Effective field theory of particle mixing

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We introduce an effective field theory to study indirect mixing of two fields induced by their couplings to a common decay channel in a medium. The extension of the method of Lee, Oehme, and Yang, the cornerstone of analysis of CP violation in flavored mesons, to include the mixing of particles with different masses provides a guide to and benchmark for the effective field theory. The analysis reveals subtle caveats in the description of mixing in terms of the widely used non-Hermitian effective Hamiltonian, more acute in the nondegenerate case. The effective field theory describes the dynamics of field mixing where the common intermediate states populate a bath in thermal equilibrium, as an open quantum system. We obtain the effective action up to second order in the couplings, where indirect mixing is a consequence of offdiagonal self-energy components. We find that if only one of the mixing fields features an initial expectation value, indirect mixing induces an expectation value of the other field. The equal time two point correlation functions exhibit an asymptotic approach to a stationary thermal state, and the emergence of long-lived bath-induced coherence which displays quantum beats as a consequence of interference of quasinormal modes in the medium. The amplitudes of the quantum beats are resonantly enhanced in the nearly degenerate case with potential observational consequences.

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I. INTRODUCTION

The dynamics of particle mixing induced by their coupling to a common intermediate state or decay channel is of broad fundamental interest within the context of CP violation and/or baryogenesis. Field mixing may also be a consequence of "portals," connecting standard model degrees of freedom to hypothetical ones via mediator particles beyond the standard model. Such portals may lead to mixing between fields on different sectors of the portal via the exchange of these mediators, namely a common intermediate state to which fields on different sides of the portals couple.

Axions, CP-odd pseudoscalar particles proposed in extensions beyond the standard model as a possible solution of the strong CP problem in quantum chromodynamics (QCD) [1–3], could be a compelling cold dark matter candidate [4–6]. However, various extensions beyond the standard model can include axionlike particles with properties similar to the QCD axion which may also be suitable dark matter candidates [7–11]. Just as the QCD axion these axionlike particles couple to photons and gluons via Chern-Simons terms such as $\vec{E}\cdot\vec{B}$ in the case of photons, or $\tilde{G}^{\mu\nu;b}G_{\mu\nu;b}$ in the case of gluons, as a consequence of the chiral anomaly. Their mutual coupling to photons and gluons entails that the various "flavors" of axions or axionlike particles may mix via a common intermediate state of photons and gluons. For example, processes such as $A \leftrightarrow \gamma\gamma \leftrightarrow A^0$, with A, A^0 being different axionlike particles, yield off-diagonal self-energy components $\Sigma_{A;A^0}$, hence an indirect mixing via the common intermediate state.

A paradigmatic example in vacuum is the mixing of $K^0 - \bar{K}^0$ or flavored meson-antimesons as a consequence of common intermediate states of two or three pions (or the weak interaction box diagram), providing dynamical observational signatures of CP violation [12–17].

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Field mixing via a common intermediate state in a thermal medium has recently been studied [18] within the context of axion-neutral pion mixing after the QCD phase transition, since the neutral pion couples to two photons precisely via a $U(1)$ Chern-Simons term as a consequence of the chiral anomaly.

Recently, it has been realized that topological materials and/or Weyl semimetals also feature emergent axionlike quasiparticles as collective excitations, which couple to electromagnetism via processes akin to the U(1) anomaly [19–28]. Therefore, these "synthetic" axions may mix with the cosmological axion in the same manner as pions or generic axionlike particles in the early universe.

This possibility motivates the study of mixing between the cosmological and the synthetic axions, which may yield

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alternative experimental avenues to probe cosmological axions with condensed matter experiments.

A. Motivations and objectives

Motivated by the ubiquity of field mixing and its broad relevance in particle physics, cosmology, and possibly in condensed matter physics, we extend the preliminary study of Ref. [18] and develop a more general effective field theory framework to study mixing as a consequence of coupling of different fields to common intermediate states or decay channels. We distinguish direct mixing as a result of explicit mixing terms in the Lagrangian, such as offdiagonal mass matrices or kinetic mixing terms, from the indirect mixing via common intermediate states leading to off-diagonal self-energy components, such as flavored meson mixing, for example, $K^0 - K^{-0}$. Our study is focused on this latter, indirect mixing case.

The theory of $K^0 - K^{-0}$ mixing via weak interaction intermediate states was advanced by Lee et al. [29] in their pioneering study of CP violation. It is based on the theory of atomic linewidths developed by Weisskopf and Wigner [30–33], and it is the cornerstone of the analysis of mixing dynamics of flavored mesons and CP violation [12–15] in terms of an effective non-Hermitian Hamiltonian.

Our main focus is to develop an effective field theory framework to study the dynamics of indirect mixing when the particles in the intermediate states are components of a thermal bath as is the case in cosmology. An advantage of the effective field theory formulation of mixing is that it allows one to obtain correlation functions in the medium, to understand their approach to thermalization, and to observe the emergence of long-lived coherence, namely off-diagonal components of the two point field correlation function that survives in the long time limit even when initially the different fields are uncorrelated.

The preliminary study of Ref. [18] focused on the particular case of axion-neutral pion mixing near the QCD phase transition where the axion was assumed to be a light or ultralight CP-odd scalar. In this case there is a large mass difference between the mixing partners leading to suppression of interference effects. Furthermore, axions and neutral pions couple to photons with the same operator $(\vec{E} \cdot \vec{B})$ but with different couplings, making this a particular case.

Instead, here we contemplate more general scenarios including that of degenerate or nearly degenerate mixing fields and coupling to intermediate states with different operators with nonvanishing correlations in the thermal bath, thereby leading to mixing via off-diagonal self-energy matrix elements. This more general situation may be relevant for CP violation in the early universe and yields far richer dynamics including nonperturbative interference

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phenomena in the form of quantum beats that plays an important role in the approach to thermalization and the dynamics of coherence, with possible observational consequences.

Unlike the case of direct mixing, such as neutrino mixing via an off-diagonal mass matrix, or kinetic mixing, indirect mixing in a medium, as is relevant in cosmology, to the best of our knowledge has not yet been studied at a deeper level.

Our objectives are (i) to provide a consistent effective field theory framework to study the dynamics of mixing via intermediate states in equilibrium in a medium; (ii) to apply this formulation to study the nonequilibrium dynamics of expectation values and correlation functions of the mixing fields; and (iii) to focus in particular on the approach to thermalization and the emergence and long time survival of coherence even when initially the mixing fields are uncorrelated.

The equations of motion obtained from the effective field theory allow one to study the dynamical evolution of expectation values and correlation functions and the emergence and evolution of coherence, hence providing an approach to the study of coherence that complements the quantum master equation [34–36]. We also recognize that the effective field theory approach to mixing may also be extended to the case of neutrinos in the mass basis, and may provide an alternative framework to study the quantum kinetics of massive neutrinos in the medium [37]. More recentlyaquantumfieldtheoreticalapproachtoaBoltzmann equation for axions consistently including misaligned condensateshasbeenintroducedinRef.[38].Theformulation of an effective field theory of mixing developed in this study mayprovideacomplementaryapproachwhendifferenttypes of axions mixing indirectly via a common intermediate state are considered.

In this article our main objective is to develop the theoretical framework in general, without specifying

particular models or applications, which will be the subject of future study.

B. Brief summary of results

As a prelude to developing the effective field theory framework, in Sec. II we extend the Lee-Oehme-Yang (LOY) theory of mixing to the case of nondegenerate mixing particles and with generic couplings to intermediate states, and solve exactly the equations for the amplitudes, which goes beyond the usual approach based on a non-HermitianeffectiveHamiltonian[12–15].Thegeneralization to the nearly degenerate and nondegenerate cases provides an extension to analyze the dynamics of mixing relaxing the assumption of validity of CPT. This study serves as a guide and benchmark toward establishing the effective field theory framework, and also reveals interesting caveats of the usual approach with a non-Hermitian Hamiltonian, which become more important in the nondegenerate case and may be relevant in precision measurements of CP violation.
Appendix A discusses the origin of some of these caveats in the case of a single species.

In Sec. III we consider indirect mixing of two bosonic fields induced by their couplings to a common decay channel in the medium. These common intermediate states populate a bath in thermal equilibrium.

We generalize the methods of Refs. [18,39] to obtain the effective action in the in-in or Schwinger-Keldysh formulation of nonequilibrium quantum field theory [40–44] up to second order in couplings. This effective action determines the time evolution of the reduced density matrix upon tracing the bath degrees of freedom, and it describes the dynamics of mixing as an open quantum system. The equations of motion obtained from the effective action are stochastic with self-energy and noise kernels obeying a generalized fluctuation dissipation relation. Indirect mixing is a consequence of off-diagonal self-energy components arising from the correlations of the coupling operators in the bath. The solution of the equations of motion yield the time evolution of expectation values and correlation functions in terms of superpositions of quasinormal modes in the medium. The cases of nondegenerate and nearly degenerate fields are studied in detail. We find that if only one of the fields has an initial nonvanishing expectation value, indirect mixing induces an expectation value for the other field. Furthermore, the equal time two points correlation function approaches a stationary thermal state independent of the initial conditions and even when initially the fields are uncorrelated exhibit an emergent long-lived bath-induced coherence, namely off-diagonal components. Both diagonal and off-diagonal correlation functions display quantum beats, as a consequence of interference of quasinormal modes. The amplitudes of the quantum beats are resonantly enhanced in the case of nearly degenerate fields. In this section we establish the correspondence between the LOY formulation of particle mixing and the effective field theory of mixing.

Several appendixes supplement technical details. Appendix A discusses the caveats associated with a nonHermitian Hamiltonian for a single species. Section IV summarizes the main results and conclusions.

II. THE LEE-OEHME-YANG THEORY OF MIXING

We begin by extending and generalizing the formulation of meson mixing pioneered by Lee et al. [29,30] to analyze CP violation in the kaon system, which is based on the Weisskopf-Wigner theory of atomic linewidths [30], to the case when particles of different masses mix via a common set of intermediate states, or common decay channel. Such a generalization will lead us to the formulation of an effective quantum field theory of mixing including the case when the particles in the intermediate states constitute a medium as is relevant in cosmology.

Consider a system whose Hamiltonian H is given as a soluble part H_0 and a perturbation H_I: $H = H_0 + H_I$.
The time evolution of states in the interaction picture of H_0 is given by

$$i\frac{d}{dt}|\Psi(t)\rangle_I = H_I(t)|\Psi(t)\rangle_I; \quad (2.1)$$

where the interaction Hamiltonian in the interaction picture is

$$H_I(t) = e^{iH_0t}H_Ie^{-iH_0t}; \quad (2.2)$$

where H_I is proportional to a set of couplings assumed to be small.

Equation (2.1) has the formal solution $|\Psi(t)\rangle_I =$

$$U(t;t_0)|\Psi(t_0)\rangle_I; \quad (2.3)$$

where the time evolution operator in the interaction picture $U(t;t_0)$ obeys

$$i\frac{d}{dt}U(t;t_0) = H_I(t)U(t;t_0): \quad (2.4)$$

Now we can expand

$$|\Psi(t)\rangle_I = \sum_n C_n(t)|n\rangle; \quad (2.5)$$

where $|n\rangle$ form a complete set of orthonormal states chosen to be eigenfunctions of H_0, namely $H_0|n\rangle = E_n|n\rangle$; in the quantum field theory case these are many-particle Fock

states. From Eq. (2.1) and the expansion (2.5) one finds the equation of motion for the coefficients $C_n(t)$, namely

$$\dot{C}_n(t) = -i\sum_m C_m(t)\langle n|H_I(t)|m\rangle; \quad (2.6)$$

Although this equation is exact, it generates an infinite hierarchy of simultaneous equations when the Hilbert space of states spanned by $\{|n\rangle\}$ is infinite dimensional. However, this hierarchy can be truncated by considering the transition between states connected by the interaction Hamiltonian at a given order in H_I.

Let us consider quantum states $|\phi_1\rangle;|\phi_2\rangle$ associated with the meson fields $\phi_{1,2}$, respectively; these may be single particle momentum eigenstates of the Fock quanta of these fields, and focus on the case when the interaction

Hamiltonian does not couple$a \neq b$. Instead, these states are connecteddirectly the states $|\phi_1\rangle;|\phi_2\rangle$, namely $\langle\phi|H|\phi\rangle = 0$

indirect coupling through the common set of intermediatejto a common set of intermediate statesϕ_1The states$,2\rangle \leftrightarrow |\kappa\rangle$and$_{1,2}\rangle$, as depicted in Fig.$\phi_2\rangle$ mix as a consequence of this$|\kappa\rangle$1by. H_I, namely

states, namely $|\phi_{1,2}\rangle \leftrightarrow |\kappa\rangle \leftrightarrow |\phi_{1,2}\rangle$, yielding an

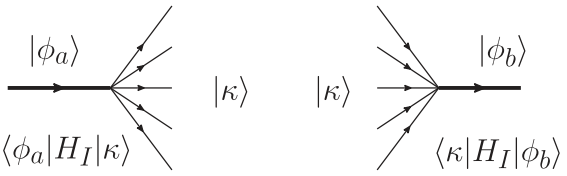

FIG. 1. Mixing between $|\phi\rangle_a$ and $|\phi\rangle_b$ mediated by a common set of intermediate states $|\kappa\rangle$.

offdiagonal self-energy matrix. If H_I has nonvanishing matrix elements $\langle\phi_i|H_I|\phi_j\rangle \neq 0$, we assume that these have been absorbed into terms in H_0 and only consider transitions betweenIn the subspace$|\phi_i\rangle$ and other states$|\phi_1\rangle;|\phi_2\rangle;|\{\kappa\}\rangle \neq$the quantum state in the$|\phi_{1,2}\rangle$ mediated by H_I.

interaction picture is given by $|\Psi_I(t)\rangle = C_1(t)|\phi_1\rangle +$

$$C_2(t)|\phi_2\rangle + \sum_\kappa C_\kappa(t)|\kappa\rangle; \quad (2.7)$$

and the set of equations (2.6) becomes

$$\dot{C}_1(t) = -i\sum_\kappa \langle\phi_1|H_I(t)|\kappa\rangle C_\kappa(t); \quad (2.8)$$

$$\dot{C}_2(t) = -i\sum_\kappa \langle\phi_2|H_I(t)|\kappa\rangle C_\kappa(t); \quad (2.9)$$

$$\dot{C}_\kappa(t) = -i[\langle\kappa|H_I(t)|\phi_1\rangle C_1(t) + \langle\kappa|H_I(t)|\phi_2\rangle C_2(t);$$

(2.10)

where the time dependent transition matrix elements are given by $\langle l|H_I(t)|m\rangle = T_{lm}e^{i(E_l-E_m)t}$; $T_{lm} = \langle l|H_I(0)|m\rangle$; (2.11)

Hermiticity of H_I entails that

$$T_{ml} = T_{lm}^*. \quad (2.12)$$

The set of equations (2.8)–(2.10) truncates the hierarchy of equations by neglecting the transitions between the states $|\{\kappa\}\rangle$ and $|\kappa_0\rangle$at a higher order in, and such transitions con-H_I and the set of equations for the amplitudes, effectively reducingare neglected up to$|\phi_{1,2}\rangle$ correspond to $K^0;\bar{K}^0$ mesons mixing via a common decay$\mathcal{O}(H_I^2)$.

Truncating the hierarchy closes

the set of states to a closed subset in the full Hilbert space.

As a familiar example, let us consider the case where

channel into two pions (there is also the three pion decay channel) so that $K^0 \leftrightarrow 2\pi \leftrightarrow K^{-0}$.

Taking the normalized initial quantum state $|\Psi(t=0)\rangle$ as a coherent linear superposition of the single particle states $|\phi_{1,2}\rangle$, it is given by $|\Psi(t=0)\rangle = (C_1(0)|\phi_1\rangle + C_2(0)|\phi_2\rangle) \otimes |0_\kappa\rangle$; (2.13) where $|0_\kappa\rangle$ is the vacuum state for the intermediate states $|\kappa\rangle$, corresponding to setting

$$C_\kappa(0) = 0; \tag{2.14}$$

for the excited $|\kappa\rangle$ states, and with normalization condition

$$|C_1(0)|^2 + |C_2(0)|^2 = 1: \tag{2.15}$$

A. Unitarity

The set of equations (2.8)–(2.10) describes unitary time evolution in the restricted Hilbert space of states $|\text{space of}\rangle$ of the theory that is closed under the equations $|\phi_1\rangle, |\phi_2\rangle, |\kappa\rangle$, which is a subset of the full Hilbert

$$\frac{d}{dt}\left(|C_1(t)|^2 + |C_2(t)|^2 + \sum_\kappa |C_\kappa(t)|^2\right) = 0; \tag{2.16}$$

and the initial conditions (2.14) and (2.15) yield $|C_1(t)|^2 +$

$$|C_2(t)|^2 + \sum_\kappa |C_\kappa(t)|^2 = 1: \tag{2.17}$$

This is the statement that time evolution within the sub-Hilbert space $\{|\phi_1\rangle, |\phi_2\rangle, |_1\kappa_{,2}\rangle\}$states decay, it follows thatis unitary. In particular, if the $|C_{1,2}(t=\infty)|^2 = 0$, and

$$\sum_\kappa |C_\kappa(t=\infty)|^2 = 1: \tag{2.18}$$

The set of equations (2.10) with the initial condition (2.14) can be integrated to yield

$$C_\kappa(t) = -i \int_0^t \langle T_{\kappa 1} e^{i(E_\kappa - E_1)t_0} C_1(t_0) + T_{\kappa 2} e^{i(E_\kappa - E_2)t_0} C_2(t_0)\rangle dt_0; \tag{2.19}$$

of motion (2.8)–(2.10). Unitarity can be seen as follows: using Eqs.$\langle H_I(t)|i\rangle$(2.8)because–(2.10)$H_I$and noticing that(t) is an Hermitian operator, it$\langle i|H_I(t)|m\rangle = \langle$follows that

where the labels 1 and 2 correspond to $\phi_{1,2}$. Inserting the solution (2.19) into Eqs. (2.8) and (2.9) leads to the following set of equations for the coefficients $C_1(t); C_2(t)$:

$$\dot{C}_1(t) = -\int_0^t \sum_\kappa \left[T_{1\kappa} T_{\kappa 2} e^{i(E_1 - E_2)t} e^{i(E_2 - E_\kappa)(t - t_0)} C_2(t_0) + T_{1\kappa} T_{\kappa 2} e^{i(E_1 - E_2)t} e^{i(E_2 - E_\kappa)(t - t_0)} C_2(t_0)\right] dt_0; \tag{2.20}$$

$$\dot{C}_2(t) = -\int_0^t \sum_\kappa \left[T_{2\kappa} T_{\kappa 1} e^{i(E_2 - E_1)t} e^{i(E_1 - E_\kappa)(t - t_0)} C_1(t_0) + T_{2\kappa} T_{\kappa 1} e^{i(E_2 - E_\kappa)(t - t_0)} C_2(t_0)\right] dt_0: \tag{2.21}$$

This procedure of solving for the amplitudes of the intermediate states plays the role of "integrating out" or "tracing over" the κ degrees of freedom, yielding an effective set of equations of motion for the amplitudes of the single particle states $|\phi_{1,2}\rangle$. Since the interaction Hamiltonian H_I is assumed to include a weak coupling, the amplitude equations (2.20) and (2.21) are exact up to second order in this coupling. Pictorially, this procedure is

equivalent to joining the legs representing the χ field together in Fig. 1, thereby forming a loop or loops that yield(s) the self-energy.

B. Exact solutions

The set of amplitude equations (2.20) and (2.21) can be solved exactly. For this purpose it is convenient to define

$$e^{-iE_1 t}C_1(t) \equiv A_1(t); \quad e^{-iE_2 t}C_2(t) \equiv A_2(t); \tag{2.22}$$

and to introduce the spectral densities

$$\mu_{ab}(k_0) \asymp \sum_{\kappa} \tfrac{1}{4}T_{a\kappa}T_{\kappa b}\delta(k_0 - E_\kappa) = \mu_{ba}(k_0); \quad a,b = 1,2; \tag{2.23}$$

where the second identity follows from Eq. (2.12). The set of equations for the amplitudes $A_{1,2}$ following from (2.20) and (2.21) is written more compactly by introducing the self-energy matrix

$$\sigma_{ab}(t - t_0) = \int_{-\infty}^{\infty} \mu_{ab}(k_0)e^{-ik_0(t-t_0)}dk_0. \tag{2.24}$$

This self-energy has an intuitive interpretation as a second order Feynman diagram wherein the lines representing the intermediate states $|\kappa\rangle$ in Fig. 1 are joined into "propagators" yielding a (multi)loop diagram, representing the self-energy up to second order in H_I.

In terms of the self-energy the set of equations (2.20) and (2.21) becomes

$$\dot{A}_a(t) + iE_aA_a(t) + \int_0^t dt' \sum_b \sigma_{ab}(t-t')A_b(t') = 0. \tag{2.25}$$

This equation can be solved via a Laplace transform. Defining the Laplace transforms for $\mathrm{Re}(s) > 0$

$$\tilde{A}_a(s) = \int_0^\infty e^{-st}A_a(t)dt; \tag{2.26}$$

$$\tilde{\sigma}_{ab}(s) = \int_0^\infty e^{-st}\sigma_{ab}(t)dt = \int_{-\infty}^{\infty} \frac{\mu_{ab}(k_0)}{s - ik_0}dk_0;$$

the set of equations (2.25) leads to

$$(M_{11} + M_{12}\tilde{A}_{12}(s))A_1(0) = AA_{12}(0), \tag{2.27}$$

$$(M_{22} + M_{21}\tilde{A}_{12}(s))A_2(0) = AA_{12}(0),$$

with matrix elements

$$M_{11} = s + iE_1 + \tilde{\sigma}_{11}(s); \tag{2.28}$$

$$M_{12} = \tilde{\sigma}_{12}(s); \qquad M_{21} = \tilde{\sigma}_{21}(s); \tag{2.29}$$

$$M_{22} = s + iE_2 + \tilde{\sigma}_{22}(s); \tag{2.30}$$

where we suppressed the dependence of the matrix elements M_{ij} on the Laplace variable s to simplify notation but it is implicit in all matrix elements.

It proves convenient to introduce

$$\bar{M} = \tfrac{1}{2}(M_{11} + M_{22}); \tag{2.31}$$

$$D = \sqrt{(M_{11} - M_{22})^2 + 4M_{12}M_{21}}/2; \tag{2.32}$$

$$\alpha = \frac{M_{11} - \bar{M}}{D}; \qquad \beta = \frac{M_{12}}{D}; \tag{2.33}$$

$$\gamma = \frac{M_{21}}{D};$$

where α, β, γ fulfill the relation

$$\alpha^2 + \beta\gamma = 1. \tag{2.34}$$

The inverse of the matrix with elements M_{ab} yields the Laplace Green's function, which is given by (see Appendix C for details)

$$AA_{12}(s) = \frac{\bar{M} - (s + \tilde{\sigma}_{12}(s) - \tilde{\sigma}_{22}(s))}{(s + \tilde{\sigma}_{12}(s) - \tilde{\sigma}_{22}(s))(s + \tilde{\sigma}_{11}(s) - \tilde{\sigma}_{12}(s)) + \bar{M}}AA_{21}(0), \tag{2.35}$$

where the projector operators (see Appendix C) are given by

$$P(s) - = \frac{1}{2}(1 \; R(s)); \qquad R(s) = \alpha\gamma((s)) - \beta\alpha((s)). \qquad (2.36)$$

Finally, the time evolution of the amplitudes $A_{1;2}(t)$ is obtained via the inverse Laplace transform,

$$\begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} = \int_C e^{st}\left[\frac{\bar{M}_-(P(s)}{(-s)D_2(s)} + \frac{\bar{M}_+(P(s)}{(s)D_2(s)}\right]\frac{ds}{2\pi i} \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix}; \qquad (2.37)$$

where the Bromwich countour C runs parallel to the imaginary axis to the right of all the singularities in the complex s plane. Stability implies that the real parts of the singularities are negative; therefore, the contour corresponds to $s = i\nu + \epsilon; -\infty \leq \nu \leq \infty; \epsilon \to 0^+$. It is convenient to change variables to $\nu = -\omega$, in terms of which

$$\tilde{\sigma}_{ab}(s = i(-\omega - i\epsilon)) \equiv i\Delta_{ab}(\omega) = i\,\mathcal{P}\int_{-\infty}^{\infty}\frac{\mu_{ab}(k_0)\,dk_0}{\omega - k_0} - i\pi\mu_{ab}(\omega); \qquad (2.38)$$

where $\mathcal{P}$ stands for the principal part. The relation (2.23) implies that

$$\Delta_{ba}(\omega) = \mathcal{P}\int_{-\infty}^{\infty}\frac{\mu_{ab}^*(k_0)\,dk_0}{\omega - k_0} - i\pi\mu_{ab}^*(\omega). \qquad (2.39)$$

Upon this analytic continuation, Eq. (2.37) becomes

$$\begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} = -\int_{-\infty}^{\infty} e^{-i\omega t}\left[\frac{P_-(\omega)}{\omega - W_-(\omega)} + \frac{P_+(\omega)}{\omega - W_+(\omega)}\right]\frac{d\omega}{2\pi i} \times \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix}; \qquad (2.40)$$

where

$$W_{\mp}(\omega) = \frac{1}{2}\{(E_1 + E_2 + \Delta_{11}(\omega) + \Delta_{22}(\omega)) \mp D(\omega)\}; \qquad (2.41)$$

$$D(\omega) = [(E_1 - E_2 + \Delta_{11}(\omega) - \Delta_{22}(\omega))^2 + 4\Delta_{12}(\omega)\Delta_{21}(\omega)]^{1/2}; \qquad (2.42)$$

$P_{\mp}(\omega)$ are the analytic continuation of $P_{\mp}(s)$ for $s \to -i(\omega + i\epsilon)$ and (2.40) has a simple interpretation: it is the

The bracket inside the integral in interpretation: it is the Dyson (geometric) resummation of the second order self-energy matrix, and the time evolution obtained from (2.40) includes this resummation of second order self-energy corrections. Note that as a consequence of the projector matrices being off-diagonal, even when one of the amplitudes vanishes initially, for example, if $A_2(0) = C_2(0) = 0$, it becomes nonvanishing at a later time. This observation will have interesting implications in the analysis of in-medium mixing in the next section.

In the weak coupling limit we invoke the Breit-Wigner approximation, valid in the intermediate time regime, where each term in (2.40) features a complex pole in the lower half ω plane at[1]

$$\omega = W_{\pm}(\omega_\pm) \equiv \varepsilon_\pm - i\frac{\Gamma_\pm}{2}; \qquad (2.43)$$

where $\varepsilon_\pm$ are the renormalized frequencies and $\Gamma_\pm$ the decay rates. In weak coupling, it follows that $\Gamma \propto H_I \ll E_{1;2}$, and we

[1] More precisely, the poles are in the second Riemann sheet, but close to the real axis in the complex ω plane.

will refer to these complex frequencies as quasinormal modes. For a vanishing damping rate, these are the usual normal modes associated with the coupling of harmonic oscillators, the "quasi" reflects their damping as a consequence of their coupling to and decay into a (common) continuum.

Evaluating (2.40) by contour integration closing in the lower half ω plane for t > 0, and expanding near the complex poles Wðωþ ¼ Wðωþ þ ðω−ωþdWðωþ= dωjω¼ω þ , we obtain the final result

$$A_{12}(t) = \tfrac{1}{2} e^{-i\omega_+ t} Z_+ P_+(\omega_+) + e^{-i\omega_- t} Z_- P_-(\omega_-)$$

$$A(t) \to A \times A_2^1 \delta^0_0 \quad (2.44)$$

where $Z = 1 - dW(\omega_+)/d\omega|_{\omega_+}^{-1}$ and

$$P(\omega_+) = -2\gamma^{\alpha}\omega\omega^{\omega} \quad 1^{\beta(\omega_\pm \mp \alpha\omega_+)}; \quad (2.45)$$

this result is general.

The Breit-Wigner approximation relies on weak coupling so that the width of the state is much smaller than its mass and that the distance between the real part of the pole and the beginning of the multiparticle cuts must be much larger than the half-width of the particle. This entails that the spectral representation of the propagator can be well approximated by a Lorentzian centered at the real part of the pole with the width determined by the imaginary part of the self-energy at the position of the pole. Furthermore, this entails that the spectral density of the self-energy is finite and smooth near the value of the pole. This is the same criterion as in Fermi's golden rule.

It is important to highlight that the Breit-Wigner approximation leading to the result (2.44) is valid only during an intermediate time regime; it is neither valid as $t \to 0$ nor at a very long time, when power law corrections emerge [33,45–48].

As analyzed in detail in these references, the asymptotic late time behavior of the integral in Eq. (2.40) is determined by the behavior of the spectral density at the threshold of multiparticle cuts, which yields a power law that emerges when the amplitude is already perturbatively small (see Ref. [49] for a specific example), and the behavior at early times, $t \to 0$, receives contributions from the full spectral density, contributing to a renormalization of the amplitude of the field. We refer to the intermediate timescale, as the scales between these two limits that depend specifically on the details of the spectral density of the self-energy. However, as is expected in the case of a weakly coupled theory, the intermediate timescale in which there is exponential decay is generically wide and is captured reliably by the usual Breit-Wigner approximation of the propagator.

Therefore, the extrapolation to $t \to 0$ is not consistent with this approximation. In fact, the wave function renormalization is a consequence of "dressing" and renormalization during an initial transient timescale describing the formation of a quasiparticle [50]; in renormalizable theories it is usually ultraviolet divergent. The timescale of formation of the quasiparticle is typically associated with the ultraviolet behavior of the spectral density, and it is in general much shorter than the typical oscillation and decay timescales of the particle [50].

In the following analysis we assume without loss of generality that $E_1 \geq E_2$, and, consistently with perturbation theory, that $E_{1,2} \gg \Delta_{ab} \propto H_I^2$. Furthermore, from the identity (2.34) we choose

$$\alpha(\omega_+) = 1 - \beta(\omega_+)\gamma(\omega_+) + \ldots \quad (2.46)$$

Hence, in the limit of vanishing coupling $\Delta_{ab} \to 0$, it follows that

$$\omega_+ \to E_1; \quad \omega_- \to E_2; \quad \alpha \to 1; \quad \beta, \gamma \to 0;$$

$$P_+ \to \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}; \quad P_- \to \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}; \quad (2.47)$$

Therefore, in this limit the amplitudes $C_{1,2}$ do not depend on time as it must be the case in the absence of interactions. Two limits are important: (i) $E_1 - E_2 \gg \Delta_{ab}$, to which we refer as the nondegenerate case, and (ii) $E_1 - E_2 \lesssim \Delta_{ab}$, to which we refer as the (nearly) degenerate case. The first case describes, for example, the mixing between axionlike particles and a neutral pseudoscalar meson as studied in Ref. [18], such as the pion, with the pion mass much larger than that of the axion. The second case includes neutral (pseudoscalar) flavored meson-antimeson mixing, such as $K^0 - K^{-0}$ under the condition of charge conjugation, parity, and time reversal (CPT) invariance [in which case $E_1 = E_2$; $\Delta_{11}(s) = \Delta_{22}(s)$]. This second case also applies to neutral meson mixing if there is a small (CPT) violation, in which case E_1, E_2 and the diagonal matrix elements Δ_{11}, Δ_{22} may

be slightly different but small compared to the individual energies $E_{1;2}$.

(I) Nondegenerate case: $E_1 - E_2 \gg \Delta_{ab}$. In this case we can approximate

$$D(\omega) \simeq E_1 - E_2 + \Delta_{11}(\omega) - \Delta_{22}(\omega) + O(\Delta^2); \quad (2.48)$$

from which it follows that to leading order $[O(\Delta)]$ $W_+(\omega)$

$$= E_1 + \Delta_{11}(\omega); \quad W_-(\omega) = E_2 + \Delta_{22}(\omega); \quad (2.49)$$

and to leading order in couplings, the complex poles are at

$$\omega_+ = E_1 + \Delta_{11}(E_1) = E_{1R} - i\Gamma_+/2; \quad E_{1R} = E_1 + \mathrm{Re}\Delta_{11}(E_1); \quad \Gamma_+ = 2\pi\rho_{11}(E_1); \quad (2.50)$$

$$\omega_- = E_2 + \Delta_{22}(E_2) = E_{2R} - i\frac{\Gamma_-}{2}; \quad E_{2R} = E_2 + \mathrm{Re}\Delta_{22}(E_2); \quad \Gamma_- = 2\pi\rho_{22}(E_2); \quad (2.51)$$

where $E_1^{R;2}$ are the renormalized energies. Up to leading orderdependent amplitudes are given by$O(\Delta)$, it is straightforward to find that the time

$$A_1(t) = Z_+A_1(0)e^{-i\omega_+ t} + 2\,{}_{12}(E_1)e^{-iE_{\omega_+ 1R}t} -$$

$$- \frac{}{E\Delta_{2R12}(E_2)e^{-i\omega_- t}A_2(0)}; \quad (2.52)$$

$$A_2(t) = Z_-A_2(0)e^{-i\omega_- t}$$

$$+ 21\Delta_{21}(E_1)e^{-iE_{\omega_+ 1R}t} = E\Delta_{R221}(E_2)e^{-i\omega_- t}A_1(0): \quad (2.53)$$

The terms in brackets in (2.52) and (2.53) are perturbatively small in this case because $\Delta_{ab} \ll E_1 - E_2$. Since

brackets, which are already of$Z$ (II) (Nearly) degenerate case$\simeq 1 + O(H^2_I)$, we neglected them in the terms in the:$O(E(\delta_1\Delta_{;2}) \gg \propto \Delta_{Hab2I};.E^1 - E^2 \lesssim \Delta_{ab}$.

In this case we write

$$E_1 + 2 \equiv E^- \gg \Delta_{ab}; \qquad E_1 - E_2 \equiv \delta \lesssim O(\Delta); \quad (2.54)$$

and to leading order in Δ the complex poles are given by

$$\omega = - \frac{1}{} \Delta_{11}\ \bar{E} \quad \Delta_{22} - E + 2(\delta) + \quad (E)$$

$$(2E^-) D$$

$$\varepsilon - \Gamma_\pm - {}_R - \bar{\Gamma} \quad \frac{(E)}{} i \quad E \quad i - \quad D$$

$$\equiv \quad 2 = \quad 2 \quad 2 \quad ; \quad (2.55)$$

where

$$E^- = E^- + 2(\mathrm{Re}\Delta_{11}(E^-) + \mathrm{Re}\Delta_{22}(E^-));$$

$$\Gamma^- = -12(\mathrm{Im}\Delta_{11}(E^-) + \mathrm{Im}\Delta_{22}(E^-)): \quad (2.56)$$

From Eqs. (2.23) and (2.38) it follows that $\Gamma^- > 0$. Since in this case $\delta \lesssim \Delta^{ab}$, we find that $\beta(\omega) \simeq \gamma(\omega) \simeq O(1)$; therefore, in this case all matrix elements of the proquentlyjectors $P \simeq O(\gamma 1)P_\omega$. However, to leading order

in$P = (E^-\gamma(E^-);\alpha(\omega) = \alpha(E^-)$, and conse-$\Delta$ we find

$$\beta(\omega) = P\beta(E^-\omega);P =$$

In this (nearly) degenerate case, the individual energies are much larger than the respective widths and the energy difference is smaller than or of the same order as the imaginary part of the self-energies evaluated at $\delta(E_1 + E_2)/2$. Therefore, in this (nearly) degenerate case, the Breit-Wigner approximation is valid, and we find to leading order in Δ the time dependent amplitudes

$$A(t) \quad 1$$

$$A_{21}(t) = 2\_\delta Z_+ e^{-i\omega_+ t} + Z_- e^{-i\omega_- t}]$$

$$+ R(E^-)(\delta Z_+ e^{-i\omega_+ t} - Z_- e^{-i\omega_- t})A_{12}(0) \quad (2.57)$$

with

$$R(E^-) = \_12\alpha\gamma(EE^-) - \beta(E^- - E^-) \quad (2.58)$$

We can now compare this result with the usual result for flavored meson-antimeson mixing, such as $K^0 - K^0$ under

purpose of comparison, we define the conditions of $(\_; \Delta_{11}(E^-) = CPT\Delta_{22})E^-$ invariance, which implies. In this case, and for the $E_1 =$

$E = E$

$$m_{ab} \quad \_\_ - i\Gamma_{ab}; \quad m_{ab} = Z\,\mathrm{P}\int_{-\infty}^{} \frac{\Gamma_{ab}}{2} dk_0;$$

$$\Delta(E) = m \quad 2 \quad -\infty \quad E$$

$$\Gamma_{ab} = 2\pi\mu_{ab}(E^-); \quad (2.59)$$

in terms of which we find

$$D(E^-) = 2m_{12} - i\_\_\Gamma_{12}m_{12} - i\Gamma_{12}/4; \quad (2.60)$$

$$\alpha(E^-) = 0; \quad (2.61)$$

$$^- m_{12}{}^2 - i\Gamma_{12}{}^2/4; \quad (2.62)$$

$$\beta(E) = m - i\frac{\Gamma}{2}$$

$$\gamma(E^-) = m_{12}{}^2 - i\Gamma_{12}{}^2/4 = \_\_\_\_\beta(E^-); \quad (2.63)$$

yielding

$$A_1(t) = \tfrac{1}{2}f_+(t)A_1(0) + \beta(E^-)f_-(t)A_2(0); \quad (2.64)$$

$$A_2(t) = \tfrac{1}{2}f_+(t)A_2(0) + \gamma(E^-)f_-(t)A_1(0);$$

(2.65) with

$$f(t) = \frac{1}{2}(Z_+ e^{-i\omega_+ t} \; Z_- e^{-i\omega_- t}): \quad (2.66)$$

Setting $Z = 1$, the expressions (2.64) and (2.65) with (2.66) are the usual ones for the case of flavored mesonantimeson mixing with (CPT) symmetry [12–15,32]. In Ref. [32] the contribution from wave function renormalization was neglected[2] but it was recognized that it would

modify the amplitudes. Therefore, with $Z \simeq 1 + O(\Delta)$ it is clear that neglecting the wave function renormalizations affects the amplitudes at second order in the interaction. This perturbative correction may become relevant for precision measurements of flavor mixing.

C. Markov approximation: The effective non-Hermitian Hamiltonian

Let us write

$$Z \quad t_0$$

$$XT_{ak}T_{kb}e^{i(E_b - E_k)(t - t_0)}dt_0 = W_{ab}(t;t_0); \quad W_{ab}(t;0) = 0$$

$$0 \quad k$$

[2] See Appendix A, footnote in page 102 in Ref. [32], where it is explicitly stated that such a contribution was neglected but would modify the amplitudes.

ð2:67Þ

so that

$$\sum_{\kappa} T_{a\kappa}T_{\kappa b}e^{i(E_b-E_\kappa)(t-t_0)} = \int d\, W_{ab}(t;t_0)$$

ð2:68Þ

Inserting this definition in (2.20) and (2.21) and integrating by parts

$$\int_0^t dt_0 \int d\, W^{ab}(t;t^0)C^b(t^0)dt^0 = W^{ab}(t;t)C^b(t) - W_{ab}(t;t_0)\int d\, C_b(t_0)dt_0; \quad \text{ð2:69Þ}$$

since $T_{a\kappa}T_{\kappa b} \propto H_I^2$, and from the evolution equations (2.20) and (2.21) it follows that $\dot{C}_a \propto H_I^2$; therefore, the second term in (2.69) is of $O(H_I^4)$ and will be neglected to leading order in the interaction, namely H_I^2.

Hence, up to $O(H_I^2)$, the evolution equations for the amplitudes (2.20) and (2.21) become

$$\dot{C}_1(t) = -\{W_{11}(t;t)C_1(t) + e^{i(E_1-E_2)t}W_{12}(t;t)C_2(t)\}; \quad \text{ð2:70Þ}$$

$$\dot{C}_2(t) = -\{e^{i(E_2-E_1)t}W_{21}(t;t)C_1(t) + W_{22}(t;t)C_2(t)\}. \quad \text{ð2:71Þ}$$

With the definitions (2.22) the amplitude equations become

$$i\dot{A}_1(t) = E_1A_1 - iW_{11}(t;t)A_1 - iW_{12}(t;t)A_2; \quad \text{ð2:72Þ}$$

$$i\dot{A}_2(t) = E_2A_2 - iW_{21}(t;t)A_2 - iW_{22}(t;t)A_2. \quad \text{ð2:73Þ}$$

[3] This approximation is also implicitly implemented in Ref. [29].

With

$$W_{ab}(t;t) = \sum_{\kappa} T_{a\kappa}T_{\kappa b}\int_0^t e^{i(E_b-E_\kappa)(t-t_0)}dt_0 = \sum_{\kappa}\int_0^{\infty} \mu_{ab}(k_0)e^{i(E_b-k_0)(t-t_0)}dk_0dt_0; \quad \text{ð2:74Þ}$$

where we used the definition of the spectral density, Eq. (2.23). We highlight that this first step in the Markov approximation is equivalent to the full set of equations consistently up to order H_I^2, since the neglected terms of $O(\Delta^2 \simeq H_I^4)$.

Because in the nearly degenerate case $E_1 - E_2 \lesssim \Delta_{ab} \propto H_I^2$, the first stage of the Markov approximation, yielding Eqs. (2.72) and (2.73) is consistent with this case.

The set of equations (2.72) and (2.73) can be written in terms of a time dependent Hamiltonian

$$i\dot{A}_1(t) = \int d\, A_1(t) = H_{\text{eff}}(t)A_1(t) \quad \text{ð2:75Þ}$$

where the matrix elements of $H_{\text{eff}}(t)$ are obtained from Eqs. (2.72) and (2.73). Unlike the case of a single species analyzed in detail in Appendix A, for two species mixing, $H_{\text{eff}}(t)$ is a 2×2 matrix, and $[H_{\text{eff}}(t), H_{\text{eff}}(t')] \neq 0$ for $t \neq t'$; therefore, the solution of the evolution equations is not a simple exponential. The usual approach, following the main approximation in the Weisskopf-Wigner method implemented in the LOY formulation [29], invokes the long time limit[3]

$$\int_0^{t\to\infty} e^{i(E_b-E_\kappa)(t-t_0)}dt_0 \to iP\frac{1}{E-E_\kappa} + \pi\delta(E_b-E_\kappa); \quad \text{ð2:76Þ}$$

yielding

$$-iW_{ab}\tfrac{1}{2}t;t \rightarrow \Delta_{ab}\delta E_b\rangle; \qquad (2.77)$$

where $\Delta_{ab}(\omega)$ is defined by Eq. (2.38). Taking this long time limit, the amplitude equations (2.75) become an effective Schrödinger equation with a time independent effective Hamiltonian $i\dot{A}_{21}(t) = H_{\text{eff}}A_{21}(t)$

(2.78) with

$$H_{\text{eff}} = E_1\Delta^{\dagger}_{21}\Delta(E_1)^{\delta}\langle E_1| E_2\Delta_{12}\Delta^{\delta}(E_2)\delta^{\dagger}\langle E_2|$$

$$\equiv \begin{matrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{matrix} = H_{\text{eff}}(\infty): \qquad (2.79)$$

This effective Hamiltonian is not Hermitian; this is a manifestation that it describes the (approximate) dynamics of a quantum open system, namely of a subset of degrees of freedom which are coupled to a continuum of other degrees of freedom whose dynamics has been "integrated out." Time evolution is not unitary in this subset, as is explicit from the unitarity condition (2.16) and (2.17), which indicates a flow of probability from the $|\phi_1\rangle;|\phi_2\rangle$ to the excited intermediate statesgrated out in the equations of motion.$\{f_k g_i\}$ which have been inte-

It proves convenient to rewrite H_{eff} as

$$H_{\text{eff}} = \frac{1}{2}(E_1 + \Delta_{11}(E_1) + E_2 + \Delta_{22}(E_2))I$$

$$- \frac{1}{2}\sqrt{(E_1 + \Delta_{11}(E_1) - E_2 - \Delta_{22}(E_2))^2 + 4\Delta_{12}(E_2)\Delta_{21}(E_1)}\,i_{1=2}$$

where I is the 2×2 identity matrix and

$$\tilde{D}(E_1;E_2) = h(E_1 + \Delta_{11}(E_1) - E_2 - \Delta_{22}(E_2))^2$$

and (2.81)

$$\tilde{R}(E_1;E_2) = \alpha\tilde{\gamma}\tilde{}(EE_{11};E;E_{22}) \qquad (2.82)$$
$$-\beta\tilde{\alpha}(EE_{11};E;E_{22});$$

with the definitions

$$\tilde{\alpha}(E_1;E_2) = (E_1 + \Delta_{11}(E_1))\overline{\delta(E - E_1)} \qquad (2.83)$$
$$\delta(E_1 - E)^{-1};E\rangle\langle E_{22}| - \Delta_{22}(E_2)|;$$

$$\tilde{\beta}(E_1;E_2) = \overline{\tilde{D2}\Delta(E_{12})}(1);EE_{22}|; \quad \tilde{\gamma}(E_1;E_2) = \overline{\tilde{D2}\Delta(E_{21})}(1);EE_{12}|: \qquad (2.84)$$

It follows from these definitions that

$$\tilde{\alpha}^2 + \tilde{\beta}\tilde{\gamma} = 1; \qquad (2.85)$$

$$\frac{1}{D}\tilde{E};E\,\tilde{R}E;E; \qquad (2.80)$$

which implies that

$$\tilde{R}^2(E_1;E_2) = I; \qquad (2.86)$$

therefore the matrixConsider the eigenvalue equation (suppressing the argu-$\tilde{R}$ features eigenvalues 1.

ments $E_{1;2}$),

$$\tilde{R}p! = pq!; \qquad (2.87)$$

$$q$$

the solution of which is

$$p = N(1 + \tilde{\alpha}); \quad q = N\tilde{\gamma}; \qquad (2.88)$$

$$p^- = N^-(1-\alpha^-); \qquad q^- = N^-\gamma^-; \qquad (2.89)$$

with N normalization factors. These are eigenvectors of $H_{\rm eff}$, namely

$$H_{\rm eff}\,q = \lambda\, q; \qquad (2.90)$$

with eigenvalues

$$\lambda = \frac{1}{2}\left[(E_1 + \Delta_{11}(E_1) + E + \tilde{\Gamma}^{\pm} - D^-(E_1, E_2) \equiv \tilde{\varepsilon}^{\pm} - \frac{i}{2}\right.$$
$$\left. + \Delta_{22}(E_2 \right]. \qquad (2.91)$$

where $\varepsilon^-;\Gamma^-$ are both real. The effective Hamiltonian can be diagonalized by introducing

$$U^{-1} = \begin{pmatrix} p & p \\ q & -q \end{pmatrix};$$

$$U = \frac{1}{2}\begin{pmatrix} p/q & p/q \\ p/q & -p/q \end{pmatrix}; \qquad (2.92)$$

satisfying $UU^{-1} = U^{-1}U = I$, and yielding

$$UH_{\rm eff}U^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \qquad (2.93)$$

Let us define

$$A_{12}(t) = U^{-1}V V_{12}^{\dagger}(t)\,; \quad A(t) = \begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix}. \qquad (2.94)$$

and the effective evolution equations for $V_{1,2}(t)$ become

$$\frac{d}{dt}V_{12}(t) = \lambda_+ V V_{21}(t) - V V_{12}(t) \Rightarrow V V_{12}(t)$$
$$V(t) = \begin{pmatrix} e^{-i\lambda_+ t} & 0 \\ 0 & e^{-i\lambda_- t} \end{pmatrix} V_{12}(0). \qquad (2.95)$$

Using the definition (2.94) evaluated at $t = 0$ yields the solution for the amplitudes

$$A_{21}(t) = U^{-1} e^{-i\lambda_+ t} \quad e^{-i\lambda_- t} U A_{12}(0). \qquad (2.96)$$

With the relations (2.85), (2.88), and (2.89) it is straightforward to find that

$$A_{21}(t) = \tfrac{1}{2} e^{-i\lambda_+ t} P^- + e^{-i\lambda_- t} P^- A_{12}(0); \qquad (2.97)$$

with the projector operators

$$P^- = \frac{1}{2}(I - R^-); P^- 2 = P^-; \qquad (2.98)$$

where R^- is given by Eq. (2.82),

or, alternatively

$$A_2^1(t) = \frac{1}{2}\frac{h}{e^{-i\lambda_+ t} + e^{-i\lambda_- t}} + (e^{-i\lambda_+ t} - e^{-i\lambda_- t}) R^- i$$
$$A(t) = \begin{pmatrix} A \\ A \end{pmatrix} \times A_2^1(0). \qquad (2.99)$$

Comparing the results via Laplace transform and BreitWigner approximation, namely (2.44) to the solution of the set of equations (2.78) obtained in the infinite time limit, namely (2.97) and (2.99), we find several sources of discrepancies: (i) The wave function renormalization constants Z in

(2.44) and (2.57) are missing in (2.97) and (2.99).

(ii) Whereas the projector operators in (2.44) depend on the values of ω, namely the complex poles, those in (2.97) depend on E_1, E_2 separately. Furthermore, the values of the complex frequencies ω (2.43) are not obviously similar to λ (2.91).

The origin of these discrepancies can be traced to taking the long time limit (2.76) and (2.77) before integrating the set of equations (2.72) and (2.73), which is equivalent to the original set of equations (2.20) and (2.21) up to order

time limits will translate into differences of $O(H_{2I})$. Any discrepancy between the order of the long $O(H_{2I})$.

In Appendix A it is shown that the discrepancy in wave function renormalization of amplitudes originates in this long time limit in the simpler case of one species. We now compare the results for the eigenvalues and eigenvectors of the Laplace transform method and the Markov approximation with the effective Hamiltonian.

1. Nondegenerate case

For $E_1 - E_2 \gg \Delta_{ab}$ we can approximate $\tilde{D}(E_1;E_2)$, given by (2.81), as

$$\tilde{D}(E_1;E_2) \simeq E_1 + \Delta_{11}(E_1) - E_2 - \Delta_{22}(E_2); \qquad (2.100)$$

yielding for the eigenvalues λ, Eq. (2.91),

$$\lambda_+ = E_1 + \Delta_{11}(E_1) \equiv E_1^R - i\frac{\Gamma_1}{2};$$

$$\lambda_- = E_2 + \Delta_{22}(E_2) \equiv E_2^R - i\frac{\Gamma_2}{2}; \qquad (2.101)$$

which agree with the eigenvalues obtained from the Laplace transform (2.50) and (2.51). For the amplitudes we now find up to $O(\Delta)$

$$A_1(t) = A_1(0)e^{-i\lambda_+ t} + \frac{\Delta_{12}(E_2)e^{-i\lambda_+ Rt}}{-\Delta_{R12}(E_2)e^{-i\lambda_- t}}\left(\frac{2}{E_1 - E_2}\right)A_2(0); \qquad (2.102)$$

$$A_2(t) = A_2(0)e^{-i\lambda_- t} + \frac{\Delta_{21}(E_1)e^{-i\lambda_+ Rt}}{-\Delta_{R21}(E_1)e^{-i\lambda_- t}}\left(\frac{2}{E_1 - E_2}\right)A_1(0): \qquad (2.103)$$

The differences with the result from the Laplace transform, Eqs. (2.52) and (2.53), are noteworthy: (i) the wave function renormalization constants multiplying the diagonal terms in (2.52) and (2.53) are missing in (2.102) and (2.103); and (ii) the differences in the arguments of Δ_{12}, Δ_{21} are in the brackets. Clearly the discrepancies are of second order in $\Delta_{ab} \propto H_I^2$, as discussed above.

2. (Nearly) degenerate case

For $E_1;E_2 \gg \Delta_{ab}$ but with $E_1 - E_2 \equiv \delta \lesssim O(\Delta)$, it follows that

$$\lambda = \bar{E} + \tfrac{1}{2}(\Delta_{11}(\bar{E}) + \Delta_{22}(\bar{E})) \pm \tilde{D}(\bar{E});$$

$$\bar{E} = \tfrac{1}{2}(E_1 + E_2); \qquad (2.104)$$

where $\tilde{D}(\bar{E})$ corresponds to setting $E_1 \simeq E_2 \simeq \bar{E}$ in the matrix elements of $\tilde{D}(E_1;E_2)$. The eigenvalues λ again coincide with ω given by Eq. (2.55). Furthermore, it is straightforward to confirm that in this case $\tilde{D}(E_1;E_2) = R(\bar{E})$ given by Eq. (2.58). Therefore, the main difference between the Laplace result (2.57) and that from the effective Hamiltonian (2.99) is the wave function renormalization Z multiplying the initial amplitudes in (2.57).

In fact, at a fundamental level, the emergence of the wave function renormalization of the amplitudes of the quasinormal modes precludes the description of their time evolution in terms of an effective non-Hermitian Hamiltonian. This can be understood from the following simple argument: the formal solution of the amplitude equation (2.78) is

$$A_2(t) = e^{-iH_{eff}t}A_1(0) \equiv U(t,0)A_1(0); \qquad (2.105)$$

which obviously does not include a wave function renormalization as prefactor of the quasinormal mode amplitudes. The wave function renormalization is an off-shell contribution that describes the dressing by virtual states of the single (quasi-)particles onshorttimescales,andyields secondorder corrections to the amplitudes. While it may be finite in the case of the box diagram contribution to flavored neutral meson mixing, it is in general ultraviolet divergent in quantum field theory.

Therefore, we conclude that the Laplace transform with the Breit-Wigner approximation provides a more accurate description of the evolution of mixing as compared to that obtained from the effective non-Hermitian Hamiltonian.

3. Quantum beats

The two orthogonal states $|\phi_1\rangle;|\phi_2\rangle$ decaying into a common channel $|\kappa\rangle$ lead to interference in the amplitudes of the decay state $|\kappa\rangle$ as a consequence of "which path" information in the decay. This is similar to the case of quantum beats in "V"-shaped three level systems, in which two higher levels radiatively decay to the lowest level [51], an ubiquitous phenomenon in quantum optics. This interference phenomenon, or quantum beats, is featured in the amplitudes of the decay products described by the states,

$|\kappa\rangle$The analysis above has focused on the time evolution, namely the coefficients $C_\kappa(t)$.

tudesof the amplitudes$C_\kappa(t)$ of the intermediate states via Eq.$C_{1;2}(t)$, which also determine the ampli-(2.23), we find$A_{(2.19)1;2}(t)$.,

Writing these coefficients in terms of the amplitudes and introducing the spectral densities

$$|C_\kappa(t)|^2 = \int_0^t dt_1 \int_0^t dt_2 \int_{-\infty}^{\infty} dk_0 \sum_{a;b} A_a(t_1)\mu_{ab}(k_0)A_b(t_2)e^{-ik_0(t_1-t_2)}: \quad (2.106)$$

It is convenient to introduce $\Theta(t_1 - t_2) + \Theta(t_2 - t_1) = 1$ inside the time integrations, use the property of the spectral density (2.23), and include the definition of the self-energy (2.24) to show that

$$|C_\kappa(t)|^2 = \sum_{a;b} \int_0^t dt_1 A_a(t_1) \int_0^t \sigma(t_1 - t_2) A_b(t_2) dt_2 + \text{c.c.}; \quad (2.107)$$

the complex conjugate (c.c.) contribution arises from the term with $\Theta(t_2 - t_1)$ upon relabeling $t_1 \leftrightarrow t_2$, $a \leftrightarrow b$ and using the property (2.23). Using the amplitude equations (2.25) we finally find

$$\sum_\kappa |C_\kappa(t)|^2 = -\sum_{a=1;2} \int_0^t dt_1 \frac{d}{dt_1}\left[\tfrac{1}{2}A_a(t_1)A_a(t_1)\right]$$

$$= |A_1(0)|^2 + |A_2(0)|^2 - \tfrac{1}{2}|A_1(t)|^2 + |A_2(t)|^2; \quad (2.108)$$

and this result is precisely the unitarity relation formally established by Eqs. (2.16) and (2.17) providing a complementary and explicit proof of unitarity exhibiting the role of the self-energy.

Following LOY [29], introducing the total population of the $\phi_1;\phi_2$ states as

$$N(t) = \tfrac{1}{2}|A_1(t)|^2 + |A_2(t)|^2 \equiv \tfrac{1}{2}|C_1(t)|^2 + |C_2(t)|^2; \quad (2.109)$$

and writing the amplitudes $A_a(t)$ as linear superpositions of the quasinormal modes, namely

$$A_a(t) = A_{a+}e^{-i\varepsilon_{+}t}e^{-\Gamma_2 t} + A_{a-}e^{-i\varepsilon_{-}t}e^{-\Gamma_2 t}; \quad a = 1;2; \quad (2.110)$$

where the coefficients A_a can be read off Eq. (2.44), it follows that

$$N_{\delta t} = X |j_{A_a}|^2 e^{-\Gamma_b t} + |j_{A_a}|^2 e^{-\Gamma_a t}$$

$$a=1,2$$

$$+ 2\mathrm{Re}\{A_a A_b^* e^{i(\epsilon_b - \epsilon_a)t} e^{-\frac{1}{2}(\Gamma_b + \Gamma_a)t}\}; \quad (2.111)$$

the last term displays the quantum beats as a consequence of the interference between the quasinormal modes. With the normalization (2.15) the unitarity relations (2.17) and (2.108) yield

$$\sum_{\kappa} |C_{\kappa}(\delta t)|^2 = 1 - N_{\delta t}; \quad (2.112)$$

displaying the quantum beats from (2.111) in the last term. Therefore unitarity entails that the quantum beats in the total population are reflected in the time evolution of the decay products.

These interference terms are, of course, well known, originally recognized in the seminal work by LOY [29], and have been experimentally observed in the decays products of flavored neutral mesons [12–15]. We note that the coefficients A_a depend on the wave function renormalization constants Z in the solutions (2.44), an important discrepancy with the usual effective non-Hermitian Hamiltonian description of particle mixing.

Our main objective in analyzing the dynamics of mixing within the framework of the LOY theory of flavored meson mixing is to provide a guide to and benchmark for the effective field theory approach to the dynamics of mixing in a medium studied in the next section.

III. THE EFFECTIVE ACTION FOR PARTICLE MIXING

The previous section extended and generalized the formulation of particle mixing, originally implemented to study CP violation in the neutral kaon system, to the case in which different particles (in general with different masses) mix via common intermediate states or decay channels. As it is clear from this analysis, such a formulation is applicable and generally applied to the case of an initial state being a pure state, and primarily, when such a state is a linear superposition of single particle states [29]. This analysis also revealed several subtleties associated with the time evolution of the amplitudes in terms of an effective Hamiltonian. It also highlighted that the nonHermiticity of the effective Hamiltonian is a hallmark of a quantum open system, namely such a Hamiltonian describes the nonunitary time evolution of a reduced subset of states which are coupled to a continuum of states that have been integrated out.

Our main objective is to provide a framework to study the dynamics of particle mixing in a medium, as it is necessary within the realm of cosmology. In this case, we are interested in the time evolution of a density matrix, describing a statistical ensemble of particles, not just a pure state of a few particles. Furthermore, we are interested in obtaining the time evolution of correlation functions and distribution functions in the medium, in particular their asymptotic behavior and possible thermalization, not on the amplitudes of single (or a few) particle states.

Rather than considering the most general case of mixing between charged bosons or fermions which necessarily add several technical complications, we consider the simpler case of real scalar or pseudoscalar bosonic fields $\phi_1;\phi_2$ interacting with degrees of freedom in thermal equilibrium denoted collectively by χ, to establish the main framework and results within a simpler setting, thus paving the way to extrapolating to a more general case.

The mixing between ϕ_1 and ϕ_2 is indirect and a consequence of a coupling to a common set of intermediate states yielding a self-energy with off-diagonal elements in the space of $\phi_{1,2}$ similar to the cases studied in the previous section.

The general Lagrangian density describing this situation is given by

$$L[\phi_1;\phi_2;\chi] = L_\phi + L_\chi + L_I; \quad (3.1)$$

where

$$L_\phi = \frac{1}{2} X; [\partial\phi_a)^2 - m_{2a}\phi_{2a}; \quad 2 \quad a=1\ 2$$

$$L_I = -\phi_1 O_1[\chi] - \phi_2 O_2[\chi]; \quad (3.2)$$

where L_χ is the Lagrangian of the χ fields. These are assumed to describe degrees of freedom in thermal equilibrium including interactions among these fields, and $O_{1,2}[\chi]$ are (composite) operators associated with the χ degrees of freedom. These operators include couplings $g_{1,2}$ assumed to be small. Indirect field mixing is a consequence of nonvanishing correlations $\langle O_1 O_2\rangle$ in the medium yielding off-diagonal self-energy matrix elements.

Let us consider the initial density matrix at a time $t = 0$ to be of the form $\hat{\rho}(0) = \hat{\rho}_\phi(0) \otimes \hat{\rho}_\chi(0)$: (3.3)

The initial density matrix $\hat{\rho}_\phi(0)$ is normalized so that $\mathrm{Tr}_\phi \hat{\rho}_\phi(0) = 1$ and that of the χ fields will be taken to describe a statistical ensemble in thermal equilibrium at a temperature $T = 1/\beta$, namely

$$\hat{\rho}_\chi(0) = e^{-\beta H_\chi}/\overline{\mathrm{Tr}_\chi e^{-\beta H_\chi}}; \qquad (3.4)$$

where H_χ is the total Hamiltonian for the fields χ and may include other fields to which χ is coupled other than the fields $\phi_{1,2}$. The χ vacuum is obtained in the limit $\beta \to \infty$.

For example, for the discussion of the previous section the initial density matrix is given by

$$\hat{\rho}(0) = |\Psi(t=0)\rangle\langle\Psi(t=0)|; \qquad (3.5)$$

where $|\Psi(t=0)\rangle$ is the state (2.13).

The factorization of the initial density matrix is an assumption often explicitly or implicitly made in the literature, it can be relaxed by including initial correlations among the various fields at the expense of daunting technical complications. In this study we will not consider this important case, assuming the factorization as in (3.4). In what follows we will refer to the set of fields $\phi_{1,2}$ collectively simply as $\phi \equiv \{\phi_1, \phi_2\}$ to simplify notation.

The main concept that anchors the framework developed below is the following: the time evolution of the full density matrix in the Schrödinger picture is given by

$$\hat{\rho}(t) = e^{-iHt}\hat{\rho}(0)e^{iHt}; \qquad (3.6)$$

where H is the total Hamiltonian

$$H = H_{0\phi} + H_\chi + \sum_{a=1}^{2} \int d^3x\, \phi_a O_a(\chi); \qquad (3.7)$$

where $H_{0\phi}$ and H_χ are the Hamiltonians for the respective fields. We will assume that the composite operators O_a include weak couplings so as to define a perturbative expansion, and second order terms in O_a imply second order in couplings, which we will denote as $O(g^2)$ with g a generic coupling.

The reduced density matrix for the $\phi_{1,2}$ degrees of freedom is obtained by tracing over the χ degrees of freedom, namely

$$\hat{\rho}_r(t) = \mathrm{Tr}_\chi \hat{\rho}(t): \qquad (3.8)$$

This reduced density matrix does not evolve unitarily in time, its time evolution is determined by a time nonlocal effective action [35,40–44]. One of our main objectives is to obtain this effective action.

It is convenient to write the density matrix in the field basis which facilitates a path integral representation of the nonequilibrium reduced density matrix [35,40–44].

In the field basis the matrix elements of $\hat{\rho}_\phi(0)$ and $\hat{\rho}_\chi(0)$ are given by

$$\langle\phi|\hat{\rho}_\phi(0)|\phi_0\rangle = \rho_{\phi;0}(\phi;\phi_0); \quad \langle\chi|\hat{\rho}_\chi(0)|\chi_0\rangle = \rho_{\chi;0}(\chi;\chi_0); \qquad (3.9)$$

and this is a functional density matrix as the fields feature spatial arguments. $\hat{\rho}_\phi(0)$ represents either a pure state, such as a coherent state, or more generally an initial statistical ensemble, whereas $\hat{\rho}_\chi(0)$ is assumed to describe a thermal ensemble and is given by Eq. (3.4).

To obtain the effective action, we follow the procedure described above: evolve the initial density matrix in time, trace over the χ degrees of freedom thereby obtaining the reduced density matrix for the ϕ fields, and determine the effective action from its time evolution. Including source terms for the fields ϕ, we can compute expectation values and correlation functions as a function of time from variational derivatives as usual.

We now follow the main methods and results of Refs. [18,39], summarizing here the main aspects pertinent to the case of mixing for consistency of presentation. The reduced density matrix is given by

$$\rho_{\phi r}(t) = \mathrm{Tr}_\chi U(t)\hat{\rho}(0)U_{-1}(t); \quad U(t) = e^{-iHt}: \qquad (3.10)$$

In field space,

$$\rho(\phi_f;\chi_f;\phi_{0f};\chi_{0f};t) = \langle\phi_f;\chi_f|U(t)\hat{\rho}(0)U^{-1}(t)|\phi_{0f};\chi_{0f}\rangle; \qquad (3.11)$$

from which the reduced density matrix elements are obtained by taking the trace on χ, namely setting $\chi^0_f = \chi_f$ and carrying out the functional integral in χ_f, $\rho^r(\phi_f;\phi_{0f};;t) = \int D\chi_f \langle\phi_f;\chi_f|U(t)\hat{\rho}(0)U^{-1}(t)|\phi_{0f};\chi_f\rangle$:

ð3:12Þ

With the functional integral representation

$$\langle\phi_f;\chi_f|U(t)\rho(0)U^{-1}(t)|\phi_{0f};\chi_{0f}\rangle$$

$$= \int D\phi_i D\chi_i D\phi_{0i} D\chi_{0i} \langle\phi_f;\chi_f|U(t)|\phi_i;\chi_i\rangle\rho_{\phi;0}(\phi_i,\phi_{0i})\otimes\rho_{\chi;0}(\chi_i,\chi_{0i})\langle\phi_{0i};\chi_{0i}|U^{-1}(t)|\phi_{0f};\chi_{0f}\rangle; \quad (3.13)$$

it follows that the reduced density matrix elements are

$$\rho_r(\phi_f;\phi_{0f};t) = \int D\chi_f \int D\phi_i D\chi_i D\phi_{0i} D\chi_{0i} \times \langle\phi_f;\chi_f|U(t)|\phi_i;\chi_i\rangle\rho_{\phi;0}(\phi_i,\phi_{0i})\otimes\rho_{\chi;0}(\chi_i,\chi_{0i})\langle\phi_{0i};\chi_{0i}|U^{-1}(t)|\phi_{0f};\chi_f\rangle: \quad (3.14)$$

The $D\phi$, etc., are functional integrals where the spatial argument has been suppressed. The matrix elements of the time evolution forward and backward can be written as path integrals, namely

$$\langle\phi_f;\chi_f|U(t)|\phi_i;\chi_i\rangle = \int D\phi_+ D\chi_+ e^{i\int_0^t d^4x L(\phi_+;\chi_+)}; \quad (3.15)$$

$$\langle\phi_{0i};\chi_{0i}|U^{-1}(t)|\phi_{0f};\chi_{0f}\rangle = \int D\phi_- D\chi_- e^{-i\int_0^t d^3x L(\phi_-;\chi_-)}; \quad (3.16)$$

where we use the shorthand notation

$$\int_0^t d^4x \equiv \int_0^t dt \int d^3x: \quad (3.17)$$

$L(\phi;\chi)$ is given by (3.1) and (3.2) and the boundary conditions on the path integrals are

$$\phi_+(\vec{x},t=0) = \phi_i(\vec{x}); \quad \phi_+(\vec{x},t) = \phi_f(\vec{x}); \quad (3.18)$$

$$\chi_+(\vec{x},t=0) = \chi_i(\vec{x}); \quad \chi_+(\vec{x},t) = \chi_f(\vec{x}); \quad (3.19)$$

$$\phi_-(\vec{x},t=0) = \phi_{0i}(\vec{x}); \quad \phi_-(\vec{x},t) = \phi_{0f}(\vec{x}); \quad (3.20)$$

$$\chi_-(\vec{x},t=0) = \chi_{0i}(\vec{x}); \quad \chi_-(\vec{x},t) = \chi_{0f}(\vec{x}): \quad (3.21)$$

The field variables ϕ and χ along the forward (+) and backward (−) evolution branches are recognized as those necessary for the in-in or Schwinger-Keldysh [40–43] closed time path approach to the time evolution of a density matrix.

The reduced density matrix for the fields ϕ_a (3.14) can be written as

$$\rho_r(\phi_f;\phi_{0f};t) = \int D\phi_i D\phi_{0i} T(\phi_f;\phi_{f0};\phi_i;\phi_{i0};t)\rho_\phi(\phi_i,\phi_{0i};0); \quad (3.22)$$

where the time evolution kernel is given by

$$T(\phi_f;\phi_i;\phi_{0f};\phi_{0i};t) = \int D\phi_+ \int D\phi_- e^{iS_{\rm eff}(\phi_+;\phi_-;t)}; \quad (3.23)$$

from which the in-in effective action out of equilibrium is identified as

$$S_{\rm eff}(\phi_+;\phi^-;t) = \int_0^t dt_0 \int d^3x \{L_0(\phi_+) - L_0(\phi^-)\} + I(\phi_+;\phi_-;t); \quad (3.24)$$

where $I(\phi^+;\phi^-;t$ is the influence action [43,44] and is obtained by tracing over the χ degrees of freedom, namely

$$e^{iI(\phi_+;\phi^-;t)} = \int D\chi_i D\chi_{i0} D\chi_f \int D\chi_+$$

$$\times Z\,D\chi_4 - e_i R\,d4x½L½\chi_b - P_a\,\phi_{ba}\,O_a½\chi_b$$

$$-iR\,d\,x½L½\chi_- - P_a\,\phi_{-a}\,O_a½\chi_-\rho_\chi\delta\chi_i;\chi_{0i};0Þ;\quad \delta3:25Þ$$

$$\times e$$

with the definition (3.17).

Note that in the influence action (3.25) the ϕ fields act as background field variables, the functional and path integrations are performed in all fields χ other than the ϕ fields. These functional integrals are obtained by expanding the terms $e^{\mp i \int d^4x \phi_a^{\pm} \mathcal{O}_a \chi}$ in power series and carrying the path and functional integrals in $\chi;\chi_i;\chi_f;\chi^0$, yielding correlation functions of the operators O_a and reexponentiating. This is depicted in Fig. 2.

From Eqs. (3.22) and (3.23) it is clear that the effective action $S_{\rm eff}$ determines the time evolution of the reduced density matrix.

The path integral representations for both $T\,½\phi_f;\phi_i;\phi^0{}_f;\phi^0{}_i;t$ and $I½\phi^b;\phi^-;t$ feature the boundary conditions in (3.18)–(3.21) except that we now set $\chi\delta x;t^{\rightarrow}Þ = \chi_f\delta x^{\rightarrow}Þ$ to trace over the χ field.

The technical steps to obtain $I½\phi^b;\phi^-;t$ in perturbation theory, up to second order in the operators O_a, up to $O\delta g^2Þ$ are available in Ref. [18]. We follow the steps in this reference to find up to second order in couplings

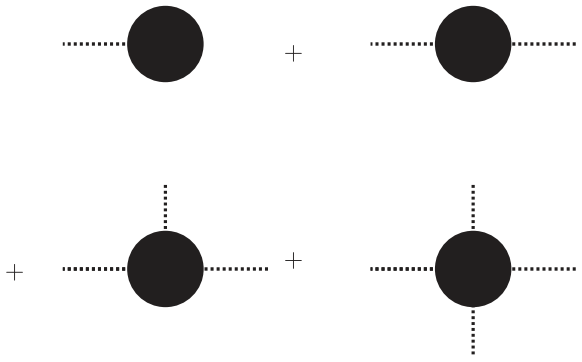

FIG. 2. Pictorial representation of the influence action $I½\phi_b;\phi_-;t$. The dashed lines are the (background) fields χ fields yielding correlation ϕ_a, and the filled circles the trace over the The second graph with two dashed lines yields the influencefunctions of the operators $O½\chi$. Each vertex carries a coupling.

function up to second order in the couplings (3.26). We assume that $hO_b;_a\phi i = 0$; hence, there is no first order contribution to $I½\phi$

$$iI½\phi^b;\phi^-;t = \bar{Z}\int d^4x_1 d^4x_2 f\phi^b{}_a\,\delta x^{\rightarrow}{}_1;t_1Þ\phi^b{}_b\,\delta x^{\rightarrow}{}_2;t_2Þ$$

$$\times G^{>}_{ab}(x_1 - x_2) + \phi^-_a(x_1,t_1)\phi^-_b(x_2,t_2)^{\rightarrow}$$
$$\times G^{<}_{ab}(x_1 - x_2) - \phi^+_a(x_1,t_1)\phi^-_b(x_2,t_2)^{\rightarrow}$$
$$\times G^{<}_{ab}(x_1 - x_2) - \phi^-_a(x_1,t_1)\phi^+_b(x_2,t_2)^{\rightarrow}$$
$$\times G^{>}_{ab}\,x_1 - x_2\quad \Theta\,t_1 - t_2\delta Þg\,\delta \qquad Þ;$$

$$\delta3:26Þ$$

where and $G^{\lessgtr}_{ab}\,x_1 - x_2\,\delta Þ$ are given by

$$G^{>}_{ab}\,x_1 - x_2\,\delta Þ = hO_a\delta x_1ÞO_b\delta x_2Þi_\chi; \qquad \delta3:27Þ$$

$$G^{<}_{ab}\,x_1 - x_2\,\delta Þ = hO_b\delta x_2ÞO_a\delta x_1Þi_\chi; \qquad \delta3:28Þ$$

and we have assumed that $hO_ai = 0$ (so that the first diagram in Fig. 2 vanishes). The operators O_a are Hermitian from which it follows that

$$G^{<}{}_{ab}\delta x_1 - x_2Þ = G^{>}{}_{ba}\delta x_2 - x_1Þ: \qquad \delta3:29Þ$$

This is the general form of the influence function up to second order in the operators $O_a½\chi$, but to all orders in

$$iS_{\rm eff}½\Phi;R = -iZ\,d^3x X_a\,R_{a;i}\delta xÞ\Phi^{\cdot}{}_a\delta x;t = 0Þ$$

the couplings of the χ fields to any other field except $\phi_{1;2}$. We can obtain expectation values and correlation functions of $\phi_{1;2}$ by including sources $J_a(x)$ with $L_0(\phi) \to L_0(\phi) - J(x)\phi(x)$ and defining the generating functional P_a

$$R_a(x;\vec{t}) = \delta\phi^b{}_a(x;\vec{t}) - \phi^-{}_a(x;\vec{t}) \quad (3.32)$$

The boundary conditions on the ϕ path integrals given by (3.31) translate into the following boundary conditions on the center of mass and relative variables:

$$+ iZ_0 dt X_{k;a} f - R_a(-k;\vec{t}) \quad \delta\dot{\Phi}_a(k;\vec{t}) + \omega^2{}_a(k)\Phi_a(k;\vec{t}) + \Phi_a(k;\vec{t})J_a(-k;\vec{t}) \quad g$$

$$\vec{}$$

$$-Z_0^t dt_1 Z_0^t dt_2 X_{ab} \frac{1}{}R_a(-k;\vec{t}) \quad {}_1N_{ab}(k;t_1 - t_2)R_b(k;t \quad \vec{} \quad {}_2 + iR_a(-k;\vec{t}) \quad {}_1\Sigma_{ab}{}^R(k;t_1 - t_2)\Phi_b(k;t_2)^{+}; \quad (3.35) \quad 2$$

$$Z[J^+;J^-] = \mathrm{Tr}\rho_r(J^+;J^-;t) \quad \Phi_a(x;\vec{t}) = 0 = \Phi_{a;i}; \quad R_a(x;\vec{t}) = 0 = R_{a;i}; \quad (3.33)$$

$$= \int D\phi_f D\phi_i D\phi_{0i} \int D\phi_+ \quad \Phi_a(x;\vec{t}) = t_f = \Phi_{a;f}(x) \vec{}; \quad R_a(x;\vec{t}) = t_f = 0: \quad (3.34)$$

$$\times \int D\phi^- e^{iS_{eff}[\phi^+;J^+;\phi^-;J^-;t]}\rho(\phi(0),\phi_i;\phi_{0i};0) \quad (3.30)$$

with the boundary conditions

$$\phi_+(x;t) = \phi_-(x;t) = \phi_f(x) \vec{}; \quad \phi_+(x;t) = \phi_-(x;t) = \phi_{0i}(x) \vec{}; \quad \phi_a(x;t) = \phi_{f;a}(x) \vec{}; \quad \phi_{a-}(x;t) = \phi_{f;a}(x) \vec{}: \quad (3.31)$$

Expectation values or correlation functions of ϕ in the reduced density matrix are obtained as usual with variational derivatives with respect to the sources J.

The effective action (3.24) may be written in a manner more suitable to exhibit the equations of motion by introducing the Keldysh [41] center of mass and relative variables

$$\Phi_a(x;\vec{t}) = \frac{1}{2}\phi_+(x;\vec{t}) + \phi_-(x;\vec{t}) \quad R_a(x;\vec{t}) = \phi_+(x;\vec{t}) - \phi_-(x;\vec{t}); \quad (3.32)$$

The boundary conditions on the ϕ path integrals given by (3.31) translate into the following boundary conditions on the center of mass and relative variables:

$$\Phi_a(x;\vec{t}) = 0 = \Phi_{a;i}; \quad R_a(x;\vec{t}) = 0 = R_{a;i}; \quad (3.33)$$

$$\Phi_a(x;\vec{t}) = t_f = \Phi_{a;f}(x) \vec{}; \quad R_a(x;\vec{t}) = t_f = 0: \quad (3.34)$$

Taking the spatial Fourier transform, the effective action (3.24) with the influence functional (3.26) becomes where $\omega^2_a(k) = k^2 + m^2_a$. To obtain the above form, we integrated by parts in time, defined $J_a(x) = \delta J_{+a}(x) - J_{-a}(x)$, and kept only the sources conjugate to Φ^a because we are interested in expectation values and correlation functions of this variable only as discussed in detail below.

The nonlocal kernels in the above effective Lagrangian are given by [18]

$$N_{ab}(k;t) + G^<_{ab}(k;t - t^0); \quad (3.36)$$

$$i\Sigma^R_{ab}(k;t - t^0) = \frac{1}{2}G^>_{ab}(k;t - t^0) - G^<_{ab}(k;t - t^0)\Theta(t - t^0)$$

$$\equiv i\Sigma_{ab}(k;t - t^0)\Theta(t - t^0); \quad (3.37)$$

where $G^{<,>}(k;t - t^0)$ are the spatial Fourier transforms of the correlation functions in (3.27) and (3.28). It is clear from these correlation functions that if $\langle O_1 O_2\rangle \neq 0$, the self-energy features nonvanishing off-diagonal matrix

elements, and these are responsible for indirect mixing between the fields ϕ_1 and ϕ_2. Since each operator is associated with a coupling g_a, the self-energy and noise kernels are of second order, and we will refer to them generically as $\Sigma \propto g^2; N_{ab} \propto g^2$ to emphasize the second order nature of these kernels.

In the exponential of the effective action $e^{iS_{\rm eff}}$, the quadratic term in the relative variables R_a can be written as a functional integral over a noise variable ξ_a as follows:

$$\exp\left[\frac{1}{2}\bar{Z}_1\bar{Z}_2 \int_a \delta(k^\rightarrow;t_1) N_{ab}\delta(k^\rightarrow;t_1-t_2) R_b\delta(k^\rightarrow;t_2)\, dt\right]$$

$$= C^\sim \int D\xi_a \exp\left[-\frac{1}{2}\int dt_1 \int dt_2 \xi_a\delta(-k^\rightarrow;t_1) N^{-1}_{ab}\delta(k^\rightarrow;t_1-t_2)\xi_b\delta(k^\rightarrow;t_2) + i\int dt \xi_a\delta(-k^\rightarrow;t) R_a\delta(k^\rightarrow;t)\right]; \quad (3.38)$$

where $C^\sim$ is a normalization factor.

The time evolution of the density matrix defines an initial value problem; consequently, we seek to obtain the equations of motion as an initial value problem rather than a boundary value problem. Since the Heisenberg equations of motion are second order in time, an initial value problem is determined by providing the initial values of the field and its canonical momentum. This suggests to consider the Wigner transform of the initial density matrix by writing it in terms of the initial center of mass and relative variables $\Phi_{a;i}$ and $R_{a;i}$,

$$\rho_\phi(\delta\phi_{a;i};\phi_{0a;i}) \equiv \rho_\phi\left(\Phi_{a;i} + \frac{1}{2}R_{a;i};\Phi_{a;i} - \frac{1}{2}R_{a;i}\right); \quad (3.39)$$

and introduce the functional Wigner transform [43,51] as a Fourier transform in the relative variable,

$$W(\Phi_{a;i};\Pi_{a;i}) = \int DR\, e^{-i\int d^3x \Pi_{a;i}\delta x^\rightarrow} R_i\delta x^\rightarrow \rho_\phi\left(\Phi_{a;i} + \frac{1}{2}R_{a;i};\Phi_{a;i} - \frac{1}{2}R_{a;i}\right); \quad (3.40)$$

which allows us to write (up to a normalization factor)

$$\int R \int R\, \rho_\phi\left(\Phi_{a;i} + \frac{1}{2}R_{a;i};\Phi_{a;i} - \frac{1}{2}R_{a;i}\right) = \int D\Pi_{a;i} e^{i\int d^3x \Pi_{a;i}\delta x^\rightarrow} R_i\delta x^\rightarrow W(\Phi_{a;i};\Pi_{a;i}); \quad (3.41)$$

$$\int R \qquad \int R$$

the variables Π_a are the momenta conjugate to the variable Φ_a, and $W(\Phi;\Pi$ yields a probability distribution in "phasespace" Φ, Π.

Gathering these results together, we now write the generating functional (3.30) in terms of the Keldysh variables (3.32), with the effective action in these variables given by Eq. (3.35). Implementing the Wigner transform (3.41) and using the representation (3.38) we obtain

$$Z[J] = \int D\Phi_f \int DR_i D\Phi_i D\Pi_i \int D\Phi DRD\xi W[\Phi_i;\Pi_i] \times P[\xi] \times \exp i\int dt X_{\vec{k}} \dot{\Phi}_a(\vec{k};t) J_a(-\vec{k};t)$$

$$\times \exp -i\int dt X_{\vec{k}} R_a(-\vec{k};t)\dot{\Phi}_a(\vec{k};t) + \omega^2_a(k,t)\Phi_a(\vec{k};t) + \int_0^t \Sigma_{ab}(\vec{k};t-t')\Phi_b(\vec{k};t') - \int_0^{t'} dt'' - \xi_a(\vec{k};t)$$

$$\times \exp i X_{\vec{k}} R_{a;i}(-\vec{k};t)[\delta\Pi_{a;i}(\vec{k};t) + -\dot{\Phi}_{a;i}(\vec{k};t)]; \quad (3.42)$$

where $\omega^2_a(k,t) = k^2 + m^2_a$ and repeated field indices are summed over. The noise probability distribution function $P[\xi_a]$ is given by

$$P[\xi_a] = \tilde{C} Y_{\vec{k}} \exp\left[-\frac{1}{2}\int dt_1 \int dt_2 \xi_a(-\vec{k};t_1)[N^{-1}_{ab}(k;t_1-t_2)]\xi_a(\vec{k};t_2)\right]. \quad (3.43)$$

The generating functional $Z[J]$ is the final form of the time evolved reduced density matrix after tracing over the bath, yield the correlation functions of the Keldysh center degrees of freedom. Variational derivatives with respect to the source of mass variables Φ.

Carrying out the functional integrals over $R_i(\vec{k},t)$ and $R_{\vec{k}}(t)$ yields a clearer form, namely

$$Z[J] \propto \int D\Phi_{a;f} \int D\Phi_{a;i} D\Pi_{a;i} \int D\Phi_a D\xi_a W[\Phi_i;\Pi_i] \times P[\xi] \times \exp i\int dt X_{a;\vec{k}} \dot{\Phi}_a(\vec{k};t) J_a(-\vec{k};t)$$

$$\times Y_{\vec{k}} \delta[\dot{\Phi}_a(\vec{k};t) + \omega^2_a(k,t)\Phi_a(\vec{k};t) + \int_0^t \Sigma_{ab}(k;t-t')\Phi_b(\vec{k};t') - \int_0^{t'} dt'' - \xi_a(\vec{k};t)]$$

$$\times Y_{a;\vec{k}} \delta[\Pi_{a;i}(\vec{k};t) + -\dot{\Phi}_{a;i}(\vec{k};t)]. \quad (3.44)$$

Obtaining expectation values and correlation functions from this generating functional is straightforward:

(i) The functional delta functions in (3.44) determine the field configurations that contribute to the generating functional $Z[J]$. These are the solutions of[39] for the stochastic Langevin equation of motion $\Phi_a(\vec{k};t)$, namely

$$\dot{\Phi}_a(\vec{k};t) + [\omega^2_a(k,t)\Phi_a(\vec{k};t) + \int_0^t \Sigma_{ab}(\vec{k};t-t')\Phi_b(\vec{k};t') - \int_0^{t'} dt'' - \xi_a(\vec{k};t)] = 0. \quad (3.45)$$

This equation of motion is retarded as it involves the retarded self-energy, thereby defining a causal

initial value problem. This is a distinct consequence of the in-in formulation of time evolution.

(ii) The expectation value and correlations of the stochastic noise $\xi_a(\vec{k};t)$ are determined by the Gaussian probability distributionnition $\langle\langle(\cdots)\rangle\rangle$ for averages with$P[\xi_a]$. Introducing the defi-$P[\xi_a]$, the Gaussian

stochastic noise features the following averages:

$$\langle\langle \xi_a(\vec{k};t)\rangle\rangle = 0;$$

$$\langle\langle \xi_a(\vec{k};t)\xi_b(\vec{k}';t')\rangle\rangle = N_{ab}(\vec{k};t)\,\delta(t-t')\,\delta_{\vec{k},-\vec{k}'}. \quad (3.46)$$

Since $P[\xi_a]$ is a Gaussian distribution function, higher order correlation functions are obtained by implementing Wick's theorem. This averaging is a manifestation of stochasticity, establishing a direct relation between nonequilibrium dynamics of quantum open systems and stochastic field theory [52,53]. (iii) The stochastic equation of motion (3.45) must be solved with the initial conditions

$$\Phi_a(\vec{k};t=0) = \Phi_{a;i}(\vec{k}); \quad \dot{\Phi}_a(\vec{k};t=0) = \Pi_{a;i}(\vec{k}); \quad (3.47)$$

and these initial conditions confirm that $\Pi_{a;i}$ are the canonical momenta conjugate to $\Phi_{a;i}$. The solution of (3.45) is a functional of the variables $\Phi_{a;i}(\vec{k})$;$\Pi_{a;i}(\vec{k})$, which are distributed according to the probability distribution function $W[\Phi_{a;i},\Pi_{a;i}]$, which in turn is determined by the initial density matrix. This is another manifestation of stochasticity, but now in the distribution of initial conditions.

We now introduce the notation $\overline{(\cdots)}$ to denote averaging over the initial conditions (3.47) with the distribution function $W[\Phi_{a;i},\Pi_{a;i}]$.

The solutions of the Langevin equation (3.45) $\Phi_a[\vec{k};t;\xi;\Phi_{a;i},\Pi_{a;i}]$ are functionals of the stochastic noise variables ξ_a and the initial conditions. Therefore correlation functions of the original field variables ϕ_a in the reduced density matrix correspond to averaging the products of the solutions over both the initial conditions with the Wigner distribution function $W[\Phi_{a;i},\Pi_{a;i}]$, and the noise with the probability distribution function $P[\xi]$. We denote such

averages by $\overline{\langle\langle(\cdots)\rangle\rangle}$ where $(\cdots)$ is any functional of the initial conditions (3.47) and ξ_a.

These stochastic averages yield the expectation values and correlation functions of functionals of Φ_a obtained from variational derivatives with respect to J_a.

In Appendix B we provide a nonperturbative spectral Lehmann representation of the correlation functions $G^{\gtrless}_{ab}(x-y)$ that enter in the definitions of the self-energy (3.37) and noise correlation function (3.36). The result is that these nonlocal kernels can be written in a dispersive representation as

$$\Sigma_{ab}(k;t-t') = -iZ\int \frac{d^2k_0}{(2\pi)^2}\rho_{ab}(k_0;k)e^{-ik_0(t-t')}; \quad (3.48)$$

$$N_{ab}(k;t-t') = \frac{1}{2}Z\int \frac{d^2k_0}{(2\pi)^2}\rho_{ab}(k_0;k)\coth\frac{\beta 2k_0}{2}e^{-ik_0(t-t')}; \quad (3.49)$$

where $\rho_{ab}(k_0;k)$ is a 2×2 matrix of spectral densities (see Appendix B for details). The representations (3.48) and (3.49) are a manifestation of a generalized fluctuation dissipation relation, a consequence of taking the χ degrees of freedom in thermal equilibrium.

The stochastic equation of motion (3.45) with initial conditions (3.47) defines an initial value problem whose solution is obtained by Laplace transform. Let us define the Laplace transforms

$$\int_0^\infty$$

$$\tilde{\Phi}_a(\vec{k};s) = \int_0^\infty e^{-st}\Phi_a(\vec{k};t)dt; \quad (3.50)$$

$$\int_0^\infty \tilde{\xi}_a(\vec{k};s) = \int_0^\infty e^{-st}\xi_a(\vec{k};t)dt; \quad (3.51)$$

$$\tilde{\Sigma}_{ab}(\vec{k};s) = \int_0^\infty e^{-st}\Sigma_{ab}(\vec{k};t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\rho_{ab}(k_0,k)}{k_0 - is}dk; \quad (3.52)$$

where in (3.52) we used the representation (3.48). The Laplace transform of the Langevin equation (3.45) with initial conditions (3.47) becomes

$$G^{-1}_{ab}(k;s)\tilde{\Phi}_b(\vec{k};s) = \Pi_{a;i}(\vec{k};s)\Phi_{a;i}(\vec{k};s) + \xi_a(\vec{k};s); \quad (3.53)$$

where

$$G^{-1}_{ab}(k;s) = (s^2 + \omega^2_a(k))\delta_{ab} + \tilde{\Sigma}_{ab}(\vec{k};s): \quad (3.54)$$

The solution in real time is obtained by inverse Laplace transform, and it is given by

$$\Phi_a(\vec{k};t) = \Phi_{ha}(\vec{k};t) + \Phi_{\xi a}(\vec{k};t); \quad (3.55)$$

where $\Phi_{ha}(\vec{k};t);\Phi_{\xi a}(\vec{k};t)$ are the homogeneous and inhomogeneous solutions, respectively, namely

$$\Phi_{ha}(\vec{k};t) = \dot{G}_{ab}(k;t)\Phi_{b;i}(\vec{k};) + G_{ab}(k;t)\Pi_{b;i}(\vec{k};);$$

$$\Phi^{\xi}_a(\vec{k};t) = \int_0^t G_{ab}(k;t-t^0)\xi_b(\vec{k};t^0)dt^0; \quad (3.56)$$

and repeated indices are summed over. Green's function is given by

$$G_{ab}(k;t) = \frac{1}{2\pi i}\int_C e^{st}G_{ab}(k;s)ds; \quad (3.57)$$

where C denotes the Bromwich contour parallel to the imaginary axis and to the right of all the singularities of $G_{ab}(k;s)$ in the complex s plane, closing along a large $\mathrm{Re}(s) < 0$. These singularities semicircle at infinity with $\mathrm{Re}(s) < 0$; thus, the contour runs parallel to the imaginary axis $s = i(\nu - i\epsilon)$, with $-\infty \le \omega \le \infty$ and $\epsilon \to 0^+$. Finally, changing variables

$$G_{ab}(k;t) = \int_{-\infty}^{\infty} G_{ab}(k;s = -i\omega + \epsilon)e^{-i\omega t}\frac{d\omega}{2\pi}; \quad (3.58)$$

and for $t > 0$ the integration contour is closed in the lower half ω plane. We obtain Green's function $G_{ab}(k;s)$ by following the steps in Appendix C. Without loss of generality we consider $m_1^2 \geq m_2^2$ and define

$$M^2(k;s) = s^2 + \tfrac{1}{2}(\omega^2_1(k) + \omega^2_2(k)) + \tilde{\Sigma}_{11}(k;s) + \tilde{\Sigma}_{22}(k;s);$$

$$D(k;s) = \tfrac{1}{2}(\omega^2_1(k) - \omega^2_2(k)) + \tilde{\Sigma}_{11}(k;s) - \tilde{\Sigma}_{22}(k;s))^2 + 4\tilde{\Sigma}_{12}(k;s)\tilde{\Sigma}_{21}(k;s))^{1/2}; \quad (3.60)$$

$$\alpha^2(k;s) = \frac{1}{2}\left[\omega^2_1(k) + \omega^2_2(k) + \tilde{\Sigma}_{11}(k;s) + \tilde{\Sigma}_{22}(k;s)\right] + D(k;s);$$

$$\beta^2(k;s) = \frac{\omega^2_1(k) - \omega^2_2(k) + \tilde{\Sigma}_{11}(k;s) - \tilde{\Sigma}_{22}(k;s)}{2D(k;s)}; \quad (3.61)$$

$$\gamma^2(k;s) = \frac{\tilde{\Sigma}_{12}(k;s)}{D(k;s)}; \quad \gamma^2(k;s) = \frac{\tilde{\Sigma}_{21}(k;s)}{D(k;s)}; \quad (3.62)$$

with the property $\alpha^2(k;s) + \beta^2(k;s) - \gamma^2(k;s) = 1 \Rightarrow \alpha(k;s)$

¼ q1 −β¯ðk;sÞγ¯ðk;sÞffi; ð3:63Þ

where we used the same argument leading to Eq. (2.46) for the choice of sign for α¯.

In terms of these variables, $G^{-}_{ab}{}^{1}$ðk;sÞ has the same form as Eq. (C3) in Appendix C, yielding

$$G(k;s) = \overline{M^{-}}\,\overline{P(k;-s)}\,D + \overline{\delta(2k;s)}\,\beta\, M^{-}\,\overline{P(k;s)}\,\beta\,\delta + \beta(k;s)\,D\,\delta(2k;s);$$

$$P = -2\delta\,1\,R; \qquad (3.64)$$

with

$$\alpha^{-} \quad \beta^{-} \qquad 2 \qquad 2$$

$$R = \gamma^{-} - \alpha^{-}; \qquad R = 1 \Rightarrow P(k;s) = P(k;s):$$

(3.65)

The analytic continuation of the self-energies is

$$\tilde{\Sigma}_{ab}(k;s = -i\omega + \epsilon)$$

$$= \frac{1}{2\pi} Z_{-\infty}^{\infty} P\,\overline{\rho_{ab}\omega\delta - k_0;k_0}\,D\,dk_0 - i\pi\rho_{ab}(\omega;k): \qquad (3.66)$$

Upon analytic continuation to ω Green's function (3.58) becomes

$$G_{ab}(k;t) = \frac{k;\omega}{-\infty\,\delta \quad \beta \quad D \qquad -\delta k; \quad D} \int_{-\infty}^{\infty} \left[\frac{\mathbb{P}_{-}(\quad)}{i\epsilon^{2} - \Omega^{2} \quad \omega} - \omega \right]$$

$$P\,2\delta(k;\omega 2)D \qquad {}_{ab}e^{-i\omega t}\,d\omega; \qquad (3.67)$$

$$\beta\,\delta\omega + i\epsilon\beta - \Omega\beta(k;\omega)D \qquad 2\pi$$

with $_{2} \quad 1 \qquad 12 \qquad 22 \qquad \tilde{}\,_{11}$ k;ωÞ

$$\Omega(k;\omega) = 2\bar{\omega}\,\delta(kD + \omega\,\delta(kD + \Sigma\,\delta$$

$$+ \tilde{\Sigma}_{22}(k;\omega)D\,\delta(k;\omega); \qquad (3.68)$$

where the functions of ω are understood as the functions of s upon analytic continuation $s = -i\omega + \epsilon$, keeping the same name for the functions to simplify notation.

The form of Green's function is similar to Eq. (2.40) of the previous section, with important differences: whereas the denominators in Eq. (2.40) are linear in ω, therefore each term features only one pole, and the denominators in (3.67) are quadratic in ω implying that each term features two poles. This discrepancy has a simple explanation: the set of amplitude equations leading up to (2.40) describe the evolution of single particle states, whereas the effective field theory yields the time evolution of the density matrix in the field basis, and a real scalar field describes positive frequency particle states and negative frequency antiparticle states. Even in the absence of perturbations, the propagator has two poles yielding the time evolution $e^{\mp i\omega(k)t}$ for the amplitudes. Furthermore, the self-energies $\tilde{\Sigma}_{ab}(k;\omega)$ have dimensions of energy squared, unlike the Δ_{ab} in the previous section that feature dimensions of energy.

The complex poles in Green's functions are at

$$\omega^{2} = \Omega^{2}(k;\omega); \qquad (3.69)$$

namely

$$\omega(s) = \Omega_{\beta}(k;\omega(s)_{\beta}); \qquad (3.70)$$

$$\beta$$

$$\omega(s)_{-} = \Omega(-k;\omega(s)_{-}); \qquad (3.71)$$

where the superscripts () denote the two roots of (3.70) and (3.71) for each subscript label$_2$ β;− corresponding to the signs of Ω in (3.68). These roots define the complex frequencies of the quasinormal modes.

Consistent with perturbation theory, we assume that $\omega_{1,2}^{2}\,k \gg \tilde{\Sigma}_{ab}\,k,\omega \propto H_{I}$ ð Þ ð Þ ² allowing one to implement

Breit-Wigner and narrow width approximations to the propagators. Just as in the case discussed in the previous section, the validity of the Breit-Wigner approximation relies on weak coupling, in particular that the distance between the real part of the poles and thresholds is much larger than the half-width of the resonance. This criterion holds for both the nondegenerate and nearly degenerate cases, because in the latter the condition of near degeneracy is that $\omega^{2}{}_{a} \gg \tilde{\Sigma}_{ab}$ and $\omega_{1}^{2} - \omega_{2}^{2} \lesssim \tilde{\Sigma}_{ab}$. This

approximation describes exponential relaxation valid in the intermediate timescale as discussed above.

In these approximations, we expand around each pole in the denominators in (3.67), namely $\omega = \omega_{\bar{\sigma}p} + \delta\omega - \omega_{\bar{\sigma}p}$;

$$\Omega_p(k;\omega) = \Omega_p(k;\omega_{\bar{\sigma}p} + \delta\omega - \omega_{\bar{\sigma}p}) \times \frac{1}{2\omega_p}\frac{d^2}{d\omega^2}\Omega_p(k;\omega)\Big|_{\omega=\omega_{\bar{\sigma}p}}\delta\omega^2 \quad (3.72)$$

and similarly for each of the other poles. Using the pole condition (3.69) yields the general form of Green's function,

$$G_{ab}(k;t) = G_{ab;p}(k;t) + G_{ab;-}(k;t); \quad (3.73)$$

where each term corresponds to the contribution of the individual quasinormal modes corresponding to the subscripts , namely

$$G_{ab;p}(k;t) = -i\,Z_{\bar{\sigma}p}\,\frac{e^{-\omega_{\bar{\sigma}p}t}}{2\omega_p}\,P_{p\bar{\sigma}}(\omega_{\bar{\sigma}p})\,P_{ab}(p)$$

$$Z_{\bar{\sigma}p} = \left(1 - \frac{d}{d\omega}\Omega_{\bar{p}}^{-2}(k;\omega_{\bar{\sigma}p})\,\omega^2\right)^{-1}_{\omega=\omega_{\bar{\sigma}p}}$$

$$G_{ab;-}(k;t) = -iZ_{\bar{\sigma}p-}\frac{e^{-i\omega_{\bar{\sigma}p-}t}}{2\omega_{\bar{\sigma}p-}}\,P_{-\bar{\sigma}}(\omega_{\bar{\sigma}p-})\,P_{ab}(p)$$

$$Z_{\bar{\sigma}-p} = \left(1 - \frac{d}{d\omega}\Omega_{-2}^{-2}(k;\omega_{\bar{\sigma}p})\,\omega^2\right)^{-1}_{\omega=\omega_{\bar{\sigma}p-}}$$

(3.76) Green's function for each quasinormal mode features both positive ($\omega^{\bar{\sigma}p}$) and negative ($\omega^{\bar{\sigma}-p}$) frequency contributions.

This is the general result for Green's function, again displaying the four poles: positive and negative frequency for each quasinormal mode, with the associated wave function renormalization constants arising from the residues at the poles in the Breit-Wigner approximation.

Note that while the result (2.44) features only positive frequency components, Green's function (3.73) features both positive ($\omega^{\bar{\sigma}p}$) and negative ($\omega^{\bar{\sigma}-p}$) frequency components. As discussed above the origin of this difference is that whereas the Weisskopf-Wigner formulation, upon which the LOY theory is based, describes the time evolution of single particle (positive frequency) or antiparticle (negative frequency) amplitudes, Green's function in the effective field theory describes the propagation of fields that include both positive and negative frequencies and describe the quasinormal modes of propagation as a consequence of mixing and decay. Although this expression looks cumbersome in its index structure, we clarify again: the superscripts () refer to the positive (particle)

refer to the two (quasi)normal modes from mixing, corre- and negative (antiparticle) frequencies, the subscripts

sponding to the in Eq. (3.68).

With the purpose of comparison with the LOY theory, we focus on the same possible scenarios as in the previous section.

A. Nondegenerate: $m_1^2 - m_2^2 \gg \tilde{\Sigma}_{ab}$

In this case it follows that

$$D(k;\omega) \simeq \omega_1^2 - k^2 - \omega_2^2 \quad (3.77)$$
$$(k) + \tilde{\Sigma}_{11}(k;\omega) - \tilde{\Sigma}_{22}(k;\omega);$$

yielding up to second order in the couplings

$$\Omega^2(k,\omega) \simeq \omega_{i}^2 + \tilde{\Sigma}_{11}(k;\omega), \quad (3.78)$$

$$\Omega^2(k,\omega) \simeq \omega_2^2 + \tilde{\Sigma}_{22}(k;\omega). \quad (3.79)$$

$$\alpha(k;\omega) \simeq 1 + O(\tilde{\Sigma}_{12}), \quad (3.80)$$

$$\beta(\omega) \simeq 2\tilde{\Sigma}_{12}(k;\omega) \ll \sqrt{\omega^2 - m_1^2 - m_2^2}, \quad (3.81)$$

$$\bar{\gamma}(k,\omega) \simeq \frac{2(k,)}{\omega^2 - m_i^2 - m_2^2} \ll 1, \quad (3.82)$$

Therefore, up to second order in couplings the complex poles are at

$$\omega_+(k) = \omega_r(k) - i\gamma_+, \quad (3.83)$$

$$\omega_-(k) = \omega_r(k) - i\gamma_-, \quad (3.84)$$

where

$$\omega_r(k) = \omega_1(k) + \delta\omega(k); \qquad \omega_r(k) = \omega_2(k) + \delta\omega(k) \quad (3.85)$$

are the renormalized frequencies of each quasinormal mode and to leading order $\delta\omega(k) = \tilde{\Sigma}_{11}(k;\omega_1(k))$; $\delta\omega(k) = \tilde{\Sigma}_{22}(k;\omega_2(k))$; $\gamma(k) = -\mathrm{Im}\tilde{\Sigma}_{12}(k;\omega_1(k))$; $\gamma(k) = -\mathrm{Im}\tilde{\Sigma}_{12}(k;\omega_2(k))$. (3.86)

$$\Gamma_+(k) = \rho_{11}(2\omega_1(k);k), \quad (3.87)$$

To obtain the above results we used the property $\rho^{aa}(-\omega;k) = -\rho^{aa}(\omega;k)$ (no sum over a) for the diagonal(B15) in matrix elements of the spectral density [see Eq. Appendix B]. The contributions $\delta\omega(k)$ are renormalizations of the bare frequencies $\omega_{1,2}$, respectively.

Green's function (3.73) with (3.74) and (3.75) is given to leading order in this case by

$$G_{ab}(k;t) = i\theta(t)\sum_r \frac{1}{2\omega_r(k)} e^{-i\omega_r(k)t} e^{-\frac{1}{2}\gamma_r(k)t} P_r(k)\delta(\omega - \omega_1(k))_{ab}; \quad (3.88)$$

$$G_{ab}(k;t) = i\theta(-t)\sum_r \frac{1}{2\omega_r(k)} e^{-i\omega_r(k)t} e^{-\frac{1}{2}\gamma_r(k)t} P_r(k)\delta(\omega + \omega_2(k))_{ab}; \quad (3.89)$$

with the projection operators given to leading order in the couplings by

$$P_r(\omega_1(k)) = \frac{1}{2}\gamma(k;\omega_1(k)) \quad (2 \quad 0 \quad 1), \quad (3.90)$$

$$P_r(\omega_2(k)) = \frac{1}{2}\gamma(k;\omega_2(k)) \quad (2 \quad 0 \quad 2) - \frac{1}{2}\beta(k;\omega_2(k)).$$

$$P_r(\omega) = \gamma(k;\omega) \quad (3.91) \quad 1$$

Since in this nondegenerate case $\beta;\gamma \propto \tilde{\Sigma}$ [see Eqs. (3.81) and (3.82)] the off-diagonal terms are perturbatively small, for these the wave function renormalization constants can be set $Z \simeq 1$ to leading order. This result agrees with those obtained in Ref. [18] for the case of pion-axion mixing.

B. (Nearly) degenerate: $m_1^2 - m_2^2 \lesssim \tilde{\Sigma}_{ab}$

In this case, it is convenient to define

$$\Omega_2\,\delta k;\omega\rangle = \omega^{-}{}_2\delta k\rangle + E\delta k;\omega\rangle; \qquad (3.92)$$

with

$$\omega^{-}\delta k\rangle = \frac{1}{2\omega_{21}\delta k\rangle + \omega_{22}\delta k\rangle_{1=2} \gg \tilde{\Sigma}_{ab};D\ ffi} \qquad (3.93)$$

and

$$1\ \tilde{}\ {}_{11} - k;\omega\rangle + \tilde{\Sigma}_{22}\delta k;\omega\rangle\ D\delta k;\omega\rangle \ll \omega^{-}{}_2 E\,\delta k;\omega\rangle = 2\Sigma\,\delta \qquad (3.94)$$

with $\tilde{\Sigma}_{ab}$ and D of the same order. In this case the complex poles are at

$$\omega_{\delta}\delta k\rangle = \omega_{r}\delta k\rangle - i\,2\delta\rangle; \qquad (3.95)$$

$$\omega_{\delta}{}_{-}\delta k\rangle = \omega_{-r}\delta k\rangle - i\,\frac{\Gamma_{\delta}}{2}\delta k\rangle; \qquad (3.96)$$

where

$$\omega_{r}\delta k\rangle = \omega^{-}\delta k\rangle + \delta\delta k\rangle \qquad (3.97)$$

$$\Gamma_{\delta}\,k$$

are the renormalized (nearly degenerate) frequencies, with

$$\delta\,\delta k\rangle = Re\tfrac{1}{2}E2\delta\omega k;^{-}\delta k\rangle\omega^{-}\,\delta k\rangle;$$

$$\Gamma_{\delta}\,\delta k\rangle = \delta\mp\rangle Im\frac{E}{\tfrac{1}{2}\delta\omega^{-}k;\delta k\rangle\omega^{-}\delta}\rangle; \qquad (3.98)$$

and both are of quadratic order in the couplings.

We assume that the decay rates $\Gamma^{\delta\rangle}$ are all positive for stability, and further properties ofthe specific details of the self-energies$\delta;\Gamma^{\delta\rangle}\tilde{\Sigma}_{ab}$, which in turnwill depend on depend on the type of operators O_a. In this case $\omega_{\flat r} - \omega_{-r} \simeq \tilde{\Sigma}$, namely the difference in the quasinormal mode frequencies are of quadratic order in the couplings. To leading order in the couplings Green's functions (3.74) and (3.75) in this nearly degenerate case are given by

$$-i\omega_{r-\flat r}te^{-ir_{2\flat}\underline{\flat}t}P_{\flat}\delta\omega_{\flat r}\rangle\ G\ \delta k;t\rangle = 2\omega_{\flat}\,Z_{\delta\flat\flat\flat}\,e_{\flat}$$

$$-Z_{\delta\flat-\flat}e^{i\omega_{\flat r}te^{-i\ 2\underline{\flat}t}}P_{\flat}\delta-\omega_{\flat r}\rangle i; \qquad (3.99)$$

$$G_{-}\delta k;t\rangle = 2\underline{\quad}\omega i_{-r}\,hZ_{i\delta\flat\flat-\text{--}rt}e^{-ii\omega_{\Gamma 2\underline{\ }\text{---}rtt}e^{-ir_{2-\underline{\flat}t}}}P_{-}\delta\omega_{-r}\rangle$$

$$-Z_{\delta\text{--}\flat e\ \omega}\quad e_{-}\,P_{-}\delta-\omega_{-r}\rangle i; \qquad (3.100)$$

where all matrix elements of P are of $O\delta 1\rangle$.

C. Expectation values and correlation functions

We seek to obtain expectation values and correlation functions of ϕ_a in the reduced density matrix. In particular, we focus on equal time correlation functions. If asymptotically at long time these become time independent, this is a signal of the emergence of a stationary state, from which
we can assess if the fields reach thermal equilibration with the bath. Furthermore, off-diagonal equal time correlation functions will also inform on the emergence and long time survivability of coherence. Therefore, we must relate these to the averages of the center of mass Keldysh fields Φ_a. To establish this relation, we begin with the path integral representations for the forward and backward time evolution operators (3.13), (3.15), and (3.16) which show that $\phi^{b}{}_{a}$

follows that the expectation value of the fields in the fullare associated with $U\delta t\rangle$ and $\phi^{-}{}_{a}$ with $U^{-1}\delta t\rangle$. Hence, it

density matrix is given by $\langle\phi_a(\vec{x},t)\rangle = \mathrm{Tr}\phi_a\rho(t) = \mathrm{Tr}\rho(t)\phi_a = \mathrm{Tr}\rho(0)\phi_a(\vec{x},t) = \mathrm{Tr}\rho(0)\Phi_a(\vec{x},t) = \langle\Phi_a(\vec{x},t)\rangle$; (3.101) whereas

$$\mathrm{Tr}R_a(\vec{x},t)\rho(0) = 0. \tag{3.102}$$

Similarly, correlation functions in the forward, backward, and mixed forward-backward branches are given by

$$\mathrm{Tr}\phi_a(\vec{k},t)\phi_b(\vec{k}',t_0)\rho(0) \equiv \mathrm{Tr}T(\phi_a(\vec{k},t)\phi_b(\vec{k}',t_0))\rho(0);$$

$$\mathrm{Tr}\phi_{-a}(\vec{k},t)\phi_{-b}(\vec{k}',t_0)\rho(0) \equiv \mathrm{Tr}\rho(0)\tilde{T}(\phi_a(\vec{k},t)\phi_b(\vec{k}',t_0));$$

$$\mathrm{Tr}\phi_a(\vec{k},t)\phi_{-b}(\vec{k}',t_0)\rho(0) \equiv \mathrm{Tr}\phi_a(\vec{k})(t,t_0)\rho(t_0)\phi_b(\vec{k}',t_0) = \mathrm{Tr}\phi_b(\vec{k}',t_0)\phi_a(\vec{k},t)\rho(t);$$

(3.103) where T and $\tilde{T}$ are the time ordering and antitime ordering symbols, respectively. Using the relations (3.103) it is straightforward to confirm that

$$\mathrm{Tr}\Phi_a(\vec{x},t)\rho\Phi_b(\vec{x}',t')\hat{\rho}(0) \equiv \frac{1}{2}\mathrm{Tr}\{\phi_a(\vec{x},t)\phi_b(\vec{x}',t_0)\rho(t_0) + \phi_b(\vec{x}',t_0)\phi_a(\vec{x},t)\rho(t)\}. \tag{3.104}$$

Upon obtaining the functional solutions of Eq. (3.45) our objective is to obtain the connected equal time correlation functions $\langle\phi_a(t)\phi_b(t)\rangle_c = \mathrm{Tr}\rho(0)\phi_a(t)\phi_b(t) - \mathrm{Tr}\rho(0)\phi_a(t)\mathrm{Tr}\rho(0)\phi_b(t)$; (3.105) and the population for each field component of wave vector $\vec{k}$, namely

$$n_a(\vec{k},t) = \frac{1}{2\omega_a(\vec{k})}\mathrm{Tr}\rho(0)\{|\dot{\phi}_a(\vec{k})|^2 + \omega_a^2(\vec{k})|\phi_a(\vec{k})|^2\} - \frac{1}{2},$$

(no sum over a). (3.106)

Establishing contact with the dynamics of the density matrix of two level systems [51], the off-diagonal components of the connected correlation function (3.105) are a manifestation of coherence. If initially the fields are uncorrelated, the off-diagonal components of the correlation function vanish. Therefore, if upon time evolution these are nonvanishing, these off-diagonal correlations between the two fields are a consequence of coherence induced by the indirect mixing through the interactions of the field with the bath.

With the definition of the Keldysh center of mass field variables Φ_a (3.32) and the relations (3.101)–(3.103), we find that the equal time connected correlation function (3.105) is given by

$$\langle\phi_a(t)\phi_b(t)\rangle_c = \overline{\langle\langle\Phi_a(t)\Phi_b(t)\rangle\rangle} - \overline{\langle\langle\Phi_a(t)\rangle\rangle}\,\overline{\langle\langle\Phi_b(t)\rangle\rangle}. \tag{3.107}$$

To obtain the population for each field (3.106) we now introduce

$$C^{>}_a(\vec{k};t;t^0) = \mathrm{Tr}\phi^{-}_a(\vec{k},t)\phi^{b}_a(-\vec{k},t^0)\rho(0);$$

$$C^{<}_a(\vec{k};t;t_0) = \mathrm{Tr}\phi_{-a}(\vec{k},t_0)\phi_{ba}(-\vec{k},t)\rho(0); \tag{3.108}$$

and the populations of the wavevector $\vec{k}$ component of each field ϕ_a (3.106) become

$$\frac{1}{2}\left(\frac{\partial}{\partial}\right.$$

$$n_a(k;t) = 4\omega_a(k)\dot{t}(t)\overline{\quad} + \omega^2 t_a(k)$$

$$> \quad < \quad 1$$

$$n_a(k;t) = 2\omega_a 1(k)\mathrm{Tr}(\rho(0)\,\Phi^{\cdot}_a(\vec{k};t)\Phi^{\cdot}_a(-\vec{k};t)) + \omega_{2a}(k)\Phi_a(\vec{k};t)\Phi_a(-\vec{k};t) - 12$$

$$= 2\omega_a 1(k)\langle\langle \Phi^{\cdot}_a(\vec{k};t)\Phi^{\cdot}_a(-\vec{k};t)\rangle\rangle + \omega_{a2}(k)\langle\langle \Phi_a(\vec{k};t)\Phi_a(-\vec{k};t)\rangle\rangle - 21: \qquad (3.110)$$

$$\times [C_a(k;t;t^0) + C_a(k;t;t^0)]_{t=t_0} \; 2: \qquad (3.109)$$

The corollary of this analysis is that we can obtain the connected correlation functions and the populations of the fields $\phi_{1;2}$ by obtaining the solutions of the Langevin equation of motion (3.45) with initial conditions (3.47) and taking the averages over the initial conditions and noise described above.

Armed with the solution of the Langevin equations (3.55) and (3.56), the above results, and the general form for Green's function (3.73), in terms of Green's functions for the quasinormal modes (3.74) and (3.75), we can now study the expectation values, connected correlation functions (3.107), and populations (3.110). The solutions (3.55) and (3.56) along with the averages (3.46) yield for the spatial Fourier transform of the fields

$$\langle\phi_a(k;\vec{t})\rangle = G^{\cdot}_{ab}(k;t)\Phi_{b;i}(\vec{k}) + \int G_{ab}(k;t)\Pi_{b;i}(\vec{k}):$$

(3.111) Similarly, the connected correlation functions (3.107) are

$$\langle\phi_a(k;\vec{t})\phi_b(-k;\vec{t})\rangle_c = \Phi^h_a(k;\vec{t})\Phi^h_b(-k;\vec{t}) - \Phi^h_a(k;\vec{t})\Phi^h_b(-k;\vec{t}) + \langle\langle \Phi^{\xi}_a(k;\vec{t})\Phi^{\xi}_b(-k;\vec{t})\rangle\rangle; \qquad (3.112)$$

where Φ^h and Φ^{ξ} are given by (3.56). These are general results for expectation values and correlation functions, from which we can obtain their time evolution.

D. Time evolution, thermalization, and bath-induced coherence

Using the definition (3.32) and the relations (3.103) it is straightforward to show that this symmetrized product yields

Taken together, the results (3.111) and (3.112) inform important aspects for the time evolution of expectation values and correlation functions:

(I) Even when initially only one of the fields, for example ϕ_1, features an expectation value, the off-diagonal components of Green's functions determined by the projector operators P in (3.74) and (3.75) induce a nonvanishing ϕ_2. This expectation value for the other field, in this case phenomenon has been noticed in Ref. [18] in the case of axion-pion mixing. In the LOY theory discussed in the previous section, a similar feature emerges at the level of the amplitudes of the single particle states $|\phi_1\rangle$ and $|\phi_2\rangle$: for example, if the initial amplitudes are $C_1(0) \neq 0$; $C_2(0) = 0$, upon time evolution a nonvanishing amplitude $C_2(t)$ is induced as a consequence of mixing. The off-diagonal components of G_{ab} are a consequence of the off-diagonal components of the self-energy matrix and a direct manifestation of the couplings of the fields to correlated operators of the bath degrees of freedom, namely "indirect" mixing.

(II) A similar phenomenon emerges for the connectedcorrelation function (3.112). Even if the fields $\phi_{1;2}$ are initially uncorrelated, a nonvanishing correlation emerges from the off-diagonal components of the noise correlation function that determines the last term in (3.112). We refer to the emergence of nonvanishing correlations as bathinduced coherence, referring as coherence to the offdiagonal connected correlation functions of the field in agreement with the description of the time evolution of a density matrix in two level systems [51].

(III) The off-diagonal components of the projectors P

are perturbatively smallwhereas they are of Oð1ÞOin the nearly degenerate case. InðΣÞ in the nondegenerate case,

turn, this implies that the induced expectation values and coherence are perturbatively small in the nondegenerate case, in agreement with the results in Ref. [18], but are of Oð1Þ in the nearly degenerate case. This expectation is confirmed by the analysis below.

The first two terms in (3.112) decay exponentially because Green's functions do, and depend explicitly on the initial conditions. The last term is independent of the initial conditions, it is completely determined by the noise term induced by the bath degrees of freedom, and, as we show below, it survives in the long time limit, hence determining the approach to a stationary state.

We now focus on this last term, which upon using the noise correlation function (3.46) and (3.49) is given by

$$\langle\langle \Phi_a(t)\Phi_b(t)\rangle\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\int_t G_{ac}(k;\tau)e^{ik_0\tau}d\tau \times \int_0^t G_{bd}(k;\tau)e^{-ik_0\tau}d\tau\,\rho_{cd}(k_0)\coth\left(\frac{\beta k_0}{2}\right). \quad (3.113)$$

Each of the G^0s in this expression is a sum of Green's functions of each quasinormal mode given by Eqs. (3.73)–(3.75); therefore, each G features four terms; and hence, there are altogether 16 terms in (3.113). Because ρ_{cd} is of second order in couplings, we will focus on the terms that are of Oð1Þ in these couplings, and these arise from the terms that feature small denominators of second order in the couplings that compensate the numerator ρ_{cd}.

Each of the terms in G features exponentials of the form $e^{-i(\omega-i\Gamma/2)t}$ where ω stands for the real part of the quasinormal mode frequencies, the describes the positive and negative frequency components, and $\Gamma > 0$ stands generically for the decay rate of these modes. Therefore, the time integral of such a typical term in the first bracket in (3.113) yields $\int_t^{\infty} e^{i(k_0\mp\omega_{\beta r}-i\Gamma_{\beta}/2)\tau}d\tau = \frac{e^{i(k_0\mp\omega_{\beta r}-i\Gamma_{\beta}/2)t}}{i(k_0\mp\omega_{\beta r}-i\Gamma_{\beta}/2)}$; (3.114)

and for the second bracket there is a similar generic contribution but with $k_0 \to -k_0$. Obviously as $t\to\infty$ these contributions remain nonvanishing, confirming that the noise contribution to the correlation functions and coherences (off-diagonal) remain finite in the long time limit. The k_0 integral in (3.113) is dominated by the poles in the complex k_0 plane. To identify these, consider the product of the positive frequency contribution for the first G with the negative frequency contribution of the second G for the same quasinormal mode, for example, that of frequency $\omega_{\beta r}$. Such a term is proportional to the product

$$\frac{\left[i(k_0-\omega_{\beta r}-i\Gamma_{\beta}/2)t-1\right]e^{i(k_0-\omega_{\beta r}-i\Gamma_{\beta}/2)t}-1}{i(k_0-\omega_{\beta r}-i\Gamma_{\beta}/2)}\cdot\frac{\left[-i(k_0+\omega_{\beta r}-i\Gamma_{\beta}/2)t-1\right]e^{-i(k_0+\omega_{\beta r}-i\Gamma_{\beta}/2)t}-1}{-i(k_0+\omega_{\beta r}-i\Gamma_{\beta}/2)}$$

$$= \frac{1}{(k_0-\omega_{\beta r}-i\Gamma_{\beta}/2)(k_0+\omega_{\beta r}-i\Gamma_{\beta}/2)}\left[1+e^{-\Gamma_{\beta}t}-2e^{-\Gamma_{\beta}t/2}\cos\left((k_0-\omega_{\beta r})t\right)\right]; \quad (3.115)$$

and we refer to these as direct terms: there are two for each quasinormal mode. These terms feature poles at $k_0 = \omega_{\beta r} - i\Gamma_{\beta}/2$ yielding for the integral a contribution proportional to

$$\propto 4\times 2\pi P_{\beta}(\omega_{\beta r})P_{ac}\left[\frac{\rho_{cd}(\omega_{+r})}{\Gamma^{+}\Gamma}\right]\mathbb{P}_{+}(-\omega_{\beta r})P_{bd}(\omega_{\beta r})\left[1+2n(\omega_{\beta r})\right]P_{\beta}\times\left[1-e^{-\Gamma_{\beta}t/2}\right]; \quad (3.116)$$

where $n(\omega)$ is the Bose-Einstein distribution function with energy ω. In this expression we kept the leading order terms

and neglected wave function renormalization constants. Furthermore, $\rho_{cd}(\omega)$ and $\coth(\beta\omega)$ are evaluated at $\omega_{\beta r}$, namely the real part of the frequency of the quasinormal mode, sinceimportant aspect of this contribution is that it is ofΓ_β are of second order in couplings. An$O(1)$, because ρ and Γ are both of second order in the couplings. Now consider the positive (or negative) frequency contributions of the same quasinormal modes in both brackets, for example, for the positive frequency whose time integrals yield a term proportional to

$$\frac{1}{4}\left[\frac{e^{i(k_0-\omega_{\beta r}+i\frac{\Gamma_\beta}{2})t}-1}{i(k_0-\omega_{\beta r})-\frac{\Gamma_\beta}{2}}\right]\left[\frac{e^{-i(k_0+\omega_{\beta r}-i\frac{\Gamma_\beta}{2})t}-1}{-i(k_0+\omega_{\beta r})-\frac{\Gamma_\beta}{2}}\right]$$

$$=\frac{1}{2}\left[\frac{1-e^{-2i\omega_{\beta r}t}e^{-\Gamma_\beta t}-e^{i(k_0-\omega_{\beta r}+i\frac{\Gamma_\beta}{2})t}+e^{-i(k_0+\omega_{\beta r}-i\frac{\Gamma_\beta}{2})t}}{(k_0-\omega_{\beta r})^2+(i\frac{\Gamma_\beta}{2})^2}\right]; \quad (3.117)$$

and the negative frequency contribution is obtained by $\omega_{\beta r}\to-\omega_{\beta r}$. We refer to these as indirect terms. These$\omega_r(\beta,\Gamma_\beta)$ feature complex poles at $k_0=\pm\omega_r-i\frac{\Gamma_\beta}{2}$, yielding terms proportional to

$$\frac{P}{\omega-\omega_{\beta r}}\rho_{cd}(2\omega,\omega_r,\beta)-\frac{P}{\omega+\omega_{\beta r}}\rho_{cd}(-2\omega,\omega_r,\beta)\frac{1-e^{-2i\omega_r t-\Gamma_\beta t}}{2i\omega_r}\times(1+2n(\omega_{\beta r},\beta)); \quad (3.118)$$

where we have used that in the narrow width approximation $\omega_r\gg\Gamma_\beta$. These indirect terms are of second order in the β couplings and therefore are subleading with respect to the direct terms.

Finally, consider the contribution of a positive frequency of one quasinormal mode in one bracket and a negative frequency of the other mode in the other bracket. We refer to these as interference terms. For example, consider the positive frequency mode ω_r in the first bracket in (3.113), and the negative frequency mode$-\omega_{-r}$ in the second bracket, where the time integrals yield a term proportional to

$$\frac{1}{4}\left[\frac{e^{i(k_0-\omega_{r\beta}+i\frac{\Gamma_\beta}{2})t}-1}{i(k_0-\omega_{r\beta})-\frac{\Gamma_\beta}{2}}\right]\left[\frac{e^{-i(k_0+\omega_{-r\beta}-i\frac{\Gamma_\beta}{2})t}-1}{-i(k_0+\omega_{-r\beta})-\frac{\Gamma_\beta}{2}}\right]$$

$$=\frac{1}{2}\left[\frac{1-e^{-2i\omega_{r\beta}t}e^{-\Gamma_\beta t}-e^{i(k_0-\omega_{r\beta}+i\frac{\Gamma_\beta}{2})t}+e^{-i(k_0+\omega_{r\beta}-i\frac{\Gamma_\beta}{2})t}}{(k_0-\omega_{r\beta})^2+(i\frac{\Gamma_\beta}{2})^2}\right]; \quad (3.119)$$

A similar analysis as for the previous terms yields the following leading order contribution to the correlation function:

$$\rho_{cd}(\omega^-)\,P_\beta(\omega_{\beta ac})\frac{1}{(\omega_{\beta r}-\omega_{-r}-i\frac{\Gamma_\beta}{2})(\omega_{\beta r}+\omega_r-i\Gamma_\beta)}\times P_-(\omega^-)_{bd}\left[1-e^{i(\omega_{\beta r}-\omega_r)t}e^{-\frac{1}{2}(\Gamma_{\beta b}+\Gamma_{\beta b})t}\right](1+2n(\omega^-)); \quad (3.120)$$

These interference terms exhibit the quantum beats, an interference phenomenon associated with the difference in the (quasi)normal mode frequencies, similar to that in the expression (2.112).

These results are general and highlight the perturbative and nonperturbative contributions to the correlation functions in the long time limit. Before we discuss the nondegenerate and nearly degenerate cases, it is convenient to compare the results above to the case of the equal time correlation functions of a free field theory in thermal equilibrium, which is given by

$$\langle\Phi_b(-\vec{k};t)\Phi_i\rangle=\frac{1}{2\omega_{\delta_{aab}}(k)}(1+2n(\omega_a,\beta)); \quad (3.121)$$

where we assumed uncorrelated fields, and the brackets stand for statistical averages in a thermal ensemble of uncorrelated fields.

It is then clear that the long time limits (3.116), (3.118), and (3.120) all feature exponential relaxation to a thermalized stationary state with the asymptotic long time limit featuring the thermal factors $1+2n(\omega,\beta)$ in terms of the real parts of the frequencies of the quasinormal modes

(the imaginary parts yield subleading contributions). This is one of the important results of this study.

Furthermore, all feature off-diagonal terms which we identify as coherence because of the similarity with two level systems [51], as discussed above. We refer to this phenomenon as bath-induced coherence because even if the fieldsareinitiallyuncorrelated,theirinteractionwiththebath inducesoff-diagonaltermsthatsurviveinthelongtimelimit. Interference terms between the two different quasinormal modes leads to the approach to the stationary thermal state with quantum beats. Thermalization, the emergence of offdiagonalcoherence inthelongtime limit, andquantumbeats from interference between the two different quasinormal modesaresomeofthemainresultsofthisstudy.InSec.IIwe highlighted that in the LOY theory, quantum beats emerged in the time evolution of the total population (2.109) and by unitarity in the amplitudes of intermediate states or decay products [see Eq. (2.110)]. In the effective field theory these interference terms are explicit in the approach to the stationarythermalstateofthecorrelationfunctionsofthefields , both the diagonal and off-diagonal (coherence) components displaying the quantum beats.

1. Nondegenerate case

In the nondegenerate case, with $\omega_r - \omega_{-r} \simeq \omega_1 - \omega_2 \gg \Gamma$ the direct terms (3.116) are the leading ones. As shown by Eqs. (3.90) and (3.91) with (3.81) and (3.82), in the nondegenerate case the off-diagonal components of the projection operators are of $O(g^2)$. Therefore, in this case, the correlation functions exhibit thermalization albeit with a perturbatively small coherence. Furthermore, because the interference terms are perturbatively small since $\rho=(\omega_1 - \omega_2) \propto \propto g^2$, the quantum beats in the approach to thermalization feature small amplitudes. This is in agreement with the results obtained in Ref. [18] for the case of (ultralight) axion-pion mixing.

2. Nearly degenerate case

In the nearly degenerate case with $\omega_r - \omega_{-r} \lesssim \Gamma$ both the direct (3.116) and the interference terms (3.120) are of $O(1)$, and all the matrix elements of the projector operators are also of $O(1)$. In this case the amplitude of the quantum beats is large, enhanced by the near resonance, and the timescale of these interference effects is similar to the relaxation timescale. In this case the offdiagonal correlations, namely the coherence, become large, amplified by the (near) resonant denominators, and could potentially be observable. This situation is akin to the case of $K^0 - K^{-}$ ^{0} mixing where the decay products exhibit quantum beats on the timescales comparable to the lifetime. This is clearly the same physical process as described by the LOY theory described in Sec. II. However, in the effective field theory approach the quantum beats are explicit in the correlation functions of the mixing fields in both the diagonal and the offdiagonal components, and in the approach to the thermal stationary state.

This large amplitude interference effect may open a window toward observation of synthetic-cosmological axion mixing via their (anomalous) coupling to photons with a Chern-Simons term. This pathway is being explored as a possible mechanism to harness synthetic axion quasiparticles in condensed matter systems to probe the cosmological axion [54].

E. Relation to the LOY formulation of mixing

The results of the effective field theory bear a similarity with those obtained from the LOYtheory in Sec. II, but also have noteworthy differences. We seek to establish a more direct correspondence between both formulations enlightening the reason for the similarities and the origin of the differences.

The main ingredient to obtain the time evolution of expectation values and correlation functions is Green's function (3.67), which is completely determined by the solutions of the Langevin equation (3.45) for the homogeneous case $\xi_a \equiv 0$, namely

$$\ddot{\Phi}_a(\vec{k};t) + \omega^2_a(k)\Phi_a(\vec{k};t) + \int_0^t \Sigma_{ab}(\vec{k};t-t')\Phi_b(\vec{k};t')dt' = 0. \quad (3.122)$$

In the absence of the self-energy, the solutions are the usual free field positive and negative frequency components with constant amplitudes. Since the self-energy is $\propto g^2$ (with g a generic coupling), we write

$$\Phi_a(\vec{k};t) = C_a(\vec{k};t)e^{-i\omega_a(k)t} + C_a(\vec{k};t)e^{i\omega_a(k)t}; \quad (3.123)$$

where the amplitudes $C_a(\vec{k};t); C_a(\vec{k};t)$ are slowly varying, namely $\dot{C}_a; \dot{C}_a \propto \propto g^2$. The equations of motion (3.122) become

$$e^{-i\omega_a(k)t}\ddot{C}_a(\vec{k};t) - 2i\omega_a(k)\dot{C}_a(\vec{k};t) + e^{-i(\omega_b(k)-\omega_a(k))t}\int_0^t \Sigma_{ab}(\vec{k};t-t')C_b(\vec{k};t')dt'$$

$$\dot{C}_a(k,t)\theta + \theta C_a(k,t)\theta + \theta e^{i(\omega_a(k)\theta - \omega_a(k)\theta)t} Z(t) \Sigma_{ab}(k,t) - t_0\theta e^{-i(\omega_b(k)\theta - \omega_a(k)\theta)(t-t_0)} C_b(k,t_0)\theta dt_0 = 0; \quad (3.124)$$

where in the last terms the sum over b is implicit. Because the terms inside the brackets are slowly varying and of $O(g^2)$, each bracket must vanish independently, yielding

$$\dot{C}_a(k,t) - 2i\omega_a(k)C_a(k,t) + \int e^{-i(\omega_b(k)\theta - \omega_a(k)\theta)t} Z(t) \Sigma_{ab}(k,t) - t_0\theta e^{i(\omega_b(k)\theta - \omega_a(k)\theta)(t-t_0)} C_b(k,t_0)\theta dt_0 = 0; \quad (3.125)$$

and the equation for C is obtained from (3.125) by replacing $\omega_{a;b} \to -\omega_{a;b}$. Let us neglect $\dot{C}$ in (3.125) for a moment; we will show below that it is subleading in the long time limit. We introduce

$$\Sigma_{ab}(k,t) = \frac{\partial}{\partial k} W_{ab}(k,t;t,0) = 0; \quad (3.126)$$

$$W(k,t;t_0) = 2\omega(k) \int_{t_0}^{t} dt_{00}\, \Sigma_{ab}(k,t) - t_{00}\theta e^{i(\omega_b(k)\theta - \omega_a(k)\theta)(t-t_{00})} dt_{00};$$

in terms of which (3.125) becomes

$$C_a(k,t)\theta = -e^{-i(\omega_b(k)\theta - \omega_a(k)\theta)t} Z(t) dt \int_{t_0}^{t} W_{ab}(k,t;t_0) C_b(k,t_0)\theta dt_0: \quad (3.127)$$

Upon integration by parts and the use of the initial condition in (3.126), the integral becomes

$$\int_{t_0}^{t} dt_0\, W_{ab}(k,t;t_0) C_b(k)\theta\, dt^0 = W_{ab}(k,t;t,t) C_b(k,t)\theta - \int_{t_0}^{t} W_{ab}(k,t;t^0)\theta dt \frac{d}{dt_0} C_b(k,t^0)\theta dt^0;$$

$$(3.128)\theta \frac{d}{dt^0}$$

because $W \propto g^2$ and $\dot{C} \propto \Sigma \propto g^2$ the second term in Eq. (3.128) is of $O(g^4)$ and will be neglected to leading order, yielding

$$\dot{C}_1(k,t)\theta = -fW_{11}(k,t;t,C_1(k,t)\theta) + \theta e^{i(\omega_1(k)\theta - \omega_2(k)\theta)t} W_{12}(k,t;t,C_2(k,t)\theta)\theta g; \quad (3.129)$$

$$\dot{C}_2(k,t)\theta = -fe^{i(\omega_2(k)\theta - \omega_1(k)\theta)t} W_{21}(k,t;t,C_1(k,t)\theta) + \theta W_{22}(k,t;t,C_2(k,t)\theta)\theta g; \quad (3.130)$$

where

$$W_{ab}(k,t) = \int_0^t dt_0 \int_{-\infty}^{\infty} \frac{dk\,\rho}{2\pi\, 2\omega\, k} \theta \delta(k - k_0)\theta P_{ab}(k_0;k,t_0)\theta e^{i(\omega_b(k)\theta - k_0\theta)(t-t_0)} dt_0: \quad (3.131)$$

Comparing the amplitude equations for the positive frequency components (3.129) and (3.130) with the amplitude equations in the LOY formulation, Eqs. (2.70) and (2.71) with $W_{ab}(k,t;t)$ given by Eq. (2.74), we see that they are exactly the same with the identifications $a_b(k) \to a_b(k)\theta \delta(k - k_0)\theta P \to \rho_{ab}(k_0;k,t)\theta \equiv \delta(k - k_0)\theta P$

θP

$$W(t;t') \equiv W(t;t'); \qquad \mu(k_0) \equiv a(k); \qquad E_{1,2}(k) = \omega_{1,2}(k): \tag{3.132}$$

$$4\pi\omega(k)$$

Furthermore, this analysis clarifies that the amplitudes C_a for the negative frequency components are also present in the effective field theory framework, but not in the LOY theory. Invoking the long time limit

$$\int_0^t e^{i(\omega_b(k)-k_0)(t-t_0)}dt_0 \to i\,\mathcal{P}\frac{1}{(\omega_b(k)-k_0)} - i\pi\delta(\omega_b(k)-k_0) \qquad t\to\infty \tag{3.133}$$

yields

$$-iW_{ab}(t;t') \to \Delta_{ab}(\omega_b(k))(t-t'); \tag{3.134}$$

where $\Delta_{ab}(\omega)$ is given by Eq. (2.39) with the identification (3.132). Since in the long time limit $W_{ab}(t;t') \to \Delta_{ab}(\omega_b(k))(t-t')$, it follows from the amplitude equations (2.70) and (2.71) that $\ddot{C}_a \propto g^4$ and can be consistently neglected, thus justifying neglecting $\ddot{C}$ in Eq. (3.125). The equations for the amplitudes (3.129) and (3.130) become exactly the same as the set of equations (2.70) and (2.71) in the LOY theory; therefore, the positive frequency components of Green's function $G_{ab}(k;t)$, Eq. (3.73), is equivalent to the bracket in Eq. (2.44), explaining the similar projector operators. However, full Green's function (3.73) includes the negative frequency components, because (3.73) describes the time evolution of fields rather than single particle amplitudes.

The solutions of the Langevin equation that determine the expectation values and correlation functions in the effective field theory, namely (3.55), feature two terms.

The homogeneous term $[\Phi^h_a(\vec{k};t)]$ in (3.56) depends on the initial conditions and corresponds to the solution (2.44) in the LOY theory, which also depends on the initial conditions. However, the inhomogeneous term $[\Phi^\xi_a(\vec{k};t)]$ in (3.56) is independent of initial conditions and is determined by the noise. It is this inhomogeneous term that determines the asymptotic behavior of the correlation functions and exhibits the approach to a thermal stationary state in the long time limit, while the homogeneous term decays exponentially at long time, in the same manner as the amplitudes in the LOY theory. This is one of the major differences between the effective field theory and LOY theory of mixing.

This analysis highlights the similarities and differences between the LOY theory and the effective field theory, the differences are noteworthy: (i) the effective field theory describes the evolution of fields, including both positive and negative frequency components of the quasinormal modes. (ii) The effective field theory description yields the correlation functions, describes the approach to a thermal steady state, as well as the emergence and long time survival of coherence, aspects that are not captured by the LOY theory. Another important difference is that in the effective field theory, the quantum beats are manifest in the approach to thermalization of the correlation function as a consequence interference of quasinormal modes, in both the diagonal (populations) and the off-diagonal (coherence) components of the correlation functions. (iii) Since the LOY method describes the evolution of the amplitudes of pure, single particle states, it cannot describe correlation functions.

IV. SUMMARY OF RESULTS AND CONCLUSIONS

A. Summary of results

We generalized the seminal theory of particle mixing pioneered by Lee, Oehme, and Yang to study CP violation in $K^0 - \overline{K^0}$ mixing. This theory is the cornerstone of all analysis of CP violation in flavored meson mixing in terms of an effective non-Hermitian Hamiltonian.

We extend this theory in two ways: (i) to include the cases in which the mixing degrees of freedom are not mass degenerate in the absence of perturbations, thereby relaxing the assumption of CPT invariance, and (ii) to treat the time evolution without resorting to the approximation of a time independent non-Hermitian effective Hamiltonian, and discuss the caveats resulting from this approximation, which become more important in the nondegenerate case. The LOY theory is only valid for pure single (or few) particle states and does not directly allow

one to obtain correlation functions of the mixing fields, nor the time evolution of multiparticle states, such as coherent states, or statistical ensembles. However, its generalization and extension provide a useful guide to and benchmark for the effective field theory which we introduce to describe indirect particle mixing as a consequence of their coupling to a common set of intermediate states or decay channels populated in a medium.

The effective action determines the time evolution of the reduced density matrix after tracing over the degrees of freedom in the medium described as a bath in thermal equilibrium. Therefore, it describes the dynamics of field mixing as a quantum open system. Indirect mixing is a result of nonvanishing correlations of the operators that couple the mixing partners to the intermediate states in the medium, and is manifest in off-diagonal components of the self-energy. The dynamics of field mixing is determined by a Langevin-like equation of motion with a dissipative selfenergy kernel and stochastic noise obeying a generalized fluctuation dissipation relation. The solution of the equations of motion determines the dynamics of expectation values and correlation functions in terms of a superposition of quasinormal modes in the medium. The off-diagonal elements of the self-energy and noise kernels lead to indirect mixing and the emergence of long-lived coherence, namely off-diagonal components of the two point correlation functions, even when initially the mixing fields are uncorrelated. We refer to this phenomenon as bath-induced coherence. We analyze in detail the cases in which the masses of the mixing particles are widely different, namely the nondegenerate case, and when they are nearly degenerate, which may describe small violations of CPT. In both cases even if one of the fields features an initial expectation value and the other does not, the latter develops an expectation value as a consequence of indirect mixing. We find the remarkable result that the equal time two point correlation functions of the fields approach a thermal stationary state and feature quantum beats as a consequence of the interference of the quasinormal modes. In the nondegenerate case these interference effects feature perturbatively small amplitudes; however, in the nondegenerate case the amplitude of the quantum beats is resonantly enhanced and nonperturbative. These interference effects may provide an observational avenue to probe cosmological axions in condensed matter systems.

We establish a direct relation between the effective field theory and the LOY theory of mixing, and highlight important differences, in particular that the effective field theory describes emergent, bath-induced long-lived coherence independent of the initial conditions that approach asymptotically a stationary thermal state.

B. Conclusions

Indirect field mixing as a consequence of common intermediate states or decay channels is of great importance in particle physics, cosmology, and possibly condensed matter physics. In particle physics indirect field mixing is at the heart of flavor meson mixing and CP violation in the standard model. Beyond the standard model it may be a consequence of intermediate messengers connecting standard model particles to degrees of freedom beyond through portals. In cosmology various axionlike particles may mix through common decay channels into photons and/or gluons, and in condensed matter synthetic axions, emergent quasiparticles in materials that feature parity breaking, such as topological insulators and Weyl semimetals, may hybridize (mix) with cosmological axions, thereby offering a way to probe the latter by exciting the former. Thus, the interdisciplinary relevance of field mixing motivates the study in this article. An important result of this study is that the equal time correlation functions feature quantum beats, as a consequence of interference of the quasinormal modes in the medium.

As demonstrated within the LOY theory, quantum beats are also manifest in the time evolution of the decay products, which may provide an observational signature of field mixing. This could be of particular relevance in the case of axion mixing.

The phenomena revealed by this study, such as bathinduced emergent coherence, induced condensates, and quantum beats, are all qualitatively general independent of the particular couplings or degrees of freedom in the medium. However, the quantitative form of the quasinormal modes, the projection operators, and the amplitudes of the quantum beats clearly will depend on the particular models and the parameters that define them.

Although we focused on field mixing in the case of bosonic fields, the general approach is also suitable to study indirect mixing for fermionic or gauge degrees of freedom. In the case of fermions the derivation of the effective field theory would require the extension of the current study to Grassman fields. One possible avenue would be to study neutrino mixing in the mass basis, where the weak interaction vertices feature flavor off-diagonal terms after diagonalizing a mass matrix in the free part of the

Lagrangian. An effective field theory description of indirect mixing (of the mass eigenstates) in a medium in which vector bosons and charged leptons are in thermal equilibrium may be a suitable application of the concepts developed in this study that may be worthwhile to study further. The effective field theory approach may complement the study of neutrinos [37] and axions in a medium including condensates [38] with kinetic or Boltzman equations and allow one to obtain off-diagonal

correlation functions, namely coherences, not just populations. We expect many features of the results found in this study to be common to other field-mixing scenarios; for example, we conjecture that the emergence of the long-lived coherence (off-diagonal correlation functions) approach to thermalization and quantum beats, as a result of the interference between (quasi)normal modes in the medium, is a robust consequence of field mixing that may yield to novel phenomena, and plausible observational consequences, worthy of further exploration.

Among further questions that remain to be addressed in future studies are the issues of renormalizability; in particular, if the off-diagonal matrix elements of the self-energy featuredivergences,renormalizingthemmaynecessitateoffdiagonal counterterms in the bare Lagrangian. This would call for direct mixing terms (such as an off-diagonal mass matrix) to be included in the bare Lagrangian. These aspects must be studied on a model dependent basis, since the renormalization aspects are directly related to the type of operators O_a½χ. Furthermore, we have assumed that $\langle O$½χi ¼ 0; however, a nonvanishing expectation value of this operator in the medium would require introducing tadpole terms that may lead to condensates of the fields ϕ_a. All of these questions, while interesting in their own right, remain for further study.

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APPENDIX A: SINGLE SPECIES

In this appendix we gather the results of the WeisskopfWigner approximation in the simpler case of one species to highlight the main aspects associated with the fulfillment of unitarity and the differences between the exact results via Laplace transform and the Markov approximation. For a single species ϕ we have

$$\dot{C}_\phi(t) = -iX_\kappa \langle \phi j H_I(t) P_j \kappa i C_\kappa(t); \quad (A1)$$

$$\dot{C}_\kappa(t) = -iC_\phi(t)\langle \kappa j H_I(t) P_j \phi i; \quad (A2)$$

where the sum over κ is over all the intermediate states coupled to $j\phi i$ via H_I.

Consider the initial value problem in which at time $t = 0$ the state of the system $j\Psi(t = 0)i = j\phi i$, i.e.,

$$C_\phi(0) = 1; \qquad C_\kappa(0) = 0: \quad (A3)$$

We can solve Eq. (A2) and then use the solution in Eq. (A1) to find

$$C_\kappa(t) = -iZ_0 \langle \kappa j H_I(t_0) P_j \phi i C_\phi(t_0) dt_0; \quad (A4)$$

$$\dot{C}_\phi(t) = -Z_0 \int^t \sigma(t - t^0) C_\phi(t^0) dt^0; \quad (A5)$$

where

$$\sigma(t - t_0) = X_\kappa \langle \phi j H_I j \kappa i j_2 e^{i(E_\phi - E_\kappa)(t-t_0)}$$

$$\equiv \int_{-\infty}^{\infty} \mu(k_0) e^{-i(k_0 - E_\phi)(t-t')} dk_0; \quad (A6)$$

and we introduced the spectral density

$$\mu(k_0) = X_\kappa \langle \phi j H_I j \kappa i j_2 \delta(k_0 - E_\kappa): \quad (A7)$$

Inserting the solution for $C_\phi(t)$ into Eq. (A4) one obtains the time evolution of amplitudes $C_\kappa(t)$ from which we can computeThe set of equations$\dot{C}_\kappa(t)$, namely the time dependent probability toκi. (A1) and (A2), together with the populate the state j

Hermiticity of the interaction Hamiltonian H_I, yields

$$\int dt |C_\phi(t)|^2 + X_\kappa |C_\kappa(t)|^2 = 0; \quad (A8)$$

which along with the initial conditions (A3) leads to the unitarity relation

$$|C_\phi(t)|^2 + X_\kappa |C_\kappa(t)|^2 = 1: \quad (A9)$$

1. Exact solution of Eq. (A5)

The integro-differential equation (A5) for $C_\phi(t)$ can be solved by Laplace transform. Introducing the Laplace variable s and the Laplace transform of $C_\phi(t)$ as $C_\phi(s)$, with the initial condition $C_\phi(t=0)=1$, we find

$$C_\phi(s) = \frac{1}{s} + \int_{-\infty}^{\infty} dk_0 \, \frac{|\mu(k_0)|^2}{s + i\omega(k_0) - E_\phi}\, C_\phi(s) \tag{A10}$$

with solution

$$C_\phi(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds \; C_\phi(s) e^{st}; \tag{A11}$$

where $\epsilon \to 0^+$ determines the Bromwich contour in the complex s plane parallel to the imaginary axis to the right of all the singularities. Writing $s = i(\omega - i\epsilon)$ we find

$$C_\phi(t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega - \omega_0(k_0) + i\epsilon}. \tag{A12}$$

The integral is carried out by closing the contour in the lower half ω plane. In the free case where $\mu(k_0)=0$, the pole is located at $\omega = -i\epsilon \to 0$, leading to a constant $C_\phi = 1$. In perturbation theory there is a complex pole very near the origin that can be obtained directly by expanding the integral in the denominator near $\omega = 0$. We find

$$\int_{-\infty}^{\infty} dk_0^0 \frac{|\mu(k_0)|^2}{\omega - \omega(k_0) + i\epsilon} \simeq \frac{|\mu(k_0)|^2}{E_\phi - \omega(k_0) + i\frac{\Gamma_\phi}{2}}; \tag{A13}$$

where

$$\Delta E_\phi = P\int_{-\infty}^{\infty} d\omega' \frac{\mu(k_0)}{E_\phi - \omega(k_0)}; \tag{A14}$$

$$\Gamma_\phi = 2\pi\mu(E_\phi); \tag{A15}$$

$$Z = \frac{1}{1 + \mu(k_0)^2}; \tag{A16}$$

$$z_\phi = P\int_{-\infty}^{\infty} dk \, (E_\phi - k)$$

and P stands for the principal part. The term ΔE_ϕ is recognized as the energy shift while Γ_ϕ is seen to be the decay rate as found from Fermi's golden rule. The long time limit of $C_\phi(t)$ is determined by this complex pole near the origin leading to the asymptotic behavior to leading order in the coupling

$$C_\phi(t) \simeq Z_\phi e^{-i\Delta E_\phi t} e^{-\frac{\Gamma_\phi}{2}t} \quad (t); \tag{A17}$$

where

$$Z_\phi = \frac{1}{1 - z_\phi} \simeq 1 - z_\phi = \frac{\partial}{\partial E_\phi}[E_\phi + \Delta(E_\phi)] \tag{A18}$$

is the wave function renormalization constant.

2. Markov approximation

The time evolution of $C_\phi(t)$ determined by Eq. (A5) is slow in the sense that the timescale is determined by a weak coupling kernel $\sigma \propto H_I^2$. This suggests to use a Markovian approximation in terms of a consistent expansion in derivatives of C_ϕ. For this purpose, let us define

$$W(t;t_0) = Z_0 \int_{t_0}^{t} \sigma(t - t')\, dt' \tag{A19}$$

so that

$$\sigma(t - t_0) = \frac{d}{dt_0} W(t;t_0); \qquad W(t;0) = 0: \tag{A20}$$

Integrating by parts in Eq. (A5) we obtain

$$Z_0 \int_0^t \sigma(t - t')C_\phi(t')dt' = W(t;t)C_\phi(t) - Z_0 \int_0^t W(t;t')\frac{d}{dt'}C_\phi(t')dt'. \tag{A21}$$

The second term on the right-hand side is formally of fourth

$$' \simeq H_I^2 \qquad \phi\ t \simeq H_I^2$$

fore, it can be neglected to

leading orderorder in H_I because $W\eth t;t\Þ$ and $C^{\cdot}\ \eth O\Þ$

$\eth H^{I}{}_2\ \Þ$. Up to; there-(A5) leading order in this Markovian

approximation Eq. becomes

$$C^{\cdot}{}_\phi\eth t\Þ\ \Þ\ W_0\eth t;t\ÞC_\phi\eth t\Þ\ ¼\ 0; \qquad \eth A22\Þ$$

with the solution

$$C_\phi\eth t\Þ\ ¼\ e^{-i}R_{0^t}{}^{E\eth t_0\Þdt_0}; \qquad E\eth t\Þ\ ¼\ -iW\eth t;t\Þ: \qquad \eth A23\Þ$$

we findNote that in general $E\eth t\Þ$ is complex. To leading order in H^2_I

$$E\eth t\Þ\ ¼\ -iZ_0{}^t\sigma\eth t-t^0\Þdt^0$$

$$¼\ Z_{-\infty}^{\infty} dk_0\,\eth E_\phi\ -\ k_0\Þ\ \frac{\mu\eth k_0\Þ}{1}{}^{½}\ -\ e^{-i\eth k_0-E_\phi\Þt} \qquad \eth A24\Þ$$

so that

$$Z_0^t E\eth t^0\Þdt^0\ ¼\ t\int_{-\infty}^{\infty} dk_0\,\frac{\mu\eth k_0\Þ}{\eth E_\phi-k_0\Þ}\ 1-\frac{\sin\eth k_0-E_\phi\Þt}{\eth k_0-E_\phi\Þt}$$

$$-\ i\int_{-\infty}^{\infty} dk_0\,\frac{\mu\eth k_0\Þ}{E_\phi-k_0}{}^{2}\,½1-\cos½\eth k_0-E_\phi\Þt\ :\ \eth$$

$$\Þ \qquad \eth A25\Þ$$

Asymptotically as $t\to\infty$, these integrals approach

$$\int_{-\infty}^{\infty} dk_0\,\mu\ \frac{\eth\ \ \Þ}{k_0}\ \ 1-\frac{\eth}{\sin}\frac{\ }{k_0-E_\phi}\ \frac{\Þt}{\Þt}$$

$$-\infty \qquad \eth E_\phi\ \ k_0\Þ \qquad \eth k_0\ \ E_\phi$$

$$\frac{P}{t\to\infty}\int_{-\infty}^{\infty} dk_0\,\frac{\mu\ k^0}{\eth E_\phi\ k_0\Þ\Þ}; \qquad \eth A26\Þ$$

$$\infty$$

$$\int_{-\infty} dk_0\,\frac{\mu\eth k_0\Þ}{\eth E_\phi\ -\ k_0\Þ^{2}}{}^{½}1\ -\ \cos½\eth k_0\ -\ E_\phi\Þt$$

$$t\to \qquad \int_{\infty}\ \ 0\ \frac{0}{-\ \ 2} \qquad \mu\eth k\ \Þ_0$$

$$: \qquad \eth A27\Þ$$

$$\longrightarrow\pi t\mu\eth E_\phi\Þ\ \Þ\ P\int_{-\infty} dk\ \eth E_\phi\ \ k\ \Þ$$

$$\infty$$

Using these results we find that in the long time limit

$$-iE\eth t^0\Þdt^0 \qquad \int_0^t \to -i\Delta E_\phi t\ -t-z_\phi; \qquad \frac{\Gamma_\phi}{2} \qquad \eth A28\Þ$$

where $\Delta E_\phi;\Gamma_\phi;z_\phi$ are given by Eqs. (A14)–(A16) and (A18). From this we read off

$$C_\phi\eth t\Þ\ ¼\ Z_\phi e^{-i\Delta E_\phi t}e^{-\ _2 t;} \qquad \eth A29\Þ$$

where we approximated $e^{-z_\phi}\simeq 1-z_\phi ¼ Z_\phi$ up to second order in perturbation theory. This is in complete agreement with the asymptotic result from the exact solution Eq. (A17) obtained via the Laplace transform.

3. Taking the long time limit before integration

We now compare the results obtained above with those obtained with yet another approximation: taking the long time limit in $W\eth t;t\Þ$ in (A22) before integrating this evolution equation:

$$W\eth t;t\Þ\overset{t\to\infty}{\longrightarrow}\int_{-\infty}^{\infty}\mu(k_0)\int_0^{\infty} e^{-i\eth k_0-E_\phi\Þ\tau}d\tau dk_0\ ¼Z\ \infty P\ \eth\ \mu\eth k_0\Þ$$

$$\Þdk_0-i\pi\mu\eth E_\phi\Þ; \qquad \eth A30\Þ\ i$$

$$-\infty \qquad E_\phi-k_0$$

therefore, under this approximation the solution of (A22) is

$$\Gamma_\Phi$$

$$C_\phi(t) = e^{-i\Delta E_\phi t} e^{-\frac{\Gamma_\phi}{2}t}. \quad \text{(A31)}$$

Obviously, the main difference with the solutions (A17) and (A22) is the lack of wave function renormalization in (A31). Therefore, we conclude that the Markov approximation leading to (A22) reproduces the exact result obtained from Laplace transform; however, the further approximation of replacing $W(t;t')$ by its infinite time limit (A30) in the Markovian equation (A22) misses the wave function renormalization.

4. Unitarity

Because of the exponential decay of the amplitude of the initial state, the unitarity condition (A9) entails that in the long time limit

$$\sum_\kappa |C_\kappa(\infty)|^2 = 1. \quad \text{(A32)}$$

We now address how this constraint is fulfilled. The coefficients $C_\kappa(t)$ are given by Eq. (A4).

Introducing the leading order result (A31) [since $Z_\phi = 1 + O(H^2)$] into Eq. (A4) for the coefficients C_κ, we find to leading order

$$|C_\kappa(\infty)|^2 = \frac{|\langle \kappa|H_I|\phi\rangle|^2}{(E_{\phi_R} - E_\kappa)^2 + \frac{\Gamma_\phi^2}{4}}. \quad \text{(A33)}$$

This expression can be interpreted as follows. If $|\kappa\rangle$ is resonant with the state $|\phi\rangle$ with, are $E_\kappa \sim$ "populated" E_{ϕ_R}, i.e., those nearly Γ_ϕ centered at" $|\phi\rangle$ with an is an unstable state, the states $|$

amplitude $\propto 1/\Gamma_\phi$ within a band of width E_ϕ^R. Furthermore,

$$\int^{\infty} \frac{\mu(k_0)}{(E_{\phi_R} - k_0)^2 + \frac{\Gamma_\phi^2}{4}}\, dk_0 \simeq 1$$

$$|C_\kappa(\infty)|^2 = \int_{-\infty}^{} \frac{\Gamma_\phi}{(E_\phi - k_0)^2 + \frac{\Gamma_\phi^2}{4}}\, \mu(k_0)\, dk_0 \quad \text{(A34)}$$

where we have written

$$\frac{1}{(E_{\phi_R} - k_0 + \frac{\Gamma_\phi}{2} i)} \frac{1}{(E_{\phi_R} - k_0 - \frac{\Gamma_\phi}{2} i)} = \frac{\Gamma_\phi}{(E_{\phi_R} - k_0)^2 + \frac{\Gamma_\phi^2}{4}}; \quad \text{(A35)}$$

and in the narrow width limit $\Gamma_\phi \to 0$ we replace

$$\frac{\Gamma_\phi}{(E_{\phi_R} - k_0)^2 + \frac{\Gamma_\phi^2}{4}} \to 2\pi\delta(E_\phi - k_0) \quad \text{(A36)}$$

and used the result (A15) to obtain (A34). Unitarity entails a probability flow from the initial toward the final excited states.

APPENDIX B: LEHMANN REPRESENTATION OF CORRELATION FUNCTIONS

The correlation functions $G^{>}_{ab}(x-y)$; $G^{<}_{ab}(x-y)$ can be written in an exact Lehmann (spectral) representation that is useful to include in the equations of motion:

$$G^{>}_{ab}(x-y) = \frac{1}{Z_\chi}\sum_\chi \mathrm{Tr}\, e^{-\beta H_\chi} O_a(x) O_b(y); \quad \text{(B1)}$$

$$G^{<}_{ab}(x-y) = \frac{1}{Z_\chi}\sum_\chi \mathrm{Tr}\, e^{-\beta H_\chi} O_b(y) O_a(x); \quad \text{(B2)}$$

where $Z_\chi = \mathrm{Tr}\, e^{-\beta H_\chi}$ and $O_a(x;t') = e^{iH_\chi t'} e^{-i\vec{P}\cdot\vec{x}} O_a(0) \times e^{i\vec{P}\cdot\vec{x}} e^{-iH_\chi t'}$. In terms of a complete set of simultaneous eigenstates of $H_\chi;\vec{P}$, namely $(H_\chi;\vec{P})|n\rangle = (E_n;\vec{P}_n)|n\rangle$, and by inserting the identity in this basis, we find

$$\frac{1}{} \quad e^{-\beta E_n} e \quad {}^{m} \quad G_{>ab}\eth x_1 - x_2 \text{Þ} \text{¼} i\eth E_n - E \text{Þ} \eth t_1 - t_2 \text{Þ} e - i\eth \vec{P}_{n} - \vec{P}_{m} \text{Þ} \cdot \eth \vec{x}_{1} - \vec{x}_{2} \text{Þ} hnjO_a \eth 0 \text{Þ} jmihmjO_b \eth 0 \text{Þ} jni; \qquad \eth B3 \text{Þ}$$

$$Z_\chi X_{n;m}$$

$$G_{<ab}\eth x_1 - x_2 \text{Þ} \text{¼} \underline{\quad} e^{-\beta E_n} e^{-i\eth E_n - E_m \text{Þ} \eth t_1 - t_2 \text{Þ} e i\eth \vec{P}_{n} - \vec{P}_{m} \text{Þ} \cdot \eth \vec{x}_{1} - \vec{x}_{2} \text{Þ} hnjO_b \eth 0 \text{Þ} jmihmjO_a \eth 0 \text{Þ} jni: \quad \eth B4 \text{Þ}$$

$$Z_\chi X_{n;m}$$

These representations may be written in terms of spectral densities, by introducing

$$\rho_{>} \quad \vec{} \quad \eth 2\pi \text{Þ}^4 \quad {}^{-\beta E_n} O \quad O \quad \delta - \quad - \quad \delta_3 \vec{} - \vec{} - \vec{}$$

$$ab\eth\ 0 \quad \text{Þ} \text{¼} \quad Z_\chi X_{n;m} \quad h\, j\, a\eth\ \text{Þ} j\, ih\, j\, b\eth\ \text{Þ} j\, i\, \eth\ 0 \quad \eth\ m \quad n\text{ÞÞ} \quad \eth \quad \eth\ m \quad n\text{ÞÞ} \quad \eth \quad \text{Þ}$$

$$\rho_{<} \quad \vec{} \quad \eth 2\pi \text{Þ}^4 \quad {}^{-\beta E_n} O \quad O \quad \delta - - \quad \delta_3 \vec{} - \vec{} - \vec{}$$

$$ab\eth\ 0 \quad \text{Þ} \text{¼} \quad Z_\chi X_{n;m} \quad h\, j\, b\eth\ \text{Þ} j\, ih\, j\, a\eth\ \text{Þ} j\, i\, \eth\ 0 \quad \eth\ n \quad m\text{ÞÞ} \quad \eth \quad \eth\ n \quad m\text{ÞÞ} \quad \eth \quad \text{Þ}$$

in terms of which

$$_{>} - \quad d_4 k\, \rho_{>} \quad \vec{} \quad {}^{-ik_0 \eth t_1 - t_2 \text{Þ}\ i\vec{k} \cdot \eth \vec{x}_{1} - \vec{x}_{2} \text{Þ}}$$

$$ab\eth\ 1 \quad 2\text{Þ} \text{¼} Z\eth 2\pi \text{Þ}^4 \quad ab\eth\ 0 \quad \text{Þ} \qquad \eth \quad \text{Þ}$$

$$_{<} - \quad d_4 k \quad _{<} \quad \vec{} \quad {}^{-ik_0 \eth t_1 - t_2 \text{Þ}\ i\vec{k} \cdot \eth \vec{x}_{1} - \vec{x}_{2} \text{Þ}}\, \rho$$

$$1$$

$$k\,;k \quad \underline{\quad} \quad e \quad n \quad 0\, m\, m \quad 0\, n \quad k \quad E \quad E \quad k \quad P \quad P \quad ; \qquad B5$$

$$k\,;k \quad \underline{\quad} \quad e \quad n \quad 0\, m\, m \quad 0\, n \quad k \quad E \quad E \quad k \quad P \quad P \quad ; \qquad B6$$

$$G \quad x \quad x \quad \underline{\quad} \quad k\,;k\, e \quad e \quad ; \qquad B7$$

$$G_{ab}(x_1, x_2) = \int \frac{d^2\pi}{(2\pi)^4} P_{ab}(k_0; k)\, e^{\qquad} e^{\qquad} : \tag{B8}$$

Relabeling $n \leftrightarrow m$ and using the k_0 delta function in (B6), we find the generalized Kubo-Martin-Schwinger condition [55]

$$\rho_{<ab}(k_0; \vec{k}) = e^{-\beta k_0} \rho_{>ab}(k_0; \vec{k}) : \tag{B9}$$

Introducing the spectral density

$$\rho_{ab}(k_0; \vec{k}) = \rho_{>ab}(k_0; \vec{k}) - \rho_{<ab}(k_0; \vec{k}) = \rho_{>ab}(k_0; \vec{k})(1 - e^{-\beta k_0}); \tag{B10}$$

it follows that

$$\rho_{ab}{}^{>}(k_0; \vec{k}) = (1 + n(k_0))\rho_{ab}(k_0; \vec{k}); \quad \rho^{<}{}_{ab}(k_0; \vec{k}) = n(k_0)\rho_{ab}(k_0; \vec{k}); \tag{B11}$$

where

$$n(k_0) = \frac{1}{e^{\beta k} - 1} : \tag{B12}$$

Therefore, the spatial Fourier transform of the self-energy matrix (3.37) and the noise kernel (3.36) can be written as

$$\Sigma_{ab}(k; t - t_0) = -i\int dk^2\pi_0 \rho_{ab}(k_0; k) e^{-ik_0(t-t_0)}; \tag{B13}$$

$$N_{ab}(k; t - t_0) = \frac{1}{2}\int (dk^2\pi_0)\rho_{ab}(k_0; k)\coth\beta \frac{}{2} k_0 e^{-ik_0(t-t_0)} : \tag{B14}$$

This is the general relation between the self-energy and the noise correlation function commonly determined by the spectral density $\rho_{ab}(k_0; k)$, a direct consequence of the fluctuation-dissipation relation as a result of the bath being in thermal equilibrium. spectral densityAssuming rotational invariance implies that $\rho_{ab}(k_0; \vec{k}) = \rho_{ab}(k_0; k)$, in particular the diagonal matrix elements of the

$$\rho_{aa}(k_0; k) = \frac{(2\pi)^4}{Z_\chi} \sum_{n,m} e^{-\beta E_n} \langle n|j(0)|m\rangle\langle m|j(0)|n\rangle \tfrac{1}{2}(2\pi)^4 \delta(k_0 - (E_m - E_n)) \delta(k_0 - (E_n - E_m)) \delta^3(\vec{k} - (\vec{P}_m - \vec{P}_n))$$
$$= -\rho_{aa}(-k_0; k); \qquad \text{no sum over a:} \tag{B15}$$

The assumption of rotational invariance also applies to correlation functions of pseudoscalar operators (relevant for axions) in a thermal equilibrium density matrix that is invariant under rotations because these are bilinear in the operators and, hence, are invariant under $\vec{k} \to -\vec{k}$.

We note that because the operators O_a are Hermitian, it follows that$_{ab}(0) = \rho^{\gtrless}_{ab}(k_0; k) = \rho^{\gtrless}_{ba}(k_0; k)$. and, consequently, $\rho(k_0; k) =$

APPENDIX C: LAPLACE GREEN'S FUNCTION

Consider the matrix

$$\mathcal{M} = \begin{matrix} M & M \\ M_{11} & M_{12} \end{matrix}; \tag{C1}$$

whose (right and left) inverse is

$$\mathcal{M}^{-1} = \det(\mathcal{M})^{-1}\begin{matrix} M_{22} & -M \\ -M & M_{11} \end{matrix}; \tag{C2}$$

In terms of the variables (2.31)–(2.33) it follows that

$$M = M^{-}_{D}{}^{2}\gamma_{D2}\,\alpha\, M^{-}_{D2} - \beta_{D2}\,\alpha;$$

$$\det[M = M^{-} - \_D2M^{-} + D\_2; \quad (C3)$$

where we used the relation (2.34).

Therefore, the inverse of the matrix (C1) is given by

$$M^{-1} = \det\_\_\_\_\_1[MM^{-} 1 - \_D2\,\alpha\gamma - \beta\alpha: \quad (C4)$$

Writing

$$M^{-} = 12\_M^{-} + \overline{D}_2 + \overline{1}_2M^{-} - \overline{D}_2;$$

$$D = M^{-} + D2M^{-} - D2M^{-} 1 - {}_{D2} - M^{-} 1 + {}_{D2}$$

yields

$$M^{-1} = M - \underline{\quad}_{D2} + M^{-}P + {}_{D2}; \qquad P = 2\_(1 - R); \quad (C5)$$

with

$$R = \begin{matrix}\alpha & \beta \\ \gamma & -\alpha;\end{matrix} \qquad R^2 = 1; \quad (C6)$$

where the last equality follows from the identity (2.34). Therefore, the matrices P are projectors, namely

$$P^2 = P; \quad (C7)$$

hence, their eigenvalues are 0,1.

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