following sufficient condition on the entirety of the constraint set $\mathcal{X}$,

$$V_j(x^+) \leq (1-\gamma)V_j(x) + \gamma, \quad \forall x \in \mathcal{X}, \tag{6a}$$

where $0 \leq \gamma \leq 1$. Additionally, the constraint admissibility condition (3c) is met if the following holds

$$V_j(x) \geq g(x), \qquad \forall x \in \mathbb{R}^{n_x} \setminus \mathcal{X}, \tag{6b}$$

where $g(x)$ from (1b) defines $\mathcal{X}$. If the conditions (6b) and (6a) hold, then $\mathcal{O}_j$ from (5) is a feasible solution of (3).

*Lemma 1:* The CAPI set conditions (3b) and (3c) hold if the conditions (6a) and (6b) hold.

*Proof:* When $x \in \mathcal{O}_j$ we have $V_j(x) \leq 1$ by definition (5), which implies $V_j(x) \leq (1-\gamma)V_j(x) + \gamma \leq 1$ since the right hand side of (6a) is a convex combination of $V_j(x) \leq 1$ and 1. Therefore, $V_j(x^+) \leq 1$ holds when $V_j(x) \leq 1$, which supports $x^+ \in \mathcal{O}_j$. Additionally, when $x \notin \mathcal{X}$ we have $1 \leq g(x) \leq V_j(x)$ from (6b) which implies $x \notin \mathcal{O}_j$. Thus $\mathcal{O}_j \subseteq \mathcal{X}$. ∎

The scalar $\gamma$ from (6a) creates a convex combination of $V_j(x)$ and 1, allowing $V_j(x^+) - V_j(x) \geq \gamma(1 - V_j(x)) \geq 0$ to increase in $V_j(x)$ inside $\mathcal{O}_j$ where $V_j(x) < 1$. To synthesize the CAPI set $\mathcal{O}_j$, we adopt the following parameterization

$$V_j(x) = \begin{cases} \theta_j^\top \phi(x) & \text{if } x \in \mathcal{X}, \\ g(x) & \text{if } x \notin \mathcal{X}, \end{cases} \tag{7}$$

where $V_j(x)$ on $\mathcal{X}$ is a linear combination of some preselected basis functions $\phi(x) = [\phi_1(x), \cdots, \phi_{n_\theta}(x)]^\top$, and $\theta$ is the parameter vector to be computed. Based on the generalized Weierstrass approximation theorem [14], the lifted linear form $\theta_j^\top \phi(x)$ is generic for synthesizing a continuous $V_j(x)$ on the compact set $\mathcal{X}$. For example, the basis $\phi(x)$ can be eigenfunctions of $V_j(x)$, and the parameters $\theta_j$ be the eigenvalues of $V_j(x)$. Additionally, the resulting set $\mathcal{O}_j$ is compact, since $\mathcal{O}_j \subseteq \mathcal{X}$ by construction, according to Lemma (1), where $\mathcal{X}$ is compact, and $V_j(x)$ is continuous on $\mathcal{X}$.

By choosing a set of basis functions $\phi$ with established generalization properties [15], we can ensure that the lifted linear form $\theta_j^\top \phi(x)$ parameterizes any continuous function $V_j(x)$ on $\mathcal{X}$.

### A. Deterministic data-driven positive invariance

In this subsection, we discuss the intractability of a deterministic approach to synthesizing CAPI sets from data, as it leads to an optimization problem with an infinite number of constraints. To synthesize a CAPI set $\mathcal{O}_j$, we may attempt to solve the following linear program (LP)

$$\min_{\theta_j} \; \theta_j^\top \int_{\mathcal{X}} \phi(x) \; \mathrm{d}\mu(x) \tag{8a}$$
$$\text{s.t. } \theta_j^\top\big((1-\gamma)\phi(x) - \phi(x^+)\big) \geq -\gamma, \quad \forall x \in \mathcal{X} \cap \mathrm{Pre}_j(\mathcal{X}), \tag{8b}$$
$$\theta_j^\top (1-\gamma)\phi(x) \geq g(x^+) - \gamma, \quad \forall x \in \mathcal{X} \setminus \mathrm{Pre}_j(\mathcal{X}), \tag{8c}$$
$$\theta_j^\top \phi(x) \geq l(x), \quad \forall x \in \mathcal{X}, \tag{8d}$$

where the weights of the linear cost (8a) involve a Lebesgue integral with some set-measure $\mu$, which can be analytically pre-computed, $g(x)$ defines the constraint set $\mathcal{X}$ from (1b), $\mathrm{Pre}_j(\mathcal{X})$ is the predecessor set of $\mathcal{X}$ under $f_j(x)$, and $l : \mathbb{R}^{n_x} \to \mathbb{R}$ lower-bounds $V_j(x)$ to make the LP (8) bounded. Trivial choices for $l(x)$ can be $l(x) = g(x)$ or $l(x) = 0$.

The following proposition shows that this approach guarantees $\mathcal{O}_j$ from (5) is CAPI.

*Proposition 1:* Let $\theta = \theta_j^\star$ be a feasible solution of (8). Then, $V_j(x)$ from (7) creates a set $\mathcal{O}_j$ from (5) that is CAPI for the closed-loop system (1).

*Proof:* The conditions (8b) and (8c) enforce (6a). Additionally, the definition $V_j(x) = g(x)$ on $\mathbb{R}^{n_x} \setminus \mathcal{X}$ from (7) enforces (6b). Thus $\mathcal{O}_j$ is CAPI according to Lemma 1. ∎

According to Proposition 1, solving the infinite-dimensional LP (8) produces a CAPI set $\mathcal{O}_j$. The heuristic cost (8a) rewards minimizing $V_j(x)$ to increase the volume $\mu(\mathcal{O}_j)$ of its level-set $\mathcal{O}_j = \{x : V_j(x) \leq 1\}$. Nonetheless, enforcing (8b)-(8d) for every $x \in \mathcal{X}$ is impossible since the set $\mathcal{X}$ has infinite cardinality. This requires an infinite amount of data, and leads to an infinite number of constraints. A common approach to resolve this challenge is to consider specific models for the dynamics, which imposes restrictive assumptions on the system (1). For instance, results from sum of squares (SOS) programming show that the problem is tractable for finite degree polynomial dynamics with polynomial constraints, where $\phi$ is a polynomial basis [16]. Similarly, quadratic basis functions can be used for stable linear time-invariant (LTI) systems, where the existence of the parameter $\theta_j$ is supported by the Kalman–Yakubovich–Popov lemma [17].

Previously, we showed in [6] that tightening the constraints (8b) and (8c) leads to a guaranteed CAPI set under less restrictive assumptions, including knowing an upper bound on the Lipschitz constant of the system and having a sufficiently dense data-set. Unfortunately, the data density requirements necessitates an excessively large data-set. Furthermore, tightening (8b) and (8c) can render the LP (8) infeasible.

### B. Probabilistic data-driven positive invariance

In this section, we present a data-driven method for synthesizing a set $\mathcal{O}_j$ with probabilistic guarantees of being CAPI. We use scenario optimization to avoid the infinite constraints of LP (8) and approximate the solution of the LP (8) using finite data. With finite data available, the LP (8) becomes an uncertain optimization. In our approach, we approximate the uncertain LP (8) by the following randomized program (RP)

$$\min_{\theta} \; \theta_j^\top \int_{\mathcal{X}} \phi(x) \; \mathrm{d}\mu(x) \tag{9a}$$
$$\text{s.t. } \theta_j^\top\big((1-\gamma)\phi(x_j^i) - \phi(x_j^{i+})\big) \geq -\gamma, \quad \forall i \in \mathbb{N}_1^{n_p}, \tag{9b}$$
$$\theta_j^\top (1-\gamma)\phi(x_j^i) \geq g(x_j^{i+}) - \gamma, \quad \forall i \in \mathbb{N}_{n_p+1}^{n_s}, \tag{9c}$$
$$\theta_j^\top \phi(x_j^i) \geq l(x_j^i), \quad \forall i \in \mathbb{N}_1^{n_s}, \tag{9d}$$

where $\mathbb{N}_1^{n_p}$ indicates the $n_p$ samples $x_j^i \in \mathcal{X}$ that map to $x_j^{i+} \in \mathcal{X}$ i.e. $x_j^i \in \mathcal{X} \cap \mathrm{Pre}_j(\mathcal{X})$. The RP (9) is an LP with a finite number $2n_s$ of constraints, hence tractable. Compared to the deterministic approach from [6], our probabilistic approach does not require the Lipschitz constant of the system, it is less data intensive, and provides some level of guarantees even when data is limited. The latter is crucial since data often comes from expensive and time consuming experiments or high-fidelity simulations.

The following theorem provides probabilistic guarantees for the CAPI set $\mathcal{O}_j$ computed through our data-driven approach i.e., the solution of Problem 1.

*Theorem 1:* Let the data $D_j$ from (2) be randomly generated according to the probability distribution $\mathbb{P}_x$. Let (9) be feasible. Let $n_s$ and $\epsilon, \beta \in [0,1]$ satisfy

$$\beta = \sum_{i=1}^{n_\theta - 1} \binom{n_s}{i} \epsilon^i (1-\epsilon)^{n_s - i}. \tag{10}$$

Then, (4b) holds i.e. the set $\mathcal{O}_j$ from (5) with optimal parameters $\theta^\star$ from (9) is CAPI with accuracy $1-\epsilon \in [0,1]$ and confidence $1-\beta \in [0,1]$.

*Proof:* First, we reformulate the optimization (8) as

$$\theta_j^\star = \arg\min_{\theta_j} \; c^\top \theta_j \tag{11a}$$

$$\text{s.t. } \theta_j \in \Theta_j(x), \quad \forall x \in \mathcal{X}, \tag{11b}$$

where $c = \int_{\mathcal{X}} \phi(x) \; \mathrm{d}\mu(x)$ and

$$\Theta_j(x) = \{\theta_j \in \mathbb{R}^{n_\theta} : \; \theta_j \text{ satisfy (8b)-(8d)}\}.$$

Since $x^+ = f_j(x)$ is a deterministic mapping, the constraint set $\Theta_j(x)$ only depends on the sampled states $x \in \mathcal{X}$. Scenario optimization [11], [18] draws random samples $x_j^i$, $i \in \mathbb{N}_1^{n_s}$ from $\mathcal{X}$ with probability $\mathbb{P}_x$, and approximates the optimization (11) by solving the RP

$$\hat{\theta}_j \in \arg\min_{\theta_j} \; c^\top \theta_j \tag{12a}$$

$$\text{s.t. } \theta_j \in \Theta_j(x_j^i), \quad i \in \mathbb{N}_1^{n_s}, \tag{12b}$$

which is equivalent to the RP (9). The set $\Theta_j(x)$ is convex and closed with respect to $\theta_j$ since it is an intersection of half-spaces defined by (8b)-(8d). Therefore, according to Theorem 1 from [19], the relation

$$\mathbb{P}_{\hat{\theta}_j}\{\mathbb{P}_x\{\hat{\theta}_j \notin \Theta_j(x)\} \le \epsilon\} \ge 1 - \sum_{i=1}^{n_\theta - 1} \binom{n_s}{i} \epsilon^i (1-\epsilon)^{n_s - i} \tag{13}$$

holds. According to Proposition 1, the condition $\hat{\theta}_j \in \Theta_j(x)$ is a sufficient for satisfying the CAPI conditions (3b) and (3c). Thus, the set-inclusion $\Theta_j(x) \subseteq \{\hat{\theta}_j \mid \mathcal{O}_j \text{ CAPI}\}$ holds. Therefore, the probability $\mathbb{P}_x\{\hat{\theta}_j \notin \Theta_j(x)\} \le \epsilon$ that we can sample $x \in \mathcal{X}$ that violates the constraints $\hat{\theta}_j \notin \Theta_j(x)$ of (9) is higher than the probability (4a) that we have not found a CAPI set i.e. $\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \le \mathbb{P}_x\{\hat{\theta}_j \notin \Theta_j(x)\} \le \epsilon$. This implies $\{\hat{\theta}_j | \mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \le \epsilon\} \supseteq \{\hat{\theta}_j | \mathbb{P}_x\{\hat{\theta}_j \notin \Theta_j(x)\} \le \epsilon\}$. Combining with (13) we obtained the desired bound $\mathbb{P}_{\hat{\theta}_j}\{\mathbb{P}_x\{\mathcal{O}_j \text{ not CAPI}\} \le \epsilon\} \ge \mathbb{P}_{\hat{\theta}_j}\{\mathbb{P}_x\{\hat{\theta}_j \notin \Theta_j(x)\} \le \epsilon\} \ge 1-\beta$. ■

According to Theorem 1, condition (10) specifies the confidence bound $1-\beta$ for a dataset of size $n_s$ and a targeted accuracy of $1-\epsilon$. Often, it is desirable to compute $n_s$ for given values of $\epsilon$ and $\beta$. A lookup table is provided in [19] to verify that the sample size $n_s$ is sufficient to satisfy condition (10). Alternatively, $n_s$ can be over-approximated [20] by

$$n_s \ge \frac{1}{\epsilon} \frac{e}{e-1} \left( \log \frac{1}{\beta} + \frac{n_\theta (n_\theta + 3)}{2} \right), \tag{14}$$

where $e$ is the Euler number. Theorem 1 implicitly relies on the geometric condition for invariance [1] to compute $\mathcal{O}_j$ from pairs $\{x_j, x_j^+\}$ of states $x_j$ and successor states $x_j$. This contrasts with related work in reachability [20] which typically requires trajectories. We note Theorem 1 holds independently of the probability $\mathbb{P}_x$ and the convexity of the set $\mathcal{X}$ [19]. However, note that accuracy is measured with respect to the probability distribution $\mathbb{P}_x$. Thus, $\mathcal{O}_j$ can be inaccurate in regions where $\mathbb{P}_x$ has low mass since it is unlikely that we will draw a sample that disproves (4a) $\mathcal{O}_j$ is CAPI.

## IV. Switched control via data-driven invariance

In this section, we will leverage our probabilistic data-driven CAPI sets $\mathcal{O}_j$ from the previous section to synthesize a switched controller that enforces constraints. To solve Problem 2, we select the switched index $j_t^\star$ of $\kappa_{j_t^\star}$ by solving the following one-step integer program (IP)

$$j_t^\star = \arg\min_{j \in \mathbb{N}_1^{n_\kappa}} \; J_j(x_t, r_t) \tag{15a}$$

$$\text{s.t. } x_t \in \mathcal{O}_j, \tag{15b}$$

which is solved in real-time at each time instance $t$. The IP (15) uses the CAPI sets $\mathcal{O}_j$, $j \in \mathbb{N}_1^{n_\kappa}$ from Section III. The following proposition demonstrates that the switched controller $\kappa_{j_t^\star}$ enforces the constraints and thus solves Problem 2.

*Proposition 2:* Let $\mathcal{O}_j$ be CAPI for every $j \in \mathbb{N}_1^{n_\kappa}$. Let the state $x_t \in \mathcal{O}_j$ for some $j \in \mathbb{N}_1^{n_\kappa}$ at the current time $t$. Let the index $j_t^\star$ of the operating controller $\kappa_{j_t^\star}$ be the optimizer of (15). Then, the online optimization (15) is recursively feasible and the constraint $x_\tau \in \mathcal{X}$ is enforced at every time instance $\tau \ge t$.

*Proof:* The assumption $x_t \in \mathcal{O}_j$ for some $j \in \mathbb{N}_1^{n_\kappa}$ renders (15) initially feasible at the current time $t$. Additionally, (15) will be feasible at time $t+1$ since the applied controller $\kappa_{j_t^\star}$ keeps $x^+ = x_{t+1}$ inside $\mathcal{O}_{j_t^\star}$. Thus, (15) is recursively feasible for $\tau \ge t$. Finally, the constraint is enforced at all time $\tau \ge t$ since $\mathcal{O}_{j_\tau^\star} \subseteq \mathcal{X}$. ■

Although our CAPI computation approach requires considerable computational resources to solve Problem 1, these computations can be performed offline. However, the one-step IP (15) can be solved online by exhaustively evaluating the costs $J_j$ and safety $\mathcal{O}_j$ at each controller $j \in \mathbb{N}_1^{n_\kappa}$.

## V. Numerical examples

We demonstrate our data-driven invariance approach through an illustrative and a practical example.

### A. Computation of CAPI sets for a linear system

In this example, we applied our CAPI set computation method to the following under-damped LTI system

$$x^+ = \begin{bmatrix} 0.8 & 0.6 \\ -0.2 & 0.8 \end{bmatrix} x, \tag{16a}$$

$$x \in \mathcal{X} = \{x \in \mathbb{R}^2 : \; \|x\|_\infty \le 1\}, \tag{16b}$$

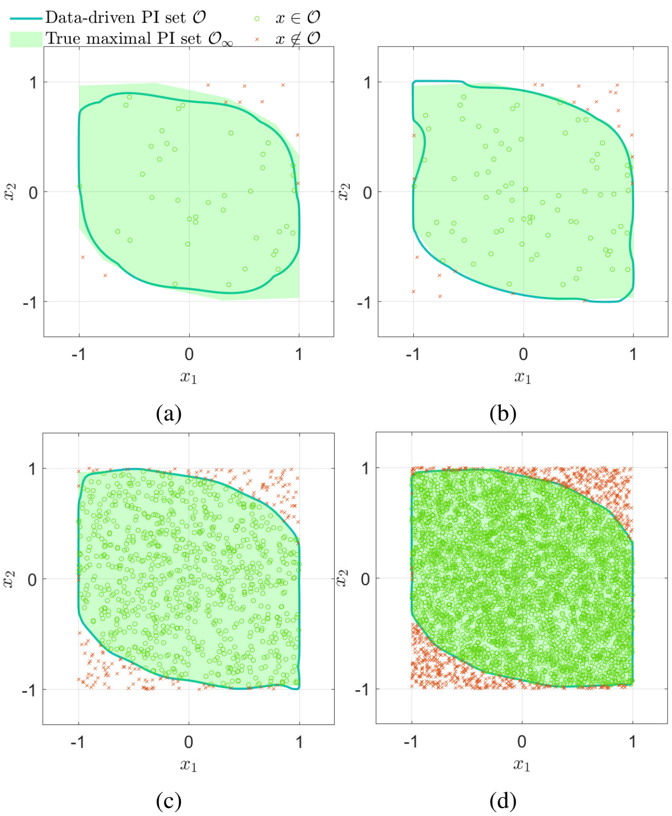


Fig. 1: Our probabilistic data-driven CAPI sets for the LTI system (16) with $n_s$ samples and $n_\theta$ basis functions. For (a)-(d), respectively, we set $n_s = 50, 100, 1000, 5000$ and $n_\theta = 3^2, 5^2, 11^2, 21^2$, and computed FNR$= 10.81, 2.74, 0.99, 0.59\%$ AND FPR$= 0.01, 1.16, 0.41, 0\%$.

where the constraint set $\mathcal{X}$ is defined by the infinity norm $g(x) = \|x\|_\infty$. We considered an LTI system with a polytopic constraint since its true max-PI set can be computed using standard algorithms [1] for comparison. To test our data-driven approach, we randomly sampled $\mathcal{X}$ using a uniform probability $\mathbb{P}_x$ over $\mathcal{X}$. Then, we applied the random samples $x_j^i$, $i \in \mathbb{N}_1^{n_s}$ to the system (16) and collected their pairs $x_j^{i+}$. For the basis functions $\phi$, a set of thin-plate spline functions $\phi_m(x) = \|x - c_m\|^2 \ln \|x - c_m\|$, $m \in \mathbb{N}_1^{n_\theta}$, was used, with centers $c_m$ positioned at the nodes of a $\sqrt{n_\theta} \times \sqrt{n_\theta}$ grid over $\mathcal{X}$. We set $\gamma = 0.4$ and $l(x) = g(x)$, and solved (9) to synthesize the CAPI set (5).

The purpose of our approach *is not* to approximate the max-PI set $\mathcal{O}_\infty$, but rather a set $\mathcal{O}$ that is CAPI with probability bounds (4b). However, the numerical results in Fig. 1 show that our CAPI set converges to the max-PI set $\mathcal{O}_\infty$ for the LTI system (16) as $n_s$ and $n_\theta$ increase. To quantify this, we used a Monte Carlo approach where we drew an additional $10^4$ random samples $x_j^i, i \in \mathbb{N}_1^{10^4}$, from the box $\mathcal{B}$, and computed the false negative rate (FNR)$=\frac{\text{number of } (x_j^i \in \mathcal{O}_\infty \setminus \mathcal{O}) \times 100\%}{\text{number of } (x_j^i \in \mathcal{O}_\infty)}$ and the false positive rate (FPR)$=\frac{\text{number of } (x_j^i \in \mathcal{O} \setminus \mathcal{O}_\infty) \times 100\%}{\text{number of } (x_j^i \notin \mathcal{O}_\infty)}$ reported in Fig. 1.

## B. Switched control for autonomous lane-keeping

We illustrate the effectiveness of our data-driven switched controller through an autonomous lane-keeping problem similar to that in [6]. For the vehicle dynamics, we used the following kinematic bicycle model from [21],

$$\dot{y} = v \sin(\psi + u), \qquad -2 \le y \le 2, \tag{17a}$$

$$\dot{\psi} = \frac{v}{l} \sin(u), \tag{17b}$$

where $y$ is the lateral position of the center of mass, $\psi$ is the inertial heading angle, $v$ is the speed, $l = 1.6m$ is the distance from the rear axle to the center of mass, and the control input $u$ is the angle between the direction of motion and the longitudinal axis. The widths of the vehicle is assumed 1.6 m and each lane is 3.6 m wide. Therefore, the allowable maneuver of the center of mass from the center of a two-lane road should belong to the constraint set $\mathcal{X} = \{x : \frac{1}{4}([1, 0]x)^2 \le 1\}$, where $x = [y, \psi]^\top$ is the state of the system.

For the controllers, we used offset state feedback controllers

$$u = \kappa_j(x) = K(x - [\bar{r}_j, 0]), \tag{18}$$

where $\bar{r}_j = -2.4 + 0.04(j-1)$, $j \in \mathbb{N}_1^{121}$. The gain $K$ is the linear–quadratic regulator (LQR) gain with the weights $Q = \mathrm{diag}([100, 10])$ and $R = 1$, derived based on the following linearization of (17) at the origin,

$$\dot{x} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} v \\ v/l \end{bmatrix} u, \tag{19}$$

where $v = 20\,\text{m/s} \approx 45\,\text{mph}$ is the nominal speed. To make the problem more challenging, we used the nominal LQR gain $K$, designed at the nominal speed, for two experiments at the nominal speed $v = 20\,\text{m/s} \approx 45\,\text{mph}$ and the overtaking speed of $v = 27\,\text{m/s} \approx 60\,\text{mph}$.

We randomly sampled the box $\mathcal{B} = \{(y, \psi) : y \in [-2, 2], \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ using a uniform probability $\mathbb{P}_x$ to draw the samples $x_j^i$, $i \in \mathbb{N}_1^{1000}$. Then, for each $j \in \mathbb{N}_1^{121}$, we simulated the kinematic bicycle model (17) in closed-loop with each of the nominal controllers $\kappa_j$ using MATLAB's ODE45 solver for one time step of duration 0.1 seconds to sample the pairs $\{x_j^i, x_j^{i+}\}$ of data (2). In practice, data can be gathered via a test-drive on a wider road. For the basis functions $\phi$, a set of thin-plate spline functions $\phi_m(x) = \|x - c_m\|^2 \ln \|x - c_m\|$, $m \in \mathbb{N}_1^{1000}$ was used with centers $c_m$ located on a $15 \times 15$ grid within the box $\mathcal{B}$. We set $\gamma = 0.4$ and $l(x) = g(x)$, and solved the RP (9) to synthesize the CAPI sets $\mathcal{O}_j$, $j \in \mathbb{N}_1^{121}$.

Fig. 2 shows our data-driven CAPI sets $\mathcal{O}_j$ for four different $j$ with respect to $\bar{r}_j$, where the blue star is the state equilibrium $x_\infty = [\bar{r}_j, 0]$. We observed that for almost every $j \in \mathbb{N}_1^{18^2}$ our data-driven CAPI set $\mathcal{O}_j$ includes the equilibrium in its interior.

The true max-PI sets $\mathcal{O}_{j\infty}$ are not known for the bicycle model (17). However, we approximated samples of $\mathcal{O}_{j\infty}$ by creating a uniform $20 \times 20$ set of initial conditions over the box $\mathcal{B}$ and labeling whether the resulting trajectory satisfied the constraints over a horizon of 4 s. We used these samples to approximate the max-PI sets $\mathcal{O}_{j\infty}$ through a nearest neighbor method, i.e., the nearest sample $x_j^i$ to the current state $x$ determines whether $x \in \mathcal{O}_{j\infty}$. These samples are shown by green circles $x_j^i \in \mathcal{O}_{j\infty}$, and red crosses $x_j^i \notin \mathcal{O}_{j\infty}$ in

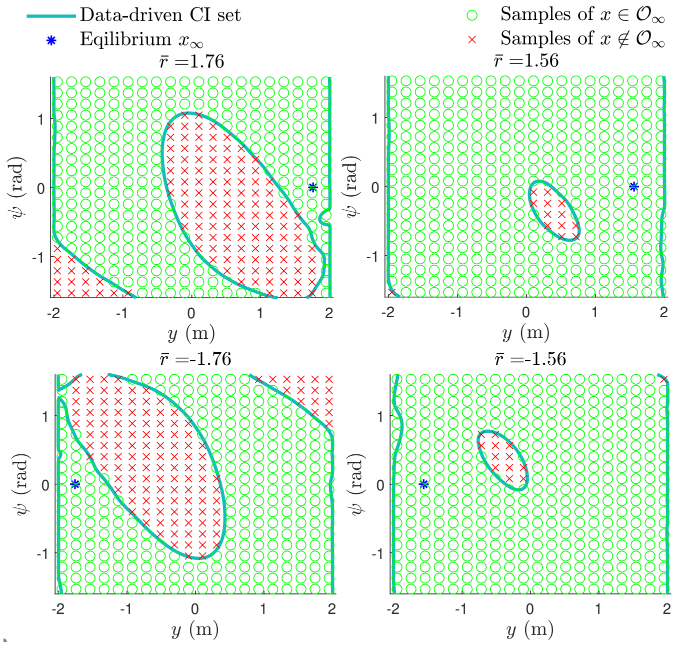


Fig. 2: Our probabilistic data-driven CAPI sets for the nonlinear car model (17) at four different values of $\bar{r}_j$. The green circles are the approximated samples of the max-PI sets $\mathcal{O}_{j\infty}$.

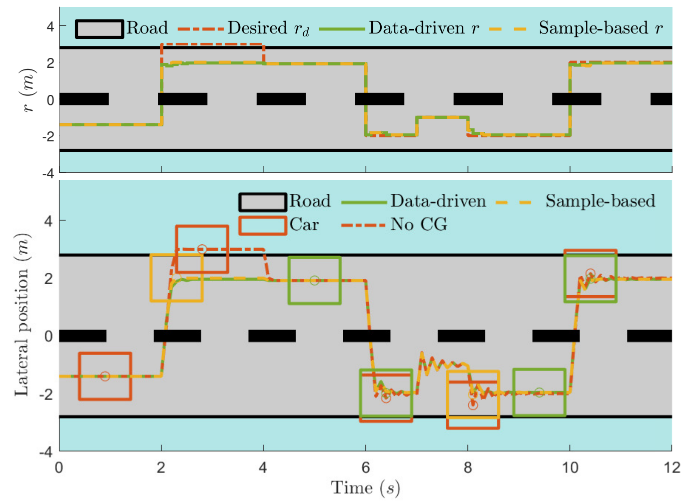


Fig. 3: Switched control of the car model (17) during overtaking maneuvers. The speed changes from 45 mph to 60 mph at $t = 5$ s. The boxes represent the car at several instances.

Fig. 2. We used the approximated max-PI sets for the switched controller (15) to compare it to our probabilistic data-driven switched controller.

Fig. 3 shows the reference tracking of the bicycle model (17) in closed-loop with our switched nominal LQR controller compared to the nominal controller $u = K(x-[\bar{r}_t, 0])$ and the sample-based controller, where $r_t$ is the desired reference. We used MATLAB's ODE45 solver and zero-order hold controllers $\kappa_{j_t^\star}$, where the switched index $j_t^\star$ is computed by solving (15) and the performance metrics $J_j(x,r) = \|r_t - r_j\|$, $j \in \mathbb{N}_1^{121}$, are used to minimize the discrepancy between the desired reference $r_t$ and the applied reference $r_{j_t^\star}$. The applied reference $r_{j_t^\star}$ is the value used to design the respective controller $\kappa_{j_t^\star}$.

We simulated an overtaking maneuver, where the speed changes from the nominal 20 m/s to a higher 27 m/s at $t = 5$ s. Fig. 3 shows that the car stays on the road with our probabilistic data-driven controller, while it leaves the road four times at $t = 2.1, 6.8, 8.1, 10.4$ s with the nominal controller, and two times at $t = 2.3, 8.1$ s with the sample-based controller.

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