Designing Control Barrier Functions for Underactuated Euler–Lagrange Systems using Dynamic Safety Margins

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***Abstract*— This letter shows how to design control barrier functions for underactuated and fully-actuated Euler-Lagrange systems subject to state and input constraints. The proposed method uses passivity-based considerations to limit the total energy available to the system and ensure constraint satisfaction. The approach can handle multiple state and input constraints regardless of relative degree.**

***Index Terms*— Constrained control, control barrier functions, Euler–Lagrange systems, Lyapunov methods,**

I. Introduction

CONTROL barrier functions (CBFs) are safety certificates that can be used to synthesize constrained control policies. While they have achieved good results in practice [1], finding a CBF for general systems remains an open question. In recent works [2]–[4], tools from the reference governor literature have been used to construct CBFs. However, these techniques rely on a prestabilizing controller, which may itself be hard to find for general systems. In this letter, we focus on Euler–Lagrange (EL) systems and show how to use energy-based considerations to build CBFs.

Other works that have studied the design of valid CBFs include [5], which uses sum of squares; [6], which uses finite-horizon predictions; and [7], which relies on a backup policy that stabilizes to a fixed point. Some works that use CBFs for EL systems include [8], which addresses the lacking relative-degree using the time to collision in their CBF expression; [9], which achieves safe control of unknown EL systems by decomposing the task in two and using a combination of barrier Lyapunov functions (BLFs) and CBFs; and [10], which proposes using input-to-state safe CBFs to safely achieve formation control for multiple EL systems. These works do not consider input constraints and only apply to fully-actuated EL systems. While [11] constructs valid CBFs for EL systems subject to interval state and input constraints, their results apply only to fully- and overactuated systems. Other constrained control approaches specialized to EL systems include explicit reference governors (ERGs) [12] and BLFs [13].

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In this letter, we consider the constrained control problem of underactuated and fully-actuated EL systems subject to state and input constraints. We recall how to design a passivity-based prestabilizing controller and show that the virtual energy of the closed-loop system is a Lyapunov function. Finally, we construct Lyapunov-based dynamic safety margins (DSMs), which were shown in [4] to be valid CBFs. The approach in [1] also employs Lyapunov functions to construct valid CBFs. While similar to our construction, they rely on control Lyapunov functions and select the relevant CBF-condition based on a switching internal state, resulting in a hybrid system. Conversely, we prestabilize the system and perform a dynamic extension on the applied reference.

The effectiveness of the constructed DSM-based CBFs is studied in simulation examples and compared to the ERG [12] and candidate exponential CBFs [14].

II. Preliminaries

A. Control Barrier Functions

Modern control barrier functions provide certificates of control invariance that can be used for optimization-based constrained control [15]. Given a control-affine system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}, \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}^n$ and $g:\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous functions, let the closed sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ describe the state and input constraints, respectively.

Definition 1: [15] A continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ is a *control barrier function* (CBF) if there exists a class $\mathcal{K}_\infty$ function $\alpha : [0,\infty) \to [0,\infty)$ such that

$$\forall \mathbf{x} \in \mathcal{C}, \quad \sup_{\mathbf{u}\in\mathcal{U}} \left[L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u}\right] \geq -\alpha\big(h(\mathbf{x})\big), \tag{2}$$

where $L_f h$ and $L_g h$ are the Lie derivatives of h along f and g, respectively, and $\mathcal{C} = \{\mathbf{x} \in \mathcal{D} \mid h(\mathbf{x}) \geq 0\} \subset \mathbb{R}^n$.
It has been shown [15] that the zero superlevel set $\mathcal{C}$ is control invariant. Provided that $\mathcal{C} \subset \mathcal{X}$, this result certifies that the solution to the following CBF-based program exists and is a valid constrained control policy

$$\begin{aligned} \min_{\mathbf{u}\in\mathcal{U}} \quad & \|\mathbf{u} - \kappa(\mathbf{x})\|^2 \\ \text{s.t.} \quad & L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} \geq -\alpha\big(h(\mathbf{x})\big), \end{aligned} \tag{3}$$

where $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is a nominal control policy with desired performance properties. Note that, if $\mathcal{U}$ is a convex set, then (3) is a convex program and can be solved efficiently. Although simple to implement, the main drawback of CBFs is that they are difficult to find in general. In [4], the authors showed how to construct CBFs using DSMs.

B. Dynamic Safety Margins

Dynamic safety margins are a key component of explicit reference governors [16]. In broad terms, they measure the distance to constraint violation of a prestabilized system.

Let $\mathbf{v} \in \mathbb{R}^m$ be a parametrization of the steady-state manifold of a system and let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a prestabilizing controller for (1) such that $\forall \mathbf{v} \in \mathbb{R}^m$, the state $\bar{x}(\mathbf{v})$ is an asymptotically stable equilibrium point of

$$\dot{\mathbf{x}} = f_\pi(\mathbf{x}, \mathbf{v}) \triangleq f(\mathbf{x}) + g(\mathbf{x})\pi(\mathbf{x}, \mathbf{v}). \tag{4}$$

The steady-state input required to maintain equilibrium is denoted $\bar{u}(\mathbf{v}) = \pi\big(\bar{x}(\mathbf{v}), \mathbf{v}\big)$.

Definition 2: A continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a *reference-dependent Lyapunov function* for the prestabilized system f_π if, $\forall \mathbf{v} \in \mathbb{R}^m$, there exists a neighborhood of $\bar{x}(\mathbf{v})$, denoted $\mathcal{D}_\mathbf{v} \subset \mathbb{R}^n$, wherein

$$V\big(\bar{x}(\mathbf{v}), \mathbf{v}\big) = 0, \tag{5a}$$

$$V(\mathbf{x}, \mathbf{v}) > 0, \quad \forall \mathbf{x} \in \mathcal{D}_\mathbf{v} \setminus \{\bar{x}(\mathbf{v})\}, \tag{5b}$$

$$\frac{\partial V}{\partial \mathbf{x}} f_\pi(\mathbf{x}, \mathbf{v}) \leq 0, \quad \forall \mathbf{x} \in \mathcal{D}_\mathbf{v}. \tag{5c}$$

DSMs can be designed using reference-dependent Lyapunov functions. To do so, let us begin by defining two sets

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^m \mid \bar{x}(\mathbf{v}) \in \mathcal{X}, \bar{u}(\mathbf{v}) \in \mathcal{U}\}, \tag{6}$$

$$\mathcal{X}_\mathbf{v} = \{\mathbf{x} \in \mathcal{X} \mid \pi(\mathbf{x}, \mathbf{v}) \in \mathcal{U}\}, \tag{7}$$

called, respectively, the steady-state admissible reference set and the reference-dependent state constraint set. We denote the complement of the reference-dependent state constraint set by $\mathcal{X}_\mathbf{v}^c = \mathbb{R}^n \setminus \mathcal{X}_\mathbf{v}$ (i.e., the unsafe set). Let us also define two important values of a reference-dependent Lyapunov function: The *safety threshold value*

$$\Gamma^*(\mathbf{v}) \triangleq \inf_{\mathbf{x} \in \mathcal{X}_\mathbf{v}^c \cap \mathcal{D}_\mathbf{v}} V(\mathbf{x}, \mathbf{v}) - \inf_{\mathbf{x} \in \mathcal{X}_\mathbf{v} \cap \mathcal{D}_\mathbf{v}} V(\mathbf{x}, \mathbf{v}), \tag{8}$$

which identifies the largest constraint-admissible level set of V; and the *stability threshold value*

$$\bar{\Gamma}(\mathbf{v}) \triangleq \inf_{\mathbf{x} \in \partial \mathcal{D}_\mathbf{v}} V(\mathbf{x}, \mathbf{v}), \tag{9}$$

which identifies the largest invariant level set of V contained in the closure of $\mathcal{D}_\mathbf{v}$.

Theorem 1: [4] Let V be a reference-dependent Lyapunov function that is strictly monotonically increasing in $\mathcal{D}_\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^m$. The function $\Delta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$\Delta(\mathbf{x}, \mathbf{v}) = \min\big(\Gamma^*(\mathbf{v}), \bar{\Gamma}(\mathbf{v})\big) - V(\mathbf{x}, \mathbf{v}), \tag{10}$$

is a dynamic safety margin for π.

As shown in [4], when differentiable, DSMs are valid CBFs for the augmented system obtained by concatenating (1) and the reference dynamics $\dot{\mathbf{v}} = \mathbf{w}$, where $\mathbf{w} \in \mathbb{R}^m$ is a virtual input. Moreover, DSMs are such that $\mathcal{C} = \{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \Delta(\mathbf{x}, \mathbf{v}) \geq 0\} \subset \mathcal{X} \times \mathcal{V}$. With this, we can obtain a constrained control policy by solving the following optimization problem

$$\begin{aligned} \min_{(\mathbf{u}, \mathbf{w}) \in \mathcal{U} \times \mathbb{R}^m} \quad & \|\mathbf{u} - \kappa(\mathbf{x})\|^2 + \eta \|\mathbf{w} - \rho(\mathbf{v})\|^2 \\ \text{s.t.} \quad & \frac{\partial \Delta}{\partial \mathbf{x}} f(\mathbf{x}) + \frac{\partial \Delta}{\partial \mathbf{x}} g(\mathbf{x})\mathbf{u} + \frac{\partial \Delta}{\partial \mathbf{v}} \mathbf{w} \geq -\alpha\big(\Delta(\mathbf{x}, \mathbf{v})\big), \end{aligned} \tag{11}$$

where $\eta > 0$ is a small scalar, α is a class $\mathcal{K}_\infty$ function, and $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is a nominal control law for the reference dynamics $\dot{\mathbf{v}} = \mathbf{w}$. A suitable choice is $\rho(\mathbf{v}) = \mathbf{r} - \mathbf{v}$, where $\mathbf{r} \in \mathbb{R}^m$ is a desirable setpoint.

Similar to (3), if $\mathcal{U}$ is convex, (11) is also convex and can be solved efficiently. The constrained control policy obtained by solving (11) guarantees safety $\mathbf{x}(t) \in \mathcal{X}$ and respects the input constraints $\mathcal{U}$. Moreover, for any pair $(\mathbf{x}, \mathbf{v})$ such that $\Delta(\mathbf{x}, \mathbf{v}) \geq 0$, the input pair $\big(\pi(\mathbf{x}, \mathbf{v}), 0\big)$ is feasible.

Remark 1: If $\mathcal{X}$ and $\mathcal{U}$ are obtained by intersecting multiple constraints, one can compute separate safety threshold values Γ_i^* and add them to the $\min$ in (10).

Remark 2: Instead of relying on differentiability of the $\min$ function in (10), the control-sharing property of CBFs [17, Definition 2] allows us to enforce as separate constraints $\Delta_0(\mathbf{x}, \mathbf{v}) = \bar{\Gamma}(\mathbf{v}) - V(\mathbf{x}, \mathbf{v})$, $\Delta_i(\mathbf{x}, \mathbf{v}) = \Gamma_i^*(\mathbf{v}) - V(\mathbf{x}, \mathbf{v})$, in (11) and retain feasibility. Moreover, if $\bar{\Gamma}$ and Γ_i^* are nonsmooth, any smooth lower-bound can be substituted.

Although finding Lyapunov functions can be challenging in general, this letter provides a closed-form solution by specializing the results to Euler–Lagrange systems.

C. Euler–Lagrange Systems

Consider the Euler–Lagrange model presented in [18]

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) + R\dot{\mathbf{q}} = B\mathbf{u}, \tag{12}$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized coordinates, $\mathbf{u} \in \mathbb{R}^m$ is the vector of inputs, $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is the mass matrix, $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is the Coriolis matrix, $G : \mathbb{R}^n \to \mathbb{R}^n$ is the gravity vector, $R \in \mathbb{R}^{n \times n}$ is the Rayleigh dissipation coefficient, and $B \in \mathbb{R}^{n \times m}$ is a full column rank matrix relating the external inputs to the generalized coordinates. In this work, we consider both fully and underactuated ($m \leq n$) EL systems. They have the following properties

Property 1: The mass matrix M is positive definite. That is, $\forall \mathbf{q} \in \mathbb{R}^n$, $M(\mathbf{q}) \succ 0$.

Property 2: For any $\mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^n$, the matrix difference $\dot{M}(\mathbf{q}, \dot{\mathbf{q}}) - 2C(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

EL systems (12) are control-affine systems (1) when we assign $\mathbf{x} = [\mathbf{q}; \dot{\mathbf{q}}]$. With this notation, $\mathbf{x} \in \mathbb{R}^{2n}$ instead of $\mathbb{R}^n$.

III. Constructing CBFs for EL Systems

In this letter, we make use of Theorem 1 to construct a DSM for EL systems, which is also a CBF. To this end, we need two pieces: A reference-dependent Lyapunov function that is strictly monotonically increasing, and its safety and stability threshold values (or smooth lower-bounds for them).

A. Reference-dependent Lyapunov Function

This section is dedicated to presenting a passivity-based prestabilizing controller π and an associated reference-dependent Lyapunov function V. Moreover, we show it is strictly monotonically increasing. While most results in this subsection are well-understood in the literature, they are rarely presented in the context of parameterized equilibrium manifolds.

To simplify the presentation, we assume without loss of generality that the actuated and unactuated channels are independent. That is, the generalized coordinates are $\mathbf{q} = [\mathbf{q}_a;\ \mathbf{q}_u]$, where $\mathbf{q}_a \in \mathbb{R}^m$ and $\mathbf{q}_u \in \mathbb{R}^{n-m}$. As a consequence, we can further assume without loss of generality that $B = \begin{bmatrix} I_m & 0 \end{bmatrix}^\top$. Given this distinction in the generalized coordinates, the steady-state manifold of (12) is

$$\bar{\mathcal{S}} \triangleq \{\mathbf{q} \in \mathbb{R}^n \mid (I_n - BB^\top)G(\mathbf{q}) = 0\}, \tag{13}$$

which collects all coordinates whose unactuated channels are critical points of the unactuated Gravity vector field. With this, we can always define an injective mapping $\bar{q} : \mathbb{R}^m \to \bar{\mathcal{S}}$ such that for any parameter $\mathbf{v} \in \mathbb{R}^m$, hereafter called a *reference*, the state-input triplet $(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}) = \big(\bar{q}(\mathbf{v}), 0, B^\top G\big(\bar{q}(\mathbf{v})\big)\big)$ is an equilibrium point of (12). The following assumption ensures the image of $\bar{q}$ is a set of stable equilibria with respect to the unactuated dynamics. This limits the approach to minimum-energy configuration targets (e.g., cart-pendulum system with downward mass).

Assumption 1: The mapping $\bar{q} : \mathbb{R}^m \to \bar{\mathcal{S}}$, with $\bar{q}(\mathbf{v}) = [\bar{q}_a(\mathbf{v}); \bar{q}_u(\mathbf{v})]$, is smooth and any point in its image is a strict local minima of the potential function associated to G along the unactuated coordinates. That is, $\forall \mathbf{v} \in \mathbb{R}^m$, there exists an open neighborhood of $\bar{q}_u(\mathbf{v})$, denoted $\mathcal{Q}^u_{\mathbf{v}} \subset \mathbb{R}^{n-m}$, such that, $\forall \mathbf{q} \in \mathbb{R}^m \times (\mathcal{Q}^u_{\mathbf{v}} \setminus \{\bar{q}_u(\mathbf{v})\})$,

$$\big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top (I_n - BB^\top)G(\mathbf{q}) > 0. \tag{14}$$

We consider the end gravity-compensated PD controller

$$\pi(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) = B^\top G\big(\bar{q}(\mathbf{v})\big) - K_P B^\top \big(\mathbf{q} - \bar{q}(\mathbf{v})\big) - K_D B^\top \dot{\mathbf{q}}, \tag{15}$$

where we require

$$K_D \succ -B^\top R B, \tag{16}$$

and $K_P = K_P^\top$ be such that $\forall \mathbf{v} \in \mathbb{R}^m$, there exists an open neighborhood of $\bar{q}_a(\mathbf{v})$, denoted $\mathcal{Q}^a_{\mathbf{v}} \subset \mathbb{R}^m$, wherein $\forall \mathbf{q} \in (\mathcal{Q}^a_{\mathbf{v}} \setminus \{\bar{q}_a(\mathbf{v})\}) \times \mathbb{R}^{n-m}$,

$$\begin{aligned}&\big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top BB^\top \big(G(\mathbf{q}) - G\big(\bar{\mathbf{q}}(\mathbf{v})\big)\big) \\ &\quad + \big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top BK_P B^\top \big(\mathbf{q} - \bar{q}(\mathbf{v})\big) > 0.\end{aligned} \tag{17}$$

This requirement, along with Assumption 1, ensures any point in the image of $\bar{q}$ is a strict local minima of the closed-loop potential function. The following remark provides a sufficient condition on K_P for (17) to hold when the actuated gravity vector field is differentiable and independent.

Remark 3: Let $\mathbf{v} \in \mathbb{R}^m$ be given. If there exists a differentiable function $G_a : \mathbb{R}^m \to \mathbb{R}^m$ such that $\forall \mathbf{q} \in \mathbb{R}^n$, $G_a(\mathbf{q}_a) = B^\top G(\mathbf{q})$, and

$$K_P \succ -\frac{\partial G_a\big(\bar{q}_a(\mathbf{v})\big)}{\partial \mathbf{q}_a}, \tag{18}$$

then there exists an open neighborhood of $\bar{q}_a(\mathbf{v})$, denoted $\mathcal{Q}^a_{\mathbf{v}} \subset \mathbb{R}^m$, wherein $\forall \mathbf{q} \in (\mathcal{Q}^a_{\mathbf{v}} \setminus \{\bar{q}_a(\mathbf{v})\}) \times \mathbb{R}^{n-m}$, (17) holds.

With this, let us define the quantities

$$\begin{aligned}\Phi(\mathbf{q}, \mathbf{v}) &= G(\mathbf{q}) - BB^\top G\big(\bar{q}(\mathbf{v})\big) + BK_P B^\top \big(\mathbf{q} - \bar{q}(\mathbf{v})\big), \\ \Psi(\mathbf{q}, \dot{\mathbf{q}}) &= (R + BK_D B^\top)\dot{\mathbf{q}},\end{aligned}$$

and note that the precompensated system can be written as

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = -\Phi(\mathbf{q}, \mathbf{v}) - \Psi(\mathbf{q}, \dot{\mathbf{q}}). \tag{19}$$

Let $\mathcal{Q}_{\mathbf{v}} \triangleq \mathcal{Q}^a_{\mathbf{v}} \times \mathcal{Q}^u_{\mathbf{v}}$. It can be shown that the functions Φ and Ψ have the following properties $\forall \mathbf{v} \in \mathbb{R}^m$,

$$\Phi(\bar{q}(\mathbf{v}), \mathbf{v}) = 0 \tag{20}$$

$$\forall \mathbf{q} \in \mathcal{Q}_{\mathbf{v}} \setminus \{\bar{q}(\mathbf{v})\},\ \big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top \Phi(\mathbf{q}, \mathbf{v}) > 0 \tag{21}$$

$$\forall \mathbf{q} \in \mathbb{R}^n,\quad \frac{\partial \Phi}{\partial \mathbf{q}} = \frac{\partial G}{\partial \mathbf{q}} + \begin{bmatrix} K_P & 0 \\ 0 & 0 \end{bmatrix} \text{ is symmetric} \tag{22}$$

$$\forall \mathbf{q} \in \mathcal{Q}_{\mathbf{v}},\ \forall \dot{\mathbf{q}} \in \mathbb{R}^n,\quad \dot{\mathbf{q}}^\top \Psi(\mathbf{q}, \dot{\mathbf{q}}) \geq 0 \tag{23}$$

Given these properties, the following holds.

Proposition 1: For all $\mathbf{v} \in \mathbb{R}^m$, the system (19), with output $\mathbf{y} = B^\top \dot{\mathbf{q}}$, is output strictly passive.

Proof: Let $\mathbf{v} \in \mathbb{R}^m$ be given and let

$$V(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) = \frac{1}{2}\dot{\mathbf{q}}^\top M(\mathbf{q})\dot{\mathbf{q}} + \int_{\bar{q}(\mathbf{v})}^{\mathbf{q}} \Phi(\boldsymbol{\xi}, \mathbf{v})^\top \mathrm{d}\boldsymbol{\xi}, \tag{24}$$

be a storage function in the set $\mathcal{D}_{\mathbf{v}} \triangleq \mathcal{Q}_{\mathbf{v}} \times \mathbb{R}^n$. Note that V is positive definite by Property 1 and (21). Property 2 yields $\dot{V}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) = -\dot{\mathbf{q}}^\top \Psi(\mathbf{q}, \dot{\mathbf{q}})$ and it follows that

$$\begin{aligned}0 &= \dot{V}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) + \dot{\mathbf{q}}^\top (R + BK_D B^\top)\dot{\mathbf{q}} \\ &\geq \dot{V}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) + \mathbf{y}^\top (B^\top R B + K_D)\mathbf{y}.\end{aligned}$$

Since $B^\top RB + K_D \succ 0$, we conclude that the system is output strictly passive. ■

Let us generalize the definition of zero-state observability [19] to systems with parametrized equilibria.

Definition 3: The system (19) with output function $h_\pi(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v})$ is said to be *steady-state observable* if, for any $\mathbf{v} \in \mathbb{R}^m$, no solution $\mathbf{q}(t), \dot{\mathbf{q}}(t)$ can stay identically in $\mathcal{S}_\pi = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^n \times \mathbb{R}^n \mid h_\pi(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}) = h_\pi\big(\bar{q}(\mathbf{v}), 0, \mathbf{v}\big)\}$, other than the steady-state solution $\mathbf{q}(t) \equiv \bar{q}(\mathbf{v})$, $\dot{\mathbf{q}}(t) \equiv 0$.

With these ingredients, the following Lemma is a trivial extension of [19, Lemma 6.7] applied to (19)

Lemma 1: If the system (19) with output $\mathbf{y} = B^\top \dot{\mathbf{q}}$ is steady-state observable, then $\forall \mathbf{v} \in \mathbb{R}^m$, the equilibrium point $(\mathbf{q}, \dot{\mathbf{q}}) = \big(\bar{q}(\mathbf{v}), 0\big)$, is asymptotically stable. Also, the storage function (24) is a reference-dependent Lyapunov function.

The following result shows that the reference-dependent Lyapunov function V defined in (24) is monotonically increasing by arguing that its gradient does not vanish.

Proposition 2: Given any $\mathbf{v} \in \mathbb{R}^m$, let $\mathcal{D}_{\mathbf{v}} \triangleq \mathcal{Q}_{\mathbf{v}} \times \mathbb{R}^n$. Then, $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{D}_{\mathbf{v}} \setminus \big\{\big(\bar{q}(\mathbf{v}), 0\big)\big\}$ is a regular point of $V(\cdot, \cdot, \mathbf{v}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Proof: Let $\mathbf{v} \in \mathbb{R}^m$ be given and, for a contradiction, let $(\mathbf{q}, \dot{\mathbf{q}}) \in \mathcal{D}_{\mathbf{v}} \setminus \big\{\big(\bar{q}(\mathbf{v}), 0\big)\big\}$ be such that $\begin{bmatrix} \frac{\partial V}{\partial \mathbf{q}} & \frac{\partial V}{\partial \dot{\mathbf{q}}} \end{bmatrix} = 0$. Since

$\frac{\partial V}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}^\top M(\mathbf{q}) = 0$ and $M(\mathbf{q})$ is nonsingular, it follows that $\dot{\mathbf{q}} = 0$. Noting that

$$\frac{\partial V}{\partial \mathbf{q}} = \frac{1}{2}\Big[\dot{\mathbf{q}}^\top \tfrac{\partial M}{\partial q_1}\dot{\mathbf{q}} \quad \cdots \quad \dot{\mathbf{q}}^\top \tfrac{\partial M}{\partial q_n}\dot{\mathbf{q}}\Big] + \Phi(\mathbf{q},\mathbf{v})^\top = \Phi(\mathbf{q},\mathbf{v})^\top,$$

we have that $\big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top \Phi(\mathbf{q},\mathbf{v}) = 0$, which contradicts (21) because $\mathbf{q} \in \mathcal{Q}_\mathbf{v} \setminus \{\bar{q}(\mathbf{v})\}$. ■

Remark 4: The presented results apply to more general EL systems as long as there exists a prestabilizing controller achieving the closed-loop form of (19), and for which properties (20)-(23) hold. The aircraft example presented in the numerical section of this letter is one such system.

B. Safety and Stability Threshold Values

The expressions for the safety (8) and stability (9) threshold values appear difficult to compute at first glance. However, they usually amount to evaluating V at the boundary of the constraints $\mathcal{X}_\mathbf{v}$ for Γ^* and at unstable equilibria for $\bar{\Gamma}$. In the most general case, the optimization problems must be solved on a case-by-case basis. However, in the special case of polyhedral state and input constraints, the structure of EL systems can be leveraged to obtain analytical expressions for Γ^* [12]. Let the state and input constraint sets be

$$\mathcal{X} = \{(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{2n} \mid W_q\mathbf{q} + W_{\dot{q}}\dot{\mathbf{q}} \le \mathbf{z}\}, \tag{25}$$
$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^m \mid M_u\mathbf{u} \le \mathbf{b}\}, \tag{26}$$

where $\mathbf{z} \in \mathbb{R}^{n_x}$ and $\mathbf{b} \in \mathbb{R}^{n_u}$. Let $\mathbf{e}_i$ be the i-th column of $I_{n_x+n_u}$. The i-th reference-dependent state constraint set is $\mathcal{X}_{\mathbf{v},i} = \big\{\mathbf{e}_i^\top H[\mathbf{q};\ \dot{\mathbf{q}}] \le \mathbf{e}_i^\top \delta(\mathbf{v})\big\}$, where

$$H = \begin{bmatrix} W_q & W_{\dot{q}} \\ -M_uK_PB^\top & -M_uK_DB^\top \end{bmatrix}, \tag{27}$$

$$\delta(\mathbf{v}) = \begin{bmatrix} \mathbf{z} \\ \mathbf{b} - M_uB^\top G\big(\bar{q}(\mathbf{v})\big) - M_uK_PB^\top \bar{q}(\mathbf{v}) \end{bmatrix}, \tag{28}$$

and the i-th steady-state admissible reference set is $\mathcal{V}_i = \{\mathbf{v} \in \mathbb{R}^m \mid \mathbf{e}_i^\top\omega(\mathbf{v}) \ge 0\}$, where $\omega(\mathbf{v}) = \delta(\mathbf{v}) - H[\bar{q}(\mathbf{v});\ 0]$. Let V be lower-bounded by a quadratic function. That is, $\forall i \in \{1,\dots,n_x+n_u\}$

$$\forall \mathbf{v} \in \mathcal{V}_i,\ \forall (\mathbf{q},\dot{\mathbf{q}}) \in \mathcal{D}_\mathbf{v}, \quad \underline{V}(\mathbf{q},\dot{\mathbf{q}},\mathbf{v}) \le V(\mathbf{q},\dot{\mathbf{q}},\mathbf{v}), \tag{29}$$

where $\underline{V}$ is given as

$$\underline{V}(\mathbf{q},\dot{\mathbf{q}},\mathbf{v}) = \frac{1}{2}\big(\mathbf{q} - \bar{q}(\mathbf{v})\big)^\top \underline{\Phi}(\mathbf{v})\big(\mathbf{q} - \bar{q}(\mathbf{v})\big) + \frac{1}{2}\dot{\mathbf{q}}^\top \underline{M}\dot{\mathbf{q}}, \tag{30}$$

with $\underline{M} \succ 0$ and $\underline{\Phi}(\mathbf{v}) \succ 0$, $\forall \mathbf{v} \in \mathbb{R}^m$.

Proposition 3: [12, Proposition 3] If $\mathcal{X}$ and $\mathcal{U}$ are given as in (25) and (26), respectively, and the reference-dependent Lypunov function satisfies (29), then the i-th ($i \in \{1,\dots,n_x+n_u\}$) safety threshold value satisfies

$$\Gamma_i^*(\mathbf{v}) \ge \frac{\mathbf{e}_i^\top\omega(\mathbf{v})|\mathbf{e}_i^\top\omega(\mathbf{v})|}{2\mathbf{e}_i^\top H\underline{Q}^{-1}(\mathbf{v})H^\top\mathbf{e}_i} \triangleq \underline{\Gamma}_i^*(\mathbf{v}), \tag{31}$$

where $\underline{Q}(\mathbf{v}) = \mathrm{diag}\,(\underline{\Phi}(\mathbf{v}), \underline{M})$.

Remark 5: Note that, without any loss of generality, the polyhedral state and input constraint can be reference-dependent, i.e. $W_q(\mathbf{v}), W_{\dot{q}}(\mathbf{v}),\ \mathbf{z}(\mathbf{v}),\ M_u(\mathbf{v}),\ \mathbf{b}(\mathbf{v})$. This is useful for performing planar embedding of convex obstacles, see for example [12, Eq. (28)].

A small difference between (31) and its counterpart in [12] is that our lower-bound for the i-th safety threshold value $\underline{\Gamma}_i^*$ becomes negative when evaluated at inadmissible references $\mathbf{v} \in \mathbb{R}^m \setminus \mathcal{V}_i$. Finally, we can apply Theorem 1 to conclude

$$\Delta(\mathbf{q},\dot{\mathbf{q}},\mathbf{v}) = \min\big(\bar{\Gamma}(\mathbf{v}), \underline{\Gamma}_1^*(\mathbf{v}), \dots, \underline{\Gamma}_{n_x+n_u}^*(\mathbf{v})\big) - V(\mathbf{q},\dot{\mathbf{q}},\mathbf{v}), \tag{32}$$

is a DSM for π and, if differentiable, a CBF with respect to the augmented system. Thus, (11) can be used to obtain an admissible constrained control policy that is always feasible and guarantees safety $\big(\mathbf{q}(t),\dot{\mathbf{q}}(t)\big) \in \mathcal{X}$. In the event Δ is not differentiable, recall Remark 2.

IV. Examples

In this section, we present two carefully selected examples: The mass-spring system features high relative degree constraints and underactuated dynamics. The aircraft example features inputs that enter nonlinearly and a non-quadratic Rayleigh dissipation function. We compare the performance of the proposed DSM-CBF with candidate exponential CBFs [14] and the ERG [12]. For an additional example, the reader is referred to the overhead crane system in the numerical section of [4], which uses the formalisms detailed in this letter.

A. Mass-spring System

Consider the EL system with two masses and a simple spring depicted in Fig. 1. Picking generalized coordinates $\mathbf{q} = [x_1; x_2]$, the mass matrix and gravity vector are

$$M(\mathbf{q}) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad G(\mathbf{q}) = k\begin{bmatrix} x_1 - x_2 \\ x_2 - x_1 \end{bmatrix}, \tag{33}$$

with $m_1 = m_2 = 1$ and $k = 4$. Also, the dissipation coefficient is $R = 0$ and the Coriolis matrix is $C(\mathbf{q},\dot{\mathbf{q}}) = 0$. The system is underactuated ($m = 1$) with $B = [1; 0]$. Let us define the equilibrium mapping $\bar{q}(\mathbf{v}) = [v; v]$ and consider the passivity-based controller (15) with $K_P = 0.2$ and $K_D = 0$. The reference-dependent Lyapunov function is

$$V(\mathbf{q},\dot{\mathbf{q}},v) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}K_P(x_1 - v)^2.$$

It can be shown that $\mathcal{D}_v = \mathbb{R}^2$ for all $v \in \mathbb{R}$. We consider constraints $x_2 \le x_{\max}$ and $|u| \le u_{\max}$, $x_{\max} = u_{\max} = 3$. The safety threshold values can be obtained analytically

$$\Gamma_1^*(v) = \frac{1}{2}\frac{kK_P}{(k + K_P)}(x_{\max} - v)|x_{\max} - v| \tag{34}$$

$$\Gamma_2^*(v) = \frac{m_1u_{\max}^2}{2(K_D^2 + K_Pm_1)}. \tag{35}$$

Here, $\Gamma_1^*(v)$ can be interpreted as the minimum potential energy required to violate the constraint $x_2 \le x_{\max}$ given a reference v. Indeed, x_2 is connected to x_1 by a spring of stiffness k, whereas x_1 is connected to v by means of a (virtual) spring of stiffness K_P. The constant that appears in (34) is nothing more than the stiffness of a spring k in series with a spring K_P. Therefore, $\Delta_i(\mathbf{q},\dot{\mathbf{q}},v) = \Gamma_i^*(v) - V(\mathbf{q},\dot{\mathbf{q}},v)$

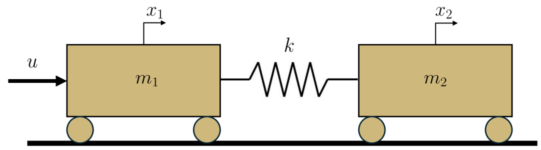

Fig. 1. Mass-spring system.

are valid, control-sharing CBFs for the augmented system. The nominal controller κ has the same form as π but with more aggressive gains $K_P^\kappa = 3$ and $K_D^\kappa = 1$. The candidate exponential CBF is $h(\mathbf{q}, \dot{\mathbf{q}}) = x_{\max} - x_2$ and has relative-degree 4. We follow the procedure in [14] to implement and tune it. The simulation results are shown in Fig. 2. The proposed DSM-CBF approach outperforms the other approaches when the system is initialized in the origin. When the initial conditions are switched to $x_1(0) = x_2(0) = -13$, the candidate exponential CBF approach becomes infeasible about 4 seconds into the simulation. This highlights the dangers of using candidate CBFs that are tuned for a specific scenario. In contrast, the DSM-CBF is formally guaranteed to remain feasible for any initial conditions that are feasible.

B. Fixed-wing Aircraft Longitudinal Dynamics

Consider the longitudinal dynamics of the fixed-wing aircraft shown in Fig. 3. The generalized coordinate is the attack angle $q = \alpha$, and evolves according to

$$J\ddot{\alpha} + d_1 L(\alpha)\cos\alpha + \mu|\dot{\alpha}|\dot{\alpha} = d_2 u \cos\alpha, \tag{36}$$

where $J = 4.5 \times 10^6\ \mathrm{kg\,m^2}$ is the moment of inertia of the aircraft, $d_1 = 8\ \mathrm{m}$ and $d_2 = 40\ \mathrm{m}$ are the lever-arm of the main wing and the elevator airfoil, respectively. The term $\mu = 2 \times 10^7\ \mathrm{N\,m\,s^2}$ is the aerodynamic damping factor. The lift curve $L(\alpha)$ is approximated with a third-order polynomial

$$L(\alpha) = k_0 + k_1\alpha - k_3\alpha^3, \tag{37}$$

with $k_0 = 2.5\times10^5\ \mathrm{N}$, $k_1 = 8.6\times10^6\ \mathrm{N/rad}$ and $k_3 = 4.35\times 10^7\ \mathrm{N/rad^3}$. The stall angle α_S is the angle of maximum lift and corresponds to $\alpha_S = \sqrt{k_1/(3k_3)} \approx 0.258\mathrm{rad} \approx 14.8\mathrm{deg}$. This system is more complex than (12) in two ways: The Rayleigh dissipation function $F(\dot{\alpha}) = \mu\dot{\alpha}^2|\dot{\alpha}|/3$ is not quadratic; and the input u does not enter the system linearly. Nonetheless, Properties 1 and 2 hold. Defining the equilibrium mapping $\bar{\alpha}(v) = v$, we consider the prestabilizing controller

$$\pi(\alpha, \dot{\alpha}, v) = \frac{d_1}{d_2}L(v) - K_P(\alpha - v) - K_D\dot{\alpha}, \tag{38}$$

with $K_P = 4.5 \times 10^5$ and $K_D = 0$. Defining quantities

$$\Phi(\alpha, v) = d_1\cos\alpha\big(L(\alpha) - L(v)\big) + d_2K_P(\alpha - v)\cos\alpha,$$
$$\Psi(\alpha, \dot{\alpha}) = (\mu|\dot{\alpha}| + d_2K_D\cos\alpha)\dot{\alpha},$$

the system can be written as (19) with $M(\alpha) = J$ and $C(\alpha, \dot{\alpha}) = 0$. Moreover, it can be shown that for any $v \in (-\alpha_S, \alpha_S)$, Φ and Ψ satisfy properties (20)–(23) with $\mathcal{Q}_v = (-\alpha_S, \alpha_S)$. Therefore,

$$V(\alpha, \dot{\alpha}, v) = \frac{1}{2}J\dot{\alpha}^2 + \int_v^\alpha \Phi(\xi, v)\mathrm{d}\xi = \frac{1}{2}J\dot{\alpha}^2 + \bar{P}(\alpha, v),$$

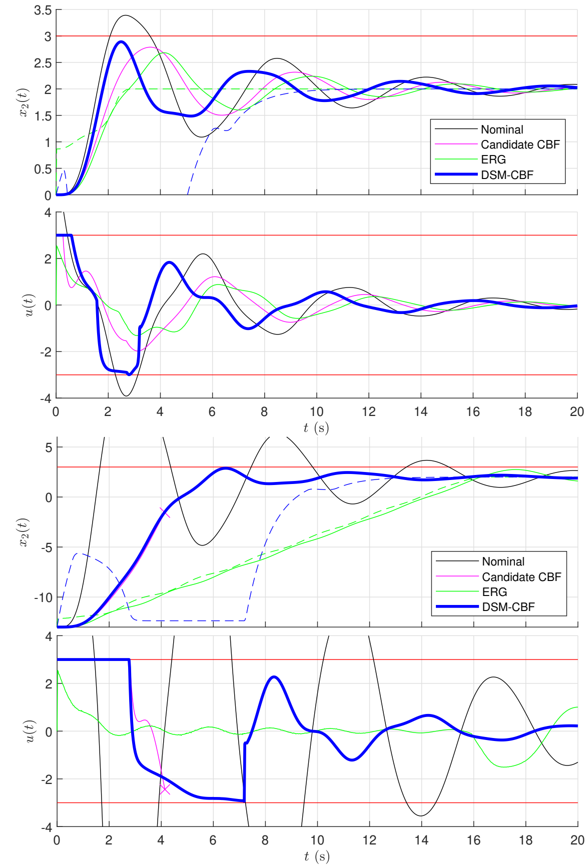

Fig. 2. Simulation results for the mass-spring system. The only difference between the top and bottom figures is the initial position of the masses. This change makes the candidate CBF approach fail while our proposed DSM-CBF remains resilient. The dashed blue line is the reference $\boldsymbol{v}$ of the prestabilized system. The DSM-CBF guarantees safety by temporarily pushing $\boldsymbol{v}$ away from the constraints to enlarge the Lyapunov level sets.

is a reference dependent Lyapunov function, where the closed-loop potential function $\bar{P}$ is

$$\begin{aligned}\bar{P}(\alpha, v) = {} & (d_1k_1 + d_2K_P)\left(\cos\alpha - \cos v + (\alpha - v)\sin\alpha\right)\\ & - 3d_1k_3\left((\alpha^2 - 2)\cos\alpha - (v^2 - 2)\cos v\right)\\ & - d_1k_3\left((\alpha^3 - 6\alpha - v^3)\sin\alpha + 6v\sin v\right).\end{aligned}$$

The stability threshold value can be computed as follows

$$\bar{\Gamma}(v) = \begin{cases}\overline{P}(\alpha_S, v), & \text{if } v \in [0, \alpha_S),\\ \overline{P}(-\alpha_S, v), & \text{if } v \in (-\alpha_S, 0).\end{cases} \tag{39}$$

We consider the constraints $\alpha \le \alpha_S$ and $u \ge u_{\min}$ with $u_{\min} = 0$. Due to the limitation $\mathcal{Q}_v = (-\alpha_S, \alpha_S)$, the state constraint is enforced intrinsically by the stability threshold value and doesn't need additional consideration. For the input constraint, the reference-dependent state constraint set is

$$\mathcal{X}_v = \{(\alpha, \dot{\alpha}) \mid K_P\alpha + K_D\dot{\alpha} \le \frac{d_1}{d_2}L(v) + K_Pv - u_{\min}\}.$$

With this, it follows that $\mathcal{V} \subset [-0.03, \alpha_S)$ for any $u_{\min} \ge 0$. It can be shown that (29) holds for $\underline{M} = J$ and

$$\underline{\Phi}(v) = \left(d_1\frac{L(\alpha_S) - L(v)}{\alpha_S - v} + d_2K_P\right)\cos\alpha_S.$$

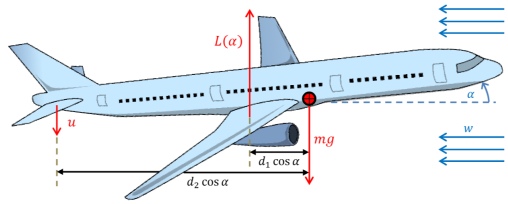

Fig. 3. Fixed-wing aircraft longitudinal dynamics model from [12].

It follows by Proposition 3 that the safety threshold value for the input constraint satisfies $\forall v \in \mathcal{V}$,

$$\Gamma^*(v) \leq \underline{\Gamma}^*(v) = \frac{\left(\frac{d_1}{d_2}L(v) - u_{\min}\right)\left|\frac{d_1}{d_2}L(v) - u_{\min}\right|}{2K_P^2\underline{\Phi}^{-1}(v) + 2K_D^2 J^{-1}}.$$

We conclude that the following are valid CBFs

$$\Delta_1(\alpha, \dot{\alpha}, v) = \overline{P}(\alpha_S, v) - V(\alpha, \dot{\alpha}, v),$$
$$\Delta_2(\alpha, \dot{\alpha}, v) = \overline{P}(-\alpha_S, v) - V(\alpha, \dot{\alpha}, v),$$
$$\Delta_3(\alpha, \dot{\alpha}, v) = \underline{\Gamma}^*(v) - V(\alpha, \dot{\alpha}, v).$$

The nominal controller κ has the same form as π but with more aggressive gains $K_P^\kappa = 4.5 \times 10^6$ and $K_D^\kappa = 4.5 \times 10^4$. The candidate exponential CBF is $h_1(\alpha, \dot{\alpha}) = \alpha_S - \alpha$, and it has relative degree 2. We follow the procedure in [14] to implement and tune it. The simulation results are shown in Fig. 4. While all approaches achieve safety, the candidate CBF requires careful tuning to work. In fact, considering more aggressive gains for the candidate CBF approach renders the underlying optimization problem infeasible.

V. Conclusion

In this letter, we designed control barrier functions for Euler–Lagrange systems. To achieve this, we rely on passivity-based controllers for Euler–Lagrange systems with parameterized equilibrium manifolds. Then, we used the associated storage functions to design dynamic safety margins. We showed that for linear constraints, when the storage function can be lower-bounded by a quadratic function, the DSMs have a closed form. The resulting DSM-based CBFs can be used to enforce multiple state and input constraints with arbitrary relative degree. Numerical simulations compare the performance of DSM-CBFs to candidate CBFs and ERGs. Future work includes assessing the robustness of the DSM-based CBFs in the presence of noise and disturbances, as well as generalizing the approach to general classes of systems.

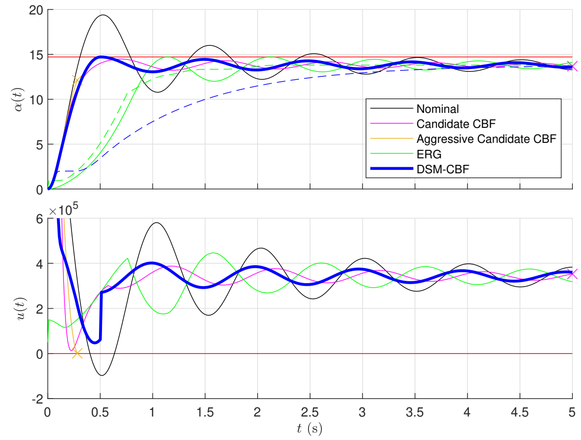

Fig. 4. Simulation results for the aircraft system. We can observe how attempting to achieve better performance for the candidate CBF by choosing more aggressive class $\mathcal{K}_\infty$ functions can lead to infeasibility of the CBF-QP. Once again, the DSM-CBF guarantees safety by maintaining the virtual reference v away from the constraints.

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