Building Robust Control Barrier Functions from Robust Maximal Output Admissible Sets

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***Abstract*— This paper addresses the constrained control of nonlinear systems subject to bounded disturbances and arbitrary state and input constraints. This is done by defining a robust discrete-time control barrier function (RDCBF) and using it to synthesize a control policy. Given that RDCBFs are certificates of robust control invariance, it is shown that robust maximal output admissible sets can be used to construct RDCBFs. By also specializing the approach to linear systems, the paper provides a step-by-step algorithm for designing a safe and recursively feasible RDCBF-based controller for linear discrete-time systems subject to bounded disturbances and polyhedral state and input constraints. Numerical examples showcase the effectiveness of the proposed controller compared to other robust constrained control approaches.**

I. Introduction

Robust control barrier functions (RCBFs) provide a simple yet effective framework for providing constraint-handling capabilities in the presence of model uncertainties [1], [2] or external disturbances [3], [4]. However, the literature on RCBFs is mostly centered around *existence*-type proofs, meaning that the systematic construction of a valid RCBF remains an open research question. In [5], the authors propose input-to-state safe (ISSf) CBFs and provide a constructive approach to synthesize them from existing CBFs. While ISSf CBFs certify the existence of a robust control invariant set, they require a priori knowledge of both a valid CBF and the ISS gains. Robust CBFs have been proposed for the discrete-time domain [6]–[8] as an extension of the discrete-time exponential CBFs introduced in [9]. In analogy to the continuous-time case, these schemes assume the *existence* of an RCBF but provide no guidance on how to obtain it.

In this work, we propose a new robust discrete-time CBF (RDCBF) definition that yields necessary and sufficient conditions for robust safety. Then, we propose an algorithm to systematically construct RDCBFs using existing tools from the command governor (CG) literature [10]. Specifically, we extend our previous work on discrete-time CBFs [11] to account for bounded disturbances. Then, we explore how robust maximal output admissible sets (RMOASs) are large robust control invariant sets for an augmented system. Finally, we use these sets to construct an RDCBF and formulate the associated robustly safe control policy. This approach is then specialized to the linear systems case subject to polyhedral state and input constraints. For this special case, we present a complete algorithm for constructing RDCBFs in closed form. Finally, examples showcase that the constructed RDCBFs yield good performance when compared to other constrained control methods such as robust model predictive control (RMPC) [12] and Robust CG (RCG) [13].

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Notation: Given two matrices $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{m\times p}$, let $[A; B] \triangleq [A^\top \ B^\top]^\top \in \mathbb{R}^{n+m\times p}$. The symbol $\subset$ denotes set inclusion, but not necessarily strict inclusion. An infinite sequence of vectors $\mathbf{x}_k \in \mathcal{C}$ is denoted by $\{\mathbf{x}_k\} \subset \mathcal{C}$.

II. Preliminaries

Given state and input domains $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$, consider a discrete-time system

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k), \tag{1}$$

where $\mathbf{x}_k \in \mathbb{X}$, $\mathbf{u}_k \in \mathbb{U}$, and $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is a continuous function. Let $\mathcal{X} \subset \mathbb{X}$ be a closed set describing state constraints and $\mathcal{U} \subset \mathbb{U}$ be a compact[1] set describing input constraints. Additionally, let $\kappa : \mathbb{X} \to \mathbb{U}$ be a nominal controller that achieves desirable performance in the absence of constraints. As detailed in [11], DCBFs can be used to generate an input sequence $\{\mathbf{u}_k\} \subset \mathcal{U}$ that mimics the performance of the nominal controller κ while also enforcing constraint satisfaction $\{\mathbf{x}_k\} \subset \mathcal{X}$.

Definition 1: [11] A continuous function $h : \mathbb{X} \to \mathbb{R}$ is a *discrete-time control barrier function* (DCBF) if

$$\sup_{\mathbf{u}\in\mathcal{U}} \Big[h\big(f(\mathbf{x}, \mathbf{u})\big)\Big] \geq 0, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{2}$$

where

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{X} \mid h(\mathbf{x}) \geq 0\}. \tag{3}$$

Lemma 1: [11] The set $\mathcal{C} \subset \mathbb{X}$ given in (3) is control invariant if and only if h is a DCBF.

Given a DCBF h such that $\mathcal{C} \subset \mathcal{X}$, the safety goal is achieved with an optimization-based control policy $\beta : \mathcal{C} \to \mathcal{U}$

$$\beta(\mathbf{x}) = \underset{\mathbf{u}\in\mathcal{K}(\mathbf{x})}{\operatorname{argmin}} \|\mathbf{u} - \kappa(\mathbf{x})\|^2, \tag{4}$$

where

$$\mathcal{K}(\mathbf{x}) = \{\mathbf{u} \in \mathcal{U} \mid h\big(f(\mathbf{x}, \mathbf{u})\big) \geq 0\}. \tag{5}$$

Since $\mathcal{K}(\mathbf{x}) \neq \emptyset$, $\forall \mathbf{x} \in \mathcal{C}$, the optimization problem (4) is always feasible and β is well-defined.

[1] As detailed in Remark 1, compactness of $\mathcal{U}$ makes DCBFs sufficient for control invariance of $\mathcal{C}$. Necessity holds even if $\mathcal{U}$ is unbounded.

III. Robust DCBF-based Control

In this section, we extend the DCBF definition to systems subject to bounded disturbances. To this end, consider

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{d}_k), \tag{6}$$

where $f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^q \to \mathbb{X}$ is a continuous function and $\mathbf{d}_k \in \mathcal{D}$ is a bounded disturbance, with $\mathcal{D} \subset \mathbb{R}^q$ compact. The constraint sets $\mathcal{X}$ and $\mathcal{U}$ are as defined above. Based on [14], a set $\mathcal{C} \subset \mathbb{X}$ is *robust control invariant* if

$$\forall \mathbf{x} \in \mathcal{C},\ \exists \mathbf{u} \in \mathcal{U} : \quad f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \in \mathcal{C},\ \forall \mathbf{d} \in \mathcal{D}. \tag{7}$$

With this characterization of robust control invariance, we have the following definition

Definition 2: A continuous function $h : \mathbb{X} \to \mathbb{R}$ is a robust discrete-time control barrier function (RDCBF) if

$$\sup_{\mathbf{u} \in \mathcal{U}} \inf_{\mathbf{d} \in \mathcal{D}} \Big[h\big(f(\mathbf{x}, \mathbf{u}, \mathbf{d})\big) \Big] \geq 0, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{8}$$

where $\mathcal{C}$ is defined as in (3).

Theorem 1: The set $\mathcal{C}$ defined in (3) is robust control invariant if and only if h is an RDCBF.

Proof: Let $\mathcal{C} \subset \mathbb{X}$ be robust control invariant. Given $\mathbf{x} \in \mathcal{C}$, there exists $\mathbf{u} \in \mathcal{U}$ such that $f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \in \mathcal{C}$, $\forall \mathbf{d} \in \mathcal{D}$. This implies $\inf_{\mathbf{d} \in \mathcal{D}} h\big(f(\mathbf{x}, \mathbf{u}, \mathbf{d})\big) \geq 0$. Thus, (8) holds, and h is an RDCBF.

Now, let h be an RDCBF and let $\mathbf{x} \in \mathcal{C}$ be given. By continuity of h, f and compactness of $\mathcal{D}$, we have that the map $\mathbf{u} \mapsto \inf_{\mathbf{d} \in \mathcal{D}} \big[h\big(f(\mathbf{x}, \mathbf{u}, \mathbf{d})\big)\big]$ is continuous. Since $\mathcal{U}$ is compact, the supremum of this map is attained at some $\mathbf{u} \in \mathcal{U}$ and this input vector is such that $f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \in \mathcal{C}$ for all $\mathbf{d} \in \mathcal{D}$. Since $\mathbf{x} \in \mathcal{C}$ was arbitrary, we have that (7) holds and we conclude that $\mathcal{C}$ is robust control invariant. ∎

Remark 1: If the input constraint set $\mathcal{U}$ is closed, but not bounded, h being a (R)DCBF is necessary, but not sufficient, for (robust) control invariance of $\mathcal{C} = \{\mathbf{x} \in \mathbb{X} : h(\mathbf{x}) \geq 0\}$. To illustrate this, consider the linear, scalar system $x_{k+1} = u_k$ and the continuous function

$$h(x) = \begin{cases} -x - 1, & x \leq 0, \\ -e^{-x}, & x > 0, \end{cases}$$

with $\mathcal{C} = (-\infty, -1]$. Given $x \in \mathcal{C}$ and $\mathcal{U} = [0, \infty)$, which is closed but not bounded, we note that

$$\sup_{u \in \mathcal{U}} [h(u)] = \max\big(-1, \sup_{u > 0} -e^{-u}\big) = 0.$$

Thus, h is a DCBF even though $\mathcal{C}$ is clearly not control invariant. Nevertheless, the loss of sufficiency is not an issue for this paper since its main focus is using $\mathcal{C}$ to construct h, as opposed to the other way around. Thus, all the remaining results in this paper are applicable when the assumptions on $\mathcal{U}$ are relaxed from being a compact set to a closed set.

Similar to the DCBF case, given an RDCBF h such that $\mathcal{C} \subset \mathcal{X}$, we can define the optimization-based robust control policy $\beta : \mathcal{C} \to \mathcal{U}$ as in (4) with the new feasible set

$$\mathcal{K}(\mathbf{x}) = \{\mathbf{u} \in \mathcal{U} \mid h\big(f(\mathbf{x}, \mathbf{u}, \mathbf{d})\big) \geq 0,\ \forall \mathbf{d} \in \mathcal{D}\}. \tag{9}$$

Again, h being an RDCBF guarantees that $\mathcal{K}(\mathbf{x}) \neq \emptyset$, $\forall \mathbf{x} \in \mathcal{C}$, so β is well-defined. By construction, the closed-loop system controlled with β is robustly safe, satisfying $\mathbf{x}_k \in \mathcal{X}$ for all $k \in \mathbb{N}$ even in spite of the unknown disturbance sequence $\{\mathbf{d}_k\} \subset \mathcal{D}$. Note that, given $\mathbf{x} \in \mathcal{C}$, verifying $\mathbf{u} \in \mathcal{K}(\mathbf{x})$ for the robust case (9) is much more difficult than in the nominal case (5) because we must verify the condition for all possible $\mathbf{d} \in \mathcal{D}$. The next remark addresses this shortcoming.

Remark 2: The following conditions greatly reduce the complexity of the set $\mathcal{K}(\mathbf{x})$:

(a) The system (6) is control-and-disturbance-affine

$$f(\mathbf{x}, \mathbf{u}, \mathbf{d}) = f_x(\mathbf{x}) + f_u(\mathbf{x})\mathbf{u} + f_d(\mathbf{x})\mathbf{d}, \tag{10}$$

with $f_x : \mathbb{X} \to \mathbb{R}^n$, $f_u : \mathbb{X} \to \mathbb{R}^n \times \mathbb{R}^m$ and $f_d : \mathbb{X} \to \mathbb{R}^n \times \mathbb{R}^q$ continuous functions;

(b) the RDCBF $h : \mathbb{X} \to \mathbb{R}$ is linear.

If the assumptions in Remark 2 hold, $\mathcal{K}(\mathbf{x})$ is equivalent to

$$\mathcal{K}(\mathbf{x}) = \{\mathbf{u} \in \mathcal{U} \,|\, h\big(f_x(\mathbf{x})\big) + h\big(f_u(\mathbf{x})\mathbf{u}\big) + h\big(f_d(\mathbf{x})\delta(\mathbf{x})\big) \geq 0\},$$

where $\delta : \mathbb{X} \to \mathbb{R}^q$ is defined as

$$\delta(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{d} \in \mathcal{D}} h\big(f_d(\mathbf{x})\mathbf{d}\big). \tag{11}$$

Note that, given the function $\delta(\mathbf{x})$, the optimization problem (4) is reduced to a quadratic program (QP).

Remark 3: While the term $\delta(\mathbf{x})$ given in (11) may seem troublesome, it is in fact decoupled from the optimization problem defining β (4). Additionally, the linearity assumption on h implies that (i) if $\mathcal{D}$ is convex, (11) is a convex program, (ii) if $\mathcal{D}$ is polyhedral, (11) is a linear program, and (iii) if the system is linear in $\mathbf{d}$, i.e. $f_d(\mathbf{x}) = E \in \mathbb{R}^n \times \mathbb{R}^q$, (11) can be computed offline and δ is a constant function.

Thanks to Theorem 1, finding an RDCBF for arbitrary state and input constraint sets is equivalent to finding a constraint-admissible, robust control invariant set $\mathcal{C} \subset \mathcal{X}$. Of course, larger sets $\mathcal{C}$ are preferred to increase performance. This realization leads us to consider the robust maximal output admissible set (RMOAS) [10] as a candidate for $\mathcal{C}$.

IV. RDCBF construction using the RMOAS

This section shows how the established theory of output admissible sets [10], [15] can be used to find a suitable $\mathcal{C}$ and an associated RDCBF.

A. Robust Maximal Output Admissible Set

To apply RMOAS results, we need the following assumption

Assumption 1: The disturbance set admits $0 \in \mathcal{D}$. Furthermore, in the absence of disturbances, the equilibrium points of (6) can be parameterized using continuous functions $\bar{x} : \mathbb{V} \to \mathbb{X}$ and $\bar{u} : \mathbb{V} \to \mathbb{U}$ such that

$$\bar{x}(\mathbf{v}) = f\big(\bar{x}(\mathbf{v}), \bar{u}(\mathbf{v}), 0\big), \quad \forall \mathbf{v} \in \mathbb{V}. \tag{12}$$

Moreover, there exists a control law $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$ and functions $\sigma \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, $\forall \mathbf{v} \in \mathbb{V}$, $\forall \mathbf{x}_0 \in \mathbb{X}$,

$$\|\mathbf{x}_k - \bar{x}(\mathbf{v})\| \leq \max\left(\sigma\left(\|\mathbf{x}_0 - \bar{x}(\mathbf{v})\|, k\right), \gamma\left(d_{\max}\right)\right),\ \forall k \in \mathbb{N}$$

where $d_{\max} = \sup_{\mathbf{d}\in\mathcal{D}} \|\mathbf{d}\|$ and $\mathbf{x}_k$ satisfies the prestabilized dynamics

$$\mathbf{x}_{k+1} = f_\pi(\mathbf{x}_k, \mathbf{v}, \mathbf{d}_k) \triangleq f\big(\mathbf{x}_k, \pi(\mathbf{x}_k, \mathbf{v}), \mathbf{d}_k\big). \quad (13)$$

Since $\mathbf{v} \in \mathbb{V}$ can be interpreted as a reference for the prestabilizing control law, we define the steady-state admissible reference set

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{V} \mid \bar{x}(\mathbf{v}) \in \mathcal{X},\ \bar{u}(\mathbf{v}) \in \mathcal{U}\}. \quad (14)$$

Let π be a prestabilizing controller as in Assumption 1. For each reference $\mathbf{v} \in \mathbb{V}$, the input constraint set $\mathcal{U}$ induces state constraints on the prestabilized system f_π. Combining these with the original state constraint set $\mathcal{X}$, we define the reference-dependent state constraint set

$$\mathcal{X}_\mathbf{v} = \{\mathbf{x} \in \mathcal{X} \mid \pi(\mathbf{x}, \mathbf{v}) \in \mathcal{U}\}. \quad (15)$$

By continuity of π on $\mathbb{V}$, it follows that $\mathcal{X}_\mathbf{v}$ is closed in $\mathbb{X}$. Then, the robust maximal output admissible set (RMOAS) is defined as follows.

Definition 3: The RMOAS of the prestabilized system f_π given in (13) is

$$\mathcal{O}_\infty = \lim_{k\to\infty} \mathcal{O}_k, \quad (16)$$

where $\mathcal{O}_k$ satisfies the set recursion

$$\mathcal{O}_{k+1} = \{(\mathbf{x}, \mathbf{v}) \in \mathcal{O}_k : \big(f_\pi(\mathbf{x}, \mathbf{v}, \mathbf{d}), \mathbf{v}\big) \in \mathcal{O}_k, \forall \mathbf{d} \in \mathcal{D}\}, \quad (17)$$

with $\mathcal{O}_0 \triangleq \{(\mathbf{x}, \mathbf{v}) \in \mathbb{X} \times \mathcal{V} \mid \mathbf{x} \in \mathcal{X}_\mathbf{v}\}$.

Note that $\mathcal{O}_{k+1} \subset \mathcal{O}_k$ for all $k \in \mathbb{N}$. Thus, if there exists a $k^* \in \mathbb{N}$ such that $\mathcal{O}_{k^*+1} = \mathcal{O}_{k^*}$, then $\mathcal{O}_\infty = \mathcal{O}_{k^*}$ and we say $\mathcal{O}_\infty$ is finitely determined [10]. Since this property is important for tractable computation, we study under what conditions $\mathcal{O}_\infty$ is finitely determined.

As a consequence of Assumption 1, for each $\mathbf{v} \in \mathbb{V}$, there exists [16] a robust forward invariant set $\mathcal{F}_\mathbf{v}$ such that

$$\mathcal{B}\big(\bar{x}(\mathbf{v}), \gamma(d_{\max})\big) = \{\mathbf{x} \in \mathbb{X} : \|\mathbf{x} - \bar{x}(\mathbf{v})\| \leq \gamma(d_{\max})\} \subset \mathcal{F}_\mathbf{v}.$$

That is, $\mathbf{x} \in \mathcal{F}_\mathbf{v}$ implies that $f_\pi(\mathbf{x}, \mathbf{v}, \mathbf{d}) \in \mathcal{F}_\mathbf{v}$, $\forall \mathbf{d} \in \mathcal{D}$. Now, we make the following assumption about the constraint sets.

Assumption 2: The set $\mathcal{X}$ is compact and there exists $\mathbf{v} \in \mathcal{V}$ such that $\mathcal{F}_\mathbf{v} \subset \mathcal{X}_\mathbf{v}$.

Proposition 1: Under Assumptions 1 and 2, there exists a compact subset $\mathcal{V}^\epsilon \subset \mathcal{V}$ such that the RMOAS inner approximation

$$\mathcal{O}^\epsilon_\infty \triangleq \mathcal{O}_\infty \cap (\mathbb{X} \times \mathcal{V}^\epsilon) \neq \emptyset, \quad (18)$$

is finitely determined

Proof: By construction, $(\mathbf{x}, \mathbf{v}) \in \mathcal{O}_n$ implies that, given $\mathbf{x}_0 = \mathbf{x}$ and any sequence $\{\mathbf{d}_0, \dots, \mathbf{d}_n\} \subset \mathcal{D}$, the solution to $\mathbf{x}_{k+1} = f_\pi(\mathbf{x}_k, \mathbf{v}, \mathbf{d}_k)$ satisfies $\mathbf{x}_k \in \mathcal{X}_\mathbf{v}$ for all $k \in \{0, \dots, n\}$. By Assumption 2, there exists a compact set $\mathcal{V}^\epsilon \subset \mathcal{V}$ such that $\forall \mathbf{v} \in \mathcal{V}^\epsilon$, $\mathcal{F}_\mathbf{v} \subset \mathcal{X}_\mathbf{v}$. Define

$$a = \max_{(\mathbf{x},\mathbf{v})\in\mathcal{X}\times\mathcal{V}^\epsilon} \|\mathbf{x} - \bar{x}(\mathbf{v})\|, \quad (19)$$

and note that $a < \infty$ by compactness of $\mathcal{X} \times \mathcal{V}^\epsilon$ and continuity of $\bar{x}$. Let us pick $k^* \in \mathbb{N}$ such that $\sigma(a, k^*) \leq \gamma(d_{\max})$.

Now, let $(\mathbf{x}, \mathbf{v}) \in \mathcal{O}^\epsilon_{k^*} = \mathcal{O}_{k^*} \cap (\mathbb{X} \times \mathcal{V}^\epsilon)$ and let $\{\mathbf{d}_0, \dots, \mathbf{d}_{k^*+1}\} \subset \mathcal{D}$ be given. We immediately have that $\mathbf{x}_k \in \mathcal{X}_\mathbf{v}$ for all $k \in \{0, \dots, k^*\}$, with $\mathbf{x}_0 = \mathbf{x}$. Let us examine $\mathbf{x}_{k^*+1}$ against the ISS condition of Assumption 1:

$$\begin{aligned}\|\mathbf{x}_{k^*+1} - \bar{x}(\mathbf{v})\| &\leq \max\left(\sigma\left(\|\mathbf{x} - \bar{x}(\mathbf{v})\|, k^*+1\right), \gamma(d_{\max})\right)\\ &\leq \max\left(\sigma(a, k^*+1), \gamma(d_{\max})\right)\\ &\leq \gamma(d_{\max}).\end{aligned}$$

This implies

$$\mathbf{x}_{k^*+1} \in \mathcal{B}\big(\bar{x}(\mathbf{v}), \gamma(d_{\max})\big) \subset \mathcal{F}_\mathbf{v} \subset \mathcal{X}_\mathbf{v}. \quad (20)$$

Thus, $(\mathbf{x}, \mathbf{v}) \in \mathcal{O}^\epsilon_{k^*+1}$, yielding that $\mathcal{O}^\epsilon_{k^*} \subset \mathcal{O}^\epsilon_{k^*+1}$ and we conclude $\mathcal{O}^\epsilon_{k^*} = \mathcal{O}^\epsilon_\infty$. Furthermore, note that any $\mathbf{v} \in \mathcal{V}^\epsilon \neq \emptyset$ is such that $\big(\bar{x}(\mathbf{v}), \mathbf{v}\big) \in \mathcal{O}^\epsilon_\infty \neq \emptyset$. ∎

Although the systematic construction of $\mathcal{O}_\infty$ (or $\mathcal{O}^\epsilon_\infty$) for general nonlinear systems remains an open research question, methods for computing polyhedral RMOAS have been proposed for linear systems [10], piecewise affine systems [17], and systems governed by DC functions [18].

B. Augmenting the System and RDCBFs

Having identified the RMOAS of the prestabilized system f_π, let us consider the augmented system

$$\begin{bmatrix}\mathbf{x}_{k+1}\\ \mathbf{v}_{k+1}\end{bmatrix} = \begin{bmatrix}f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{d}_k)\\ \mathbf{w}_k\end{bmatrix}, \quad (21)$$

where the augmented state is $(\mathbf{x}_k, \mathbf{v}_k) \in \mathbb{X} \times \mathbb{V}$ and the augmented input is $(\mathbf{u}_k, \mathbf{w}_k) \in \mathbb{U} \times \mathbb{V}$. The augmented state constraint set is $\mathcal{X} \times \mathbb{V}$ and the augmented input constraint set is $\mathcal{U} \times \mathbb{V}$. We have the following results:

Lemma 2: The RMOAS $\mathcal{O}_\infty \subset \mathcal{X} \times \mathcal{V}$ and the inner approximation $\mathcal{O}^\epsilon_\infty \subset \mathcal{X} \times \mathcal{V}^\epsilon$ are robust control invariant sets for the augmented system (21).

Proof: Let $(\mathbf{x}, \mathbf{v}) \in \mathcal{O}_\infty$ and note that $\mathbf{x} \in \mathcal{X}_\mathbf{v}$, which implies that $\pi(\mathbf{x}, \mathbf{v}) \in \mathcal{U}$. The augmented input $\big(\pi(\mathbf{x}, \mathbf{v}), \mathbf{v}\big)$ is such that $\big(f(\mathbf{x}, \pi(\mathbf{x}, \mathbf{v}), \mathbf{d}), \mathbf{v}\big) \in \mathcal{O}_\infty$, $\forall \mathbf{d} \in \mathcal{D}$. Thus, $\mathcal{O}_\infty$ is robust control invariant. The same argument holds for $\mathcal{O}^\epsilon_\infty$. ∎

Remark 4: Projecting the RMOAS $\mathcal{O}_\infty$ and the inner approximation $\mathcal{O}^\epsilon_\infty$ onto the state-space yields robust control invariant sets for the original system (6). These projections are denoted, respectively, by $\mathrm{Proj}_x\mathcal{O}_\infty \subset \mathcal{X}$ and $\mathrm{Proj}_x\mathcal{O}^\epsilon_\infty \subset \mathcal{X}$.

Corollary 1: Any continuous function $h : \mathbb{X} \times \mathbb{V} \to \mathbb{R}$ such that $\mathcal{O}_\infty = \{(\mathbf{x}, \mathbf{v}) \in \mathbb{X} \times \mathbb{V} \mid h(\mathbf{x}, \mathbf{v}) \geq 0\}$ is an RDCBF. The result also holds for $\mathcal{O}^\epsilon_\infty$.

Proof: See Lemma 2 and Theorem 1. ∎

Having found an RDCBF h for the augmented system, we can implement RDCBF-based control as follows. Let $\mathbf{r} \in \mathbb{V}$ be a desired reference and let $\kappa : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$ be a nominal controller that steers the system to $\bar{x}(\mathbf{r})$ without taking constraints into account. The safe control policy $\beta : \mathcal{C} \times \mathbb{V} \to \mathcal{U} \times \mathbb{V}$ becomes

$$\beta(\mathbf{x}, \mathbf{v}) = \operatorname*{argmin}_{(\mathbf{u},\mathbf{w})\in\mathcal{K}(\mathbf{x},\mathbf{v})} \|\mathbf{u} - \kappa(\mathbf{x}, \mathbf{r})\|^2 + \eta\|\mathbf{w} - \mathbf{r}\|^2, \quad (22)$$

where $\eta \geq 0$, and

$$\mathcal{K}(\mathbf{x},\mathbf{v}) = \{(\mathbf{u},\mathbf{w}) \in \mathcal{U}\times\mathbb{V} : h\big(f(\mathbf{x},\mathbf{u},\mathbf{d}),\mathbf{w}\big) \geq 0, \forall \mathbf{d}\in\mathcal{D}\}.$$

Remark 5: Although the additional term $\eta\|\mathbf{w}-\mathbf{r}\|^2$ in (22) is not necessary for safety, its inclusion makes the objective function strictly convex, which is helpful for numerical solvers. In practice, we observed that, as $\eta \to \infty$, the performance of the constructed RDCBF degrades to that of the RCG.

V. RDCBF Design for Linear Systems

This section specializes the previous results to linear systems subject to polyhedral state and input constraints and bounded disturbances. The construction of the RMOAS differs from that in [10] because they consider the RMOAS as a subset of the state-space ($\mathcal{O}_\infty \subset \mathbb{R}^n$), while we augment the RMOAS to include references ($\mathcal{O}_\infty \subset \mathbb{R}^{n+m}$). This augmentation is a nontrivial extension that leads to better-performing RDCBFs. Consider the discrete-time, linear system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k + E\mathbf{d}_k, \tag{23}$$

where $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{u}_k \in \mathbb{R}^m$, $\mathbf{d}_k \in \mathcal{D}$, with $\mathcal{D} \subset \mathbb{R}^q$ compact. In this case, we assume the state $\mathcal{X} \subset \mathbb{R}^n$ and input $\mathcal{U} \subset \mathbb{R}^m$ constraint sets are polytopes (bounded polyhedra) containing the origin and described by

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid W\mathbf{x} \leq \mathbf{z}\}, \quad \mathcal{U} = \{\mathbf{u} \in \mathbb{R}^m \mid M\mathbf{u} \leq \mathbf{b}\}. \tag{24}$$

With this setup, we will construct an RDCBF-based constrained control law by retracing all the steps laid out in previous sections. Sample MATLAB code for generating the proposed control law can be found on GitHub[2].

A. Design the Prestabilizing Controller

Assumption 1 implies that the pair (A, B) is stabilizable. Furthermore, we assume that B is full column rank. With this, $\ker\begin{bmatrix}A - I_n & B\end{bmatrix}$ admits a basis in the form $G = [G_x; G_u]$ with $G_x \in \mathbb{R}^{n\times m}$ and $G_u \in \mathbb{R}^{m\times m}$. Assumption 1 also ensures that it is possible to design a gain $K \in \mathbb{R}^{m\times n}$ such that $A - BK$ is Schur. We consider a prestabilizing policy

$$\pi(\mathbf{x},\mathbf{v}) = G_u\mathbf{v} - K(\mathbf{x} - G_x\mathbf{v}), \tag{25}$$

where $\mathbf{v} \in \mathbb{R}^m$ is the reference and $\bar{x}(\mathbf{v}) = G_x\mathbf{v}$ is the equilibrium point of the undisturbed system associated to $\mathbf{v}$. The dynamics of the prestabilized system are then captured by the closed-loop matrices $A_\pi = A - BK$ and $B_\pi = B(G_u + KG_x)$.

Remark 6: As detailed in [19], Assumption 2 can be relaxed by defining a set of system outputs $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$ and a compact output constraint set $\mathcal{Y}$ that captures all the state $\mathcal{X}$ and input $\mathcal{U}$ constraints, such that the pair $(A_\pi, C - DK)$ is observable and $0 \in \operatorname{Int}(\mathcal{Y})$. In this work, as is customary in the CBF literature, we present the result directly in state-input space and abstain from defining system outputs.

[2]https://github.com/ROCC-Lab-CU-Boulder/Robust-CBF-for-DT-LTI

B. Closed Form RMOAS

Let $\epsilon \in [0, 1]$ and consider the set

$$\mathcal{V}^\epsilon = \left\{\mathbf{v} \in \mathbb{R}^m \;\middle|\; \begin{bmatrix}W & 0\\ 0 & M\end{bmatrix} G\mathbf{v} \leq (1-\epsilon)\begin{bmatrix}\mathbf{z}\\ \mathbf{b}\end{bmatrix}\right\} \subset \mathcal{V}. \tag{26}$$

Since B has full column rank, so does G_x, making the map $\bar{x}$ injective. Thus, $\mathcal{V}$ is bounded by compactness of $\mathcal{X}$ and $\mathcal{U}$. It then follows that the closed subset $\mathcal{V}^\epsilon \subset \mathcal{V}$ is also compact. Next, let us define the recursions

$$L_{k+1} = L_k \begin{bmatrix}A_\pi & B_\pi\\ 0 & I_m\end{bmatrix}, \quad \mathbf{a}_{k+1} = \mathbf{a}_k - \boldsymbol{\phi}_k, \tag{27}$$

where

$$L_0 = \begin{bmatrix}0 & WG_x\\ 0 & MG_u\\ W & 0\\ -MK & M(G_u + KG_x)\end{bmatrix}, \quad \mathbf{a}_0 = \begin{bmatrix}(1-\epsilon)\mathbf{z}\\ (1-\epsilon)\mathbf{b}\\ \mathbf{z}\\ \mathbf{b}\end{bmatrix},$$

and the i-th component of the vector $\boldsymbol{\phi}_k$ is

$$[\boldsymbol{\phi}_k]_i = \max_{\mathbf{d}\in\mathcal{D}} \mathbf{e}_i^\top L_k[E; 0]\mathbf{d}, \tag{28}$$

where $\mathbf{e}_i$ is the i-th canonical basis vector of appropriate dimension such that $[\boldsymbol{\phi}_k]_i = \mathbf{e}_i^\top\boldsymbol{\phi}_k$. The following results show that the RMOAS for linear systems has a closed form representation, as opposed to the recursive form in Def. 3.

Lemma 3: Let $\epsilon \in [0, 1]$. Given any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$, and $k \in \mathbb{N}$, the following statements are equivalent:

(a) $L_{k+1}[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_{k+1}$.
(b) $L_{k+1}[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_k - L_k[E; 0]\mathbf{d}, \quad \forall \mathbf{d} \in \mathcal{D}$.

Proof: Let $\ell \in \mathbb{N}$ denote the dimension of the vector $\mathbf{a}_0$ and let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^m$ and $k \in \mathbb{N}$ be given. Suppose (a) holds and let $i \in \{1, \ldots, \ell\}$ be given. Then, $\mathbf{e}_i^\top\mathbf{a}_{k+1} = \mathbf{e}_i^\top\mathbf{a}_k - [\boldsymbol{\phi}_k]_i = \mathbf{e}_i^\top\mathbf{a}_k - \max_{\mathbf{d}\in\mathcal{D}}\ \mathbf{e}_i^\top L_k[E; 0]\mathbf{d}$, which is a lower-bound on $\mathbf{e}_i^\top\big(\mathbf{a}_k - L_k[E; 0]\mathbf{d}\big)$ for any $\mathbf{d} \in \mathcal{D}$. Since $i \in \{1, \ldots, \ell\}$ was arbitrary, the vector inequality in statement (b) follows. Suppose now that (b) holds and consider the vector inequality element-wise. Note that, for any $i \in \{1, \ldots, \ell\}$, it follows from compactness of $\mathcal{D}$ that there exists a $\mathbf{d} \in \mathcal{D}$ such that $\mathbf{e}_i^\top L_k[E; 0]\mathbf{d} = \max_{\mathbf{d}\in\mathcal{D}}\ \mathbf{e}_i^\top L_k[E; 0]\mathbf{d} = [\boldsymbol{\phi}_k]_i$. It then follows that $\mathbf{e}_i^\top L_{k+1}[\mathbf{x}; \mathbf{v}] \leq \mathbf{e}_i^\top(\mathbf{a}_k - [\boldsymbol{\phi}_k]_i)$, for all i. This implies $L_{k+1}[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_k - \boldsymbol{\phi}_k = \mathbf{a}_{k+1}$, and (a) holds. ∎

Proposition 2: Let $\epsilon \in [0, 1]$. For any $k \in \mathbb{N}$,

$$\mathcal{O}_k^\epsilon = \{(\mathbf{x},\mathbf{v}) \mid L_i[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_i, i = 0, \ldots, k\}. \tag{29}$$

Proof: For the base case $k = 0$, $L_0[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_0$ implies that $\mathbf{v} \in \mathcal{V}^\epsilon$, $\mathbf{x} \in \mathcal{X}$ and $\pi(\mathbf{x},\mathbf{v}) \in \mathcal{U}$, which further implies that $\mathbf{x} \in \mathcal{X}_\mathbf{v}$. So, the statement holds by definition of $\mathcal{O}_0^\epsilon$. For the inductive step, assume the statement holds at $k \in \mathbb{N}$ and consider the case $k + 1$. By definition, $\mathcal{O}_{k+1}^\epsilon = \{(\mathbf{x},\mathbf{v}) \in \mathcal{O}_k^\epsilon \mid (A_\pi\mathbf{x} + B_\pi\mathbf{v} + E\mathbf{d}, \mathbf{v}) \in \mathcal{O}_k^\epsilon, \forall \mathbf{d} \in \mathcal{D}\}$. From the assumed description of $\mathcal{O}_k^\epsilon$, we can write this set as

$$\mathcal{O}_{k+1}^\epsilon = \mathcal{O}_k^\epsilon \cap \{(\mathbf{x},\mathbf{v}) \mid L_{k+1}[\mathbf{x}; \mathbf{v}] \leq \mathbf{a}_k - L_k[E; 0]\mathbf{d}, \forall \mathbf{d} \in \mathcal{D}\},$$

Applying Lemma 3, the proof is complete by induction. ∎

Note that the previous results allow for $\epsilon = 0$, and that we make a slight abuse of notation by considering $\mathcal{O}_k^0 =$

$\mathcal{O}_k$. With this, the inner approximation $\mathcal{O}_k^\epsilon$ can be made arbitrarily close to $\mathcal{O}_k$. In the next section, we will see that $\epsilon \in [0, 1]$ is related to the number of iterations needed to describe $\mathcal{O}_\infty^\epsilon$.

C. Algorithmic Computation of RMOAS

Given $\epsilon \in (0, 1]$, Algorithm 1 iteratively computes $\mathcal{O}_k^\epsilon$ and checks redundancy of the new constraints in $\mathcal{O}_{k+1}^\epsilon \subset \mathcal{O}_k^\epsilon$. The algorithm terminates at iteration k^* if all new constraints are redundant, i.e. if $\mathcal{O}_{k^*}^\epsilon = \mathcal{O}_{k^*+1}^\epsilon = \mathcal{O}_\infty^\epsilon$. Nevertheless, it should be noted that the algorithm may not converge for an arbitrary choice of $\epsilon \in (0, 1)$. To systematically construct the RMOAS, we recommend testing the algorithm with $\epsilon = 1$. Notably, since $\mathcal{V}^1 = \{0\}$, only two results are possible:

- If there exists a forward invariant set $\mathcal{F}_0 \subset \mathcal{X}_0$, we can apply Proposition 1 to show that the algorithm converges. In this case, we recommend using a line search method to identify for the smallest value $\underline{\epsilon} \in (0, 1)$ such that $\mathcal{O}_\infty^\epsilon$ is finitely determined $\forall \epsilon \in [\underline{\epsilon}, 1]$.
- If there is no forward invariant set $\mathcal{F}_0 \subset \mathcal{X}_0$, we can apply [10, Remark 6.2] to show that the algorithm converges because $\mathcal{O}_\infty^1 = \emptyset$. In this case, we can deduce that the disturbance set $\mathcal{D}$ is too large for the prestabilized system. Although changing the prestabilizing controller may help, it is possible that the problem is ill-posed, meaning that the constraint set is incompatible with the disturbances.

Algorithm 1 Compute k^* and $\mathcal{O}_\infty^\epsilon$

1: Initialize $k = 0$, $H = L_0$ and $\mathbf{c} = \mathbf{a}_0$
2: **repeat**
3: $\quad \mathcal{O}_k^\epsilon \leftarrow \{\mathbf{z} \in \mathbb{R}^{n+m} \mid H\mathbf{z} \leq \mathbf{c}\}$
4: $\quad$ Compute L_{k+1}, $\mathbf{a}_{k+1}$ with (27)
5: $\quad$ Initialize empty L^+, $\mathbf{a}^+$
6: $\quad$ **for** $i \in \{1, \ldots, \text{length}(\mathbf{a}_{k+1})\}$ **do**
7: $\quad\quad$ **if** $\max_{\mathbf{z} \in \mathcal{O}_k^\epsilon} [\mathbf{e}_i^\top L_{k+1}\mathbf{z}] > \mathbf{e}_i^\top \mathbf{a}_{k+1}$ **then**
8: $\quad\quad\quad L^+ \leftarrow \begin{bmatrix} L^+ \\ \mathbf{e}_i^\top L_{k+1} \end{bmatrix}$, $\mathbf{a}^+ \leftarrow \begin{bmatrix} \mathbf{a}^+ \\ \mathbf{e}_i^\top \mathbf{a}_{k+1} \end{bmatrix}$
9: $\quad\quad$ **end if**
10: $\quad$ **end for**
11: $\quad$ **if** $\mathbf{a}^+$ is not empty **then**
12: $\quad\quad k \leftarrow k + 1$
13: $\quad\quad H \leftarrow [H;\ L^+]$, $\mathbf{c} \leftarrow [\mathbf{c};\ \mathbf{a}^+]$
14: $\quad$ **end if**
15: **until** $\mathbf{a}^+$ is empty
16: $k^* \leftarrow k$.
17: $\mathcal{O}_\infty^\epsilon \leftarrow \mathcal{O}_k^\epsilon$

Remark 7: The representation for $\mathcal{O}_\infty^\epsilon$ obtained in Algorithm 1 is not necessarily minimal. In fact, the algorithm can be made more efficient by replacing the computation of L_{k+1} and $\mathbf{a}_{k+1}$, in line 4, with the assignment

$$L_{k+1} \leftarrow L^+ \begin{bmatrix} A_\pi & B_\pi \\ 0 & I_m \end{bmatrix}, \quad \mathbf{a}_{k+1} \leftarrow \mathbf{a}^+ - \tilde{\boldsymbol{\phi}}_k, \tag{30}$$

where $\tilde{\boldsymbol{\phi}}_k$ is computed as in (28) but replacing $L_k \to L^+$. This is because when a constraint becomes redundant, it will remain redundant in all future iterations. See [20, Proposition V.1] for more details.

D. RMOAS-based RDCBF

We can now formulate a closed-form RDCBF that can be used in (4) to synthesize a safe controller.

Theorem 2: Let $\epsilon \in (0, 1]$ be such that $\mathcal{O}_\infty^\epsilon$ is finitely determined. Consider the augmented system

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{v}_{k+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{x}_k + B\mathbf{u}_k + E\mathbf{d}_k \\ \mathbf{w}_k \end{bmatrix}, \tag{31}$$

with augmented state constraint set $\mathcal{X} \times \mathbb{R}^m$ and augmented input constraint set $\mathcal{U} \times \mathbb{R}^m$. Define the set $\mathcal{I} = \{1, \ldots, \ell\} \times \{0, \ldots, k^*\}$, where ℓ is the dimension of $\mathbf{a}_0$ and k^* is the finite index of $\mathcal{O}_\infty^\epsilon$. Then, the continuous function

$$h(\mathbf{x}, \mathbf{v}) = \min_{(i,k) \in \mathcal{I}} \mathbf{e}_i^\top (\mathbf{a}_k - L_k[\mathbf{x}; \mathbf{v}])$$

is an RDCBF with $\mathcal{C} = \mathcal{O}_\infty^\epsilon \subset \mathcal{X} \times \mathcal{V}^\epsilon$.

Proof: See Proposition 2 and Corollary 1. ∎

E. Implementing RDCBF-based control

The following result shows that the RDCBF-based policy in (22) reduces to a simple QP.

Lemma 4: The feasible set of the RDCBF-based policy $\beta(\mathbf{x}, \mathbf{v})$ is a polyhedron

$$\mathcal{K}(\mathbf{x}, \mathbf{v}) = \mathcal{U} \times \mathbb{R}^m \cap \{M_k[\mathbf{u};\ \mathbf{w}] \leq \mathbf{b}_k(\mathbf{x}), k = 0, \ldots, k^*\},$$

$$M_k = L_k \begin{bmatrix} B & 0 \\ 0 & I_m \end{bmatrix}, \quad \mathbf{b}_k(\mathbf{x}) = \mathbf{a}_k - L_k \begin{bmatrix} A \\ 0 \end{bmatrix} \mathbf{x} - \boldsymbol{\phi}_k. \tag{32}$$

Proof: Given $(\mathbf{x}, \mathbf{v}) \in \mathcal{O}_\infty^\epsilon$, note that $\mathcal{K}(\mathbf{x}, \mathbf{v}) \neq \emptyset$ because h is an RDCBF. Let $(\mathbf{u}, \mathbf{w}) \in \mathcal{K}(\mathbf{x}, \mathbf{v}) = \{(\mathbf{u}, \mathbf{w}) \in \mathcal{U} \times \mathbb{R}^m \mid h(A\mathbf{x} + B\mathbf{u} + E\mathbf{d}, \mathbf{w}) \geq 0,\ \forall \mathbf{d} \in \mathcal{D}\}$. Let $k \in \{0, \ldots, k^*\}$ be given and note that $\forall \mathbf{d} \in \mathcal{D}$, $\forall i \in \{1, \ldots, \ell\}$, $\mathbf{e}_i^\top (\mathbf{a}_k - L_k[A\mathbf{x} + B\mathbf{u} + E\mathbf{d};\ \mathbf{w}]) \geq h(A\mathbf{x} + B\mathbf{u} + E\mathbf{d}, \mathbf{w}) \geq 0$. Applying the definition of $\boldsymbol{\phi}_k$ and considering a vector inequality (stacking the ℓ rows), we arrive at $M_k[\mathbf{u};\ \mathbf{w}] \leq \mathbf{b}_k(\mathbf{x})$.

For the other direction, assume that $(\mathbf{u}, \mathbf{w}) \in \mathcal{U} \times \mathbb{R}^m$ are such that $M_k[\mathbf{u};\ \mathbf{w}] \leq \mathbf{b}_k(\mathbf{x})$ for all $k \in \{0, \ldots, k^*\}$. Let $\mathbf{d} \in \mathcal{D}$ and $(i, k) \in \mathcal{I}$ be given and note that $\mathbf{e}_i^\top M_k[\mathbf{u};\ \mathbf{w}] = \mathbf{e}_i^\top L_k[B\mathbf{u};\ \mathbf{w}] \leq \mathbf{e}_i^\top \mathbf{b}_k(\mathbf{x}) = \mathbf{e}_i^\top (\mathbf{a}_k - L_k[A\mathbf{x};\ 0] - \boldsymbol{\phi}_k)$. Applying the definition of $\boldsymbol{\phi}_k$ and rearranging, we obtain that $\mathbf{e}_i^\top (\mathbf{a}_k - L_k[A\mathbf{x} + B\mathbf{u} + E\mathbf{d};\ \mathbf{w}]) \geq 0$. Thus, by definition of $h(\mathbf{x}, \mathbf{v})$, we have that $\forall \mathbf{d} \in \mathcal{D}$, $h(A\mathbf{x} + B\mathbf{u} + E\mathbf{d}, \mathbf{w}) \geq 0$, completing the proof. ∎

VI. Examples

In this section, we present two examples to demonstrate the performance of RDCBFs synthesized from the RMOAS for robust constrained control. We compare our approach with the robust MPC [12] and the robust command governor (RCG) [13]. In all examples, we solve the optimization problems in MATLAB using YALMIP [21] with MOSEK [22]. Projections and set operations are computed with MPT3 [23]. All computations are performed in a laptop PC running Windows 11 with an Intel i5 @ 1.60 GHz CPU and 16 GB RAM.

TABLE I
OPTIMIZATION PROBLEM SOLVE TIMES

	Double integrator		F-16 pitch pointing	
	Avg [ms]	Max [ms]	Avg [ms]	Max [ms]
RDCBF	0.80	1.01	1.31	1.77
RMPC	1.41	1.93	-	-
RCG	0.45	0.55	1.07	1.31

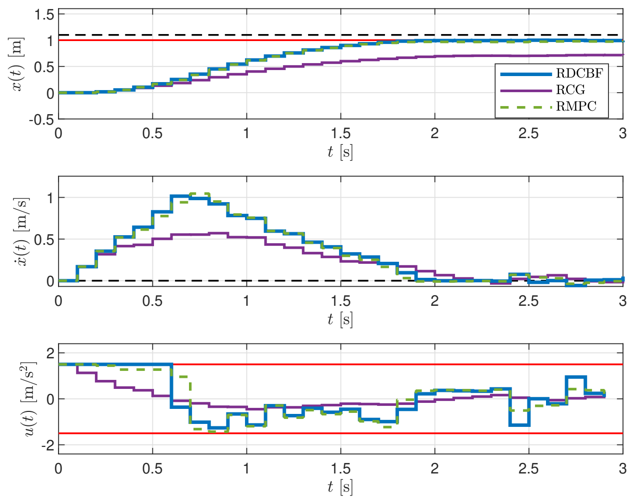

Fig. 1. Simulation example for the double integrator system. The red lines represent the position $\mathcal{X}$ and input $\mathcal{U}$ constraints, whereas the dashed black line is the desired reference.

Example 1: Let $\mathbf{x} = [x; \dot{x}]$ and $u = \ddot{x}$ be the state and input vectors, respectively, of a double integrator system sampled at 10 Hz. The system has the form of (23) with matrices

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

and sets $\mathcal{X} = \{\mathbf{x} : |x| \leq 1\}$, $\mathcal{U} = \{u : |u| \leq 1.5\}$, and $\mathcal{D} = \{d : |d| \leq 0.4\}$. The nominal controller is $\kappa(\mathbf{x}, r) = -24.84(x - r) - 11.87\dot{x}$ and the prestabilizing controller is $\pi(\mathbf{x}, v) = -2(x - v) - 2.2\dot{x}$. Using the linesearch approach detailed in Subsection V-C, we found $\epsilon = 0.25$, leading to the set $\mathcal{V}^\epsilon = [-0.75, 0.75]$. The desired reference is $r = 1.1$ and we use $\eta = 0.1$. Figs. 1 and 2 compare the performance of the RMOAS-based RDCBF to an RMPC [12] and an RCG [13]. To ensure fairness, all three controllers are designed to behave identically in the unconstrained case $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{U} = \mathbb{R}$. In the presence of constraints, the RDCBF achieves closed-loop performance comparable to RMPC, while having lower computational cost (Table I). Since the desired reference was intentionally placed outside the constraint set $\mathcal{X}$, the RDCBF and RMPC successfully steer the system close to the constraint boundary (without ever violating it). The same is not true for the RCG, which instead steers the system to a strictly steady-state admissible equilibrium, associated to some $\mathbf{v} \in \mathcal{V}^\epsilon$.

Example 2: Let the constrained F-16 pitch dynamics detailed in [11, Example 2] be augmented with the disturbance matrix $E = [0.0157\mathbf{e}_2 \; 0.0524\mathbf{e}_4 \; 0.0524\mathbf{e}_5]$, where $\mathbf{e}_i$ is the i-th canonical basis vector in $\mathbb{R}^5$, and disturbance set

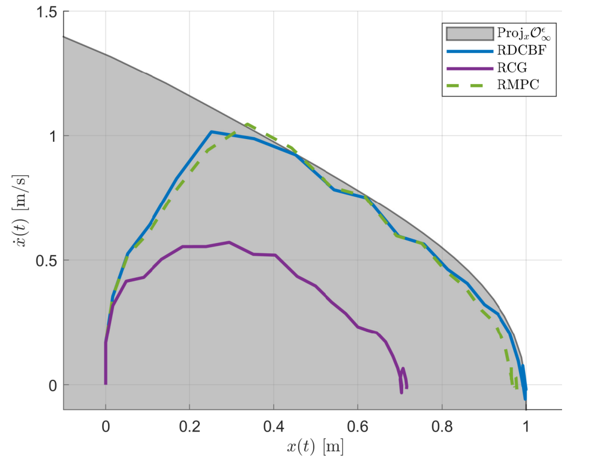

Fig. 2. State-space trajectory generated by each approach. Also shown is the projection of the RMOAS onto the state-space $\text{Proj}_x\mathcal{O}^\epsilon_\infty$ (see Remark 4). The RMPC is not restricted to the set $\text{Proj}_x\mathcal{O}^\epsilon_\infty$ and can therefore be more aggressive than the RDCBF. However, the difference between the two is negligible compared to the difference in computational cost (see Table I).

$\mathcal{D} = \{\mathbf{d} \in \mathbb{R}^3 \mid \|\mathbf{d}\|_\infty \leq 1\}$. Note that the state vector is $\mathbf{x} = [\theta; q; \alpha; \delta_e; \delta_f]$, collecting the pitch, pitch rate, angle of attack, elevator deflection and flaperon deflection, respectively. The input vector is $\mathbf{u} = [\delta_{ec}; \delta_{fc}]$, collecting the elevator and flaperon deflection commands, respectively. The task is reaching a reference pitch angle $\theta_r = \pi/20$ and a reference flight path angle $\gamma_r = 13\pi/360$, with $\gamma = \theta - \alpha$. The constraint sets are $\mathcal{X} = \{\mathbf{x} : |\alpha| \leq 4\pi/180, |\delta_e| \leq 25\pi/180, |\delta_f| \leq 20\pi/180\}$ and there are no input constraints as they are virtual commands (i.e., $\mathcal{U} = \mathbb{R}^2$). The gain matrices of the nominal and prestabilizing controllers were obtained using an LQR with state penalty $Q = \text{diag}([10 \; 0 \; 10 \; 0 \; 0])$ and input penalty $R = 10^{-3}I_2$ for the nominal controller κ, but $R = 10^{-2}I_2$ for the prestabilizing controller π. Using the linesearch approach detailed in Subsection V-C, we found $\epsilon = 0.15$. Fig. 3 compares the response obtained using the RDCBF and RCG. As in the previous example, the RDCBF achieves faster convergence than the RCG in exchange for a modest increase in computational cost (Table I). This example does not include an RMPC comparison because the algorithm for computing the minimum disturbance invariant set [12] was intractable for this 5-dimensional state. This highlights that the proposed RDCBF scheme is easier to implement than RMPC. In the absence of disturbances, [11, Example 2] showed that DCBFs achieve closed-loop performance comparable to MPC.

VII. CONCLUSION

In this paper, we defined robust discrete-time control barrier functions (RDCBFs) and showed that they can be used to synthesize a constrained control policy that is robust with respect to bounded disturbances. Next, we showed how to synthesize RDCBFs by leveraging the theory of robust maximal output admissible sets (RMOASs). We then

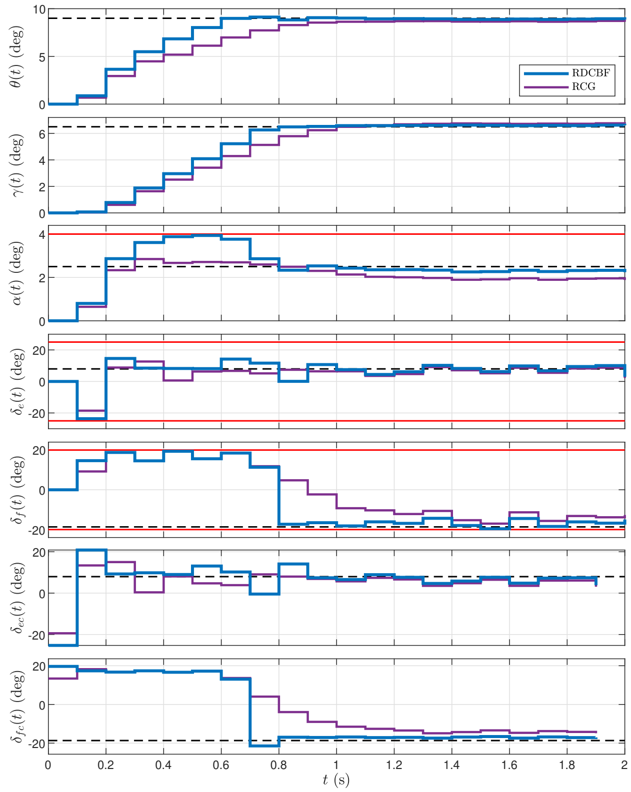

Fig. 3. Simulation of the F-16 aircraft model. It can be seen that both the RDCBF and RCG approaches successfully enforce the safety constraints (dashed red lines) on the angle of attack α and control surfaces deflection δ_e, δ_f even in the presence of disturbances.

specialized the approach to linear systems and provided a systematic algorithm to construct RDCBFs and implement the derived control policy. Numerical simulations showed that the proposed approach can achieve closed-loop performance comparable to robust MPC, while having a lower computational footprint and being substantially easier to implement.

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