Nonequilibrium dynamics of axionlike particles: The quantum master equation

Shuyang Cao* and Daniel Boyanovsky†

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA

(Received 9 December 2022; accepted 24 February 2023; published 13 March 2023)

We study the nonequilibrium dynamics of axionlike particles (ALP) coupled to Standard Model degrees of freedom in thermal equilibrium. The quantum master equation (QME) for the ALP reduced density matrix is derived to leading order in the coupling of the ALP to the thermal bath, but to all orders of the bath couplings to degrees of freedom within or beyond the Standard Model other than the ALP. The QME describes the damped oscillation dynamics of an initial misaligned ALP condensate, thermalization with the bath, decoherence, and entropy production within a unifying framework. The ALP energy density $E(t)$ features two components: a "cold" component from the misaligned condensate and a "hot" component from thermalization with the bath, with $E(t) = E_c e^{-\gamma(T)t} + E_h(1-e^{-\gamma(T)t})$ thus providing a "mixed dark matter" scenario. Relaxation of the ALP condensate, thermalization, decoherence, and entropy production occur on similar timescales. An explicit example with ALP-photon coupling, valid post recombination yields a relaxation rate $\gamma(T)$ with a substantial enhancement from thermal emission and absorption. A misaligned condensate is decaying at least since recombination and on the same timescale thermalizing with the cosmic microwave background (CMB). Possible consequences for birefringence of the CMB and ALP contribution to the effective number of ultrarelativistic species and galaxy formation are discussed.

DOI: 10.1103/PhysRevD.107.063518

I. INTRODUCTION

The axion, introduced in quantum chromodynamics (QCD) as a solution of the strong CP problem [1–3] may be produced nonthermally in the early Universe, for example by a misalignment mechanism and is recognized as a potentially viable cold dark matter candidate [4–6]. Extensions beyond the standard model can accommodate pseudoscalar particles with properties similar to the QCD axion, namely axionlike particles (ALP) which can also be suitable dark matter candidates [7–11], in particular as candidates for ultra light dark matter [12,13]. Constraints on the mass and couplings of ultralight ALP [9–11,14] are being established by various experiments [15–17]. There are two important features that characterize ALP, (i) a misalignment mechanism results in damped coherent oscillations of the expectation value of the ALP field which gives rise to the contribution to the energy density as a cold dark matter component [4–6,9–11,18], (ii) its pseudoscalar nature leads to an interaction between the ALP and photons or gluons via pseudoscalar composite operators of gauge fields, such as $\vec{E}\cdot\vec{B}$ in the case of the ALP-photon interaction and $G^{\mu\nu;b}\tilde{G}_{\mu\nu;b}$ in the case of gluons, which allows an ALP to decay into two photons or gluons. The effect of this decay process in the evolution of ALP condensates has been studied in Refs. [19–22] including stimulated decay in a photon background. The damping of an ALP condensate via a "friction" term in its equation of motion has been studied in Refs. [23–25], and thermalization of ALP has been studied in Refs. [26,27], these references focused on either damping via friction or thermalization as unrelated independent processes. A recent study [28] has recognized the common origin of these two seemingly different processes by obtaining the nonequilibrium effective action that determines the time evolution of the reduced ALP density matrix. This study showed that damping of a misaligned ALP condensate and thermalization are two complementary aspects and are linked by the fluctuation dissipation relation, a fundamental and ubiquitous property of a bath in thermal equilibrium. This reference also established that both

* shuyang.cao@pitt.edu
† boyan@pitt.edu

processes contribute to the ALP energy density, an important aspect if the ALP are suitable dark matter candidates.

Decay and thermalization of an ALP condensate post recombination may have a profound impact on birefringence of the cosmic microwave background (CMB) if its origin is the electromagnetic coupling of a pseudoscalar ALP [29–32].

In Ref. [28], the nonequilibrium dynamics of ALP was studied to leading order in the coupling of the ALP to other degrees of freedom treated as a bath in thermal equilibrium by implementing the in-in Schwinger-Keldysh formulation of nonequilibrium quantum field theory to obtain the effective action. The equations of motion for the ALP

2470-0010=2023=107(6)=063518(15)

© 2023 American Physical Society

obtained from the effective action are retarded and akin to a Langevin equation with a friction term determined by the retarded self-energy and a noise term related to the self energy via the fluctuation-dissipation relation. This relation is a consequence of the bath degrees of freedom being in thermal equilibrium. An important result of the Langevin nature of the effective equations of motion is a direct relationship between the damping of an ALP coherent condensate and thermalization of its fluctuations. This result was found to be general to leading order in the ALP coupling to the bath degrees of freedom but to all orders in the couplings of these "environmental" fields to any other field within or beyond the standard model other than the ALP and is a corollary of the fluctuation dissipation relation. An analysis of the coupling of ALP to the CMB post recombination in this article also revealed a substantial enhancement of the damping and thermalization rates if the ALP is an ultralight dark matter candidate as well as unexpected possible phase transitions and exotic new phases.

A. Motivation and objectives

The results in Ref. [28] and their possible cosmological consequences, motivate us to seek a complementary formulation of the nonequilibrium dynamics of ALP coupled to "environmental" degrees of freedom in equilibrium that does not rely on the in-in Schwinger-Keldysh approach to the effective action, thereby offering an alternative and independent assessment of the nonequilibrium dynamics of ALP coupled to a thermal environment.

In this article we adapt methods of quantum optics and quantum information to study the nonequilibrium dynamics of ALP fields implementing a quantum master equation approach ubiquitous in the treatment of quantum open systems [33–37]. The quantum master equation describes the time evolution of the ALP reduced density matrix, it has been implemented in particle physics [38–44] and cosmology [45–50] and has proven to be a powerful and reliable method to study nonequilibrium dynamics.

The main objectives of this article are: (i) to scrutinize the results obtained in Ref. [28] with an alternative and independent method, (ii) to inquire on complementary aspects of the time evolution of the reduced density matrix, in particular the evolution of coherences, which yield supplementary information on thermalization and decoherence, and (iii) to compare the timescales of decoherence to those of damping of the misaligned condensate and thermalization.

In this study we are not concerned with bounds on couplings and or masses of the putative ALP but focus on fundamental aspects of the nonequilibrium evolution of its density matrix including misaligned initial states. Furthermore, while our ultimate objective is to study the nonequilibrium dynamics in an expanding cosmology, we initiate this program as a prelude by focusing on Minkowski space time.

B. Brief summary of results

We consider an ALP field in interaction with Standard Model degrees of freedom which are considered to be in thermal equilibrium. In Sec. II, we obtain the QME for the reduced density matrix of the ALP up to second order in the coupling of the ALP to these degrees of freedom, but to all orders in the coupling of the bath degrees of freedom to fields within or beyond the Standard Model different from the ALP under well defined approximations. The resulting QME is of the Lindblad form [33–37], it is obtained up to second order in the ALP coupling to Standard Model degrees of freedom and to all orders in the couplings of these degrees of freedom to any other field within or beyond the Standard Model except for the ALP, and includes misaligned initial conditions for the ALP field. The QME describes the damping of the misaligned condensate, thermalization with the bath and decoherence with a concomitant entropy production. The ALP energy density describes a "mixed" dark matter scenario with a cold component E_c from the misaligned coherent condensate, and a "hot" component E_h from thermalization with the bath, with the total energy density interpolating between the cold and hot components as $E(t) \simeq E_c e^{-\gamma(T)t} + E_h(1 - e^{-\gamma(T)t})$, where the relaxation rate $\gamma(T)$ also describes the decoherence rate. We study in detail ALP coupling to the CMB post recombination, if the ALP is an ultralight dark matter candidate there is a substantial enhancement of the relaxation rate, its long wavelength limit is given by $\gamma(T) = g^2 m_a^2 T/16\pi$. The results suggest that if $\gamma(T) < H_0$ the misaligned condensate has been decaying at least since recombination and thermalizing with the CMB

onasimilartimescale.Therefore,ifcosmicbirefringenceisa consequence of the CMB coupling to a pseudoscalar ALP, the rotation angle since the surface of last scattering should feature a thermal spectrum of fluctuations.

II. THE QUANTUM MASTER EQUATION

We study the time evolution of the reduced density matrix of an axionlike field $a(x)$ coupled to generic fields $\chi(x)$ to which we refer as "environmental" fields via a pseudoscalar operator $O_\chi(x)$, with the Lagrangian density

$$\mathcal{L}(a;a,\chi) = \frac{1}{2}\partial_\mu a\,\partial^\mu a - \frac{1}{2}m_a^2 a^2(x) - g a(x) O_\chi(x) + \mathcal{L}_\chi \tag{2.1}$$

where $\mathcal{L}_\chi$ is the Lagrangian density describing the "environmental" fields χ, these fields could be the electromagnetic field, fermion or gluon fields and themselves be coupled to other degrees of freedom within or beyond the Standard Model.

The Lagrangian density (2.1) describes several relevant couplings of ALP, with possible operators $O_\chi(\vec{x})$ being

$$O_\chi(\vec{x}) = \vec{E}(x)\cdot\vec{B}(x); G_{\mu\nu;b}(x)\tilde{G}^{\mu\nu;b}(x); \bar{\Psi}(x)\gamma_{;b5}\Psi(x)_{;b} \tag{2.2}$$

where $\vec{E}$, $\vec{B}$ are the electromagnetic fields, $G^{\mu\nu}$; $\tilde{G}^{\mu\nu}$ are the gluon field strength tensor and its dual respectively, and $\Psi(x)$ a fermionic field. These degrees of freedom are assumed to be in thermal equilibrium. We will first treat these fields generically denoting them as χ fields, and after obtaining the general form of the quantum master equation up to $O(g^2)$, we will focus on the relevant case with $O_\chi(\vec{x}) = \vec{E}(x)\cdot\vec{B}(x)$ since the interaction of ALP fields with the CMB could have potentially observable consequences, such as birefringence [29–32], a rotation of the polarization plane which, in contrast to Faraday rotation, is independent of the frequency with tantalizing detection possibilities [30–32].

The interaction of ALP with photons and gluons via couplings of the form $g a(x)\vec{E}(x)\cdot\vec{B}(x); g_s a(x) G^{\mu\nu;b}(x)$ $\tilde{G}^{\mu\nu;b}(x)$ are not renormalizable because the respective couplings $g; g_s$ feature dimensions $1/(\text{energy})$, an aspect that has important consequences [28] discussed below, that at the fundamental level, indicate that the Lagrangian density (2.1) describes an effective field theory valid below some cutoff scale.

Upon evolving the total initial density matrix in time, the degrees of freedom χ with the generic operator O_χ are traced over to obtain a reduced density matrix for $a(x)$ which obeys a quantum master equation. We obtain this equation in the general case valid to order g^2 in the ALP coupling to the bath, and to all orders in the couplings of the bath degrees of freedom to any other degree of freedom within or beyond the standard model except for the ALP under a set of approximations that are spelled out in detail. Whereas our ultimate objective is to pursue this approach in an expanding cosmology, here we begin this program by first carrying it out in Minkowski space time.

The quantum master equation in a Lindblad form [35–38] has recently received attention in applications to high energy physics [39–44] and cosmology [45–49]. This formulation begins with the time evolution of an initial density matrix that describes the total system of fields a, χ, which is given by

$$\hat{\rho}(t) = e^{-iHt}\hat{\rho}(0)e^{iHt}; \tag{2.2}$$

with H the total Hamiltonian

$$H = H_{0a} + H_\chi + H_I \equiv H_0 + H_I; \tag{2.3}$$

where H_{0a} is the free field Hamiltonian for the ALP, H_χ is the Hamiltonian of the χ degrees of freedom including their couplings to degrees of freedom within or beyond the Standard Model except the ALP, and $H_I = g\int d^3x a(x) O_\chi(x)$ is the coupling between the ALP and the bath degrees of freedom obtained from the Lagrangian density (2.1).

We consider an initial factorized density matrix

$$\rho(0) = \rho_a(0)\otimes\rho_\chi(0); \tag{2.4}$$

where the χ fields are in thermal equilibrium at temperature $T = 1/\beta$, namely

$$e^{-\beta H_\chi}$$

$$\rho_\chi(0) = \mathrm{Tr} e^{-\beta H_\chi}; \quad (2.5)$$

and for the ALP field we chose an initial density matrix describing a "misaligned" initial condition with a nonvanishing expectation value of the ALP field. This is implemented in terms of coherent states of free fields as follows. Quantizing the free ALP field at the initial time $t = 0$ in a finite volume V as

$$a(\vec{x}, t=0) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[b_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right];$$

$$[b_{\vec{k}}, b^\dagger_{\vec{k}_0}] = \delta_{\vec{k},\vec{k}_0}; \quad (2.6)$$

and the vacuum state defined as

$$b_{\vec{k}} |0\rangle = 0: \quad (2.7)$$

A coherent state is given by

$$|\Delta\rangle = \prod_{\vec{k}} e^{-\frac{1}{2}|\Delta_{\vec{k}}|^2} e^{-\Delta_{\vec{k}} b^\dagger_{\vec{k}}} |0\rangle; \quad (2.8)$$

it is an eigenstate of the annihilation operator,

$$b_{\vec{k}} |\Delta\rangle = \Delta_{\vec{k}} |\Delta\rangle; \quad (2.9)$$

and describes a Poisson distribution of quanta of the free ALP field. The expectation values of the ALP field and its canonical momentum in this coherent state are $\langle\Delta|a(\vec{x}, 0)|\Delta\rangle = a(\vec{x}, 0)$

$$\frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[\Delta_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \Delta^*_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right]; \quad (2.10)$$

$$\langle\Delta|\pi(\vec{x}, 0)|\Delta\rangle = \pi(\vec{x}, 0)$$

$$-i \sum_{\vec{k}} \sqrt{\frac{\omega_k}{2}} \left[\Delta_{\vec{k}} - \Delta^*_{\vec{k}} \right] e^{i\vec{k}\cdot\vec{x}}; \quad (2.11)$$

Hence we choose the initial density matrix for the ALP field to describe this "misaligned" initial state, namely

$$\rho_a(0) = |\Delta\rangle\langle\Delta|; \quad (2.12)$$

yielding

$$\mathrm{Tr} b_{\vec{k}} \rho_a(0) = \Delta_{\vec{k}}; \quad \mathrm{Tr} b^\dagger_{\vec{k}} b_{\vec{k}} \rho_a(0) = N_q(0) = |\Delta_{\vec{q}}|^2;$$

$$\mathrm{Tr} b_{\vec{k}} b_{-\vec{k}} \rho_a(0) = \Delta_{\vec{k}} \Delta_{-\vec{k}}; \quad \text{etc:} \quad (2.13)$$

We refer to the off-diagonal ALP density matrix elements in the occupation number basis (eigenstates of $b^\dagger_{\vec{k}} b_{\vec{k}}$), $_k$ for example $\mathrm{Tr} b_{\vec{k}} b_{-\vec{k}} \rho_a(0) = \Delta_{\vec{k}} \Delta_{-\vec{k}}$ as coherences [33,34].

A hallmark of a thermal density matrix is that these coherences vanish and the density matrix is diagonal in the occupation number basis. This observation will become important as a diagnosis of thermalization and its link to decoherence studied below. Translational invariance entails that

$$\Delta_{\vec{k}} = V \tilde{\Delta} \delta_{\vec{k},\vec{0}}; \quad (2.14)$$

therefore

$$N_q(0) = V |\tilde{\Delta}|^2 \delta_{\vec{q},\vec{0}}; \quad (2.15)$$

$$p$$

and

$$\frac{1}{2}\pi^2 + m_a^2 a^2 = |\tilde{\Delta}|^2: \quad (2.16)$$

$$2m_a$$

In the quantum master equation approach [33,34] the time evolution of the density matrix is considered in the interaction picture. With the full density matrix $\hat{\rho}(t)$ given by Eq. (2.2) the density matrix in the interaction picture is given by $\hat{\rho}_I(t) = e^{iH_0 t} \hat{\rho}(t) e^{-iH_0 t}$; (2.17) whose time evolution obeys $\dot{\hat{\rho}}_I(t) = -i[H_I(t), \hat{\rho}_I(t)]$; (2.18)

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture, $H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}$. The formal solution of Eq. (2.18) is given by

$$\hat{\rho}_I(t) = \hat{\rho}_I(0) - i\int_0^t dt_0 [H_I(t_0), \hat{\rho}_I(t_0)]. \quad (2.19)$$

This solution is inserted back into (2.18) leading to the iterative equation

$$\dot{\hat{\rho}}_I(t) = -i[H_I(t), \hat{\rho}_I(0)] - \int_0^t [H_I(t), [H_I(t_0), \hat{\rho}_I(t_0)]]dt_0. \quad (2.20)$$

This QME cannot be solved exactly, and several approximations are usually invoked, based on the following assumptions [33–36]:

(i) Factorization: the total density matrix factorizes into a direct product of the density matrix for the a field, $\hat{\rho}_a(t)$ and that of the bath of χ fields, $\hat{\rho}_\chi$, namely,

$$\hat{\rho}_I(t) = \hat{\rho}_a(t) \otimes \hat{\rho}_\chi(0); \quad (2.21)$$

where

$$\hat{\rho}_\chi(0) = \frac{e^{-\beta H_\chi}}{\mathrm{Tr} e^{-\beta H_\chi}}; \quad (2.22)$$

this assumption which implies that the bath degrees of freedom remain in thermal equilibrium, relies on that the bath is a reservoir with a large number of degrees of freedom and is not modified by its coupling to the system, hence the density matrix of the bath does not depend on time. This assumption also relies on weak coupling: if the initial density matrix is factorized, correlations between the system and the reservoir will build as a consequence of the interaction, therefore such correlations will be small for very weak coupling and may only contribute in higher orders. Factorization and its possible caveats are discussed further in Sec. IV.

The reduced density matrix for the ALP field a is obtained by taking the trace of the full density matrix over the bath degrees of freedom, which by assumption remains in thermal equilibrium, therefore

$$\hat{\rho}_a(t) = \mathrm{Tr}_\chi \hat{\rho}_I(t). \quad (2.23)$$

Upon taking the trace over the χ degrees of freedom the first term on the right-hand side of Eq. (2.20) vanishes under the assumption that the thermal density matrix of the environmental fields is even under parity, hence $\mathrm{Tr} O_\chi \hat{\rho}_\chi(0) = 0$, and we find the evolution equation for the reduced density matrix for the ALP field a in the interaction picture,

$$\begin{aligned}\dot{\hat{\rho}}_a(t) = -g^2 \int_0^t dt' \int d^3x \int d^3x' f_a(x) f_a(x') \\ \times \big[\hat{\rho}_a(t') G^>(x-x') \\ \hat{\rho}_a(t') a_I(x') G^<(x-x') \\ - a_I(x') \hat{\rho}_a(t') a_I(x') G^<(x-x') \\ - a_I(x') \hat{\rho}_a(t') a_I(x') G^>(x-x')\big] \quad (2.24)\end{aligned}$$

where we use the shorthand convention $x \equiv (\vec{x}; t)$; $x_0 \equiv (\vec{x}_0; t_0)$, and introduced the bath correlation functions

$$G^>(x-x') = \mathrm{Tr}_\chi \hat{\rho}_\chi(0) O_\chi(x) O_\chi(x'), \quad (2.25)$$

$$G^<(x-x') = \mathrm{Tr}_\chi \hat{\rho}_\chi(0) O_\chi(x') O_\chi(x). \quad (2.26)$$

The ALP field in the interaction picture $a_I(\vec{x}; t) = \int (d^3q) \int dq_0 \delta(\quad)$ features free field time evolution, namely

$$\frac{1}{(2\pi)^3} \frac{1}{2\pi} \int \rho(q_0, \vec{q}) \times e^{-iq_0(t-t_0)} e^{i\vec{q}\cdot(\vec{x}-\vec{x}_0)} \quad (2.31)$$

$$a_I(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} [b_{\vec{k}} e^{-i\omega_k t} e^{i\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}} e^{i\omega_k t} e^{-i\vec{k}\cdot\vec{x}}]; \quad (2.27)$$

$$G(x - x') = \int \frac{d^4q}{(2\pi)^4} \frac{\rho(q_0, \vec{q})}{(q_0 + i\epsilon)^2 - \vec{q}^2 - m^2}$$

$$\times e_{-iq_0\delta t-t^0\rangle}e_{iq^{\rightarrow}\cdot\delta x^{\rightarrow}-x^{\rightarrow}{}^0\rangle}; \qquad \delta^{2:32}\rangle$$

where the operators $b_{k^{\rightarrow}};b_{k^{\rightarrow}}$ do not depend on time,

and $\omega_k \overset{¼}{<} pk_2 \rangle m_{2a}ffi.$ where the spectral densities obey the relation>

(ii) Markov approximation the second approximation entails replacing $\rho_{Ia}\delta t^0\rangle \rightarrow \rho_{Ia}\delta t\rangle$ in the time inte- $\varrho\ \delta-q_0;q^{\rightarrow}\rangle ¼ \varrho\ \delta q_0;q^{\rightarrow}\rangle;$ ð2:33Þ gral. This is usually referred to as a Markov

approximation and is justified in weak coupling, andfulfilltheKubo-Martin-Schwingercondition[51] as can be seen by considering the first term in (2.24)

as an example. It can be written as $\varrho^{<}\delta q_0;q^{\rightarrow}\rangle ¼ e^{-\beta q_0}\varrho^{>}\delta q_0;q^{\rightarrow}\rangle;$ ð2:34Þ

$$-g^2a\delta x;t^{\rightarrow}\rangle Z^{t}\overline{dKdt\delta_0 t_0\rangle}\rho\hat{}_{Ia}\delta^{t_0}\rangle dt_0;$$

which is a consequence of the fieldsthermal equilibrium. Introducing the spectral densityχ being in

$$\delta\ \rangle \quad Z^{0} \quad Kt^0\equiv a\delta x^{\rightarrow}{}^0;t^{00}\rangle G^{>}\delta x^{\rightarrow}-x^{\rightarrow}{}^0;t-t^{00}\rangle dt^{00} \qquad \delta 2:28\rangle\varrho\delta q_0;q^{\rightarrow}\rangle ¼ \varrho\ \delta q_0;q^{\rightarrow}\rangle -\varrho$$

$$\delta q_0;q^{\rightarrow}\rangle; \qquad \delta 2:35\rangle$$

which upon integration by parts yields

the Kubo-Martin-Schwinger condition (2.34) leads to the following relations

$$-g^2a\delta x;t^{\rightarrow} \quad Z^{t_0} \quad \rangle K\delta t\rangle\overline{\rho\hat{}_{Ia}\delta t\rangle}\rangle g^2a\delta x;t^{\rightarrow} \qquad \rangle^{t}K\delta t^0\rangle^{d\rho\hat{}_{Ia}\delta^{t_0}\rangle dt^0} \quad \varrho^{>}\delta q_0;q^{\rightarrow}\rangle ¼ ½1\rangle n\delta q_0\rangle\varrho\delta q_0;q^{\rightarrow}\rangle \delta 2:36\rangle$$

$$0 \qquad dt_0$$

$$\delta 2:29\rangle \qquad \varrho^{<}\delta q_0;q^{\rightarrow}\rangle ¼ n\delta q_0\rangle\varrho\delta q_0;q^{\rightarrow}\rangle \qquad \delta 2:37\rangle$$

in the second term $d\rho\hat{}_I\Phi\delta t^0\rangle=dt^0 \propto g^2$ so this term where $n\delta q_0\rangle ¼ ½e^{\beta q_0}-1^{-1}$ is the Bose-Einstein yields a contribution that is formally of order g^4 and distribution function at temperature $T ¼ 1=\beta$. The can be neglected to second order. The same analysis above relations are proven in Appendix A, they are is applied to all the other terms in (2.24) with the general, nonperturbative and rely only on that the conclusion that in weak coupling and to leading reservoir is in thermal equilibrium.

order $\delta g^2\rangle$ the Markovian approximation $\rho\hat{}_{Ia}\delta t_0\rangle \rightarrow$ (iii) Rotating wave approximation: in writing the prod$\rho\hat{}_{Ia}\delta t\rangle$ is justified. ucts $a_I\delta x;t^{\rightarrow}\ \rangle a_I\delta x^{\rightarrow}{}^0;t^0\rangle$ of interaction picture field

Therefore in the Markov approximation the quan- operators (2.27) in (2.24) there are two types of tum master equation becomes terms with very different time evolution. Terms of the form

ρ_ˆlaðtÞ ¼ −g² Z0ᵗ dt⁰ Z d³x Z d³x⁰faIðxÞaIðx⁰Þ b†q⃗ bq⃗ eiωqðt−t0Þ; ð2:38Þ

× ρˆlaðtÞG>ðx − x0Þ are "slow," and terms of the form þ ρˆlaðtÞaIðx0ÞaIðxÞG<<ðx − x0Þ

i qt i q t t

Ið Þ lað Þ Ið Þ ð Þ b†q⃗ b†−q⃗ e2 ω e ω ð − 0Þ;

− a x ρˆ t a x0 G x − x0

− aIðx0Þρˆ laðtÞaIðxÞG>ðx − x0Þg: ð2:30Þ bq⃗ b−q⃗ e−2iωqte−iωqðt−t0Þ; ð2:39Þ

are fast, the extra rapidly varying phases e^{2iωqt} lead

The correlation functions G>ðx − x0Þ;G<ðx − x0Þ to rapid dephasing on timescales ≃1=ωq and do not are obtained in Appendix A in terms of nonpertur- yield resonant (nearly energy conserving) contribubative Lehmann representations to all orders in the tions. Neglecting these terms is tantamount to coupling of the environmental fields χ to any other neglecting nonresonant terms that average out field in thermal equilibrium except for the ALP. over the longer timescales of relaxation ≫ 1=ωq.

They are given by These terms only give perturbatively small transient contributions and are discussed in Sec. IV. Keeping only the slow terms which dominate the long time dynamics for t ≫ 1=ωq and neglecting the fast oscillatory terms defines the "rotating wave approximation" ubiquitous in quantum optics [33,34].

We will adopt these approximations and comment in Sec. IVon the corrections associated with keeping the fast terms as well as caveats in the factorization approximation and limitations of the QME.

Implementing the Markov approximation ρˆlaðt0Þ → ρterms of the formˆlaðtÞ, and the rotating wave approximation (keeping onlyb†b;bb†) using the spectral representa-

tion of the correlators (2.31), (2.32) and carrying out the spatial and temporal integrals we obtain the Lindblad form [33–38] of the quantum master equation,

ρ_ˆlaðtÞ ¼ Xq −iΔqðtÞ½b†q⃗ bq⃗ ;ρˆlaðtÞ

−Γq>2ðtÞ½b†q⃗ bq⃗ ρˆlaðtÞþρˆlaðtÞb†q⃗ bq⃗ −2bq⃗

ρˆlaðtÞb†q⃗ Γq<ðtÞ½bq⃗ b†q⃗ ρˆlaðtÞþρˆlaðtÞbq⃗

b†q⃗ −2b†q⃗ ρˆlaðtÞbq⃗ ;

−

2

ð2:40Þ

where

ΔqðtÞ¼ 2ωq Z 2π0 ϱðq0;qÞ½1−cosðω½ðq ω−qq−0Þq0Þt; ð2:41Þ g2 dq

Γq>ðtÞ Z ¼ ωg2qdq2π0 ϱðq0;qÞ½1 þ nðq Þsinð½ðωωqq−−qq0Þ0Þt; 0 ð2:42Þ

Γq<ðtÞ ¼ ωq Z 2π0 ϱðq0;qÞnðq0Þsinð½ðωωqq−−qq0Þ0Þt; ð2:43Þ g2 dq

and we introduce

$$\Gamma_q(t) = \Gamma_{>q}(t) - \Gamma_{<q}(t) = \frac{\omega_q}{2\pi_0}\int_{\Omega_{q_0;q}} \sin^2(\omega_{qq_0}t)\, g^2\, dq \quad (2.44)$$

The second and third lines in (2.40) are called the dissipator [33], these are non-Hamiltonian, purely dissipative terms, however it follows from the QME (2.40) that the trace of the reduced density matrix is conserved. It is argued in Refs. [35–38] that the Eq. (2.40) is the most general linear evolution equation that preserves unit trace and Hermiticity of the density matrix.

Expectation values of ALP operators in the interaction picture are obtained by taking the trace of such operators with the reduced density matrix, for example

$$\langle a_I(x,t)\rangle = \mathrm{Tr}[a_I(x,t)\rho(t)] = \frac{1}{2V\omega}\sum_q \left[\langle b_q(t)\rangle e^{i(\omega_q t - q\cdot x)} + \langle b^\dagger_q(t)\rangle e^{-i(\omega_q t - q\cdot x)}\right] \quad (2.45)$$

where

$$\langle b_q(t)\rangle = \mathrm{Tr}[b_q\rho(t)] = \mathrm{Tr}[b_q\rho(0)]\, ; \quad \langle b^\dagger_q(t)\rangle = \mathrm{Tr}[b^\dagger_q\rho(t)] = \mathrm{Tr}[b^\dagger_q\rho(0)]. \quad (2.46)$$

For any interaction picture operator A associated with the ALP field

$$\frac{d}{dt}\langle A_I\rangle(t) = \mathrm{Tr}[A_I(t)\dot\rho(t)] = \mathrm{Tr}[A_I(t)\rho(t)]. \quad (2.47)$$

where the average $\langle\hat O\rangle = \mathrm{Tr}[\hat O\rho(t)]$. Because $b_q, b^\dagger_q$ are time independent in the interaction picture, the time derivative of their expectation value is given solely by the second term on the right-hand side of Eq. (2.47), hence the expectation value of the number operator

$$N_q(t) = \mathrm{Tr}[\rho(t)b^\dagger_q b_q] \quad (2.48)$$ obeys

the quantum kinetic equation

$$\frac{d}{dt}\langle N_q(t)\rangle = \mathrm{Tr}[\rho(t)b^\dagger_q b_q] = \Gamma_q(t)N_q(t) - \Gamma_{<q}(t). \quad (2.49)$$

Similarly, we also find the evolution equation for the averages

$$\frac{d}{dt}\langle b_k(t)\rangle = -i\Delta_k(t)\langle b_k(t)\rangle - \Gamma_k(t)\langle b_k(t)\rangle$$

$$\frac{d}{dt}\langle b^\dagger_k(t)\rangle = i\Delta_k(t)\langle b^\dagger_k(t)\rangle - \Gamma_k(t)\langle b^\dagger_k(t)\rangle$$

and for the off-diagonal coherences,

$$\frac{d}{dt}\langle b^\dagger_k(t)\, b_{-k}(t)\rangle = -\left[\Delta_k(t) + i\Gamma_k(t)\right]\langle b^\dagger_k(t)\, b_{-k}(t)\rangle \quad (2.50)$$

$$\frac{d}{dt}\langle b_k(t)\, b_{-k}(t)\rangle = -\left[\Delta_k(t) + i\Gamma_k(t)\right]\langle b_k(t)\, b_{-k}(t)\rangle$$

From the evolution equations (2.50), (2.51) it is clear that $\Delta_k(t)$ is a time dependent renormalization of the frequency ω_k. To obtain the solutions of the above equations in the long time limit we need the following integrals

$$\int_0^t dt' \int_{\Omega_{q_0;q}} \frac{g^2}{\omega_q - q}\, \rho(q_0;q)\left[1 - \frac{\sin(\omega_q - q_0)t}{\omega_q - q}\right] dq = \Delta(t) = \frac{\omega}{2}\int_{-\infty}^{\infty} \rho(q_0;q)\, dq_0 + \frac{\omega}{2\pi}\int_{-\infty}^{\infty} \rho(q_0;q)\, \mathrm{P}\left(\frac{1}{\omega_q - q_0}\right) dq_0 \quad (2.52)$$

$$t\to\infty \quad t\delta\omega_q$$

where

$$\Gamma(t) = \frac{\omega}{2}\int_{-\infty}^{\infty} \rho(q_0;q)\, dq_0 - \frac{\omega}{2\pi}\int_{-\infty}^{\infty} \rho(q_0;q)\, \mathrm{P}\left(\frac{1}{\omega_q - q_0}\right) dq_0 \quad (2.53)$$

is a renormalization of the frequency ω_q and P stands for the principal part, and

$$\int_0^t \gamma_q(t')dt' = \frac{g^2}{2}\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\frac{\rho(q_0,q)}{\omega_q}\left[\frac{1-\cos[(\omega_q-\omega_q)t]}{(\omega_q-\omega_q)^2}\right] \to \pi g^2\int_{-\infty}^{\infty}dq_0\,\rho(q_0,q)\,\delta(\omega_q-\omega_q)\,t \quad (t\to\infty)$$

$$= \pi g^2\,\frac{\rho(\omega_q,q)}{2\omega_q}\,t \equiv \gamma_q t\,, \qquad \gamma_q = \Gamma_q(\omega_q) = \frac{\pi}{2}g^2\omega_q\,\rho(\omega_q,q)\,, \tag{2.55}$$

is the decay rate in agreement with Fermi's golden rule. In the long time limit, the solution of Eqs. (2.50), (2.51) are

$$\langle b_{\vec{k}}(t)\rangle = Z e^{-i\delta\omega_q t}e^{-\gamma_q t}\langle b_{\vec{k}}(0)\rangle\,, \tag{2.56}$$

$$\langle b^\dagger_{\vec{k}}(t)\rangle = Z e^{i\delta\omega_q t}e^{-\gamma_q t}\langle b^\dagger_{\vec{k}}(0)\rangle\,,$$

$$\langle b_{\vec{k}}b_{-\vec{k}}(t)\rangle = Z^2 e^{-2i\delta\omega_q t}e^{-\gamma_q t}\langle b_{\vec{k}}b_{-\vec{k}}(0)\rangle\,, \tag{2.57}$$

$$\langle b^\dagger_{\vec{k}}b^\dagger_{-\vec{k}}(t)\rangle = Z^2 e^{2i\delta\omega_q t}e^{-\gamma_q t}\langle b^\dagger_{\vec{k}}b^\dagger_{-\vec{k}}(0)\rangle\,,$$

where to leading order in the coupling,

$$Z = 1 - \frac{g^2}{2}\int_0^\infty \frac{dq_0}{\pi}\,\mathrm{P}\,\frac{\rho(q_0,q)}{\omega_q^2-\omega_{q_0}^2} \tag{2.58}$$

is the wave function renormalization.

If the initial averages $\langle b_{\vec{k}}\rangle(0) = 0$; $\langle b_{\vec{k}}b_{-\vec{k}}\rangle(0) = 0$ such values remain as fixed points of the evolution equations. However for a "misaligned" initial condition (2.12), (2.9) yielding the initial averages (2.13), it follows that in the long time limit the solutions of Eqs. (2.50), (2.51) are, respectively

$$\langle b_{\vec{q}}\rangle(t) = Z e^{-i\delta\omega_q t}e^{-\gamma_q t}\Delta_{\vec{q}}\,; \tag{2.59}$$

$$\langle b_{\vec{q}}b_{-\vec{q}}\rangle(t) = Z^2 e^{-2i\delta\omega_q t}e^{-\gamma_q t}\Delta_{\vec{q}}\Delta_{-\vec{q}}\,; \tag{2.60}$$

along with their Hermitian conjugates.

Absorbing $\delta\omega_q$ into the renormalization of the frequency and with the initial expectation values given by (2.10), (2.11), (2.14) we find that the expectation value of the ALP field is given by

$$\langle a\rangle(t) = e^{-\gamma_0 t}a_0(0)\cos(m_{aR}t) + \pi_0(0)\sin(m_{aR}t) + O(g^2);$$

$$m_{aR}$$

(2.61)

where m_{aR} is now the renormalized ALP mass and we have neglected (nonsecular) terms of order g^2 associated with the wave function and mass renormalizations. Equations (2.59), (2.60), (2.61) indicate that the expectation values and off-diagonal coherences decay in time, leading to a reduced density matrix diagonal in the number representation, this is the hallmark of decoherence. These results imply that the damping of the ALP condensate is directly linked to decoherence.

Neglecting perturbatively small nonsecular terms of $O(g^2)$ in the long time limit yields in this limit $\langle b^\dagger_{\vec{q}}\,b_{\vec{q}}\rangle(t) \equiv N_q(t) = N_q(0)e^{-\gamma_q t} + n(\omega_q)(1-e^{-\gamma_q t})$;

$$N_q(0) = |\Delta_{\vec{q}}|^2\,; \qquad n(\omega_q) = \frac{1}{e^{\beta\omega}-1}_q \tag{2.62}$$

which describes thermalization, and an exponential approach to the thermal fixed point of the quantum kinetic equation.

Taken together, the results given by (2.59)–(2.62) summarize some of the main results from the QME: damping of the ALP condensate, decoherence and thermalization are all related, the decoherence rate is the same as the thermalization rate as well as the damping rate of the misaligned component to the energy density. For $t \gg 1/\gamma_q$ the density matrix becomes diagonal in the occupation number basis and the misaligned condensate has relaxed to zero. The ALP has reached thermal equilibration with the bath.

From Eqs. (2.62), (2.15), and (2.16) we obtain the time evolution of the ALP energy density neglecting a time independent zero point contribution, it is given by

$$\rho(t) = \frac{1}{V}\sum_{\vec{q}} E_q(t)\,N_q(t)$$

$$= \frac{1}{2}\,m^2 a^{-2} e^{-\gamma_0 t}\,\pi^- + \int \frac{d^3q}{(2\pi)^3}\,\omega_q\,\left(1-e^{-\gamma_q t}\right)\,\left(f(\omega_q)+\frac{1}{2}\right) \tag{2.63}$$

The first term in (2.63) describes the decay of the condensate from the misaligned initial state, whereas the second term describes the thermalization of the ALP degrees of freedom.

This analysis highlights that the contribution from a misaligned condensate to the energy density, thermalization with the bath and decoherence as described by the decay of the off-diagonal components in the ALP occupation number pointer basis, all occur on similar timescales, which is completely determined by the relaxation rate γ_q.

The results (2.61), (2.63) are in complete agreement with those of Ref. [28] which were obtained with a very different approach based on the nonequilibrium Schwinger-Keldysh effective action. Furthermore, the general expression for the frequency renormalization (2.53) and wave function renormalization (2.58) are also in agreement with the general results found in Ref. [28] in the strict perturbative regime, although they cannot reproduce nonperturbative aspects which are revealed by the effective action and are discussed in section IV.

A. Decoherence and entropy production

The evolution equations (2.59), (2.60) describe the decay of the coherences, in other words, the emergence of decoherence, whereby the density matrix becomes diagonal in the pointer basis of the eigenstates of the occupation number operator $b^\dagger_q b_q$ for $t \gg 1/\gamma_q$. Furthermore, the timescale of decoherence is similar to the relaxation rate of the misaligned component of the energy density and that of thermalization. Decoherence and the evolution towards a diagonal reduced density matrix in the occupation number basis, in turn imply entropy production. At long time when the off diagonal terms are negligible, and the reduced density matrix becomes diagonal in the occupation number basis, with thermal populations, the total entropy becomes

$$\mathcal{S}(\infty) = \sum_q \left[(f+1)\ln(f+1) - f\ln f\right]_{\omega_q} \tag{2.64}$$

Since the initial ALP density matrix (2.12) describes a pure state, hence vanishing entropy, $\mathcal{S}(\infty) > 0$ implies entropy production for the ALP as a consequence of environmentinduced decoherence [52]. This is an important bonus of the QME which unambiguously describes decoherence via the decay of the coherences (2.59), (2.60), over the usual Boltzmann equation approach to thermalization, wherein entropy production is inferred via Boltzmann's H-theorem from the time evolution of a classical $H(t)$ function which inputs solely the occupation number evolution but which does not have any information on off-diagonal coherence.

III. ALP-PHOTON INTERACTIONS

The results obtained in the previous section are general up to $O(g^2)$ and to all orders in the couplings of the bath field χ to any other field except for the ALP. Whereas our study addresses the nonequilibrium dynamics of ALP fields, the results also apply to any field with an interaction of the form (2.1) and initial conditions that allow for the evolution of a coherent condensate [18]. However, although the results are generic, the relaxation rate γ_q, frequency and wave function renormalizations depend on the spectral properties of the bath correlations.

In this section we focus on ALP interaction with photons via the coupling

$$L_I = -g a(x)\,\vec{E}(x)\cdot\vec{B}(x) \tag{3.1}$$

We consider the thermal bath of CMB blackbody radiation of free massless photons, neglecting electromagnetic interactions with charged leptons and quarks. This restricts the validity of our treatment to temperatures well below the masses of these other degrees of freedom and under conditions when the electron density in particular is vanishingly small, therefore there is no (gauge invariant) thermal mass or plasma frequency for the photons. These conditions are certainly fulfilled in cosmology after recombination at temperatures $T \simeq$ eV when the free electron density vanishes rapidly and the distribution functions of quarks and charged leptons are thermally suppressed at these temperatures.

Ref.The spectral density[28] and summarized in Appendix$\rho(q_0;\vec{q})$ has been obtained inB for consistency of presentation, it is given by

$$\rho(q;\vec{q}) = \frac{2}{32\pi}\frac{1-e^{-\beta\omega}}{2}\left[\frac{2}{\beta q}\ln\left|\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right| - \Theta(-Q^2)\,\mathrm{sign}(q_0)\right]; \quad (3.2)$$

where

$$Q^2 = q_0^2 - q^2; \quad \omega(l) = q_0; \quad l = \frac{q}{|\vec{q}|}: \quad (3.3)$$

andFrom Eq.$\beta = 1/T$(2.55)with Twe obtain the relaxation ratethe temperature of the radiation bath.

$$\gamma_q(0) = \frac{2}{16\pi\omega_q}\frac{1-e^{-\beta\omega}}{2}\left[1+\frac{2}{\beta q}\ln\left|\frac{1-e^{-\beta(\omega+q)}}{1-e^{-\beta(\omega-q)}}\right|\right];$$

$$\gamma_q(0) = \frac{g^2 m_a^4}{64\pi\omega_q}: \quad (3.4)$$

The zero temperature contribution $\gamma_q(0)$ is recognized as the ALP decay rate into two photons [9], whereas the finite temperature contribution is a consequence of stimulated emission and absorption processes in the radiation bath. In the long-wavelength limit we find

$$\gamma_q(T) = \frac{g^2 m_a^3}{64\pi}\left(1+\frac{2}{e^{m_a/2T}-1}\right); \quad (3.5)$$

which in the high temperature limit $T \gg m_a$ yields

$$\gamma_q(T) = \frac{g^2 m_a^3}{16\pi}\frac{T}{m_a}. \quad (3.6)$$

For example, if T corresponds to the temperature of the cosmic microwave background today $T \simeq 10^{-4}$ eV the finite temperature correction yields a very large enhancement over the zero temperature rate if the ALP is an ultralight candidate with $m_a \lesssim 10^{-22}$ eV. A substantial relaxation rate of the ALP post recombination may yield important cosmological consequences, for example in birefringence if it is caused by the coupling of CMB photons to a pseudoscalar ALP [29–32] (see discussion below).

From the results of Appendix B, the frequency renormalization given by Eq. (2.53) is found to be

$$\delta\omega_q = \delta\omega_q(0) + \delta\omega_q(T); \quad (3.7)$$

spectral densitywhere $\delta\omega_q(0)$ is obtained from the(3.2) and by introducing an ultravioletT = 0 contribution to the cutoff Λ, it is given by

$$\delta\omega_q(0) = -\frac{128 g^2 \pi^2 \omega_q}{12\Lambda^4}\left(2m_a^2\Lambda^2 + (m_a^2)^2\right)\ln\frac{\Lambda}{m_e^2 a^{3=2}}: \quad (3.8)$$

In Appendix C the finite temperature contribution in the high temperature limit $T \gg \omega_q$ is found to be

$$\delta\omega_q(T) = -\frac{g^2 \pi \omega^2 T_q^4}{15\pi m^2 T^2}\left(O(m_a^4/T^4)\right): \quad (3.9)$$

The frequency renormalization (3.7) is identified as a temperature dependent mass renormalization by writing the renormalized frequency $\Omega_q = \omega_q + \delta\omega_q$ up to $O(g^2)$ as

$$\Omega_q = q + m_R^2(T)/q = \omega_q + \Delta m^2(T)/\omega_q + \Delta m^2\omega_q^2(T) + \delta\omega_q; \quad (3.10)$$

from which we find the finite temperature renormalized mass up to $O(g^2)$

$$m_R^2(T) = m_R^2(0)\left[1 - \frac{T^2}{4c^2}\right];$$

$$m_R^2(0) = m_a^2\left[\frac{g^2}{64\pi^2}\right]\left[1 + \frac{2m_a^2}{\Lambda^2}\left(\ln\frac{\Lambda^2}{m_a^2} + \frac{3}{2}\right)\right]; \quad (3.11)$$

where

$$T_c \simeq 1.1\, s\, m_R(0)/g; \quad (3.12)$$

and we kept the leading order in the high temperature limit $T/m_a \gg 1$ in the finite temperature correction. The result (3.11) agrees with Ref. [28] which obtained a similar finite temperature mass from the nonequilibrium effective action, and indicates that $m_R^2(T)$ becomes negative for $T > T_c$ suggesting a long wavelength instability and the possibility of an inverted phase transition as discussed in Ref. [28]. However, within the context of the quantum master equation there is a caveat on this interpretation because the result for the frequency renormalization has been obtained in strict perturbation theory and the renormalized frequency $\omega_q + \delta\omega_q$ does not yield any instability. This caveat is discussed in more detail in Sec. IV.

IV. DISCUSSION AND CAVEATS

A. Counterrotating terms

In the derivation of the quantum master equation (2.40) we neglected terms of the form

$$b_{\vec{q}}\, b_{-\vec{q}}\, e^{-2i\omega_q t} e^{i\omega_q(t-t_0)}; \quad b^\dagger_{\vec{q}}\, b_{-\vec{q}}\, e^{2i\omega_q t} e^{-i\omega_q(t-t_0)}: \quad (4.1)$$

The time integral over t^0 can be carried out following the steps leading to Eq. (2.40) yielding contributions of the form $b_{\vec{q}}\, b_{-\vec{q}}\, e^{-2i\omega_q t}\rho(\vec{q}_0;q)\rho(q)_{la}(t)$ etc. The contribution of these terms to the equations of motion for linear or bilinear forms of $b;b^\dagger$ are straightforward to obtain, they do not yield terms that grow secularly in time because the rapid dephasing of the oscillatory terms average out in the time integrals. These are nonresonant terms and yield perturbatively small subleading contributions of the form $\delta\omega_q/\omega_q \ll 1$; $\gamma_q/\omega_q \ll 1$ in weak coupling, as compared to those obtained from Eq. (2.40) which captures the secular growth in time because of the resonances and describes the leading behavior in the long time dynamics.

B. Factorization

Factorization of the full density matrix (2.21) is one of the main assumptions in the derivation of the Lindblad form of the quantum master equation [33,34]. This assumption neglects correlations between ALP field and the thermal bath as discussed above, it may be justified for weak coupling: assuming an initial factorization, correlations will build up upon time evolution but will remain perturbatively small, hence they may be neglected to leading order in the coupling g. The assumption that the total density matrix remains factorized with the bath in thermal equilibrium which remains unaffected by the coupling to the ALP at all times is consistent with the interpretation of the bath as a reservoir. However, as the ALP population builds up as a consequence of thermalization, it is plausible that correlations between the ALP and the bath become stronger as the ALP population reaches a thermal state, leading up to a possible breakdown of the factorization assumption. Such a scenario merits deeper scrutiny which is beyond the scope of this study.

C. Nonequilibrium effective actionvs quantum master equation

In Ref. [28] the time evolution of an initial density matrix was studied by implementing the in-in Schwinger-Keldysh formulation of nonequilibrium field theory. In this formulation the time evolution is described by the in-in effective action that leads to a Langevin equation of motion for the ALP field in terms of the retarded self-energy Σ and a noise term both related by the fluctuation dissipation

relation. The solution of the Langevin equation of motion inputs the full propagator including the self-energy correction, and the (complex) poles in the propagator at

$$\omega^2_P(q_P) = \omega^2(q_P) + \Sigma(\omega_P, q_P); \quad (4.2)$$

determine the frequency and lifetime of ALP oscillations. The finite temperature effective mass is obtained from the real part of the solution of the Eq. (4.2) for $q = 0$, namely $m^2(T) = \mathrm{Re}[\omega^2_P(q=0)]$.

In the case of ALP-photon interaction with the coupling (3.1), after absorbing the zero temperature, ultraviolet divergent contributions into a definition of the zero temperature renormalized mass m_{aR}, the solution of the pole equation (4.2) at $q = 0$ yields precisely the result (3.11) to leading order in the high temperature expansion $T \gg m_{aR}$. The finite temperature mass, as properly defined by the position of the pole in the propagator at zero momentum, indicates the possibility of an instability and an inverted phase transition for $T > T_c$ as advocated in Ref. [28], a conclusion that does not rely on an expansion near the bare frequency.

In contrast, the quantum master equation approach yields a perturbative correction to the bare frequency in the form $\omega_q + \delta\omega_q$ with a real $\delta\omega_q$ which obviously does not entail any instability. The main reason for this discrepancy between the effective action and the quantum master equation can be traced to the fact that in the latter approach the time integrals in (2.30), convolve the spectral representations (2.31), (2.32) with the time dependence (2.38) featuring the external ALP frequency ω_q. Therefore, the rates (2.41)–(2.43) in the Lindblad QME (2.40), are effectively evaluated at the frequencies ω_q yielding strictly perturbative corrections for the frequency and wave function renormalizations. At heart, this is a consequence of the perturbative nature of the QME in interaction picture.

Another important difference with the nonequilibrium effective action, is that as found in Ref. [28], the zero temperature contribution to the real part of the self energy is

$$\Sigma_{R0}(\omega; q) = -\frac{g^2}{12\pi^2}\left[\Lambda^4 + 2Q^2\Lambda^2 + Q^2\ln\left(\frac{\Lambda}{|Q|}\right)^2\right]; \quad (4.3)$$

where $Q^2 = \omega^2 - q^2$; the logarithmic divergence multiplying Q^2 implies that the renormalized effective action requires a new higher derivative term $\propto \phi\partial^\mu\partial_\mu\partial^2 a^2\phi$ to absorb the logarithmic divergence from the self-energy. This is a consequence of the nonrenormalizable interaction (3.1) since the coupling g has dimensions of $[\mathrm{energy}]^{-1}$.

In contrast, the QME yields the frequency renormalization $\delta\omega^0_{q\,0}$ (3.8), which is proportional to the real part of the self-energy (4.3) evaluated on the (bare) mass shell, namely for $Q^2 = m^2_a$. Again this is a consequence of the time integrals leading to the QME in Lindblad form, and can be traced to the interaction picture representation of the density matrix.

Therefore, the QME confirms the damping of the misaligned ALP condensate, thermalization, and that the ALP energy density features a mixture of a "cold" component from the damped misaligned condensate and a "hot" component from thermalization, and that damping of the cold and thermalization of the hot components and decoherence occur on similar timescales.

An instability and possible phase transition cannot be captured by the QME which relies on a perturbative expansion in interaction picture field theory, assuming a well-defined mass shell and stable oscillations of the various degrees of freedom. An instability will lead to a breakdown of most approximations: certainly the Markov and rotating wave approximations, since the former relies on a wide separation of timescales and the second on well defined mass shells associated with the oscillation frequencies. Therefore, an instability associated with a possible phase transition and novel phases for $T > T_c$ is well beyond the realm of validity of the QME and should not be expected to be described reliably by it.

While the QME cannot directly confirm the possibility of an inverted phase transition and the emergence of exotic phases both described by the Schwinger-Keldysh effective action and truly nonperturbative aspects, it does allow to understand the sources of these discrepancies in strict perturbation theory.

An important advantage of the QME is that it allows to obtain the time evolution of coherences and populations in a more direct manner thereby establishing that thermalization and decoherence with the concomitant entropy production occur on similar timescales.

D. On the similarity of timescales

An important result is that the timescales of damping of the condensate (2.61), decoherence (2.60), and thermalization (2.62) are all very similar and simply related. This similarity originates in the form of the Lindblad QME, Eq. (2.40), which solely inputs bilinears of the form $b^\dagger_q b_q$ (one annihilation and one creation operator),

and this form of the QME unequivocally leads to the quantum kinetic equations (2.49)–(2.51) whose solutions display the timescales of damping, decoherence, and thermalization in terms of the same function $\Gamma_k(t)$. In turn, the particular form of the Lindblad QME is a consequence of the linear coupling of the axion to the composite operators O_χ as described by the Lagrangian density (2.1). Although we have not studied nonlinear axion couplings, it is quite possible that in the case of nonlinear couplings the timescales could be quite different. Investigating this possibility would merit further study beyond the scope of our objectives.

E. Possible cosmological consequences

Although we have studied the nonequilibrium dynamics of ALP's in Minkowski space time, the results allow us to provide a preliminary extrapolation to cosmology.

A pseudoscalar ALP coupled to photons as in Eq. (3.1) leads to cosmic birefringence, namely a frequency independent rotation (in contrast to Faraday rotation) in the polarization angle Ψ between the surface of last scattering and today [29–32]. For a homogeneous misaligned condensate slowly varying in time $\langle\dot{\sigma}(t)\rangle$ such a change is given by [29]

$$\Delta\Psi = \frac{g}{2}\left[\sigma(t_{LSS}) - \sigma(t_0)\right]: \qquad (4.4)$$

The amplitude of the misaligned condensate decays as a consequence of the ALP interaction with the CMB photons, therefore the condensate is decaying during the cosmological expansion since recombination, and as described above the ALP fluctuations are thermalizing with the radiation bath on similar timescales. This hitherto unappreciated fact has important consequences.

If the lifetime $1/\gamma(T) \ll 1/H_0$ the amplitude of the condensate $\langle\dot{\sigma}(t_0)\rangle \simeq 0$, however, the ALP thermalizes on the same timescale as the condensate decays, therefore, if the misaligned condensate has completely decayed between the surface of last scattering and today, we conjecture that the fluctuations of Ψ would feature a thermal power spectrum as a consequence of thermalization of axion fluctuations with the CMB.

This conjecture is motivated precisely by the similarity of the condensate damping and thermalization timescales revealed by the Lindblad QME. The arguments in Ref. [29] leading up to the result (4.4) hinge on the change in the photon frequency for the different polarizations as a consequence of the coupling to the axion condensate, namely the expectation value of the axion field. However, as the solution from the Lindblad QME shows, the fluctuations of the axion thermalize on the same timescale as damping of the condensate, therefore we expect that the polarization angle will feature thermal fluctuations, since it is modified by the axion field. Rather than focusing solely on the change in frequency for the different polarization as a consequence of the dynamical axion condensate, the dynamics of the polarization post recombination should be described by the Stokes parameters which involve combinations of the transverse components of the electric field squared. At the quantum level the electric field is associated with a quantum operator, whose Heisenberg equation of motion involves the full axion field [29–32] both its expectation value as well as the fluctuating component. Therefore we conjecture that the square of the electric field operator will depend on the square of the axion field which includes the fluctuations of the axion field. As described by the Lindblad QME the fluctuations thermalize with the CMB on the same timescale as the mean-field (expectation value) decays. Hence, this reasoning leads us to expect that fluctuations in the Stokes parameters, which describe the polarization field, should feature a thermal spectrum. At this stage, this remains as a plausible conjecture which merits deeper scrutiny on its own, which, however, is well beyond the original scope of this article.

If the ALP lifetime is much shorter and the misaligned condensate decays prior to the last scattering surface, then it has reached full thermalization with the CMB and if it is an ultralightdarkmatter candidate itcontributesto the effective numberofultrarelativisticdegreesoffreedom.Ifthelifetime is of the order of $1/H_0$ then the ALP contributes as "mixed" dark matter, with a cold component with weight $e^{-\gamma(T)/H_0}$ and a hot (thermal) component with weight $(1 - e^{-\gamma(T)/H_0})$. This latter possibility opens a window to an interesting scenario, where the cold dark matter component would dominate at earlier time during galaxy formation and the hot component would dominate later, with a larger velocity dispersion, hence a larger free streaming length, featuring a crossover between cold and hot components on timescales that depend on the coupling and ALP mass. This scenario brings interesting and hitherto unexplored consequences for galaxy formation that merit further study.

V. CONCLUSIONS AND FURTHER QUESTIONS

We studied the nonequilibrium dynamics of ALP's motivated by the possibility that these particles belonging to a sector beyond the Standard Model may be suitable dark matter candidates. A hallmark of ALP's is their coupling to pseudoscalar composite operators associated with Standard

Model degrees of freedom, and in particular their coupling to electromagnetism may lead to cosmic birefringence, namely the rotation of the plane of polarization of the CMB with tantalizing possibilities of detection. In this article we consider generic couplings of the ALP field [aðxÞ] of the form gaðxÞOχðxÞ where Oχðx⃗Þ are pseudoscalar composite operators of Standard Model degrees of freedom (χ) assumed to be a bath in thermal equilibrium, and derive a quantum master equation that describes the time evolution of the ALP reduced density matrix upon tracing the χ degrees of freedom. The QME is obtained up to $O(g^2)$ but to all orders in the coupling of the χ (Standard Model) degrees of freedom to any other degree of freedom within or beyond the Standard Model except for the ALP. The initial ALP density matrix allows for a misaligned condensate. The QME describes the damping of the misaligned condensate, thermalization with the bath and decoherence, namely the damping of the off-diagonal reduced density matrix elements in the occupation number basis within a unified framework.

The ALP time dependent energy density EðtÞ features two components: a cold (c) component from the misaligned condensate and a hot (h) component from thermalization with the bath,with $E(t) \simeq E_c e^{-\gamma(T)t} + E_h(1 - e^{-\gamma(T)t})$ where the relaxation rate γðTÞ also describes the decoherence rate. Therefore,the damping of the misaligned condensate energy density, the approach to thermalization with the bath and decoherence all occur on the same timescales. We focus on the particular example of the ALP coupling to electromagnetism where $O_\chi(x) = \vec{E}(x) \cdot \vec{B}(x)$ where the radiation field describes the CMB post recombination. The long wavelength relaxation rate is enhanced by emission and absorption in the photon bath and at high temperature $T \gg m_a$ and

in the long wavelength limit is given by $\gamma(T) = \frac{g^2}{16\pi} m_a^2 T$ featuring a substantial enhancement over the zero temperature rate. These results are in agreement with those of Ref. [28] but obtained with an independent method.

Thetimedependenceoftheenergydensitysuggeststhatif the ALP is a dark matter candidate and interacts with Standard Model degrees of freedom in (local) thermal equilibrium, it provides a "mixed" dark matter scenario where the "warmth" depends on time: at earlier times it describesacolddarkmattercomponentandatlatetimesahot component, with potentially profound implications on galaxyformation.IftheALPisanultralightcandidate,andifthe misaligned condensate has completely decayed prior to the last scattering surface the thermal component contributes to theeffectivenumberorultrarelativisticdegreesoffreedom.If its lifetime is smaller than the Hubble time, $\gamma(T) \ll 1/H_0$, the misaligned condensate is decaying at least since after recombination and thermalizing with the CMB on similar timescales. Therefore, if cosmic birefringence is a consequenceofthecouplingofphotonstoapseudoscalarALP,t he rotation angle since the last scattering surface should include thermal features in its fluctuation spectrum.

These extrapolations to the cosmological setting must be taken as indicative based on the results in Minkowski space time. The next step of the program is to include cosmological expansion and assess if and how it modifies the conclusions above, we expect to report on this aspect in forthcoming studies.

ACKNOWLEDGMENTS

The authors gratefully acknowledge support from the U.S. National Science Foundation through Grant No. NSF 2111743.

APPENDIX A: ENVIRONMENTAL CORRELATION FUNCTIONS: LEHMANN AND SPECTRAL REPRESENTATIONS

The dynamics and dissipative processes depend on the correlation functions $G^{\lessgtr}$ of the environment in Eqs. (2.24)–(2.26).

Because the bath is in thermal equilibrium, its initial density matrix is $\rho_\chi(0) = e^{-\beta H_\chi}/\mathrm{Tr}e^{-\beta H_\chi}$ which is spacetime translationally invariant, and the Heisenberg picture operators associated with the bath are given by $O_\chi(x;\vec{t}) = e^{iH_\chi t} O_\chi(x;\vec{0}) e^{-iH_\chi t}$ we can write

$$G^{>}(\vec{x} - \vec{x}^{\,0};t - t^0) = \langle O_\chi(x;\vec{t}) O_\chi(\vec{x}^{\,0};t^0)\rangle_\chi$$

$$= \int \frac{d^4k}{(2\pi)^4} \rho_>(k;\vec{k}_0) e^{-ik_0(t-t^0)} e^{i\vec{k}\cdot(\vec{x}-\vec{x}^{\,0})} \tag{A1}$$

$$G^{<}(\vec{x} - \vec{x}^{\,0};t - t^0) = \langle O_\chi(\vec{x}^{\,0};t^0) O_\chi(x;\vec{t}\,)\rangle_\chi$$

$$= \int \frac{d^4k}{(2\pi)^4} \rho_<(k;\vec{k}) e^{-ik_0(t-t^0)} e^{i\vec{k}\cdot(\vec{x}-\vec{x}^{\,0})}:$$

$2\pi)^4$ (A2) These representations are obtained by writing $O_\chi(\vec{x},t) = e^{iH_\chi t}e^{-i\vec{P}\cdot\vec{x}}O_\chi(\vec{0},0)e^{-iH_\chi t}e^{i\vec{P}\cdot\vec{x}}$ and introducing a complete set of simultaneous eigenstates of H_χ and the total momentum operator $\vec{P}$, $H_\chi|\vec{P}|n\rangle = E_n|\vec{P}|n\rangle$, from which we obtain the following Lehmann representations,

$$\rho_>(k_0,\vec{k}) = \frac{(2\pi)^4}{Z}\sum_{m,n} e^{-\beta E_n}\langle n|O_\chi(\vec{0},0)|m\rangle\langle m|O_\chi(\vec{0},0)|n\rangle \times \delta(k_0 - E_m + E_n)\delta^3(\vec{k}-\vec{P}_m+\vec{P}_n) \quad (A3)$$

$$\rho_<(k_0,\vec{k}) = \frac{(2\pi)^4}{Z}\sum_{m,n} e^{-\beta E_n}\langle m|O_\chi(\vec{0},0)|n\rangle\langle n|O_\chi(\vec{0},0)|m\rangle \times \delta(k_0 - E_n + E_m)\delta^3(\vec{k}-\vec{P}_n+\vec{P}_m): \quad (A4)$$

Upon relabeling $m \leftrightarrow n$ in the sum in the definition (A4) and recalling that O_χ is a Hermitian operator, we find the Kubo-Martin-Schwinger relation [51] $\rho_<(k_0,k) = e^{-\beta k_0}\rho_>(k_0,k)$: (A5)

The spectral density is defined as $\rho(k_0,k) = \rho_>(k_0,k) - \rho_<(k_0,k) = \rho_>(k_0,k)(1 - e^{-\beta k_0})$ (A6)

therefore

$$\rho_>(k_0,k) = \rho(k_0,k)(1+n(k_0)),$$
$$\rho_<(k_0,k) = \rho(k_0,k)n(k_0), \quad (A7)$$

where

$$n(k_0) = \frac{1}{e^{\beta k}-1}. \quad (A8)$$

Furthermore, from the first equality in (A5) it follows that $\rho(-k_0,k) = -\rho(k_0,k)$; (A9) $\rho(k_0,k) > 0$ for $k_0 > 0$: (A10)

We emphasize that these are exact relations, the "environmental" fields χ may be coupled to other fields, for example, in the case of the ALP interaction with the electromagnetic fields, the gauge field also interacts with electrons, charged leptons, and quarks. The Lehmann representations (A3), (A4) are nonperturbative and unambiguously yield the relations (A5)–(A10) which are general, nonperturbative statements relying on thermal equilibrium and space-time translational invariance and do not depend on these couplings.

APPENDIX B: SPECTRAL DENSITY FOR $\vec{E}\cdot\vec{B}$ CORRELATION FUNCTIONS

We begin with the quantization of the gauge field within a volume V eventually taken to infinity,

$$\vec{A}(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k},\lambda=1,2}\hat{\epsilon}_{\vec{k},\lambda}\sqrt{\frac{\hbar}{2\omega_k}}\left[a_{\vec{k},\lambda}e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda}e^{ik\cdot x}\right]; \quad (B1)$$

where $\hat{\epsilon}_{\vec{k},\lambda}$ are the transverse polarizaton vectors chosen to be real. From Eqs. (A1) and (A2) we need the correlation functions

$$G_>(x-y) = \langle \vec{E}(x)\cdot\vec{B}(x)\vec{E}(y)\cdot\vec{B}(y)\rangle; \quad (B2)$$

$$G_<(x-y) = \langle \vec{E}(y)\cdot\vec{B}(y)\vec{E}(x)\cdot\vec{B}(x)\rangle = G_>(y-x); \quad (B3)$$

where we now refer to $\langle(\cdots)\rangle$ as averages in the thermal density matrix of free field photons.

In the thermal ensemble the expectation value $\langle\vec{E}(x)\cdot\vec{B}(x)\rangle = \langle\vec{E}(x)\rangle\cdot\langle\vec{B}(x)\rangle = 0$ correlation function becomes by parity invariance. Using Wick's theorem, the

$\vec{\cdot}\,\vec{B}$

$$\frac{\pi}{2}\int d^3k\,\left(1-\frac{k}{k}\cdot\frac{}{}\right)^2 \vec{q}-\vec{k}\,\langle \vec{E}(x)\cdot\vec{B}(y)\vec{B}(x)\cdot\vec{B}(y)\rangle = \sum_{i,j}\{\langle E_i(x)E_j(y)\rangle\langle B_i(x)B_j(y)\rangle$$

$$+\langle E_i(x)B_j(y)\rangle\langle B_i(x)E_j(y)\rangle\}. \quad (B4)$$

A straightforward calculation yields $\langle E_i(x)E_j(y)\rangle$

$= \langle B_i(x)B_j(y)\rangle$

$$= \frac{1}{2V}\sum_k \left(\delta_{ij}-\hat{k}_i\hat{k}_j\right)$$

$$\times \{(1+n(k))e^{-ik\cdot(x-y)}+n(k)e^{ik\cdot(x-y)}\};$$

(B5)

$$\rho_>(q_0;q) = \int (2\pi)^3 k|\vec{q}-\vec{k}|\,|\vec{q}-\vec{k}|\,\{(1+n(k))(1+n(|\vec{q}-\vec{k}|))\delta(q_0-k-|\vec{q}-\vec{k}|)$$

$$+n(k)n(|\vec{q}-\vec{k}|)\delta(q_0+k+|\vec{q}-\vec{k}|)\}\left(1+\frac{k}{k}\cdot\frac{\vec{q}-2\vec{k}}{|\vec{q}-\vec{k}|}\right)^2$$

$$\times\{(1+n(k))n(|\vec{q}-\vec{k}|)\delta(q_0+k-|\vec{q}-\vec{k}|)$$

$$+n(k)(1+n(|\vec{q}-\vec{k}|))\delta(q_0-k+|\vec{q}-\vec{k}|)\}:$$

(B9)

similarly $\langle E_i(x)B_j(y)\rangle =$

$-\langle B_i(x)E_j(y)\rangle$

$$= -2\frac{1}{V}\sum_k k\delta(\hat{\epsilon}_{ik;\vec{1}}\hat{\epsilon}_{k;j\vec{2}}-\hat{\epsilon}_{ik;\vec{2}}\hat{\epsilon}_{jk;\vec{1}})$$

$$\times\{(1+n(k))e^{-ik\cdot(x-y)}+n(k)e^{ik\cdot(x-y)}; \quad (B6)$$

(where $n(k) = 1/(e^{\beta k}-1)$. Combining the two terms in (B4) we find

$$G_>(x-y) = \int \frac{}{}\frac{1}{2}\sum_k\sum_p\left(\hat{k}\cdot\hat{p}\right)^2$$

$$\times\{(1+n(k))n(p)e^{-ip\cdot(x-y)}+n(k)(1+n(p))e^{ip\cdot(x-y)}\}. \quad (B7)$$

Expanding the product, we perform the following change of variables in the various terms:

(1) in the term $n(k)n(p)$: $\vec{k}\to-k;\vec{p}\to-\vec{p}$;

(2)(3) in the term within the term withπ_3 we obtain $n(1+k)p(1+n(1+|k\vec{p}|))$:: $kp\vec{\to}\to-1-=\vec{k}\vec{p}V.;$ $P_{\vec{q}\to}$

Taking the infinite volume limit with $\sum_k \to$

$$\int \frac{d^3k}{(2\pi)^3}$$

$$\rho_>(q_0;q) = \int (2\pi)^3 k|\vec{q}-\vec{k}|\,|\vec{q}-\vec{k}|\,\{(1+n(k))(1+n(|\vec{q}-\vec{k}|))\delta(q_0-k-|\vec{q}-\vec{k}|)$$

$$+n(k)n(|\vec{q}-\vec{k}|)\delta(q_0+k+|\vec{q}-\vec{k}|)\}\left(1+\frac{k}{k}\cdot\frac{\vec{q}-2\vec{k}}{|\vec{q}-\vec{k}|}\right)^2$$

$$\times\{(1+n(k))n(|\vec{q}-\vec{k}|)\delta(q_0+k-|\vec{q}-\vec{k}|)$$

$$+n(k)(1+n(|\vec{q}-\vec{k}|))\delta(q_0-k+|\vec{q}-\vec{k}|)\}:$$

(B9)

$$G_>(x-y) = \int \frac{dq}{}\frac{}{}2\pi_0\int (d2\pi)^{3q}p_3$$

$$\rho_>(q_0;q)e^{-iq_0\delta t-t^0)e^{iq\vec{}\cdot(x\vec{}-y\vec{})}}; \quad (B8)$$

where
Writing

$$G_<(x-y) = \int \frac{}{}\frac{}{}dq2\pi_0\int (d2\pi)^{3q}p_3$$

$$\rho_<(q_0;q)e^{-iq_0\delta t-t^0)e^{iq\vec{}\cdot(x\vec{}-y\vec{})}};$$

(B10)

and using the relation (B3) we find that $\rho_<(q_0;\vec{q}) =$

ρsated by>ð−q0;−k⃗ q⃗→Þ, however the sign change in−k⃗ inside the k-integral with the final resultq⃗ can be compen-

$$\rho^{<}(q_0;\vec{q}\,) = \rho^{>}(-q_0;\vec{q}\,); \qquad (B11)$$

using the various delta functions in the definition offurthermore, using the identity ð1 þ nðwÞÞ ¼ $e^{\beta w}$nðwρÞ>and$_{we}$

find

$$\rho^{<}(q_0;\vec{q}\,) = e^{-\beta q_0}\rho^{>}(q_0;\vec{q}\,); \qquad (B12)$$

which is the Kubo-Martin-Schwinger relation [51], thereby confirming the general results (A5). The spectral density is given by [see Eq. (A6)] $\rho(q_0;q) = \rho^{>}(q_0;q) - \rho^{<}(q_0;q)$ with

$$= -\frac{k^w}{k}\,\rho(q_0;q) = \pi^2 Z\,(2d\pi^3 k^3 w\,1\,f(\beta_2 \cdot \vec{q}\,)\tfrac{1}{2}[n(k)+n(w)]$$

$$\times\delta(\delta q_0 - k - w) - \delta(q_0 + k + w)\,(k\cdot w + k^2)$$

$$(k\cdot\vec{q}\,)(n(w) - n(k))\times\delta(\delta q_0$$

$$-k\cdot w) - \delta(q_0 + k\cdot w)\Big];$$

$$w = |\vec{q} - \vec{k}|: \qquad (B13)$$

The spectral density is calculated by implementing the following steps:

$$\int d^3k = \int_0^\infty dk \int_0^\pi 2\pi\sin\theta\, d\theta \int_{-1}^{1} d(\cos\theta) = 4\pi\int_0^\infty k^2\, dk;$$

$$w = |\vec{q} - \vec{k}| = \sqrt{q^2 + k^2 - 2kq\cos\theta};$$

$$dw = \frac{k\,d(\cos\theta)}{w}: \qquad (B14)$$

Carrying out the integrations, which are facilitated by the delta function constraints we find

$$\rho(q_0;\vec{q}\,) = \frac{\pi}{4q}\,\Theta(Q^2)\,(1+\beta\cdot\beta)\,\sqrt{1-\frac{4m^2}{Q^2}}\,\Big[1 - e^{-\beta\omega}\Big]^{-1}\Theta(-Q\cdot\beta)\,\mathrm{sign}(q_0\cdot\beta); \qquad (B15)$$

where

$$Q^2 = q_0^2 - q^2; \quad \omega(\beta) = |q_0\cdot\beta|; \quad |q\cdot\beta| = |\vec{q}\cdot\vec{\beta}|. \qquad (B16)$$

APPENDIX C: FINITE TEMPERATURE CONTRIBUTION TO $\delta\omega_q$

$$\delta\omega \equiv \frac{\sqrt{64\pi^2 q\,\omega_q}\,g^2 T}{q\,2q_0}\,I(\beta q); \qquad \omega = \sqrt{q^2 - 4m^2}: \qquad (C1)$$

$$I(\beta q) = \int_{-\infty}^{\infty}\frac{(q_0^2 - q^2)}{\omega_q - q_0/2}\,\ln\Big|1 - e^{-\beta(\omega_\beta - q_0/2)}\Big|\,dq_0$$

Since the argument of the logarithm is odd under $q_0 \to -q_0$, it follows that I can be written as

$$I(\beta q) = \int_0^\infty 2q_0(q_0^2 - q^2)\ln\Big|\frac{1 - e^{-\beta(\omega_q - q_0/2)}}{1 - e^{-\beta(\omega_q + q_0/2)}}\Big|\,dq_0: \qquad (C2)$$

Using the results

$$\int_0^\infty x^n \ln|1 - e^{-(x+y)}|\,dx = -\Gamma(n+1)\,\mathrm{Li}_{n+2}(e^{-y}) \qquad (C3)$$

$$\int_0^\infty x^n \ln|1 - e^{-(x-y)}|\,dx$$

$$= (-1)^n\Gamma(n+1)\,\mathrm{Li}_{n+2}(e^{-y})$$

$$- 2\sum_{i=1}^{n+1}(-1)^i\,\frac{n!}{(n+1-i)!}\,y^{n+1-i}\,\mathrm{Li}_{i+1}(e^{-y}); \qquad (C4)$$

where $\mathrm{Li}_\nu(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^\nu}$ is the polylogarithm, along with the identities

$$\int_0^\infty \frac{dx}{e^{x+y} - 1} = \sum_{n=1}^{\infty} e^{-ny}\,y^{-1} = \mathrm{Li}_1(e^{-y}) \qquad (C5)$$

Pðþ Þ ¼ ne x þ z n

P Z k dx −1e−nðk−γÞ

0 x þz n

¼ −n1e−nðγþzÞ½−Eiðnz Þ þ Eiðnðγ þ z Þ Þ ðC6Þ

$$Z \quad \frac{dx}{} \quad -\frac{1}{} e^{-n(x-} \qquad \infty^{\gamma} \qquad 1^{\,n\,k\,z}$$

P kþ þ ¼ −ne ðþ Þ Eiðnðγ þ z Þ Þ; x

z n

ðC7Þ

and the representation of the exponential integral function

$$\infty \; x^{n}$$

$$\text{Eiðx Þ} = \gamma + \ln(jx| + \sum_{n=1} \frac{}{n\,n!}; \qquad (C8)$$

¼

where γ is Euler's constant, we find in the high temperature limit $T \gg \omega_q$

δωðqTÞ ¼ −g30ζπω2Tq4 1þ2415πm2Ta22 þOðm4a=T4Þþ:ðC9Þ

[1] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38, 1440 (1977); Phys. Rev. D 16, 1791 (1977).
[2] S. Weinberg, Phys. Rev. Lett. 40, 223 (1978).
[3] F. Wilczek, Phys. Rev. Lett. 40, 279 (1978).
[4] J. Preskill, M. B. Wise, and F. Wilczek, Phys. Lett. 120B, 127 (1983).
[5] L. F. Abbott and P. Sikivie, Phys. Lett. 120B, 133 (1983).
[6] M. Dine and W. Fischler, Phys. Lett. 120B, 137 (1983).
[7] T. Banks and M. Dine, Nucl. Phys. B479, 173 (1996).
[8] A. Ringwald, Phys. Dark Universe 1, 116 (2012).
[9] D. J. E. Marsh, Phys. Rep. 643, 1 (2016); F. Chadha-Day, J. Ellis, and D. J. E. Marsh, Sci. Adv. 8, abj3618 (2022); D. J. E. Marsh, arXiv:1712.03018; A. Diez-Tejedor and D. J. E. Marsh, arXiv:1702.02116; J. E. Kim and D. J. E. Marsh, Phys. Rev. D 93, 025027 (2016).
[10] P. Sikivie, Rev. Mod. Phys. 93, 015004 (2021).
[11] P. Sikivie, Lect. Notes Phys. 741, 19 (2008).
[12] W. Hu, R. Barkana, and A. Gruzinov, Phys. Rev. Lett. 85, 1158 (2000).
[13] L. Hui, J. P. Ostriker, S. Tremaine, and E. Witten, Phys. Rev. D 95, 043541 (2017).
[14] N. Banik, A. J. Christopherson, P. Sikivie, and E. M. Todarello, Phys. Rev. D 95, 043542 (2017).
[15] CAST Collaboration, Nat. Phys. 13, 584 (2017).
[16] ADMX Collaboration, Phys. Rev. Lett. 127, 261803 (2021).
[17] P. W. Graham, I. G. Irastorza, S. K. Lamoreaux, A. Lindner, and K. A. van Bibber, Annu. Rev. Nucl. Part. Sci. 65, 485 (2015).
[18] M. S. Turner, Phys. Rev. D 28, 1243 (1983); 33, 889 (1986).
[19] Z. G. Berezhiani, M. Yu. Khlopov, and R. R. Khomeriki, Sov. J. Nucl. Phys. 52, 65 (1990); Z. G. Berezhiani, A. S. Sakharov, and M. Yu. Khlopov, Sov. J. Nucl. Phys. 55, 1063 (1992).
[20] P. Carenza, A. Mirizzi, and G. Sigl, Phys. Rev. D 101, 103016 (2020).
[21] A. Arza, T. Schwetz, and E. Todarello, J. Cosmol. Astropart. Phys. 10 (2020) 013.
[22] D. S. Lee and K-W.Ng, Phys. Rev. D 61, 085003 (2000).
[23] L. D. McLerran, E. Mottola, and M. E. Shaposhnikov, Phys. Rev. D 43, 2027 (1991).
[24] A. Papageorgiou, P. Quílez, and K. Schmitz, J. High Energy Phys. 01 (2023) 169.
[25] K. Choi, S. H. Im, H. J. Kim, and H. Seong, J. High Energy Phys. 02 (2023) 180.
[26] M. Bolz, A. Brandenburg, and W. Buchmuller, Nucl. Phys. B606, 518 (2001); B790, 336(E) (2008).
[27] E. Masso, F. Rota, and G. Zsembinszki, Phys. Rev. D 66, 023004 (2002).
[28] S. Cao and D. Boyanovsky, Phys. Rev. D 106, 123503 (2022).
[29] D. Harari and P. Sikivie, Phys. Lett. B 289, 67 (1992).
[30] E. Komatsu, Nat. Rev. Phys. 4, 452 (2022).
[31] Y. Minami and E. Komatsu, Phys. Rev. Lett. 125, 221301 (2020).
[32] P. Diego-Palazuelos et al., J. Cosmol. Astropart. Phys. 01 (2023) 044; arXiv:2203.04830.
[33] N. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2007).
[34] C. Gardiner and P. Zoeller, Quantum Noise (SpringerVerlag, Berlin, 2010).
[35] G. Lindblad, Commun. Math. Phys. 48, 119 (1976).
[36] V. Gorini, A. Kossakowski, and E. C. D. Sudarshan, J. Math. Phys. (N.Y.) 17, 821 (1976).
[37] G. Pearle, Eur. J. Phys. 33, 805 (2012).
[38] S. Weinberg, Phys. Rev. A 90, 042102 (2014); S. Weinberg, Phys. Rev. A 93, 032124 (2016); S. Weinberg, Phys. Rev. A 94, 042117 (2016).
[39] T. Banks, L. Susskind, and M. Peskin, Nucl. Phys. B244, 125 (1984).
[40] C. Burrage, C. Käding, P. Millington, and J. Minár,ˇ Phys. Rev. D 100, 076003 (2019).
[41] Y. Akamatsu, Prog. Part. Nucl. Phys. 123, 103932 (2022); 91, 056002 (2015).
[42] X. Yao, Int. J. Mod. Phys. A 36, 2130010 (2021).

[43] Y. Akamatsu and T. Miura, EPJ Web Conf. 258, 01006 (2022).
[44] N. Brambilla, M. A. Escobedo, J. Soto, and A. Vairo, Phys. Rev. D 96, 034021 (2017); N. Brambilla, M. A. Escobedo, J. Soto, and A. Vairo, Phys. Rev. D 97, 074009 (2018).
[45] D. Boyanovsky, New J. Phys. 17, 063017 (2015); Phys. Rev. D 106, 045019 (2022); 92, 023527 (2015).
[46] T. J. Hollowood and J. I. McDonald, Phys. Rev. D 95, 103521 (2017).
[47] S. Shandera, N. Agarwal, and A. Kamal, Phys. Rev. D 98, 083535 (2018).
[48] S. Brahma, A. Berera, and J. Calderón-Figueroa, J. High Energy Phys. 08 (2022) 225.
[49] T. Colas, J. Grain, and V. Vennin, Eur. Phys. J. C 82, 1085 (2022).
[50] A. D. Hammou and N. Bartolo, arXiv:2211.07598.
[51] R. Kubo, J. Phys. Soc. Jpn. 12, 570 (1957); P. C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).
[52] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003); Ann. Phys. (Leipzig) 9, 855 (2000).