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Directional Acoustic Bulk Waves in a 2D Phononic Crystal

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Abstract: We used the transfer matrix method to investigate the conditions supporting the existence of directional bulk waves in a two-dimensional (2D) phononic crystal. The 2D crystal was a square lattice of unit cells composed of rectangular subunits constituted of two different isotropic continuous media. We established the conditions on the geometry of the phononic crystal and its constitutive media for the emergence of waves, which, for the same handedness, exhibited a non-zero amplitude in one direction within the crystal's 2D Brillouin zone and zero amplitude in the opposite direction. Due to time-reversal symmetry, the crystal supported propagation in the reverse direction for the opposite handedness. These features may enable robust directional propagation of bulk acoustic waves and topological acoustic technology.

Keywords: phononic crystal; directional waves; topological acoustics

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1. Introduction

Unconventional topological features can endow waves in acoustic metamaterials and phononic crystals with exotic properties [1]. Breaking time-reversal symmetry of the equations governing the propagation of acoustic waves in a medium may lead to conditions for which a bulk wave or an edge/interface wave (localized at a surface or interface) propagates in only one direction. These one-way propagating waves are robust against backscattering due to obstacles, such as defects in the medium, as a reflected wave is not supported by the medium. In this scenario, physically realizing one-way propagation of acoustic waves requires the use of active media in which one injects energy to break time-reversal symmetry. Examples of such an approach include media subjected to spatiotemporal modulation of their physical properties [2–7] or media supporting time-reversal symmetry-breaking elements such as circulators [8] or gyroscopes [9].

A less demanding approach to achieving some form of immunity to backscattering of acoustic waves is to employ media structured in such a way that leads to breaking inversion or parity symmetry [10,11]. Conditions in these media arise for the existence of edge waves propagating in opposite directions at surfaces or interfaces that are orthogonal to each other, and hence support a form of resilience against back reflections. The conversion of an incident wave impinging on an obstacle to a reflected wave (propagating in the opposite direction) is more difficult in these media than in conventional materials, as it requires an obstacle shape that couples the two otherwise orthogonal waves. Immunity to backscattering is therefore possible for a range of defects; however, this approach does not lead to robust immunity to backscattering.

An interesting opportunity in this context arises when we combine broken inversion/parity symmetry with Fabry–Pérot resonances, leading to directional bulk waves that are more robust to backscattering [12,13]. These waves, called DRAK (Deymier, Runge, Alù, Khanikaev) modes, exhibit non-zero amplitudes when propagating in one direction but zero amplitude when propagating in the opposite direction for the same handedness. So far, DRAK modes have been studied in continuous or discrete one-dimensional superlattices. These superlattices are composed of periodic arrays of alternating layers of two different materials. The DRAK mode arises when a Fabry–Pérot resonance [14] of one type of layer becomes incompatible with the translational periodicity of the superlattice, that is, the resonance becomes incompatible with the Bloch wave character of waves in periodic media. While scattering of orthogonal edge modes does not offer robust immunity against backscattering [15,16], DRAK modes have been shown to exhibit robust immunity to backscattering by general scattering potentials [13].

The current paper extends previous work on DRAK modes in 1D superlattices to 2D phononic crystals. Here, we derive conditions for the existence of DRAK modes in the 2D Brillouin zone of a square lattice of unit cells composed of rectangular subunits constituted of two different continuous media. This work demonstrates that DRAK modes may not be limited to one-way propagation in low dimensionality phononic structures but also to propagation in composite structures with dimensionality higher than one.

We introduce the 2D model system in Section 2. In Section 3, we use the transfer matrix method [17] to establish relations between the amplitudes of the constitutive subunits of the phononic crystal. Section 4 illustrates some of the conditions which may lead to DRAK modes in the 2D Brillouin zone of the phononic crystal. Finally, some conclusions are drawn in Section 5 regarding the relevance of this work in the context of reducing reflection loss in acoustic devices.

2. Model System

We consider a two-dimensional (2D) phononic crystal composed of two different types of materials (Figure 1). The phononic crystal is periodic in the x and y directions. The square unit cell is constituted of four subunits labelled 1^l and 1^u for material 1 and 2^l and 2^u for material 2. The upper scripts "l" and "u" stand for lower and upper quadrants of the unit cell. The length of the edges of the subunits are labeled d_1 and d_2. The length of the edge of the unit cell is $L = d_1 + d_2$. The origin of a unit cell in the (x,y) system of coordinates is given by (nL, mL), where n and m are integers.

The interfaces between subunits belonging to the different unit cells (nL, mL) and $((n-1)L, mL)$ are labelled (I) and (II). The interfaces between subunits belonging to the unit cells (nL, mL) and $(nL, (m-1)L)$ are labelled (III) and (IV). Interfaces between subunits within the same unit cell (nL, mL) are indicated by (V), (VI), (VII), and (VIII).

We consider elastic shear waves polarized in the direction perpendicular to the phononic crystal plane and seek solutions for the displacement field, u, taking the general form of Bloch waves.

Within a medium of type 1, the displacement field is written as

$$u_1^l(x,y) = e^{iq_x nL}\left(A_+^l e^{ik_x^{(1)}(x-nL)} + A_-^l e^{-ik_x^{(1)}(x-nL)}\right) \times e^{iq_y mL}\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) \tag{1a}$$

$$u_1^l(x,y) = e^{iq_x nL}\left(A_+^l e^{ik_x^{(1)}(x-nL)} + A_-^l e^{-ik_x^{(1)}(x-nL)}\right) \times e^{iq_y mL}\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right) \tag{1b}$$

For medium 2, we have

$$u_2^l(x,y) = e^{iq_x nL}\left(C_+^l e^{ik_x^{(2)}(x-nL-d_1)} + C_-^l e^{-ik_x^{(2)}(x-nL-d_1)}\right) \times e^{iq_y mL}\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{2a}$$

$$u_2^u(x,y) = e^{iq_x nL}\left(C_+^u e^{ik_x^{(2)}(x-nL)} + C_-^u e^{-ik_x^{(2)}(x-nL)}\right) \times e^{iq_y mL}\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) \tag{2b}$$

In Equations (1) and (2), q_x and q_y are wave numbers in the x and y directions, and the displacements $u_{1,2}^{l,u}$ are time-dependent and multiplied by the factor $e^{i\omega t}$, where ω is the wave angular frequency. The waves' displacement in media 1 and 2, assumed homogeneous and isotropic, obey the equation of motions of the form $\rho_j \frac{\partial^2 u_j}{\partial t^2} = \mu_j \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_j$ with $j = 1,2$, where ρ_j is the mass density and μ_j is the stiffness. Considering the ansatz $u_j = u_{0j} e^{ik_x^{(j)}} e^{ik_y^{(j)}} e^{i\omega t}$, the wave numbers $k_x^{(j)} > 0$, $k_y^{(j)} > 0$ satisfy the dispersion relations in media 1 and 2, $\omega = c_j \sqrt{\left(k_x^{(j)}\right)^2 + \left(k_y^{(j)}\right)^2}$ where $c_j = \sqrt{\frac{\mu_j}{\rho_j}}$ is the speed of shear waves in medium j.

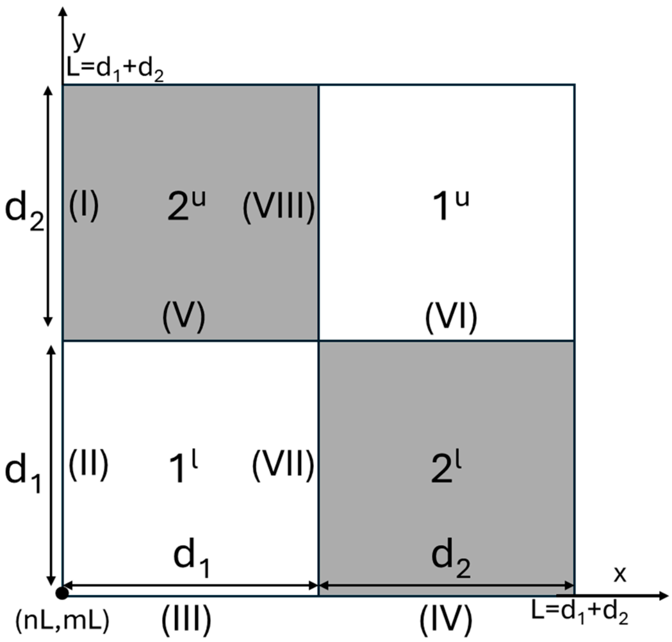

Figure 1. Schematic illustration of the unit cell of the model 2D periodic phononic crystal.

A_+^l, A_-^l, B_+^l, B_-^l, A_+^u, A_-^u, B_+^u, B_-^u, C_+^l, C_-^l, D_+^l, D_-^l, C_+^u, C_-^u, D_+^u, D_-^u are 16 unknown amplitudes to be determined using displacement and stress continuity conditions at the eight interfaces.

The various components of stress in the subunits are obtained from Equations (1a,b) and (2a,b) as follows. For medium 1, we get

$$c_1^2 \frac{\partial u_1^l(x,y)}{\partial x} = \rho_1 c_1^2 e^{iq_x nL} ik_x^{(1)}\left(A_+^l e^{ik_x^{(1)}(x-nL)} - A_-^l e^{-ik_x^{(1)}(x-nL)}\right) \times e^{iq_y mL}\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) \tag{3a}$$

$$\rho_1 c_1^2 \frac{\partial u_1^l(x,y)}{\partial y} = \rho_1 c_1^2 e^{iq_x nL} \left(A_+^l e^{ik_x^{(1)}(x-nL)} + A_-^l e^{-ik_x^{(1)}(x-nL)} \right) \times e^{iq_y mL} ik_y^{(1)} \left(B_+^l e^{ik_y^{(1)}(y-mL)} - B_-^l e^{-ik_y^{(1)}(y-mL)} \right) \tag{3b}$$

$$\rho_1 c_1^2 \frac{\partial u_1^u(x,y)}{\partial x} = \rho_1 c_1^2 e^{iq_x nL} ik_x^{(1)} \left(A_+^u e^{ik_x^{(1)}(x-nL-d_1)} - A_-^u e^{-ik_x^{(1)}(x-nL-d_1)} \right) \times e^{iq_y mL} \left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)} \right) \tag{3c}$$

$$\rho_1 c_1^2 \frac{\partial u_1^u(x,y)}{\partial y} = \rho_1 c_1^2 e^{iq_x nL} \left(A_+^u e^{ik_x^{(1)}(x-nL-d_1)} + A_-^u e^{-ik_x^{(1)}(x-nL-d_1)} \right) \times e^{iq_y mL} ik_y^{(1)} \left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} - B_-^u e^{-ik_y^{(1)}(y-mL-d_1)} \right). \tag{3d}$$

For medium 2, we have

$$\rho_2 c_2^2 \frac{\partial u_2^l(x,y)}{\partial x} = \rho_2 c_2^2 e^{iq_x nL} ik_x^{(2)} \left(C_+^l e^{ik_x^{(2)}(x-nL-d_1)} - C_-^l e^{-ik_x^{(2)}(x-nL-d_1)} \right) \times e^{iq_y mL} \left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)} \right) \tag{4a}$$

$$\rho_2 c_2^2 \frac{\partial u_2^l(x,y)}{\partial y} = \rho_2 c_2^2 e^{iq_x nL} \left(C_+^l e^{ik_x^{(2)}(x-nL-d_1)} + C_-^l e^{-ik_x^{(2)}(x-nL-d_1)} \right) \times e^{iq_y mL} ik_y^{(2)} \left(D_+^l e^{ik_y^{(2)}(y-mL)} - D_-^l e^{-ik_y^{(2)}(y-mL)} \right) \tag{4b}$$

$$\rho_2 c_2^2 \frac{\partial u_2^u(x,y)}{\partial x} = e^{iq_x nL} ik_x^{(2)} \left(C_+^u e^{ik_x^{(2)}(x-nL)} - C_-^u e^{-ik_x^{(2)}(x-nL)} \right) \times e^{iq_y mL} \left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)} \right) \tag{4c}$$

$$\rho_2 c_2^2 \frac{\partial u_2^u(x,y)}{\partial y} = e^{iq_x nL} \left(C_+^u e^{ik_x^{(2)}(x-nL)} + C_-^u e^{-ik_x^{(2)}(x-nL)} \right) \times e^{iq_y mL} ik_y^{(2)} \left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} - D_-^u e^{-ik_y^{(2)}(y-mL-d_1)} \right). \tag{4d}$$

where $\rho_j c_j^2 = \mu_j$ is the stiffness of medium j.

3. Conditions of Continuity

3.1. Interfaces (II) and (VII)

To begin solving for the modal amplitudes, we write the conditions of continuity of displacement and stress at interfaces (II) and (VII). At interface (II), continuity of displacement takes the form u_1^l in unit cell $\{n,m\}$ = u_2^l in unit cell $\{n-1,m\}$, or:

$$\left(A_+^l + A_-^l \right) \left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)} \right) = e^{-iq_x L} \left(C_+^l e^{ik_x^{(2)} d_2} + C_-^l e^{-ik_x^{(2)} d_2} \right) \times \left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)} \right) \tag{5a}$$

To obtain Equation (5), we have used the fact that $L - d_1 = d_2$.

The condition of continuity of stress at interface (II) takes the form

$$F_x\left(A_+^l - A_-^l\right)\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) = e^{-iq_xL}\left(C_+^l e^{ik_x^{(2)}d_2} - C_-^l e^{-ik_x^{(2)}d_2}\right) \times\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{5b}$$

At interface (VII), for displacement we obtain u_1^l in unit cell {*n*,*m*} = u_2^l in unit cell {*n*,*m*}

$$\left(A_+^l e^{ik_x^{(1)}d_1} + A_-^l e^{-ik_x^{(1)}d_1}\right)\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) = \left(C_+^l + C_-^l\right) \times\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{6a}$$

Continuity of stress at interface (VII) leads to

$$F_x\left(A_+^l e^{ik_x^{(1)}d_1} - A_-^l e^{-ik_x^{(1)}d_1}\right)\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) = \left(C_+^l - C_-^l\right) \times\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{6b}$$

In Equations (5b) and (6b), we have introduced the quantity $F_x = \frac{\rho_1 c_1^2 k_x^{(1)}}{\rho_2 c_2^2 k_x^{(2)}}$ by redefining

$$A_+ = A_+^l\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) \tag{7a}$$

$$A_- = A_-^l\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) \tag{7b}$$

$$C_+ = C_+^l\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{7c}$$

$$C_- = C_-^l\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{7d}$$

In Equation (7a–d), we assume that the quantities in parentheses are not zero. Note that the parentheses in Equation (7a,b) are the same, as are those in Equation (7c,d).

The conditions of continuity given by Equations (5a,b) and (6a,b) can be arranged in matrix form:

$$\begin{pmatrix} \alpha_{11} & \beta_{11} & -1 & -1 \\ F_x\alpha_{11} & -F_x\beta_{11} & -1 & 1 \\ 1 & 1 & -e^{-iq_xL}\alpha_{22} & -e^{-iq_xL}\beta_{22} \\ F_x & -F_x & -e^{-iq_xL}\alpha_{22} & e^{-iq_xL}\beta_{22} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \\ C_+ \\ C_- \end{pmatrix} = 0 \tag{8}$$

We introduced the notation $\alpha_{11} = \frac{1}{\beta_{11}} = e^{ik_x^{(1)}d_1}$ and $\alpha_{22} = \frac{1}{\beta_{22}} = e^{ik_x^{(2)}d_2}$.

We can solve for the amplitudes A_+, A_-, C_+, C_- using the transfer matrix approach. For this we rewrite a part of Equation (8) as

$$\begin{pmatrix} 1 & 1 \\ F_x & -F_x \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = e^{-iq_xL}\begin{pmatrix} \alpha_{22} & \beta_{22} \\ \alpha_{22} & -\beta_{22} \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} \tag{9}$$

Equation (9) can be reformulated as

$$e^{+iq_xL}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \frac{1}{2F_x}\begin{pmatrix} (F_x+1)\alpha_{22} & (F_x-1)\beta_{22} \\ (F_x-1)\alpha_{22} & (F_x+1)\beta_{22} \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} \tag{10}$$

The remaining part of Equation (8) gives:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \beta_{11} \\ F_x\alpha_{11} & -F_x\beta_{11} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \tag{11}$$

This latter equation results in:

$$\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+F_x)\alpha_{11} & (1-F_x)\beta_{11} \\ (1-F_x)\alpha_{11} & (1+F_x)\beta_{11} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \tag{12}$$

Combining Equations (12) and (10) yields:

$$e^{+iq_xL}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \frac{1}{2F_x}\begin{pmatrix} (F_x+1)\alpha_{22} & (F_x-1)\beta_{22} \\ (F_x-1)\alpha_{22} & (F_x+1)\beta_{22} \end{pmatrix}\frac{1}{2}\begin{pmatrix} (1+F_x)\alpha_{11} & (1-F_x)\beta_{11} \\ (1-F_x)\alpha_{11} & (1+F_x)\beta_{11} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \tag{13}$$

or

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_{n+1} = \frac{1}{2F_x}\begin{pmatrix} (F_x+1)\alpha_{22} & (F_x-1)\beta_{22} \\ (F_x-1)\alpha_{22} & (F_x+1)\beta_{22} \end{pmatrix}\frac{1}{2}\begin{pmatrix} (1+F_x)\alpha_{11} & (1-F_x)\beta_{11} \\ (1-F_x)\alpha_{11} & (1+F_x)\beta_{11} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_n \tag{14}$$

where we have used the Bloch wave definition $\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_n = e^{+iq_xnL}\begin{pmatrix} A_+ \\ A_- \end{pmatrix}$.

Calculating the transform matrix relating the A_+, A_- amplitudes between two adjacent unit cells leads to

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_{n+1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_n \tag{15}$$

The components of the transfer matrix are defined as

$$T_{11} = T_{22}^* = \frac{1}{4F_x}\alpha_{11}\left[(F_x+1)^2\alpha_{22} - (F_x-1)^2\beta_{22}\right] \tag{16a}$$

$$T_{12} = T_{21}^* = \frac{-1}{4F_x}\beta_{11}(F_x+1)(F_x-1)[\alpha_{22} - \beta_{22}] \tag{16b}$$

The "*" in Equation (16a,b) stands for the complex conjugate.

Using the Bloch theorem, we are seeking eigenvalues of the transfer matrix taking the form $\lambda = e^{iq_xL}$, and λ satisfies the second order equation:

$$\lambda^2 - \lambda(T_{11} + T_{11}^*) + T_{11}T_{11}^* - T_{12}T_{21}^* = 0$$

There exist two solutions given by

$$\lambda = \frac{T_{11} + T_{11}^*}{2} \pm \frac{1}{2}\sqrt{(T_{11} - T_{11}^*)^2 + 4T_{12}T_{21}^*} \tag{17}$$

We now investigate the condition $\sin k_x^{(2)}d_2 = 0$, which corresponds to a Fabry–Pérot resonance [12,14] of medium 2^l in the x direction. This condition is satisfied if $k_x^{(2)}d_2 = p\pi$, where p is an integer. There are two cases to consider, even and odd multiples of π.

To solve for the amplitudes, we rewrite Equation (15) in the form of an eigenvalue problem:

$$\begin{pmatrix} T_{11} - e^{+iq_xL} & T_{12} \\ T_{21} & T_{22} - e^{+iq_xL} \end{pmatrix}\begin{pmatrix} A_+ \\ A_- \end{pmatrix}_n = 0 \tag{18}$$

where e^{+iq_xL} is an eigenvalue, then we get

$$\left(T_{11} - e^{+iq_xL}\right)A_+ = -T_{12}A_- \tag{19}$$

We can solve for the amplitudes:

$$A_+ = -T_{12} \tag{20a}$$

$$A_- = T_{11} - e^{iq_xL} \tag{20b}$$

We first consider the case when p is odd. Under this condition, $\alpha_{22} = \beta_{22} = -1$ and $T_{12} = T_{21}^* = 0$. At the Fabry–Pérot resonance, $A_+ = 0$. Provided that the dispersion relation of the superlattice satisfies $\cos q_xL = -\cos k_x^{(1)}d_1$, we have $T_{11} = -\alpha_{11}$ and also $A_- = -i\sin k_x^{(1)}d_1 - i\sin q_xL$. $A_- = 0$ when $q_xL = k_x^{(1)}d_1 + l\pi$ where the integer l is odd. Note that this mode may be outside the first Brillouin zone. We recall that $k_x^{(1)}d_1 > 0$. Due to the periodicity in the wave vector space, q_xL is therefore located on the negative side of the first Brillouin zone, $q_xL \in [-\pi, 0]$ and we can rewrite $q_xL = -k_x^{(1)}d_1 < 0$. When $A_+ = A_- = 0$, Equation (12) ensures that $C_+ = C_- = 0$.

We now consider the case when p is even. Under this condition, $\alpha_{22} = \beta_{22} = +1$ and $T_{12} = T_{21}^* = 0$. At the Fabry–Pérot resonance, we still have, $A_+ = 0$. Provided that the dispersion relation of the superlattice satisfies $\cos q_xL = \cos k_x^{(1)}d_1$, we have $T_{11} = +\alpha_{11}$ and $A_- = i\sin k_x^{(1)}d_1 - i\sin q_xL = 0$ when $q_xL = k_x^{(1)}d_1 + l\pi$, where the integer l is even. q_xL is therefore located on the positive side of the first Brillouin zone, $q_xL \in [0, \pi]$. Due to the periodicity in the wave vector space, we can rewrite $q_xL = k_x^{(1)}d_1 > 0$. Again, $A_+ = A_- = 0$, implying that $C_+ = C_- = 0$.

3.2. Interfaces (I) and (VIII)

The conditions of continuity of displacement at (I):

$$\begin{aligned}\left(C_+^u + C_-^u\right)\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) &= e^{-iq_xL}\left(A_+^u e^{ik_x^{(1)}d_2} + A_-^u e^{-ik_x^{(1)}d_2}\right)\\ &\times\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right)\end{aligned} \tag{21a}$$

The condition of continuity of stress at interface (I) takes the form of

$$\begin{aligned}\left(C_+^u - C_-^u\right)\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) &= e^{-iq_xL}F_x\left(A_+^u e^{ik_x^{(1)}d_2} - A_-^u e^{-ik_x^{(1)}d_2}\right)\\ &\times\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right)\end{aligned} \tag{21b}$$

At interface (VIII), for displacement we obtain

$$\begin{aligned}\left(C_+^u e^{ik_x^{(2)}d_1} + C_-^u e^{-ik_x^{(2)}d_1}\right)\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) &= \left(A_+^u + A_-^u\right)\\ &\times\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right)\end{aligned} \tag{22a}$$

The continuity of stress at interface (VIII) leads to

$$\begin{aligned}\left(C_+^u e^{ik_x^{(2)}d_1} - C_-^u e^{-ik_x^{(2)}d_1}\right)\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) &= F_x\left(A_+^u - A_-^u\right)\\ &\times\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right)\end{aligned} \tag{22b}$$

By redefining

$$C'_+ = C_+^u\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) \tag{23a}$$

$$C'_- = C_-^u\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) \tag{23b}$$

$$A'_{+} = A^{u}_{+}\left(B^{u}_{+}e^{ik_y^{(1)}(y-mL-d_1)} + B^{u}_{-}e^{-ik_y^{(1)}(y-mL-d_1)}\right) \tag{23c}$$

$$A'_{-} = A^{u}_{-}\left(B^{u}_{+}e^{ik_y^{(1)}(y-mL-d_1)} + B^{u}_{-}e^{-ik_y^{(1)}(y-mL-d_1)}\right) \tag{23d}$$

we introduced the notation $\alpha_{12} = \frac{1}{\beta_{12}} = e^{ik_x^{(1)}d_2}$ and $\alpha_{21} = \frac{1}{\beta_{21}} = e^{ik_x^{(2)}d_1}$.

The conditions of continuity given by Equations (21a,b) and (22a,b) can be arranged in matrix form:

$$\begin{pmatrix} e^{-iq_xL}\alpha_{12} & e^{-iq_xL}\beta_{12} & -1 & -1 \\ e^{-iq_xL}F_x\alpha_{12} & -e^{-iq_xL}F_x\beta_{12} & -1 & 1 \\ 1 & 1 & -\alpha_{21} & -\beta_{21} \\ F_x & -F_x & -\alpha_{21} & \beta_{21} \end{pmatrix}\begin{pmatrix} A'_{+} \\ A'_{-} \\ C'_{+} \\ C'_{-} \end{pmatrix} = 0 \tag{24}$$

Following Section 3.1, we introduce a transfer matrix such that

$$\begin{pmatrix} C'_{+} \\ C'_{-} \end{pmatrix}_{n+1} = \begin{pmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{pmatrix}\begin{pmatrix} C'_{+} \\ C'_{-} \end{pmatrix}_{n} \tag{25}$$

The components of the transfer matrix are defined as

$$T'_{11} = T'^{*}_{22} = \frac{1}{4F_x}\alpha_{21}\left[(F_x+1)^2\alpha_{12} - (F_x-1)^2\beta_{12}\right] \tag{26a}$$

$$T'_{12} = T'^{*}_{21} = \frac{1}{4F_x}\beta_{21}(F_x+1)(F_x-1)[\alpha_{12} - \beta_{12}] \tag{26b}$$

The Bloch theorem implies that:

$$C'_{+} = -T'_{12} \tag{27a}$$

$$C'_{-} = T'_{11} - e^{iq_xL} \tag{27b}$$

In Section 3.1. we considered the resonant condition $\sin k_x^{(2)}d_2 = 0$. This condition is not sufficient to lead to zero amplitudes (e.g., $\alpha_{12} - \beta_{12} = 0$). Here, one needs to consider the condition $\sin k_x^{(1)}d_2 = 0$. This is the condition $k_x^{(1)}d_2 = p'\pi$, where p' can be odd or even.

We first consider the case when p' is odd. Under this condition, $\alpha_{12} = \beta_{12} = -1$ and $T'_{12} = T'^{*}_{21} = 0$. At this resonance, $C'_{+} = 0$. Provided that the dispersion relation of the superlattice satisfies $\cos q_xL = -\cos k_x^{(2)}d_1$, we have $T'_{11} = -\alpha_{21}$ and also $C'_{-} = -i\sin k_x^{(2)}d_1 - i\sin q_xL$. $C'_{-} = 0$ when $q_xL = k_x^{(2)}d_1 + l'\pi$ where the integer l' is odd, that is $q_x < 0$ but not when $q_x > 0$. When $C'_{+} = C'_{-} = 0$, we also get $A'_{+} = A'_{-} = 0$.

We now consider the case when p' is even. Under this condition, $\alpha_{12} = \beta_{12} = +1$ and $T'_{12} = T'^{*}_{21} = 0$. At the resonance we still have, $C'_{+} = 0$. Provided that the dispersion relation of the superlattice still satisfies $\cos q_xL = \cos k_x^{(2)}d_1$, we have $T'_{11} = +\alpha_{21}$ and $C'_{-} = i\sin k_x^{(2)}d_1 - i\sin q_xL = 0$ when $q_xL = k_x^{(1)}d_1 + l'\pi$ where l' is even, i.e., when $q_x > 0$ but not when $q_x < 0$. Again, when $C'_{+} = C'_{-} = 0$, implying that $A'_{+} = A'_{-} = 0$.

3.3. Interfaces (III) and (V)

At interface (III), the condition of continuity of displacement reads:

$$(B_{+} + B_{-}) = e^{-iq_yL}\left(D_{+}e^{ik_y^{(2)}d_2} + D_{-}e^{-ik_y^{(2)}d_2}\right) \tag{28a}$$

At interface (V), displacement continuity yields

$$\left(B_+ e^{ik_y^{(1)} d_1} + B_- e^{-ik_y^{(1)} d_1}\right) = (D_+ + D_-) \tag{28b}$$

The continuity of stress at interfaces (III) and (V) gives

$$F_y(B_+ - B_-) = e^{-iq_y L}\left(D_+ e^{ik_y^{(2)} d_2} - D_- e^{-ik_y^{(2)} d_2}\right) \tag{29a}$$

$$F_y\left(B_+ e^{ik_y^{(1)} d_1} - B_- e^{-ik_y^{(1)} d_1}\right) = (D_+ - D_-) \tag{29b}$$

where we have defined the quantity $F_y = \frac{\rho_1 c_1^2 k_y^{(1)}}{\rho_2 c_2^2 k_y^{(2)}}$.

In Equations (28) and (29), we have defined:

$$B_\pm = B_\pm^l\left(A_+^l e^{ik_x^{(1)}(x-nL)} + A_-^l e^{-ik_x^{(1)}(x-nl)}\right) \tag{30a}$$

$$D_\pm = D_\pm^u\left(C_+^u e^{ik_x^{(2)}(x-nL)} + C_-^u e^{-ik_x^{(2)}(x-nl)}\right) \tag{30b}$$

Equations (28) and (29) are reformulated in matrix form:

$$\begin{pmatrix} \eta_{11} & \delta_{11} & -1 & -1 \\ F_y\eta_{11} & -F_y\delta_{11} & -1 & 1 \\ 1 & 1 & -e^{-iq_yL}\eta_{22} & -e^{-iq_yL}\delta_{22} \\ F_y & -F_y & -e^{-iq_yL}\eta_{22} & e^{-iq_yL}\delta_{22} \end{pmatrix}\begin{pmatrix} B_+ \\ B_- \\ D_+ \\ D_- \end{pmatrix} = 0 \tag{31}$$

where we used the notation $\eta_{11} = \frac{1}{\delta_{11}} = e^{ik_y^{(1)} d_1}$ and $\eta_{22} = \frac{1}{\delta_{22}} = e^{ik_y^{(2)} d_2}$. Equation (31) is isomorphic to Equation (8) but for the direction y.

We now investigate the condition $\sin k_y^{(2)} d_2 = 0$ which corresponds to a Fabry–Pérot resonance of medium 2^u in the y direction. This condition is satisfied if $k_y^{(2)} d_2 = r\pi$, where r is an integer. There are two cases to consider, even and odd multiples of π.

Solving for the amplitudes, we first consider the case when r is odd. Under this condition, $\eta_{22} = \delta_{22} = -1$. At the Fabry–Pérot resonance, $B_+ = 0$. Provided that the dispersion relation of the superlattice satisfies $\cos q_y L = -\cos k_y^{(1)} d_1$, and also $A_- = -i\sin k_y^{(1)} d_1 - i\sin q_y L$. $A_- = 0$ when $q_y L = k_y^{(1)} d_1 + s\pi$ where the integer s is odd, that is, when $q_y < 0$ but not when $q_y > 0$. The condition $B_+ = B_- = 0$ leads to $D_+ = D_- = 0$.

We now consider the case when r is even. Under this condition, $\eta_{22} = \delta_{22} = +1$. At the Fabry–Pérot resonance we still have, $B_+ = 0$. Provided that the dispersion relation of the superlattice still satisfies $\cos q_y L = \cos k_y^{(1)} d_1$, $B_- = i\sin k_y^{(1)} d_1 - i\sin q_y L = 0$ when $q_y L = k_y^{(1)} d_1 + s\pi$ where s is even. This corresponds to a wave number in the negative region of the Brillouin zone $q_y > 0$. Again, when $B_+ = B_- = 0$, $D_+ = D_- = 0$.

3.4. Interfaces (IV) and (VI)

The conditions of displacement and stress continuity at interfaces (IV) and (VI) can be arranged in matrix form:

$$\begin{pmatrix} e^{-iq_yL}\eta_{12} & e^{-iq_yL}\delta_{12} & -1 & -1 \\ e^{-iq_yL}F_x\eta_{12} & -e^{-iq_yL}F_x\delta_{12} & -1 & 1 \\ 1 & 1 & -\eta_{21} & -\delta_{21} \\ F_y & -F_y & -\eta_{21} & \delta_{21} \end{pmatrix}\begin{pmatrix} B'_+ \\ B'_- \\ D'_+ \\ D'_- \end{pmatrix} = 0 \tag{32}$$

Here, we introduced the notation $\eta_{12} = \frac{1}{\delta_{12}} = e^{ik_y^{(1)} d_2}$ and $\eta_{21} = \frac{1}{\delta_{21}} = e^{ik_y^{(2)} d_1}$. We have also defined

$$B'_{\pm} = B^u_{\pm}\left(A^u_+ e^{ik_x^{(1)}(x-nL-d_1)} + A^u_- e^{-ik_x^{(1)}(x-nl-d_1)}\right) \tag{33a}$$

$$D'_{\pm} = D^l_{\pm}\left(C^l_+ e^{ik_x^{(2)}(x-nL-d_1)} + C^l_- e^{-ik_x^{(2)}(x-nl-d_1)}\right) \tag{33b}$$

We can perform the same analysis as was done for the pair of interfaces (I) and (VIII) but for propagation along the y direction.

We now investigate the resonant condition $\sin k_y^{(1)} d_2 = 0$. This condition is satisfied if $k_y^{(1)} d_2 = r'\pi$, where r' is an integer. There are two cases to consider, even and odd multiples of π.

Solving for the amplitudes, we first consider the case when r' is odd. Under this condition, $\eta_{12} = \delta_{12} = -1$. At resonance, $B'_+ = 0$. Provided that the dispersion relation of the superlattice satisfies $\cos q_y L = -\cos k_y^{(2)} d_1$, and also $B'_- = -i\sin k_y^{(2)} d_1 - i\sin q_y L$. $B'_- = 0$ when $q_y L = k_y^{(2)} d_1 + s'\pi$ where the integer s' is odd. This corresponds to $q_y < 0$ but not $q_y > 0$. The condition $B'_+ = B'_- = 0$ leads to $D'_+ = D'_- = 0$.

We now consider the case when r' is even. Under this condition, $\eta_{12} = \delta_{12} = +1$. At the resonance we still have, $B'_+ = 0$. Provided that the dispersion relation of the superlattice still satisfies $\cos q_y L = \cos k_y^{(2)} d_1$, $B'_- = i\sin k_y^{(2)} d_1 - i\sin q_y L = 0$ when $q_y L = k_y^{(2)} d_1 + s'\pi$ where s is even, corresponding to the positive side of the Brillouin zone in the y direction, that is, $q_y > 0$ but not $q_y < 0$. Again, $B'_+ = B'_- = 0$ implies that $D'_+ = D'_- = 0$.

Table 1 summarizes these findings.

Table 1. Resonant conditions resulting from continuity conditions at interfaces leading to zero amplitudes. See the text for details.

Interfaces	Resonance	Dispersion	Bloch Wave Number	Amplitudes
(II) & (VII)	$k_x^{(2)} d_2 = p\pi$ p odd	$\cos q_x L = -\cos k_x^{(1)} d_1$	$q_x L = k_x^{(1)} d_1 + l\pi$ l odd	$A_+ = A_- = 0$ $C_+ = C_- = 0$ $q_x < 0$
	$k_x^{(2)} d_2 = p\pi$ p even	$\cos q_x L = \cos k_x^{(1)} d_1$	$q_x L = k_x^{(1)} d_1 + l\pi$ l even	$A_+ = A_- = 0$ $C_+ = C_- = 0$ $q_x > 0$
(I) & (VIII)	$k_x^{(1)} d_2 = p'\pi$ p' odd	$\cos q_x L = -\cos k_x^{(2)} d_1$	$q_x L = k_x^{(2)} d_1 + l'\pi$ l' odd	$C'_+ = C'_- = 0$, $A'_+ = A'_- = 0$ $q_x < 0$
	$k_x^{(1)} d_2 = p'\pi$ p' even	$\cos q_x L = \cos k_x^{(2)} d_1$	$q_x L = k_x^{(2)} d_1 + l'\pi$ l' even	$C'_+ = C'_- = 0$, $A'_+ = A'_- = 0$ $q_x > 0$
(III) & (V)	$k_y^{(2)} d_2 = r\pi$ r odd	$\cos q_y L = -\cos k_y^{(1)} d_1$	$q_y L = k_y^{(1)} d_1 + s\pi$ s odd	$B_+ = B_- = 0$ $D_+ = D_- = 0$ $q_y < 0$
	$k_y^{(2)} d_2 = r\pi$ r even	$\cos q_y L = \cos k_y^{(1)} d_1$	$q_y L = k_y^{(1)} d_1 + s\pi$ s even	$B_+ = B_- = 0$ $D_+ = D_- = 0$ $q_y > 0$
(IV) & (VI)	$k_y^{(1)} d_2 = r'\pi$ r' odd	$\cos q_y L = -\cos k_y^{(2)} d_1$	$q_y L = k_y^{(2)} d_1 + s'\pi$ s' odd	$B'_+ = B'_- = 0$ $D'_+ = D'_- = 0$ $q_y < 0$
	$k_y^{(1)} d_2 = r'\pi$ r' even	$\cos q_y L = \cos k_y^{(2)} d_1$	$q_y L = k_y^{(2)} d_1 + s'\pi$ s' even	$B'_+ = B'_- = 0$ $D'_+ = D'_- = 0$ $q_y > 0$

4. DRAK Modes

In this section, we identify sets of conditions which lead to directional modes, i.e., DRAK modes based on the findings of the previous section.

4.1. Case I

Within a medium of type 1, the displacement field can be written as

$$u_1^l(x,y) = e^{iq_x nL} e^{iq_y mL}\left(A_+ e^{ik_x^{(1)}(x-nL)} + A_- e^{-ik_x^{(1)}(x-nL)}\right) \tag{34a}$$

$$u_1^u(x,y) = e^{iq_x nL} e^{iq_y mL}\left(A'_+ e^{ik_x^{(1)}(x-nL-d_1)} + A'_- e^{-ik_x^{(1)}(x-nL-d_1)}\right) \tag{34b}$$

In medium 2, we can write

$$u_1^u(x,y) = e^{iq_x nL} e^{iq_y mL}\left(A'_+ e^{ik_x^{(1)}(x-nL-d_1)} + A'_- e^{-ik_x^{(1)}(x-nL-d_1)}\right) \tag{35a}$$

$$u_2^u(x,y) = e^{iq_x nL} e^{iq_y mL}\left(C'_+ e^{ik_x^{(2)}(x-nL)} + C'_- e^{-ik_x^{(2)}(x-nL)}\right) \tag{35b}$$

For the amplitudes, we recall Equation (7a–d):

$$A_\pm = A_\pm^l\left(B_+^l e^{ik_y^{(1)}(y-mL)} + B_-^l e^{-ik_y^{(1)}(y-mL)}\right) \tag{36a}$$

$$C_\pm = C_\pm^l\left(D_+^l e^{ik_y^{(2)}(y-mL)} + D_-^l e^{-ik_y^{(2)}(y-mL)}\right) \tag{36b}$$

$$C'_\pm = C_\pm^u\left(D_+^u e^{ik_y^{(2)}(y-mL-d_1)} + D_-^u e^{-ik_y^{(2)}(y-mL-d_1)}\right) \tag{36c}$$

$$A'_\pm = A_\pm^u\left(B_+^u e^{ik_y^{(1)}(y-mL-d_1)} + B_-^u e^{-ik_y^{(1)}(y-mL-d_1)}\right) \tag{36d}$$

To find DRAK modes, we seek conditions for the displacement field $\left\{u_1^l, u_1^u, u_2^l, u_2^u\right\}$, that is, for the corresponding amplitudes, to vanish for wave vectors located on one side of the 2D Brillouin zone and not the other. For this, we use Table 1.

Let us consider the resonant conditions for medium 2^l corresponding to $\sin k_x^{(2)} d_2 = 0$. For instance, we chose $k_x^{(2)} d_2 = p\pi$, with p being an odd integer. The dispersion relation that must be satisfied is $\cos q_x L = -\cos k_x^{(1)} d_1$, with $q_x L = k_x^{(1)} d_1 + l\pi$ and l an odd integer. Under these conditions we have $A_\pm = 0$ and $C_\pm = 0$ for $q_x < 0$. This resonance implies that $u_1^l(x,y) = u_2^l(x,y) = 0$.

We also consider the additional resonance of medium 1^u, $\sin k_x^{(1)} d_2 = 0$, such that $k_x^{(1)} d_2 = p'\pi$ with being p' an odd integer. The corresponding dispersion relation that must be satisfied by the wave vector is $\cos q_x L = -\cos k_x^{(2)} d_1$ with $q_x L = k_x^{(2)} d_1 + l'\pi$ and l' odd. Under these conditions, we have $C'_+ = C'_- = 0$, and $A'_+ = A'_- = 0$, leading to $u_1^u(x,y) = u_2^u(x,y) = 0$.

These conditions for the existence of a DRAK mode are independent of q_y. The motion associated with the joint resonances in the x direction of the subunits 2^l and 1^u is incompatible with the translational periodicity of Bloch waves in that same direction, leading to zero amplitudes.

A DRAK mode will exist when the 2D dispersion relation of the phononic crystal, $\omega(q_x, q_y)$, satisfies specific conditions. In the current case, these conditions are independent of q_y. When both resonances are satisfied, the wave number in the x direction is defined as

$$q_x L = k_x^{(1)} d_1 + l\pi = k_x^{(2)} d_1 + l'\pi$$

Which, using the resonant conditions for the wave number in the x direction, leads to

$$\left(p' - p\right)\frac{d_1}{d_2} = \left(l' - l\right) \tag{37}$$

Equation (37) determines the ratio of geometric parameters $\frac{d_1}{d_2}$ yielding the wave number, q_x, which corresponds to specific resonances.

We recall that the angular frequency of the Bloch waves is given by

$$\omega = c_1\sqrt{\left(k_x^{(1)}\right)^2 + \left(k_y^{(1)}\right)^2} = c_2\sqrt{\left(k_x^{(2)}\right)^2 + \left(k_y^{(2)}\right)^2} \tag{38}$$

At the specific frequency $\omega(q_x, q_y)$, the relation of Equation (38) imposes $k_y^{(1)}$ and $k_y^{(2)}$ to take the values

$$k_y^{(1)} = \sqrt{\frac{\omega^2}{c_1^2} - \left(\frac{p'\pi}{d_2}\right)^2} \tag{39a}$$

and

$$k_y^{(2)} = \sqrt{\frac{\omega^2}{c_2^2} - \left(\frac{p\pi}{d_2}\right)^2} \tag{39b}$$

These relationships can be used to determine the speeds of sound of the constitutive materials which are compatible with the dispersion relation and the resonance conditions of the 2D phononic crystal. For example, combining Equation (39a,b) and taking into account that $k_y^{(1)} > 0$ and $k_y^{(2)} > 0$, we get $\frac{c_2}{c_1} < \frac{p'}{p}$.

4.2. Case II

The displacement of the four subunits forming the unit cell given by Equation (7a–d) can be rewritten as

$$u_1^l(x,y) = e^{iq_x nL} e^{iq_y mL}\left(B_+ e^{ik_y^{(1)}(y-mL)} + B_- e^{-ik_y^{(1)}(y-mL)}\right) \tag{40a}$$

$$u_1^u(x,y) = e^{iq_x nL} e^{iq_y mL}\left(B'_+ e^{ik_y^{(1)}(y-mL-d_1)} + B'_- e^{-ik_y^{(1)}(y-mL-d_1)}\right) \tag{40b}$$

and

$$u_2^l(x,y) = e^{iq_x nL} e^{iq_y mL}\left(D'_+ e^{ik_y^{(2)}(y-mL)} + D'_- e^{-ik_y^{(2)}(y-mL)}\right) \tag{41a}$$

$$u_2^u(x,y) = e^{iq_x nL} e^{iq_y mL}\left(D_+ e^{ik_y^{(2)}(y-mL-d_1)} + D_- e^{-ik_y^{(2)}(y-mL-d_1)}\right) \tag{41b}$$

The resonances associated with $\sin k_y^{(2)} d_2 = 0$ (subunit 2^u) and $\sin k_y^{(1)} d_2 = 0$ (subunit 1^u) may lead to $B_+ = B_- = 0$, $D_+ = D_- = 0$, $B'_+ = B'_- = 0$, and $D'_+ = D'_- = 0$ for $q_y < 0$, yielding displacements that vanish.

For instance, we may choose $k_y^{(2)} d_2 = r\pi$ with r being an odd integer. The dispersion relation that must be satisfied is $\cos q_y L = -\cos k_y^{(1)} d_1$, with $q_y L = k_y^{(1)} d_1 + s\pi$ and s an odd integer. Under these conditions we have $B_\pm = 0$ and $D_\pm = 0$ for $q_y < 0$. This resonance implies that $u_1^l(x,y) = u_2^u(x,y) = 0$.

We also consider the additional resonance associated with $\sin k_y^{(1)} d_2 = 0$, such that $k_1^{(1)} d_2 = r'\pi$ with r' an odd integer. The corresponding dispersion relation that must be satisfied by the wave vector is $\cos q_y L = -\cos k_y^{(2)} d_1$ with $q_y L = k_y^{(2)} d_1 + s'\pi$, and s' odd. Under these conditions, we have $B'_+ = B'_- = 0$ and $D'_+ = D = 0$, leading to $u_1^u(x,y) = u_2^l(x,y) = 0$ for $q_y < 0$.

These conditions for the existence of a DRAK mode are independent of q_x. The motion associated with these joint resonances in the y direction of the subunits 1^u and 2^u is incompatible with the translational periodicity of Bloch waves in that same direction leading to zero amplitudes.

A DRAK mode will exist when the 2D dispersion relation of the phononic crystal, $\omega(q_x, q_y)$, satisfies some specific conditions. In the current case, these conditions are independent of q_x. When both resonances are satisfied, the wave number in the x direction is defined as

$$q_y L = k_y^{(1)} d_1 + s\pi = k_y^{(2)} d_1 + s'\pi$$

Which, using the resonant conditions for the wave number in the y direction, leads to:

$$\left(r' - r\right) \frac{d_1}{d_2} = \left(s' - s\right) \tag{42}$$

Equation (42) determines the ratio of geometric parameters $\frac{d_1}{d_2}$, yielding the wave number, q_y, which corresponds to specific resonances.

We recall that the angular frequency of Bloch waves is given by

$$\omega = c_1 \sqrt{\left(k_x^{(1)}\right)^2 + \left(k_y^{(1)}\right)^2} = c_2 \sqrt{\left(k_x^{(2)}\right)^2 + \left(k_y^{(2)}\right)^2} \tag{43}$$

At the specific frequency $\omega(q_x, q_y)$, the relation of Equation (42) imposes $k_x^{(1)} > 0$ and $k_x^{(2)} > 0$ to take the values

$$k_x^{(1)} = \sqrt{\frac{\omega^2}{c_1^2} - \left(\frac{r'\pi}{d_2}\right)^2} \tag{44a}$$

and

$$k_x^{(2)} = \sqrt{\frac{\omega^2}{c_2^2} - \left(\frac{r\pi}{d_2}\right)^2} \tag{44b}$$

Again, these relationships can be used to determine the speeds of sound of the constitutive materials which are compatible with the dispersion relation and the resonance conditions of the 2D phononic crystal.

5. Conclusions

We used the transfer matrix method to investigate the conditions leading to the existence of one-way propagating bulk waves, called DRAK modes, in a two-dimensional (2D) phononic crystal. The model system investigated here was a 2D phononic crystal taking the form of a square lattice of unit cells composed of rectangular subunits constituted of two different continuous media. We sought conditions where local resonances of the Fabry–Pérot type of the subunits became incompatible with the translational periodicity associated with the Bloch theorem. We established conditions on the geometry of the phononic crystal and the physical properties of the constitutive media for the existence of waves that exhibit a non-zero amplitude in one direction within the crystal's 2D Brillouin zone and zero amplitude in the opposite direction for the same handedness. Because of time-reversal symmetry, we can expect a twin mode with a non-zero amplitude and opposite handedness propagating in the reverse direction. This work extends previous studies that have focused on DRAK modes in one-dimensional discrete and continuous superlattices. The existence of directional DRAK waves in 2D phononic structures opens new avenues in the design of acoustic devices which may exhibit robust immunity to scattering by obstacles such as defects, imperfections, or impedance mismatch between different parts of the devices, expectedly making them less prone to back reflections for a broader range of obstacles that do not couple modal handedness. Indeed, acoustic wave devices such as bulk acoustic wave (BAW) and surface acoustic wave (SAW) devices find many applications in radio frequency telecommunication [18,19] and sensors of various physical quantities [20]. However, the performance of these devices is strongly affected by loss such as return loss, which measures the amount of reflected signal caused by the impedance mismatch between the transduction parts of the device and that of the delay line part. DRAK modes may help overcome return loss in such acoustic devices beyond what can be achieved with current technologies.

Future work will involve numerical studies of DRAK modes in more complex phononic structures than the idealized "checkerboard" system studied here, as well as phononic structures with different symmetries or phononic structures more amenable to contemporary fabrication techniques.

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