

Uniformization and the Yamabe Problem

Stephen E. McKeown and Cheikh Birahim Ndiaye

Introduction

Conformal geometry is the subfield of differential geometry that studies manifolds on which infinitesimal angles are defined, but not lengths; that is, the crossing angle between any two intersecting curves is well-defined, but their lengths are not.

Given a manifold M , recall that a Riemannian metric g is a (smoothly varying) choice $\langle \cdot, \cdot \rangle_g$ of inner product on each tangent space $T_p M$. Now, given any inner product $\langle \cdot, \cdot \rangle$ on a vector space V , the angle θ between two vectors $X, Y \in V$ may be defined by

$$\cos(\theta) = \frac{\langle X, Y \rangle}{\sqrt{\langle X, X \rangle \langle Y, Y \rangle}}.$$

The right-hand side remains unaltered if $\langle \cdot, \cdot \rangle$ is replaced by $c\langle \cdot, \cdot \rangle$ ($c > 0$). Thus, angles are well-defined given only a ray of inner products. For this reason, a conformal manifold is usually defined as a manifold M equipped with an equivalence class $[g]$ of metrics, where $\tilde{g} \sim g$ if $\tilde{g} = e^{2\omega}g$, with $\omega \in C^\infty(M)$ a smooth function. (The mode of expression is conventional: the exponential forces the scaling factor to be positive, while the factor 2 is merely for convenience. Other conventions are also popular.)

One of the numerous reasons for the ubiquity of conformal geometry within differential geometry is that a conformal change—replacing g by $e^{2\omega}g$ —is about the simplest nontrivial transformation that can be performed on a metric while preserving some of its geometric content. That the change is scalar also makes it analytically quite practical to study. Just what is preserved under such a change leads to one large set of questions within the field. Thus, for example, the Weyl tensor, which is the totally tracefree part of the Riemann curvature tensor, transforms according to the simple rule $\tilde{W} = e^{2\omega}W$. We will focus on another

set of questions: what can be changed in a useful way? Put differently: given a Riemannian metric g , is there a conformal metric $e^{2\omega}g$ with especially nice properties? (Can it perhaps be used to study g ?) This question has been extremely influential in geometric analysis, and we will be concerned with several versions of it.

Beginnings: Two Dimensions

Conformal geometry in two dimensions is a very different subject from that in higher dimensions, because of its intimate relationship with complex analysis. Every nonsingular holomorphic function between regions of the complex plane is angle-preserving (conformal). Thus, if it is also bijective, it is in fact a conformal equivalence—from the point of view of conformal geometry, two regions connected by such a function are equivalent. The Riemann mapping theorem may thus be viewed as saying that every simply connected proper open subset of the plane is conformally equivalent to the unit disk.

A natural question is whether the same might in some sense be true of a compact Riemann surface (Σ^2, g) . Of course, topological considerations forbid the existence of a homeomorphism between Σ and the disk, but might we at least expect the existence of a smooth function $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega}g$ has vanishing Gaussian curvature? (This would imply that $\tilde{g}$ is at least locally isometric to the disk.) In fact, the answer is certainly no: the Gauss-Bonnet theorem asserts that the Gaussian curvature K_g of any metric satisfies

$$2\pi\chi(\Sigma) = \int_{\Sigma} K_g dV_g,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ . Thus, unless $\chi(\Sigma) = 0$ (i.e., unless Σ is a torus), there can be no such function. However, a significant classical result asserts that we can obtain the best that Gauss-Bonnet allows.

Theorem 1 (Uniformization theorem). *Let (Σ, g) be a compact Riemann surface. There exists $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega}g$ has constant Gaussian curvature $K_{\tilde{g}}$. Assuming the normalization $\text{vol}_{\tilde{g}}(\Sigma) = 1$, we have $K_{\tilde{g}} = 2\pi\chi(\Sigma)$.*

There are many proofs; we mention [3]. Note that, because Gaussian curvature fully determines the curvature in two dimensions, it follows from this theorem that every

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surface is conformal to one that is locally isometric to either the plane, the sphere, or hyperbolic space.

One might very well ask what the situation is for a surface with boundary. If Σ has a boundary, Gauss-Bonnet reads

$$2\pi\chi(\Sigma) = \int_{\Sigma} K_g dV_g + \oint_{\partial\Sigma} \kappa_g ds,$$

where κ_g is the geodesic curvature of the boundary. Thus, in this case, Gauss-Bonnet does *not* forbid obtaining $K_g \equiv 0$, as long as κ_g is nonzero. Sure enough, the uniformization theorem for surfaces with boundary states that a conformal change lets us push the topological data to the boundary, and make it constant there.

Theorem 2 (Uniformization theorem for surfaces with boundary). *Let (Σ, g) be a compact Riemann surface with nonempty boundary. There exists $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega}g$ has vanishing Gaussian curvature $K_{\tilde{g}}$ and the boundary has constant geodesic curvature $\kappa_{\tilde{g}}$.*

See, for example, [1].

Higher Dimensions: The Yamabe Problem

In higher dimensions, complex analysis no longer plays an important role in conformal geometry, and the methods of proof are often quite different. Indeed, two dimensions is much more “flexible,” since the Liouville theorem shows that the space of conformal maps of $\mathbb{R}^n$ for $n \geq 3$ is finite dimensional, in contrast to the two-dimensional case. Yet it is natural to ask very similar questions. One of the most celebrated problems of twentieth-century differential geometry is the Yamabe problem, raised by Hidehiko Yamabe in 1960. This asks whether the uniformization theorem holds for compact manifolds M^n when $n \geq 3$. That is, given a Riemannian manifold (M^n, g) , does there exist $\omega \in C^\infty(M)$ such that $\tilde{g} = e^{2\omega}g$ has constant scalar curvature $R_{\tilde{g}}$? (To look for constant higher-rank curvature would yield a vastly overdetermined problem, since a conformal change is itself a scalar.) Actually, it is more common in this context to write the conformal change as $\tilde{g} = u^{\frac{4}{n-2}}g$, requiring $u > 0$; this is merely conventional, and is because the problem becomes easier to express and study when written in terms of

$$u = e^{(n-2)\omega/2}. \quad (1)$$

Yamabe believed he had proven that the answer is yes, but seven years after his tragically early death, Neil Trudinger discovered a significant error in his path-breaking paper. Partial solutions to the Yamabe problem were given over the next ten years by Trudinger and by Thierry Aubin, and in 1984, Richard Schoen finally solved all the remaining cases using the positive mass theorem.

Theorem 3 (Yamabe, Trudinger, Aubin, Schoen). *Suppose (M^n, g) is a compact Riemannian manifold, $n \geq 3$. Then there*

exists a positive $u \in C^\infty(M)$ such that $\tilde{g} = u^{\frac{4}{n-2}}g$ has constant scalar curvature $R_{\tilde{g}}$.

Note that in higher dimensions, the Chern-Gauss-Bonnet theorem says nothing about $\int_M R_g dV_g$, so whether the constant given by this theorem is positive, negative, or zero is a property more of the conformal geometry of $[g]$ than of the topology (although the two are not unrelated).

The condition $R_{\tilde{g}} = c$ (with c constant) is equivalent to u solving the PDE

$$P_2 u = c u^{\frac{n+2}{n-2}}, \quad (2)$$

where $P_2 u = 4 \frac{n-1}{n-2} \Delta_g u + R_g u$ is the *conformal Laplacian* of u . (The conformal Laplacian is so named because it satisfies the relatively nice conformal transformation property $\tilde{P}_2(u^{-1}f) = u^{-\frac{n+2}{n-2}} P_2(f)$. Here $\tilde{P}_2$ is the conformal Laplacian of $\tilde{g} = u^{\frac{4}{n-2}}g$.) One thus wishes to find a strictly positive solution to (2). The original approach to solving the Yamabe problem is variational: (2) is the Euler-Lagrange equation of the functional

$$\mathcal{E}(u) = \frac{\int_M \left(\frac{4(n-1)}{(n-2)} |du|^2 + R_g u^2 \right) dV_g}{\left(\int_M |u|^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{2n}}}, \quad u \in C^\infty(M), u > 0.$$

It follows easily by Hölder’s inequality that $\mathcal{E}$ is bounded below, so one can take a minimizing sequence; the challenge is then to show that it converges to a solution to (2), and that said solution is positive. The analytical and geometric ideas that go toward showing that this is the case are ponderous. First, one replaces the exponent $\frac{n+2}{n-2}$ in (2) by $\frac{n+2}{n-2} - \varepsilon$, as the former is the “critical exponent” for which the relevant Sobolev injection fails to be compact. This process of *subcritical regularization* yields a compact variational problem. Finding a solution u_ε via the direct method in the calculus of variations, the question then becomes whether this converges to a solution u as $\varepsilon \rightarrow 0$. This is addressed by studying the Euler equation associated to u_ε .

Let $Y([g]) = \inf_u \mathcal{E}(u)$, which one may show is a conformal invariant. It was shown by Trudinger and Aubin that if u_ε does not converge to a solution u , then we must have $Y([g]) \geq Y([S^n, \tilde{g}])$, where the latter is the conformal round sphere; so a solution exists if $Y([g]) < Y([S^n, \tilde{g}])$. The problem is thus reduced to showing that the latter inequality holds. This can be done in many cases by constructing clever test functions u that realize the inequality. In the work of Aubin, the test function is constructed using local geometry. Schoen’s approach in solving the remaining cases used global geometry to construct test functions. Indeed, one of his key ideas in completing the proof was to use the Green’s function of P_2 itself as a conformal change

factor; since it blows up at a point, this transforms the manifold to a noncompact but asymptotically flat space. A careful study of the asymptotics of the Green's function allows the problem then to be solved by appealing to the positive mass theorem of Schoen-Yau, which arose from General Relativity. The full details, including all the original references, are given in the well-known survey [7]. We will refer to the manner of argument described here as the Aubin-Schoen minimization technique. There is another variational approach to the Yamabe problem by Bahri-Coron and Bahri-Brezis, called the barycenter technique, using algebraic topological tools.

Once again, there are natural related problems to ask in the case of compact manifolds with boundary. The one we will consider here can be considered a natural analog of the Riemann mapping theorem and of Theorem 2. In higher dimensions, the appropriate boundary curvature to consider is the mean curvature H_g , rather than the geodesic curvature. The following theorem was first studied by Pascal Cherrier and was established in most cases by José Escobar in 1992; the remaining cases were treated by Sergio Almaraz, Sophie Chen, Fernando Coda Marques, and Martin Mayer and Cheikh Ndiaye.

Theorem 4. *Let (M^n, g) be a compact manifold with boundary, where $n \geq 3$. There exists $u \in C^\infty(M)$ with $u > 0$ such that $\tilde{g} = u^{\frac{4}{n-2}}g$ has vanishing scalar curvature ($R_{\tilde{g}} = 0$) and constant mean curvature: $H_{\tilde{g}} = \text{const}$.*

(Escobar also treated the reverse problem, where R_g is constant and $H_g = 0$.) The works of Escobar, Almaraz, Chen, and Marques used the Aubin-Schoen minimization approach. Indeed, numerous new techniques were developed in the proof of these theorems. Escobar developed a highly refined asymptotic expansion of the Green's function of the conformal Laplacian near the boundary, as well as a positive mass theorem in the bounded setting. Subsequent developments were often based on ever-finer constructions of local or global test functions. The final step, by Mayer and Ndiaye, used the Bari-Coron barycenter technique. Of the many papers we could cite in this line, we mention [5].

Fourth Order

The scalar curvature is not the only scalar function that is important to understanding the curvature of a Riemannian manifold. Since it was introduced by Tom Branson and Bent Ørsted, the so-called Q -curvature has been an object of intense study. This is a local curvature invariant. Define $J = \frac{R}{2(n-1)}$ and the Schouten tensor by $P = \frac{1}{n-2}(\text{Ric} - Jg)$. Then Q is defined by $Q = -\Delta_g J - 2|P|_g^2 + \frac{nJ}{2}$; thus, it is fourth-order in derivatives of the metric tensor. We will mention here two of its numerous interesting properties.

First, let the Paneitz operator be the fourth-order operator defined by $P_4 u = \Delta_g^2 + \delta((4P - (n-1))Jg)du + \frac{n-3}{2}Qu$, where δ is the adjoint (with respect to g) of the exterior derivative, and the two-tensor in the second term acts as an endomorphism on one-forms via g^{-1} . Now, this linear elliptic operator is a pointwise conformal invariant in the sense that, if $\tilde{g} = e^{2\omega}g$, then $\tilde{P}_4 u = e^{(-n/2-2)\omega}P_4(e^{(n/2-2)\omega}u)$; this is analogous to the conformal change property of the conformal Laplacian. The first interesting property of the Q -curvature, then, is that under a conformal change, when $n = 4$, the Q -curvature transforms by $\tilde{Q} = e^{-4\omega}(Q + P_4\omega)$, which is exactly analogous to the relationship between the Gaussian curvature and the Laplacian when $n = 2$; while if $n \geq 5$, it transforms by $\tilde{Q} = u^{\frac{n+4}{n-4}}P_4 u$, which is analogous to the relationship (for $n > 2$) between the scalar curvature R and P_2 . Here, $u = e^{(n-4)\omega/2}$, much as in (1). The second interesting property is that, in dimension four, the Chern-Gauss-Bonnet formula can be written in such a way as to include Q :

$$4\pi^2\chi(M^4) = \frac{1}{2} \int_M \left(\frac{1}{4}|W|_g^2 + Q \right) dV_g,$$

where W is the Weyl tensor of g . Now, $|W|_g^2 dV_g$ is an absolute pointwise conformal invariant, so $\int_M Q dV_g$ is a global conformal invariant. Given these properties, it becomes irresistible to see Q playing a rather exact fourth-order analog to the second-order R , at least in conformal geometry.

The question then arises: can one make a conformal transformation $\tilde{g} = e^{2\omega}g$ so that $Q_{\tilde{g}}$ is a constant? In the case $n = 4$, due to the Chern-Gauss-Bonnet formula and the conformal invariance of total Q , this has the flavor of the uniformization theorem. In higher dimension, it is the fourth-order analog of the Yamabe problem. The affirmative answer to the fourth-order uniformization question in dimension four was given in two steps in the papers [2, 4] by Alice Chang and Paul Yang and by Zindine Djadli and Andrea Malchiodi, which are landmarks of fourth-order nonlinear geometric analysis.

Theorem 5. *Suppose (M^4, g) is a Riemannian manifold, and let $k_p = \int_M Q dV_g$. Suppose that $\ker P_4 \cong \mathbb{R}$ and that $k_p \notin 16\pi^2\mathbb{N}$. Then there exists a function $\omega \in C^\infty(M)$ such that $\tilde{g} = e^{2\omega}g$ has constant Q -curvature.*

The work [2] can be seen as a Q -curvature version of the Aubin-Schoen minimization approach. Elaborate additional ideas were needed to approach the problem, using the Adams(-Moser-Trudinger) inequality. The work [4], by contrast, can be seen as a min-max analog of the Bahri-Coron algebraic topological argument, based on an improved version of the Adams inequality.

Just as mean curvature is a natural first-order boundary curvature associated to scalar curvature, in four dimensions there is a natural third-order boundary curvature

associated to Q , the so-called T -curvature defined by Alice Chang and Jie Qing. This curvature had leading part $T = -\frac{1}{12} \frac{\partial}{\partial n} R_g$, where $\frac{\partial}{\partial n}$ is the inward unit normal field. One can define a linear boundary operator $P_3 : C^\infty(M) \rightarrow C^\infty(\partial M)$ with leading part $P_3 u = \frac{1}{2} \frac{\partial}{\partial n} \Delta_g u + \Delta_\partial \frac{\partial}{\partial n} u$ that is conformally invariant in the sense that $\tilde{P}_3 = e^{-3\omega} P_3$, and such that it controls the conformal behavior of T according to the formula $\tilde{T} = e^{-3\omega} (T + P_3 \omega)$. The curvature T appears in the Chern-Gauss-Bonnet formula with Q according to the equation

$$4\pi^2 \chi(M^4) = \frac{1}{2} \int_M \left(\frac{1}{4} |W|_g^2 + Q \right) dV_g + \oint_{\partial M} (\mathcal{L} + T) dV_\partial,$$

where $\mathcal{L}$ is a pointwise conformally invariant curvature quantity. A natural fourth-order generalization of the boundary uniformization problem, in four dimensions, is thus to ask whether we can make a conformal change that will set Q to zero and T to a constant. The answer is generically yes, as shown by Ndiaye [9].

Theorem 6. *Suppose (M^4, g) is a Riemannian manifold with boundary. Let H_g be the mean curvature, and $k_P = \frac{1}{2} \int_M Q dV_g + \int_{\partial M} T dV_\partial$; and suppose that $H_g = 0$, that $k_P \notin 4\pi^2 \mathbb{N}$, and that only constant solutions exist to the system*

$$\begin{cases} P_4 u = 0 & \text{in } M, \\ P_3 u = 0, & \text{on } \partial M, \\ \frac{\partial}{\partial n} u = 0, & \text{on } \partial M. \end{cases}$$

Then there exists ω such that $\tilde{g} = e^{2\omega} g$ has $Q_{\tilde{g}} = 0$, $T_{\tilde{g}} = \text{const}$, and $H_{\tilde{g}} = 0$.

The proof uses the min-max method of Djadli-Malchiodi.

Extending the fourth-order problem to higher dimensions than four required significant new techniques. One of the challenges of studying fourth-order elliptic equations, as opposed to second-order ones, is that there is no maximum principle in the case of the former. It was thus a significant achievement when Matthew Gursky and Malchiodi [6] showed that, under certain circumstances, the Paneitz operator on a closed manifold *does* satisfy a maximum principle:

Theorem 7. *Suppose that (M^n, g) is a closed Riemannian manifold, with $n \geq 5$, and that*

$$\begin{cases} Q_g \geq 0 \text{ with } Q_g > 0 \text{ somewhere; and} \\ R_g \geq 0. \end{cases} \quad (3)$$

If $u \in C^4$ satisfies $P_4 u \geq 0$, then either $u > 0$ or $u \equiv 0$.

Moreover, under these hypotheses, P_4 is a positive operator.

In the same paper, they also proved a fourth-order positive mass theorem for the Green's function of the Paneitz

operator for the global case. Using these remarkable results, they were able to prove a fourth-order Yamabe theorem in higher dimensions.

Theorem 8. *Suppose that (M^n, g) is a closed Riemannian manifold with $n \geq 5$, and that (3) holds. Then there exists $u \in C^\infty(M)$, $u > 0$, such that $h = u^{\frac{4}{n-4}} g$ has constant positive Q -curvature.*

The proof in [6] is by a nonlocal flow. One can also use a variational proof à la Aubin-Schoen using Theorem 7 and the positive mass theorem in [6]; this was pointed out quickly by Emmanuel Hebey and Frédéric Robert. Shortly thereafter, Fengbo Hang and Yang weakened the condition on scalar curvature to nonnegative Yamabe constant. This improvement by Hang-Yang and the observation by Hebey-Robert were made even before final publication of [6]; see the discussion and references in that paper.

Jeffrey Case introduced boundary curvatures analogous to T in dimension higher than 4; the third-order one, he calls T_3 . There is also an associated conformally invariant boundary operator of third order, which we also call B_3 . An argument in [10] enabled the proof of the following fourth-order Escobar theorem in higher dimensions.

Theorem 9. *Suppose that (M^n, g) , $n \geq 5$, is a compact Riemannian manifold with boundary. If $H = 0$ and the Green's function defined by*

$$\begin{cases} P_4 u(x, y) = 0 & \text{in } M \\ B_3 u(x, y) = \delta_x & \text{on } \partial M \\ \frac{\partial}{\partial n} u(x, y) = 0 & \text{on } \partial M \end{cases} \quad (4)$$

is positive, then there exists a conformal metric $\tilde{g} = u^{\frac{4}{n-4}} g$ satisfying $\tilde{Q} = 0$, $\tilde{T}_3 = \text{const}$, and $\tilde{H} = 0$.

The equation (4) is dual to the problem for the fractional curvature $Q_{3/2}$ in the Poincaré-Einstein setting. The argument in [10] works for the case of (4) as well. The technique of proof is similar to that of Mayer-Ndiaye.

Higher Order

The existence of conformally invariant operators and associated curvatures such as P_2, P_4 and R, Q is not an isolated phenomenon. In 1992, Robin Graham, Ralph Jenne, Lionsel Mason, and George Sparling showed that there exist conformally covariant operators P_{2k} of every order $2k \leq \dim M$, having principal part Δ_g^k . Branson defined an associated Q_{2k} curvature so that, in dimension $2k$, the Q_{2k} -curvature transforms by

$$\tilde{Q}_{2k} = e^{-2k\omega} (Q + P_{2k}\omega)$$

(with a somewhat more complicated transformation formula in higher dimensions). Branson's original construction was by analytic continuation in the dimension. The

GJMS operators, meanwhile, were defined via the so-called ambient-metric construction of Charles Fefferman and Graham. This allows the study of a conformal manifold by situating it in a Lorentzian manifold of two dimensions higher, much as the sphere could be studied by situating it in the light cone in Minkowski space of two higher dimensions. Relationships between the two were illuminated when Graham and Maciej Zworski showed that each could be defined in terms of the scattering operator on an asymptotically hyperbolic (formally) Einstein space whose boundary was the given manifold.

The higher-order uniformization problem for Q_{2k} (sometimes called the *critical case*), where $\dim M = 2k$, has been solved by Ndiaye [8] using the min-max argument of [4].

Theorem 10. *Suppose (M^{2k}, g) is a closed Riemannian manifold, that $\ker P_{2k} = \mathbb{R}$, and that $\int_M Q_{2k} dV_g \notin (2k-1)! \omega_{2k} \mathbb{N}$, where ω_{2k} is the volume of the $2k$ -sphere. Then there exists $\omega \in C^\infty(M)$ such that $e^{2\omega} g$ has constant Q_{2k} -curvature.*

As for the lower-order curvatures, in the critical dimension there is again a Chern-Gauss-Bonnet formula, due to Spyros Alexakis, connecting the total Q_{2k} -curvature and the topology.

The higher-order Yamabe problem (sometimes called the noncritical case), where $\dim M > 2k \geq 6$, was solved quite recently by Saikat Mazumdar and Jérôme Vétois in the setting where the local test function argument works, or if the positive mass theorem holds, using the Aubin-Schoen minimization technique. For example, we mention the following:

Theorem 11. *Suppose (M^n, g) is a closed Riemannian manifold, that $1 < k < \frac{n}{2}$, and that the Green's function of P_{2k} is positive. Assume moreover that the $2k$ th Yamabe constant Y_{2k} satisfies*

$$Y_{2k} := \inf_{\tilde{g} \in [g]} \left(\text{vol}_{\tilde{g}}(M)^{-\frac{n-2k}{n}} \int_M Q_{2k, \tilde{g}} dV_{\tilde{g}} \right) > 0.$$

If $2k+1 \leq n \leq 2k+3$ or (M, g) is locally conformally flat, then assume also that the mass of the Green's function of P_{2k} is positive somewhere on M . Then there exists $u \in C^\infty(M)$ such that $u^{\frac{4}{n-2k}} g$ has constant Q_{2k} -curvature.

The mass of the Green's function mentioned here is the constant term in the asymptotic expansion (in especially nice coordinates) of the Green's function; it is so named because of its relationship, in the $k = 1$ case, with the Arnowitt-Deser-Misner mass of general relativity. One expects that the Bahri-Coron argument should work to yield the remaining cases of this theorem, still assuming the Green's function is positive.

Conclusion

The Yamabe problem and its generalizations have been among the driving forces of conformal geometry in the past sixty years. A natural class of problems, they have required the development of new and powerful insights both analytic and geometric for their solution, as well as novel algebraic-topological arguments. Nor are they of interest merely as curiosities: as the ever-growing number of theorems whose hypotheses include some assumption on the Yamabe constant show, once proved they have become one of the most useful tools of the conformal geometer. Leverage in problems can be obtained by making a conformal change so that an appropriate curvature quantity is constant or vanishing.

There remains a large landscape of problems not yet solved, which will require yet new insights; and we have not even discussed all those that have already been attacked. Fully nonlinear Yamabe problems, nonlocal fractional curvatures, higher order boundary curvatures, CR Yamabe questions, extrinsic curvature on corners, and many other problems have seen enormous work and remain at the focus of bustling activity. The source of a whole vast literature, the Yamabe problem stands as a powerful reminder of the power of asking the right question.

References

- [1] Simon Brendle, *A family of curvature flows on surfaces with boundary*, Math. Z. **241** (2002), no. 4, 829–869, DOI 10.1007/s00209-002-0439-1. MR1942242
- [2] Sun-Yung A. Chang and Paul C. Yang, *Extremal metrics of zeta function determinants on 4-manifolds*, Ann. of Math. (2) **142** (1995), no. 1, 171–212, DOI 10.2307/2118613. MR1338677
- [3] X. X. Chen, *Calabi flow in Riemann surfaces revisited: a new point of view*, Internat. Math. Res. Notices **6** (2001), 275–297, DOI 10.1155/S1073792801000149. MR1820328
- [4] Zidine Djadli and Andrea Malchiodi, *Existence of conformal metrics with constant Q -curvature*, Ann. of Math. (2) **168** (2008), no. 3, 813–858, DOI 10.4007/annals.2008.168.813. MR2456884
- [5] José F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. (2) **136** (1992), no. 1, 1–50, DOI 10.2307/2946545. MR1173925
- [6] Matthew J. Gursky and Andrea Malchiodi, *A strong maximum principle for the Paneitz operator and a non-local flow for the Q -curvature*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 9, 2137–2173, DOI 10.4171/JEMS/553. MR3420504
- [7] John M. Lee and Thomas H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91, DOI 10.1090/S0273-0979-1987-15514-5. MR888880
- [8] Cheikh Birahim Ndiaye, *Constant Q -curvature metrics in arbitrary dimension*, J. Funct. Anal. **251** (2007), no. 1, 1–58, DOI 10.1016/j.jfa.2007.06.008. MR2353700

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- [9] Cheikh Birahim Ndiaye, *Constant T -curvature conformal metrics on 4-manifolds with boundary*, Pacific J. Math. **240** (2009), no. 1, 151–184, DOI 10.2140/pjm.2009.240.151. MR2485478
- [10] Cheikh Birahim Ndiaye, Yannick Sire, and Liming Sun, *Uniformization theorems: between Yamabe and Paneitz*, Pacific J. Math. **314** (2021), no. 1, 115–159, DOI 10.2140/pjm.2021.314.115. MR4329973



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