

Brownian axionlike particles

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We study the nonequilibrium dynamics of a pseudoscalar axionlike particle (ALP) weakly coupled to degrees of freedom in thermal equilibrium by obtaining its reduced density matrix. Its time evolution is determined by the in-in effective action which we obtain to leading order in the ALP coupling but to all orders in the couplings of the bath to other fields within or beyond the standard model. The effective equation of motion for the ALP is a Langevin equation with noise and friction kernels obeying the fluctuation dissipation relation. A “misaligned” initial condition yields damped coherent oscillations, however, the ALP population increases towards thermalization with the bath. As a result, the energy density features a mixture of a cold component from misalignment and a hot component from thermalization with proportions that vary in time $\delta_{\text{cold}} \propto e^{-\Gamma t} \propto \delta_{\text{hot}} \propto 1 - e^{-\Gamma t}$, providing a scenario wherein the “warmth” of the dark matter evolves in time from colder to hotter. As a specific example we consider the ALP-photon coupling $g_a E^\mu \cdot B^\nu$ to lowest order, valid from recombination onwards. For $T \gg m_a$ the long-wavelength relaxation rate is substantially enhanced $\Gamma_T \propto \frac{g_a}{16} \frac{m_a}{\pi^2} T$. The ultraviolet divergences of the ALP self-energy require higher-order derivative terms in the effective action. We find that at high temperature, the finite-temperature effective mass of the ALP is $m_a^2 \delta T \propto m_a^2 \delta 0 \propto 1 - \delta T = T_c \propto T_c^4$, with $T_c \propto \sqrt{pm_a \delta 0} = g f_i$, suggesting the possibility of an inverted phase transition, which when combined with higher derivatives may possibly indicate exotic new phases. We discuss possible cosmological consequences on structure formation, the effective number of relativistic species and birefringence of the cosmic microwave background.

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I. INTRODUCTION

The axion, introduced in quantum chromodynamics (QCD) as a solution of the strong CP problem [1–3] may be produced nonthermally in the early Universe, for example by a misalignment mechanism and is recognized as a potentially viable cold dark matter candidate [4–6]. Extensions beyond the standard model can accommodate pseudoscalar particles with properties similar to the QCD axion, namely axionlike particles (ALPs) which can also be suitable dark matter candidates [7–11], in particular as candidates for ultra light dark matter [12,13]. Constraints

on the mass and couplings of ultralight ALPs [9–11,14] are being established by various experiments [15–17]. There are two important features that characterize ALPs: (i) a misalignment mechanism results in coherent oscillations of the expectation value of the ALP field which gives rise to the contribution to the energy density as a cold dark matter component [4–6,9–11,18], and (ii) its pseudoscalar nature leads to an interaction between the

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ALP and photons or gluons via pseudoscalar composite operators of gauge fields, such as $E^\mu \cdot B^\nu$ in the case of the ALP-photon interaction and $G^{\mu\nu;b} \tilde{G}_{\mu\nu;b}$ in the case of

gluons, which allows an ALP to decay into two photons or gluons. The effect of this decay process in the evolution of ALP condensates has been studied in Refs. [19–21] including stimulated decay in a photon background.

A. Motivation and objectives

In this article we study the nonequilibrium dynamics of coherent oscillations of ALPs coupled to generic environmental fields in thermal equilibrium by obtaining the nonequilibrium in-in effective action from which we derive the effective equations of motion of ALP condensates.

A simple example highlights our main motivation and objectives; consider the textbook situation of a particle in an harmonic potential immersed in a heat bath in equilibrium. The interaction of the particle with the bath degrees of freedom induce two main modifications to the equations of motion of the particle; (i) a friction term arising from energy transfer with the bath degrees of freedom, and (ii) a

stochastic noise term arising from the random “kicks” that the environment gives the particle. This is the basis of Brownian motion and the effective equation of motion of the Brownian particle is a Langevin equation, $\ddot{x} = -\gamma \dot{x} + \xi$

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$\xi = \omega_0 \sqrt{\gamma} \cos(\omega_0 t + \phi)$

with ξ a stochastic noise with a (generally) Gaussian probability distribution function yielding the (classical) averages and correlations

$$\langle \xi \rangle = 0; \quad \langle \xi \xi \rangle = 2\gamma k_B T \delta(t) - \langle \dot{\xi} \dot{\xi} \rangle = \langle \dot{\xi} \dot{\xi} \rangle$$

The relation between the noise correlation function and the friction coefficient in (1.2) is the (classical) fluctuation dissipation relation, a direct consequence of the bath degrees of freedom being in thermal equilibrium. As a result, whereas the (stochastic) average $\langle x \rangle$

$$x(t) = x_0 e^{-\gamma t/2} \cos\left[\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} t\right], \quad \text{the mean square fluctuation}$$

$\langle x^2 \rangle = \frac{4\gamma k_B T}{\omega_0^2 + \gamma^2}$, this is simply classical equipart $\gg 1$. In this case, the Brownian particle reaches thermal equilibration with the bath on a relaxation time scale $\propto 1/\gamma$.

This simple illustrative example motivates our study in this article, namely to understand the effective dynamics of ALPs when they are coupled to a bath of other degrees of freedom in (local) thermal equilibrium. The familiar example of a Brownian particle in a heat bath suggests that the effective equations of motion of a coherent ALP

condensate should be akin to a Langevin equation with a friction and noise term related by a fluctuation dissipation relation as a consequence of the bath degrees of freedom with which the ALP interacts being in thermal equilibrium. Our objective is precisely to derive, and solve such equation and explore its consequences by implementing the methods of nonequilibrium field theory. For this purpose, we adapt the seminal formulation of quantum Brownian motion [22–25] to the realm of nonequilibrium quantum field theory [26–30]. This is achieved in the in-in or Schwinger-Keldysh [26,31–33] formulation of time evolution in quantum field theory. Unlike the in-out formulation, the in-in formulation yields causal, retarded equations of motion [30,34–36].

The objectives of this study are twofold: (i) to obtain the time evolution of a reduced density matrix, nonequilibrium effective action, equations of motion and correlation functions for ALPs weakly coupled to degrees of freedom in thermal equilibrium. We first consider a generic model with coupling of the form $\propto g O$ with g a weak coupling, O the ALP field and O a composite pseudoscalar operator associated with the bath degrees of freedom. We obtain the effective action to leading order in the coupling ($O \delta g^2 \propto$), and to all orders of the couplings of the environmental

© 2022 American Physical Society fields to other degrees of freedom within or beyond the standard model, and (ii) to apply the general results to the relevant case of ALP-photon interaction with $O \propto E^\mu B_\mu$ where the radiation field in thermal equilibrium is identified with the cosmic microwave background (CMB).

In this article we address these objectives in Minkowski space-time, obtaining the (nonequilibrium) effective action and effective equations of motion for ALPs to order g^2 in the (weak) coupling g and arbitrary ALP mass m_a as a prelude to extending the methods to an expanding cosmology and exploring phenomenological consequences and constraints in future work. Furthermore, in this study we do not adopt a particular set of parameters for ALP couplings and mass, nor on possible bounds on these. Our main focus is to study the general aspects of the effective dynamics resulting from these interactions under the sole assumption of weak coupling between the ALP and degrees of freedom of the standard model and that the latter are in thermal equilibrium.

B. Summary of results

We study the time evolution of an initially prepared density matrix describing a misaligned initial state for the ALP and an equilibrium thermal bath of generic fields coupled to the ALP. Tracing over the bath fields yields a reduced density matrix for the ALP field whose time

evolution is determined by the nonequilibrium effective action which is obtained up to $O(\delta g^2)$ but to all orders in the couplings of the bath degrees of freedom to other degrees of freedom within or beyond the standard model.

The ALP equations of motion obtained from the nonequilibrium effective action are stochastic of the Langevin type (1.1) with a friction kernel determined by the retarded ALP self-energy (a manifestation of radiation reaction) and a Gaussian noise whose two-point correlation function is related to the self-energy via a generalized fluctuation dissipation relation. This is a consequence of the bath degrees of freedom being in thermal equilibrium. Hence, the notion of Brownian ALP's. In Ref. [37] a local friction coefficient for the equation of motion of the expectation value of the axion field was obtained as a consequence of sphaleronlike transitions in high-temperature QCD. Thermal friction from the ALP coupling to high-temperature plasmas has also been discussed in Refs. [38,39], and thermalization has been studied in Refs. [40,41]. Our approach is very different in that we obtain the in-in nonequilibrium effective action which allows us to obtain the full equation of motion including the noise term and directly show that the selfenergy contribution which yields the “friction” term and the noise correlation functions are related by fluctuation dissipation. The noise term is of paramount importance in obtaining correlations of the ALP field, and as a consequence of the noise term we find that the processes that lead to “friction” and damping of the misaligned expectation value are the same as those leading to thermalization with the environment on similar time scales, thereby providing a direct bridge between damping of a coherent condensate and thermalization. To the best of our knowledge, this approach, which explains both aspects of ALP's nonequilibrium dynamics, has not yet been implemented for ALPs. A corollary of this important result, relevant for dark matter, is that the energy density features a mixture of a “cold” and “hot” components whose relative weight vary in time; $E \approx \delta \text{cold} \rho e^{-\Gamma t} + \delta \text{hot} \rho e^{-\Gamma t} - e^{-\Gamma t} \rho$ where the “cold” component corresponds to the damped coherent oscillations arising from a misaligned initial condition and the hot component to the approach to thermalization and is a consequence of the stochastic noise. The relaxation rate Γ is determined by the imaginary part of the ALP self-energy and is a result of (stimulated) emission and absorption processes with the bath.

After obtaining the general results, we focus on the interaction of ALPs with photons via the coupling $g \mathbf{E} \cdot \mathbf{B}$. The coupling g has dimensions of $1 = \text{energy} \cdot \text{length}^2$ resulting in a nonrenormalizable interaction. Ultraviolet divergences necessitate the introduction of higher-derivative terms of the (ALP) fields. Emission and absorption processes such

as $a \leftrightarrow 2\gamma$ yield a relaxation rate that is enhanced at high temperatures $T \gg m_a$ by a factor $\propto T = m_a$. Furthermore, we find that the finite temperature contribution to the selfenergy yields a temperature dependent effective ALP mass $m_{2a} \delta T \approx m_{2a} \delta 0 \propto 1 - \delta T = T_c \rho^4$ with $T_c \propto \rho m_a \delta 0 = g f_i$, this behavior of the effective mass suggests an inverted phase transition and combined with the necessity for higher-derivative terms in the effective Lagrangian points to the possibility of novel phases and Lifshitz phase transitions [42].

In Sec. II we obtain the effective action out of equilibrium for ALP fields implementing the in-in Schwinger-Keldysh formulation of nonequilibrium field theory for a generic interaction of the form $g a \delta x \rho O \delta x \rho$, with an initial density matrix for the ALP field that implements a misaligned initial condition and a thermal density matrix for the environmental fields. In this section we obtain general results; the Langevin equation of motion for the ALP field, and the time-dependent energy density with a cold component from misalignment and a hot component from thermalization. In Sec. III we focus on the interaction with photons, obtain the one-loop, self-energy, and noise-correlation function at finite temperature, discuss renormalization issues and study their high- and low-temperature limits.

In Sec. IV we discuss the main aspects of the results and point out some caveats. In this section we argue that the results obtained in the case of photon interactions are valid after recombination in the early Universe and discuss possible cosmological consequences. Our conclusions are summarized in Sec. V. Several appendices include technical details.

II. THE EFFECTIVE ACTION OUT OF EQUILIBRIUM

We study the nonequilibrium effective action of an axionlike field “environmental δx ” coupled to generic fields “ χ ” via an operator $\chi \delta x \rho$ to which $O_\chi \delta x \rho$, with the Lagrangian density

$$L_{\text{eff}} = \frac{1}{2} \partial_\mu a \cdot x \partial^\mu a \cdot x - \frac{1}{2} m_a^2 a^2 \cdot x - \frac{1}{2} \delta \rho \cdot 2 \delta \rho g a \delta x \rho O_\chi \delta x \rho \delta L_\chi, \quad \delta \rho = 2 \delta \rho g a \delta x \rho O_\chi \delta x \rho \delta L_\chi, \quad \delta L_\chi = \delta \rho g a \delta x \rho O_\chi \delta x \rho \delta L_\chi, \quad \delta \rho = 2 \delta \rho g a \delta x \rho O_\chi \delta x \rho \delta L_\chi,$$

where L_χ is the Lagrangian density describing the “environmental” fields χ , these fields could be the electromagnetic field, fermion or gluon fields. We will first treat these fields generically to leading order in the coupling to exhibit the general form and properties of the

effective action for ALP and then we will focus specifically on the case of the pseudoscalar coupling to the electromagnetic field, a hallmark of ALPs.

The Lagrangian density (2.1) describes several relevant couplings of ALPs, such as

$$\mathcal{L} \supseteq \frac{1}{4} - g_a \partial x^\mu E^\nu \partial x^\mu B^\nu - g_s a \partial x^\mu G^{\mu\nu; b} \partial x^\nu G^{\sim \nu; b} \partial x^\mu;$$

$$\mathcal{L} \supseteq \frac{1}{4} - g_\psi a \partial x^\mu \bar{\psi} \partial x^\nu \psi \partial x^\mu;$$
δ2.2p

where \vec{E} , \vec{B} are the electromagnetic fields, $G^{\mu\nu; b}$, $\tilde{G}^{\sim \mu\nu; b}$ are the gluon field strength tensor and its dual respectively, and $\psi \partial x^\mu$ a fermionic field. Therefore the interaction in (2.1) describes a wide range of possible interactions of the ALP with other degrees of freedom which in this study are assumed to be in thermal equilibrium initially.

Whereas the photon and gluon interactions are not renormalizable because the respective couplings g_s, g_a feature dimensions $1 = \text{dimensions of energy}$, the coupling g_ψ is dimensionless so the interaction with the fermionic pseudoscalar is renormalizable. This aspect will have important consequences as discussed below in Sec. III.

Upon evolving the total initial density matrix in time, the degrees of freedom χ with the generic operator O_χ will be traced over to obtain a reduced density matrix for $a \partial x^\mu$. We achieve this to leading order in the coupling g , but to all orders in the couplings of the χ fields with themselves or with other degrees of freedom within or beyond the standard model, except for the ALP.

Although we are ultimately interested in obtaining an effective quantum field theory by tracing out these degrees of freedom in an expanding cosmology, in this study we focus on Minkowski space time as a first step towards extending these methods to cosmology. We consider the generic fields χ as a bath in thermal equilibrium.

The main strategy is to begin with an initial density evolve it in timematrix $\rho^\wedge \partial 0 \rho$ describing the ALP field and the environment, $\rho^\wedge \partial t \rho \supseteq U \partial t \rho^\wedge \partial 0 \rho U^{-1} \partial t \rho$ with $U \partial t \rho$ the

unitary time evolution operator for the ALP-environment, and trace over the environmental degrees of freedom yielding a reduced density matrix for the ALP fields, namely $\rho_a^\wedge \partial t \rho \supseteq \text{Tr}_\chi \rho^\wedge \partial t \rho$. This is the in-in or Schwinger-

Keldysh [26,31–33] formulation of nonequilibrium quantum field theory, the time evolution of the reduced density matrix is determined by a nonequilibrium effective action that includes the effects of the environment via a nonlocal term known as the influence functional [22] in the

theory of quantum Brownian motion. This effective action yields causal equations of motion [34,35], which turn out to be stochastic, akin to a Langevin equation with noise and dissipation terms that are related by a general fluctuation dissipation relation, a consequence of the environmental bath being in thermal equilibrium.

The reduced density matrix can be represented by a path integral in terms of the nonequilibrium effective action that includes the influence functional. This method has been used previously to study quantum Brownian motion [22–28,30] and for studies of quantum kinetics beyond the Boltzmann equation [26,43,44].

Let us consider the initial density matrix at a time $t \supseteq 0$ to be of the form $\rho^\wedge \partial 0 \rho \supseteq \rho^\wedge a \partial 0 \rho \otimes \rho^\wedge \chi \partial 0 \rho$:

The initial density matrix $\rho^\wedge a \partial 0 \rho$ is normalized so that $\text{Tr}_a \rho^\wedge a \partial 0 \rho \supseteq 1$ and that of the χ fields will be taken to describe a statistical ensemble in thermal equilibrium at a temperature $T \supseteq 1 = \beta$, namely

$$\rho^\wedge \chi \partial 0 \rho \supseteq \frac{e^{-\beta H_\chi}}{\text{Tr}_\chi e^{-\beta H_\chi}};$$
δ2.4p

where H_χ is the total Hamiltonian for the fields χ , and may include other fields to which χ is coupled other than the ALP; this possibility will be discussed further below.

The factorization of the initial density matrix is an assumption often explicitly or implicitly made in the literature, it can be relaxed by including initial correlations at the expense of daunting technical complications. We will not consider here this important case, relegating it to future study.

In the field basis the matrix elements of $\rho^\wedge a \partial 0 \rho$ and $\rho^\wedge \chi \partial 0 \rho$ are given by

$$h a j \rho^\wedge a \partial 0 \rho j a o i \supseteq \rho_{a; o} \partial a; a o \rho; \quad h \chi j \rho^\wedge \chi \partial 0 \rho j \chi o i \supseteq \rho_{\chi; o} \partial \chi; \chi o \rho;$$
δ2.5p

we emphasize that this is a functional density matrix as the field has spatial arguments. The density matrix for the ALP field a represents either a pure state or more generally an initial statistical ensemble, whereas $\rho^\wedge \chi \partial 0 \rho$ is given by Eq. (2.4).

The physical situation described by (2.4) is that of a field (or fields) in thermal equilibrium at a temperature $T \supseteq 1 = \beta$,

namely a heat bath, which is put in contact with another system, here represented by the field a . Once the system and bath are put in contact their mutual interaction will evolve the initial state out of equilibrium because the initial density matrix does not commute with the total Hamiltonian with interactions.

To obtain the effective action out of equilibrium for the ALP field a we evolve the initial density matrix in time and it follows that the reduced density matrix elements are

from which the reduced density matrix elements are

$$\rho_{\delta a f; a o f; t} \not\propto Z D\chi_f h a f; \chi_f j U \delta t \rho^* \delta O \rho U^{-1} \delta t \rho j a o f; \chi_f i$$

δ2:11p

$$\rho_{\delta a f; a o f; t} \not\propto Z D\chi_f Z D a i D \chi_i D a o i D \chi_o i h a f; \chi_f j U \delta t \rho j a i; \chi_i p a; \delta a i; a o i \rho$$

$$\otimes \rho_{\chi; 0} \delta \chi_i; \chi_{0 i} p h a o i; \chi_{0 i j} U^{-1} \delta t \rho j a o f; \chi_f i$$

δ2:13p

trace over the “bath” degrees of freedom, leading to a reduced density matrix for the field a , from which we can compute its expectation values or correlation functions as a function of time.

The time evolution of the initial density matrix is given by $\rho^* \delta t \rho \not\propto U \delta t \rho^* \delta O \rho U^{-1} \delta t \rho$; δ2:6p

where

$$U \delta t \rho \not\propto e^{-iHt}; \quad \delta2:7p$$

The total Hamiltonian H is given by

$$H \not\propto H_{0a} + H_\chi + g Z d^3 x \delta x \rho O_\chi \delta x \rho; \quad \delta2:8p$$

and H_{0a}, H_χ are the Hamiltonians for the respective fields.

The reduced density matrix for the ALP field is obtained by tracing over the χ degrees of freedom as

$$\rho_{ra \delta t} \not\propto \text{Tr}_\chi U \delta t \rho^* \delta O \rho U^{-1} \delta t \rho; \quad \delta2:9p$$

To extract the nonequilibrium effective action for the ALP it is more convenient to obtain the density matrix elements in field space, namely $\rho_{\delta a f; \chi f; a o f; \chi o f; t} \not\propto h a f; \chi_f j U \delta t \rho^* \delta O \rho U^{-1} \delta t \rho j a o f; \chi_f i$;

δ2:10p

With the functional integral representation

$$h a f; \chi_f j U \delta t \rho^* \delta O \rho U^{-1} \delta t \rho j a o f; \chi_f i$$

$$\not\propto Z D a i D \chi_i D a o i D \chi_o i h a f; \chi_f j U \delta t \rho j a i; \chi_i p a; \delta a i; a o i \rho$$

$$\otimes \rho_{\chi; 0} \delta \chi_i; \chi_{0 i} p h a o i; \chi_{0 i j} U^{-1} \delta t \rho j a o f; \chi_f i; \quad \delta2:12p$$

The etc, are functional integrals where the spatial argument has been suppressed. The matrix elements of the R

time evolution forward and backward can be written as path integrals, namely $h a f; \chi_f j U \delta t \rho j a i; \chi_i \not\propto Z D a p D \chi_{p i} e i R$

$$d^4 x L \not\propto a p; \chi_p; \quad \delta2:14p$$

$$h a o i; \chi_{0 i j} U^{-1} \delta t \rho j a o f; \chi_f i \not\propto Z D a - D \chi - e - i R d^3 x L \not\propto a - \chi; \quad \delta2:15p$$

where we use the shorthand notation

$$Z d^4 x \equiv Z_0^t dt Z d^3 x; \quad \delta2:16p$$

$L \not\propto a - \chi$ is given by (2.1) and the boundary conditions on the path integrals are

where the time evolution kernel is given by

$$\begin{aligned} a \delta x; t' \not\propto 0 &\propto a_i \delta x' \not\propto; & a^b \delta x; t' \not\propto 0 &\propto a_f \delta x' \not\propto; & \delta 2:17 \not\propto \\ \chi_b \delta x; t' \not\propto 0 &\propto \chi_i \delta x' \not\propto; & \chi_b \delta x; t' \not\propto 0 &\propto \chi_f \delta x' \not\propto; & \delta 2:18 \not\propto \\ a^- \delta x; t' \not\propto 0 &\propto a_0 \delta x' \not\propto; & a^- \delta x; t' \not\propto 0 &\propto a^0_f \delta x' \not\propto; & \delta 2:19 \not\propto \\ \chi^- \delta x; t' \not\propto 0 &\propto \chi_0 \delta x' \not\propto; & \chi^- \delta x; t' \not\propto 0 &\propto \chi_0 \delta x' \not\propto; & \delta 2:20 \not\propto \end{aligned}$$

The field variables $a; \chi$ along the forward ($\not\propto$) and backward ($-$) evolution branches are recognized as those

necessary for the in-in or Schwinger-Keldysh [26,31–33] closed time path approach to the time evolution of a density matrix.

The reduced density matrix for the light field a (2.13), can be written as

$$\rho \delta a_f; a_0; t \not\propto 0 \propto Z D a_i D a_0 T \not\propto a_f; a_0; a_i; a_0; t \not\propto a \delta a_i; a_0; 0 \not\propto;$$

$\delta 2:21 \not\propto$

from which the in-in effective action out of equilibrium is identified as

$$S_{\text{eff}} \not\propto a; a^- \not\propto Z \int_0^t dt Z \int_0^t d\chi^b L \not\propto a \not\propto - L \not\propto a^- \not\propto - \not\propto a; a^-;$$

$\delta 2:23 \not\propto$

where $L \not\propto a; a^-$ is the influence action [22] obtained by tracing over the χ degrees of freedom,

$$e^{i \not\propto a^b; a^- \not\propto} Z D \chi_i D \chi_0 D \chi_f Z D \chi^b Z D \chi^- e^{-i R d^4 x \not\propto L \not\propto \chi^b - g a^b O_\chi^b e^{-i R d^4 x \not\propto L \not\propto \chi^- - g a^- O_\chi^-} \rho_\chi \delta \chi_i; \chi_0; 0 \not\propto}; \quad \delta 2:24 \not\propto$$

The path integral representations for both $T \not\propto a_f; a_i; a^0_f; a^0_i; t$ and $L \not\propto a^b; a^-$ feature the boundary conditions in (2.17)–(2.20) except that we now set $\chi \delta x; t' \not\propto 0 \propto \chi_f \delta x' \not\propto$ to trace over χ field.

In the above path integral defining the influence action Eq. (2.24), the ALP fields $a \delta x \not\propto$ act as external sources (c-number) coupled to the operator O_χ . Therefore, it is straightforward to conclude that the right hand side of Eq. (2.24) is the path integral representation of the trace over the environmental fields coupled to external sources a , namely

$$e^{i \not\propto a; a^- \not\propto} \text{Tr} \chi \not\propto U \delta t; a \not\propto \rho_\chi \delta 0 \not\propto U^{-1} \delta t; a \not\propto; \quad \delta 2:25 \not\propto$$

where $U \delta t; a \not\propto$ is the time evolution operator in the χ sector in presence of external sources a , i.e.,

$$U \delta t; a \not\propto \not\propto T \delta e^{-i R_0} H_\chi \not\propto a \not\propto \delta t \not\propto \not\propto; \quad U^{-1} \delta t; a \not\propto \not\propto T \delta e^{-i R_0} H_\chi \not\propto a \not\propto \delta t \not\propto \not\propto; \quad \delta 2:26 \not\propto$$

with

$$H_\chi \not\propto a \not\propto \not\propto \not\propto \not\propto H_\chi \not\propto g Z d^3 x a \not\propto \delta x; t' \not\propto \not\propto \not\propto O_\chi \delta x' \not\propto; \quad \delta 2:27 \not\propto$$

and T^\sim is the antitime evolution operator describing evolution backward in time, it is defined by $T^\sim \delta A \delta t_1 \delta B \delta t_2 \dots$

The calculation of the influence action is facilitated by passing to the interaction picture for the Hamiltonian $t^1 \delta B \delta t^2 \delta A \delta t^2 - t^1 \delta B \delta t^2 \delta A \delta t^1 \delta B \delta t^1 - t^2 \delta A \delta t^2 \delta B \delta t^1 \delta A \delta t^1 \delta B \delta t^2$, defining

$$U \delta t; a \delta t \sim e^{-iH_\chi t} U_{ip} \delta t; a \delta t$$

δ2:28p

and the δt for the $iH_\chi t$ cancel out in the trace in χ sector, it follows that (2.25), since $U \delta t; a \delta t$ is the time evolution operator in presence of external sources

$$a \delta x; t$$

$$U_{ip} \delta t; a \delta t \sim 1 - ig Z d^4 x^0 a^b \delta x; t^\sim \delta O_\chi \delta x; t^\sim \delta b$$

δ2:29p

$$- g_1^2 Z d^4 x_1 Z d^4 x_2 T \delta a^b \delta x^\sim_1; t_1 \delta O_\chi \delta x^\sim_1; t_1 \delta a^b \delta x^\sim_2; t_2 \delta O_\chi \delta x^\sim_2; t_2 \delta b \delta b ; 2$$

$$U_{ip}^{-1} \delta t; a \delta t \sim 1 + ig Z d^4 x^0 a^- \delta x^\sim 0; t^0 \delta O_\chi \delta x^\sim 0; t^0 \delta b$$

δ2:30p

$$- g_2^2 Z d^4 x_1 Z d^4 x_2 T^\sim \delta a^- \delta x^\sim_1; t_1 \delta O_\chi \delta x^\sim_1; t_1 \delta a^- \delta x^\sim_2; t_2 \delta O_\chi \delta x^\sim_2; t_2 \delta b \delta b ;$$

where $O_\chi \delta x; t^\sim \delta b$ is in the Heisenberg picture of H_χ .

Now the trace (2.25) can be obtained systematically in perturbation theory in g from which we obtain the influence functional. Up to $O(g^2)$ we find

$$I \delta a \delta b; a^- \sim -g Z d^4 x \delta a \delta b \delta x \delta b - a^- \delta x \delta b \delta h O_\chi \delta x \delta b \delta \chi$$

$$+ ig Z d^4 x_1 Z d^4 x_2 f a^b \delta x_1 \delta a^b \delta x_2 \delta b G^b_c \delta x_1 - x_2 \delta b \delta a^- \delta x_1 \delta a^- \delta x_2 \delta b G^c_c \delta x_1 - x_2 \delta b$$

δ2:31p

$$- a^b \delta x_1 \delta a^- \delta x_2 \delta b G^b_c \delta x_1 - x_2 \delta b - a^- \delta x_1 \delta a^b \delta x_2 \delta b G^b_c \delta x_1 - x_2 \delta b ;$$

which is confirmed by expanding the left-hand side of (2.25) and comparing to the right-hand side. In this expression the connected correlation functions in the initial density matrix of the χ fields, namely $\rho_\chi \delta 0 \delta b$ are given by

$$G_{-c} \delta x_1 - x_2 \delta b \sim h O_\chi \delta x_1 \delta O_\chi \delta x_2 \delta b \delta \chi - h O_\chi \delta x_1 \delta \chi h O_\chi \delta x_2 \delta b \delta \chi \sim G_{>c} \delta x_1 - x_2 \delta b ;$$

δ2:32p

$$G_{b-c} \delta x_1 - x_2 \delta b \sim h O_\chi \delta x_2 \delta O_\chi \delta x_1 \delta b \delta \chi - h O_\chi \delta x_2 \delta \chi h O_\chi \delta x_1 \delta b \delta \chi \sim G_{<c} \delta x_1 - x_2 \delta b ;$$

δ2:33p

$$G_{\mu\mu c} \delta x_1 - x_2 \not{p} \not{q} G_{c>} \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_1 - t_2 \not{p} \not{p} G_{cc} \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p} \not{p} \quad \delta 2:34p$$

$$G_{--c} \delta x_1 - x_2 \not{p} \not{q} G_{c>} \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p} \not{p} G_{cc} \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_1 - t_2 \not{p} \not{p} \quad \delta 2:35p$$

in terms of fields in the Heisenberg picture of H_χ , where

$$h \not{p} \not{i} \not{q} \text{Tr}_\chi \not{p} \not{p} \not{\rho}_\chi \not{0} \not{p} \not{p} \quad \delta 2:36p$$

Furthermore, for the case of hermitian operators O_χ as considered here it follows that

$$G_c \delta x_1 - x_2 \not{p} \not{q} G^< \delta x_2 - x_1 \not{p} \not{p} \quad \delta 2:37p$$

We highlight that the correlation functions $G^>; G^<$ are exact, namely to all orders in the couplings of the environmental fields χ that enter in O to all other fields to which it couples except the ALPs.

In the cases under consideration, we assume that the initial density matrix for the bath, $\rho_\chi \not{0} \not{p} \not{q} e^{-\beta H_\chi}$ is CP invariant, for example in quantum electrodynamics where $\rho_\chi \not{0} \not{p} \not{p}$ describes blackbody radiation, for which $h^{E \rightarrow} \cdot B \rightarrow i \not{p} \not{q} 0$.

Therefore $h^O \delta x_{1,2} \not{p} \not{i} \not{q} 0$ in the connected correlation functions (2.32)–(2.35), hence in what follows we suppress the subscript “c” in the correlation functions.

The influence action (2.31) becomes simpler by writing it solely in terms of the two correlation functions $G^<$, this is achieved by implementing the following steps:

(i) In the term with $a^b \not{\delta} x_1 \not{p} a^b \not{\delta} x_2 \not{p} \not{p}$: in the contribution

$G^< \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p}$ [see Eq. (2.34)] relabel $x_1 \leftrightarrow x_2$ and use the property (2.37).

(ii) In the term with $a^b \not{\delta} x_1 \not{p} a^b \not{\delta} x_2 \not{p} \not{p}$: in the contribution

$G^> \delta x_1 - x_2 \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p}$ [see Eq. (2.35)] relabel $x_1 \leftrightarrow x_2$ and use the property (2.37).

(iii) In the term with $a^b \not{\delta} x_1 \not{p} a^b \not{\delta} x_2 \not{p} \not{p}$: multiply $G^< \delta x_1 - x_2 \not{p} \not{p}$

by $\not{\Theta} \not{t}_1 - t_2 \not{p} \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p} \not{p} \not{q} 1$ and in the term with $\not{\Theta} \not{t}_2 - t_1 \not{p} \not{p}$ relabel $x_1 \leftrightarrow x_2$ and use the property (2.37).

(iv) In the term with $a^b \not{\delta} x_1 \not{p} a^b \not{\delta} x_2 \not{p} \not{p}$: multiply $G^> \delta x_1 - x_2 \not{p} \not{p}$ by $\not{\Theta} \not{t}_1 - t_2 \not{p} \not{p} \not{\Theta} \not{t}_2 - t_1 \not{p} \not{p} \not{q} 1$ and in the term with $\not{\Theta} \not{t}_2 - t_1 \not{p} \not{p}$ relabel $x_1 \leftrightarrow x_2$ and use the property (2.37).

We find

$$1/2 a^b; a^b \not{q} i g^2 Z d^4 x_1 d^4 x_2 f a^b \not{\delta} x_1; t_1 \not{p} a^b \not{\delta} x_2; t_2 \not{p} G^> \delta x_1 - x_2 \not{p} \not{p}$$

$$\not{p} a^b \not{\delta} x_1; t_1 \not{p} a^b \not{\delta} x_2; t_2 \not{p} G^< \delta x_1 - x_2 \not{p} \not{p}$$

$$1; t_1 \not{p} a^b \not{\delta} x_2; t_2 \not{p} G^< \delta x_1 - x_2 \not{p} \not{p}$$

$$-a^b \not{\delta} x_1; t_1 \not{p} a^b \not{\delta} x_2; t_2 \not{p} G^> \delta x_1 - x_2 \not{p} g^O \not{\delta} t_1 - t_2 \not{p} \not{p}$$

$\delta 2:38p$

where $G^<$ are given by Eqs. (2.32) and (2.33). This is the general form of the influence function up to second order in the ALP-environment coupling but to all orders in the couplings of the environmental fields that enter the composite operator O to any other field. Notice that $1/2 a^b; a^b \not{q} a^b \not{q} 0$ consistently with its definition given by Eq. (2.25). A graphical depiction of the influence action $1/2 a^b; a^b$ is displayed in Fig. 1.

For example, for the ALP-photon interaction in Eq. (2.2), some of the correlations included in the influence action are displayed in Fig. 2; the one-loop diagram features free-

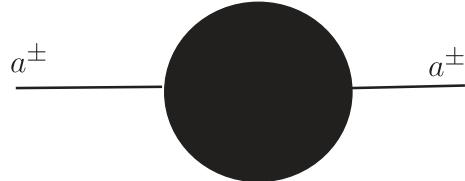


FIG. 1. A graphical depiction of $\langle a^b a^- \rangle$. The black circle denotes the correlation functions G_S to all orders in the couplings to degrees of freedom other than the ALP.

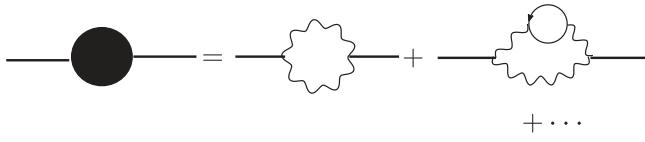


FIG. 2. $\langle a^b a^- \rangle$ for the ALP-photon interaction in (2.2). The one-loop diagram features free photon propagators, the two-loop diagram includes the polarization from e^+e^- pairs in the thermal plasma, etc.

photon propagators, the two-loop diagram features a polarization correction to one of the propagators with electron-positron pairs in the thermal bath, this two-loop diagram features an extra power of α the fine structure constant. Similar diagrams with quarks and gluon loops are included for the ALP-gluon interaction in (2.2). The black bath degrees of freedom in thermal equilibrium to all orders “bubble” symbolizes the $hOOi$ correlation functions of the

in their interactions, $\alpha; \alpha_s$, etc.

We can obtain expectation values and correlation functions of the ALP fields by including sources δa^b and defining the generator J^b with

functional

$$Z \int d^4x J^b \int d^4x' a^b(t) a^-(t') = \int d^4x J^b(t) a^-(t)$$

$$\times Z \int d^4x a^b(t) a^-(t) = Z \int d^4x a^b(t) a^-(t)$$

02:39 p

with the boundary conditions

$$\begin{aligned} a^b \delta x; t' &= 0; \quad a^b \delta x; t' = 0; \\ a^- \delta x; t' &= 0; \quad a^- \delta x; t' = 0; \end{aligned}$$

02:40 p

Expectation values or correlation functions of a in the reduced density matrix are obtained as usual with variational derivatives with respect to the sources J .

A. Effective equations of motion: Langevin equation

The effective action (2.23) may be written in a manner more suitable to exhibit the equations of motion by introducing the Keldysh [32] variables

$$A \delta x; t' = \frac{1}{2} \delta a^b \delta x; t' + \delta a^- \delta x; t' = 0;$$

$$R \delta x; t' = \delta a^b \delta x; t' - \delta a^- \delta x; t' = 0;$$

02:41 p

The boundary conditions on the a path integrals given by (2.40) translate into the following boundary conditions on the center of mass and relative variables

$$A \delta x; t' = 0; \quad R \delta x; t' = 0; \quad 02:42 p$$

$$A \delta x; t' = 0; \quad R \delta x; t' = 0; \quad 02:43 p$$

In terms of the center of mass and relative field variables, the effective action (2.23) with the influence functional (2.31) becomes with $\omega^2 k \propto m^2 a \propto k^2$

$$i S_{\text{eff}}[A; R] = -i \int d^3x R \delta x \int d^4t A^b \delta x; t = 0 = i \int d^3x R \int d^4t A^b \delta x; t = 0$$

→

$$- Z_0^t \int d^3x_1 Z_0^t \int d^3x_2 \frac{1}{2} \int d^4t_1 \int d^4t_2 N_{k^-} \delta t_1 - t_2 R_{k^-} \delta t_2 = R_{k^-} i \Sigma^{R_{k^-}} \delta t_1 - t_2 A_{k^-} \delta t_2; \quad 02:44 p$$

where we have integrated by parts and defined $J\delta x \nabla \frac{1}{4} \delta J \delta x - J \delta x \nabla \delta J$, keeping solely the source conjugate to A because we are interested in expectation values and correlation functions of this variable only as discussed in detail below. The kernels in the above effective Lagrangian are given by [see Eqs. (2.32)–(2.35)]

$$N_k \delta t - t^0 \nabla \frac{1}{4} g \nabla^2 \frac{1}{2} G^> \delta k; t - t^0 \nabla \delta k; t - t^0 \nabla; \quad \delta 2:45p$$

$$i \Sigma^R_k \delta t - t^0 \nabla \frac{1}{4} g^2 \frac{1}{2} G^> \delta k; t - t^0 \nabla - G^< \delta k; t - t^0 \nabla \Theta \delta t - t^0 \nabla \equiv i \Sigma_k \delta t - t^0 \nabla \Theta \delta t - t^0 \nabla; \quad \delta 2:46p$$

where the effective action $G^< \delta k; t - t^0 \nabla$ is the spatial Fourier transform of the correlation functions g_{eff} , the quadratic term in the relative variable R can be written as a functional integral over a noise (2.32)–(2.35). In the exponential of variable ξ as follows:

$$\exp -\frac{1}{2} \int dt_1 \int dt_2 R_{-k} \delta t_1 \nabla N_k \delta t_1 - t_2 \nabla R_k \delta t_2 \nabla \\ \frac{1}{4} C \int D\xi \exp -\frac{1}{2} \int dt_1 \int dt_2 \xi_{-k} \delta t_1 \nabla N_k \delta t_1 - t_2 \nabla \xi_k \delta t_2 \nabla \int dt \xi_{-k} \delta t \nabla R_k \delta t; \quad \delta 2:47p$$

where C is a normalization factor.

For the initial density matrix $\rho_a \delta a_i; a^0; 0$ in (2.39) it proves convenient to write it in terms of the initial center of mass and relative variables A_i, R_i as

$$\rho_a \delta a_i; a^0; 0 \equiv \rho_a A_i \nabla^2; A_i = 2i; 0 \quad \delta 2:48p$$

and introduce the functional Wigner transform [45]

$$W^< A_i; \pi \frac{1}{4} \int DR i e^{-i R} \int dx \pi \delta x \nabla^2 R \delta x \nabla^2 \rho_a A_i \nabla^2 R^2; A_i = 2i; 0 \quad \delta 2:49p$$

which allows us to write (up to a normalization factor)

$$\rho_a A_i \nabla^2; A_i = 2i; 0 \frac{1}{4} \int D\pi i e^i R \int dx \pi \delta x \nabla^2 R \delta x \nabla^2 W^< A_i; \pi; \quad \delta 2:50p$$

As it will become clear below, the Wigner transform naturally leads to an initial value problem wherein the evolution of the field is determined from initial conditions on its value and first time derivative.

Gathering these results together, we now write the generating functional (2.39) in terms of the Keldysh variables (2.41), with the effective action in these variables given by Eq. (2.44), implementing the Wigner transform (2.50) and using the representation (2.47)

$$Z/J \propto \int D\mathbf{A}_f D\mathbf{R}_i D\mathbf{A}_i D\pi_i Z DADRD\xi W/A_i; \pi_i \times P/\xi \times \exp i \int dt X_k A_k \delta t \mathbf{P} J_{-k} \delta t \mathbf{P}$$

$$\times \exp -i \int dt X_k R_{-k} \delta t \mathbf{P} \dot{A}_k \delta t \mathbf{P} \delta \omega_{2k} A_k \delta t \mathbf{P} \delta Z_{0t} \Sigma_k \delta t - t_0 \mathbf{P} A_k \delta t \mathbf{P} \delta \dot{A}_k \delta t \mathbf{P}$$

$$\times \exp i X_k R_{-k} \delta t \mathbf{P} \delta \pi \delta k \delta t \mathbf{P} - A \dot{i} \delta k \delta t \mathbf{P}:$$

02:51 p

where the noise probability distribution function

$$P/\xi \propto N \sim Y_k \exp -21 \int dt_1 \int dt_2 \xi_{-k} \delta t_1 \mathbf{P} N_{-k}^{-1} \delta t_1 - t_2 \mathbf{P} \xi_k \delta t_2 \mathbf{P}:$$

02:52 p

The generating functional Z/J is the final form of the time evolved reduced density matrix after tracing over the bath, yield the correlation functions of the Keldysh center degrees of freedom. Variational derivatives with respect to the source of mass variable A .

Carrying out the functional integrals over $R_i \delta k \delta t$ and $R_k \delta t \mathbf{P}$ yields a more clear form, namely

$$Z/J \propto \int D\mathbf{A}_f D\mathbf{R}_i D\mathbf{A}_i D\pi_i Z DAD\xi W/A_i; \pi_i \times P/\xi \times \exp i \int dt X_k A_k \delta t \mathbf{P} J_{-k} \delta t \mathbf{P}$$

$$\times Y_k \delta A_k \delta t \mathbf{P} \delta \omega_{2k} A_k \delta t \mathbf{P} \delta Z_{0t}$$

$$\Sigma_k \delta t$$

$$- t_0 \mathbf{P} A_k \delta t \mathbf{P} \delta \dot{A}_k \delta t \mathbf{P} \times Y_k \delta \pi \delta k \delta t \mathbf{P} - A \dot{i} \delta k \delta t \mathbf{P}:$$

02:53 p

The functional delta functions clearly determine the field configurations that contribute to the generating functional Z/J :

(i) The equation of motion of $A_k \delta t \mathbf{P}$ is a stochastic Langevin equation, namely

$$\tilde{A}_{k^*} \delta t \propto \omega_{2k^*} A_{k^*} \delta t \propto Z \quad \Sigma_{k^*} \delta t - t_0 A_{k^*} \delta t \propto \xi_{k^*} \delta t:$$

0

02:54p

Note that this equation of motion involves the retarded self-energy, thereby defining a causal initial value problem, this is a distinct consequence of the in-in formulation of time evolution. (ii) The initial conditions of A_{k^*} satisfy

$$A_{k^*} \delta t \propto 0 \propto A_{i;k^*}; A_{i;k^*} \delta t \propto 0 \propto \pi_{i;k^*}; \quad 02:55p$$

$A_{i;k^*}, \pi_{i;k^*}$ are drawn from the distribution function

$W\%A_i;\pi_i$ (i.e., the initial density matrix). This is one of the manifestations of stochasticity,

and we use δP to denote averaging over the initial conditions (2.55) with the distribution function $W\%A_i;\pi_i$.

(iii) The expectation value and correlations of the stochastic noise $\xi_{k^*} \delta t \propto$ are determined by a Gaussian probability distribution $P\%\xi$, yielding

$$\langle\langle \xi_{k^*} \delta t \propto \rangle\rangle \propto 0; \langle\langle \xi_{k^*} \delta t \propto \xi_{k^*} \delta t \propto \rangle\rangle \propto N_{k^*} \delta t - t_0 \delta_{k^*} \delta_{k^*} \propto; \quad 02:56p$$

Since functions are obtained by implementing Wick where $P\%\langle\langle \xi \rangle\rangle$ is Gaussian, higher-order correlation means averaging weighted by $P\%\xi_s$.

theorem. This averaging is the second manifestation of stochasticity.

Therefore, averaging over both the initial conditions with the Wigner distribution function, and the noise with $P\%\xi$, is

now denoted by $\langle\langle \delta t \propto \rangle\rangle$ and $\xi \delta t \propto$. These stochastic averages is any functional of the initial conditions (2.55)

yield the expectation values and correlation functions of functionals of A obtained from variational derivatives with respect to J .

It remains to relate observables to correlation functions of the Keldysh center of mass variable A . The path integral representations for the forward and backward time evolution operators (2.12), (2.14), and (2.15) show that a^b is

associated with that inside the path integral operators in the forward, $U \delta t \propto$ and a^- with $U^{-1} \delta t \propto$; hence, it follows

backward and mixed forward-backward branches,

$$A^b B^b \rightarrow \text{Tr} A B \propto; \quad A^- B^- \rightarrow \text{Tr} B A \propto; \quad A^b B^- \rightarrow \text{Tr} A \propto B; \quad 02:57p$$

etc. Therefore, from the cyclic property of the trace the expectation value of the ALP field in the total density matrix is

$$\begin{aligned} \text{ha} \delta x; t \propto \text{Tr} \delta x; t \propto \delta 0 \propto \text{Tr} \delta 0 \delta x; t \propto \propto \\ \propto \text{Tr} A \delta x; t \propto \delta 0 \propto \overline{\langle\langle A \delta x; t \propto \rangle\rangle}; \quad 02:58p \end{aligned}$$

whereas

$$\text{Tr} R \delta x; t \propto \delta 0 \propto 0; \quad 02:59p$$

We now introduce

$$C_{>k} \delta t; t_0 \propto \text{Tr} a_{-k^*} \delta t \delta a_{-k^*} \delta t \propto \delta 0 \propto;$$

$$C_{<k} \delta t; t_0 \propto \text{Tr} a_{-k^*} \delta t \delta a_{-k^*} \delta t \propto \delta 0 \propto; \quad 02:60p$$

and the energy per mode of wave vector k^*

$$E_k \propto \frac{1}{4} \frac{\partial \delta t}{\partial t \partial t^0} \propto \Omega_k^2 \frac{1}{2} C_{>k} \delta t; t_0 \propto C_{<k} \delta t; t_0 \propto; \quad 02:61p$$

where we anticipate a renormalization of the frequency $\omega_k \rightarrow \Omega_k$, which will be addressed in detail below. Using the definition (2.41) and the relations (2.57) it is

straightforward to show that this symmetrized product yields

$$E \frac{1}{2} \text{Tr} \delta A^\alpha \delta A^\beta \frac{1}{2} \delta t^\alpha \delta t^\beta \langle \Omega_{2k} \delta A^\alpha \delta t^\beta A^\beta \delta t^\alpha \delta A^\alpha \rangle$$

$$= \frac{1}{2} \langle \delta A^\alpha \delta A^\beta \rangle$$

which is the average energy per mode, a component of the energy momentum tensor. This analysis confirms that at least for the time evolution of the expectation values of the ALP field and its energy (momentum tensor) only the center of mass Keldysh variable A is needed.

B. General properties of environmental correlation functions

The dynamics and dissipative processes depend on the correlation functions of the environment and crucially on their spectral density, these correlation functions determine the self-energy Σ and the noise-correlation function N .

Because the bath is in thermal equilibrium, its initial density matrix is $\rho_0 \propto e^{-\beta H_b} = \text{Tr} e^{-\beta H_b}$ which is spacetime translationally invariant, and the Heisenberg picture operators associated with the bath are given by $iH_b t$ $O_b \delta x; t^\alpha \propto$

$e^{iH_b t} O_b \delta x; t^\alpha e^{-iH_b t}$ we can write

$$G^> \delta x^\alpha - x^\alpha(0; t) - t^0 \propto \text{Tr} O_b \delta x; t^\alpha O_b \delta x^\alpha(0; t^0) \propto$$

$$Z = \frac{1}{2} \int \frac{d\pi^4 k}{2\pi} \rho^>(\vec{k}) e^{-ik_0 t} e^{i\vec{k} \cdot \vec{x}} \propto$$

$$-x^\alpha(0) \propto$$

$$G^< \delta x^\alpha - x^\alpha(0; t) - t^0 \propto \text{Tr} O_b \delta x^\alpha(0; t^0) O_b \delta x; t^\alpha \propto$$

$$Z = \frac{1}{2} \int \frac{d^4 k}{2\pi} \rho^<(\vec{k}) e^{-ik_0 t} e^{i\vec{k} \cdot \vec{x}} \propto$$

$$-x^\alpha(0) \propto$$

These representations are obtained by writing $O_b \delta x; t^\alpha \propto e^{iH_b t} e^{-iH_b t} O_b \delta x^\alpha$ and introducing a complete set of simultaneous eigenstates of H_b and the total momentum operator P^α , $\delta H_b; P^\alpha \propto j_{ni} \propto \delta E_n; P^\alpha \propto j_{ni}$, from which we obtain the following Lehmann representations,

$$\rho^>(\vec{k}) \propto \frac{1}{2} \text{Tr} \int \frac{d\pi^4 k}{2\pi} \delta H_b; P^\alpha X_{m;n} e^{-\beta E_n} j_{hnj} O_b \delta x^\alpha$$

$\propto j_{mj} j_{nj}$

$$\propto \delta \delta k_0 - \delta E_m - E_n \rho^>(\vec{k}) \propto -\delta P_m - P_n \rho^>(\vec{k}) \propto$$

$$\rho^<(\vec{k}) \propto \frac{1}{2} \text{Tr} \int \frac{d\pi^4 k}{2\pi} \delta H_b; P^\alpha X_{m;n} e^{-\beta E_n} j_{hmj} O_b \delta x^\alpha$$

$$\propto \delta \delta k_0 - \delta E_n - E_m \rho^<(\vec{k}) \propto -\delta P_n - P_m \rho^<(\vec{k}) \propto$$

Upon relabeling $m \leftrightarrow n$ in the sum in the definition (2.66) and recalling that O is an Hermitian operator, we find the Kubo-Martin-Schwinger relation [46–50] $\rho^<(\vec{k}_0) \propto \rho^>(\vec{k}_0) \propto e^{-\beta k_0} \rho^>(\vec{k}_0) \propto$

$$\propto \delta \delta k_0 - \delta E_n - E_m \rho^>(\vec{k}_0) \propto -\delta P_n - P_m \rho^>(\vec{k}_0) \propto$$

The spectral density is defined as $\rho(\vec{k}_0) \propto \rho^>(\vec{k}_0) \propto$

$$\rho^<(\vec{k}_0) \propto \rho^>(\vec{k}_0) \propto 1 - e^{-\beta k_0}$$

$$\propto \delta \delta k_0 - \delta E_n - E_m \rho^<(\vec{k}_0) \propto -\delta P_n - P_m \rho^<(\vec{k}_0) \propto$$

therefore,

$$\rho(\vec{k}_0) \propto \rho^>(\vec{k}_0) \propto 1 - e^{-\beta k_0}$$

$$\rho^<(\vec{k}_0) \propto \rho^>(\vec{k}_0) \propto \frac{1}{1 - e^{-\beta k_0}}$$

$$\propto \delta \delta k_0 - \delta E_n - E_m \rho^<(\vec{k}_0) \propto -\delta P_n - P_m \rho^<(\vec{k}_0) \propto$$

where

$$\frac{1}{1 - e^{-\beta k_0}} \propto \frac{1}{e^{\beta k_0} - 1} \propto$$

Furthermore, from the first equality in (2.67) it follows that

$$\rho^<(\vec{k}_0) \propto -\rho^>(\vec{k}_0) \propto$$

$$\rho^>(\vec{k}_0) > 0 \quad \text{for } k_0 > 0$$

In terms of the spectral densities we find

$$\frac{1}{2}G^>\delta x^>-x^>^0;t-t^0p = G^<\delta x^>-x^>^0;t-t^0p$$

$$\frac{1}{4}Z \delta \int \frac{2d\pi^4 k p}{4} \rho \delta k; k p e^{-ik_0 \delta t - t^0 p} e^{ik^> \cdot \delta x^>-x^>^0 p} \quad \text{d2:73p}$$

which determines the self-energy $\Sigma \delta t - t^0 p$ Eq. (2.46), and

$$\frac{1}{2}G^>\delta x^>-x^>^0;t-t^0p \int G^<\delta x^>-x^>^0;t-t^0p \equiv Z \delta \int \frac{2d\pi^4 k p}{4} \rho \delta k; k p e^{-ik_0 \delta t - t^0 p} e^{ik^> \cdot \delta x^>-x^>^0 p} \quad \text{d2:74p Kd}$$

which determines the noise correlation function $N \delta t - t^0 p$, Eq. (2.45), where

$$K^>\delta k_0; k p \frac{1}{4} \rho \delta k_0; k p \coth \frac{\beta}{2} k_0; \quad \text{d2:75p}$$

Equation (2.75) is the general form of the fluctuation dissipation relation. Note that $\rho \delta k_0; k p$ is odd whereas

$K^>\delta k_0; k p$ is even “environmental” in k_0 . We emphasize that these are exact “fields” χ may be coupled to relations, the other fields, for example, in the case of the ALP interaction with the electromagnetic fields as in Eq. (2.2) the gauge field also interacts with electrons, charged leptons and quarks, and similarly with the possible interaction with fermionic fields in Eq. (2.2), these interact with other gauge fields. The relations (2.67)–(2.75) are general, nonperturbative statements relying on thermal equilibrium and spacetime translational invariance and do not depend on these couplings.

The general expressions (2.73)–(2.74) allow us to write function the self-energy $N \delta t - \Sigma t_k^0 \delta p t$ (2.45)– $t^0 p$ (2.46) as and the noise correlation

$$\Sigma \delta t - t^0 p \frac{1}{4} - i g Z \delta \int dk 2\pi^4 p \delta k; k p e^{-ik_0 \delta t - t^0 p}; \quad \text{d2:76p}$$

$$N \delta t - t^0 p \frac{1}{4} g \int dk 2\pi^4 p \delta k; k p \coth \beta \frac{2k_0 e^{-ik_0 \delta t - t^0 p}}{2k_0 e^{-ik_0 \delta t - t^0 p}}; \quad \text{d2:77p}$$

this is the general relation between the self-energy and the noise-correlation function commonly determined by the spectral density $\rho \delta k_0; k p$, a direct consequence of the fluctuation-dissipation relation as a result of the bath being in thermal equilibrium.

C. Misaligned initial conditions

The initial density matrix for the ALP field is determined by initial conditions. We consider an initial density matrix describing a pure state compatible with a “misalignment” mechanism whereby the expectation value of the ALP field is nonvanishing initially and also allow a nonvanishing expectation value of its canonical momentum. This is achieved by considering a coherent state of the form $j \Delta i \frac{1}{4} \int k e^{\Delta k^> b^> k - \Delta k^> b_k^> j 0 i}; \quad \text{d2:78p}$

where $j 0 i$ is the free field (ALP) vacuum state, $b^> k$; $b_k^>$ are ALP free-field creation and annihilation operators, and $\Delta k^>$ are complex c-number coefficients that determine the initial values for A_k , π_k . In the Schrödinger representation, the state (2.78) is represented by the coherent state wavefunctional

$$\Psi \frac{1}{4} e i R \int dx^> i \delta x^> \rho a \delta x^> \Psi \frac{1}{4} a - \bar{A} i; \quad \text{d2:79p}$$

where Ψ_0 is the ground state wavefunctional of a free ALP field theory. Such a wavefunctional is Gaussian and yields an average momentum $\bar{\pi}_i$ and expectation value of the field given by \bar{A}_i whose Fourier expansion is determined by the complex coefficients $\Delta k^>$ in Eq. (2.78). The pure state density matrix describing this coherent state as representative of the “misaligned” initial condition is

$\rho_{a\bar{a}}(a, \bar{a}, 0) \propto \Psi_{a\bar{a}}(0)$

02:80p

and its Wigner transform is given by

$$W_{a\bar{a}}(A_i, \bar{A}_i) \propto N \Pi_{k^*} e^{-\Omega_{k^*} \delta A_i(k^*) - \bar{A}_i(k^*) \delta \bar{A}_i(k^*) - \bar{A}_i(k^*) \rho}$$

$$\times e^{-\frac{1}{\Omega_{k^*}} \delta \bar{A}_i(k^*) - \bar{\pi}_i(k^*) \delta \bar{\pi}_i(k^*) - \bar{\pi}_i(k^*) \rho};$$

02:81p

with N a normalization factor and Ω_{k^*} the covariance which will be related to the renormalized effective frequency (see below). Translational invariance imposes that the expectation value of the (pseudo) scalar (ALP) field be independent of the momentum, therefore we write in a finite but large quantization volume V

$$\bar{A}_i(k^*) \propto \bar{A}_i p V f f_i \delta_{k^* 0}; \quad \bar{\pi}_i(k^*) \propto \bar{\pi}_i p V f f_i \delta_{k^* 0}; \quad 02:82p$$

where $\bar{A}_i, \bar{\pi}_i$ are the space-time constant expectation values of the field and canonical momentum in the translational invariant initial state. With this Wigner probability distribution function we find the averages over the initial conditions

$$\overline{\delta A_i(k^*)} \propto \bar{A}_i p V f f_i \delta_{k^* 0}; \quad \overline{\delta A_i(k^*)} = 0$$

$$\overline{\delta \pi_i(k^*)} \propto \bar{\pi}_i p V f f_i \delta_{k^* 0};$$

$$\overline{\delta \pi_i(k^*) - \bar{\pi}_i(k^*) \delta \bar{\pi}_i(k^*) - \bar{\pi}_i(k^*) \rho} = 0$$

2:83
02:82p

with higher-order correlations obtained via Wick's theorem.

This is a simple realization of the “misalignment” mechanism whereby the initial state is a coherent state that features a nonvanishing expectation value of the field and its canonical momentum, these define the initial value problem.

D. The solution of the Langevin equation

The solution of the Langevin (stochastic) equation (2.54) is obtained by Laplace transform, define the Laplace transforms

$$A_{k^*} \delta s \propto Z_0 e^{-st} A_{k^*} \delta t p dt; \quad 02:84p$$

$$\xi_{k^*} \delta s \propto Z_0 e^{-st} \xi_{k^*} \delta t p dt; \quad 02:85p$$

$$\Sigma_{k^*} \delta s \propto Z_0 \int_{-\infty}^{\infty} e^{-st} \Sigma_{k^*} \delta t p dt \propto -\frac{g^2}{k_0 - is} \int_0^{\infty} \rho(k) dk; \quad 02:86p$$

where in (2.86) we used the dispersive representation (2.76).

With the initial conditions (2.55) the solution of the Laplace transform of the Langevin equation is

$$A_{k^*} \delta s \propto \pi_{i, k^*} \frac{p}{2} \delta s \omega_{2k^*} \delta \pi_{i, k^*} \delta s \propto \delta s \delta s; \quad 02:87p$$

The solution in real time is obtained by inverse Laplace transform, it is given by

$$A_{k^*} \delta t p \propto A_{k^*} \delta t p + A_{k^*} \delta t p; \quad 02:88p$$

where $A_{k^*} \delta t p$, $A_{k^*} \delta t p$ are the homogeneous and inhomogeneous solutions respectively, namely

$$A_{k^*} \delta t p \propto A_{i, k^*} G_{k^*} \delta t p + \pi_{i, k^*} G_{k^*} \delta t p$$

$$A_{k^*} \delta t p \propto Z_0 t G_{k^*} \delta t p - t \delta t p \xi_{k^*} \delta t p dt; \quad 02:89p$$

and the Green's function is given by

yielding the Kramers-Kronig relation

$$G_k \delta t \frac{1}{4} = \frac{2\pi i}{\omega_2 k \epsilon s \delta t} \int_{\Sigma} \delta s \delta t ds; \quad \delta t: 90 \text{ p} \quad 1$$

Σ denotes the Bromwich contour parallel to the imaginary axis and to the right of all the singularities of $\delta s_2 \delta \omega_2 \delta \Sigma \delta k$ in the complex s -plane and closing along a large δ

correspond to poles and multiparticle branch cuts with semicircle at infinity with $\text{Re} \delta s < 0$. These singularities axis $\text{Re} \delta s < 0$, thus the contour runs parallel

to the imaginary $\delta v - i\epsilon$, with $-\infty \leq v \leq \infty$ and $\epsilon \rightarrow 0$.

Therefore,

$$G_k \delta t \frac{1}{4} = \int_{-\infty}^{\infty} G^* \delta v \delta t e^{i v t} 2 \text{d}v; \quad \delta t: 91 \text{ p}$$

where

$$G^* \delta v \delta t \frac{1}{4} = \delta v - i\epsilon - 1 \omega_2 - \Sigma \delta v; \quad \delta t: 92 \text{ p}$$

The self energy in frequency space is given by the dispersive form

$$\Sigma \delta v; \quad \frac{2}{2} \frac{Z}{\pi} \int_{-\infty}^{\infty} \frac{\rho \delta k_0; \delta t}{\sqrt{k_0}} \sum_{i \in \delta k_0} \delta v; \quad \delta t: 93 \text{ p}$$

with the real and imaginary parts given by

$$\Sigma \delta v; \quad 2g\pi^2 \int_{-\infty}^{\infty} \rho \delta k_0^0; \delta k_0 \delta t; \quad \delta t: 94 \text{ p}$$

$$\Sigma \delta v; \quad 2 \rho \delta v; \quad \delta t: 95 \text{ p}$$

$$\Sigma_R \delta v; \quad \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta v \delta k_0 \delta k' \delta k_0 \delta k' \delta t; \quad \delta t: 96 \text{ p}$$

To obtain the above representations we have used the relation $\rho \delta k_0; \delta t \frac{1}{4} = -\rho \delta k_0; \delta t$ [see Eq. $\Sigma_R \delta v$ (2.72); $\delta t \frac{1}{4}$], as a consequence of which it follows that

complex poles corresponding to the solution of the equation $\Sigma \delta v; \delta t \frac{1}{4} = -\Sigma \delta v; \delta t$. $G^* \delta v \delta t$ given by Eq. (2.92) features

$$\omega_2 \rho \delta k_0 \frac{1}{4} = \omega_2 \delta k_0 \delta t \omega_2 \delta k_0; \quad \delta t: 97 \text{ p}$$

to leading order in g^2 we find

$$\omega_2 \rho \delta k_0 \frac{1}{4} = \Omega_k \delta k_0 \delta t \Gamma_2; \quad \delta t: 98 \text{ p}$$

where

$$\Omega_k \frac{1}{4} = \omega_k \delta k_0 \Sigma_R \delta v \delta t \delta k_0; \quad \Gamma_k \frac{1}{4} = \Sigma \delta v \delta t \delta k_0; \quad 2g\omega_2 \rho \delta k_0; \quad \delta t: 99 \text{ p}$$

Writing in the denominator of the integrand in (2.91) that near each pole, $\Sigma \delta v; \delta t \frac{1}{4} = \Sigma \delta \omega \delta k_0; \delta t \delta G^* \delta v \delta t$ can be written in a Breit form $-\Sigma \delta \omega \delta k_0; \delta t \delta G^* \delta v \delta t$ we find

Wigner form as

$$G^* \delta v \delta t \frac{1}{4} = \frac{2 \omega_2 \rho \delta k_0 \delta v \delta t}{2 \omega_2 \rho \delta k_0 \delta v \delta t} \Gamma_2 \Omega_k - i \Gamma_2 \delta t; \quad \delta t: 100 \text{ p}$$

with the wave function renormalization constant

$$-1 = 1 - \frac{\Sigma'(\omega_p, k)}{\omega_p} = 1 - \frac{Z \partial \Omega \partial g^2}{2};$$

$$\langle\langle A_{k^-} \partial t | A_{-k^-} \partial t \rangle\rangle \propto A_{k^-} \partial t | A_{-k^-} \partial t | 4\pi^2 Z \propto \partial \Omega | k |$$

$$\Sigma^0 \partial \omega_p; k \propto \frac{d}{d \omega_p} \Sigma^0 \partial \omega_p; k \propto \omega_p;$$

δ2:101p

$$- \times \coth \beta 2k \propto Z \partial \Omega \partial g^2 \propto \partial \Omega \partial g^2$$

2dko: τp e δ2:105p

To leading order in g^2 we find

$$G_k \partial t \propto e^{-\Gamma_k t} \sin \Omega_k t \partial t \propto \partial \Omega \partial g^2; \quad \delta2:102p$$

where we have assumed a narrow width $\Gamma_k = \Omega_k \propto g^2 \ll 1$ and neglected terms of this order. Using this result in Eq. (2.89) we find

$$\langle\langle A_{k^-} \partial t \rangle\rangle \propto e^{-\Gamma_k t} \bar{A}_{i;k^-} \propto \cos \Omega_k t \partial t - 2\Gamma_k \sin \Omega_k t \partial t$$

$$\partial \bar{A}_{i;k^-} \sin \Omega_k t \partial t \propto \partial \Omega \partial g^2; \quad \delta2:103p$$

$$\langle\langle A_{k^-} \partial t \rangle\rangle \propto e^{-\Gamma_k t} \Omega_k \bar{A}_{i;k^-} - \sin \Omega_k t \partial t - 2\Gamma_k \cos \Omega_k t \partial t$$

$$\bar{A}_{i;k^-} \cos \Omega_k t \partial t - 2\Gamma_k \langle\langle A_{k^-} \partial t \rangle\rangle \propto \partial \Omega \partial g^2; \quad \delta2:104p$$

where we used (2.56), and $\bar{A}_{i;k^-}$ are the average of the initial conditions with the Wigner distribution function (2.83). We have explicitly displayed the terms $\propto \Gamma_k = \Omega_k$ to exhibit that they arise from the derivative of the exponential damping term, however, these terms are of $\partial \Omega \partial g^2$ and must be neglected for consistency as we are also neglecting terms of the same order from wave function renormalization. Similarly, we find

$$g \propto$$

Using the leading order result (2.102) for $G_k \partial t$ the integral in (2.105) is straightforward. Inserting the result into (2.105) yields four terms, the resulting integrals are performed by contour integration in the complex k_0 -plane: in the narrow width approximation the two direct terms feature residues $\propto 1 = \Gamma_k$, whereas the interference terms feature residues $\propto 1 = \partial \Omega \partial g^2$, these latter terms and the poles at frequencies $k_0 \propto 2\pi i m = \beta$ yield contributions of $\propto 0; 1; 2$, namely the Matsubara $\partial \Omega \partial g^2$ and will be

neglected, whereas the terms with residues $\propto 1 = \Gamma_k \propto 1 = g^2$ yield the leading contributions. Using the definition of Γ_k (2.99) and keeping solely the leading-order terms in (2.103) we obtain

$$\begin{aligned} \langle\langle A_{k^-} \partial t | A_{-k^-} \partial t \rangle\rangle & \propto \\ & \frac{1}{k \partial t} \propto \frac{1}{k \partial t} \propto \frac{1}{-k \partial t} \propto \frac{1}{2\Omega_k k \partial t} \propto \frac{1}{2n \partial \Omega_k \partial t} \\ & \times \delta(1 - e^{-\Gamma_k t}) \propto \partial \Omega \partial g^2; \quad \delta2:106p \end{aligned}$$

where

$$H_k \partial t \propto A_{i;k^-} \cos \Omega_k t \partial t \propto \bar{A}_{i;k^-} \frac{\sin \Omega_k t}{\partial \Omega \partial g^2};$$

$$\delta2:107p \Omega_k$$

and $n \partial \Omega_k \partial t$ is the Bose-Einstein distribution function.

This is a noteworthy result; for $\Gamma_k t \gg 1$ the surviving term is precisely the free-field expectation value $\delta h b_{k^-} b_{-k^-}$ $\propto b_{k^-} b_{-k^-} i \propto = 2\Omega_k$ of (ALP) operators, where the average is

¹ The residue for $m \neq 0$ vanishes because the spectral density vanishes at $k_0 \neq 0$.

in a thermal equilibrium statistical ensemble; namely, at long time $t \gg 1/\Gamma_k$ the ALPs thermalize with the bath.

A similar calculation, implementing the same approximations yields for the average energy per mode (2.62)

$$E_k \approx e^{-\Gamma_k t} h^{-k} \delta t \propto \Omega_k^2 H_k^{-k} \delta t \propto \Omega_k^2 \delta t$$

$$\propto \Omega_k^2 \delta t = 2n \delta \Omega_k \delta t = e^{-\Gamma_k t} \propto \Omega_k^2 \delta t$$

This result confirms thermalization at long time, the second term, which survives for $t \gg 1/\Gamma_k$ is identified as the expectation value of the free field Hamiltonian in a thermal density matrix, namely the internal energy. These results are a manifestation of thermalization in the same manner as a Brownian oscillator as mentioned in the introduction; whereas the average of the coordinate relaxes to the minimum of the potential, the mean square root fluctuations reveal thermalization with the bath. This is ultimately a consequence of the fluctuation-dissipation relation manifest in the relation (2.75) between the noise and the self-energy (friction) kernels, a corollary of the Kubo-Martin-Schwinger condition (2.67) as a consequence of the equilibrium bath correlations.

Therefore, for the Wigner distribution function (2.81) describing a misaligned initial condition with the averages given by Eq. (2.83) and neglecting a zero-point contribution, we find the energy density

$$E \approx 1 - \propto k E_k \approx e^{-\Gamma_k t} h^{-2} \propto m_a^2 \bar{A}_2$$

$$V V^2$$

Z

$\propto \delta^2 d\pi^3 k \Omega_k \delta \Omega_k \delta t = e^{-\Gamma_k t} \propto \Omega_k^2 \delta t$ oscillations arising from a “misaligned” initial condition as in the usual case of an ALP, whereas the second term yields a hot dark matter contribution from the approach to thermalization, each weighted by the damping exponentials. Whereas the first term depends on the initial conditions of the ALP, the second term is completely determined by the noise, namely the thermal bath. This is one of the important results of this study.

Therefore, if the ALP relaxes on cosmological time scales at a given time t its contribution to dark matter is a mixture of cold and hot components, with a fraction determined by the relaxation rate Γ_k and the time scale t . This result suggests a scenario where the “warmth” of the dark matter evolves in time from colder to hotter.

The result (2.109) is general, it is valid to order g^2 for any ALP interaction of the form $g \delta x \propto O \delta x$ and to all orders in the interactions of the bath fields with other fields besides the ALP. This is an important corollary of the results in this section: the processes that yield the friction in the equation of motion of the misaligned expectation value are the same processes that lead to thermalization. Therefore, whereas several studies focused on the friction term in the equation of motion of the coherent condensate [37–39] and other studies focused on thermalization [40,41] our results show that both processes are related by the fluctuation dissipation relation, occur on similar time scales and both contribute to the evolution of the energy density of the ALP field. Therefore, the time evolution of the energy density given by Eq. (2.109) is one of the important results of our study, it applies to all dissipative processes resulting from interactions of the ALP with other degrees of freedom and is a direct consequence of the fluctuation dissipation relation.

III. ALP INTERACTING WITH PHOTONS

The results obtained in the previous section are general, although the focus is on ALP fields, the results also apply to any field with an interaction of the form (2.1) and initial conditions that allow for the evolution of a coherent condensate [18]. These results have a clear physical significance in terms of the nonequilibrium manifestation of Brownian fluctuations; a bath in equilibrium induces both a self-energy (friction) and a noise term in the effective equations of motion, the spectral properties of

The first term is identified with a cold dark matter contribution and originates in the damped coherent both are related by the generalized fluctuation dissipation relation, a hallmark of a bath in thermal equilibrium. Although the results are general, the details, namely relaxation times, frequency renormalization etc., depend on the spectral properties of the bath correlations. In this section we focus on ALP interaction with photons via the coupling

$$L \approx -g \delta x \cdot E \cdot \delta x \cdot B \cdot \delta x; \quad \delta^3 \propto$$

as in Eq. (2.2). The main assumption invoked in our study is that we consider free massless photons neglecting interactions with charged leptons and quarks. The one

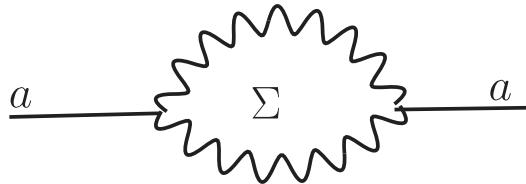


FIG. 3. One loop ALP self-energy from the coupling $L \propto -ga\delta x^\mu E^\nu \delta x^\mu \cdot B^\nu \delta x^\mu$.

loop contribution to the ALP self-energy is displayed in Fig. 3.

The regime of validity of this assumption is discussed in detail in Sec. IV. An important aspect of the coupling (3.1) is that this interaction is nonrenormalizable because the coupling g has dimensions of 1=energy. As a result the loop corrections associated with the self-energy feature ultraviolet divergences which cannot be absorbed into the parameters of the Lagrangian and the theory must be interpreted as a low-energy effective field theory.

In Appendix A we obtain the spectral density from the thermal correlation functions of the composite operator $E^\nu \delta x^\mu \cdot B^\nu \delta x^\mu$, it is given by [see Eq. (A15)],

$$\rho \delta q_0; q^\nu \propto \Theta Q^2 2\pi p_2 \frac{1}{1 - \beta \omega q \ln(1 - \beta \omega q)} e^{-\beta \omega \beta \omega p_1} \Theta \delta Q_2 p^\nu$$

$$2 \quad 1 - e^{-\beta \omega p_1}$$

$$\frac{\beta \omega q \ln(1 - e^{-\beta \omega p_1}) \Theta \delta Q_2 p^\nu \text{sign} \delta q_0 p^\nu}{\Theta Q^2 2\pi p_2}$$

$$Q_2 \propto q_0 - q_2; \quad \omega \delta p \propto j q_0 j 2 q; \quad \omega \delta p \propto \frac{1}{4} q j 2 q j$$

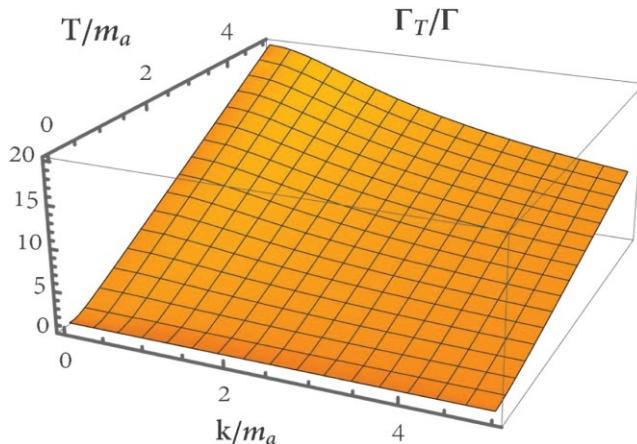
namely emission and absorption of photons with the reverse. The terms with $\Theta \delta Q^2 p^\nu$ arise from the processes $a \leftrightarrow 2\gamma$, or recombination process $2\gamma \rightarrow a$ a consequence of the heat bath, these processes feature support on the ALP mass shell for massive ALPs. The contribution proportional to only features support below the light cone and describes off- $\Theta \delta Q^2 p^\nu$ shell processes $\gamma a \leftrightarrow \gamma$. This interpretation stems from the delta functions in the expressions for the spectral density equation. [see the second line in Eq. (A13)].

From the definition of the relaxation rate (2.99) and with the result (3.2) we find

$$2 \quad 1 - e^{-\beta \omega p_1} \quad g_2 m_a$$

$$\Gamma_T \propto \Gamma_1 \beta \omega q \ln(1 - e^{-\beta \omega p_1}) \frac{1}{k_0 \Omega_k}; \quad \Gamma \propto 64 \frac{\pi \Omega_k}{\Theta Q^2 2\pi p_2} \Theta \delta Q_2 p^\nu; \quad 03:3p$$

the first contribution is the zero temperature ALP decay rate, and the second is the finite temperature contribution which is a consequence of stimulated emission and absorption in the heat bath. The ratio Γ_T/Γ is displayed in Fig. 4 as a function of the dimensionless ratios $T=m_a/k=m_a$.



$$\Sigma \delta R \partial \delta v; k \not= 0 \not= -64g\pi^2 2 Z_{-K\Lambda^2} P \alpha \not= 2K_2 \not= \delta K \alpha_2 P_2 d\alpha; \dots$$

$$K^2 \not= v^2 - k^2;$$

$$\delta 3:7p$$

with the result

$$\Sigma \delta R \partial \delta v; k \not= 0 \not= -64g\pi^2 2 Z_{-K\Lambda^2} P \alpha \not= 2K_2 \not= \delta K \alpha_2 P_2$$

$$\Lambda^2$$

$$\not= \delta K \alpha_2 P_2 \ln \frac{1}{2} j K_2 j; \dots$$

$$\delta 3:8p$$

This result clearly exhibits the nonrenormalizability of the effective field theory of ALPs; the Λ^4 term is

FIG. 4. Ratio Γ_T/Γ vs $T=m_a$; $k=m_a$.

absorbed into a mass renormalization, the $\delta v_2 - k_2 p$

Λ^2 term yields an ultraviolet divergent wave function renormalization as per Eq. (2.101), however the

ment over the zero temperature case for The finite temperature contribution yields a large enhance- $T \gg m_a/k$. For logarithmic divergence $v_2 - k_2 p_2 \ln \frac{1}{2} \Lambda$ cannot be δ absorbed into the renormalization of parameters and example in the long-wavelength limit $k \ll m_a$ we find field redefinitions of the original Lagrangian, which then must be appended with a new higher-derivative

$$\Gamma_T \not= g 64^2 m \pi^3 a^{-1} p^{2n} m^2 a$$

$\delta 3:4$ term be renormalized by the term with the logarithmic $C \delta \partial_\mu \partial^\mu a p^2$ where C is a new

coefficient that will

divergence. Therefore, the effective action necessary which in the high-temperature limit $T \gg m_a$ yields states the addition of a higher derivative term to absorb the ultraviolet divergences. While such extension of the effective field theory is both necessary

$$\Gamma_T \not= g 16^2 m \pi^3 a^{-1} m T_a : \dots$$

$\delta 3:5$ minimal ALP effective field theory to establish and interesting

on its own, here we focus on the

contact with the more familiar ALP Lagrangians,

For example, if T corresponds to the temperature of the background today $T \simeq 10^{-4}$ eV the finite $C_{1/4}^0$. In Sec. IV we comment on possible effects thereby we set the new renormalized coupling cosmic microwave

temperature correction yields a large enhancement for associated with the higher-derivative terms.

$m_a \ll \mu\text{eV}$, and an enormous one if the ALP is an ultralight (ii) $T \neq 0$: The finite temperature contribution to the candidate with $\approx m_a 10^{-22} \text{ eV}$ with potentially relevant self-energy is ultraviolet finite and is studied in detail in Appendix B; the results of this appendix

cosmological consequences discussed in Sec. IV.

allow to obtain its high- and low-temperature behavior. For $T \gg \Omega_k$ we find to leading orders in the

A. Real part of the self-energy

high-temperature expansion

(i) The spectral density $\Sigma_{RT}(\omega)$: The real part of the self-energy is given by, with the zero-temperature contribution to (3.2) we find $\Sigma_{RT}(\omega) = g^2 T^4 - 15\pi^2 - 24m_{Ta}^2 + 32m_T^4 a_4(1 - \gamma)$

Eq.

ΣδR0pδv;kþ ¼ 64gπ2 20PZ0 $\frac{\infty \delta k^2 - k^2 p^2}{v - k_0} \Theta \delta v$ 0 - k2 k2pdko
 δ ln4mπaT -Ωkk⁻ lnΩk⁻mþa kþ δ3:9p;

and for $T \ll m_a$ we find

integral is ultraviolet divergent, introducing an upper relabeling $k_0 \rightarrow -k_0$ in the second

integral, the total

$$\sum_{\text{TRD}} \Omega_k = \frac{1}{4} \cdot 2\pi^2 k^2 \cdot 32\pi^6 3^4 m_a \cdot 1 \cdot 4m k^2 \cdot a$$

frequency cutoff Λ delimiting the range of validity of

16 4 T2

k

$$\beta 5 m^4 - m^2 \beta$$

the effective ALP field theory, and changing integration variables to $\alpha \sqrt{k_0^2 - k^2} \rightarrow \delta v^2 - k^2 b$, we find

Defining the effective finite temperature mass as the $k \rightarrow 0$ limit of the dispersion relation (2.97), the high temperature limit $T \gg m_a$ (3.9) yields an effective, temperature dependent mass

$$m_{2a}\delta T \frac{1}{4} \quad m_{2aR} 1 - \frac{1}{15g_2 T_{2aR} 4};$$

$$\delta 3:11 \mathbf{p} \pi_2 \mathbf{m}$$

where m_{aR} is the renormalized mass absorbing the zero temperature renormalization. Equation (3.11) can be written in a more illuminating form as

$$m_{2a}\delta T \frac{1}{4} m_{a2}\delta 0 \mathbf{p} 1 - \frac{T_c}{T} \quad \frac{4}{g}; \quad \frac{T_c}{T} \frac{1}{4}$$

$$1.11s^{m_a\delta 0} \mathbf{p} \text{ff}i: \quad T \quad g$$

$$\delta 3:12 \mathbf{p}$$

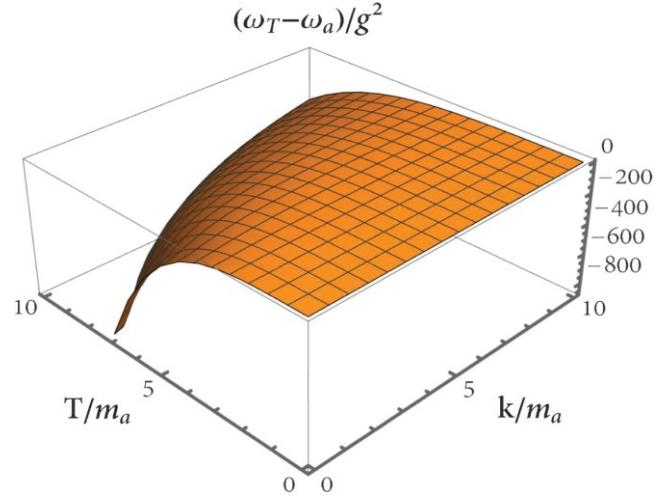
This result suggests the possibility of an inverted phase transition at a temperature $T \nless T_c$: for $T > T_c$ the effective squared mass is negative signaling an instability, whereas it is positive for $T < T_c$. This situation is the opposite of the usual phase transition where $m^2\delta T \frac{1}{4} > 0$ for $T > T_c$ indicating an ordered phase and symmetry restoration, and $m^2\delta T \frac{1}{4} < 0$ for $T < T_c$ indicating symmetry breaking. This intriguing result is a consequence of the high-temperature behavior of the real part of the self-energy, which to the best of our knowledge has not been studied before.

For $T < T_c$, we define the finite temperature correction to the dispersion relation (2.99) as

$$\omega_T - \omega_a \equiv \sum \delta \Omega_k; \mathbf{k} \mathbf{p}; \quad \delta 3:13 \mathbf{p}$$

$$2\Omega_k$$

in Fig. 5 we display the finite temperature correction to the dispersion relation $\delta \omega_T - \omega_a \mathbf{p} = g^2$ in units of m^3_a vs $T = m_a; \mathbf{k} = m_a$.



For $k \nless 0$ the figure clearly shows the fast drop in the effective mass as T increases in agreement with the analysis yielding Eq. (3.11).

The possible high temperature instability as a consequence of $m^2\delta T \frac{1}{4} < 0$ for $T > T_c$ indicates that the results obtained in the previous section for the energy density

(2.109) are valid only for $T < T_c$ since the solution of the Green's function (2.102) implied real frequencies Ω_k and a perturbative correction to the position of the poles. The instability for $T > T_c$ yields an imaginary frequency Ω_k in the solution which translates into a growing exponential.

IV. DISCUSSION AND CAVEATS:

A. The Gaussianity of noise correlations

The noise variable ξ is described by a Gaussian probability distribution function (PDF) given by Eq. (2.52). The Gaussianity is a consequence of the nonequilibrium effective action of the ALP field being quadratic [26]. However, this Gaussian PDF does not entail that either the ALP or the environmental fields are free. As per the discussion in Sec. II B the Lehmann representation of the environmental correlation functions is to all orders in the couplings of the environmental fields to other fields within or beyond the standard model other than the ALP field. The fluctuation dissipation relation (2.75) is a consequence of the Lehmann representation, the self-energy (2.76)–(2.86), which enters in the full propagator (2.92) is the sum (to all orders) of one particle irreducible diagrams such as those displayed in Fig. 2. Therefore the spectral density $\rho \delta k_0; \mathbf{k} \mathbf{p}$ that determines the noise correlation functions are also to all orders in such

couplings. Hence, even when the noise PDF is Gaussian, this does not entail that either the ALP or the environmental fields are free.

FIG. 5. Finite temperature correction to the dispersion relation $\delta\omega_T - \omega_a b = g^2$ in units of m_a^3 vs $T = m_a$; $k = m_a$.

after recombination for $T \lesssim 0.3$ eV. For a light ALP with $m_a \lesssim \mu\text{eV}$ even the temperature of the CMB today $T_{\text{cmb}} \simeq 10^{-4}$ eV is such that $T = m_a \gg 1$ and there is a large finite temperature enhancement to the relaxation rate, which becomes quite substantial for an ultralight (ALP) with $m_a \lesssim 10^{-20}$ eV.

2. Thermalization

Thermalization of ALPs in the early Universe has been studied previously [40,41,52]. However, our method and results go much further. The nonequilibrium effective action yields the effective equation of motion for ALP fields which is a Langevin equation with “friction” and noise contributions that satisfy the fluctuation dissipation relation. The solution of this Langevin equation allows us to study the evolution of ALP condensates from misaligned initial conditions along with thermalization which is shown to be a consequence of the noise term and the fluctuation-dissipation relation. The effective action also allows us to study renormalization aspects and the finite temperature corrections to the (ALP) mass arising from the real part of

1. ALPs coupled to photons: Region of validity

In the case of ALP interaction with photons, we have assumed that photons constitute a thermal bath of blackbody radiation, having in mind the cosmic microwave background. At high temperatures, for relativistic electrons, namely $T \gg m_e$ with m_e the electron mass, photons acquire a gauge invariant plasma mass $\simeq eT/3$ via hard thermal-loop corrections to the photon self-energy [47,48,51]. For a light or ultralight (ALP) this plasma mass would shut off the lowest-order emission/absorption channel $a \leftrightarrow 2\gamma$. When electrons become nonrelativistic, but there is a free-electron density n , the plasma frequency becomes $\frac{4\pi}{m} n e^2$ which would also shut off this channel for light or ultralight ALP. However, after recombination, the free electron density vanishes precipitously as electrons combine with protons into neutral hydrogen. Photons are effectively massless as evidenced by the nearly perfect blackbody spectrum of the CMB. Since we have assumed massless photons in the calculation of the spectral density, our assumptions are valid

the self-energy (the thermalization rate is related to the imaginary part of the self-energy on the mass shell). The effective action has been obtained up to second order in (ALP) coupling, but to all orders in the couplings of the “environmental” fields to any other fields to which they couple other than the (ALP). For example the study of thermalization in Refs. [40,41,52] in which the ALP is coupled to quarks or other Standard Model degrees of freedom correspond to obtaining the two loop contributions to $G^>;G^<$ in Eqs. (2.45) and (2.46), hence they are included in the general considerations of Sec. II. To see this, let us consider the ALP-gluon interaction vertex $g_s a \delta x^\mu b G^{\mu\nu b} \delta x^\nu b G^<_{\mu\nu b} \delta x^\mu b$. The process $a \beta$ gluon $\leftrightarrow q\bar{q}$ is contained in the correlation function $h G G G^< G^> i$ at two loops, with one gluon propagator featuring a q^4 self-energy loop, this is the QCD equivalent of the second diagram in Fig. 2 featuring a fermion loop correction to the propagator of the gauge boson. Using Cutkosky’s cutting rules it is a simple exercise to see that the rate for the scattering process $a \beta$ gluon $\leftrightarrow q\bar{q}$ is given by the imaginary part of the two loop diagram where the cut goes through the $q\bar{q}$ loop. Similarly for the processes $a \beta$ gluon $\leftrightarrow 2\text{gluons}$

which corresponds to a gluon loop for a gluon self-energy. This is the thermalization rate that enters in the Boltzmann equation in Ref. [41] or the cross section in Ref. [52].

3. Mixed cold and hot components

An important corollary of the Langevin-like equation of motion (2.54) are the general results (2.108) and (2.109) which entail that the energy density of ALPs feature a mixture of cold and hot components, the cold component is determined by oscillatory coherent condensate resulting from misaligned initial conditions and the hot corresponds to the thermalized part, which is determined by the “noise” term in the Langevin equation, with proportions varying in time as $\tilde{\delta} \text{cold} \propto e^{-rt} \propto \tilde{\delta} \text{hot} \propto 1 - e^{-rt}$. The damping of the cold component is a consequence of the “friction” term in the equations of motion determined by the imaginary part of the self-energy, and the growth rate of the hot component, namely the thermalization rate, is related to the damping rate of the cold component by the fluctuation dissipation relation. The cold component originates in the coherent oscillations resulting from a “misaligned” initial condition, whereas the hot component results from the approach to thermal equilibration with the bath.

4. Novel exotic phases

For ALP-photon coupling, the real part of the self-energy reveals two important features: because the coupling g has dimensions $1 = \text{energy} \propto$ the interaction Lagrangian density is nonrenormalizable. As a result we find that the effective action must necessarily include higher derivative terms of the form $C \partial_\mu \partial^\mu a \partial_\nu \partial^\nu p^2$ with C a constant that absorbs the logarithmic ultraviolet divergence (3.8). We have (arbitrarily) set $C \neq 0$ to establish contact with the usual Lagrangian proposed for ALPs, but this clearly implies a fine-tuning. Furthermore, the finite temperature part of the self-energy features the high temperature limit (3.9) which yields an effective temperature dependent mass squared given by Eq. (3.12). The power of temperature $\propto T^4$ is a consequence of the nonrenormalizable coupling with mass dimension -2 . The negative sign yields the opposite behavior compared to scalar theories with (second-order) phase transitions, the physical origin of the negative sign eludes these authors. We conjecture that the sign is a result of the coupling to a pseudoscalar composite operator with vector fields, but such conjecture awaits confirmation by comparing to other pseudoscalar couplings such as those shown in Eq. (2.2), which is beyond the original scope of this study. This

effective mass squared suggests the possibility of an inverted phase transition with $m^2 \partial T \propto < 0$ for $T > T_c$ signalling an instability towards a phase of lower free energy. Such instability entails that nonlinearities in the ALP effective Lagrangian are relevant, these may be associated with a potential for the ALP field, or from higher orders in the effective action, for example a term of the form $\propto g^4 a^4$ (with the various branch labels), which because of the nonrenormalizable nature of the coupling will feature the largest scale in the loop to the fourth power and may conspire with the quadratic term to stabilize the theory. The emergence of these nonlinearities in higher orders of the effective action merit further study. The main result of the energy density (2.109) is valid only for $T < T_c$ because the analysis relies on the perturbative renormalization of the frequencies, so that Ω_k in (2.109) is real. Taken together, this instability in combination with higher-derivative terms may lead to novel exotic inhomogeneous phases for $T > T_c$ of the Lifshitz type [42]. The possibility of high-temperature instabilities and novel phases are worthy of a more detailed and deeper study including other types of pseudoscalar interactions, which is beyond the scope of this article.

5. QED vs QCD

Although this discussion has focused on ALP-photon coupling, a similar conclusion can be drawn for ALP-gluon coupling $g_s a \partial_\mu \partial^\mu a \partial_\nu \partial^\nu G_{\mu\nu} G^{\mu\nu}$, since g_s also has dimensions of energy^{-1} . To lowest order in the strong coupling α_s the correlation function $h G G G^* G^*$ is a gluon loop and yields a similar high temperature dependence of the effective mass squared $\propto g_s^2 T^4$ on dimensional grounds, and a zero-temperature logarithmic ultraviolet divergence which requires a higher-derivative counterterm. Although similar to the QED case, the actual contributions from gluon loops must be studied in detail because the non-Abelian nature may lead to cancellations which these simple arguments may not capture. The study of the QCD contribution from gluons must necessarily focus on temperatures scales above the deconfinement temperature $\simeq 150$ MeV, which requires hard-thermal loop resummations [48,51] since the light quarks are ultrarelativistic in this temperature range whereas for $T < 150$ MeV (ALP) interact with neutral pions. The study of these processes is well beyond the scope of this article but clearly merit further study.

6. Possible cosmological consequences

While the results obtained above are valid in Minkowski space time, we can conjecture on their possible implications in cosmology. The effective squared mass at high temperature (3.12) suggests a high temperature inverted phase transition with $m^2 \delta T \propto <0$ for $T > T_c$, becoming positive for $T < T_c$, the opposite of the usual behavior in (second-order) phase transitions. This in turn implies that the nonlinearities in the ALP (effective) potential are important in the evolution of the coherent condensate, furthermore, the necessity of introducing higher-order derivatives to absorb logarithmic ultraviolet divergences when combined with the high-temperature instability may lead to novel inhomogeneous phases, such as Lifshitz phases [42] with the possible generation of inhomogeneities associated with the dark matter component that are not a consequence of inflationary fluctuations. The time evolution of the energy density yielding a mixture of a cold and a hot component (2.109) gives rise to the interesting possibility that the “warmth” of this dark matter candidate evolves in time from a colder to a hotter component, the weight of each component is determined by the relaxation rate and the time scale. Hence, it is possible that for a specific set of parameters (coupling and mass) the dark matter component is cold at the time of recombination but warms up as time evolves towards a warmer component, thereby yielding ALPs as a warm dark matter candidate in the most recent Universe. This possibility has potentially important consequences for galaxy formation since an ALP which is a warm dark matter candidate may help to solve the core vs cusp problem in dwarf galaxies. Furthermore, if the ALP is an ultralight dark matter candidate, it can become an ultrarelativistic component even for a temperature $\simeq T_{\text{cmb}} \simeq 0.1 \text{ eV}$ at the time of recombination, which then contributes to N_{eff} the effective number of relativistic species. As the interaction with the cosmic microwave background continues after recombination until today, the decay of the coherent condensate component and thermalization may affect the signal on birefringence if it is a consequence of the interaction of the CMB with a pseudoscalar field [53].

7. Caveats

In this article we have studied the effective action and its consequence in Minkowski space-time as a prelude towards a more comprehensive study including cosmological expansion which will be addressed in future work. Cosmological expansion introduces several important modifications; in the evolution of the condensates (coherent states) from misaligned initial

conditions, dilution of the population and time dependent relaxation rates [54] among the most obvious ones. In the regime when the cosmological expansion rate $H \delta t \propto$ is much smaller than the relaxation rate, we expect an adiabatic treatment (see Ref. [54]) to be reliable. However, in this case we would expect that ALPs would completely thermalize with the CMB after recombination and would feature the CMB temperature today. In obtaining the effective action we have traced over the CMB degrees of freedom therefore we cannot assess at this stage whether the back reaction $a \rightarrow 2\gamma$ would induce distortions in the CMB power spectrum. Such distortion would impose severe constraints on the coupling and mass of the ALP fields since these determine the relaxation rate. If, on the other hand the relaxation rate is much smaller than $H \delta t \propto$ we would expect that the thermal (hot) ALP population today would be rather small. In our treatment we have assumed the initial ALP density matrix to describe a misaligned vacuum state, described by a coherent state of a free field vacuum. This initial state neglects any population that could have been produced earlier, such as a produced thermally from QCD processes [40,41,52] or even processes beyond the standard model or during inflation. A thermal initial condition can be accounted for, including misalignment, simply by proposing a coherent state built from a thermal density matrix. Such modification will result in new contributions to the correlation functions and energy density from the initial averages with the Wigner function or alternatively with the initial density matrix. In particular this scenario would yield another thermal contribution to the energy density originating in the initial density matrix of the ALP field; therefore, the results obtained in this study provide a lower bound on the ALP energy density.

V. CONCLUSIONS

We studied the nonequilibrium dynamics of a pseudoscalar ALP weakly coupled to “environmental” degrees of freedom in thermal equilibrium in Minkowski space-time as a prelude towards extending the methods to cosmology. We considered a generic coupling $g a \delta x \propto O \delta x \propto$ with O a pseudoscalar composite operator of the bath degrees of freedom without adopting a particular set of parameters, couplings and ALP mass or bounds on them but only assuming a weak coupling between the ALP and the standard model degrees of freedom. Our focus in this article is to obtain the ALP effective action and equations of motion and to explore their consequences for general couplings and mass.

By considering the time evolution of an initial density matrix for the ALP and environmental fields in the in-in or Schwinger-Keldysh formulation, we obtained the reduced density matrix for the ALP by tracing over the environmental fields. The time evolution of the ALP reduced density matrix is determined by the nonequilibrium effective action, which we obtain up to $O(\delta g^2 p)$ in the weak coupling g but to all orders in the couplings of the environmental fields to any other field (different from the ALP) within or beyond the standard model. The effective equations of motion for the ALP field obtained from the in-in effective action are causal Langevin equations with a (nonlocal) self-energy and a Gaussian stochastic noise term whose power spectra fulfill the fluctuation-dissipation relation. The initial density matrix for the ALP field implements a “misaligned” initial condition. The effective Langevin equations of motion show that the processes that lead to the damping of the coherent condensate are the same that lead to thermalization with the environment as a direct result of the fluctuation dissipation relation. Whereas previous studies either focused on the “friction” term in the equations of motion of the coherent condensate, or on thermalization via Boltzmann equations, the nonequilibrium effective action and Langevin equation obtained in this study establishes a bridge between both aspects linking them via the fluctuation dissipation relation, a hitherto unrecognized but important aspect of coupling to an environment and shows that both occur on similar time scales. Damping of the coherent misaligned expectation value and thermalization with the environment emerge naturally from the effective Langevin equations of motion, and for generic environments we find that the total energy density features a mixture of a cold and hot components; $E \delta T \propto \delta \text{cold} p e^{-\Gamma t} + \delta \text{hot} p \delta(1 - e^{-\Gamma t})$ the cold component is a consequence of the coherent oscillations from misalignment and the hot component from thermalization with the bath. The relaxation rate Γ is determined by the imaginary part of the self-energy. The damping of the cold and the growth of the hot components are a direct consequence of the fluctuation-dissipation relation.

This time dependent energy density may provide a compelling dark matter scenario wherein the “warmth” of the dark matter evolves in time from colder to hotter. This is one of the important results of our study.

As a specific example we study ALP-photon coupling with $O(\delta E \cdot B)$ where the radiation field represents the CMB after recombination when photons can be treated as free and massless (vanishing plasma frequency). This is a

nonrenormalizable interaction, the one-loop contribution to the ALP self-energy features ultraviolet divergences that necessitate higher-derivative terms in the effective action, of the form $C \delta \partial_{3\mu} \partial_\mu a \delta x p p^2$. The long wavelength

relaxation rate $\Gamma = \frac{g^2 m_a}{64\pi} (1 - 2\sqrt{\frac{1}{4} + \frac{m_a^2}{\delta m_a^2}})^{1/2}$ features a large enhancement for $T \gg m_a$ which is substantial even for the CMB temperature $\simeq 10^{-4}$ eV if the ALP is a light dark matter candidate with $m_a \lesssim \mu\text{eV}$ and even more so if it is an ultralight candidate with $m_a \simeq 10^{-20}$ eV. We find that the high-temperature limit of the self-energy yields a temperature-dependent effective mass squared $m_a^2 \delta T p^{1/2} \propto m_a^2 \delta T^{1/2} (1 - \delta T/T_c)^{1/2}$ with $T_c \simeq p m_a \delta T = g f_i$ suggesting a possible inverted phase transition with a negative mass squared for $T > T_c$ which when combined with higher-derivative terms in the effective action may lead to the possibility of novel exotic phases.

This study has revealed aspects that have not been previously discussed, such as the necessity of higher derivative operators, the high-temperature correction to the mass which suggests a possible inverted phase transition, and that a misaligned initial condition naturally leads to an energy density that features a mixture of cold and hot components with fractions that depend on time through the relaxation rate, with the cold component diminishing and the hot component increasing in time. If ALPs are suitable dark matter candidates this mixed cold-hot component may lead to interesting cosmological consequences: for structure formation the “warmth” of the dark matter, a consequence of the cold and hot components, may help in solving the core vs cusp problem, furthermore, the hot component may provide a contribution to the effective number of relativistic degrees of freedom at recombination, and the continued interaction between the ALP and the CMB postrecombination until today may affect a birefringence signature if it is a consequence of a coupling of the CMB to a pseudoscalar field. These results may also point to possibly alternative bounds on the couplings and mass of ALPs.

The next step is to extend the methods implemented here to the realm of an expanding cosmology as well as other possible interactions which will be the focus of future work.

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APPENDIX A: SPECTRAL DENSITY FOR $\mathbf{E}^\rightarrow \cdot \mathbf{B}^\rightarrow$ COUPLING

We begin with the quantization of the gauge field within a volume V eventually taken to infinity,

$$A_\rightarrow \delta x^\mu \frac{1}{4} \int_V dk^\nu \epsilon_{\mu\nu 1;2} \epsilon^\lambda \frac{1}{2} \lambda \delta k^\nu \frac{1}{2} dk^\lambda \lambda e^{-ik \cdot x} \delta k^\lambda \lambda e^{ik \cdot x}; \quad \text{DA1p}$$

where ϵ^λ are the transverse polarization vectors chosen to be real. From Eqs. (2.63) and (2.64) we need the correlation functions

$$G_{>} \delta x^\mu \frac{1}{4} h E^\rightarrow \delta x^\nu \cdot B^\rightarrow \delta y^\mu E^\rightarrow \delta y^\nu \cdot B^\rightarrow \delta y^\nu; \quad \text{DA2p}$$

$$G_{<} \delta x^\mu \frac{1}{4} h E^\rightarrow \delta y^\nu \cdot B^\rightarrow \delta y^\mu E^\rightarrow \delta x^\nu \cdot B^\rightarrow \delta x^\nu; \quad \text{DA3p}$$

where we now refer to $h \delta \mu$ as averages in the thermal density matrix of free field photons. $E^\rightarrow \delta x^\mu \cdot B^\rightarrow \delta x^\mu \neq 0$ by parity invariance. Using Wick's theorem the

In the thermal ensemble the expectation value h
correlation function

A straightforward calculation yields

1 6A5P
hEiðxþEjðyþi ¼ hBiðxþBjðyþi ¼ 2___V Xk kððij - k[→]ik[→]jþ½ð1 þ nðkþþe-ik·ðx-yþ þ nðkþeik·ðx-yþ;

similarly

2V $k;1 k^*$

where $n\delta k \approx 1 = \delta e^{\beta k} - 1$. Combining the two terms in (A4) we find

Expanding the product, we perform the following change of variables in the various terms: (1) in the term $\vec{p} \rightarrow -\vec{k}$; (2) in the term with $\delta_1 \nabla \vec{n} \delta k \nabla \vec{n} \delta p \nabla p: \vec{p} \rightarrow -\vec{p}$, (3) in the term with $\nabla \delta k \nabla \delta_1 \nabla \vec{n} \delta p \nabla p: k \delta k \nabla \vec{n} \delta p \nabla p: k \rightarrow -k$, yielding in

the infinite volume limit

$G > \delta x - y^{\frac{1}{4}}$ $z = \frac{dq}{dt} 2\pi^0 z \delta d 2\pi^3 q p_3 \rho > \delta q_0; q p e^{-i q_0 \delta t - t^0 p e i q^*} \delta x^* - y^* p$; $\delta A 8 p$

where

$\rho \rightarrow \delta q_0; q \rho \frac{1}{4} - \pi^2 Z \delta 2 d \pi^3 k \rho_3 k j q \rightarrow - k \rightarrow j 1 - - \pi^2 k k \rightarrow \cdot j q q \rightarrow - - k k \rightarrow j \rightarrow 2 \frac{1}{2} \delta 1 \rho n \delta k \rho \rho \delta 1 \rho n \delta j q \rightarrow - k \rightarrow j \rho \rho \delta q_0 - k - j q \rightarrow - k \rightarrow j \rho$

þ nðkþnðjq → - k → jþððqo þ k þ jq → - k → jþ þ 1 þ kk_ _____ → · qq → -- kk → j 2½ð1 þ

$$n\delta k \mathbb{P} \mathbb{P} n\delta j q \rightarrow - k \rightarrow j \mathbb{P} \delta \delta q_0 - k \mathbb{P} j q \rightarrow - k \rightarrow j \mathbb{P} j$$

þ nðkþð1 þ nðjq[→] – k[→] jþþððqo þ k – jq[→] – k[→] jþ:

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Writing

$$G \propto \delta x - y^{\frac{1}{4}} Z \propto -dq_2 \pi^0 Z \propto \delta d_2 \pi^3 q^{\frac{1}{4}} \rho \propto \delta q_0; q \propto e^{-iq_0 \delta t - t^0} e^{i \vec{q} \cdot \vec{x} - \vec{y} \cdot \vec{p}}; \quad \delta A 10 \text{p}$$

$$\rho^{<\partial q_0; q} \rightarrow \rho^{>\partial -q_0; q} \rightarrow \rho^{>\partial q_0; q} \rightarrow \rho^{<\partial q_0; q} \quad \text{dA11p}$$

furthermore, using the identity $\delta(1-p) \delta(\partial^\mu p) \delta(\partial_\mu p) \delta(\partial^\nu p) \delta(\partial_\nu p)$ and using the various delta functions in the definition of $\rho^>$ we find

$$\rho < \delta q_0; q \rightarrow p \frac{1}{4} e^{-\beta q_0 p} \delta q_0; q \rightarrow p; \quad \delta A 12 p$$

which is the Kubo-Martin-Schwinger relation, thereby confirming the general results (2.67). The spectral density is given by [see Eq. (2.68)] $\rho\delta q_0; q \propto \rho^>\delta q_0; q - \rho^<\delta q_0; q$ with $\rho\delta q_0; q \propto \pi_2 Z \delta_2 d\pi_3 k_3 k_w 1/2 \cdot q^> \propto 1 \propto n\delta k \propto n\delta w \delta\delta q_0$.

þ ɔkw - k2 þ k' - q' þ2ðnðwþ - nðkþþððqo - k þ wþ - ððqo þ k - wþþg; w ½ jq' - k' j: ðA13þ

The spectral density is calculated by implementing the following steps:

$$\frac{d^3k}{k - w\beta - \delta\delta q_0} \frac{d\theta}{\beta k \beta w\beta\beta f\delta k w \beta k - k} \frac{d\cos\theta}{d\omega}$$

$$Z_8 \pi^3 \frac{1}{4} Z_0 k^2 \frac{1}{2} d \cos \theta_{4\pi} \delta \delta \text{pp}; \quad w \frac{1}{4} j \vec{q} - \vec{k} \vec{j} \frac{1}{4} \vec{q} \vec{q}_2 \vec{p} k_2 - \vec{2k} q \cos \delta \theta \text{ppf};$$

$$\delta w \text{pp} \frac{1}{4} - kq : \quad \delta A 14 \vec{p}$$

Carrying out the integrations, which are facilitated by the delta function constraints we find

$$\delta_0 \quad \frac{1}{\beta \sqrt{32\pi}} \beta \beta q \quad 1 - e^{-\beta\omega_{-1}} \quad \delta \quad \beta \beta q \quad 1 - e^{-\beta\omega_{-2}} \delta \quad \beta \quad \delta \beta \quad \delta \quad \beta$$

where

$$\delta A 16 \beta$$

$Q_2 \neq q_{20} - q_2; \omega \delta \beta \neq j q_0 j 2 - q; \omega \delta \beta \neq j 2 q_0 j$: APPENDIX B: FINITE TEMPERATURE CONTRIBUTION TO Σ_R

$$\rho q ; q^2 \quad \delta 2 \beta_2 \frac{1}{\text{sign } q} \quad \frac{2 \ln 1 - e^{-\beta\omega_{p1}} \Theta Q_2}{A 15 Q} \quad - \quad \frac{2 \ln 1 - e^{-\beta\omega_{p2}} \Theta - Q_2}{A 15 Q}$$

$$\Sigma_{R \delta T \beta} \delta v; k \beta \neq 32 g_2 \pi T_2 k \beta Z_{-\infty} \frac{\frac{2}{0} - k^2}{\delta k v - k \beta} \ln \frac{1}{1 - e^{-\beta\omega_{p1}} \delta k_0} \equiv 32 g_2 \pi T_2 k \delta v; k \beta; \omega \neq -k^2 k_0 : \delta B 1 \beta$$

Since the argument of the logarithm is odd under $k_0 \rightarrow -k_0$, it follows that I can be written as

$$I \delta v; k \beta \neq P Z_0 \delta k k_0 - k_2 \beta_2 \beta \ln 1 - e^{\frac{1}{2} \beta k_0} \frac{1}{1 - e^{-\frac{1}{2} \beta k_0}} \quad \delta B 2 \beta \approx 2k$$

where

$$I_1 \neq P_0 \quad Z \quad \frac{\infty 1 - e^{-\frac{1}{2} k_0}}{2k_0 k_0^2 - \nu^2} \quad \frac{\infty 1 - e^{-\frac{1}{2} k_0}}{2 \nu^2 - k^2} \quad \ln \delta \quad \beta \delta \quad \beta \beta \quad \frac{1 - e^{-\frac{1}{2} (k_0)}}{1 - e^{-\frac{1}{2} (k_0)}} \\ \beta k \beta \delta k_0 \\ I_2 \neq P Z_0 \quad \frac{k_2}{0} \frac{0}{\nu^2} \ln \frac{k_2}{1 - e^{-\frac{1}{2} \beta k_0}} \beta k \beta \delta k_0 : \delta B 3 \beta$$

Using the results

$$Z \approx \int_0^\infty x^n \ln \frac{1}{1 - e^{-\beta x}} dx \approx -\Gamma(n+1) \beta^n \Gamma(n+1) e^{-\beta} \quad \delta B 4 \beta$$

p

2i ∫ 1/2 ∫ 2i p ∫ 1/2 ∫ 2i p ∫ 1/2 ∫ 2i p

where Li is the polylogarithm, we find

$$I_1 \int \frac{3\pi\beta_2 k \ln v_2 - k_2 p}{\ln B6p} = 85\pi\beta_{22} \quad ;$$

For I_2 , we first write it as

$$I_2 \int \frac{1}{k_0 - v} \int \frac{1}{0} \int \frac{1}{p v} \int \frac{2}{\ln 6} \frac{1 - \exp(-\beta \frac{ik_0 - ki}{2})}{4} \frac{3}{1 - \exp(-\beta \frac{kbk_0}{2})} \frac{7}{5} dk \quad ;$$

$$dB7p$$

Note that

$$\ln \frac{1}{1 - x} = \sum_{n=1}^{\infty} \frac{1}{n x^n} \quad ;$$

$$dB8p$$

and

$$Z = \int_{-\infty}^{\infty} \frac{dx}{x} = -\frac{1}{2} e^{-\frac{1}{2}}$$

$$P_0 = \frac{1}{n!} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-n(\int_{-\infty}^{\infty} dz + \int_{-\infty}^{\infty} dy + \int_{-\infty}^{\infty} dz)} = \frac{1}{n!} e^{-n} \quad ;$$

$$dB9p$$

$$x \int z \quad n$$

$$P_{Z0k} dx = \frac{1}{k!} e^{-k} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-k(\int_{-\infty}^{\infty} dz + \int_{-\infty}^{\infty} dy + \int_{-\infty}^{\infty} dz)} = \frac{1}{k!} e^{-k} \quad ;$$

$$dB10p x \int z \quad n$$

$$Z = \int_{-\infty}^{\infty} \frac{dx}{x} = -\frac{1}{2} e^{-n(x - \frac{1}{2})}$$

$$P_k = \frac{1}{k!} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-k(\int_{-\infty}^{\infty} dz + \int_{-\infty}^{\infty} dy + \int_{-\infty}^{\infty} dz)} = \frac{1}{k!} e^{-k} \quad ;$$

$$dB11p$$

The exponential integral function features a useful representation,

$$\int_{-\infty}^{\infty} x^n$$

where γ is Euler's constant. This expansion allows us to extract the low- and high-temperature limits, yielding the

$$\Sigma_{RT} \propto \Omega_k k^p \frac{1}{4} g_2 T^4 = 15 - 24 T_{a2} + 32 T_{a4} 1 - \gamma \ln m_a - k k^* \ln \frac{4\pi T}{m_a} : \quad dB13p$$

In the low-temperature limit $T \ll m_a/k$ we find

$$\Sigma_{RT} \propto \Omega_k k^p \frac{1}{4} g_2 T^4 = \frac{4\pi}{\pi^2 m_2 m_4} 45_2 k_{22} + 32\pi 63_4 m_a 1 + 4 \frac{4\pi T \Omega}{m_a} + k_{22} + 165 k_{44} T_{22} : \quad dB14p$$

high-temperature behavior for $T \gg \Omega_k$

$$\pi^2 m_2 m_4 \quad 4\pi T \Omega \quad \Omega k$$

$$m_a \quad m_a \quad m_a \quad m_a$$

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