ON THE TOTAL PERIMETER OF PAIRWISE DISJOINT CONVEX BODIES

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Abstract. In this note we introduce a pseudometric on closed convex planar curves based on distances between normal lines and show its basic properties. Then we use this pseudometric to give a shorter proof of the theorem by Pinchasi that the sum of perimeters of k convex planar bodies with disjoint interiors contained in a convex body of perimeter p and diameter d is not greater than $p + 2(k-1)d$.

1. Introduction

For a convex body C, we denote its perimeter by $\mathrm{per}(C)$ and its diameter by $\mathrm{diam}(C)$. Given a convex body C, it is natural to find the maximal total perimeter of k disjoint convex bodies confined to C. Glazyrin and Morić studied this problem in [6] and conjectured that the upper bound is always $\mathrm{per}(C) + 2(k-1)\mathrm{diam}(C)$. They proved this bound for some particular cases and made partial progress towards the general conjecture by showing the upper bound $1.22195\mathrm{per}(C) + 2(k-1)\mathrm{diam}(C)$. In [10], Pinchasi proved the general conjecture.

Theorem 1.1 (Pinchasi)**.** *If convex planar bodies C_i, $1 \leq i \leq k$, with disjoint interiors are contained in a convex planar body C, then*

$$\sum_{i=1}^{k} per(C_i) \leq per(C) + 2(k-1)\,diam(C).$$

In this note we provide a shorter and (almost) self-contained proof of Theorem 1.1 using the construction of a pseudometric on convex curves (see Section 2). Apart from the proof, we find this pseudometric and its properties inherently interesting. The constructed pseudometric is particularly nice for *curves of constant width*, that is, closed planar convex curves whose width (distance between parallel supporting lines) is the same for all directions. As a simplest nontrivial example, we mention the Reuleaux triangle formed by the intersection of three circular disks, each having a center on the boundary of the other two.

The proof of Theorem 1.1 in the paper is based on the general idea of Pinchasi. However, it deviates from the proof of Pinchasi quite significantly in tools and details. The proofs coincide only for partitioning a disk and differ already for the general partitioning case. The proof of Pinchasi is combinatorial in nature and uses halving lines for discrete sets of points. Our proof generalizes the case of a disk in a different direction and, essentially, uses ideas from differential geometry by utilizing curves of constant width. This is a very natural direction for extension because curves of constant width possess many of the properties of a disk and any convex curve can be extended to a curve of constant width of the same diameter. In particular, this approach allows us to obtain a short and transparent proof of the partitioning case that works in the same clear manner as the proof of Pinchasi for disks (see the end of Section 3). The proof of the general case is more involved and requires several technical steps and ideas, different from those used by Pinchasi. These are covered in Lemma 4.1 and presented in Section 4.

For the sake of simplicity, some statements will be formulated for *strictly convex curves*, that is, curves that have exactly one common point with each supporting line. In particular, all curves of constant widths are strictly convex curves (see, for instance, [9, Theorem 3.1.1]). All statements are true for general convex curves as well, with minor modifications. Throughout the whole paper by convex curves we always mean closed convex curves and use the term almost interchangeably with convex bodies. We will say that a point is inside a convex curve if it belongs to the interior of the corresponding convex body and it is outside a convex curve if it does not belong to the convex body.

2. Pseudometric on convex curves

In this section we define the pseudometric on the set of convex planar curves and adjacent notions. We fix a unit vector u_0 in the plane and by u_θ, $\theta \in [0, 2\pi]$, define a unit vector that is obtained by rotating u_0 by angle θ counterclockwise. Given a convex curve C, each supporting line of C partitions the plane into two half-planes, one containing the interior of C and one having no interior points of C. We choose the half-plane containing C and consider a unit vector u orthogonal to the supporting line such that the chosen half-plane is in the direction of the vector with respect to the supporting line. For a common point v of the supporting line and C, we then say that u is an inward normal vector to C at v. Note that for each unit vector u, there is always a unique corresponding supporting line but u may be an inward normal vector at many points in the boundary of C if C is not strictly convex. Alternatively, a unit vector u is an inward normal vector to C at the boundary point v if and only if the linear functional $u \cdot x$, $x \in C$, attains its minimum at v, where $\cdot$ is the standard dot product in the plane.

Let v be a point in the boundary of C with the inward normal vector u_θ. By a normal line $\ell_\theta(C)$ we denote the line through v in the direction of u_θ. Normal lines are uniquely defined for all θ when C is strictly convex. For general convex curves, ℓ_θ are uniquely defined for all but countably many values of θ (this follows, for instance, from the result in [3]). Therefore, the integrals in the definitions below are properly defined for all pairs of convex curves.

Definition 2.1. For two convex curves C_1 and C_2, we define

$$\operatorname{pdist}(C_1, C_2) = \frac{1}{2}\int_0^{2\pi} \operatorname{dist}(\ell_\theta(C_1), \ell_\theta(C_2))d\theta,$$

where dist is the standard Euclidean distance between parallel lines.

Remark 2.2. It is easy to show that if instead of dist we take the signed distance between lines, then the integral in the right hand side is zero.

In the following proposition we will show that pdist is a pseudometric on the space of convex planar curves. In other words, it is a non-negative real function defined on pairs of curves and satisfying 1) the triangle inequality, 2) the symmetry condition $\operatorname{pdist}(C_1, C_2) = \operatorname{pdist}(C_2, C_1)$, 3) the condition $\operatorname{pdist}(C, C) = 0$.

Proposition 2.3. *The space of all convex curves equipped with the distance from Definition 2.1 is a pseudometric space. Moreover, the distance is convex with respect to Minkowski addition, that is,*

$$pdist(tC_1 + (1-t)C_2, D) \leq t\, pdist(C_1, D) + (1-t)\, pdist(C_2, D),$$

for all convex curves C_1, C_2, D and for any $t \in [0, 1]$. The equality for $t \in (0, 1)$ holds only if $\ell_\theta(C_1)$ and $\ell_\theta(C_2)$ are on the same side of $\ell_\theta(D)$ for all θ.

Proof. The triangle inequality follows immediately from the one-dimensional triangle inequality for each θ, that is, from

$$\mathrm{dist}(\ell_\theta(C_1), \ell_\theta(C_3)) \le \mathrm{dist}(\ell_\theta(C_1), \ell_\theta(C_2)) + \mathrm{dist}(\ell_\theta(C_2), \ell_\theta(C_3)).$$

The remaining conditions of a pseudometric follow trivially from the definition.

For convexity, consider θ such that all three normal lines $\ell_\theta(C_1)$, $\ell_\theta(C_2)$, and $\ell_\theta(tC_1 + (1-t)C_2)$ are uniquely defined (as mentioned above, there are only countably many values of θ when it is not true). Note that if the linear functional $u_\theta \cdot x$ attains its unique minimum on C_1 at v_1 and on C_2 at v_2, then its unique minimum on $tC_1 + (1-t)C_2$ is clearly attained at $tv_1 + (1-t)v_2$. This immediately implies that the normal line of a Minkowski sum is the Minkowski sum of normal lines of the summands:

$$\ell_\theta(tC_1 + (1-t)C_2) = t\ell_\theta(C_1) + (1-t)\ell_\theta(C_2).$$

Now it remains to use the fact that for each θ the distance to a given line $\ell_\theta(D)$ is a convex function (in other terms, for a fixed real a, the real function $|x-a|$ is convex). If $\ell_\theta(C_1)$ and $\ell_\theta(C_2)$ are in different open half-planes bounded by $\ell_\theta(D)$ then, due to continuity, there is an interval of θ where this holds and the inequality must be strict. □

A single point can be considered a degenerate strictly convex curve. Definition 2.1 and Proposition 2.3 work in this case just the same. Moreover, for two points v_1 and v_2, $\mathrm{pdist}(v_1, v_2)$ coincides with the regular Euclidean distance multiplied by 2.

The distance from Definition 2.1 does not define a metric because there are curves that are different but have the same normal line bundles, for example, concentric circles.

Similarly, we can generalize a perimeter of a convex curve.

Definition 2.4. For convex curves C and D,

$$\mathrm{pper}_D(C) = \frac{1}{2}\int_0^{2\pi} (\mathrm{dist}(\ell_\theta^r(C), \ell_\theta(D)) + \mathrm{dist}(\ell_\theta^l(C), \ell_\theta(D)))d\theta,$$

where $\ell_\theta^r(C)$ and $\ell_\theta^l(C)$ are supporting lines to C parallel to u_θ.

Note that the standard perimeter (curve length) $\mathrm{per}(C)$ can be obtained by a similar formula with $\mathrm{dist}(\ell_\theta^r(C), \ell_\theta^l(C))$ under the integral. This is essentially a version of the Crofton formula [12, Section 1.5] in the case of a convex closed curve.

Lemma 2.5 (Crofton formula)**.** *For a convex curve C,*

$$per(C) = \frac{1}{2}\int_0^{2\pi} dist(\ell_\theta^r(C), \ell_\theta^l(C))d\theta,$$

where $\ell_\theta^r(C)$ and $\ell_\theta^l(C)$ are supporting lines to C parallel to u_θ.

The Crofton formula immediately implies the monotonicity of perimeters for nested convex curves: if a convex curve C is in inside a convex curve C', then $\mathrm{per}(C) \le \mathrm{per}(C')$. We will use this property throughout the paper. We have to note though that a similar property does generally not work for pper, that is, $\mathrm{pper}_D(C)$ may be larger than $\mathrm{pper}_D(C')$ for some convex curve D.

The triangle inequality implies that $\mathrm{pper}_D(C) \ge \mathrm{per}(C)$ and the equality holds if and only if each normal line of D intersects C. This happens, for instance, when D is a disk containing C and the center of D is inside C so pper is indeed a generalization of the standard Euclidean perimeter.

It immediately follows from the definitions that for any point v and any curve D,

$$\mathrm{pper}_D(v) = 2\mathrm{pdist}(v, D).$$

For curves of constant width, the definitions above become even more convenient. In particular, we can substitute $\frac{1}{2}\int_0^{2\pi}$ by $\int_0^{\pi}$ both in Definition 2.1 and Definition 2.4 because $\ell_\theta = \ell_{\theta+\pi}$ for each $\theta \in [0, \pi]$. Let us give a brief explanation for the latter fact (see also [9, Theorem 3.1.1]).

If D is a curve of constant width, then its width in any direction is equal to the diameter of D. Now, let ℓ and ℓ' be parallel supporting lines touching D at v and v', respectively. On the one hand, $|vv'| \geq \mathrm{dist}(\ell, \ell') = \mathrm{diam}(D)$. On the other hand, $|vv'| \leq \mathrm{diam}(D)$. Therefore, $|vv'| = \mathrm{dist}(\ell, \ell')$ and vv' must be orthogonal to both supporting lines so vv' is a normal line both at v and v'. We conclude that $\ell_\theta = \ell_{\theta+\pi}$ for each θ. Since $|vv'| = \mathrm{diam}(D)$, we will call such points v and v' diametrically opposite.

If D is a circle, then $\mathrm{pdist}(v, D) = 2|vo|$, where o is the center of D. Hence all points v in the circle D satisfy $\mathrm{pdist}(v, D) = \mathrm{diam}(D)$. The following lemma extends this property to all curves of constant width.

Lemma 2.6. *For a constant width curve D and a point v, $\mathrm{pdist}(v, D) = \mathrm{diam}(D)$ for $v \in D$, $\mathrm{pdist}(v, D) < \mathrm{diam}(D)$ for v inside D, and $\mathrm{pdist}(v, D) > \mathrm{diam}(D)$ for v outside D.*

In order to prove this lemma, we will need the following result about convex curves. Assume γ is a strictly convex curve parametrized by inward normal vectors, that is, $\gamma(\theta)$ is a point on the curve, where the inward normal vector is u_θ (γ is not injective for singular points of the curve). We do not need a precise form of this parametrization but the mere fact that $\gamma(\theta)$ is almost everywhere differentiable (this follows from [9, Theorem 5.1.1] or from the theorem of Aleksandrov [2]). For each θ, we define a unit tangent vector w_θ by rotating u_θ by $\pi/2$ clockwise.

Lemma 2.7. *For all strictly convex curves γ and for any pair of θ_1, θ_2 such that $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \gamma(\theta) \cdot w_\theta \, d\theta = \gamma(\theta_1) \cdot u_{\theta_1} - \gamma(\theta_2) \cdot u_{\theta_2}.$$

Proof. Differentiating $\gamma(\theta) \cdot u_\theta$ we get $\dot\gamma \cdot u_\theta + \gamma \cdot (-w_\theta) = -\gamma \cdot w_\theta$ for almost all θ so

$$\int_{\theta_1}^{\theta_2} \gamma(\theta) \cdot w_\theta \, d\theta = \int_{\theta_1}^{\theta_2} -\frac{d}{d\theta}\left(\gamma \cdot w_\theta\right) \, d\theta = \gamma(\theta_1) \cdot u_{\theta_1} - \gamma(\theta_2) \cdot u_{\theta_2},$$

□

Remark 2.8. This lemma may be extended to general convex curves by taking into account line segments in D or carefully approximating D with strictly convex curves.

We also note that all curves of constant width are naturally characterized by their parametrizations using the support function [7], [8], [9, Theorem 5.3.5].

The dot product $\gamma(\theta) \cdot w_\theta$ is precisely the distance from the origin 0 to the normal line ℓ_θ taken with a sign. The integral calculated in Lemma 2.7 can be used to calculate a part of $\mathrm{pdist}(0, \gamma)$ as $|\gamma(\theta_1) \cdot u_{\theta_1} - \gamma(\theta_2) \cdot u_{\theta_2}|$ when the origin lies on the same side of all normal lines ℓ_θ for all $\theta \in [\theta_1, \theta_2]$. This is exactly what we are going to do in the proof of Lemma 2.6.

Proof of Lemma 2.6. Let $v \in D$ and assume that u_{θ_1} is the inward normal vector to D at v. Without loss of generality, $v = 0$ and $\theta_1 = 0$. Parametrizing the curve as above, $v = \gamma(0)$. Let $v' = \gamma(\pi)$. Note that v' is diametrically opposite to v on D.

Clearly, v lies on the same side of all ℓ_θ for $\theta \in (0, \pi)$ (v may belong to an infinite set of these lines if it is a singular point of the curve). By Lemma 2.7,

$$\operatorname{pdist}(v, D) = \int_0^\pi \operatorname{dist}(v, \ell_\theta)\, d\theta = |\gamma(0) \cdot u_0 - \gamma(\pi) \cdot u_\pi| = |\gamma(\pi)| = \operatorname{diam}(D).$$

Let v be inside D. Assume v_1 is the point in D closest to v. Then the line $v_1 v$ is a normal line of D. Let v_2 be the second point of intersection of D and $v_1 v$. v_1 and v_2 are diametrically opposite points of D. There is a normal line $\ell_\theta(D)$ such that v_1 and v_2 are in different open half-planes with respect to this line (for example, when u_θ is orthogonal to the line $v_1 v_2$). By the convexity part of Proposition 2.3, $\operatorname{pdist}(v, D)$ is strictly smaller than $t \operatorname{pdist}(v_1, D) + (1-t) \operatorname{pdist}(v_2, D)$ for some $t \in (0, 1)$. Due to the first part of the proof, both $\operatorname{pdist}(v_1, D)$ and $\operatorname{pdist}(v_2, D)$ are equal to $\operatorname{diam}(D)$. Therefore, $t \operatorname{pdist}(v_1, D) + (1 - t) \operatorname{pdist}(v_2, D) = \operatorname{diam}(D)$ for some $t \in (0, 1)$. and, subsequently, $\operatorname{pdist}(v, D)$ is strictly smaller than $\operatorname{diam}(D)$.

Let v be outside D. Connect it to any point v_1 inside D. Let v_2 be the point of intersection of vv_1 and D. Then $\operatorname{pdist}(v_2, D) = \operatorname{diam}(D)$ and $\operatorname{pdist}(v_1, D) < \operatorname{diam}(D)$. By the convexity part of Proposition 2.3, $\operatorname{pdist}(v_2, D) \le t \operatorname{pdist}(v_1, D) + (1-t) \operatorname{pdist}(v, D)$. We conclude that

$$\operatorname{pdist}(v, D) \ge \frac{1}{1-t} (\operatorname{pdist}(v_2, D) - t \operatorname{pdist}(v_1, D)) > \operatorname{diam}(D).$$

□

In the last lemma of this section, we prove the connection between perimeters of convex bodies in a partition and distances from vertices of this partition.

Lemma 2.9. *If a convex body with boundary C is partitioned into convex bodies with boundaries C_i with all partition vertices v_j of degree 3, then for any convex curve D,*

$$\sum_i \mathit{pper}_D(C_i) = \mathit{pper}_D(C) + \sum_j \mathit{pdist}(v_j, D).$$

Proof. There are countably many values of θ such that C or one of C_i has a supporting line parallel to u_θ with more than one common point with it. It is, therefore, sufficient to check that for all other θ, the integrand is the same in both sides of the suggested equality. Integrands in both sides contain distances from lines parallel to u_θ to $\ell_\theta(D)$. We just need to carefully check that the lines are the same in the right and the left hand sides.

First, we take into account lines that do not go through vertices of the partition. They show up as supporting lines of C in the right hand side and then they show up in the left hand side too, as supporting lines of C_i.

All the remaining lines are going through vertices of the partition. There are two possible scenarios: all partition edges from a vertex v_j are on one side with respect to a line parallel to u_θ or three edges are split into two non-empty groups. In the former case, v_j is necessarily in C and the line through v_j contributes to the left hand side twice, in two $\operatorname{pper}(C_i)$, and to the right hand side twice, in $\operatorname{pper}(C)$ and one $\operatorname{pdist}(v_j)$. In the latter case, the line through v_j contributes only to one of $\operatorname{pper}(C_i)$ and to one $\operatorname{pdist}(v_j)$. □

3. Preparatory work and partitions

In this section, we show preliminary work to reduce the problem to curves of constant width and provide a short proof of Theorem 1.1 for the case of partitions. For the first step, we use the structural result from the paper of Pinchasi.

Proposition 3.1. [10] *For disjoint convex bodies C_1, C_2,..., C_k inside the convex body D, there exist a partition of D into convex bodies C'_1, C'_2, ..., C'_{k+l}, $l \geq 0$, such that $C_i \subseteq C'_i$ for all $1 \leq i \leq k$, and all C'_j, $k+1 \leq j \leq k+l$, have no common points with each other and with the boundary of D.*

Sets C'_j, $k+1 \leq j \leq k+l$, are called *holes* of the partition. This technical result is not explicitly formulated in [10] but it directly follows from Claim 1 and Claim 2 in [10], which we refer to for the rigorous proof. Here we provide a quick, and straightforward, idea of the proof. We consider any maximal by inclusion convex extension of the bodies C_i into convex bodies C'_i. We need to show, that the holes, connected components of $D \setminus \cup C'_i$, are convex. Note that any side of a body cannot be extended inside a hole only if its part is "blocked" by another body. After a brief analysis, it is easy to see that the boundary of a hole is formed by bodies that are cyclically bounding each other (see Figure 2). In the case of a nonconvex hole, there is no blocking at a nonconvex vertex, so the cycle cannot be formed (see Figure 1). A similar observation shows that holes cannot have common points with the boundary of D and with each other.

Note that $\sum_1^k \operatorname{per}(C'_i) \geq \sum_1^k \operatorname{per}(C_i)$ so it is sufficient to prove the bound of Theorem 1.1 for the extended case only.

For the next step, we explain how to extend this partition to a partition of a constant width body D', $D \subseteq D'$. We will need the following lemma.

Lemma 3.2. [1, Lemma 4] *Any finite convex partition of a convex body in $\mathbb{R}^2$ can be extended to a convex partition of $\mathbb{R}^2$.*

We extend the partition of D to the partition $\tilde{C}_1$, ..., $\tilde{C}_{k+l}$ of the plane by Lemma 3.2 and use it to get a partition of D' into $C''_1 = \tilde{C}_1 \cap D'$, ..., $C''_{k+l} = \tilde{C}_{k+l} \cap D'$. Note that this extension does not change any convex parts that were strictly inside D including all holes. Let l_1, l_2, ..., l_n be the lengths of new line segments added to the extended parts of the partition (see Figure 2). Then

$$\sum_{i=1}^{k} \operatorname{per}(C'_i) - \operatorname{per}(D) = \sum_{i=1}^{k} \operatorname{per}(C''_i) - \operatorname{per}(D') - 2\sum_{j=1}^{n} l_j.$$

If D' has the same diameter as D, then it is sufficient to prove the bound for D'. For any convex body D, there is a constant width body D' such that $D \subseteq D'$ and $\operatorname{diam}(D') = \operatorname{diam}(D)$ [5, Theorem 54] so it is sufficient for us to prove the bound of Theorem 1.1 for the case when the large convex body has constant width.

Boundaries of the partition sets form a plane graph. By adding vertices and introducing degenerate edges and faces we can assume that all vertices in this graph have degree 3. Here we provide a brief sketch of the explanation that this assumption is valid. In the first scenario, assume there is a vertex p of the plane graph surrounded by non-holes and connected to vertices p_1, ..., p_r, consecutively. Then we can introduce a degenerate hole $p'_1 \dots p'_r$ (all vertices of the hole geometrically coincide with p and are distinguishable as planar graph vertices only) with degenerate edges $p'_1p'_2$, ..., $p'_rp'_1$ so that $p_1p'_1$, ..., $p_rp'_r$ are also edges of the plane graph. In the second scenario, assume there is a vertex p of degree at least 4 on the boundary of the body D or p is a vertex of one of the holes. Let p be connected to p_1, ..., p_r, consecutively, with pp_1 and pp_r on the boundary of

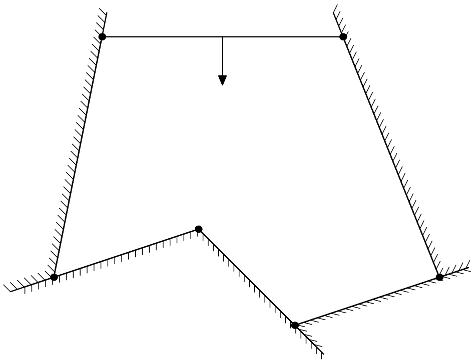

FIGURE 1. Extension of a non-convex hole

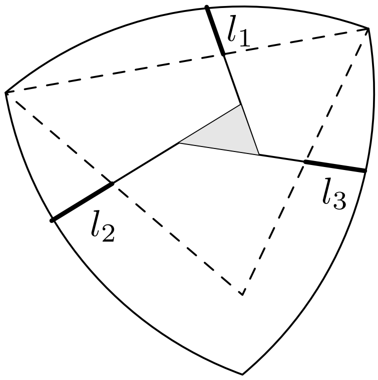

FIGURE 2. Extending a partition of a triangle to the Reuleaux triangle of the same diameter

D/hole. Then we can add vertices $p'_1, \dots, p'_r$ (again geometrically coinciding with p) with degenerate edges $p'_1p'_2$, …, $p'_{r-1}p'_r$ lying consecutively on the boundary of D/hole so that $p_1p'_1$, …, $p_rp'_r$ are also edges of the plane graph.

After all reductions, Theorem 1.1 follows from the following theorem.

Theorem 3.3. *A constant width body D is partitioned into convex bodies C_1, …, C_k, H_1, …, H_l such that all H_j, $1 \le j \le l$, are pairwise disjoint and have no common points with the boundary of D and all vertices of the graph defined by the partition have degree 3. Then*

$$\sum_{i=1}^{k} pper(C_i) \le per(D) + 2(k-1)diam(D).$$

At the end of this section we give a very short proof of Theorem 3.3 in the case when the partition of D contains no holes, that is, D is partitioned into convex bodies C_1, …, C_k. This proof, in particular, implies Theorem 1.1 for the case when C_i partition C.

Proof for the case of partitions. By Euler's formula, the number of partition vertices is $2(k-1)$. Then be Lemma 2.9,

$$\sum_{i=1}^{k} \mathrm{pper}_D(C_i) = \mathrm{per}(D) + \sum_{j=1}^{2(k-1)} \mathrm{pdist}(v_j, D)$$

By Lemma 2.6, all $\mathrm{pdist}(v_j, D) \le \mathrm{diam}(D)$, so

$$\sum_{i=1}^{k} \mathrm{pper}_D(C_i) \le \mathrm{per}(D) + 2(k-1)\mathrm{diam}(D).$$

□

4. PROOF OF THE GENERAL CASE

Before proving Theorem 3.3 we give a proof to a key lemma.

Lemma 4.1. *For any convex polygon P with vertices a_i, $1 \le i \le m$, inside a constant width body D,*

$$\sum_1^m pdist(a_i, D) - pper_D(P) \le (m-2)\,diam(D).$$

Proof. The proof of this lemma is fairly involved and consists of several steps and ideas so we first provide a brief plan describing major steps.

(1) By induction, we show that it is sufficient to prove the lemma for $m = 3$.
(2) In the case of triangle, we reformulate the problem via interjacent lines of the triangle and show that, due to convexity, the maximum of the left hand side of the inequality is necessarily attained on a triangle $\triangle a_1a_2a_3$ with at least two vertices, a_1 and a_2, on the boundary of D.
(3) We explain why it is sufficient to prove the bound for smooth curves of constant width with curvatures bounded from above.
(4) We analyze how the expression under consideration varies when vertices a_1 and a_2 are perturbed by moving them along the smooth curve. A straightforward, but lengthy, calculus of variations shows that the maximum is possible only for $\triangle a_1a_2a_3$ such that the normal lines at a_1 and a_2 bisect the corresponding interior angles of the triangle.
(5) In the final step we use a geometric lemma of Balitskiy to tackle the case of angle bisectors as normals.

(1) First, we show by induction that it is sufficient to prove this lemma for triangles. Indeed, if we already know that the statement holds for all polygons with not more than $m-1$ sides, $m \ge 4$, then we can partition an m-gon P into an $(m-1)$-gon P_1 with the set of vertices S_1 and a triangle P_2 with the set of vertices S_2. By the inductive hypothesis,

$$\sum_{a\in S_1} \mathrm{pdist}(a, D) + \sum_{a\in S_2} \mathrm{pdist}(a, D) \le \mathrm{pper}_D(P_1) + (m-3)\mathrm{diam}(D) + \mathrm{pper}_D(P_2) + \mathrm{diam}(D).$$

By Lemma 2.9,

$$\mathrm{pper}_D(P_1) + \mathrm{pper}_D(P_2) = \mathrm{pper}_D(P) + \sum_{a\in S_1\cap S_2} \mathrm{pdist}(a, D).$$

Substituting this sum in the inequality above we get the required bound for the m-gon P.

(2) For a triangle $a_1a_2a_3$,

$$\sum_1^3 \mathrm{pdist}(a_i, D) - \mathrm{pper}_D(a_1a_2a_3) = \int_0^\pi \mathrm{dist}(\ell_\theta^{mid}, \ell_\theta(D))d\theta, \tag{4.1}$$

where ℓ_θ^{mid} is the interjacent line of the triangle in direction of u_θ, that is, ℓ_θ^{mid} goes through a vertex and crosses the triangle. If $a_1a_2a_3$ is a degenerate triangle, one point or a line segment, the integral above is the same as $\mathrm{pdist}(a, D)$ for one of the vertices a so it is not larger than $\mathrm{diam}(D)$ by Lemma 2.6.

For a nondegenerate triangle $a_1a_2a_3$, we consider a family of triangles with sides parallel to the sides of $\triangle a_1a_2a_3$. For this family, the integral in (4.1) is convex with respect to Minkowski sums because expressions under the integral are convex (the argument is analogous to the one from the proof of Proposition 2.3). Therefore, among all triangles in the family, the maximum of (4.1) is necessarily attained on a triangle with at least two vertices on the boundary of D.

(3) Each curve of constant width can be approximated by a smooth curve of constant width (see, for instance, [11, 13]) so we assume that D has a smooth boundary. We can also assume that the curvature of the boundary is bounded from above (for instance, by approximating D with $D+\varepsilon\omega$, where ω is a circle with center at the origin and unit radius). The function $\operatorname{pdist}(a_1, D)+\operatorname{pdist}(a_2, D)+\operatorname{pdist}(a_3, D)-\operatorname{pper}_D(a_1a_2a_3)$ is continuous with respect to a_1, a_2, $a_3 \in D$ so it reaches its maximum at some triple of points. We consider the triple where it is maximal and, using the argument from above, assume a_1 and a_2 are on the boundary of D. We also assume that the boundary of D is parametrized by $\gamma(\theta)$.

(4) In this step of the proof, we show that the normal lines through a_1 and a_2 are necessarily interior angle bisectors of $\triangle a_1a_2a_3$. This is the longest and the most involved part of the proof so we first provide a quick sketch of the argument.

Assume the normal line at a_1 is not the angle bisector of $\triangle a_1a_2a_3$. The idea of the proof is to move a_1 along the curve by a small distance Δ and show that the integral can increase with this move. When doing so, all interjacent lines ℓ_θ^{mid} through a_1 change linearly in terms of Δ and we will show that this change entails the linear part of the change in 4.1. At the same time, the measure of θ such that lines ℓ_θ^{mid} change a vertex they pass through is also of linear size in terms of Δ so in total the change of this kind gives only an $O(\Delta^2)$ error. Overall, the change is linear in Δ so we can move a_1 to increase the integral.

Let us now write this argument in more detail. Denote by ℓ_θ^1, ℓ_θ^2, ℓ_θ^3 the lines with direction of u_θ through a_1, a_2, a_3, respectively. Assume ℓ_θ^1 is an interjacent line when $\theta \in [\theta_1, \theta_2]$, ℓ_θ^2 is an interjacent line when $\theta \in [\theta_2, \theta_3]$, and ℓ_θ^3 is an interjacent line when $\theta \in [\theta_3, \theta_1 + \pi]$. Also assume ℓ_{θ^*} is a normal line through a_1 and $\theta^* \in (\theta_1, \theta_2)$ but $\theta^* \neq (\theta_1 + \theta_2)/2$ (the case when $\theta^* \notin (\theta_1, \theta_2)$ will be considered later). Under this notation,

$$\int_0^\pi \operatorname{dist}(\ell_\theta^{mid}, \ell_\theta(D))d\theta = \int_{\theta_1}^{\theta_2} \operatorname{dist}(\ell_\theta^1, \ell_\theta(D))d\theta + \int_{\theta_2}^{\theta_3} \operatorname{dist}(\ell_\theta^2, \ell_\theta(D))d\theta + \int_{\theta_3}^{\theta_1+\pi} \operatorname{dist}(\ell_\theta^3, \ell_\theta(D))d\theta.$$

Now we vary a_1 by moving it along the boundary of D to a point a_1' so that $|a_1a_1'| = \Delta$ and denote by $\ell_\theta^{1'}$ a line in direction of u_θ through a_1'. At this point our goal is to show that the change of $\int_0^\pi \operatorname{dist}(\ell_\theta^{mid}, \ell_\theta(D))d\theta$ is linear with respect to Δ. First, we denote new interjacent lines by $\ell_\theta^{mid'}$ and note that $\operatorname{dist}(\ell_\theta^{mid}, \ell_\theta(D)) - \operatorname{dist}(\ell_\theta^{mid'}, \ell_\theta(D)) \in O(\Delta)$ for all θ. Second, we assume that new angles where interjacent lines change are θ_1', θ_2', and θ_3' and note that $\theta_i - \theta_i' \in O(\Delta)$ for $i = 1, 2, 3$. Then

$$\int_0^\pi \operatorname{dist}(\ell_\theta^{mid'}, \ell_\theta(D))d\theta = \int_{\theta_1'}^{\theta_2'} \operatorname{dist}(\ell_\theta^{1'}, \ell_\theta(D))d\theta + \int_{\theta_2'}^{\theta_3'} \operatorname{dist}(\ell_\theta^2, \ell_\theta(D))d\theta + \int_{\theta_3'}^{\theta_1'+\pi} \operatorname{dist}(\ell_\theta^3, \ell_\theta(D))d\theta$$

$$= \int_{\theta_1}^{\theta_2} \operatorname{dist}(\ell_\theta^{1'}, \ell_\theta(D))d\theta + \int_{\theta_2}^{\theta_3} \operatorname{dist}(\ell_\theta^2, \ell_\theta(D))d\theta + \int_{\theta_3}^{\theta_1+\pi} \operatorname{dist}(\ell_\theta^3, \ell_\theta(D))d\theta + O(\Delta^2)$$

so the linear part of the change may show up only in the first integral.

Using Lemma 2.7 we get

$$\int_{\theta_1}^{\theta_2} \operatorname{dist}(\ell_\theta^1, \ell_\theta(D))d\theta = \int_{\theta_1}^{\theta^*} (a_1 - \gamma(\theta)) \cdot w_\theta \, d\theta + \int_{\theta^*}^{\theta_2} (\gamma(\theta) - a_1) \cdot w_\theta \, d\theta$$

$$= (a_1 - \gamma(\theta_1)) \cdot u_{\theta_1} - 2(a_1 - \gamma(\theta^*)) \cdot u_{\theta^*} + (a_1 - \gamma(\theta_2)) \cdot u_{\theta_2}$$
$$= a_1 \cdot (u_{\theta_1} - 2u_{\theta^*} + u_{\theta_2}) - \gamma(\theta_1) \cdot u_{\theta_1} + 2\gamma(\theta^*) \cdot u_{\theta^*} - \gamma(\theta_2) \cdot u_{\theta_2}.$$

Assume $\ell_{\theta^{**}}$ is the normal line at a_1', $\theta^{**} \in (\theta_1, \theta_2)$. Note that $\theta^{**} - \theta^* \in O(\Delta)$ because the curvature of the boundary of D is bounded from above. Then

$$\int_{\theta_1}^{\theta_2} \mathrm{dist}(\ell_\theta^{1'}, \ell_\theta(D))d\theta = \int_{\theta_1}^{\theta^{**}} (a_1' - \gamma(\theta)) \cdot w_\theta \, d\theta + \int_{\theta^{**}}^{\theta_2} (\gamma(\theta) - a_1') \cdot w_\theta \, d\theta$$
$$= \int_{\theta_1}^{\theta^*} (a_1' - \gamma(\theta)) \cdot w_\theta \, d\theta + \int_{\theta^*}^{\theta_2} (\gamma(\theta) - a_1') \cdot w_\theta \, d\theta + O(\Delta^2)$$
$$= a_1' \cdot (u_{\theta_1} - 2u_{\theta^*} + u_{\theta_2}) - \gamma(\theta_1) \cdot u_{\theta_1} + 2\gamma(\theta^*) \cdot u_{\theta^*} - \gamma(\theta_2) \cdot u_{\theta_2} + O(\Delta^2).$$
$$= \int_{\theta_1}^{\theta_2} \mathrm{dist}(\ell_\theta^1, \ell_\theta(D))d\theta + (a_1' - a_1) \cdot (u_{\theta_1} - 2u_{\theta^*} + u_{\theta_2}) + O(\Delta^2).$$

Due to the smoothness of the boundary of D, $a_1' - a_1 = \pm\Delta \, w_{\theta^*} + O(\Delta^2)$ so

$$\int_{\theta_1}^{\theta_2} \mathrm{dist}(\ell_\theta^{1'}, \ell_\theta(D))d\theta - \int_{\theta_1}^{\theta_2} \mathrm{dist}(\ell_\theta^1, \ell_\theta(D))d\theta = \pm\Delta \, w_{\theta^*} \cdot (u_{\theta_1} - 2u_{\theta^*} + u_{\theta_2}) + O(\Delta^2).$$

The dot product $w_{\theta^*} \cdot (u_{\theta_1} - 2u_{\theta^*} + u_{\theta_2})$ is not zero since $\theta^* \neq (\theta_1 + \theta_2)/2$ so there is a linear component in the change of the integral. We choose the direction of the move depending on the sign of the dot product to ensure that the difference is positive and the integrals increases. Overall, we conclude the maximal value cannot be attained on a triangle unless the normal line through a_1 is the interior angle bisector of $\triangle a_1 a_2 a_3$.

In the case $\theta^* < \theta_1$, there is a direction along the boundary of D such that moving a_1 in this direction increases all $\mathrm{dist}(\ell_\theta^1, \ell_\theta(D))$ linearly in Δ. This move entails a linear increase of the integral from (4.1). The other cases for θ^* are analogous with additional $O(\Delta^2)$ terms for $\theta^* = \theta_1$ or θ_2 (because θ^{**} may jump inside the interval (θ_1, θ_2)).

We have showed that the normal line at a_1 is the angle bisector of the triangle. Analogously, the normal line at a_2 must be the interior angle bisector as well.

(5) For the last step of the proof, we will use the following geometric fact by Balitskiy.

Lemma 4.2. [4, Proof of Theorem 4.1] *Let points y_1 and y_2 be chosen on the interior angle bisectors of $\triangle x_1 x_2 x_3$ from x_1 and x_2, respectively. If $|x_1 y_1| = |x_2 y_2| \geq \frac{1}{2} per(x_1 x_2 x_3)$, then $|y_1 y_2| > |x_1 y_1|$.*

Now we choose the points b_1 and b_2 in D diametrically opposite to a_1 and a_2, respectively. We know that $|a_1 b_1| = |a_2 b_2| = \mathrm{diam}(D)$ and $|b_1 b_2| < \mathrm{diam}(D)$. By Lemma 4.2, $\mathrm{per}(a_1 a_2 a_3) \geq 2|a_1 b_1| = 2\mathrm{diam}(D)$. Then, as required,

$$\sum_{1}^{3} \mathrm{pdist}(a_i, D) - \mathrm{pper}_D(a_1 a_2 a_3) \leq 3\mathrm{diam}(D) - \mathrm{per}(a_1 a_2 a_3) \leq \mathrm{diam}(D).$$

□

Proof of Theorem 3.3. By Euler's formula, the number of partition vertices is $2(k+l-1)$. Then by Lemma 2.9,

$$\sum_{i=1}^{k} \mathrm{pper}_D(C_i) = \mathrm{per}(D) + \sum_{j=1}^{2(k+l-1)} \mathrm{pdist}(v_j, D) - \sum_{i=1}^{l} \mathrm{pper}_D(H_i)$$

Using Lemma 4.1 for all holes we get

$$\sum_{i=1}^{k} \mathrm{pper}_D(C_i) \leq \mathrm{per}(D) + \sum_{v_j \notin \cup H_i} \mathrm{pdist}(v_j, D) + \sum_{v_j \in \cup H_i} \mathrm{diam}(D) - 2l \ \mathrm{diam}(D)$$

Finally, by Lemma 2.6, all $\mathrm{pdist}(v_j, D) \leq \mathrm{diam}(D)$, so

$$\sum_{i=1}^{k} \mathrm{pper}_D(C_i) \leq \mathrm{per}(D) + 2(k + l - 1)\mathrm{diam}(D) - 2l \ \mathrm{diam}(D)$$
$$= \mathrm{per}(D) + 2(k - 1)\mathrm{diam}(D).$$

□

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