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THE BREZIS-NIRENBERG PROBLEM ON NON-CONTRACTIBLE BOUNDED DOMAINS OF $\mathbb{R}^3$

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Abstract. In this paper, we study the Brezis-Nirenberg problem on bounded smooth domains of $\mathbb{R}^3$. Using the algebraic topological argument of Bahri-Coron[3] as implemented in [9] combined with the Brendle[5]-Schoen[14]'s bubble construction, we solve the problem for non-contractible bounded smooth domains.

1. **Introduction and statement of the results.** In their seminal paper [6], Brezis and Nirenberg initiated the study of nonlinear elliptic equations of the form

$$\begin{cases} -\Delta u + qu = u^{\frac{m+2}{m-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain (with domain meaning open and connected) of $\mathbb{R}^m$ with a connected and smooth boundary $\partial\Omega$ (with smooth boundary meaning $\overline{\Omega}$ the closure of Ω is a smooth m-dimensional manifold with smooth boundary $\partial\Omega$), $m \geq 3$ and q is a bounded and smooth function defined on Ω. In this paper, we revisit the Boundary Value problem (BVP) (1) in the 3-dimensional case, namely when $m = 3$. Thus, we will be dealing with the BVP

$$\begin{cases} -\Delta u + qu = u^5 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^3$ with a connected and smooth boundary. It is well known that a necessary condition for the existence of positive solution to (2) is that the first eigenvalue of $-\Delta + q$ under zero Dirichlet boundary condition is

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positive, see [7]. Moreover, we will assume that the Green's function G of $-\Delta + q$ under zero Dirichlet boundary condition defined by (11) is positive in Ω and hence $-\Delta+q$ under zero Dirichlet boundary condition verifies the strong maximum principle. Thus the BVP (2) has a variational structure, since thanks to the strong maximum principle and standard elliptic regularity theory solutions of (2) can be found by looking at critical points of the Brezis-Nirenberg functional

$$J_q(u) := \frac{\langle u, u\rangle_q}{(\int_\Omega u^6\, dx)^{\frac{1}{3}}}, \quad u \in H_0^{1,+}(\Omega) := \{u \in H_0^1(\Omega): \quad u \geq 0 \text{ and } u \neq 0 \text{ in } \Omega\}, \tag{3}$$

where

$$\langle u, u\rangle_q = \int_\Omega (|\nabla u|^2 + qu^2)\, dx \tag{4}$$

and $H_0^1(\Omega)$ is the usual Sobolev space of functions which are L^2-integrable on Ω together with their first derivatives and with zero trace on $\partial\Omega$.

Existence of solutions of (2) under a Positive Mass type assumption has been obtained in an unpublished work by McLeod as discussed in the work of Brezis[7]. In this work, we use the Barycenter Technique of Bahri-Coron[3] to remove the Positive Mass type assumption of McLeod and replace it by the non-contractibility of the domain. Precisely, we prove the following theorem.

Theorem 1.1. Assuming that $\Omega \subset \mathbb{R}^3$ is a non-contractible bounded domain with a connected and smooth boundary, q is a smooth and bounded function defined on Ω, the first eigenvalue of the operator $-\Delta + q$ under zero Dirichlet boundary condition on $\partial\Omega$ is positive, and the Green's function G of $-\Delta + q$ under zero Dirichlet boundary condition on $\partial\Omega$ defined by (11) is positive in Ω, then the BVP (2) has a least one solution.

Remark 1.2. We would like to make a comment about the non-contractibility assumption in Theorem 1.1. It implies (a well-known result) that some homology with $\mathbb{Z}_2$ coefficient and positive order d is non-zero (see Lemma 2.1 below and Remark 2.2) and that is what is needed for the application of the Barycenter Technique of Bahri-Coron[3].

To prove Theorem1.1, we will use the Algebraic topogical argument of Bahri-Coron[3] which is possible since as already observed by McLeod (see [7]), the problem under study is a Global one (for the definition of "Gobal" for Yamabe type problems, see [12]). Indeed, as in [12], we will follow the scheme of the Barycenter technique as performed in the work [9] of the second author and Mayer (see also [1], [10], and [12]). One of the main difficulty with respect to the works [9] and [12] is the presence of the linear term "qu" and the lack of conformal invariance. Such a difficulty has already been encountered by Bahri-Brezis[4] on closed Riemannian manifolds. To deal with such a difficulty, Bahri-Brezis[4] have used the bubble construction of Bahri-Coron[3] recalling that their scheme of the Barycenter Technique follows the original one of Bahri-Coron[3]. However, here we use the Brendle[5]-Schoen[14]' s bubble construction and have to deal with that difficulty in a way different from the work of Bahri-Brezis[4].

2. **Notations and preliminaries.** In this section, we fix some notation and discuss some preliminaries. We start with fixing some notation. $\mathbb{N}$ denotes the set of

non-negative integers and $\mathbb{N}^*$ denotes set of the positive integers. For $a \in \mathbb{R}^3$ and $\delta > 0$, $B_a(\delta) = B(a,\delta)$ denotes the Euclidean Ball with radius δ centered at a. 1_A denotes the characteristic function of A. ∇ denotes the Euclidean gradient and Δ denotes the Euclidean Laplacian. All integrations are with respect to dx the standard Lebesque measure on $\mathbb{R}^3$ with $x = (x_1, x_2, x_3)$ the standard coordinate system of $\mathbb{R}^3$. $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote respectively the standard norm and scalar product on $\mathbb{R}^3$. We also use $|\cdot|$ to denote the absolute value on $\mathbb{R}$. For $E \subset \mathbb{R}^3$ and $p \in \mathbb{N}^*$, $L^p(E)$ denotes the usual Lebesgue space of order p with respect to dx. For $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function, $\chi^{'}$ and $\chi^{''}$ denotes respectively the first derive and second derivative of χ. For $a \in K \subset \Omega$, K compact, and $0 \leq d_1 < d_2 \leq \infty$, we set $\{d_1 \leq |x-a| \leq d_2\} = \{x \in \Omega : \quad d_1 \leq |x-a| \leq d_2\}$. To simplify notation, we write $d_1 \leq |x-a| \leq d_2$ instead of $\{d_1 \leq |x-a| \leq d_2\}$ if there is no possible confusion. Similarly, for $0 \leq d_1 < d_2 \leq \infty$, we set $\{d_1 \leq |y| \leq d_2\} = \{y \in \mathbb{R}^3 : \quad d_1 \leq |y| \leq d_2\}$. To simplify notation, we write $d_1 \leq |y| \leq d_2$ instead of $\{d_1 \leq |y| \leq d_2\}$ if there is no possible confusion.

Next, we introduce the standard bubbles of the variational problem under study. For $a \in \mathbb{R}^3$ and $\lambda > 0$, we denote by $\delta_{a,\lambda}$ the standard bubble on $\mathbb{R}^3$ with center a and scaling parameter λ, namely

$$\delta_{a,\,\lambda}(x) = c_0 \left(\frac{\lambda}{1+\lambda^2|x-a|^2} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^3, \tag{5}$$

where $c_0 > 0$ is such that $\delta_{a,\lambda}$ satisfies

$$-\Delta\delta_{a,\lambda} = \delta_{a,\lambda}^5 \quad \text{on} \quad \mathbb{R}^3. \tag{6}$$

The $\delta_{a,\lambda}$'s verify the following relations

$$\int_{\mathbb{R}^3} |\nabla\delta_{a,\lambda}|^2 = \int_{\mathbb{R}^3} \delta_{a,\lambda}^6 = \int_{\mathbb{R}^3} |\nabla\delta_{0,1}|^2 = \int_{\mathbb{R}^3} \delta_{0,1}^6 \tag{7}$$

and

$$\mathcal{S} = \frac{\int_{\mathbb{R}^3} |\nabla\delta_{a,\lambda}|^2}{(\int_{\mathbb{R}^3} \delta_{a,\lambda}^6)^{\frac{1}{3}}}, \tag{8}$$

where

$$\mathcal{S} = \inf_{u \in D^1(\mathbb{R}^3),\ u \neq 0} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} u^6)^{\frac{1}{3}}} \tag{9}$$

with

$$D^1(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \quad |\nabla u| \in L^2(\mathbb{R}^3)\}.$$

We set

$$c_3 = \int_{\mathbb{R}^3} \left(\frac{1}{1+|y|^2} \right)^{\frac{5}{2}}. \tag{10}$$

For $a \in \Omega$, let $G(a,x)$ be the unique solution of (see [7])

$$\begin{cases} -\Delta G(a,x) + qG(a,x) = 4\pi\delta_a(x), & x \in \Omega \\ G(a,x) = 0, & x \in \partial\Omega. \end{cases} \tag{11}$$

$G(a,x)$ satisfies the following estimates

$$\left| G(a,x) - \frac{1}{|x-a|} \right| \leq C \quad \text{for} \quad x \neq a \in \Omega, \tag{12}$$

and

$$\left|\nabla\left(G(a,x)-\frac{1}{|x-a|}\right)\right| \leq \frac{C}{|x-a|} \quad \text{for} \quad x \neq a \in \Omega. \tag{13}$$

Moreover, under the assumption of Theorem 1.1, we have $G > 0$ in Ω. For estimates of the form (12) and (13) in the fractional setting, see [11].

As observed in Remark 1.2, the non-contractibility of Ω implies that some homology with Z_2 coefficient of Ω is non-trivial. Indeed, denoting (respectively) by $\pi_k(\Omega)$ and $H_k(\Omega, F)$ the k-th homotopy group of Ω and the k-th homology group of Ω with coefficient in F ($k \in \mathbb{N}$ and F a field), we have :

Lemma 2.1. Assuming that $\Omega \subset \mathbb{R}^3$ is a non-contractible bounded domain with a connected and smooth boundary, then there exists $n \in \{1, 2, 3\}$ such that

$$H_n(\Omega, \mathbb{Z}_2) \neq 0.$$

Proof. We divide the proof in two parts.

A): Ω simply-connected or not simply-connected and not perfect fundamental group

Since Ω is a bounded and smooth domain of $\mathbb{R}^3$, then Ω has the homotopy type of a CW-complex. Also, since Ω is a domain of $\mathbb{R}^3$, then it is open and connected. Hence Ω is path connected. Thus by Whitehead theorem Ω non-contractible implies Ω is weakly not-contractible. Hence there exists $n \in \{1, 2, 3\}$ such that $\pi_n(\Omega) \neq 0$. Without loss of generality, we can assume $\pi_k(\Omega) = 0$ for $k < n$ by replacing n with $n_0 := \inf\{0 \leq k \leq n / \ \pi_k(\Omega) \neq 0\}$. To continue we will consider 3 cases depending on if Ω is not simply connected ($n = 1$), Ω is simply connected and not 2-connected ($n = 2$), and Ω is 2-connected and $\pi_3(\Omega) \neq 0$.

Case 1: $n = 1$.

Since Ω is path connected, then by Hurewicz Theorem, $H_1(\Omega, \mathbb{Z}) \simeq \pi_1(\Omega)^{ab}$, where $\pi_1(\Omega)^{ab}$ is the abelianization of $\pi_1(\Omega)$. Thus since $\pi_1(\Omega) \neq 0$, then Ω is not simply connected implying in this case that $\pi_1(\Omega)$ is not perfect which is equivalent to $\pi_1(\Omega)^{ab} \neq 0$. Hence $H_1(\Omega, \mathbb{Z}) \neq 0$, and using again the assumption Ω path-connected (which implies $H_0(\Omega, \mathbb{Z}) = \mathbb{Z}$) combined with the Universal Coefficient Theorem and $Tor(\mathbb{Z}, \mathbb{Z}_2) = 0$, we get

$$H_1(\Omega, \mathbb{Z}_2) \simeq H_1(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \oplus Tor(H_0(\Omega, \mathbb{Z}), \mathbb{Z}_2) \simeq H_1(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \neq 0.$$

where $Tor(\cdot, \cdot)$ is the first Tor functor. We observe that we could conclude the non triviality of $H_1(\Omega, Z_2)$ just from $H_1(\Omega, \mathbb{Z}) \neq 0$ and $H_1(\Omega, \mathbb{Z}_2) \simeq H_1(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \oplus Tor(H_0(\Omega, \mathbb{Z}), \mathbb{Z}_2)$

Case 2: $n = 2$.

In this case, we have Ω is 1-connected ($\pi_k(\Omega) = 0 \ k < 2$), then by Hurewicz Theorem, $H_2(\Omega, \mathbb{Z}) \simeq \pi_2(\Omega)$. Thus $\pi_2(\Omega) \neq 0$ implies $H_2(\Omega, \mathbb{Z}) \neq 0$. Using the Universal Coefficient Theorem we obtain

$$H_2(\Omega, \mathbb{Z}_2) \simeq H_2(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \oplus Tor(H_1(\Omega, \mathbb{Z}), \mathbb{Z}_2).$$

Applying Hurewicz Theorem as in the case $n = 1$ gives $H_1(\Omega, \mathbb{Z}) = 0$, since $\pi_1(\Omega) = 0$. Thus we have $Tor(H_1(\Omega, \mathbb{Z}), \mathbb{Z}_2) = 0$. Hence, we obtain

$$H_2(\Omega, \mathbb{Z}_2) \simeq H_2(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \neq 0.$$

Similar to the case $n = 1$ here also we could conclude the non-triviality of $H_2(\Omega, Z_2)$ just from $H_2(\Omega, \mathbb{Z}) \neq 0$ and $H_2(\Omega, \mathbb{Z}_2) \simeq H_2(\Omega, \mathbb{Z}) \otimes \mathbb{Z} \oplus Tor(H_1(\Omega, \mathbb{Z}), \mathbb{Z}_2)$.

Case 3: $n = 3$.

In this case, we have Ω is 2-connected ($\pi_k(\Omega) = 0 \; k < 3$), then by Hurewicz Theorem, $H_3(\Omega,\mathbb{Z}) \simeq \pi_3(\Omega)$. So $\pi_3(\Omega) \neq 0$ implies $H_3(\Omega,\mathbb{Z}) \neq 0$. Using the Universal Coefficient Theorem we get

$$H_3(\Omega,\mathbb{Z}_2) \simeq H_2(\Omega,\mathbb{Z}) \otimes \mathbb{Z} \oplus Tor(H_2(\Omega,\mathbb{Z}),\mathbb{Z}_2).$$

Applying Hurewicz Theorem as in the case $n = 2$ gives $H_2(\Omega,\mathbb{Z}) = 0$. Therefore we have $Tor(H_2(\Omega,\mathbb{Z}),\mathbb{Z}_2) = 0$. Hence, we obtain

$$H_3(\Omega,\mathbb{Z}_2) \simeq H_3(\Omega,\mathbb{Z}) \otimes \mathbb{Z} \neq 0.$$

Similar to the cases $n = 1, 2$, here also we could conclude the non-triviality of $H_3(\Omega,\mathbb{Z}_2)$ just from $H_3(\Omega,\mathbb{Z}) \neq 0$ and $H_3(\Omega,\mathbb{Z}_2) \simeq H_3(\Omega,\mathbb{Z})\otimes\mathbb{Z}\oplus Tor(H_2(\Omega,\mathbb{Z}),\mathbb{Z}_2)$.

B): Ω not simply connected and perfect fundamental group

In this part, we present an argument which works also for part **A)**. We decide to present both cases separately, since for part **A)**, the argument we presented use only elementary algebraic topology, while the one we are going to present use more advanced tools and results in algebraic topology and surface theory. Moreover, we will emphasize the role of the smoothness and boundedness of Ω and more importantly of the assumption dimension of Ω $m = 3$. To better point out the role of $m = 3$, we will carry the argument with $m \geq 3$ until the assumption $m = 3$ is needed which will be only at the conclusion of the argument. We argue by contradiction. Indeed assuming that

$$H_k(\Omega, Z_2) = 0 \;\text{ for }\; 1 \leq k \leq m-1. \tag{14}$$

we are going to look for a contradiction. Observe the restriction $k \leq m-1$ which leads to a stronger conclusion as discussed in Remark 2.2 below. Coming back to our proof, we have since Ω is a smooth m-dimensional compact manifold with smooth boundary, then by Lefschetz duality (Poincare duality for smooth compact manifolds with boundary), we get

$$H_k(\Omega,\partial\Omega,\mathbb{Z}_2) \simeq H^{m-k}(\Omega,\mathbb{Z}_2) \;\text{ for }\; 1 \leq k \leq m-1. \tag{15}$$

with $H^k(\cdot,\mathbb{Z}_2)$ denoting the kth-cohomology with Z_2 coefficients. Moreover, since $\mathbb{Z}_2$ is a field, the by the duality between $\mathbb{Z}_2$-homology and $\mathbb{Z}_2$-cohomology,

$$H^k(\Omega,\mathbb{Z}_2) \simeq H_k(\Omega,\mathbb{Z}_2)^{'} \quad\text{for }\; 1 \leq k \leq m-1. \tag{16}$$

with $()^{'}$ denoting the dual. Furthermore, we have the following part of the long exact sequence for relative homology of the pair $(\Omega,\partial\Omega)$

$$... \to H_{k+1}(\Omega,\partial\Omega,\mathbb{Z}_2) \to H_k(\partial\Omega,\mathbb{Z}_2) \to H_k(\Omega,\mathbb{Z}_2) \to H_k(\Omega,\partial\Omega,\mathbb{Z}_2) \to .. \tag{17}$$

Thus combining (14)-(17), we get

$$H_k(\partial\Omega,\mathbb{Z}_2) = 0, 1 \leq k \leq m-2. \tag{18}$$

On the other hand, recalling that $\partial\Omega$ is a closed $(m-1)$-dimensional connected smooth manifold, we have

$$H_k(\partial\Omega,\mathbb{Z}_2) \simeq \mathbb{Z}_2, k \in \{0, m-1\}. \tag{19}$$

Therefore, combining (18) and (19) we have the $\mathbb{Z}_2$- homology of $\partial\Omega$ is the $\mathbb{Z}_2$-homology of a $(m-1)$-sphere. Now come to play the condition $m = 3$. Indeed, if $m = 3$, then by the classification of closed surfaces $\partial\Omega$ is an embedded 2-sphere in $\mathbb{R}^3$. Hence the generalized Schoenflies Theorem implies Ω is contractible leading to a contradiction to the assumption Ω is non-contractible. $\blacksquare$.

Remark 2.2. We would like to emphasize that from the proof given when Ω has non trivial fundamental group with trivial abelianization (Part **B)**), under the assumption that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth and connected boundary, we always have $H_d(\Omega, \mathbb{Z}_2) \neq 0$ for some $d \in \{1, 2\}$.

Now, let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function satisfying

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2. \end{cases} \tag{20}$$

Using χ, for $a \in \Omega$, and $\delta > 0$ and small, we define

$$\chi_\delta^a(x) = \chi\left(\frac{|x-a|}{\delta}\right), \quad x \in \Omega. \tag{21}$$

Moreover, using χ_δ^a and the Green's function $G(a, \cdot)$, we define the Brendle[5]-Schoen[14]'s bubble

$$u_{a,\lambda,\delta} = \chi_\delta^a \delta_{a,\lambda} + (1 - \chi_\delta^a)\frac{c_0}{\sqrt{\lambda}} G(a, x). \tag{22}$$

For $K \subset \Omega$ compact, we set

$$\varrho_0 = \varrho_0^K := \frac{dis(K, \partial\Omega)}{4} > 0. \tag{23}$$

Thus, for $\forall a \in K$ and $\forall 0 < 2\delta < \varrho_0$ we have

$$u_{a,\lambda} := u_{a,\lambda,\delta} \in H_0^1(\Omega), \quad \text{and} \quad u_{a,\lambda} > 0 \ \text{ in } \ \Omega. \tag{24}$$

For $a_i, a_j \in \Omega$ and $\lambda_i, \lambda_j > 0$, we define

$$\varepsilon_{ij} = \left[\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G^{-2}(a_i, a_j)}\right]^{\frac{1}{2}}. \tag{25}$$

Moreover, for $a_i, a_j \in K$, $0 < 2\delta < \varrho_0$, and $\lambda_i, \lambda_j > 0$, we define

$$\epsilon_{ij} = \int_\Omega u_{a_i,\lambda_i}^5 u_{a_j,\lambda_j} \tag{26}$$

and

$$e_{ij} = \int_\Omega (-\Delta + q) u_{a_i,\lambda_i} u_{a_j,\lambda_j}. \tag{27}$$

Using (6) and (11), we estimate the deficit of $u_{a,\lambda}$ being a solution of BVP (2).

Lemma 2.3. Let $K \subset \Omega$ be compact, $m > 0$ be a large integer, and $\theta > 0$ be small. Then there exists $C > 0$ such that $\forall a \in K$, $\forall 0 < 2\delta < \varrho_0$ and $\forall 0 < \frac{1}{\lambda} \leq \theta\delta^m$, we have

$$\left|-\Delta u_{a,\lambda} + q u_{a,\lambda} - u_{a,\lambda}^5\right| \leq C\left[\frac{1}{\delta^2\sqrt{\lambda}} 1_{\{\delta \leq |x-a| \leq 2\delta\}} + \delta_{a,\lambda} 1_{\{|x-a| \leq 2\delta\}}\right] + C\delta_{a,\lambda}^5 1_{\{|x-a| \geq \delta|\}},$$

where ϱ_0 is as in (23).

Proof. First of all, to simplify notation, let us set $\chi_\delta := \chi_\delta^a$,

$$G_a(x) := G(a, x) \ \text{ and } \ \bar{G}_a = c_0 G_a.$$

Then, we have

$$u_{a,\lambda} = \chi_\delta \delta_{a,\lambda} + (1-\chi_\delta)\frac{\bar{G}_a}{\sqrt{\lambda}} = \chi_\delta \left(\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right) + \frac{\bar{G}_a}{\sqrt{\lambda}}.$$

This implies

$$(-\Delta + q)\, u_{a,\lambda} = (-\Delta + q)\left[\chi_\delta \left(\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right)\right] + \frac{(-\Delta + q)\,\bar{G}_a}{\sqrt{\lambda}}.$$

Clearly the lemma is true for $x = a$. Now, since for $x \neq a$, we have

$$(-\Delta + q)\,\bar{G}_a = 0,$$

then for $x \neq a$ we get

$$\begin{aligned}(-\Delta + q)\, u_{a,\lambda} = & -\Delta\chi_\delta \left[\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right] - 2\nabla\chi_\delta\nabla\left[\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right] \\ & - \chi_\delta \Delta\delta_{a,\lambda} + q\chi_\delta\delta_{a,\lambda}\,.\end{aligned}$$

This implies (for $x \neq a$)

$$(-\Delta + q)\, u_{a,\lambda} - u_{a,\lambda}^5 = \sum_{i=1}^{4} J_i$$

with

$$J_1 = -\Delta\chi_\delta \left[\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right],$$

$$J_2 = -2\left\langle \nabla\chi_\delta, \nabla\left[\delta_{a,\lambda} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right]\right\rangle,$$

$$J_3 = q\chi_\delta\delta_{a,\lambda}\,,$$

and

$$J_4 = -\chi_\delta\Delta\delta_{a,\lambda} - u_{a,\lambda}^5\,.$$

Now, we are going to estimate separately each J_i's. For J_1, we first write

$$J_1 = -\Delta\chi_\delta \left[\delta_{a,\lambda} - \frac{c_0}{\sqrt{\lambda}|x-a|} + \frac{c_0}{\sqrt{\lambda}|x-a|} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right]. \tag{28}$$

Next, using (5) and (12), we derive

$$\left|\delta_{a,\lambda} - \frac{c_0}{\sqrt{\lambda}|x-a|}\right| \leq \frac{C}{\sqrt{\lambda}}, \tag{29}$$

and

$$\left|\frac{c_0}{\sqrt{\lambda}|x-a|} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right| \leq \frac{C}{\sqrt{\lambda}}. \tag{30}$$

For $\Delta\chi_\delta$, we have

$$\nabla\chi_\delta = \chi^{'}\left(\frac{|x-a|}{\delta}\right)\frac{(x-a)}{\delta|x-a|}. \tag{31}$$

This implies

$$\Delta\chi_\delta = \chi^{''}\left(\frac{|x-a|}{\delta}\right)\frac{1}{\delta^2} + 2\chi^{'}\left(\frac{|x-a|}{\delta}\right)\frac{1}{\delta|x-a|}. \tag{32}$$

Thus, recalling the definition of χ (see (20)), we have (32) implies

$$|\Delta\chi_\delta| \leq \frac{C}{\delta^2} 1_{\{\delta\leq|x-a|\leq 2\delta\}}. \tag{33}$$

Hence, combining (28), (29), (30), and (33), we get

$$|J_1| \leq \frac{C}{\delta^2\sqrt{\lambda}} 1_{\{\delta\leq|x-a|\leq 2\delta\}}. \tag{34}$$

To estimate J_2, we first write

$$J_2 = -2\left\langle \nabla\chi_\delta,\ \nabla\left[\delta_{a,\lambda} - \frac{c_0}{\sqrt{\lambda}|x-a|} + \frac{c_0}{\sqrt{\lambda}|x-a|} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right]\right\rangle. \tag{35}$$

Next, using (5) and (13), we derive

$$\left|\nabla\left[\delta_{a,\lambda} - \frac{c_0}{\sqrt{\lambda}|x-a|}\right]\right| \leq \frac{C}{\sqrt{\lambda}|x-a|}, \tag{36}$$

and

$$\left|\nabla\left[\frac{c_0}{\sqrt{\lambda}|x-a|} - \frac{\bar{G}_a}{\sqrt{\lambda}}\right]\right| \leq \frac{C}{\sqrt{\lambda}|x-a|}. \tag{37}$$

On the other hand, using (31) and recalling (20), we obtain

$$|\nabla\chi_\delta| \leq \frac{C}{\delta} 1_{\{\delta\leq|x-a|\leq 2\delta\}}. \tag{38}$$

Hence, combining (35)-(38), we get

$$|J_2| \leq \frac{C}{\delta^2\sqrt{\lambda}} 1_{\{\delta\leq|x-a|\leq 2\delta\}}. \tag{39}$$

For J_3, since q is bounded then using (20) and (21), we clearly obtain

$$|J_3| \leq C\delta_{a,\lambda}\, 1_{\{|x-a|\leq 2\delta\}}. \tag{40}$$

Finally to estimate J_4, we observe that for $|x-a|\leq\delta$,

$$\chi_\delta(x) = 1.$$

Thus

$$J_4 = -\chi_\delta\Delta\delta_{a,\lambda} - u_{a,\lambda}^5 = -\Delta\delta_{a,\lambda} - \delta_{a,\lambda}^5 = 0 \tag{41}$$

on $\{|x-a|\leq\delta\}$. On the other hand on $\{|x-a|>\delta\}$, we clearly have

$$|u_{a,\lambda}| \leq C\delta_{a,\lambda}. \tag{42}$$

Therefore, (41) and (42) imply

$$|J_4| \leq \delta_{a,\lambda}^5\, 1_{\{|x-a|\geq\delta\}}. \tag{43}$$

Hence, the result follows from (34), (39), (40), and (43). ■

3. **PS-sequences and Deformation lemma.** In this section, we recall the analysis of Palais-Smale (PS) sequence for J_q defined by (3), see [7]. We also introduce the neighborhood of potential critical points at infinity of J_q and the associated selection maps. As in other applications of the Barycenter technique of Bahri-Coron[3], we also recall the associated Deformation lemma.

We start with the analysis of Palais-Smale (PS) sequence for J_q. By some arguments which are classical by now see for example [7] (see also [2], [8], [13], and [15]), we have the following the profile decomposition for (PS)-sequences of J_q.

Lemma 3.1. Suppose that $(u_k) \subset H_0^{1,+}(\Omega)$ is a PS-sequence for J_q, that is $\nabla J_q(u_k) \to 0$ and $J_q(u_k) \to c$ up to a subsequence, and $\int_\Omega u_k^6 = c^{\frac{3}{2}}$, then up to a subsequence, we have have there exists $u_\infty \geq 0$, an integer $p \geq 0$, a sequence of points $a_{i,k} \in \Omega$, $i = 1, \cdots, p$, and a sequence of positive numbers $\lambda_{i,k}$, $i = 1, \cdots p$, such that

1)

$$-\Delta u_\infty + q u_\infty = u_\infty^5.$$

2)

$$||u_k - u_\infty - \sum_{i=1}^{p} u_{a_{i,k},\lambda_{i,k}}||_q \longrightarrow 0.$$

3)

$$J_q(u_k)^{\frac{3}{2}} \longrightarrow J_q(u_\infty)^{\frac{3}{2}} + pS^{\frac{3}{2}}.$$

4)
For $i \neq j = 1, \cdots, p$,

$$\frac{\lambda_{i,k}}{\lambda_{j,k}} + \frac{\lambda_{j,k}}{\lambda_{i,k}} + \lambda_{i,k}\lambda_{j,k}G^{-2}(a_{i,k}, a_{jk}) \longrightarrow +\infty$$

5)
For $i = 1, \cdots, p$,

$$\lambda_{i,k} dist(a_{i,k}, \partial\Omega) \longrightarrow +\infty,$$

where $||\cdot||_q$ is the norm associated to the scalar product $\langle \cdot, \cdot \rangle_q$ defined by (4).

We discuss now the neighborhoods of potential critical points at infinity of J_q. To introduce the latter, we first fix

$$\varepsilon_0 > 0 \quad \text{and} \quad \varepsilon_0 \simeq 0. \tag{44}$$

Furthermore, we choose

$$\nu_0 > 1 \quad \text{and} \quad \nu_0 \simeq 1. \tag{45}$$

Then for $p \in \mathbb{N}^*$, and $0 < \varepsilon \leq \varepsilon_0$, we define $V(p, \varepsilon)$ the (p, ε)-neighborhood of potential critical points at infinity of J_q by

$$V(p,\varepsilon) = \{u \in H_0^{1,+}(\Omega) : \ \exists a_1, \cdots, a_p \in \Omega, \ \ \alpha_1, \cdots, \alpha_p > 0, \ \ \lambda_1, \cdots, \lambda_p > 0,$$
$$\lambda_i \geq \frac{1}{\varepsilon} \text{ for } i = 1 \cdots, p, \quad \lambda_i dist(a_i, \partial\Omega) \geq \frac{1}{\varepsilon} \text{ for } i = 1 \cdots, p,$$
$$\|u - \sum_{i=1}^{p} \alpha_i u_{a_i,\lambda_i}\|_q \leq \varepsilon, \ \ \frac{\alpha_i}{\alpha_j} \leq \nu_0 \text{ and } \varepsilon_{i,j} \leq \varepsilon \text{ for } i \neq j = 1, \cdots, p\}.$$

For the sets $V(p,\varepsilon)$ (see [3] and [7]), for every $p \in \mathbb{N}^*$ there exists $0 < \varepsilon_p \leq \varepsilon_0$ such that for every $0 < \varepsilon \leq \varepsilon_p$, we have

$$\begin{cases} \forall u \in V(p,\varepsilon) \text{ the minimization problem } \min_{B^p_\varepsilon} \|u - \sum_{i=1}^p \alpha_i u_{a_i,\lambda_i}\|_q \\ \text{has a solution } (\bar{\alpha}, A, \bar{\lambda}) \in B^p_\varepsilon, \text{which is unique up to permutations,} \end{cases} \tag{46}$$

where B^p_ε is defined as

$$B^p_\varepsilon = \{(\bar{\alpha} = (\alpha_1, \cdots, \alpha_p), A = (a_1, \cdots, a_p), \bar{\lambda} = (\lambda_1, \cdots, \lambda_p)) \in \mathbb{R}^p_+ \times \Omega^p \times \mathbb{R}^p_+$$

$$\lambda_i \geq \frac{1}{\varepsilon},\ \lambda_i dist(a_i, \partial\Omega) \geq \frac{1}{\varepsilon}, i = 1, \cdots, p,\ \ \frac{\alpha_i}{\alpha_j} \leq \nu_0 \text{ and } \varepsilon_{i,j} \leq \varepsilon, i \neq j = 1, \cdots, p\},$$

with $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^p_+$ the cartesian product of p copies of $\mathbb{R}_+$.
Denoting by σ_p the permutation group of order $p \in \mathbb{N}^*$, we define the selection map s_p via

$$s_p : V(p,\varepsilon) \longrightarrow (\Omega)^p/\sigma_p : u \longrightarrow s_p(u) = [A] \text{ and } A \text{ is given by (46)},$$

where $[A]$ is the class of A under the action of σ_p.

To finish this section we state the Deformation Lemma needed for the application of the algebraic topological argument of Bahri-Coron[3]. To do that, we first set

$$W_p := \{u \in :\ J_q(u) \leq (p+1)^{\frac{2}{3}} \mathcal{S}\}, \tag{47}$$

for $p \in \mathbb{N}$.
With the latter notation, as in [3], [9], and [12], we have Lemma 3.1 implies the following Deformation Lemma (see [4] and [3]).

Lemma 3.2. Assuming that J_q has no critical points, then for every $p \in \mathbb{N}^*$, up to taking ε_p given by (46) smaller, we have that for every $0 < \varepsilon \leq \varepsilon_p$, the topological pair $(W_p,\ W_{p-1})$ retracts by deformation onto $(W_{p-1} \cup A_p,\ W_{p-1})$ with $V(p,\ \tilde{\varepsilon}) \subset A_p \subset V(p,\ \varepsilon)$ where $0 < \tilde{\varepsilon} < \frac{\varepsilon}{4}$ is a very small positive real number and depends on ε.

4. **Self-action estimates.** In this section, we derive some sharp estimates needed for application of the Barycenter technique of Bahri-Coron[3]. We start with the numerator of J_q. Indeed, we have

Lemma 4.1. Assuming that $K \subset \Omega$ is compact, $m > 0$ is a large integer, and $\theta > 0$ is small, then there exists $C > 0$ such that $\forall a \in K$, $\forall 0 < 2\delta < \varrho_0$, and $\forall 0 < \frac{1}{\lambda} \leq \theta\delta^m$, we have

$$\int_\Omega (-\Delta + q) u_{a,\lambda}\, u_{a,\lambda} \leq \int_\Omega u^6_{a,\lambda} + \frac{C}{\lambda}\left(1 + \delta + \frac{1}{\lambda^2\delta^3}\right).$$

PROOF. Setting

$$I = \int_\Omega (-\Delta + q) u_{a,\lambda}\, u_{a,\lambda}\,,$$

we get

$$I = \int_\Omega u^6_{a,\lambda} + \underbrace{\int_\Omega \left[(-\Delta + q) u_{a,\lambda} - u^5_{a,\lambda}\right] u_{a,\lambda}}_{I_1}.$$

To continue, let us estimate I_1. Using Lemma 2.3, we get

$$\begin{aligned}|I_1| &\le \int_\Omega \left|(-\Delta+q)u_{a,\lambda} - u_{a,\lambda}^5\right| u_{a,\lambda} \\ &\le \frac{C}{\delta^2\sqrt{\lambda}} \int_\Omega u_{a,\lambda} 1_{\{\delta\le|x-a|\le 2\delta\}} \\ &+ C\int_\Omega \delta_{a,\lambda}\, u_{a,\lambda}\, 1_{\{|x-a|\le 2\delta\}} \\ &+ C\int_\Omega \delta_{a,\lambda}^5\, u_{a,\lambda}\, 1_{\{|x-a|\ge\delta\}}.\end{aligned}$$

To obtain our goal, we are going to estimate the three parts of the right hand side the latter formula. For the first term, we have

$$\begin{aligned}\int_\Omega u_{a,\lambda}\, 1_{\{\delta\le|x-a|\le 2\delta\}} &\le C\int_{\delta\le|x-a|\le 2\delta} \left[\frac{\lambda}{1+\lambda^2|x-a|^2}\right]^{\frac{1}{2}} \\ &\le C\int_{\delta\le|x-a|\le 2\delta} \frac{1}{\sqrt{\lambda}|x-a|} \\ &\le \frac{C}{\sqrt{\lambda}}\int_\delta^{2\delta} r\,dr \\ &\le C\frac{\delta^2}{\sqrt{\lambda}}.\end{aligned}$$

For the second term, we obtain

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}\, u_{a,\lambda}\, 1_{\{|x-a|\le 2\delta\}} &\le C\int_{|x-a|\le 2\delta} \left[\frac{\lambda}{1+\lambda^2|x-a|^2}\right] \\ &\le \frac{1}{\lambda}\int_0^{2\delta} dr \\ &\le C\frac{\delta}{\lambda}.\end{aligned}$$

Finally for the last term, we get

$$\begin{aligned}\int_\Omega \delta_{a,\lambda}^5\, u_{a,\lambda}\, 1_{\{|x-a|\ge 2\delta\}} &\le C\int_{|x-a|\ge 2\delta} \delta_{a,\lambda}^6 \\ &\le C\int_{|x-a|\ge 2\delta} \left[\frac{\lambda}{1+\lambda^2|x-a|^2}\right]^3 \\ &\le \frac{C}{\lambda^3}\int_{\{|x-a|\ge 2\delta\}} \frac{1}{|x-a|^6} \\ &\le \frac{C}{\lambda^3}\int_{2\delta}^{+\infty} r^{-4}\,dr \\ &\le \frac{C}{\lambda^3\delta^3}.\end{aligned}$$

Thus, collecting all we have

$$|I_1| \le C\left[\frac{1}{\lambda} + \frac{\delta}{\lambda} + \frac{1}{\lambda^3\delta^3}\right].$$

Hence, we obtain

$$\int_\Omega (-\Delta + q)u_{a,\lambda}\, u_{a,\lambda} \le \int_\Omega u^6_{a,\lambda} + \frac{C}{\lambda}\left(1 + \delta + \frac{1}{\lambda^2\delta^3}\right),$$

thereby ending the proof. ■

We turn now to the denominator of J_q and obtain the following estimate.

Lemma 4.2. Assuming that $K \subset \Omega$ is compact, $m > 0$ is a large integer, and $\theta > 0$ is small, then there exists $C > 0$ such that $\forall a \in K$, $\forall 0 < 2\delta < \varrho_0$, and $\forall 0 < \frac{1}{\lambda} \le \theta\delta^m$, we have

$$\int_\Omega u^6_{a,\lambda} = \int_{\mathbb{R}^3} \delta^6_{a,\lambda} + O\left(\frac{1}{\lambda^3\delta^3}\right).$$

Proof. We have

$$\int_\Omega u^6_{a,\lambda} = \int_{|x-a|\le\delta} u^6_{a,\lambda} + \int_{\delta<|x-a|\le 2\delta} u^6_{a,\lambda} + \int_{|x-a|>2\delta} u^6_{a,\lambda}.$$

Now, we estimate each term of the right hand side of the latter formula. For the first term, we obtain

$$\begin{aligned}\int_{|x-a|\le\delta} u^6_{a,\lambda} &= \int_{|x-a|\le\delta} \delta^6_{a,\lambda}\\ &= \int_{\mathbb{R}^3} \delta^6_{a,\lambda} - \int_{|x-a|>\delta} \delta^6_{a,\lambda}\\ &= \int_{\mathbb{R}^3} \delta^6_{a,\lambda} + O\left(\frac{1}{\lambda^3\delta^3}\right).\end{aligned}$$

For the second term, we derive

$$\begin{aligned}\int_{\delta<|x-a|\le 2\delta} u^6_{a,\lambda} &\le C\int_{\delta\le|x-a|\le 2\delta} \left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^3\\ &\le \frac{C}{\lambda^3}\int_\delta^{2\delta} r^{-4}\, dr\\ &\le \frac{C}{\lambda^3\delta^3}.\end{aligned}$$

For the last term, using (12) we get

$$\begin{aligned}\int_{|x-a|\ge 2\delta} u^6_{a,\lambda} &= \int_{|x-a|\ge 2\delta} \left(\frac{1}{\sqrt{\lambda}G_a}\right)^6\\ &= \frac{C}{\lambda^3}\int_{|x-a|\ge 2\delta} G^6_a\\ &\le \frac{C}{\lambda^3}\int_{|x-a|\ge 2\delta} \frac{1}{|x-a|^6}\\ &\le \frac{C}{\lambda^3\delta^3}.\end{aligned}$$

Therefore, we have

$$\int_\Omega u^6_{a,\lambda} = \int_{\mathbb{R}^3} \delta^6_{a,\lambda} + O\left(\frac{1}{\lambda^3\delta^3}\right).$$

■

Finally, we derive the J_q-energy estimate of $u_{a,\lambda}$ needed for the application of the Barycenter technique of Bahri-Coron[3].

Corollary 4.3. Assuming that $K \subset \Omega$ is compact, $m > 0$ is a large integer, and $\theta > 0$ is small, then there exists $C > 0$ such that $\forall a \in K$, $\forall 0 < 2\delta < \varrho_0$, and $\forall 0 < \frac{1}{\lambda} \leq \theta\delta^m$, we have

$$J_q(u_{a,\lambda}) \leq \mathcal{S}\left(1 + C\left[\frac{1}{\lambda} + \frac{\delta}{\lambda} + \frac{1}{\delta^3\lambda^3}\right]\right).$$

Proof. It follows from the properties of $\delta_{a,\lambda}$ (see (7)-(9)), Lemma 4.1 and Lemma 4.2. ∎

5. **Interaction estimates.** In this section, we derive sharp inter-action estimates needed for the algebraic topological argument for existence. Recalling (25), (26) and (27), we start with the following one relating ε_{ij}, e_{ij} and ϵ_{ji}. For this, we start with the following auxiliary estimate.

Lemma 5.1. Assuming that $K \subset \Omega$ is compact, $m > 0$ is a large integer, and $\theta > 0$ is small, then there exists $C > 0$ such that $\forall a_i, a_j \in K$, $\forall 0 < 2\delta < \varrho_0$, and $\forall 0 < \frac{1}{\lambda_i}, \frac{1}{\lambda_j} \leq \theta\delta^m$, we have

$$\int_\Omega u_{a_i,\lambda_i}\left|(-\Delta + q)u_{a_j,\lambda_j} - u^5_{a_j,\lambda_j}\right| \leq C\left(\delta + \frac{1}{\lambda_j^2\delta^2}\right)\left(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{-1}{2}}.$$

Proof. Using Lemma 2.3, we have

$$\underbrace{\left|(-\Delta + q)u_{a_j,\lambda_j} - u^5_{a_j,\lambda_j}\right|}_{L_j} \leq C\left[\frac{1}{\delta^2\sqrt{\lambda_j}}1_{\delta \leq |x-a_j| \leq 2\delta} + \delta_{a_j,\lambda_j}1_{|x-a_j| \leq 2\delta}\right] + C\delta^5_{a_j,\lambda_j}1_{|x-a_j| \geq \delta}. \tag{48}$$

On the set $\{|x - a_j| \leq 2\delta\}$, we have

$$\delta^2_{a_j,\lambda_j}(x) \geq c_0^2\left(\frac{\lambda_j}{1 + 4\lambda_j^2\delta^2}\right) \geq \frac{c_0^2}{\lambda_j\delta^2}\left[1 + O(\frac{1}{\lambda_j^2\delta^2})\right] \geq \frac{1}{2}\frac{c_0^2}{\lambda_j\delta^2}.$$

This implies

$$\frac{1}{\sqrt{\lambda_j}\delta} \leq \frac{\sqrt{2}}{c_0^2}\delta_{a_j,\lambda_j}.$$

Thus, we get

$$L_j \leq C\left(1 + \frac{1}{\delta}\right)\delta_{a_j,\lambda_j}1_{|x-a| \leq 4\delta} + C\delta^5_{a_j,\lambda_j}1_{|x-a| \geq \frac{\delta}{2}},$$

where L_j is as in (48). Hence, we obtain

$$\int_\Omega u_{a_i,\lambda_i} L_j \leq C\left(1+\frac{1}{\delta}\right)\underbrace{\int_{|x-a_j|\leq 4\delta}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_1}$$
$$+ C\underbrace{\int_{|x-a_j|\geq\frac{\delta}{2}}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_2}. \tag{49}$$

Now, we estimate I_1 as follows.

$$\begin{aligned} I_1 &= \int_{|x-a_j|\leq 4\delta}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}} \\ &= \underbrace{\int_{(2|x-a_i|\leq\frac{1}{\lambda_j}+|a_i-a_j|)\cap(|x-a_j|\leq 4\delta)}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_1^1} \\ &+ \underbrace{\int_{(2|x-a_i|>\frac{1}{\lambda_j}+|a_i-a_j|)\cap(|x-a_j|\leq 4\delta)}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_1^2}. \end{aligned}$$

To continue, we first estimate I_1^1. Indeed, using triangle inequality we have

$$\begin{aligned} I_1^1 &\leq C\int_{|x-a_i|\leq 8\delta}\left(\frac{\lambda_j}{1+\lambda_j^2|a_i-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}} \\ &\leq C\frac{\sqrt{\frac{\lambda_j}{\lambda_i}}}{\left(1+\lambda_j^2|a_i-a_j|^2\right)^{\frac{1}{2}}}\int_{|x-a_i|\leq 8\delta}\frac{1}{|x-a_i|}. \end{aligned}$$

This implies

$$I_1^1 = O\left(\delta^2\left(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{\frac{-1}{2}}\right). \tag{50}$$

For I_1^2, we derive

$$\begin{aligned} I_1^2 &\leq C\int_{|x-a_j|\leq 4\delta}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_i}{1+\lambda_j^2|a_i-a_i|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_j}{\lambda_i}\right) \\ &\leq C\frac{\sqrt{\frac{\lambda_i}{\lambda_j}}\left(\frac{\lambda_j}{\lambda_i}\right)}{\left(1+\lambda_j^2|a_i-a_j|^2\right)^{\frac{1}{2}}}\int_{|x-a_j|\leq 4\delta}\frac{1}{|x-a_j|}. \end{aligned}$$

Thus for I_1^2, we obtain

$$I_1^2 = O\left(\delta^2 \left(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{-1}{2}}\right). \tag{51}$$

Hence, combining (50) and (51), we get

$$I_1 = O\left(\delta^2 \left(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{-1}{2}}\right). \tag{52}$$

Next, let us estimate I_2. For this, we first write

$$\begin{aligned}
I_2 &= \int_{|x-a_j|\geq\frac{\delta}{2}} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}} \\
&= \underbrace{\int_{\{2|x-a_i|\leq\frac{1}{\lambda_j}+|a_i-a_j|\}\cap(|x-a_j|\geq\frac{\delta}{2})} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_2^1} \\
&+ \underbrace{\int_{(2|x-a_i|>\frac{1}{\lambda_j}+|a_i-a_j|)\cap(|x-a_j|\geq\frac{\delta}{2})} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}}}_{I_2^2}.
\end{aligned}$$

Setting $\mathcal{D} = \{2|x-a_i| \leq \frac{1}{\lambda_j} + |a_i - a_j|\} \cap \{|x-a_j| \geq \frac{\delta}{2}\}$, we estimate I_2^1 as follows

$$\begin{aligned}
I_2^1 &\leq \frac{C}{\lambda_j^{\frac{-5}{2}}} \int_{\mathcal{D}} \left(\frac{1}{1+\lambda_j^2|a_i-a_j|^2}\right)^{\frac{3}{2}} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}} \left(\frac{1}{1+\lambda_j^2|a_i-a_j|^2}\right) \\
&\leq C\left(\frac{1}{\sqrt{\lambda_i}}\right)\left(\frac{1}{\lambda_j^2\delta^2}\right) \frac{\lambda_j^{\frac{5}{2}}}{\left(1+\lambda_j^2|a_i-a_j|^2\right)^{\frac{3}{2}}} \int_{2|x-a_i|\leq\frac{1}{\lambda_j}+|a_i-a_j|} \frac{1}{|x-a_i|} \\
&\leq C\left(\sqrt{\frac{\lambda_j}{\lambda_i}}\right)\left(\frac{1}{\delta^2}\right) \frac{1}{\left(1+\lambda_j^2|a_i-a_j|^2\right)^{\frac{3}{2}}} \left(1+\lambda_j^2|a_i-a_j|^2\right)^2 \left(\frac{1}{\lambda_j^2}\right) \\
&\leq C\left(\sqrt{\frac{\lambda_j}{\lambda_i}}\right)\left(\frac{1}{\lambda_j^2\delta^2}\right) \frac{1}{\left(1+\lambda_j^2|a_i-a_j|^2\right)^{\frac{1}{2}}}.
\end{aligned}$$

This implies

$$I_1^2 = O\left(\frac{1}{\lambda_j^2\delta^2} \left(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{-1}{2}}\right). \tag{53}$$

Next, we estimate I_2^2 as follows

$$\begin{aligned} I_2^2 &\leq C\frac{\lambda_j}{\sqrt{\lambda_i}}\int_{|x-a_j|\geq\frac{\delta}{2}}\left(\frac{1}{1+\lambda_j^2|a_i-a_j|^2}\right)^{\frac{1}{2}}\left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \\ &\leq C\frac{\lambda_j}{\sqrt{\lambda_i}}\left(\frac{1}{1+\lambda_j^2|a_i-a_j|^2}\right)^{\frac{1}{2}}\frac{1}{\lambda_j^{\frac{5}{2}}}\int_{|x-a_j|\geq\frac{\delta}{2}}\frac{1}{|x-a_j|^5} \\ &\leq C\left(\sqrt{\frac{\lambda_j}{\lambda_i}}\right)\left(\frac{1}{\lambda_j^2\delta^2}\right)\frac{1}{(1+\lambda_j^2|a_i-a_j|^2)^{\frac{1}{2}}}. \end{aligned}$$

This gives

$$I_2^2 = O\left(\frac{1}{\lambda_j^2\delta^2}\left(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{\frac{-1}{2}}\right). \tag{54}$$

Therefore, using (53) and (54), we obtain

$$I_2 = O\left(\frac{1}{\lambda_j^2\delta^2}\left(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{\frac{-1}{2}}\right). \tag{55}$$

Hence, combining (49), (73) and (55), we get

$$\int_\Omega u_{a_i,\lambda_i}L_j \leq C\left(\delta+\frac{1}{\lambda_j^2\delta^2}\right)\left(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{\frac{-1}{2}},$$

thereby ending the proof of the lemma. ■

Clearly Lemma 5.1 implies the following sharp interaction-estimate relating e_{ij}, ϵ_{ij}, and ε_{ij} (for their definitions, see (25)-(27)).

Corollary 5.2. Assuming that $K\subset\Omega$ is compact, $m>0$ is a large integer, $\theta>0$ is small, and $\mu_0>0$ is small, then $\forall a_i,a_j\in K$, $\forall 0<2\delta<\varrho_0$, and $\forall 0<\frac{1}{\lambda_j}\leq\frac{1}{\lambda_i}\leq\theta\delta^m$ such that $\varepsilon_{ij}\leq\mu_0$, we have

$$e_{ij}=\epsilon_{ij}+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\varepsilon_{ij}.$$

Next, we present a lemma that provides a sharp inter-action estimate relating ϵ_{ij} and ε_{ij}. Indeed, we have.

Lemma 5.3. Assuming that $K\subset\Omega$ is compact, $m>0$ is a large integer $\theta>0$ is small, and μ_0 is small, then $\forall a_i,a_j\in K$, $\forall 0<2\delta<\varrho_0$, and $\forall 0<\frac{1}{\lambda_i}\leq\frac{1}{\lambda_j}\leq\theta\delta^m$ such that $\varepsilon_{ij}\leq\mu_0$, we have

$$\begin{aligned} \epsilon_{ij} =& c_0^6c_3\varepsilon_{ij}\left[\left(1+O\left(\delta+\frac{1}{\lambda_j^2\delta^2}\right)\right)\left(1+o_{\varepsilon_{ij}}(1)+O(\varepsilon_{ij}^2(\delta^{-2}+\log\varepsilon_{ij}^{-1}))\right)\right] \\ &+c_0^6c_3\varepsilon_{ij}\left[O\left(\varepsilon_{ij}^2\frac{1}{\delta^6}\right)\right]. \end{aligned}$$

Proof. By definition, we have

$$u_{a_i,\lambda_i}=\chi_\delta\delta_{a_i,\lambda_i}+(1-\chi_\delta)\frac{c_0}{\sqrt{\lambda}}G_{a_i},$$

with $\chi_\delta := \chi_\delta^{a_i}$. On the other hand, by definition of the standard bubble $\delta_{a,\lambda}$, we have

$$\chi_\delta \delta_{a_i,\lambda_i} = c_0 \chi_\delta \left[\frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2} \frac{|x-a_i|^2}{G_{a_i}^{-2}}} \right]^{\frac{1}{2}}.$$

Next, for $|x - a_i| \leq 2\delta$, we have

$$\begin{aligned} 1 + \lambda_i^2 G_{a_i}^{-2} \frac{|x-a_i|^2}{G_{a_i}^{-2}} &= 1 + \lambda_i^2 G_{a_i}^{-2} (1 + O(\delta)) \\ &= 1 + \lambda_i^2 G_{a_i}^{-2} + O\left(\lambda_i^2 G_{a_i}^{-2} \delta\right) \\ &= \left(1 + \lambda_i^2 G_{a_i}^{-2}\right) \left[1 + O\left(\frac{\lambda_i^2 G_{a_i}^{-2} \delta}{1 + \lambda_i^2 G_{a_i}^{-2}}\right)\right] \\ &= \left(1 + \lambda_i^2 G_{a_i}^{-2}\right) [1 + O(\delta)]. \end{aligned}$$

So, for $\chi_\delta \delta_{a_i,\lambda_i}$ we have

$$\chi_\delta \delta_{a_i,\lambda_i} = c_0 \chi_\delta \left[\frac{\lambda_i}{\left(1 + \lambda_i^2 G_{a_i}^{-2}\right)[1 + O(\delta)]} \right]^{\frac{1}{2}} = c_0 \chi_\delta \, [1 + O(\delta)] \left[\frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}}. \tag{56}$$

We have also

$$c_0 (1 - \chi_\delta) \left[\frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}} = (1 - \chi_\delta) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} \left[\frac{1}{1 + \lambda_i^{-2} G_{a_i}^2} \right]^{\frac{1}{2}}.$$

Since on $\{|x - a_i| \geq \delta\}$, we have

$$\frac{1}{1 + \lambda_i^{-2} G_{a_i}^2} = 1 + O\left(\frac{G_{a_i}^2}{\lambda_i^2}\right) = 1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right),$$

then we get

$$c_0 (1 - \chi_\delta) \left[\frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}} = (1 - \chi_\delta) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} \left(1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right)\right).$$

This implies

$$(1 - \chi_\delta) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} = c_0 (1 - \chi_\delta) \left[\frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}} \left(1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right)\right). \tag{57}$$

Thus, combining (56) and (57), we get

$$u_{a_i,\lambda_i} = c_0 \left[(1 + O(\delta)) \chi_\delta + (1 - \chi_\delta) \left(1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right)\right) \right] \left[\frac{\lambda}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}}.$$

Hence, we obtain

$$u_{a_i,\lambda_i} = c_0 \left[1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \left[\frac{\lambda}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^{\frac{1}{2}}. \tag{58}$$

Now, we are going to use (58) to achieve our goal. First of all, we write

$$\int_\Omega u_{a_j,\lambda_j}^5 u_{a_i,\lambda_i} = \int_{B(a_j,\delta)} u_{a_j,\lambda_j}^5 u_{a_i,\lambda_i} + \int_{\Omega - B(a_j,\delta)} u_{a_j,\lambda_j}^5 u_{a_i,\lambda_i}.$$

For the second term in the right hand side of the latter formula, we have

$$\begin{aligned}
\int_{\Omega-B(a_j,\delta)} u^5_{a_j,\lambda_j} u_{a_i,\lambda_i} &\leq C\int_{\Omega-B(a_j,\delta)} \left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^5 u_{a_i,\lambda_i} \\
&\leq C\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^5 \int_{\Omega-B(a_j,\delta)} u_{a_i,\lambda_i} \\
&\leq C\left(\frac{1}{\lambda_j\delta^2}\right)^{\frac{5}{2}} \left[\int_{\Omega-(B(a_j,\delta)\cup B(a_i,\delta))} u_{a_i,\lambda_i}\right] \\
&+ C\left(\frac{1}{\lambda_j\delta^2}\right)^{\frac{5}{2}} \left[\int_{(\Omega-B(a_j,\delta))\cap B(a_i,\delta)} u_{a_i,\lambda_i}\right] \\
&\leq C\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^6 \frac{1}{\sqrt{\lambda_i}} \\
&+ C\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^5 \int_{B(a_i,\delta)} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{1}{2}} \\
&\leq C\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^6 \frac{1}{\sqrt{\lambda_i}} + C\delta^2\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^5 \frac{1}{\sqrt{\lambda_i}} \\
&\leq C\left(\frac{1}{\lambda_j}\right)^{\frac{5}{2}} \left(\frac{1}{\delta}\right)^6 \frac{1}{\sqrt{\lambda_i}} \left(1+\delta^3\right).
\end{aligned}$$

Thus, we get

$$\int_\Omega u^5_{a_j,\lambda_j} u_{a_i,\lambda_i} = \int_{B(a_j,\delta)} u^5_{a_j,\lambda_j} u_{a_i,\lambda_i} + O\left(\frac{1}{\lambda_j^{\frac{5}{2}}\sqrt{\lambda_i}\delta^6}\right). \tag{59}$$

For the first term in the right hand side of (59), using (58) we have

$$
\begin{aligned}
\int_{B(a_j,\delta)} u^5_{a_j,\lambda_j} u_{a_i,\lambda_i} &= c_0^6 \int_{B(a_j,\delta)} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} [1+O(\delta)] \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}}\right]^{\frac{1}{2}} \\
&+ c_0^6 \int_{B(a_j,\delta)} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} O\left(\frac{1}{\lambda_i^2\delta^2}\right) \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}}\right]^{\frac{1}{2}} \\
&= c_0^6 [1+O(\delta)] \int_{B(a_j,\delta)} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}}\right]^{\frac{1}{2}} \\
&+ c_0^6 O\left(\frac{1}{\lambda_i^2\delta^2}\right) \int_{B(a_j,\delta)} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}}\right]^{\frac{1}{2}} \\
&= c_0^6 \frac{1}{\sqrt{\lambda_j}} \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}} \\
&+ c_0^6 \frac{O(\delta)}{\sqrt{\lambda_j}} \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}} \\
&+ c_0^6 \frac{O\left(\frac{1}{\lambda_i^2\delta^2}\right)}{\sqrt{\lambda_j}} \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_i}{1+\lambda_i^2 G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}} \\
&= c_0^6 \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}} \\
&+ O(\delta) \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}} \\
&+ O\left(\frac{1}{\lambda_i^2\delta^2}\right) \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Recalling that $\lambda_i \le \lambda_j$, then for $\varepsilon_{ij} \sim 0$, we have
1) Either $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$,

2) or $\varepsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$.
To continue, let

$$
\mathcal{A} = \left(\left\{\left|\frac{y}{\lambda_j}\right| \le \epsilon\, G_{a_i}^{-1}(a_j)\right\} \cap B(0,\delta\lambda_j)\right) \cup \left(\left\{\left|\frac{y}{\lambda_j}\right| \le \frac{\epsilon}{\lambda_i}\right\} \cap B(0,\delta\lambda_j)\right),
$$

with $\epsilon > 0$ very small. Then by Taylor expansion on $\mathcal{A}$, we have

$$\begin{aligned}\left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j} + a_j\right)\right]^{\frac{-1}{2}} &= \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}} \\ &+ \left(\frac{-1}{2}\nabla G_{a_i}^{-2}(a_j)\lambda_i y\right)\left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-3}{2}} \\ &+ O\left[\left(\frac{\lambda_i}{\lambda_j}\right)|y|^2\right]\left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-3}{2}}.\end{aligned}$$

Thus, we have

$$\int_{B(a_j,\delta)} u_{a_j,\lambda_j}^5 u_{a_i,\lambda_i} = c_0^6\left[1 + O(\delta) + O\left(\frac{1}{\lambda_i^2\delta^2}\right)\right]\left(\sum_{i=1}^{4} I_i\right), \tag{60}$$

with

$$I_1 = \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}}\int_{\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}},$$

$$I_2 = -\frac{1}{2}\left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-3}{2}}\int_{\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}}\left[\nabla G_{a_i}^{-2}(a_j)\lambda_i y\right],$$

$$I_3 = \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-3}{2}}\int_{\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} O\left[\left(\frac{\lambda_i}{\lambda_j}\right)|y|^2\right],$$

and

$$I_4 = \int_{B(0,\lambda_j\delta)-\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}}\left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j} + a_j\right)\right]^{\frac{-1}{2}}.$$

Now, let us estimate I_1. We have

$$I_1 = \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}}\left[c_3 + \int_{\mathbb{R}^3-\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}}\right],$$

where c_3 is as in (10). On the other hand, we estimate

$$\begin{aligned}\int_{\mathbb{R}^3-\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} &\leq \int_{\mathbb{R}^3-B(0,\delta\lambda_j)}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \\ &+ \int_{\mathbb{R}^3-B(0,\lambda_j\epsilon G_{a_i}^{-1}(a_i,a_j))}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}}\end{aligned}$$

if $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$, and

$$\begin{aligned}\int_{\mathbb{R}^3-\mathcal{A}}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} &\leq \int_{\mathbb{R}^3-B(0,\delta\lambda_j)}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \\ &+ \int_{\mathbb{R}^3-B(0,\epsilon\frac{\lambda_j}{\lambda_i})}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}}\end{aligned}$$

if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$. We also have

$$\int_{\mathbb{R}^3-B(0,\delta\lambda_j)}\left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} = O\left(\frac{1}{\lambda_j^2\delta^2}\right).$$

Moreover, if $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, then

$$\int_{\mathbb{R}^3 - B(0,\lambda_j \epsilon G_{a_i}^{-1}(a_j))} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} = O\left(\frac{1}{\lambda_j^2 \epsilon^2 G_{a_i}^{-2}(a_j)}\right) = O\left(\frac{1}{\lambda_j \lambda_i G_{a_i}^{-2}(a_j)}\right) = O\left(\varepsilon_{ij}^2\right).$$

Furthermore if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$, then

$$\int_{\mathbb{R}^3 - B(0,\epsilon \frac{\lambda_j}{\lambda_i})} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} = O\left(\varepsilon_{ij}^2\right).$$

This implies

$$\int_{\mathbb{R}^3 - \mathcal{A}} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} = O\left(\varepsilon_{ij}^2 + \frac{1}{\lambda_j^2 \delta^2}\right) = O\left(\varepsilon_{ij}^2 + \varepsilon_{ij}^2 \frac{1}{\delta^2}\right) = O\left(\varepsilon_{ij}^2 \frac{1}{\delta^2}\right).$$

Thus, we get

$$\begin{aligned} I_1 &= \left[\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}} \left[c_3 + O\left(\varepsilon_{ij}^2 \frac{1}{\delta^2}\right)\right] \\ &= \varepsilon_{ij}\left(1 + o_{\varepsilon_{ij}}(1)\right)\left[c_3 + O\left(\varepsilon_{ij}^2 \frac{1}{\delta^2}\right)\right]. \end{aligned}$$

Hence, we obtain

$$I_1 = c_3 \varepsilon_{ij}\left[1 + o_{\epsilon_{ij}}(1) + O\left(\varepsilon_{ij}^2 \frac{1}{\delta^2}\right)\right]. \tag{61}$$

By symmetry, we have

$$I_2 = 0. \tag{62}$$

Next, to estimate I_3 we first observe that

$$\begin{aligned} \int_{\mathcal{A}} \frac{|y|^2}{(1+|y|^2)^{\frac{5}{2}}} &\le \int_{B(0,\epsilon\lambda_j G_{a_i}^{-1}(a_j)) - B(0,1)} \frac{|y|^2}{(1+|y|^2)^{\frac{5}{2}}} \\ &\quad + \int_{B(0,\epsilon \frac{\lambda_j}{\lambda_i}) - B(0,1)} \frac{|y|^2}{(1+|y|^2)^{\frac{5}{2}}} + O(1) \\ &= O\left(\log(\epsilon \lambda_j G_{a_i}^{-1}(a_j)) + \log(\epsilon \frac{\lambda_j}{\lambda_i})\right) + O(1). \end{aligned}$$

Thus, we have

$$\begin{aligned} I_3 &= \varepsilon_{ij}^3 \left(\frac{\lambda_i}{\lambda_j}\right)\left(1 + o_{\varepsilon_{ij}}(1)\right)\left[O\left(\log(\epsilon \lambda_j G_{a_i}^{-1}(a_j)) + \log(\epsilon \frac{\lambda_j}{\lambda_i})\right) + O(1)\right] \\ &= \varepsilon_{ij}^3 \left(1 + o_{\varepsilon_{ij}}(1)\right)\left[O\left(\log(\lambda_i \lambda_j G_{a_i}^{-2}(a_j)) + \log(\frac{\lambda_j}{\lambda_i})\right) + O(1)\right]. \end{aligned}$$

Hence, we obtain

$$I_3 = O\left(\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1})\right). \tag{63}$$

Finally, we estimate I_4 as follows.
If $\varepsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$, then

$$I_4 \le C\varepsilon_{ij} \int_{B(0,\lambda_j \delta) - \mathcal{A}} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \le C\varepsilon_{ij}\left(\frac{\lambda_j}{\lambda_i}\right)^{-2} \le C\varepsilon_{ij}^5. \tag{64}$$

If $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$, then we argue as follows. In case $|a_i - a_j| \geq 2\delta$, since

$$G_{a_i}\left(\frac{y}{\lambda_j} + a_j\right) \leq C\delta^{-1}$$

for $y \in B(0, \lambda_j\delta)$, then we have

$$\begin{aligned}
I_4 &\leq C\int_{B(0,\lambda_j\delta)-\mathcal{A}} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \frac{1}{\sqrt{\lambda_i\lambda_j}\delta} \\
&\leq \frac{C}{\sqrt{\lambda_i\lambda_j}\delta}\left(\frac{1}{\lambda_j^2 G_{a_i}^{-2}(a_j)}\right) \\
&\leq \frac{C}{\lambda_i\lambda_j G_{a_i}^{-2}(a_j)} \frac{G_{a_i}^{-1}(a_j)}{\sqrt{\lambda_i\lambda_j} G_{a_i}^{-1}(a_j)\delta} \\
&\leq C\varepsilon_{ij}^2\varepsilon_{ij}\frac{1}{\delta}.
\end{aligned}$$

Thus, when $|a_i - a_j| \geq 2\delta$ we get

$$I_4 = O\left(\varepsilon_{ij}^3 \frac{1}{\delta}\right). \tag{65}$$

In case $|a_i - a_j| < 2\delta$, we first observe that

$$B(0, \lambda_j\delta) \setminus \mathcal{A} \subset A_1 \cup A_2$$

with

$$A_1 = \left\{\epsilon\lambda_j G_{a_i}^{-1}(a_j) \leq |y| \leq E\lambda_j G_{a_i}^{-1}(a_j)\right\}$$

and

$$A_2 = \left\{E\lambda_j G_{a_i}^{-1}(a_j) \leq |y| \leq \lambda_j\delta\right\},$$

where $0 < \epsilon < E$. Thus, we have

$$I_4 \leq I_4^1 + I_4^2, \tag{66}$$

with

$$I_4^1 = \int_{A_1} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j} + a_j\right)\right]^{\frac{-1}{2}}$$

and

$$I_4^2 = \int_{A_2} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j} + a_j\right)\right]^{\frac{-1}{2}}.$$

We estimate I_4^1 as follows:

$$\begin{aligned} I_4^1 &\leq C\left[1+\lambda_j^2 G_{a_i}^{-2}(a_j)\right]^{\frac{-5}{2}} \int_{|y|\leq E\lambda_j G_{a_i}^{-1}(a_j)} \left[\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)\right]^{\frac{-1}{2}} \\ &\leq C\left[1+\lambda_j^2 G_{a_i}^{-2}(a_j)\right]^{\frac{-5}{2}} \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1}{2}} \int_{|y|\leq E\lambda_j G_{a_i}^{-1}(a_j)} \left[1+\lambda_i^2 G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)\right]^{\frac{-1}{2}} \\ &\leq C\left[1+\lambda_j^2 G_{a_i}^{-2}(a_j)\right]^{\frac{-5}{2}} \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1}{2}} \int_{|y|\leq E\lambda_j G_{a_i}^{-1}(a_j)} \left[1+\lambda_i^2 \left|\frac{y}{\lambda_j}+a_i-a_j\right|^2\right]^{\frac{-1}{2}} \\ &\leq C\left[1+\lambda_j^2 G_{a_i}^{-2}(a_j)\right]^{\frac{-5}{2}} \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1}{2}} \left(\frac{\lambda_j}{\lambda_i}\right)^{3} \int_{|z|\leq \bar{E}\lambda_i G_{a_i}^{-1}(a_j)} \left[\frac{1}{1+|z|^2}\right]^{\frac{1}{2}} \\ &\leq C\left[\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-5}{2}} \left(\lambda_i^2 G_{a_i}^{-2}(a_j)\right) \\ &\leq C\varepsilon_{ij}^5\left(\lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right), \end{aligned}$$

where $\bar{E}$ is a positive constant. So we obtain

$$I_4^1 = O\left(\varepsilon_{ij}^3\right). \tag{67}$$

For I_4^2, we have

$$\begin{aligned} I_4^2 &= \int_{A_2} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}\left(\frac{y}{\lambda_j}+a_j\right)\right]^{\frac{-1}{2}} \\ &\leq C\int_{|y|\geq E\lambda_j G_{a_i}^{-1}(a_j)} \left(\frac{1}{1+|y|^2}\right)^{\frac{5}{2}} \left[\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}} \\ &\leq C\left[\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j G_{a_i}^{-2}(a_j)\right]^{\frac{-1}{2}} \left(\frac{1}{\lambda_j^2 G_{a_i}^{-2}(a_j)}\right). \end{aligned}$$

This implies

$$I_4^2 = O\left(\varepsilon_{ij}^3\right). \tag{68}$$

Thus, combining (66)-(68), we derive that if $|a_i-a_j|<2\delta$, then

$$I_4 = O\left(\varepsilon_{ij}^3\right). \tag{69}$$

Now, using (65) and (69), we infer that in case $\varepsilon_{i,j}^{-2}\simeq\lambda_i\lambda_j G_{a_i}^{-2}(a_j)$,

$$I_4 = O\left(\varepsilon_{i,j}^3\frac{1}{\delta}\right). \tag{70}$$

Finally combining (64) and (70), we get

$$I_4 = O\left(\varepsilon_{ij}^3\frac{1}{\delta}\right).$$

Collecting all we obtain

$$\begin{aligned} \int_{B(a_j,\lambda_j\delta)} u_{a_j,\lambda_j}^5 u_{a_i,\lambda_i} &= c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}\left(1+o_{\varepsilon_{i,j}}(1)\right)\right] \\ &\quad + c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}O(\varepsilon_{i,j}^2(\delta^{-2}+\log\varepsilon_{ij}^{-1}))\right]. \end{aligned} \tag{71}$$

Therefore using (59) and (71), we arrive to

$$\begin{aligned}\int_\Omega u^5_{a_j,\lambda_j}u_{a_i,\lambda_i} &= c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}\left(1+o_{\varepsilon_{i,j}}(1)\right)\right]\\ &+c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}O(\varepsilon_{i,j}^2(\delta^{-2}+\log\varepsilon_{ij}^{-1}))\right]\\ &+O\left(\frac{1}{\lambda_j^{\frac{5}{2}}\sqrt{\lambda_i}\delta^6}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_\Omega u^5_{a_j,\lambda_j}u_{a_i,\lambda_i} &= c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}\left(1+o_{\varepsilon_{i,j}}(1)\right)\right]\\ &+c_0^6\left[1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right]\left[c_3\varepsilon_{ij}O(\varepsilon_{i,j}^2(\delta^{-2}+\log\varepsilon_{ij}^{-1}))\right]\\ &+O\left(\varepsilon_{ij}^3\frac{1}{\delta^6}\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\int_\Omega u^5_{a_j,\lambda_j}u_{a_i,\lambda_i} =& c_0^6c_3\varepsilon_{i,j}\left[\left(1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right)\left(1+o_{\varepsilon_{i,j}}(1)\right)\right]\\ &+c_0^6c_3\varepsilon_{i,j}\left[\left(1+O\left(\delta+\frac{1}{\lambda_i^2\delta^2}\right)\right)O(\varepsilon_{i,j}^2(\delta^{-2}+\log\varepsilon_{ij}^{-1}))\right]\\ &+O\left(\varepsilon_{ij}^3\frac{1}{\delta^6}\right).\end{aligned}\tag{72}$$

Hence by switching the index i and j, the result follows from (26), (72), and the symmetry $\varepsilon_{ij}=\varepsilon_{ji}$. ■

We present now some sharp high-order inter-action estimates needed for the application of the algebraic topological argument for existence. We start with the following balanced high-order inter-action estimate.

Lemma 5.4. Assuming that $K\subset\Omega$ is compact, $m>0$ is a large integer, $\theta>0$ is small, and μ_0 is small, then $\forall a_i,a_j\in K$, $\forall 0<2\delta<\varrho_0$, and $\forall 0<\frac{1}{\lambda_j},\frac{1}{\lambda_i}\le\theta\delta^m$ such that $\varepsilon_{ij}\le\mu_0$, we have

$$\int_\Omega u^3_{a_i,\lambda_i}u^3_{a_j,\lambda_j}=O\left(\varepsilon_{ij}^3\delta^{-6}\log\left(\frac{1}{\varepsilon_{ij}\delta}\right)\right).$$

Proof. By symmetry, we can assume without loss of generality (w.l.o.g) that $\lambda_j\le\lambda_i$. Thus we have
1) Either $\varepsilon_{ij}^{-2}\sim\lambda_i\lambda_jG_{a_i}^{-2}(a_j)$
2) Or $\varepsilon_{ij}^{-2}\sim\frac{\lambda_i}{\lambda_j}$.

Now, if $|a_i - a_j| \geq 2\delta$, then we obtain

$$\begin{aligned} I &= \int_\Omega u^3_{a_i,\lambda_i} u^3_{a_j,\lambda_j} \\ &\leq C \int_{B(a_i,\delta)} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\frac{3}{2}} \left(\frac{\lambda_j}{1+\lambda_j^2 G^{-2}_{a_j}(x)}\right)^{\frac{3}{2}} \\ &+ \frac{C}{\lambda_i^{\frac{3}{2}}\delta^3} \int_{B(a_j,\delta)} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\frac{3}{2}} + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \\ &\leq \underbrace{C \int_{B(0,\lambda_i\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}} \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j G^{-2}_{a_j}\left(a_i+\frac{y}{\lambda_i}\right)}\right)^{\frac{3}{2}}}_{I_1} \qquad (73) \\ &+ \frac{C}{\delta^3}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \int_{B(0,\lambda_j\delta)} \left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}} + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \\ &\leq CI_1 + \frac{C}{\delta^3}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \left[\log(\lambda_j\delta)+C\right] + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \\ &\leq CI_1 + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \log(\lambda_j). \end{aligned}$$

Now, we estimate I_1 as follows.
If $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$, then we get

$$I_1 \leq C\varepsilon_{ij}^3 \left[\log(\lambda_i\delta)+C\right].$$

So, for I we have

$$\begin{aligned} I &\leq C\varepsilon_{ij}^3\left[\log(\lambda_i\delta)+C\right] + \frac{C}{\delta^6}\varepsilon_{ij}^3\log(\lambda_i\lambda_j) \\ &\leq \frac{C}{\delta^6}\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-2}G^2_{a_i}(a_j)) \\ &= O\left(\frac{\varepsilon_{ij}^3\log(\varepsilon_{ij}^{-1}\delta^{-1})}{\delta^6}\right). \end{aligned}$$

If $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G^{-2}_{a_i}(a_j)$, then we estimate

$$I_1 \leq \frac{C}{\delta^3}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}}\left[\log(\lambda_i\delta)+C\right].$$

So, for I we get

$$I \leq \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}}\log(\lambda_i\lambda_j).$$

This implies

$$I \leq \frac{C}{\delta^6}\varepsilon_{ij}^3\log(\varepsilon_{ij}^{-2}G^2_{a_i}(a_j)).$$

Hence, for $|a_i - a_j| \geq 2\delta$, we obtain

$$I = O\left(\frac{\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1}\delta^{-1})}{\delta^6}\right). \tag{74}$$

On the other hand, arguing as above, if $|a_i - a_j| < 2\delta$ then we have also

$$\begin{aligned} I &\leq I_1 + \frac{C}{\delta^3}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \log(\lambda_j) + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \\ &\leq I_1 + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\frac{3}{2}} \log(\lambda_j\lambda_i), \end{aligned}$$

where I_1 is as in (73). Thus, if $\varepsilon_{i,j}^{-2} \simeq \frac{\lambda_i}{\lambda_j}$ then

$$I \leq I_1 + \frac{C}{\delta^6}\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{3}{2}} \frac{1}{\lambda_j^3}\left[\log(\frac{\lambda_i}{\lambda_j}) + \log(\lambda_j^2)\right].$$

This implies

$$I \leq I_1 + \frac{C}{\delta^6}\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1}).$$

Next, if $\varepsilon_{i,j}^{-2} \simeq \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$ then we get

$$\begin{aligned} I &\leq I_1 + \frac{C}{\delta^6}\left(\frac{1}{\lambda_i\lambda_j G_{a_i}^{-2}(a_j)}\right)^{\frac{3}{2}} \left[\log(\lambda_i\lambda_j G_{a_i}^{-2}(a_j) + \log(G_{a_i}^2(a_j)))\right] G_{a_i}^{-3}(a_j) \\ &\leq I_1 + \frac{C}{\delta^6}\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1}). \end{aligned}$$

Now, to continue, we are going to estimate I_1. For this, we start by defining the following sets:

$$A_1 = \left\{|y| \leq \epsilon\lambda_i\sqrt{G_{a_j}^{-2}(a_i) + \frac{1}{\lambda_j^2}}\right\}$$

$$A_2 = \left\{\epsilon\lambda_i\sqrt{G_{a_j}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \leq |y| \leq E\lambda_i\sqrt{G_{a_j}^{-2}(a_i) + \frac{1}{\lambda_j^2}}\right\}$$

$$A_3 = \left\{E\lambda_i\sqrt{G_{a_j}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \leq |y| \leq 4\lambda_i\delta\right\},$$

with $0 < \epsilon < E < \infty$. Clearly by the definition of I_1 (see (73)), we have

$$I_1 \leq \int_{A_1} L_{ij} + \int_{A_2} L_{ij} + \int_{A_3} L_{ij},$$

where

$$L_{ij} = \left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}} \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j G_{a_j}^{-2}\left(a_i + \frac{y}{\lambda_i}\right)}\right)^{\frac{3}{2}}.$$

For $\int_{A_1} L_{ij}$, we obtain

$$\begin{aligned}\int_{A_1} L_{ij} &\le C\varepsilon_{ij}^3 \int_{A_1}\left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}}\\ &\le C\varepsilon_{ij}^3 \log\left(\sqrt{\frac{\lambda_i}{\lambda_j}}\sqrt{\lambda_i\lambda_j G_{a_j}^{-2}(a_i)+\frac{\lambda_i}{\lambda_j}}\right)\\ &\le C\varepsilon_{ij}^3\log(\varepsilon_{ij}^{-1}).\end{aligned}$$

For $\int_{A_2} L_{i,j}$, we get

$$\begin{aligned}\int_{A_2} L_{ij} &\le C\left(\frac{1}{\left(\frac{\lambda_i}{\lambda_j}\right)^2+\lambda_i^2 G_{a_j}^{-2}(a_i)}\right)^{\frac{3}{2}}\int_{A_2}\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j G_{a_j}^{-2}\left(a_i+\frac{y}{\lambda_i}\right)}\right)^{\frac{3}{2}}\\ &\le C\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{3}{2}}\varepsilon_{ij}^3\int_{|y|\le E\lambda_i\sqrt{G_{a_j}^{-2}(a_i)+\frac{1}{\lambda_j^2}}}\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}\left|y+\lambda_i(a_i-a_j)\right|^2}\right)^{\frac{3}{2}}\\ &\le C\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{3}{2}}\varepsilon_{ij}^3\int_{|y|\le\bar{E}\lambda_i\sqrt{G_{a_j}^{-2}(a_i)+\frac{1}{\lambda_j^2}}}\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}|y|^2}\right)^{\frac{3}{2}}\\ &\le C\left(\frac{\lambda_j}{\lambda_i}\right)^{3}\left(\frac{\lambda_j}{\lambda_i}\right)^{-3}\varepsilon_{ij}^3\int_{|y|\le\bar{E}\lambda_j\sqrt{G_{a_j}^{-2}(a_i)+\frac{1}{\lambda_j^2}}}\left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}}\\ &\le C\varepsilon_{ij}^3\log(\varepsilon_{ij}^{-1}).\end{aligned}$$

For $\int_{A_3} L_{i,j}$, we derive

$$\begin{aligned}\int_{A_3} L_{ij} &\le \int_{A_3}\left(\frac{1}{1+|y|^2}\right)^{\frac{3}{2}}\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}|y|^2}\right)^{\frac{3}{2}}\\ &\le C\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{3}{2}}\int_{A_3}\frac{1}{|y|^6}\\ &\le C\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{3}{2}}\left(\frac{1}{\lambda_i^2 G_{a_j}^{-2}(a_i)+\left(\frac{\lambda_i}{\lambda_j}\right)^2}\right)^{\frac{3}{2}}\\ &\le C\varepsilon_{ij}^3.\end{aligned}$$

Therefore, we obtain

$$I_1 \le C\varepsilon_{ij}^3\log(\varepsilon_{ij}^{-1}).$$

This implies for $|a_i-a_j|<2\delta$, we have

$$I = O\left(\frac{\varepsilon_{ij}^3}{\delta^6}\log(\varepsilon_{ij}^{-1})\right).$$

Hence, combining with the estimate for $|a_i - a_j| \geq 2\delta$ (see (74)), we get

$$\int_\Omega u^3_{a_i,\lambda_i} u^3_{a_j,\lambda_j} = O\left(\frac{\varepsilon^3_{ij}}{\delta^6}\log(\varepsilon^{-1}_{ij}\delta^{-1})\right).$$

■

Finally, we present a sharp unbalanced high-order inter-action estimate needed for the application of the Barycenter technique of Bahri-Coron[3].

Lemma 5.5. Assuming that $K \subset \Omega$ is compact, $m > 0$ is a large integer, $\theta > 0$ is small, and μ_0 is small, then $\forall a_i, a_j \in K$, $\forall 0 < 2\delta < \varrho_0$, and $\forall 0 < \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \leq \theta\delta^m$ such that $\varepsilon_{ij} \leq \mu_0$, we have

$$\int_\Omega u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j} = O\left(\frac{\varepsilon^\beta_{ij}}{\delta^6}\right),$$

where $\alpha + \beta = 6$ and $\alpha > 3 > \beta > 1$.

Proof. Let $\hat{\alpha} = \frac{1}{2}\alpha$ and $\hat{\beta} = \frac{1}{2}\beta$. Then we get $\hat{\alpha} + \hat{\beta} = 3$. Now, since $\lambda_j \leq \lambda_i$, then we have
1) Either $\varepsilon^{-2}_{ij} \sim \lambda_i\lambda_j G^{-2}_{a_i}(a_j)$

2) Or $\varepsilon^{-2}_{ij} \sim \frac{\lambda_i}{\lambda_j}$.
To continue, we write

$$\int_\Omega u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j} = \underbrace{\int_{B_{a_i}(\delta)} u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j}}_{I_1} + \underbrace{\int_{\Omega - B_{a_i}(\delta)} u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j}}_{I_2}$$

and estimate I_1 and I_2. For I_2, we obtain

$$\begin{aligned} I_2 &= \int_{\left(\Omega - B_{a_i}(\delta)\right)\cap B_{a_j}(\delta)} u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j} + \int_{\Omega - \left(B_{a_i}(\delta)\cup B_{a_j}(\delta)\right)} u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j} \\ &\leq C\int_{\left(\Omega - B_{a_i}(\delta)\right)\cap B_{a_j}(\delta)} \left(\frac{\lambda_i}{1+\lambda_i^2 G^{-2}_{a_i}(x)}\right)^{\hat{\alpha}} \left(\frac{\lambda_j}{1+\lambda_j^2|x-a_j|^2}\right)^{\hat{\beta}} \\ &+ C\int_{\Omega - \left(B_{a_i}(\delta)\cup B_{a_j}(\delta)\right)} \left(\frac{\lambda_i}{1+\lambda_i^2 G^{-2}_{a_i}(x)}\right)^{\hat{\alpha}} \left(\frac{\lambda_j}{1+\lambda_j^2 G^{-2}_{a_j}(x)}\right)^{\hat{\beta}} \\ &\leq \frac{C}{\lambda_i^{\hat{\alpha}}\lambda_j^{3-\hat{\beta}}\delta^\alpha}\int_{B_0(\lambda_j\delta)}\left(\frac{1}{1+|y|^2}\right)^{\hat{\beta}} + \frac{C}{\lambda_i^{\hat{\alpha}}\lambda_j^{\hat{\beta}}\delta^6} \\ &\leq \frac{C}{\lambda_i^{\hat{\alpha}}\lambda_j^{3-\hat{\beta}}\delta^\alpha}\left(\frac{1}{\lambda_j\delta}\right)^{2\hat{\beta}-3} + \frac{C}{\lambda_i^{\hat{\alpha}}\lambda_j^{\hat{\beta}}\delta^6}. \end{aligned}$$

Thus, we have for I_2

$$I_2 \leq \frac{C}{\lambda_i^{\hat{\alpha}}\lambda_j^{\hat{\beta}}\delta^6}. \tag{75}$$

Next, for I_1 we derive

$$I_1 = \int_{B_{a_i}(\delta)} \left(\frac{\lambda_i}{1+\lambda_i^2|x-a_i|^2}\right)^{\hat{\alpha}} \left(\frac{\lambda_j}{1+\lambda_j^2 G_{a_j}^{-2}(x)}\right)^{\hat{\beta}}$$

$$= \int_{B_0(\lambda_i\delta)} \left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}} \left[\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j G_{a_j}^{-2}\left(a_i + \frac{y}{\lambda_i}\right)}\right]^{\hat{\beta}}.$$

Thus, if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$ then

$$I_1 \le C\varepsilon_{ij}^{2\hat{\beta}} \left[\left(\frac{1}{\lambda_i\delta}\right)^{2\hat{\alpha}-3} + C\right]$$

$$\le C\varepsilon_{ij}^{\beta}.$$

If $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$ and $|a_i - a_j| \ge 2\delta$, then we estimate

$$I_1 \le C\left(\frac{1}{\lambda_i\lambda_j\delta^2}\right)^{\hat{\beta}} \left[\left(\frac{1}{\lambda_i\delta}\right)^{2\hat{\alpha}-3} + C\right]$$

$$\le C\frac{1}{\delta^3}\left(\frac{1}{\lambda_i\lambda_j}\right)^{\hat{\beta}} \le C\frac{1}{\delta^3}\left[\left(\frac{1}{\lambda_i\lambda_j G_{a_i}^{-2}(a_j)}\right)^{\frac{1}{2}}\right]^{\beta}$$

$$\le C\frac{1}{\delta^3}\varepsilon_{ij}^{\beta}.$$

Now, if $\varepsilon_{ij}^{-2} \sim \lambda_i\lambda_j G_{a_i}^{-2}(a_j)$ and $|a_i - a_j| < 2\delta$, then we get

$$I_1 \le C\int_{B_0(\lambda_i\delta)} \left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}} \left[\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\left|a_i + \frac{x}{\lambda_i} - a_j\right|^2}\right]^{\hat{\beta}}.$$

Next, we define

$$B = \left\{\frac{1}{2}|a_i - a_j| \le \frac{|y|}{\lambda_i} \le 2|a_i - a_j|\right\}$$

and have

$$I_1 \le C\int_B \left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}} \left[\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\left|a_i + \frac{x}{\lambda_i} - a_j\right|^2}\right]^{\hat{\beta}}$$

$$+C\int_{B_0(\lambda_i\delta)-B} \left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}} \left[\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\left|a_i + \frac{x}{\lambda_i} - a_j\right|^2}\right]^{\hat{\beta}}.$$

For the second term, we have

$$\begin{aligned}\int_{B_0(\lambda_i\delta)-B}\left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}}\left[\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\left|a_i+\frac{x}{\lambda_i}-a_j\right|^2}\right]^{\hat{\beta}} &\leq C\varepsilon_{ij}^{\beta}\left(\frac{1}{\lambda_i\delta}\right)^{\alpha-3}\\ &\quad+C\varepsilon_{ij}^{\beta}\\ &\leq C\varepsilon_{ij}^{\beta}.\end{aligned}$$

For the first term, we obtain

$$\begin{aligned}&\int_{B}\left(\frac{1}{1+|y|^2}\right)^{\hat{\alpha}}\left[\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\left|a_i+\frac{x}{\lambda_i}-a_j\right|^2}\right]^{\hat{\beta}}\\ &\leq C\left(\frac{1}{1+\lambda_i^2|a_i-a_j|^2}\right)^{\hat{\alpha}}\int_{|y|\leq 2\lambda_i|a_i-a_j|}\left[\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}\left|y+\lambda_i(a_i-a_j)\right|^2}\right]^{\hat{\beta}}\\ &\leq C\left(\frac{1}{1+\lambda_i^2|a_i-a_j|^2}\right)^{\hat{\alpha}}\int_{|z|\leq 4\lambda_i|a_i-a_j|}\left[\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}|z|^2}\right]^{\hat{\beta}}\\ &\leq C\left(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\frac{\alpha}{2}}\int_{|z|\leq 4\lambda_j|a_i-a_j|}\left[\frac{1}{1+|z|^2}\right]^{\hat{\beta}}.\end{aligned}$$

If $\lambda_j|a_i-a_j|$ is bounded, then we get

$$\begin{aligned}I_1 &\leq C\left(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\frac{\alpha}{2}}\\ &\leq C\varepsilon_{ij}^{\beta}.\end{aligned}$$

If $\lambda_j|a_i-a_j|$ is unbounded, then we estimate

$$\begin{aligned}I_1 &\leq C\left(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\frac{\alpha}{2}}(\lambda_j|a_i-a_j|)^{3-2\hat{\beta}}\\ &\leq C\left(\frac{1}{1+\lambda_i^2|a_i-a_j|^2}\right)^{\hat{\alpha}+\hat{\beta}-\frac{3}{2}}\left(\frac{\lambda_i}{\lambda_j}\right)^{\hat{\beta}}\\ &\leq C\left(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\hat{\beta}}\left(\frac{1}{1+\lambda_i^2|a_i-a_j|^2}\right)^{\hat{\alpha}-\frac{3}{2}}\\ &\leq C\varepsilon_{ij}^{\beta}.\end{aligned}$$

Thus, we have for I_1, obtain

$$I_1\leq\frac{C}{\delta^3}\varepsilon_{ij}^{\beta}. \tag{76}$$

On the other hand, using the estimate for I_2 (see (75)), we get

$$I_2=O\left(\frac{\varepsilon_{ij}^{\beta}}{\delta^6}\right). \tag{77}$$

Hence, combining (76) and (77), we have

$$\int_\Omega u^\alpha_{a_i,\lambda_i} u^\beta_{a_j,\lambda_j} = O\left(\frac{\varepsilon^\beta_{ij}}{\delta^6}\right).$$

■

6. **Algebraic topological argument.** In this section, we present the algebraic topological argument for existence. We start by fixing some notation from algebraic topology. For a topological space Z, $H_*(Z)$ denotes the singular homology of Z with $\mathbb{Z}_2$ coefficients. If Y is a subspace of Z, then $H_*(Z,Y)$ stands for the relative homology with $\mathbb{Z}_2$ coefficients of the topological pair (Z,Y). For a map $f : Z \to Y$ with Z and Y topological spaces, f_* denotes the induced map in homology. If $f : (Z,Y) \longrightarrow (W,X)$ is a map with (Z,Y) and (W,X) topological pairs, then f_* denotes the induced map in relative homology. Furthermore, we discuss some algebraic topological tools needed for our application of the Barycentre technique of Bahri-Coron[3]. We start with the following observation. Since Ω is a smooth bounded domain of $\mathbb{R}^3$ which is non-contractble, then there exists $n \in \{1,2,3\}$ such that $H_n(\Omega)$ is not trivial, see Lemma 2.1 (see also Remark 2.2) or [3] (see page 1 just after Theorem 1). Hence, as in [3] (see beginning of page 263), we have there exists M a smooth compact connected n-dimensional manifold without boundary and a continuous map

$$h \ : \ M \longrightarrow \Omega \tag{78}$$

such that if we denote by $[M]$ the class of orientation (modulo 2) of M, then $h_*([M]) \neq 0$. Moreover, we have clearly the existence of a compact smooth manifold with boundary K_0 such that

$$h(M) \subset K_0 \subset \Omega. \tag{79}$$

We recall the space of formal barycenter of M that we need for our Barycenter technique for existence. For $p \in \mathbb{N}^*$, the set of formal barycenters of M of order p is defined as

$$B_p(M) = \{\sum_{i=1}^p \alpha_i\delta_{a_i} \ : \ a_i \in M, \ \alpha_i \geq 0, \ \ i = 1, \cdots, p, \ \sum_{i=1}^p \alpha_i = 1\}, \ \ B_0(M) = \emptyset, \tag{80}$$

where δ_a for $a \in M$ is the Dirac measure at a. we have the existence of $\mathbb{Z}_2$ orientation classes (see [3])

$$w_p \in H_{(n+1)p-1}(B_p(M), B_{p-1}(M)), \qquad p \in \mathbb{N}^*. \tag{81}$$

Now to continue, we fix δ small such that $0 < 2\delta \leq \varrho_0$ where ϱ_0 as in (23) with K is replaced by K_0 and K_0 is given by (79). Moreover, we choose $m > 0$ a large integer and $\theta_0 > 0$ and small. After this, we let λ varies such that $0 < \frac{1}{\lambda} \leq \theta_0\delta^m$ and associate for every $p \in \mathbb{N}^*$ the map

$$f_p(\lambda) : B_p(M) \longrightarrow H^{1,+}_0(\Omega)$$

defined by the formula

$$f_p(\lambda)(\sigma) = \sum_{i=1}^p \alpha_i u_{h(a_i),\lambda}, \quad \sigma = \sum_{i=1}^p \alpha_i\delta_{a_i},$$

where h is as in (78) and $u_{h(a_i),\lambda}$ is as (24) (with a replaced by $h(a_i)$).

As in Proposition 3.1in [9] and Proposition 6.3 in [12], using Corollary 5.2, Corollary 5.3, Corollary 4.3, Lemma 5.4, and Lemma 5.5, we have the following multiple-bubble estimate.

Proposition 6.1. There exist $\bar{C}_0 > 0$ and $\bar{c}_0 > 0$ such that for every $p \in \mathbb{N}^*$, $p \geq 2$ and every $0 < \varepsilon \leq \varepsilon_0$, there exists $\lambda_p := \lambda_p(\varepsilon)$ such that for every $\lambda \geq \lambda_p$ and for every $\sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(M)$, we have

1. If $\sum_{i \neq j} \varepsilon_{i,j} > \varepsilon$ or there exist $i_0 \neq j_0$ such that $\frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu_0$, then
$$J_q(f_p(\lambda)(\sigma)) \leq p^{\frac{2}{3}} \mathcal{S}.$$
2. If $\sum_{i \neq j} \varepsilon_{i,j} \leq \varepsilon$ and for every $i \neq j$ we have $\frac{\alpha_i}{\alpha_j} \leq \nu_0$, then
$$J_q(f_p(\lambda)(\sigma)) \leq p^{\frac{2}{3}} \mathcal{S} \left(1 + \frac{\bar{C}_0}{\lambda} - \bar{c}_0 \frac{(p-1)}{\lambda} \right),$$
where ε_{ij} is as in (25) with (a_i, a_j) replaced by $(h(a_i), h(a_j))$ and $\lambda_i = \lambda_j = \lambda$, ε_0 is as in (44) and ν_0 is as in (45).

As in Lemma 4.2 in [9] and Lemma 6.4 in [12], we have the selection map s_1 (see (46)), Lemma 3.2 and Corollary 4.3 imply the following topological result.

Lemma 6.2. Assuming that J_q has no critical points, then there exists $\bar{\lambda}_1 > 0$ such that for every $\lambda \geq \bar{\lambda}_1$,
$$f_1(\lambda) \,:\, (B_1(M),\, B_0(M)) \longrightarrow (W_1,\, W_0)$$
is well defined and satisfies
$$(f_1(\lambda))_*(w_1) \neq 0 \ \text{ in } \ H_n(W_1,\, W_0).$$

We would like to make a comment about Lemma 6.2 which provides a consequence which does not require the use of homology and sufficient for the way we are going to apply the Barycenter technique. The consequence is as follows: Lemma 6.2 implies that the map $f_1(\lambda) \,:\, (B_1(M),\, B_0(M)) \longrightarrow (W_1,\, W_0)$ is not homotopic to a constant map $\bar{f}_1(\lambda) \,:\, (B_1(M),\, B_0(M)) \longrightarrow (W_1,\, W_0)$, for $\lambda \geq \bar{\lambda}_1$.

Next, as in Lemma 4.3 in [9] and Lemma 6.5 in [12], we have the selection map s_p (see (46)), Lemma 3.2 and Proposition 6.1 imply the following recursive topological result.

Lemma 6.3. Assuming that J_q has no critical points, then there exists $\bar{\lambda}_p > 0$ such that for every $\lambda \geq \bar{\lambda}_p$,
$$f_{p+1}(\lambda) : (B_{p+1}(M),\, B_p(M)) \longrightarrow (W_{p+1},\, W_p)$$
and
$$f_p(\lambda) : (B_p(M),\, B_{p-1}(M)) \longrightarrow (W_p,\, W_{p-1})$$
are well defined and satisfy
$$(f_p(\lambda))_*(w_p) \neq 0 \ \text{ in } \ H_{(n+1)p-1}(W_p,\, W_{p-1})$$
implies
$$(f_{p+1}(\lambda))_*(w_{p+1}) \neq 0 \ \text{ in } \ H_{(n+1)(p+1)-1}(W_{p+1},\, W_p).$$

As in the case of $p = 1$, we would like to make a remark about Lemma 6.3 which provides an implication which use only homotopy and sufficient for our purpose. To do that, we first observe that combing Lemma 6.2 and Lemma 6.3, we have that $(f_p(\lambda))_*(w_p) \neq 0$ in $H_{(n+1)p-1}(W_p,\ W_{p-1})$ for every $p \geq 1$ and for every $\lambda \geq \bar{\lambda}_p$. Hence, as in the case $p = 1$, we have the map $f_p(\lambda) : (B_p(M),\ B_{p-1}(M)) \longrightarrow (W_p,\ W_{p-1})$ is not homotopic to a constant map $\bar{f}_p(\lambda) : (B_p(M),\ B_{p-1}(M)) \longrightarrow (W_p,\ W_{p-1})$ for every $p \geq 1$ and for every $\lambda \geq \bar{\lambda}_p$.

Finally, as in Corollary 3.3 in [9] and Lemma 6.6 in [12], we clearly have that Proposition 6.1 implies the following result.

Lemma 6.4. Setting

$$p_0 := [1 + \frac{\bar{C}_0}{\bar{c}_0}] + 3$$

with $\bar{C}_0$ and $\bar{c}_0$ as in Proposition 6.1 and recalling (47), we have there exists $\hat{\lambda}_{p_0} > 0$ such that $\forall \lambda \geq \hat{\lambda}_{p_0}$,

$$f_{p_0}(\lambda)(B_{p_0}(M), B_{p_0-1}(M)) \subset (W_{p_0-1}, W_{p_0-2}).$$

In the same spirit as the comments after Lemma 6.2 and Lemma 6.3, we have that Lemma 6.4 implies the following remark using homotopy. The observation is: Lemma 6.4 implies that the map $H_{p_0}(\lambda)$ $(\lambda \geq \hat{\lambda}_{p_0})$ defined by

$$H_{p_0}(\lambda)(t, \sigma) = t f_1(\lambda)(\delta_{a_0}) + (1-t) f_{p_0-1}(\lambda)(\sigma) = f_{p_0}(\lambda)(\sigma_t), (t, \sigma) \in [0, 1] \times B_{p_0-1}(M),$$

with $\sigma_t := t\delta_{a_0} + (1-t)\sigma$ and seen as a map from $[0, 1] \times (B_{p_0-1}(M), B_{p_0-2})$ is into (W_{p_0-1}, W_{p_0-2}) and defines an homotopy in (W_{p_0-1}, W_{p_0-2}) between the map $f_{p_0-1}(\lambda) : (B_{p_0-1}(M),\ B_{p_0-2}(M)) \longrightarrow (W_{p_0-1},\ W_{p_0-2})$ and the constant map

$$\bar{f}_{p_0-1}(\lambda) = f_1(\delta_{a_0}) : (B_{p_0-1}(M),\ B_{p_0-2}(M)) \longrightarrow (W_{p_0-1},\ W_{p_0-2}).$$

PROOF of Theorem 1.1
As in [9] and [12], the theorem follows by a contradiction argument from Lemma 6.2 - Lemma 6.4. Indeed, assuming that J_q has no critical points, then on one hand we have the map $f_p(\lambda) : (B_p(M),\ B_{p-1}(M)) \longrightarrow (W_p,\ W_{p-1})$ is not homotopic in $(W_p,\ W_{p-1})$ to a constant map $\bar{f}_p(\lambda) : (B_p(M),\ B_{p-1}(M)) \longrightarrow (W_p,\ W_{p-1})$ for every $p \geq 1$ and $\lambda \geq \bar{\lambda}_p$, and on the other hand for $p = p_0 - 1$, we have the map $f_{p_0-1}(\lambda) : (B_{p_0-1}(M),\ B_{p_0-2}(M)) \longrightarrow (W_{p_0-1},\ W_{p_0-2})$ is homotopic in $(W_{p_0-1},\ W_{p_0-2})$ to the constant map

$$\bar{f}_{p_0-1}(\lambda) = f_1(\sigma_{a_0}) : (B_{p_0-1}(M),\ B_{p_0-2}(M)) \longrightarrow (W_{p_0-1},\ W_{p_0-2}),$$

$\forall \lambda \geq \hat{\lambda}_{p_0}$. Hence, we reach a contradiction for $\lambda \geq \max\{\hat{\lambda}_{p_0}, \bar{\lambda}_{p_0-1}\}$. ■

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