



A Toda Bracket Convergence Theorem for Multiplicative Spectral Sequences

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Abstract

Moss' theorem, which relates Massey products in the E_r -page of the classical Adams spectral sequence to Toda brackets of homotopy groups, is one of the main tools for calculating Adams differentials. Working in an arbitrary symmetric monoidal stable simplicial model category, we prove a general version of Moss' theorem which applies to spectral sequences that arise from filtrations compatible with the monoidal structure. This involves the study of Massey products and Toda brackets in a non-strictly associative context. The theorem has broad applications, e.g., to the computation of the motivic slice spectral sequence and other colocalization towers.

Keywords Toda bracket · Massey product · Spectral sequence · Motivic homotopy theory · Slice spectral sequence

Mathematics Subject Classification Primary 55T05 · 55S20; Secondary 55S30 · 14F42

1 Introduction

In homological algebra, Massey products [16] are operations that encode higher multiplicative structure on a differential graded algebra. Analogously in homotopy theory, Toda brackets (e.g., [14, 26]) encode higher multiplicative structure on homotopy classes of maps, for example on the homotopy groups of a ring spectrum.

In a nice enough spectral sequence computing homotopy groups of a ring spectrum, the multiplicative structure on E_r -pages of the spectral sequence agrees with the

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multiplicative structure on the homotopy groups. A theorem by Moss [20, Theorem 1.2] states that in the Adams spectral sequence, there is a correspondence of higher multiplicative structures as well: given a Massey product of permanent cycles in the E_r -page, such that the corresponding Toda bracket is defined, and assuming a “crossing differentials” hypothesis in the relevant degrees, the Massey product contains a permanent cycle that converges to an element of the Toda bracket. This correspondence of Toda brackets and Massey products is used heavily in determining the differentials of Adams spectral sequences; for modern work with classical references, see [11, 12].

We generalize Moss’ theorem in two ways. While Moss’ proof only applies to the Adams spectral sequence, our theorem holds for a wide range of spectral sequences, with the motivic effective slice spectral sequence as a motivating example. Furthermore, while the Adams spectral sequence is strictly multiplicative, our theorem applies in many cases where the filtered object associated with the spectral sequence is only homotopy associative. This adds generality even when the ring R of interest is not strictly associative, as strict associativity of R does not imply strict associativity of a given filtration R_* of R . We defer the discussion of hypotheses on the spectral sequence to Sect. 1.2 and state the main theorems next.

Although Theorem 1.1 discusses Toda brackets $\langle \alpha'', \alpha', \alpha \rangle$ of classes $\alpha'', \alpha', \alpha \in \pi_*(R)$ for a ring spectrum R , the generality used in the text also includes the case $\alpha'', \alpha' \in \pi_*(R)$ and $\alpha \in \pi_*(M)$ for an R -module M .

Theorem 1.1 (Theorem 4.11) *Let $R = R_0 \overset{i}{\leftarrow} R_1 \overset{i}{\leftarrow} R_2 \leftarrow \dots$ be a filtered object with a multiplication μ satisfying the setup in Sect. 1.2, and consider the spectral sequence $\{E_r(R)\}_r$ associated with this filtered object. Fix $r \geq 2$. Suppose $\mathfrak{a}, \alpha', \alpha'' \in E_r(R)$ are permanent cycles converging to elements $\alpha, \alpha', \alpha'' \in \pi_*(R)$ such that $\alpha' \mathfrak{a}$ and $\alpha'' \alpha'$ are 0 in the E_r -page, and $\mu(\alpha', \alpha)$ and $\mu(\alpha'', \alpha')$ are null. Suppose the spectral sequence is weakly convergent (see Definition 2.3) and the crossing differentials hypothesis (see Definition 2.4) is satisfied in degrees $(r-1, |\alpha' \mathfrak{a}|)$ and $(r-1, |\alpha'' \alpha'|)$. Then, the Massey product $\langle \alpha'', \alpha', \mathfrak{a} \rangle \subseteq E_r(R)$ contains a permanent cycle that converges to an element of the Toda bracket $\langle \alpha'', \alpha', \alpha \rangle \subseteq \pi_*(R)$.*

If $r \geq 2$, then $E_r(R)$ is the homology of the differential graded algebra $(E_{r-1}(R), d_{r-1})$. In the $r = 1$ case, we consider Toda brackets on $E_1(R)$ by regarding them as the homotopy groups of the associated graded objects $R_{*,1}$ of the filtration on R .

Theorem 1.2 (Theorem 4.16) *Assume the setup of Theorem 1.1 with $r = 1$. Then, the Toda bracket $\langle \alpha'', \alpha', \mathfrak{a} \rangle \subseteq \pi_*(R_{*,1}) = E_1(R)$ contains a permanent cycle that converges to an element of the Toda bracket $\langle \alpha'', \alpha', \alpha \rangle \subseteq \pi_*(R)$.*

These theorems hold for the motivic effective slice spectral sequence, as well as spectral sequences arising from more general colocalization towers (see Sect. 5.2). In Sect. 5.3, we give examples of \mathbb{R} -motivic slice spectral sequence computations using both theorems. Our theorems also hold for the Adams spectral sequence constructed in a general context (see Example 5.1).

1.1 Toda Brackets in a General Setting

This work is about multiplicative Toda brackets associated with a triple (X'', X', X) where X' is an X'' -algebra and X is an X' -module. Closely related is the *composition Toda bracket*, associated with a diagram $Y \rightarrow X'' \rightarrow X' \rightarrow X$ where consecutive compositions are nullhomotopic. The construction of both types of brackets in the stable homotopy category is classical; see [13] for references and a comparison. The literature is less straightforward about the existence of Toda brackets in a symmetric monoidal stable model category, though such a construction is commonly assumed in practice. Christensen and Frankland [2] give several general constructions of composition Toda brackets, along with a comparison.

Our version of the definition of a multiplicative Toda bracket (Definition 3.7) has an extra complication, because we do not require strict associativity of the system (X'', X', X) . In addition to providing extra generality, this is also natural to consider: even in the case that (X'', X', X) is strictly associative, the filtered system (X''_*, X'_*, X_*) may not be strictly associative. Instead, in our motivating application (the motivic slice spectral sequence), the framework given by [8] implies a structure on the filtered object that is more like an A_∞ structure; see Sect. 5.2 for details. Since the proof of the main theorem uses Toda bracket-like constructions on the filtered object (see Sect. 4.1), we consider Toda brackets in a homotopy associative setting. No real complexity is then added in passing to the setting where the original objects (X'', X', X) are also assumed to be only homotopy associative. Instead, we find that this clarifies the setup as well as being of independent interest.

1.2 Summary of Setup and Assumptions

For simplicity, we state the case where $X'' = X' = X$ is a ring object R ; see Sect. 3.5 for the most general setup we consider, covering ring actions on modules as well. The assumptions below were designed to fit the known structure of the motivic slice spectral sequence as described in [8], specifically the precise associativity properties of the filtered object R_* . Since [8] studies general colocalization towers, this is also a useful framework for other spectral sequences arising from such towers.

Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal stable simplicial model category with cofibrant unit S and consider an A_3 -ring R ; this is a homotopy associative ring object (R, μ) with a choice of homotopy $\mu(\mu(-, -), -) \simeq \mu(-, \mu(-, -))$. Suppose there is a filtration $R = R_0 \xleftarrow{i} R_1 \xleftarrow{i} R_2 \xleftarrow{i} \dots$ and let $R_{s,r} = \text{hocolim}(R_{s+r} \rightarrow R_s)$. Consider the spectral sequence

$$E_1(R) = \pi_*(R_{f,1}) \implies \pi_*(R),$$

where $\pi_*(-)$ denotes $\pi_*(\text{Map}_{\mathcal{C}}(S, -))$. Assume that there are pairings $\mu_{s,t} : R_s \otimes R_t \rightarrow R_{s+t}$ and $\mu_{s,t,1} : R_{s,1} \otimes R_{t,1} \rightarrow R_{s+t,1}$ for $s, t \geq 0$. We make the following assumptions:

1. The pairings make $\iota : R_* \rightarrow R$ a map of A_3 -rings, and the natural map $p : R_* \rightarrow R_{*,1}$ commutes with the pairings. That is, there are strictly commutative diagrams

$$\begin{array}{ccc}
 R_s \otimes R_t & \xrightarrow{\mu_{s,t}} & R_{s+t} < \\
 \downarrow \iota & & \downarrow \iota & \\
 R \otimes R & \xrightarrow{\mu} & R &
 \end{array}
 \qquad
 \begin{array}{ccc}
 R_s \otimes R_t & \xrightarrow{\mu_{s,t}} & R_{s+t} \\
 \downarrow p & & \downarrow p \\
 R_{s,1} \otimes R_{t,1} & \xrightarrow{\mu_{s,t,1}} & R_{s+t,1}
 \end{array}$$

and the associativity homotopies for R_* and R are strictly compatible.

2. A condition that is less strict than asking

$$\begin{array}{ccc}
 R_s \otimes R_t & \xrightarrow{\mu_{s,t}} & R_{s+t} \\
 \downarrow i & & \downarrow i \\
 R_s \otimes R_{t-1} & \xrightarrow{\mu_{s,t-1}} & R_{t-1}
 \end{array}
 \qquad
 \begin{array}{ccc}
 R_s \otimes R_t & \xrightarrow{\mu_{s,t}} & R_{s+t} \\
 \downarrow i & & \downarrow i \\
 R_{s-1} \otimes R_t & \xrightarrow{\mu_{s-1,t}} & R_{t-1}
 \end{array}$$

to commute, but stricter than asking these diagrams to commute up to homotopy; see Assumption 3.14 (2) and (3).

3. The pairing $\mu_{*,1}$ induces a pairing of spectral sequence E_r -pages.

Some other results on the multiplicativity of spectral sequences (e.g., [3, 9]) are formulated in terms of Cartan–Eilenberg systems, which makes assumptions only on the system of quotients $\{R_{s,r}\}$. Our setting also includes assumptions on the multiplicativity of the filtration R_* itself. Our setting is more general than the setting of monoids in filtered spectra; see Remark 3.15.

The idea behind the proof of Moss’ theorem (Theorems 1.1 and 1.2) is to lift $a, a', a'' \in E_r(R)$ to $\alpha_*, \alpha'_*, \alpha''_*$ in $\pi_*(R_*)$ and show that there is a comparison of brackets

$$\langle a'', a', a \rangle \xleftarrow{p} \langle \alpha''_*, i^r \alpha'_*, \alpha_* \rangle \xrightarrow{\iota} \langle \alpha'', \alpha', \alpha \rangle,$$

using the crossing differentials hypothesis (see Definition 2.4, Assumption 3.17) to ensure that the middle term is defined. In practice, the crossing differentials hypothesis is satisfied in many cases.

This strategy essentially works for Theorem 1.2, but for Theorem 1.1, we need to replace the element of $\langle \alpha''_*, i^r \alpha'_*, \alpha_* \rangle$ with a more complicated element, corresponding to the extra technical conditions in Assumption 3.14 (2) and (3).

1.3 Outline

In Sect. 2, we introduce notation for the spectral sequence associated with a tower, including a review of the crossing differentials criterion. In Sect. 3, we give notation for the setup of the main theorem, namely a pairing of spectral sequences satisfying certain properties along with certain elements that will form the relevant Toda brackets and Massey products. We define threefold Toda brackets and Massey products in our general setting. In Sect. 4, we prove our versions of Moss’ theorem (Theorems 4.11 and 4.16). In Sect. 5, we show that the assumptions in our setup apply to colocalization

towers, and give some example applications of these theorems to specific computations in the \mathbb{R} -motivic slice spectral sequence.

The reader who wishes to jump to the precise statements of the theorems should read Assumption 3.14, the notation at the beginning of Sect. 3.5.2, and Assumption 3.17, and then skip to Sect. 4.

2 The Spectral Sequence Associated with a Tower

2.1 The Spectral Sequence for a Tower of Homotopy Cofiber Sequences

We work in a stable symmetric monoidal simplicial model category $(\mathcal{C}, \otimes, S)$ with cofibrant unit. Suppose we have a tower

$$\begin{array}{ccccccc}
 X = X_0 & \xleftarrow{i} & X_1 & \xleftarrow{\dots} & X_f & \xleftarrow{i} & X_{f+1} & \xleftarrow{i} & \dots & (2.1) \\
 \downarrow p & \nearrow \kappa & & & \downarrow p & \nearrow \kappa & & & & \\
 X_{0,1} & & & & X_{f,1} & & & & &
 \end{array}$$

in \mathcal{C} , where $X_{f,r}$ for $r \geq 1$ is defined to be the homotopy cofiber

$$X_{f,r} := X_f \coprod_{X_{f+r}} CX_{f+r}.$$

We abuse notation, so i refers to any of the maps $X_{f+1} \rightarrow X_f$ or any of their iterates.

Applying $\pi_*(-)$, we have a spectral sequence

$$E_1^{n,f}(X) = \pi_n(X_f/X_{f+1}) \Rightarrow \pi_n(X).$$

We will use the bigrading convention (n, f) where n is the stem and f is the filtration. It will be convenient for us to have a way to view cycles and boundaries as subsets of E_1 for $r \geq 2$, let

$$\begin{aligned}
 Z_r^{*,f} &= \{a \in \pi_*(X_{f,1}) : \kappa(a) \in \pi_{*-1}(X_{f+1}) \text{ lifts to } \pi_{*-1}(X_{f+r})\}, \text{ and} \\
 B_r^{*,f} &= \begin{cases} \{a \in \pi_*(X_{f,1}) : a \text{ lifts to some } \alpha \in \ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-r+1}))\}, & f \geq r - 1, \\ 0, & f < r - 1. \end{cases}
 \end{aligned}$$

Say that an element of $Z_r^{*,*}$ is a d_{r-1} -cycle, and say that an element of $B_r^{*,*}$ is a d_{r-1} -boundary. Note that there is an inclusion $B_r^{*,f} \subseteq Z_r^{*,f}$, since $\kappa(a) = 0$ for $a \in B_r^{*,f}$. Let

$$E_r^{*,*} = Z_r^{*,*}/B_r^{*,*}.$$

Given $x \in Z_r^{*,*}$, let $[x]_r$ denote its image in $E_r^{*,*}$. Then, there is a map $d_r : E_r^{n,f} \rightarrow E_r^{n-1,f+r}$ defined as follows: given $a \in Z_r^{n,f}$, let $x \in \pi_{n-1}(X_{f+r})$ be a lift of $\kappa(a)$, and define $d_r(a) = px$. (To see that this is well defined, observe that the difference of two lifts is in $\ker(\pi_*(X_{f+r}) \rightarrow \pi_*(X_{f+1}))$.)

Remark 2.2 This is not the usual definition of cycles and boundaries, but we claim that the E_r -page as defined above is isomorphic to the usual E_r -page, defined as a subquotient of E_1 . A cycle in the usual sense is an equivalence class of elements $a \in E_1(X)$ in the kernel of d_i for all $i \leq r - 1$, modulo d_i -boundaries for $i < r - 1$. Our $Z_r^{*,*}$ consists of all such elements a , since $\kappa(a)$ lifts to $\pi_{*-1}(X_{f+r})$ if and only if its image in $\pi_{*-1}(X_{f+r-1,1})$ is zero. Our $B_r^{*,*}$ consists of all d_i -boundaries for $i \leq r - 1$: a differential $d_i(b) = a$ means that there is a class $\tilde{a} \in \pi_*(X_f)$ that projects to $a \in \pi_*(X_{f,1})$ and whose image in $\pi_*(X_{f-i+1})$ equals $\kappa(b)$; hence, the image of \tilde{a} in $\pi_*(X_{f-i})$ is zero.

Definition 2.3 (see [18, Definition 3.8]) Let

$$R_f = \bigcap_{r \geq 0} \Im(\pi_*(X_{f+r}) \rightarrow \pi_*(X_f)).$$

Say that a spectral sequence $\{E_*(X)\}$ is *weakly convergent* if the natural map $R_{f+1} \rightarrow R_f$ is an injection for all f .

2.2 The Crossing Differentials Condition

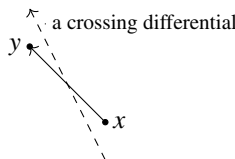
If $a \in E_r^{n,f}$ is a d_r -boundary, then, by definition, it has a lift to $\pi_*(X_f)$ (i.e., $\alpha \in \pi_n(X_f)$ with $p\alpha = a$) that is in $\ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-r}))$. Proposition 2.6 says that every lift is in the kernel, as long as we assume a criterion called the crossing differentials hypothesis.

Definition 2.4 Say that a spectral sequence $\{E_*(X)\}$ *satisfies the crossing differentials hypothesis for (r, n, f)* if

$$\text{every element in } E_{f-m+1}^{n+1,m}(X) \text{ is a permanent cycle for } 0 \leq m \leq f - r - 1. \quad (2.5)$$

If an element y is in stem n and filtration f , write $(r, |y|) := (r, n, f)$.

In the context of a differential $d_r(x) = y$ for $x \in E_r^{n+1,f-r}(X)$, $y \in E_r^{n,f}(X)$, the crossing differentials hypothesis is satisfied in degree $(r, |y|)$ if there are no differentials from stem $n + 1$ to stem n with source filtration $< f - r$ and target filtration $> f$.



Proposition 2.6 [20, Proposition 6.3] *Assume that $\{E_*(X)\}$ is weakly convergent. Suppose $a \in \pi_*(X_{f,1})$ is a d_r -boundary, and suppose $\alpha \in \pi_*(X_f)$ is any lift of a , such that α is in $\ker(\pi_*(X_f) \rightarrow \pi_*(X))$. If the crossing differentials hypothesis is satisfied in degree $(r, |a|)$, then α is in $\ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-r}))$, i.e., the composite $S \xrightarrow{\alpha} X_f \xrightarrow{i} X_{f-r}$ is nullhomotopic.*

Proof Let $b \in \pi_{*+1}(X_{f-r,1})$ be the source of the differential that hits a . Thus, we have a lift $\beta \in \pi_*(X_f)$ of $\kappa(b)$. If $\beta = \alpha$, the result follows from the long exact sequence. Otherwise, denote the nonzero element $\alpha - \beta \in \pi_*(X_f)$ by γ . Since $\alpha \in \ker(\pi_*(X_f) \rightarrow \pi_*(X))$, there is a minimal k for which $\alpha \in \ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-k}))$. Assume for the sake of contradiction that $k > r$. Since $\beta \in \ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-r}))$, we have $\gamma \in \ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-k}))$ and $\gamma \notin \ker(\pi_*(X_f) \xrightarrow{i} \pi_*(X_{f-k+1}))$. Therefore, there exists $c \in \pi_*(X_{f-k,1})$, such that $\kappa(c) = i \circ \gamma$.

By the weak convergence criterion (Definition 2.3), there is a maximal n , such that $i(\gamma) \in \pi_*(X_{f-k+1})$ lifts to $\gamma' \in \pi_*(X_{f+n})$. Since $\gamma = \alpha - \beta \in \ker(\pi_*(X_f) \rightarrow \pi_*(X_{f,1}))$, we have $n \geq 1$. Then, $0 \neq p\gamma' \in \pi_*(X_{f+n,1})$, and we conclude that $d_{k+n}(c) = p\gamma'$. This contradicts the crossing differentials hypothesis, concluding the proof, as long as the target is nonzero in the E_{k+n} -term.

If $p\gamma'$ were a d_i -boundary for $i \leq k + n - 1$, then it would have a lift $\gamma'' \in \ker(\pi_*(X_{f+n}) \rightarrow \pi_*(X_{f-k+1}))$. Then, $\gamma' - \gamma''$ lifts to $\delta \in \pi_*(X_{f+n+1})$ whose image in $\pi_*(X_{f-k+1})$ equals $i(\gamma') = i(\gamma)$, contradicting the definition of n . \square

Proposition 2.6 is used in Lemma 3.18, where we have a specified lift (given as a product) to $\pi_*(X_f)$ of a certain class in $\pi_*(X_{f,1})$; Proposition 2.6 guarantees that the chosen lift becomes zero in $\pi_*(X_{f-r})$, whereas a priori we only know that there exists a lift with this property.

2.3 Gluing Nullhomotopies

We assume functorial cones, i.e., if $f : X \rightarrow Y$ is a map, then there is a natural map $f : CX \rightarrow CY$. We assume that S is cofibrant, which implies a familiar model for CS , but not any particular model for CX for general X .

Notation 2.7 Since \mathcal{C} is a simplicial model category, a model for CS is $(S \times \Delta^1)/(S \times \{0\})$, which we write as $(S \times I)/(S \times \{0\})$ (where I represents the unit interval). Given a homotopy $F : S \times I \rightarrow X$, let \bar{F} denote the homotopy with reversed orientation $\bar{F}(x, t) = F(x, 1 - t)$. We use the convention that ΣX denotes the Kan suspension CX/X . Let

$$S^\diamond = (S \times I)/(S \times \{0\}) \coprod_S (S \times I)/(S \times \{1\}).$$

This is the mapping cone of the inclusion $S \rightarrow CS$, and $S^\diamond \simeq \Sigma S$.

In a model category, a homotopy between $f, g : X \rightarrow Y$ is a map $\text{Cyl}(X) \rightarrow Y$ satisfying an appropriate diagram, and if X is cofibrant in a simplicial model category,

we may take $X \otimes I$ as the cylinder object $\text{Cyl}(X)$ (see, e.g., [7, Lemma II.3.5]). We will refer to a map $X \otimes I \rightarrow Y$ as a homotopy, which is justified by the fact that we are ultimately concerned with the composition

$$S \otimes I \rightarrow X \otimes I \rightarrow Y$$

which is a homotopy, due the assumption that S is cofibrant.

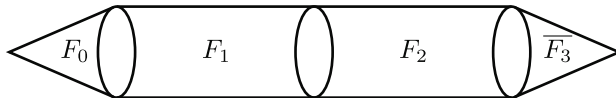
Definition 2.8 Suppose we have maps $F_0, F_n : CS \rightarrow X$ and $F_i : S \times I \rightarrow X$ for $1 \leq i \leq n - 1$, such that

$$F_i|_{S \times \{0\}} = F_{i-1}|_{S \times \{1\}}$$

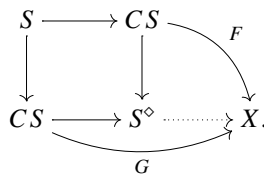
for $1 \leq i \leq n - 1$ and $F_n|_{S \times \{1\}} = F_{n-1}|_{S \times \{1\}}$. Define $d(F_0, \dots, F_n) : \Sigma S \rightarrow X$ to be

$$\Sigma S \cong (S \times I)/(S \times \{0\}) \coprod_S (S \times I) \coprod_S \dots \coprod_S (S \times I)/(S \times \{1\}) \xrightarrow{(F_0, \dots, \bar{F}_n)} X.$$

We present the following figure to illustrate the definition.



Example 2.9 Given nullhomotopies $F, G : CS \rightarrow X$, the induced map out of the pushout S^\diamond in the following diagram is $d(F, G)$:



Lemma 2.10 Given the setup of Definition 2.8

$$d(F_0, \dots, F_n) = -d(F_n, F_{n-1}^-, \dots, \bar{F}_1, F_0).$$

The reason that the endpoints do not need their orientation reversed is the convention that $d(F_0, \dots, F_n)$ refers to the map with components (F_0, \dots, \bar{F}_n) .

Proof To simplify the notation, we will show that $S \times [0, 2] \xrightarrow{(F, \bar{F})} X$ is nullhomotopic for any $F : CS \times I \rightarrow X$. The full statement is proved the same way. Define $H :$

$S \times [0, 2] \times [0, 1] \rightarrow X$ by

$$H(x, s, t) = \begin{cases} F(x, 0) & \text{if } s \leq t, \\ F(x, s - t) & \text{if } t < s \leq 1, \\ F(x, 2 - s - t) & \text{if } 1 < s < 2 - t, \\ F(x, 0) & \text{if } s \geq 2 - t. \end{cases}$$

This is a homotopy from (F, \bar{F}) to the constant map at $F(x, 0)$. □

Lemma 2.11 *Given F_0, \dots, F_n as in Definition 2.8 and $g : X \rightarrow Y$, we have*

$$g \circ d(F_0, \dots, F_n) = d(g \circ F_0, \dots, g \circ F_n).$$

Proof This is by definition of the $d(-)$ construction. □

Using this notation, we obtain an expression for the source of a d_r differential with given target.

Lemma 2.12 *Suppose $\mathbf{a} \in \pi_*(X_{f,1})$ is a d_r -boundary and $\alpha \in \pi_*(X_f)$ is a lift in $\ker(\pi_*(X_f) \rightarrow \pi_*(X_{f-r}))$. Let $F : CS \rightarrow X_{f-r}$ be any nullhomotopy making the diagram commute*

$$\begin{array}{ccccc} S & \xrightarrow{\alpha} & X_f & \xrightarrow{i} & X_{f-r+1} \\ \downarrow & & \downarrow & & \downarrow \\ CS & \xrightarrow{F} & X_{f-r} & \xlongequal{\quad} & X_{f-r} \\ \downarrow & & & & \downarrow p \\ S^\diamond & \xrightarrow{\beta} & & & X_{f-r,1}, \end{array} \tag{2.13}$$

and let $\beta = d(pF, jC\alpha)$ where $j : CX_f \rightarrow CX_{f-r+1} \rightarrow X_{f-r,1} = X_{f-r} \coprod_{X_{f-r+1}} CX_{f-r+1}$ is the natural map. Then

$$d_r([\beta]_r) = [\mathbf{a}]_r.$$

Recall that we abbreviate iterates of i by just i .

Proof From the diagram

$$\begin{array}{ccccccc} X_{f-r} & \longleftarrow & X_{f-r+1} & \longleftarrow & \cdots & \longleftarrow & X_f \\ \downarrow & & \nearrow \kappa & & & & \downarrow \\ X_{f-r,1} & & & & & & X_{f,1}, \end{array}$$

we see that we need to check that $\kappa \circ \beta = i \circ \alpha$ as elements of $\pi_*(X_{f-r+1})$. By construction, β is the induced map from the homotopy cofiber of $S \rightarrow CS$ to the

homotopy cofiber of $X_{f-r+1} \rightarrow X_{f-r}$. Therefore, the two vertical cofiber sequences in (2.13) can be extended as follows:

$$\begin{array}{ccc}
 S^\diamond & \xrightarrow{\beta} & X_{f-r,1} = X_{f-r} \coprod_{X_{f-r+1}} CX_{f-r+1} \\
 \cong \downarrow & & \downarrow \kappa \\
 \Sigma S & \xrightarrow{i \circ \alpha} & \Sigma X_{f-r+1}.
 \end{array}$$

□

3 Toda Bracket and Massey Product Setup

Let $(\mathcal{C}, \otimes, S)$ be a symmetric monoidal stable simplicial model category with cofibrant unit.

3.1 Pairings

Our eventual goal is to discuss higher multiplicative structures; we start by giving notation for the ordinary multiplication of homotopy classes of maps.

Definition 3.1 Given a pairing $\mu : X' \otimes X \rightarrow X$ and classes $\alpha : \Sigma^n S \rightarrow X$ and $\alpha' : \Sigma^m S \rightarrow X'$, the product $\mu(\alpha', \alpha)$ is defined as the composition $\Sigma^{m+n} S \cong \Sigma^m S \otimes \Sigma^n S \xrightarrow{\alpha' \otimes \alpha} X' \otimes X \xrightarrow{\mu} X$, where the first isomorphism is the unit isomorphism from the symmetric monoidal structure, which commutes with suspension.

In the rest of the paper, we will make the simplifying assumption that all maps have degree zero, i.e., have source S . The result is still true in the general case; see Remark 4.12 for details about adjustments needed to the text.

Notation 3.2 We abbreviate the composition $X' \cong X' \otimes S \xrightarrow{1 \wedge \alpha} X' \otimes X \xrightarrow{\mu} X$ as $X' \xrightarrow{\alpha} X$.

Remark 3.3 Given a pairing $\mu : X' \otimes X \rightarrow X$ and classes $\alpha : S \rightarrow X$ and $\alpha' : S \rightarrow X'$, the pairing $S \xrightarrow{\alpha' \otimes \alpha} X' \otimes X \xrightarrow{\mu} X$ can be factored using the unit axiom in two different ways

$$\begin{array}{c}
 S \xrightarrow{\alpha'} \underbrace{X' \otimes S \rightarrow X' \rightarrow X' \otimes S}_{\text{Id}} \xrightarrow{\alpha} X' \otimes X \xrightarrow{\mu} X, \\
 S \xrightarrow{\alpha} \underbrace{S \otimes X \rightarrow X \rightarrow S \otimes X}_{\text{Id}} \xrightarrow{\alpha'} X' \otimes X \xrightarrow{\mu} X,
 \end{array}$$

and so, we can regard $\mu(\alpha', \alpha)$ as a map $S \xrightarrow{\alpha'} X' \xrightarrow{\alpha} X$, or equivalently $S \xrightarrow{\alpha} X \xrightarrow{\alpha'} X$. (This is not a commutativity statement.)

If $F : CS \rightarrow X'$ and $G : CS \rightarrow X$ are nullhomotopies, we define $\mu(F, \alpha)$ and $\mu(\alpha', G)$ as the compositions

$$\begin{aligned} \mu(F, \alpha) &: CS \xrightarrow{F} X' \xrightarrow{\alpha} X, \\ \mu(\alpha', G) &: CS \xrightarrow{G} X \xrightarrow{\alpha'} X. \end{aligned}$$

Given a pairing $E_r(X') \otimes E_r(X) \rightarrow E_r(X)$, we define the product of classes $\alpha' \in E_r(X')$ and $\alpha \in E_r(X)$ to be the image along the pairing.

3.2 A_3 -Structures on a Triple

The motivation is to study ring objects in \mathcal{C} with homotopy associative filtrations. To package the necessary compatibilities and introduce the generality of module actions in addition to ring multiplications, we use the language of operads. More precisely, let \mathcal{K} denote the Stasheff associahedron operad¹ and recall that an A_n -algebra is a partial action of \mathcal{K} ignoring the structure maps involving $\mathcal{K}(m)$ for $m > n$ (see, e.g., [24]). Since $\mathcal{K}(1) = \mathcal{K}(2) = *$ and $\mathcal{K}(3) = I$, an A_3 -algebra in \mathcal{C} is simply a homotopy associative ring with a choice of homotopy $\mu(\mu(-, -), -) \simeq \mu(-, \mu(-, -))$. Given an A_3 -ring R , the notion of an A_3 -module M over R can be defined in terms of colored operads (see [8, §A.1.1]), but this boils down to the data of a map $\mu : R \otimes M \rightarrow M$ and the choice of a homotopy between $(R \otimes R) \otimes M \rightarrow M$ and $R \otimes (R \otimes M) \rightarrow M$. We work in somewhat more generality.

Definition 3.4 Given objects X'', X' , and X in \mathcal{C} , an A_3 -structure on the triple (X'', X', X) consists of the following pieces of structure:

- a map $\mu : X' \otimes X \rightarrow X$,
- a map $\mu' : X'' \otimes X' \rightarrow X'$,
- a map $\mu'' : X'' \otimes X \rightarrow X$, and
- a map $h : X'' \otimes X' \otimes X \otimes I \rightarrow X$ whose restrictions to $X'' \otimes X' \otimes X \otimes \{0\}$ and $X'' \otimes X' \otimes X \otimes \{1\}$ are $\mu(\mu'(-, -), -)$ and $\mu''(-, \mu(-, -))$, respectively.

A map $(X'', X', X) \rightarrow (Y'', Y', Y)$ of A_3 -triples consists of maps $X'' \rightarrow Y''$, $X' \rightarrow Y'$, and $X \rightarrow Y$, such that the obvious diagrams commute strictly.

To avoid notational clutter, we will use the same symbol μ to denote μ, μ' , and μ'' . We also write (μ, h) to abbreviate the full tuple of structure data (μ, μ', μ'', h) .

Remark 3.5 In a model category, a homotopy between two maps $X'' \otimes X' \otimes X \rightarrow X$ is a map out of $\text{Cyl}(X'' \otimes X' \otimes X)$, but this is not a problem for us as we will eventually precompose with a map out of the unit object S , and cofibrancy of the unit implies that $\text{Cyl}(S) = S \otimes I$.

Example 3.6 In the case $X = X' = X''$, an A_3 -ring structure on X gives rise to an A_3 -structure on (X, X, X) , and a map $X \rightarrow Y$ of A_3 -rings gives rise to a morphism of

¹ In this paper, all operads are unreduced, as we are only focused on associativity, not unitality.

A_3 -structures $(X, X, X) \rightarrow (Y, Y, Y)$. More generally, if $X'' \rightarrow X'$ is a map of A_3 -rings and X is an A_3 -module over X' , then this endows (X'', X', X) with the structure of an A_3 -triple.

3.3 Toda Brackets

We define the multiplicative Toda bracket $\langle \alpha, \alpha', \alpha'' \rangle$ following [14, Definitions 2.2.1 and 2.2.2], modified to allow for the possibility that the multiplications are not strictly associative. Let CX denote the homotopy cofiber of $X \xrightarrow{\text{Id}} X$.

Definition 3.7 Suppose (μ, h) is an A_3 -structure on the triple (X'', X', X) and let $\alpha'' : S \rightarrow X'', \alpha' : S \rightarrow X',$ and $\alpha : S \rightarrow X$ be maps, such that $\mu(\alpha', \alpha)$ and $\mu(\alpha'', \alpha')$ are nullhomotopic. Let H be a homotopy from $\mu(\mu(\alpha'', \alpha'), \alpha)$ to $\mu(\alpha'', \mu(\alpha', \alpha))$ defined by

$$H : S \otimes I \xrightarrow{\alpha'' \otimes \alpha' \otimes \alpha} X'' \otimes X' \otimes X \otimes I \xrightarrow{h} X.$$

The *Toda bracket* $\langle \alpha'', \alpha', \alpha \rangle$ is the set of homotopy classes of maps

$$d(\mu(F, \alpha), H, \mu(\alpha'', G)),$$

where $F : CS \rightarrow X'$ is any nullhomotopy of $\mu(\alpha'', \alpha')$ and $G : CS \rightarrow X$ is any nullhomotopy of $\mu(\alpha', \alpha)$. (See Remark 3.3 for multiplication of nullhomotopies.)

It is helpful to represent this using the following diagram:

$$\begin{array}{ccccc}
 CS & \xrightarrow{F} & X' & \xrightarrow{1 \otimes \alpha} & X' \otimes X \xrightarrow{\mu} X \\
 \uparrow & & \downarrow & & \parallel \\
 S \otimes \{0\} & \xrightarrow{\mu(\alpha'', \alpha')} & X' & \xrightarrow{1 \otimes \alpha} & X' \otimes X \xrightarrow{\mu} X \\
 \downarrow & & \downarrow & & \parallel \\
 S \otimes I & \xrightarrow{\alpha'' \otimes \alpha' \otimes \alpha} & (X'' \otimes X' \otimes X) \otimes I & \xrightarrow{h} & X \\
 \uparrow & & \downarrow & & \parallel \\
 S \otimes \{1\} & \xrightarrow{\mu(\alpha', \alpha)} & X & \xrightarrow{\alpha'' \otimes 1} & X'' \otimes X \xrightarrow{\mu} X \\
 \downarrow & & \downarrow & & \parallel \\
 CS & \xrightarrow{G} & X & & X
 \end{array} \tag{3.8}$$

Remark 3.9 If (X'', X', X) is strictly associative, we take h to be the trivial homotopy, in which case H is trivial and this reduces to the usual definition of $\langle \alpha'', \alpha', \alpha \rangle$ as the set of maps

$$d(\mu(F, \alpha), \mu(\alpha'', G)),$$

where F, G are nullhomotopies as in Definition 3.7.

Remark 3.10 In Definition 3.7, if two homotopies $H, H' : S \otimes I \rightarrow X$ are homotopic relative to endpoints, then every element in the Toda bracket defined using H is homotopic to an element in the Toda bracket defined using H' .

3.4 Massey Products

Definition 3.11 Given d_r -cycles $a \in E_r(X), a' \in E_r(X'), a'' \in E_r(X'')$ such that $a'a$ and $a''a'$ are boundaries, define the Massey product

$$\langle a'', a', a \rangle = \{b'a - a''b : d_r(b') = a''a' \text{ and } d_r(b) = a'a\}.$$

Remark 3.12 The reader may notice two subtle differences from the usual definition (e.g., see [16]) of the Massey product of classes in the E_{r+1} -page. First, we have defined $a, a',$ and a'' as cycles in the E_r -page. If the pairing on E_r satisfies the Leibniz rule, then our Massey product clearly represents a class in E_{r+1} . Our results are only expected to be useful in cases where the Leibniz rule holds, but we do not need to make this assumption for the proofs. Second, for the missing signs, recall that we are simplifying notation by assuming that all classes have degree zero.

3.5 Setup

We now state our assumptions on the multiplicative structure of the filtrations and introduce notation for the elements in the various brackets. This material will be assumed for the rest of the paper. The reader who wishes to skip to the statements of the main theorems should read Assumption 3.14, the notation at the beginning of Sect. 3.5.2, and Assumption 3.17.

3.5.1 Assumptions on Pairings

Now, we fix notation and assumptions that will be used in the rest of the paper. Fix objects $X, X',$ and X'' in \mathcal{C} along with filtrations (as in (2.1)) on each object, which we notate as X_*, X'_*, X''_* . We view the filtered objects X_*, X'_*, X''_* as belonging to the graded category $\mathcal{C}^{\mathbb{Z}}$, where $X_i = X_0$ if $i < 0$. (Here, \mathbb{Z} is a discrete category (no non-identity morphisms), which means that the image of X_* in $\mathcal{C}^{\mathbb{Z}}$ forgets the structure maps $i : X_s \rightarrow X_{s-1}$; the compatibility with the structure maps is addressed below.) We endow $\mathcal{C}^{\mathbb{Z}}$ with a symmetric monoidal structure via Day convolution. More explicitly, given $X_*, Y_* \in \mathcal{C}^{\mathbb{Z}}$ define $(X \otimes Y)_q = \bigsqcup_{i+j=q} X_i \otimes Y_j$, where \bigsqcup denotes the colimit in \mathcal{C} .

Note that an object $X \in \mathcal{C}$ can be viewed as a graded object via the diagonal $d(X) \in \mathcal{C}^{\mathbb{Z}}$, defined by $d(X)_s = X$ for all s , and an A_3 -structure on (X'', X', X) naturally gives rise to an A_3 -structure on $(d(X''), d(X'), d(X))$.

The setup (especially (2) and (3) below) would be much less complicated if we assumed that the diagrams

$$\begin{array}{ccc}
 X'_* \otimes X_* & \xrightarrow{\mu_0} & X_* \\
 \downarrow & & \downarrow \\
 X'_{*-r} \otimes X_* & \xrightarrow{\mu_0} & X_{*-r}
 \end{array} \tag{3.13}$$

(and similarly for the other pairings) commuted strictly. However, this need not be the case in our desired application (the motivic slice spectral sequence and, more generally, colocalization towers in the framework of [8]), and instead, we must choose assumptions that are weak enough to be satisfied by colocalization towers, but strong enough to make the main argument work. In particular, our assumptions are stronger than simply asking the above diagram to commute up to homotopy.

Assumption 3.14 Fix A_3 -structures (μ, h) on (X'', X', X) and (μ_0, h_*) on (X'', X', X_*) . Moreover:

- (1) Assume there is a map of A_3 -structures $(X''_*, X'_*, X_*) \xrightarrow{l} (d(X''), d(X'), d(X))$.
- (2) Assume that there exist maps $\mu_{-r}, \mu_r, \mu^{-r}, \mu^r$ for $r \geq 1$ making the following diagrams commute:

$$\begin{array}{ccccc}
 X''_* \otimes X'_* & \xrightarrow{\mu_0} & X'_* & & X'_{*+r} \otimes X_* & \xrightarrow{\mu^r} & X_{*+r} & & X''_* \otimes X_{*+r} & \xrightarrow{\mu_r} & X_{*+r} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow i \\
 X''_* \otimes X'_{*-r} & \xrightarrow{\mu_{-r}} & X'_{*-r} & & X'_* \otimes X_* & \xrightarrow{\mu_0} & X_* & & X''_* \otimes X_* & \xrightarrow{\mu_0} & X_* \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X'' \otimes X' & \xrightarrow{\mu} & X' & & X'_{*-r} \otimes X_* & \xrightarrow{\mu^{-r}} & X_{*-r} & & X'' \otimes X & \xrightarrow{\mu} & X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X' \otimes X & \xrightarrow{\mu} & X & & X' \otimes X & \xrightarrow{\mu} & X & & & &
 \end{array}$$

- (3) Assume that there are homotopies \tilde{H}_1 from μ^r to μ_0 and \tilde{H}_2 from μ_0 to μ_r such that

$$\begin{array}{ccc}
 X'_* \otimes X_* \otimes I & \xrightarrow{\tilde{H}_1} & X_* \\
 \downarrow & & \downarrow \iota \\
 X'_* \otimes X_* & & \\
 \downarrow & & \downarrow \\
 X' \otimes X & \xrightarrow{\mu} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X''_* \otimes X_* \otimes I & \xrightarrow{\tilde{H}_2} & X_* \\
 \downarrow & & \downarrow \iota \\
 X''_* \otimes X_* & & \\
 \downarrow & & \downarrow \\
 X'' \otimes X & \xrightarrow{\mu} & X
 \end{array}$$

commute. (Note that μ^r and μ_r have been reindexed to be maps $X'_* \otimes X_* \rightarrow X_*$ and $X''_* \otimes X_* \rightarrow X_*$, respectively.)

(4) Assume there are maps $\mu_{*,1}$ making the following diagrams commute:

$$\begin{array}{ccc}
 X'_* \otimes X_* & \xrightarrow{\mu_0} & X_* \\
 \downarrow & & \downarrow p \\
 X'_{*,1} \otimes X_{*,1} & \xrightarrow{\mu_{*,1}} & X_{*,1}
 \end{array}
 \qquad
 \begin{array}{ccc}
 X''_* \otimes X_* & \xrightarrow{\mu_0} & X_* \\
 \downarrow & & \downarrow p \\
 X''_{*,1} \otimes X_{*,1} & \xrightarrow{\mu_{*,1}} & X_{*,1}
 \end{array}$$

(5) There are pairings of spectral sequences

$$\begin{aligned}
 E_r(X') \otimes E_r(X) &\rightarrow E_r(X), \\
 E_r(X'') \otimes E_r(X) &\rightarrow E_r(X)
 \end{aligned}$$

induced by $\mu_{*,1}$ and $\mu_{*,1}$. (Recall that $E_1^{*,f}(X) = \pi_*(X_{f,1})$ and the E_r -page is a subquotient of the E_1 -page.)

Remark 3.15 The synthetic perspective [1, 5, 6, 22] views spectral sequences as filtered spectra, and synthetic tools are well equipped to address questions about \mathbb{A}_n -algebras in the ∞ -category of filtered spectra. Work in progress by Yuchen Wu and Shangjie Zhang discusses in detail the Moss convergence theorem in the synthetic homotopy theory context.

This is not directly comparable to our setting. For example in the ring spectrum case $X''_* = X'_* = X_*$, we assume that X_* is an A_3 -monoid in graded spectra, but not necessarily in filtered spectra; the latter would imply commutativity of (3.13). In this special case, assumptions (2) and (3) hold trivially.

Remark 3.16 Davis and Snaith [3, Theorem 2.2] gave an abstract proof of the differential and extension theorem for Massey products in spectral sequences. Their theorem has a hypothesis called the “exchange condition”. If (3.13) commutes, one can show that our setup implies the exchange condition.

3.5.2 The Homotopy Classes $\alpha, \alpha', \alpha''$

Suppose there are maps

$$\alpha : S \rightarrow X, \quad \alpha' : S \rightarrow X', \quad \alpha'' : S \rightarrow X''$$

with nullhomotopic products $\mu(\alpha', \alpha) \simeq 0 \simeq \mu(\alpha'', \alpha')$. Furthermore, suppose that α, α' , and α'' are detected by nonzero permanent cycles

$$\alpha \in E_r^{*,f-f'}(X), \quad \alpha' \in E_r^{*,f'-f''}(X'), \quad \alpha'' \in E_r^{*,f''}(X''),$$

respectively, such that $\alpha' \alpha$ and $\alpha'' \alpha'$ are d_r -boundaries. Then, there are maps

$$\alpha_{**} : S \rightarrow X_{f-f',1}, \quad \alpha'_{**} : S \rightarrow X'_{f'-f'',1}, \quad \alpha''_{**} : S \rightarrow X''_{f'',1}$$

representing α , α' , and α'' , and since these are permanent cycles, they lift to

$$\alpha_* : S \rightarrow X_{f-f'}, \quad \alpha'_* : S \rightarrow X'_{f'-f''}, \quad \alpha''_* : S \rightarrow X''_{f''}$$

satisfying $\alpha = \iota\alpha_*$ and $\alpha_{**} = p\alpha_*$ (and similarly for α' , α'') where $\iota : X_* \rightarrow X$ and $p : X_* \rightarrow X_{*,1}$ are the maps of graded objects guaranteed in Assumption 3.14 (1) and (4).

Strict compatibility of ι and p with the multiplicative structures (Assumption 3.14 (1) and (4)) implies that $\iota\mu_0(\alpha'_*, \alpha_*) = \mu(\alpha', \alpha)$ and $p\mu_0(\alpha'_*, \alpha_*) = \mu_{*,1}(\alpha'_{**}, \alpha_{**})$, and Assumption 3.14 (5) says that $\mu_{*,1}(\alpha'_{**}, \alpha_{**}) : S \rightarrow X_{f-f'',1}$ descends to the E_r product $\alpha'a$.

Beware that we need not have $\mu_0(\alpha''_*, \alpha'_*) = 0 = \mu_0(\alpha'_*, \alpha_*)$ or $\mu_{*,1}(\alpha''_{**}, \alpha'_{**}) = 0 = \mu_{*,1}(\alpha'_{**}, \alpha_{**})$; compare with Lemma 3.18.

Assumption 3.17 In the rest of the paper, assume that $\{E_*(X)\}$ satisfies the crossing differentials hypothesis (Definition 2.4) in degree $(r, |a'|)$ and $\{E_*(X')\}$ satisfies the crossing differentials hypothesis in degree $(r, |a''a'|)$. Moreover, we assume that the spectral sequences $\{E_*(X)\}$ and $\{E_*(X')\}$ are weakly convergent (see Definition 2.3).

Lemma 3.18 *Given Assumption 3.17, the following compositions are nullhomotopic:*

$$S \xrightarrow{\mu_0(\alpha'_*, \alpha_*)} X_{f-f''} \xrightarrow{i} X_{f-f''-r},$$

$$S \xrightarrow{\mu'_0(\alpha''_*, \alpha'_*)} X'_{f'} \xrightarrow{i} X'_{f'-r}.$$

Proof By Assumption 3.14 (1), $\mu(\alpha', \alpha) : S \rightarrow X$ factors as $\iota\mu_0(\alpha'_*, \alpha_*)$. Since this is null and (using Assumption 3.14 (4) and (5)) $\mu_0(\alpha'_*, \alpha_*)$ lifts $\alpha'a$ which is assumed to be a boundary, Proposition 2.6 implies that $i\mu_0(\alpha'_*, \alpha_*) : S \rightarrow X_{f-f''-r}$ is null. \square

4 Comparison Statements

We assume the notation in Sect. 3.5.

4.1 Comparing the Toda Bracket and Massey Product

The goal of this section is to prove Theorem 4.11, which compares the Toda bracket $\langle \alpha'', \alpha', \alpha \rangle$ with the Massey product $\langle a'', a', a \rangle$. The general idea is to define another nonempty bracket $\langle \alpha''_*, i\alpha'_*, \alpha_* \rangle$ and show that there are maps

$$\langle a'', a', a \rangle \xleftarrow{p} \langle \alpha''_*, i\alpha'_*, \alpha_* \rangle \xrightarrow{\iota} \langle \alpha'', \alpha', \alpha \rangle,$$

where $\iota : (X''_*, X'_*, X_*)$ is the map from Assumption 3.14 (1) and $p : (X''_*, X'_*, X_*) \rightarrow (X''_{*,1} \rightarrow X'_{*,1}, X_{*,1})$ is from Assumption 3.14 (4). This would be straightforward if the diagram (3.13) commuted strictly: given nullhomotopies F_* and G_* of $\mu_0(\alpha''_*, i\alpha'_*)$

and $\mu_0(i\alpha', \alpha_*)$, respectively, and a homotopy H_* between $\mu_0(\mu_0(\alpha''_*, \alpha'_*), \alpha_*)$ and $\mu_0(\alpha''_*, \mu_0(\alpha'_*, \alpha_*))$, we would consider

$$d(\mu_0(F_*, \alpha_*), iH_*, \mu_0(\alpha''_*, G_*)) \in \langle \alpha''_*, i\alpha_*, \alpha_* \rangle. \tag{4.1}$$

However, with our current assumptions, this element is not well defined, as $\mu_0(F_*, \alpha_*) : CS \rightarrow X_*$ restricted to $S \hookrightarrow CS$ is $\mu_0(\mu_0(\alpha''_*, i\alpha'_*), \alpha_*)$, which is not an endpoint for iH_* . Instead, we must use the variety of multiplication maps on (X''_*, X'_*, X_*) described in Assumption 3.14 (2) and (3) to write down a homotopy between (variants of) these two elements, that restricts to the trivial homotopy after applying ι .

For ease of notation, let

$$\begin{aligned} \alpha_{21} &= \mu_0(\alpha''_*, \alpha'_*), \\ \alpha_{10} &= \mu_0(\alpha'_*, \alpha_*). \end{aligned}$$

From Lemma 3.18, we have nullhomotopies

$$\begin{aligned} F_* : CS &\rightarrow X'_{*-r} \text{ of } i\alpha_{21}, \text{ and} \\ G_* : CS &\rightarrow X_{*-r} \text{ of } i\alpha_{10}. \end{aligned}$$

The setup ensures that the following diagrams commute strictly, except for the dotted arrows:

The first diagram (top) shows a commutative diagram with nodes S , CS , $X''_* \otimes X'_* \otimes X_*$, $X''_* \otimes X_{*-r}$, $X''_* \otimes X_*$, $X'_{*-r} \otimes X_*$, X_{*-r} , $X'_{*-r,1} \otimes X_{*,1}$, and $X_{*-r,1}$. It includes maps $(\alpha''_*, \alpha'_*, \alpha_*)$, (α_{21}, α_*) , (μ_0, Id) , i , μ^r , (μ_{-r}, Id) , μ_0 , (F_*, α_*) , p , and $\mu_{*,1}$. A dotted arrow μ_0 is labeled $\uparrow H_1$.

The second diagram (bottom) is similar, with nodes S , CS , $X''_* \otimes X'_* \otimes X_*$, $X''_* \otimes X_{*-r}$, $X''_* \otimes X_*$, $X''_* \otimes X_{*-r}$, X_{*-r} , $X''_{*,1} \otimes X_{*-r,1}$, and $X_{*-r,1}$. It includes maps $(\alpha''_*, \alpha'_*, \alpha_*)$, $(\alpha''_*, \alpha_{10})$, (Id, μ_0) , i , μ_r , (Id, μ^{-r}) , μ_0 , (α''_*, G_*) , p , and $\mu_{*,1}$. A dotted arrow μ_0 is labeled $\downarrow H_2$.

Assumption 3.14 (1) and (3) allow us to fix the following data:

- a homotopy H_* from $\mu_0(\mu_0(\alpha''_*, \alpha'_*), \alpha_*) = \mu_0(\alpha_{21}, \alpha_*)$ to $\mu_0(\alpha''_*, \mu_0(\alpha'_*, \alpha_*)) = \mu_0(\alpha''_*, \alpha_{10})$ whose composition with $\iota : X_* \rightarrow X$ recovers the homotopy H defined by the A_3 -structure on the triple (X'', X', X) ;
- a homotopy H_1 from $\mu^r(\alpha_{21}, \alpha_*)$ to $\mu_0(\alpha_{21}, \alpha_*)$ whose composition with $\iota : X_* \rightarrow X$ is constant at $\mu(\mu(\alpha'', \alpha'), \alpha)$;
- a homotopy H_2 from $\mu_0(\alpha''_*, \alpha_{10})$ to $\mu_r(\alpha''_*, \alpha_{10})$ whose composition with $\iota : X_* \rightarrow X$ is constant at $\mu(\alpha'', \mu(\alpha', \alpha))$.

Using all of these data, we may define the crucial element

$$\gamma = d(\mu_0(F_*, \alpha_*), iH_1, iH_*, iH_2, \mu_0(\alpha''_*, G_*)) : S \rightarrow X_{*-r}. \tag{4.3}$$

If (3.13) was required to commute, then H_1 and H_2 could be taken to be trivial, and γ could be replaced by (4.1). This strategy indeed works in Sect. 4.2, where the Massey product is replaced by a Toda bracket of E_1 classes.

To state the comparison with the Massey product, we need the following elements:

$$\begin{aligned} \beta^+ &= d(pF_*, j' C \alpha_{21}), \\ \beta^- &= d(pG_*, j C \alpha_{10}), \end{aligned}$$

where $j : CX_* \rightarrow X_{*-r,1}$ and $j' : CX'_* \rightarrow X'_{*-r,1}$ are as in Lemma 2.12. That lemma implies that β^+ is the source of the differential hitting $p\alpha_{21} = \mu_{*,1}(\alpha''_{**}, \alpha'_{**})$ and β^- is the source of the differential hitting $p\alpha_{10} = \mu_{*,1}(\alpha'_{**}, \alpha_{**})$. Thus, we have (using Assumption 3.14 (5)):

$$\mu_{*,1}(\beta^+, \alpha_{**}) - \mu_{*,1}(\alpha''_{**}, \beta^-) \in \langle \alpha'', \alpha', \alpha \rangle. \tag{4.4}$$

The next lemma identifies these two summands with terms that we can more easily relate to γ .

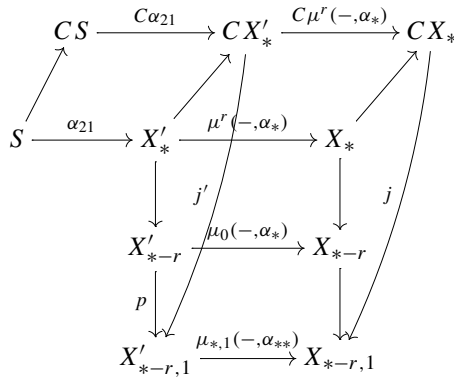
Lemma 4.5 *We have*

$$\begin{aligned} \mu_{*,1}(\beta^+, \alpha_{**}) &\simeq d(\mu_{*,1}(pF_*, \alpha_{**}), j C \mu^r(\alpha_{21}, \alpha_*)), \\ \mu_{*,1}(\alpha''_{**}, \beta^-) &\simeq d(\mu_{*,1}(\alpha''_{**}, pG_*), j C \mu_r(\alpha''_*, \alpha_{10})). \end{aligned}$$

Proof Applying Lemma 2.11 to $\mu_{*,1}(-, \alpha_{**}) : X'_{*-r,1} \rightarrow X_{*-r,1}$, we have

$$\begin{aligned} \mu_{*,1}(\beta^+, \alpha_{**}) &= \mu_{*,1}(d(pF_*, j' C \alpha_{21}), \alpha_{**}) \\ &= d(\mu_{*,1}(pF_*, \alpha_{**}), \mu_{*,1}(j' C \alpha_{21}, \alpha_{**})). \end{aligned}$$

The commutative diagram



shows that $\mu_{*,1}(j' C\alpha_{21}, \alpha_{**}) = j C\mu^r(\alpha_{21}, \alpha_*)$ as nullhomotopies $CS \rightarrow X_{*-r,1}$. The other statement is similar, using the factorization $S \xrightarrow{\alpha_{10}} X_* \xrightarrow{\mu_r(\alpha''_*, -)} X_*$ of $\mu_r(\alpha''_*, \alpha_{10})$. \square

Next, we relate γ to a sum involving terms that look like the right-hand side in Lemma 4.5.

Lemma 4.6 *We have*

$$p \circ \gamma \simeq d(p\mu_0(F_*, \alpha_*), jC\mu^r(\alpha_{21}, \alpha_*)) - d(p\mu_0(\alpha''_*, G_*), jC\mu_r(\alpha''_*, \alpha_{10})),$$

where γ is defined in (4.3).

Proof Let $f_1 = \mu^r(\alpha_{21}, \alpha_*)$ and $g_1 = \mu_r(\alpha''_*, \alpha_{10})$. We will show that

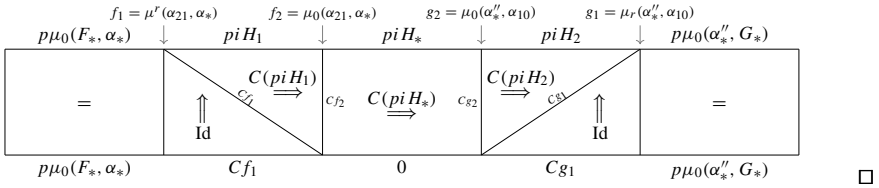
$$p \circ \gamma \simeq d(p\mu_0(F_*, \alpha_*), jCf_1, 0, jCg_1, p\mu_0(\alpha''_*, G_*)); \tag{4.7}$$

this suffices, because it is homotopic to $d(p\mu_0(F_*, \alpha_*), jCf_1) + d(jCg_1, p\mu_0(\alpha''_*, G_*))$, which is homotopic to the desired difference by Lemma 2.10. Recall that for any $f : S \rightarrow X_*$, we have a nullhomotopy

$$CS \xrightarrow{Cf} CX_* \xrightarrow{j} X_{*-r} \coprod_{X_*} CX_* = X_{*-r,r} \rightarrow X_{*-r,1}$$

of $p \circ i \circ f$, denoted by Cf . Moreover, the nullhomotopy Cf is functorial in f ; therefore, for homotopy $piH_1 : f_1 \implies f_2$, we have $C(piH_1) : Cf_1 \implies Cf_2$. Similarly, we have $C(piH_*)$ and $C(piH_2)$.

We represent the desired homotopy in (4.7) as follows, where the top row represents $p \circ \gamma$ and the bottom row represents the right-hand side of (4.7). For ease of notation, we omit j and $p : CX_{*-r} \rightarrow X_{*-r,1}$ from the notation in some places.



Proposition 4.8 We have $p \circ \gamma \in \langle \alpha'', \alpha', \alpha \rangle$.

Proof Combine (4.4) with Lemmas 4.5 and 4.6. □

Now that we have shown compatibility of γ with the Massey product; it remains to show compatibility with the Toda bracket.

Proposition 4.9 We have $\iota \circ \gamma \in \langle \alpha'', \alpha', \alpha \rangle$, where $\iota : X_* \rightarrow X$ is the map from Assumption 3.14 (1).

Proof We have

$$\begin{aligned}
 \iota \circ \gamma &= d(\iota\mu_0(F_*, \alpha_*), \iota H_1, \iota H_*, \iota H_2, \mu_0(\alpha''_*, G_*)) \\
 &= d(\mu(\iota F, \alpha), \text{Id}, \iota H_*, \text{Id}, \mu(\alpha'', \iota G_*)) \\
 &\simeq d(\mu(\iota F, \alpha), \iota H_*, \mu(\alpha'', \iota G_*)),
 \end{aligned}
 \tag{4.10}$$

where the first equality is an application of Lemma 2.11 and the second uses the assumptions on $\iota H_1 = \iota H_1$ and $\iota H_2 = \iota H_2$ and Assumption 3.14 (1) for the compatibility of μ with μ_0 . By Assumption 3.14 (1), ιH_* is the homotopy in the A_3 structure on (X'', X', X) , and ιF_* and ιG_* are nullhomotopies of $\mu(\alpha'', \alpha')$ and $\mu(\alpha', \alpha)$, respectively. Thus, (4.10) is an element of the Toda bracket. □

Now, we obtain the main theorem. The indexing differs slightly from the version in Theorem 1.1.

Theorem 4.11 Fix $r \geq 1$. Given the setup in Sects. 3.5.1 and 3.5.2 (including Assumption 3.17), there exists an element of the Massey product $\langle \alpha'', \alpha', \alpha \rangle$ that is a permanent cycle converging to an element of $\langle \alpha'', \alpha', \alpha \rangle$.

Proof Define $\gamma \in \pi_*(X_{f-r})$ as in (4.3). By Proposition 4.8, the projection of γ to the E_1 -page is an element of the Massey product $\langle \alpha'', \alpha', \alpha \rangle$. On the other hand, this element detects $\iota\gamma$, which is in $\langle \alpha'', \alpha', \alpha \rangle$ by Proposition 4.9. □

Remark 4.12 We indicate the changes needed to the previous two sections to accommodate the case where the classes have nonzero degree. Suppose we have $\alpha : S^k \rightarrow X$, $\alpha' : S^m \rightarrow X'$, $\alpha'' : S^n \rightarrow X''$ and corresponding E_r -term elements $\alpha, \alpha', \alpha''$ as in Sect. 3.5.2. The main comparison Lemma 4.6 is correct as stated, but the right-hand side needs the minus sign changed to $(-1)^{1+n}$ to agree with the Massey product (see [16, p.146]),² and we claim the left-hand side also needs to be corrected to agree with

² The sign in [16] differs from the sign in [23, §A1.4].

the Toda bracket. To define the Toda bracket in this case, replace (3.8) by a similar diagram where the top and bottom compositions are as follows:

$$\begin{aligned}
 C(S^n \otimes S^m \otimes S^k) &\xrightarrow{F} X' \otimes S^{k-1} \otimes^{\alpha} X' \otimes X \xrightarrow{\mu} X, \\
 C(S^n \otimes S^m \otimes S^k) &\xrightarrow{G} S^n \otimes X \xrightarrow{\alpha'' \otimes 1} X'' \otimes X \xrightarrow{\mu} X.
 \end{aligned}$$

However, diagram (4.2) assumes that the second composition is instead

$$S^n \otimes C(S^m \otimes S^k) \xrightarrow{G} S^n \otimes X \xrightarrow{\alpha'' \otimes 1} X'' \otimes X \xrightarrow{\mu} X.$$

This discrepancy amounts to a sign change that matches the sign in the Massey product.

4.2 Toda Brackets in the E_1 -page

In the previous section, we worked with E_r -page classes a, a', a'' for $r \geq 1$ such that $a'a = 0 = a''a'$ in $E_{r+1}(X)$. In this section, we consider $\alpha, \alpha', \alpha''$ with $\mu(\alpha', \alpha) = 0 = \mu(\alpha'', \alpha')$ that are detected by a, a', a'' in $E_1^{f-f'}(X), E_1^{f'-f''}(X')$, and $E_1^{f''}(X'')$, respectively, such that $a'a = 0 = a''a'$ in $E_1(X)$.

In this case, we cannot define a corresponding Massey product. Instead, we recall that $E_1^f(X) = \pi_*(X_{f,1})$ by definition, and our setup allows us to consider Toda brackets of such homotopy elements. More precisely, there is a Toda bracket $\langle a'', a', a \rangle$ given by the classes $S \xrightarrow{\alpha''} X''_{f'',1}, S \xrightarrow{\alpha'} X'_{f'-f'',1},$ and $S \xrightarrow{\alpha} X_{f-f',1},$ where the associativity homotopy is given by composing the associativity homotopy H_* used to define $\langle \alpha''_*, \alpha'_*, \alpha_* \rangle$ with $p : X_* \rightarrow X_{*,1}$. The goal of this section is to prove Theorem 4.16, which is the analog of Theorem 4.11 where the Massey product is replaced by $\langle a'', a', a \rangle$. We still make Assumption 3.17 in degrees $(0, |a'a|)$ and $(0, |a''a'|)$; despite the motivation about crossing differentials, this does not require the definition of a d_0 differential.

Proposition 2.6 does not make sense in this case; the replacement is Lemma 4.13.

Lemma 4.13 *Suppose $\zeta \in \pi_n(X_f)$ is a class whose images in both $\pi_n(X)$ and $\pi_n(X_{f,1})$ are zero. Suppose $\{E_*(X)\}$ is weakly convergent and satisfies the crossing differentials hypothesis in degree $(0, n, f)$. Then, $\zeta = 0$.*

Proof If not, then there exists $k \leq f$, such that the image of ζ in $\pi_n(X_k)$ is nonzero but its image in $\pi_n(X_{k-1})$ is zero. This means that there exists $z \in \pi_{n+1}(X_{k-1,1})$, such that $\kappa(z)$ lifts to $\zeta \in \pi_n(X_f)$. Then, $d_r(z) \neq 0$ for some r : if not, then ζ lifts to $\pi_n(X_{f+r})$ for all r , and in the notation of Definition 2.3, we would have $i\zeta \in \ker(R_k \rightarrow R_{k-1})$, contradicting weak convergence. Moreover, since $p(\zeta) \in \pi_n(X_{f,1})$ is zero, ζ lifts to $\pi_n(X_{f+1})$, and so, $r > f - (k - 1)$. Therefore, z is a class in $E_{f-k+2}^{n+1,k-1}$ that is not a permanent cycle, contradicting the crossing differentials hypothesis. \square

Proposition 4.14 *Given Assumption 3.17 with $r = 0, \mu_0(\alpha'_*, \alpha_*) : S \rightarrow X_{f-f'}$ and $\mu_0(\alpha''_*, \alpha'_*) : S \rightarrow X'_{f'}$ are null. For every element $\gamma_* \in \langle \alpha''_*, \alpha'_*, \alpha_* \rangle$, the composition $S \xrightarrow{\gamma_*} X_f \xrightarrow{\iota} X$ is an element of $\langle \alpha'', \alpha', \alpha \rangle$.*

Proof The first sentence follows by Lemma 4.13. Let F_* and G_* be nullhomotopies of these two classes, and let H_* be a homotopy between $\mu_0(\mu_0(\alpha''_*, \alpha'_*), \alpha_*)$ and $\mu_0(\alpha''_*, \mu_0(\alpha'_*, \alpha_*))$. Let

$$\gamma_* = d(\mu_0(F_*, \alpha_*), H_*, \mu_0(\alpha''_*, G_*)).$$

By Assumption 3.14 (1), we have $\iota \circ \gamma_* = d(\mu(\iota F_*, \alpha), \iota H_*, \mu(\alpha'', \iota G_*))$, which is an element of the Toda bracket $\langle \alpha'', \alpha', \alpha \rangle$. □

Proposition 4.15 *Given Assumption 3.17 with $r = 0$, every class in the Toda bracket $\langle \alpha''_*, \alpha'_*, \alpha_* \rangle$ projects to a class in the Toda bracket $\langle \alpha'', \alpha', \alpha \rangle$.*

Proof This holds using the same straightforward argument as Proposition 4.14, along with Assumption 3.14 (4). □

Theorem 4.16 *Suppose $\mu(\alpha', \alpha) \simeq 0 \simeq \mu(\alpha'', \alpha')$ and $\mu_{*,1}(\alpha', \mathfrak{a}) \simeq 0 \simeq \mu_{*,1}(\alpha'', \mathfrak{a})$ where $\mathfrak{a} \in \pi_*(X_{f-f',1})$ is an E_1 representative of $\alpha \in \pi_*(X)$, and similarly, for α' and α'' . Given Assumption 3.17 with $r = 0$, there is a permanent cycle in the Toda bracket $\langle \alpha'', \alpha', \mathfrak{a} \rangle$ converging to an element of $\langle \alpha'', \alpha', \alpha \rangle$.*

Proof Combine Propositions 4.14 and 4.15 as in the proof of Theorem 4.11. □

5 Applications

5.1 Strict Compatibility of i and μ

For many important examples, the diagram (3.13) is strictly commutative, which implies Assumption 3.14 (2) and (3) by setting \tilde{H}_1 and \tilde{H}_2 to be constant, and $\mu_0 = \mu_{-r} = \mu_r = \mu^{-r} = \mu^r$.

Example 5.1 Let E be a commutative, associative ring object in \mathcal{C} and consider the E -based Adams spectral sequence for computing the homotopy of an A_3 -ring object R of \mathcal{C} . This arises from the tower

$$\begin{array}{ccccccc} R & \longleftarrow & \bar{E} \otimes R & \longleftarrow & \bar{E}^{\otimes 2} \otimes R & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ E \otimes R & & E \otimes \bar{E} \otimes R & & E \otimes \bar{E}^{\otimes 2} \otimes R, & & \end{array}$$

where \bar{E} is the homotopy fiber of the unit map $S \rightarrow E$. Write $X_s = \bar{E}^{\otimes s} \otimes R$. Then, the multiplication maps on the filtered object

$$\bar{E}^{\otimes s} \otimes R \otimes \bar{E}^{\otimes t} \otimes R \rightarrow \bar{E}^{\otimes s} \otimes \bar{E}^{\otimes t} \otimes R \otimes R \rightarrow \bar{E}^{\otimes s+t} \otimes R$$

arise from commuting R past $\bar{E}^{\otimes t}$ and using the multiplication on R . It is straightforward to see that the diagram (3.13) is strictly commutative, and that Assumption 3.14

(1) and (4) hold. The induced pairing of E_r -pages (Assumption 3.14 (5)) is standard; see, e.g., [23, Theorem 2.3.3].

Example 5.2 The May spectral sequence

$$E_1 = \mathbb{F}_2[h_{i,j} : i > 0, j \geq 0] \implies \text{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

where A is the dual Steenrod algebra arises from a filtration of the cobar complex $A^{\otimes*}$ defined by the grading $|\xi_i^{2^j}| = 2i - 1$ (see [23, Proof of Theorem 3.2.3]). The cobar complex is strictly associative, and the structure maps in the filtration arise from inclusion of subcomplexes, and so, (3.13) commutes. The other parts of Assumption 3.14 are straightforward, and Theorem 4.11 recovers the May Convergence Theorem [17, Theorem 4.1].

5.2 Colocalization Towers

Our setting is designed to fit with the theorems proved by Gutiérrez, Röndigs, Spitzweck, and Østvær [8] on multiplicativity of colocalization towers.

Suppose \mathcal{M} is a stable combinatorial simplicial symmetric monoidal proper model category with cofibrant unit. Let

$$\cdots \supseteq \mathcal{C}_{-1} \supseteq \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$$

be a family of full subcategories of $\mathbf{Ho}(\mathcal{M})$ satisfying the axioms in [8, §2.1]. In particular, \mathcal{C}_i is the homologically non-negative part of some t -structure on $\mathbf{Ho}(\mathcal{M})$ (see [8, §2.1]), and so, there are corresponding colocalization and localization functors c_i and l_i . By [8, Lemma 2.1], these functors can be lifted to colocalization and localization functors on \mathcal{M} . For example ([8, §3.4]), if \mathcal{C} has an accessible t -structure where $\mathcal{C}_{\geq 0}$ is a symmetric monoidal subcategory, then the collection $\{\mathcal{C}_{\geq 0}[i] : i \in \mathbb{Z}\}$ satisfies the axioms.

Our goal is to prove the following.

Proposition 5.3 *Let X', X'' be A_3 -rings with an A_3 -map $X'' \rightarrow X'$, and let X be an A_3 - X' -module. Assume the existence of colocalization and localization functors c_i, l_i as above, and consider the filtrations on X'', X' , and X defined by $X''_i = c_i X''$ and so on. Then, (X''_*, X'_*, X_*) can be given a multiplicative structure satisfying Assumption 3.14 (1)–(4).*

We use the machinery of colored operads; see, e.g., [8, Appendix A] for an introduction. Let \mathcal{K} denote the Stasheff A_3 -operad. We encode the structure on the triple (X'', X', X) in Proposition 5.3 as an algebra structure over the following operad. Define the colored operad \mathcal{K}^{mod} with colors $\{x'', x', x\}$ as follows:

$$\mathcal{K}^{\text{mod}}(t_1, \dots, t_n; t) = \begin{cases} \mathcal{K}(n) & \text{if } t = x'' \text{ and } t_i = x'' \text{ for all } i, \\ \mathcal{K}(n) & \text{if } t = x' \text{ and } t_i \in \{x', x''\} \text{ for all } i, \\ \mathcal{K}(n) & \text{if } t = x \text{ and exactly one } t_i \text{ equals } x, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the map $X'' \rightarrow X'$ comes from the canonical element in $\mathcal{K}^{\text{mod}}(x''; x') = \mathcal{K}(1)$, and the homotopy between $\mu(\mu(-, -), -)$ and $\mu(-, \mu(-, -))$ comes from $\mathcal{K}^{\text{mod}}(x'', x', x; x) = \mathcal{K}(3) = \Delta^1$. Then, triples (X'', X', X) satisfying the hypotheses of Proposition 5.3 are \mathcal{K}^{mod} -algebras.

The strategy in [8, §2.3] for studying graded objects is to apply the general setup to the categories $\mathcal{D}_r = \prod_{i \in \mathbb{Z}} \mathcal{C}_{i+r} \subseteq \mathcal{C}^{\mathbb{Z}}$. Let $d : \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{Z}}$ denote the diagonal functor. There are corresponding colocalization and localization functors c'_r and l'_r where $c'_r d(X)_n = c_{r+n} X$. We consider the graded object X_* where $X_s = c_s X$, and write $X_{*+r} := c'_r d(X)$. Note that $(d(X''), d(X'), d(X))$ is an \mathcal{K}^{mod} -algebra in $\mathcal{C}^{\mathbb{Z}}$. For ease of notation, we denote this as simply (X'', X', X) .

The next technical lemma contains the main input we need from [8]. It requires the following notation. Given a colored operad \mathcal{O} and compatibly indexed collections \mathbf{X}, \mathbf{Y} , let $\text{Hom}_{\mathcal{O}}(\mathbf{X}, \mathbf{Y})$ be the colored collection with

$$\text{Hom}_{\mathcal{O}}(\mathbf{X}, \mathbf{Y})(t_1, \dots, t_n; t) = \begin{cases} \text{Hom}(\mathbf{X}(t_1) \otimes \dots \otimes \mathbf{X}(t_n); \mathbf{Y}(t)) & \text{if } \mathcal{O}(t_1, \dots, t_n; t) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.4 [8, Theorem 4.1 and Proof of Proposition 5.14] *Suppose \mathcal{O} is a \mathcal{J} -colored operad, and for every color $t \in \mathcal{J}$, there is a colocalization $c_{s(t)}$ (with $s(t) \in \mathbb{Z}$), such that $\mathcal{O}(t_1, \dots, t_n; t) = 0$ if $s(t_1) + \dots + s(t_n) < s(t)$. Given $\mathbf{X} \in \mathcal{C}^{\mathcal{J}}$, define $Q\mathbf{X}(t) = c_{s(t)}\mathbf{X}(t)$, and define the \mathcal{J} -colored collection \mathcal{P} , so $\mathcal{P}(t_1, \dots, t_n; t)$ is the space of commutative diagrams*

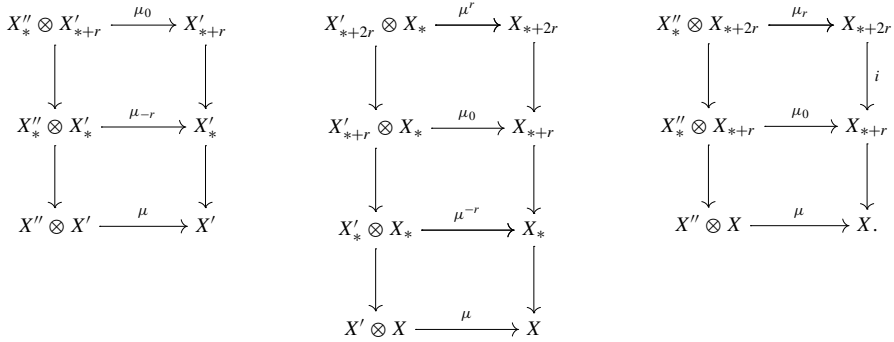
$$\begin{array}{ccc} Q\mathbf{X}(t_1) \otimes \dots \otimes Q\mathbf{X}(t_n) & \longrightarrow & Q\mathbf{X}(t) \\ \downarrow & & \downarrow \\ \mathbf{X}(t_1) \otimes \dots \otimes \mathbf{X}(t_n) & \longrightarrow & \mathbf{X}(t). \end{array}$$

Then, the natural map $\tau : \mathcal{P} \rightarrow \text{Hom}_{\mathcal{O}}(\mathbf{X}, \mathbf{X})$ is a trivial fibration in the semi-model structure from [8, §A.2]. Let \mathcal{O}_{∞} be a cofibrant resolution for \mathcal{O} . If \mathbf{X} is an \mathcal{O} -algebra, then there is an \mathcal{O}_{∞} -algebra structure ψ on $Q\mathbf{X}$ that is compatible with the \mathcal{O} -algebra structure on \mathbf{X} in the sense that there is a commutative diagram of colored collections

$$\begin{array}{ccccc} & & & & \mathcal{P} \\ & & & \nearrow \psi & \downarrow \tau \\ \mathcal{O}_{\infty} & \longrightarrow & \mathcal{O} & \longrightarrow & \text{Hom}_{\mathcal{O}}(\mathbf{X}, \mathbf{X}). \end{array}$$

Lemma 5.5 *Given the setup above, there is an A_3 -structure on (X''_*, X'_*, X_*) satisfying Assumption 3.14 (1) and (2).*

Proof Reindex Assumption 3.14 (2) so we are looking for the following diagrams along with homotopy associativity for the two multiplications $X''_* \otimes X'_{*+r} \otimes X_* \rightarrow X_{*+r}$:



The strategy is as follows: the reindexing enables us to first construct μ_{-r} and μ^{-r} by constructing a \mathcal{K}^{mod} -structure on (X''_*, X'_*, X_*) . Then, we construct μ_0 compatible with this using a new operad \mathcal{O} inspired by the morphism operad $\text{Mor}_{\mathcal{K}^{\text{mod}}}$ of [8, A.1.2] to obtain the appropriate multiplicative structure on the colocalization map $(X''_*, X'_{*+r}, X_*) \rightarrow (X''_*, X'_*, X_*)$.

We will focus on the middle diagram, as the other two diagrams are analogous. Moreover, we only need to construct the bottom two squares; for the top square, $X'_{*+2r} \otimes X_* \rightarrow X'_{*+r} \otimes X_* \xrightarrow{\mu_0} X_{*+r}$ factors through X_{*+2r} by the universal property of the colocalization. (Note that we cannot simply construct μ^{-r} and μ_0 in this way, because we also need to show μ_0 is homotopy associative after appropriate indexing; this will come from the subsequent operad arguments.) By Lemma 5.4 with $s(t) = 0$ for all colors t , we obtain a $\mathcal{K}^{\text{mod}}_\infty$ -algebra structure ψ_1 on (X''_*, X'_*, X_*) compatible with (X'', X', X) . Choose $\bar{\mu}^{-r} \in \mathcal{K}^{\text{mod}}_\infty(x', x; x)$ and let $\mu^{-r} = \psi_1(\bar{\mu}^{-r})$.

To construct μ_0 compatible with μ^{-r} , let \mathcal{O} be the colored operad with color set

$$\mathcal{J} = \{x''_0, x'_{2r}, x'_r, x'_0, x_{2r}, x_r, x_0\}$$

and

$$\mathcal{O}(t_1, \dots, t_n; t) = \begin{cases} \mathcal{K}^{\text{mod}}_\infty(t_1, \dots, t_n; t) & \text{if } \sum_i \text{sub}(t_i) \geq \text{sub}(t), \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{sub}(x_r) = r$, $\text{sub}(x_0) = 0$, and so on. An \mathcal{O} -algebra is a tuple $(A_0, B_{2r}, B_r, B_0, M_{2r}, M_r, M_0)$ with contractible spaces of structure maps such as $M_r \rightarrow M_0$ and $B_r \otimes M_0 \rightarrow M_r$.

Since the structure maps for \mathcal{O} come from $\mathcal{K}^{\text{mod}}_\infty$, the $\mathcal{K}^{\text{mod}}_\infty$ -algebra structure on (X''_*, X'_*, X_*) can be repackaged as an \mathcal{O} -algebra structure on $\mathbf{X}_* :=$

$(X''_*, X'_*, X'_*, X'_*, X_*, X_*, X_*)$. Lemma 5.4 applies where s is the subscript function sub, and gives an \mathcal{O}_∞ -algebra structure ψ on

$$QX_* := (X''_*, X'_{*+2r}, X'_{*+r}, X'_*, X_{*+2r}, X_{*+r}, X_*).$$

This parametrizes a family of multiplications $X'_{*+r} \otimes X_* \rightarrow X_{*+r}$, and we need to check that there exists one making the following diagram commute:

$$\begin{CD} X'_{*+r} \otimes X_* @>\mu_0>> X_{*+r} \\ @VVV @VVV \\ X'_* \otimes X_* @>\mu^{-r}>> X_* \end{CD} \tag{5.6}$$

This amounts to showing that $X'_{*+r} \otimes X_* \rightarrow X'_* \otimes X_* \xrightarrow{\mu^{-r}} X_*$ factors through X_{*+r} , where $\mu^{-r} = \psi_1(\tilde{\mu}^{-r})$. Since $\mathcal{O}_\infty(x'_0, x_0; x_0) \rightarrow \mathcal{O}(x'_0, x_0; x_0) = \mathcal{K}_\infty^{\text{mod}}(x', x; x)$ is a trivial fibration and the point is cofibrant, we can lift $\tilde{\mu}^{-r}$ to $\tilde{\mu}^{-r} \in \tilde{\mathcal{O}}_\infty(x', x_0; x_0)$ with $\psi(\tilde{\mu}^{-r}) = \mu^{-r}$. One can find $a \in \mathcal{O}_\infty(x_r; x_0)$ and $a' \in \mathcal{O}_\infty(x'_r; x'_0)$, such that $\psi(a)$ and $\psi(a')$ are the colocalization maps $X_{*+r} \rightarrow X_*$ and $X'_{*+r} \rightarrow X'_*$, since $\mathcal{O}_\infty \rightarrow \mathcal{O}$ is a levelwise trivial fibration and this is true for \mathcal{O} .

$$\begin{CD} (\tilde{\mu}^{-r}, a') \in \mathcal{O}_\infty(x'_0, x_0; x_0) \otimes \mathcal{O}_\infty(x'_r; x'_0) @>\psi>> \mathcal{P}(x'_0, x_0; x_0) \otimes \mathcal{P}(x'_r; x'_0) \\ @VVV @VVV \\ \mathcal{O}_\infty(x'_r, x_0; x_0) @>\psi>> \mathcal{P}(x'_r, x_0; x_0) \ni f \end{CD}$$

$$\begin{CD} \{a\} \otimes \mathcal{O}_\infty(x'_r, x_0; x_r) @>\psi>> \mathcal{P}(x_r; x_0) \otimes \mathcal{P}(x'_r, x_0; x_r) \\ @. @VVV \\ * @>\psi>> \mathcal{O}_\infty(x'_r, x_0; x_0) @>\psi>> \mathcal{P}(x'_r, x_0; x_0) \ni f. \\ @. @VV\gamma V @VVV \end{CD}$$

Again using the fact that $\mathcal{O}_\infty \rightarrow \mathcal{O}$ is a levelwise trivial fibration, it is easy to check that γ is a trivial fibration, and hence, the lift exists. Define $f \in \mathcal{P}(x'_r, x_0; x_0)$ as the image of $(\tilde{\mu}^{-r}, a')$ under the first diagram; this represents the bottom-left composition in (5.6). This determines the bottom row of the second diagram, and if $\tilde{\mu}_0 \in \mathcal{O}_\infty(x'_r, x_0; x_r)$ is the image of the lift in the second diagram, then we take $\mu_0 = \psi(\tilde{\mu}_0)$. The other maps $\mu_0 : X''_* \otimes X'_{*+r} \rightarrow X'_{*+r}$ and $\mu_0 : X''_* \otimes X_{*+r} \rightarrow X_{*+r}$ can be defined similarly, and the associativity follows from the \mathcal{O}_∞ -algebra structure. □

Lemma 5.7 Given $\mu : X' \otimes X \rightarrow X$ and a pair of maps $\mu_0, \mu_1 : X'_* \otimes X_* \rightarrow X_*$ making the following diagram in $\mathcal{C}^{\mathbb{Z}}$ commute for $i = 0, 1$:

$$\begin{array}{ccc}
 X'_* \otimes X_* & \xrightarrow{\mu_i} & X_* \\
 \downarrow & & \downarrow \\
 X' \otimes X & \xrightarrow{\mu} & X,
 \end{array} \tag{5.8}$$

there is a homotopy $\mu_* : I \rightarrow \text{Hom}(X'_* \otimes X_*, X_*)$ between them, such that (5.8) commutes for $i \in I$.

Proof Let \mathcal{P} be as in Lemma 5.4 with $\mathbf{X} = (X'', X', X)$, the operad \mathcal{K}^{mod} , and the colocalization $s(t) = 0$ for all colors t . The lemma says that $\mathcal{P}(x'; x) \rightarrow \text{Hom}_{\mathcal{K}^{\text{mod}}}(\mathbf{X}, \mathbf{X})(x', x; x) = \text{Hom}(X' \otimes X, X)$ is a trivial fibration, and since all simplicial sets are cofibrant, we have a lifting diagram as follows:

$$\begin{array}{ccc}
 & \mathcal{P}(x', x; x) & \\
 & \nearrow & \downarrow \\
 * & \xrightarrow{\mu} & \text{Hom}(X' \otimes X, X).
 \end{array}$$

Then, μ_0 and μ_1 represent two lifts, and it is a standard model category fact that any two lifts are homotopic via a homotopy over $\text{Hom}(X' \otimes X, X)$. □

Proof of Proposition 5.3 Parts (1) and (2) are proved as Lemma 5.5. Part (3) follows from this using Lemma 5.7 and reindexing μ^r as a map $X'_* \otimes X_* \rightarrow X_*$ and similarly for μ_r .

We now turn to Part (4). Following the definition of the full slice s_* in [8, §5], consider $(l'_1 c'_0 d(X''), l'_1 c'_0 d(X'), l'_1 c'_0 d(X))$. A straightforward application of [8, Theorem 4.2] (similar to the proof of [8, Theorem 5.16]) shows that this object is a \mathcal{K}^{mod} -algebra, with strict comparison to $(d(X''), d(X'), d(X))$. □

The equivariant regular slice spectral sequence is an example of a tower of localizations; Moss’ theorem has been proved in this context by Ullmann [27]. The main example in [8] is the motivic slice spectral sequence, which we discuss next; Moss’ theorem in this context is new.

5.3 The \mathbb{R} -Motivic Effective Slice Spectral Sequence

Our motivation for writing this paper is to study Toda brackets in the \mathbb{R} -motivic effective slice spectral sequence for the sphere at $p = 2$. This is the spectral sequence associated with Voevodsky’s effective slice filtration: given an effective motivic spec-

trum X , there is a tower

$$\begin{array}{ccccccc}
 X = f_0 X & \longleftarrow & f_1 X & \longleftarrow & f_2 X & \cdots & \\
 \downarrow & & \downarrow & & & & \\
 s_0 X & & s_1 X & & \cdots & &
 \end{array}$$

which gives rise to the effective slice spectral sequence. By [21], if X is a ring spectrum, we get a multiplicative pairing of spectral sequences $E_r(X) \otimes E_r(X) \rightarrow E_r(X)$. The quotients $s_q X$ are called *slices*. See [15, 25] for more details.

Lemma 5.9 *Let X'', X' be motivic (strict) ring spectra with a ring map $X'' \rightarrow X'$, and let X be an X' -module. There are multiplications on the triples (X'', X', X) , $(f_*(X''), f_*(X'), f_*(X))$, and $(s_*(X''), s_*(X'), s_*(X))$ satisfying Assumption 3.14.*

For example, we are particularly interested in the case where $X'' = X' = X$ are the sphere spectrum S , which is strictly commutative.

Proof By [8, §6], the setup of [8, §2.1] applies to the effective slice filtration on the category $\mathcal{M} := \mathbf{Sp}_T^\Sigma(k)$ of symmetric motivic spectra, and so, Assumptions 3.14 (1)–(4) follow from Proposition 5.3. For Assumption 3.14 (5), Pelaez [21, Theorem 3.6.16] shows that there is a natural multiplication on the E_r -page, defined using the usual definition of E_r as a subquotient of E_1 . \square

Röndigs, Spitzweck, and Østvær [25] calculated the slices of the motivic sphere spectrum:

$$s_*(S) = H\mathbb{Z} \otimes \text{Ext}_{MU_* MU}^{*,*}(MU_*, MU_*), \tag{5.10}$$

where the notation means that a copy of $\mathbb{Z}/2^n$ in the Ext term corresponds to an (appropriately degree-shifted) $H\mathbb{Z}/2^n$ summand, where $H\mathbb{Z}/2^n$ is the motivic Eilenberg–MacLane spectrum. In the identification (5.10), (higher) multiplicative data about the Adams–Novikov E_2 -term $\text{Ext}_{MU_* MU}^{*,*}(MU_*, MU_*)$ carries over to the slice E_1 -page $\pi_{*,*}(s_*(S))$. In practice, we work one prime at a time, and focus on the $p = 2$ case. Combining results of [25] and Hu, Kriz, and Ormsby [10], the 2-completed slice spectral sequence for the \mathbb{R} -motivic sphere converges to the homotopy groups of the 2-completed sphere, which we denote by $\pi_{*,*}^{\mathbb{R}}(S)$.

We illustrate how one uses the Toda bracket convergence theorem to compute relations and permanent cycles in the \mathbb{R} -motivic slice spectral sequence.

Example 5.11 There is a strictly defined Adams–Novikov Massey product $\langle \alpha_1, 2, \alpha_1 \rangle = 2\alpha_{2/2}$. The class α_1 is a permanent cycle converging to $\eta \in \pi_{1,1}^{\mathbb{R}}(S)$ and $\alpha_{2/2}$ is a permanent cycle converging to $\nu \in \pi_{3,2}^{\mathbb{R}}(S)$. We have $2 \cdot \alpha_1 = 0$ in the slice E_1 -page, but $2 \cdot \eta$ is nonzero in $\pi_{*,*}^{\mathbb{R}}(S)$. However, the class $\omega = 2 + \rho\eta$ is also detected by 2 in the slice spectral sequence (as $\rho\eta$ has higher slice filtration), and we have $(2 + \rho\eta)\eta = 0 = (2 + \rho\eta)\rho$ (see [4, 19]).³ Hence, there is a Toda bracket $\langle \alpha_1, 2, \alpha_1 \rangle$ in the slice E_1 -page containing $2\alpha_{2/2}$ and there is a corresponding Toda

³ There is abuse of notation here; both $2 \in \pi_{0,0}^{\mathbb{R}}(S)$ and ω are detected by $2 \in E_1^{0,0,0}$.

bracket $\langle \eta, \omega, \eta \rangle$ in homotopy that is also strictly defined. Then, Theorem 4.16 implies that $2\alpha_{2/2}$ is a permanent cycle in the \mathbb{R} -motivic slice spectral sequence converging to $\omega\nu$ which is contained in the homotopy Toda bracket $\langle \eta, \omega, \eta \rangle$.

While this example is in low enough stems that the Toda bracket argument is not needed, this argument applies to a wealth of other Adams–Novikov Massey products.

Example 5.12 In $\pi_{*,*}^{\mathbb{R}}(H\mathbb{Z}/2^n)$, there is a strictly defined Toda bracket $\langle \rho, 2, 2^{n-1} \rangle = 2^{n-1}\tau$, which can be seen using the spectral sequence associated with the cofiber sequence of motivic Eilenberg–Maclane spectra

$$H\mathbb{Z}/2 \rightarrow H\mathbb{Z}/2^n \rightarrow H\mathbb{Z}/2^{n-1}.$$

This is another source of slice E_1 -page Toda brackets: if $x \in \text{Ext}_{MU_*MU}^{*,*}(MU_*, MU_*)$ generates a copy of $\mathbb{Z}/2^n$, then one may consider the E_1 -page Toda bracket $\langle \rho, 2, 2^{n-1}x \rangle \ni 2^{n-1}\tau x$. For example, there is a strictly defined E_1 -page Toda bracket $\langle \rho, 2, \alpha_1 \rangle = \tau\alpha_1$. Using the notation in Example 5.11, Theorem 4.16 implies that the $\pi_{*,*}^{\mathbb{R}}(S)$ Toda bracket $\langle \rho, \omega, \eta \rangle$ contains $\tau\eta$.

Example 5.13 There are also slice Massey products in higher pages which converge to Toda brackets. For example, the slice differential $d_1(\tau^2) = \rho^2\tau\alpha_1$ implies that $\langle 2, \rho^2, \tau\alpha_1 \rangle$ is strictly defined as a slice E_2 -page Massey product consisting of $2\tau^2$. The products $\omega \cdot \rho^2$ and $\rho^2 \cdot \tau\eta$ are zero in homotopy (see Example 5.11), and so, the Toda bracket $\langle \omega, \rho^2, \tau\eta \rangle$ in $\pi_{*,*}^{\mathbb{R}}(S)$ is defined and contains $\omega\tau^2$. Then, we may deduce the hidden multiplication $\eta \cdot \omega\tau^2$ using a shuffle argument as follows. We have

$$\eta \cdot \langle \omega, \rho^2, \tau\eta \rangle = \langle \eta, \omega, \rho^2 \rangle \tau\eta.$$

By Example 5.12, the right-hand side contains $\rho(\tau\eta)^2$. Moreover, the right-hand side is strictly defined: $\langle \eta, \omega, \rho^2 \rangle$ contains indeterminacy generated by $\rho^3\nu$, but this is annihilated by $\tau\eta$, since $\rho^2\tau\eta = 0$. Thus, we have

$$\eta \cdot \omega\tau^2 = \rho(\tau\eta)^2.$$

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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