

Kodaira dimension & the Yamabe problem, II

MICHAEL ALBANESE AND CLAUDE LEBRUN

For compact complex surfaces (M^4, J) of Kähler type, it was previously shown [30] that the sign of the Yamabe invariant $\mathscr{Y}(M)$ only depends on the Kodaira dimension $\text{Kod}(M, J)$. In this paper, we prove that this pattern in fact extends to all compact complex surfaces **except** those of class VII. In the process, we also reprove a result from [2] that explains why the exclusion of class VII is essential here.

The *Yamabe invariant* $\mathscr{Y}(M)$ of a smooth compact n -manifold M is a diffeomorphism invariant, introduced by [24] and [36], that largely captures the global behavior of the scalar curvature $s = s_g$ for all possible Riemannian metrics g on M . A cartoon version of the idea would be to just take the supremum of the scalar curvatures of all unit-volume constant-scalar-curvature metrics on M . However, in the precise definition of the invariant, one only considers those constant-scalar-curvature metrics that are **Yamabe minimizers**; this does not change the **sign** of the supremum, but it does in particular guarantee that it is always finite. Thus the **Yamabe invariant** of M is defined to be

$$(1) \quad \mathscr{Y}(M) := \sup_{\gamma} \inf_{g \in \gamma} \frac{\int_M s_g \, d\mu_g}{\left[\int_M d\mu_g\right]^{\frac{n-2}{n}}}$$

where $d\mu_g$ denotes the metric volume measure and γ varies over all conformal classes of Riemannian metrics g on M . With this convention, $\mathscr{Y}(M) > 0$ iff M admits a metric of scalar curvature $s > 0$, whereas $\mathscr{Y}(M) \geq 0$ iff M admits a unit-volume metric of scalar curvature $s > -\epsilon$ for every $\epsilon > 0$.

When $n = 4$ and M is the underlying smooth manifold of a compact complex surface (M^4, J) , our purpose here is to study the interplay between the Yamabe invariant and a complex-analytic invariant of (M, J) called the

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Kodaira dimension. For a compact complex m -manifold (M^{2m}, J) , the Kodaira dimension [3, 17, 27] is by definition

$$\mathrm{Kod}(M, J) = \limsup_{j \rightarrow +\infty} \frac{\log \dim H^0(M, \mathcal{O}(K^{\otimes j}))}{\log j}$$

where $K = \Lambda^{m,0}$ is the *canonical line bundle* of (M, J) . The value of this invariant always belongs to $\{-\infty, 0, 1, \dots, m\}$, because the Kodaira dimension actually coincides with the largest complex dimension of the image of $M \dashrightarrow \mathbb{P}[H^0(M, \mathcal{O}(K^{\otimes j}))^*]$ among all the possible “pluricanonical” maps associated with the line bundles $K^{\otimes j}$, $j \in \mathbb{Z}^+$, subject to the rather unusual convention of setting $\dim \emptyset := -\infty$.

In complex dimensions $m \geq 3$, the Kodaira dimension is not a diffeomorphism invariant [10, 29, 34], and hence is essentially unrelated to the Yamabe invariant. By contrast, the situation is completely different when $m = 1$ or 2 . When $m = 1$, classical Gauss-Bonnet immediately implies that (1) simplifies to become $\mathscr{V}(M^2) = 4\pi\chi(M^2)$, while classical Riemann-Roch implies that the Kodaira dimension is determined by the sign of the Euler characteristic. When $m = 2$, by combining results from Seiberg-Witten theory with constructions of special sequences of Riemannian metrics, the second author was able to prove the following analogous result [30]:

Theorem (LeBrun). *Let M be the smooth 4-manifold underlying a compact complex surface (M^4, J) of Kähler type. Then*

$$\begin{aligned} \mathscr{V}(M) > 0 &\iff \mathrm{Kod}(M, J) = -\infty, \\ \mathscr{V}(M) = 0 &\iff \mathrm{Kod}(M, J) = 0 \text{ or } 1, \\ \mathscr{V}(M) < 0 &\iff \mathrm{Kod}(M, J) = 2. \end{aligned}$$

Here the Kähler-type condition is equivalent [3, 9, 40] to requiring that $b_1(M)$ be even. In the present paper, we will prove a partial generalization of this result that allows for the possibility that $b_1(M)$ might be odd:

Theorem A. *Let M be the underlying smooth 4-manifold of any compact complex surface (M^4, J) of Kodaira dimension $\neq -\infty$. Then*

$$\begin{aligned} \mathscr{V}(M) = 0 &\iff \mathrm{Kod}(M, J) = 0 \text{ or } 1, \\ \mathscr{V}(M) < 0 &\iff \mathrm{Kod}(M, J) = 2. \end{aligned}$$

One cornerstone of Kodaira’s classification of complex surfaces [3, 17] is the *blow-up* operation, which replaces a point of a complex surface Y

with a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(-1)$; this then produces a new complex surface M that is diffeomorphic to $Y \# \overline{\mathbb{CP}}_2$, where $\overline{\mathbb{CP}}_2$ denotes the smooth oriented 4-manifold obtained by equipping $\mathbb{CP}_2$ with the non-standard orientation. Conversely, any complex surface M containing a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(-1)$ can be *blown down* to produce a new complex surface Y of which M then becomes the blow-up. In principle, this blow-down procedure can then be iterated, but the process must terminate after finitely many steps, because each blow-down decreases b_2 by 1. When a complex surface X cannot be blown down, it is called *minimal*, and the upshot is that any complex surface M can be obtained from a minimal complex surface X by blowing up finitely many times. In this situation, one then says that X is a *minimal model* of M . Blowing up or down always leaves the Kodaira dimension unchanged. Moreover, the minimal model of a complex surface is actually *unique* whenever $\text{Kod} \neq -\infty$.

Our proof of Theorem [A](#) also yields the following additional main result, which was previously proved in [\[28, 30\]](#) when $b_1(M)$ is even:

Theorem B. *Let (M, J) be a compact complex surface with $\text{Kod} \neq -\infty$, and let (X, J') be its minimal model. Then*

$$\mathscr{V}(M) = \mathscr{V}(X).$$

Given the results previously proved in [\[30\]](#), Theorems [A](#) and [B](#) just require one to show that no properly elliptic complex surface with b_1 odd can admit Riemannian metrics of positive scalar curvature. In [§1](#) below, we give a covering argument that reduces this claim to a lemma asserting that blow-ups of T^2 -bundles over high-genus Riemann surfaces cannot admit positive-scalar-curvature Riemannian metrics. The subsequent two sections then give two entirely different proofs of this lemma. Our first proof, detailed in [§2](#) and previously sketched in [\[31\]](#), uses the stable-minimal-hypersurface method of Schoen and Yau [\[38\]](#). Our second proof, laid out in [§3](#) below, instead deduces the lemma from a curvature estimate implicitly proved by Kronheimer [\[25\]](#), and is closer in spirit to [\[28, 30\]](#) because of the leading role played by the Seiberg-Witten equations. We then go on, in [§4](#), to explain why Theorems [A](#) and [B](#) require the exclusion of the $\text{Kod} = -\infty$ case, while in the process giving a simplified proof of the main result of [\[2\]](#). We then conclude, in [§5](#), with a discussion of pertinent related results and various open problems.

1. Reducing the problem to a lemma

Any complex surface of $\text{Kod} = 2$ is algebraic [3], and hence of Kähler type. Since the Kähler-type versions of Theorems A and B were previously proved in [30] Theorems A and 2], we therefore only need to address the cases of complex surfaces (M^4, J) with b_1 odd and $\text{Kod} = 0$ or 1. Any such complex surface is necessarily elliptic [3], in the sense that it must admit a holomorphic map to a complex curve with generic fiber diffeomorphic to $\mathbb{T}^2$. It therefore follows [30] Corollary 1] that any such (M, J) satisfies $\mathscr{V}(M) \geq 0$, because one can construct sequences of Riemannian metrics on such spaces with $\int s^2 d\mu \rightarrow 0$. However, any complex surface with b_1 odd and $\text{Kod} = 0$ is finitely covered by some blow-up of a primary Kodaira surface, which is then a symplectic 4-manifold with $b_+ = 2$; since a celebrated result of Taubes [41] then implies that this finite cover carries a Seiberg-Witten basic class, and therefore does not admit metrics of positive scalar curvature, one may therefore conclude [30] p. 153] that $\mathscr{V}(M) = 0$. To prove Theorems A and B, it therefore suffices to prove that if $\text{Kod}(M^4, J) = 1$ and $b_1(M) \equiv 1 \pmod{2}$, then M cannot admit a Riemannian metric of positive scalar curvature. We will deduce this from the following narrower statement:

Lemma C. *Let Σ denote a compact Riemann surface of genus ≥ 2 , and let $N \rightarrow \Sigma$ be a principal $\mathbf{U}(1)$ -bundle of non-zero Chern class. Set $X = N \times S^1$, and let $M = X \# k\overline{\mathbb{CP}}_2$ for some integer $k \geq 0$. Then M does not carry any Riemannian metric g of positive scalar curvature.*

In §§2–3 below, we will prove this lemma twice, in two entirely different ways. In the meantime, though, we begin by carefully explaining why this lemma suffices to imply our main results.

Proposition 1. *Lemma C implies Theorems A and B*

Proof. Per the above discussion, it suffices to show that whenever $b_1(M)$ is odd and $\text{Kod}(M, J) = 1$, the smooth 4-manifold M cannot admit metrics of positive scalar curvature. Let (X, J') denote the minimal model of (M, J) . Because $\text{Kod}(X, J') = \text{Kod}(M, J) = 1$, normalization of some pluricanonical map defines an elliptic fibration $\varpi : X \rightarrow \Sigma$, where Σ is a smooth connected complex curve. Because $b_1(X) = b_1(M)$ is odd, there must be an element of $H^1(X, \mathcal{O})$ that is non-trivial on some fiber. An argument due to Brînzănescu [8] therefore shows that the first direct image sheaf $\varpi_*^1 \mathcal{O}$ must be a holomorphically trivial line bundle on Σ , because its degree is *a priori*

non-positive by [3 Theorem III.18.2]. Hence no fiber can just be a union of rational curves, and ϖ can therefore at worst have multiple fibers with smooth reduction. We now equip Σ with an orbifold structure by giving each point a weight equal to the multiplicity of the corresponding fiber. Because $\text{Kod}(X, J') = \text{Kod}(M, J) = 1$, we must have $\chi^{\text{orb}}(\Sigma) \leq 0$, because (X, J') would otherwise [16 §2.7] be a Hopf surface, and so have $\text{Kod} = -\infty$. In particular, Σ must be a *good orbifold* in the sense of Thurston [43]. Thus $\Sigma = \widehat{\Sigma}/\Gamma$, where $\widehat{\Sigma}$ is a smooth complex curve of positive genus, and where the finite group Γ acts biholomorphically on $\widehat{\Sigma}$. Pulling X back to $\widehat{\Sigma}$ then produces an (unramified) cover $\widehat{X} \rightarrow X$ equipped with a holomorphic submersion $\widehat{\varpi} : \widehat{X} \rightarrow \widehat{\Sigma}$. However, the j -invariant of the fibers now defines a holomorphic map $\widehat{\Sigma} \rightarrow \mathbb{C}$, and this map is of course constant because $\widehat{\Sigma}$ is compact. Thus $\widehat{\varpi}$ is locally holomorphically trivial, with fibers isomorphic to some elliptic curve E , and the obstruction to this being a principal E -bundle is then measured by the rotational monodromy map $\pi_1(\widehat{\Sigma}) \rightarrow \text{Aut}(E)/E$. However, $\text{Aut}(E)$ is compact, so $\text{Aut}(E)/E = \pi_0(\text{Aut}(E))$ is finite, and we may therefore kill this monodromy simply by replacing $\widehat{\Sigma}$ with a finite cover if necessary. This shows [8, 45] that there is a finite cover $\widehat{X}$ of X that is just a principal $[\mathbf{U}(1) \times \mathbf{U}(1)]$ -bundle over a compact complex curve $\widehat{\Sigma}$. Since $\text{Kod}(\widehat{X}) = 1$, the base $\widehat{\Sigma}$ must necessarily have genus ≥ 2 . However, the Chern classes of the two $\mathbf{U}(1)$ -factors of our principal $[\mathbf{U}(1) \times \mathbf{U}(1)]$ -bundle $\widehat{X} \rightarrow \widehat{\Sigma}$ are necessarily linearly dependent over $\mathbb{Q}$. Thus, by passing to a higher cover $\widetilde{X}$ fibering over $\widehat{\Sigma}$ and then changing basis as necessary, we may arrange for one of these Chern classes to be zero. However, they cannot both be zero, because $b_1(\widetilde{X})$ is necessarily odd; otherwise, we could produce a forbidden Kähler metric on X by taking fiber-averages of local push-forwards of some Kähler metric on the cover. It thus follows that X has an unbranched cover $\widetilde{X} \approx N \times S^1$, where $N \rightarrow \widehat{\Sigma}$ is a principal $\mathbf{U}(1)$ -bundle of non-zero degree over a Riemann surface of genus ≥ 2 . Since M is obtained from X by blowing up points, we therefore obtain an induced unbranched cover $\widetilde{M} \rightarrow M$ with $\widetilde{M} \approx (N \times S^1) \# k\overline{\mathbb{CP}}_2$. Lemma [C] now asserts that $\widetilde{M}$ cannot carry a metric of positive scalar curvature; and since we can pull Riemannian metrics back via $\widetilde{M} \rightarrow M$, this then implies that M cannot carry a positive-scalar-curvature metric either. Hence Lemma [C] implies that $\mathscr{Y}(M) = 0$. Moreover, since the same argument in particular applies to X , Lemma [C] also implies $\mathscr{Y}(M) = \mathscr{Y}(X)$. Hence Lemma [C] implies both Theorem [A] and Theorem [B] as claimed. $\square$

2. Proof via stable minimal hypersurfaces

In this section, we give a proof of Lemma [C](#) largely based on the Schoen-Yau stable-minimal-hypersurface method [\[38\]](#). For the broader context, see [§5](#).

Let (M, g) be a smooth compact oriented Riemannian $(\ell + 1)$ -manifold, $\ell \leq 6$, and let $\mathbf{a} \in H^1(M, \mathbb{Z})$ be a non-trivial cohomology class. Compactness results in geometric measure theory [\[15\]](#) guarantee that there is a mass-minimizing integral current $\mathcal{Z}$ that represents the Poincaré dual homology class in $H_\ell(M, \mathbb{Z})$, and, because $\ell < 7$, regularity results [\[39\]](#) then guarantee [\[26\]](#) that this current is just a sum

$$\mathcal{Z} = \sum n_j Z_j$$

of disjoint smooth embedded compact connected oriented ℓ -dimensional hypersurfaces $Z_j \subset M$ with positive integer multiplicities.

Because $\mathcal{Z}$ is mass-minimizing, each Z_j must be a stable minimal hypersurface. Thus, if $Z = Z_j$ for some j , and if $Z(t)$ is any smooth family of hypersurfaces in M with $Z(0) = Z$, the ℓ -dimensional volumes $\mathcal{A}(t)$ of the $Z(t)$ must satisfy

$$\mathcal{A}'(0) = 0, \quad \mathcal{A}''(0) \geq 0.$$

If h and $\mathbb{I}$ denote the induced metric and second fundamental form of Z , the vanishing of $\mathcal{A}'(0)$ is of course equivalent to the vanishing of the mean curvature $H = h^{ij} \mathbb{I}_{ij}$. On the other hand, if the normal component of $Z'(0)$ for this family is given by $u \vec{n}$, where $\vec{n}$ is the unit normal vector of Z , then, in conjunction with the Gauss-Codazzi equations, Jim Simons' second-variation formula [\[39\]](#) Theorem 3.2.2] tells us that

$$\mathcal{A}''(0) = \int_Z \left[|\nabla u|^2 + \frac{1}{2}(s_h - s_g - |\mathbb{I}|^2)u^2 \right] d\mu_h,$$

so that the homologically-mass-minimizing property of $\mathcal{Z}$ implies that

$$(2) \quad \int_Z [2|\nabla u|^2 + s_h u^2] d\mu_h \geq \int_Z s_g u^2 d\mu_h$$

for every smooth function u on Z .

Now assume that (M, g) has positive scalar curvature $s_g > 0$, and observe that [\(2\)](#) then implies that

$$(3) \quad \int_Z [2|\nabla u|^2 + s_h u^2] d\mu_h > 0$$

for every smooth function $u \neq 0$ on Z . If $\ell = 2$, plugging $u \equiv 1$ into (3) immediately implies that $\chi(Z) > 0$ by classical Gauss-Bonnet, and classical uniformization therefore tells us that there is a conformal diffeomorphism between (Z^2, h) and $(S^2, \hat{h})$, where $\hat{h}$ is the standard unit-sphere metric of scalar curvature $+2$. Similarly, when $\ell \geq 3$, the induced metric h is also conformal to a metric $\hat{h}$ of positive scalar curvature. Indeed, if $p = \frac{2\ell}{\ell-2}$ and if $u > 0$, any conformally related metric $\hat{h} = u^{p-2}h$ has volume form $d\hat{\mu} = u^p d\mu$ and scalar curvature

$$\hat{s} = u^{1-p} [(p+2)\Delta + s] u,$$

where $\Delta = d^*d = -\nabla \cdot \nabla$, so (3) implies that any $\hat{h}$ conformal to h satisfies

$$\int_Z \hat{s} d\hat{\mu} = \int_Z [(p+2)|\nabla u|^2 + s_h u^2] d\mu_h \geq \int_Z [2|\nabla u|^2 + s_h u^2] d\mu_h > 0.$$

It therefore suffices to recall that any conformal class $[h]$ on Z contains a metric $\hat{h} = u^{p-2}h$ whose scalar curvature does not change sign. Here we could, for example, invoke Schoen's proof [35] of the Yamabe conjecture, and take $\hat{h}$ to be a Yamabe metric, which in particular has constant scalar curvature. Or we could opt for the more elementary trick, due to Trudinger [44], of just taking $u > 0$ to belong to (and hence span) the lowest eigenspace of the Yamabe Laplacian $(p+2)\Delta + s_h$.

These ideas suffice to provide a simple, clean proof of Lemma C. To see why, we begin by introducing the following useful concept:

Definition 1. A smooth compact connected oriented 3-manifold N will be called *expansive* if there is a smooth map $\phi : N \rightarrow V$ of non-zero degree to a compact oriented aspherical 3-manifold V with $b_1(V) \neq 0$.

Here a compact manifold is called *aspherical* if it is an Eilenberg-MacLane space $K(\pi, 1)$. Thus, a compact manifold is aspherical if and only if its universal cover is contractible.

One immediate consequence of Definition 1 is the following:

Lemma 1. *If a smooth compact connected 3-manifold N' admits a map of non-zero degree to an expansive manifold N , then N' is itself expansive.*

Proof. If $\psi : N' \rightarrow N$ and $\phi : N \rightarrow V$, then $\deg(\phi \circ \psi) = \deg(\phi) \deg(\psi)$. $\square$

By contrast, the following key consequence is distinctly less trivial:

Lemma 2. *Expansive 3-manifolds never admit metrics of positive scalar curvature.*

Proof. If N is an expansive 3-manifold, then, by definition, there is a smooth map $\phi : N \rightarrow V$ of $\deg(\phi) \neq 0$, where V is a $K(\pi, 1)$ with $b_1(V) \neq 0$. Since $b_1(V) \neq 0$, there exists some non-zero $\mathbf{a} \in H^1(V, \mathbb{Z})$. Poincaré duality then guarantees the existence of some $\mathbf{b} \in H^2(V, \mathbb{Z})$ with $\int_V \mathbf{a} \cup \mathbf{b} \neq 0$.

We now proceed by contradiction. Suppose that N admits a Riemannian metric g of positive scalar curvature. Let $\mathcal{Z}$ be a mass-minimizing integral current representing the Poincaré dual of $\phi^*\mathbf{a}$ in $H_2(N, \mathbb{Z})$, so that

$$\mathcal{Z} = \sum n_j Z_j$$

for some collection of compact embedded oriented surfaces Z_j and integer coefficients n_j . Since g has $s > 0$, the above second-variation argument implies that each Z_j is a 2-sphere. However, since $\pi_2(V) = 0$, this means that $\phi_*([Z_j]) \in H_2(V, \mathbb{Z})$ must vanish for every j . Hence

$$0 = \sum_j n_j \int_{\phi_*[Z_j]} \mathbf{b} = \int_{\sum_j n_j [Z_j]} \phi^*\mathbf{b} = \int_N \phi^*\mathbf{a} \cup \phi^*\mathbf{b} = \deg(\phi) \int_V \mathbf{a} \cup \mathbf{b} \neq 0.$$

This contradiction therefore shows that such a metric cannot exist. $\square$

Similar ideas therefore now allow us to deduce the following result:

Theorem 1. *Let N^3 be an expansive 3-manifold, and let X^4 be a smooth compact oriented 4-manifold that admits a smooth submersion $\phi : X \rightarrow S^1$ with fiber N . Let P be any connected smooth compact oriented 4-manifold, and let $M = X \# P$. Then M does not admit Riemannian metrics of positive scalar curvature, and therefore satisfies $\mathcal{Y}(M) \leq 0$.*

Proof. Let $\bar{\phi} : X \# P \rightarrow X$ be the smooth “blowing down” map that collapses $(P - B^4) \subset M$ to a point, and let

$$f = \phi \circ \bar{\phi} : M \rightarrow S^1 = \mathbf{U}(1)$$

be the induced projection. We now recall that

$$(4) \quad H^1(M, \mathbb{Z}) = C^\infty(M, \mathbf{U}(1)) / \exp [2\pi i C^\infty(M, \mathbb{R})],$$

because there is short exact sequence of sheaves of Abelian groups

$$0 \rightarrow \mathbb{Z} \rightarrow C^\infty(_, \mathbb{R}) \xrightarrow{\exp 2\pi i \cdot} C^\infty(_, \mathbf{U}(1)) \rightarrow 0,$$

where $C^\infty(_, \mathbb{R})$ is a fine sheaf. The pre-image $f^{-1}(z)$ of a regular value is thus a copy of N in M whose homology class $[N] \in H_3(M, \mathbb{Z})$ is Poincaré dual to $[f] \in H^1(M, \mathbb{Z})$. Given any metric g on M , we now represent the homology class $[N]$ by a mass-minimizing rectifiable current $\mathcal{Z}$, which can then be written as a sum of disjoint smooth oriented connected compact hypersurfaces $Z_1, \dots, Z_k$ with positive integer multiplicities; in particular,

$$(5) \quad [N] = \sum_{j=1}^k n_j [Z_j], \quad n_j \in \mathbb{Z}^+.$$

We now choose a pairwise-disjoint collection of closed tubular neighborhoods $U_j \approx Z_j \times [-1, 1]$ of the Z_j , and then express each Z_j as $f_j^{-1}(1)$ for a smooth map $f_j : M \rightarrow \mathbf{U}(1)$ that is $\equiv -1$ outside of U_j , and given on U_j by $f_j = e^{i\Upsilon_j}$ for a smooth orientation-compatible defining function $\Upsilon_j : U_j \rightarrow [-\pi, \pi]$ for Z_j that is $\equiv \pm\pi$ near each boundary component of U_j . Next, we define

$$\widehat{f} := \prod_j f_j^{n_j} : M \rightarrow \mathbf{U}(1)$$

where the product is of course taken point-wise in $\mathbf{U}(1)$. Since each $[f_j]$ is Poincaré dual to $[Z_j]$, it follows from (5) that $[\widehat{f}]$ is therefore Poincaré dual to $\sum n_j [Z_j] = [N]$. Since f and $\widehat{f}$ therefore represent the same class in $H^1(M, \mathbb{Z})$, equation (4) therefore tells us that

$$f = e^{2\pi i w} \widehat{f}$$

for some smooth real valued function $w : M \rightarrow \mathbb{R}$, and we thus obtain an explicit homotopy of $\widehat{f}$ to f by setting $f_t = e^{2\pi i t w} \widehat{f}$ for $t \in [0, 1]$. However, $\widehat{f}$ is constant on each Z_j by construction, so $f|_{Z_j} \sim \widehat{f}|_{Z_j}$ must induce the zero homomorphism $\pi_1(Z_j) \rightarrow \pi_1(S^1)$, and the inclusion map $\imath_j : Z_j \hookrightarrow M$ therefore lifts to an embedding $\widetilde{\imath}_j : Z_j \hookrightarrow \widetilde{M}$ of this hypersurface in the covering space $\widetilde{M} \rightarrow M$ corresponding to the kernel of $f_* : \pi_1(M) \rightarrow \pi_1(S^1)$.

On the other hand, we can identify X with the mapping torus

$$\mathfrak{Y}_\varphi := (N \times \mathbb{R}) / \langle (x, t) \mapsto (\varphi(x), t + 2\pi) \rangle$$

of some diffeomorphism $\varphi : N \rightarrow N$ by simply choosing a vector field on X that projects to $\partial/\partial\vartheta$ on S^1 , and then following the flow. Since the diffeotype of $\mathfrak{Y}_\varphi \rightarrow S^1$ only depends on the isotopy class of φ , we may also assume that φ has a fixed-point $p \in N$. The flow-line $\{p\} \times \mathbb{R}$ then covers an embedded

circle $S^1 \hookrightarrow X$, which we may moreover take to avoid the ball where surgery is to be performed to construct $M = X \# P$. This circle in X then also defines an embedded circle in M , which we will call the *reference circle*; and by first making a small perturbation, if necessary, we may assume that this reference circle $S^1 \hookrightarrow M$ is also transverse to the Z_j . Since N has intersection $+1$ with the reference circle, equation (5) tells us that at least one of the hypersurfaces Z_j has non-zero intersection with the reference circle. Setting $Z = Z_j$ for some such j , we will henceforth denote the corresponding inclusion map $\mathfrak{u}_j$ by $\mathfrak{u} : Z \hookrightarrow M$, and its lift $\tilde{\mathfrak{u}}_j$ by $\tilde{\mathfrak{u}} : Z \hookrightarrow \tilde{M}$. Our mapping-torus model of X now gives us a diffeomorphism $\tilde{M} = (N \times \mathbb{R}) \# (\#_{\ell=1}^{\infty} P)$, along with a blow-down map $\hat{\mathfrak{b}} : \tilde{M} \rightarrow N \times \mathbb{R}$ that lifts $\mathfrak{b} : M \rightarrow X$. However, $\{p\} \times \mathbb{R} \subset N \times \mathbb{R}$ now meets $\hat{\mathfrak{b}} \circ \tilde{\mathfrak{u}}(Z)$ transversely, in a set whose oriented count exactly computes the homological intersection number $\mathfrak{n} \neq 0$ of our reference circle S^1 with Z . Thus, if $\pi : N \times \mathbb{R} \rightarrow N$ denotes the first-factor projection, the smooth map $\pi \circ \hat{\mathfrak{b}} \circ \tilde{\mathfrak{u}} : Z \rightarrow N$ has degree $\mathfrak{n} \neq 0$.

Since N is expansive by hypothesis, this shows that Z is also expansive by Lemma 1. Hence the given metric g cannot possibly have positive scalar curvature. Indeed, if it did, the induced metric h on $Z = Z_j$ would be conformal to a metric $\hat{h}$ of positive scalar curvature by the second-variation argument detailed above. But since Z is expansive, Lemma 2 forbids the existence of a positive-scalar-curvature metric $\hat{h}$ on Z . Hence M cannot admit a metric g of positive scalar curvature either, and $\mathscr{V}(M) \leq 0$, as claimed. $\square$

Theorem 1 now immediately implies Lemma C. Indeed, let $N \rightarrow \Sigma$ be any non-trivial circle bundle over a hyperbolic Riemann surface. Then N^3 is aspherical, because its universal cover $\mathcal{H}^2 \times \mathbb{R} \approx \mathbb{R}^3$ is contractible. Moreover, $b_1(N) = b_1(\Sigma) \neq 0$. Hence N is expansive, because the identity $N \rightarrow N$ is a degree-one map to a $K(\pi, 1)$ with $b_1 \neq 0$. We next set $X = N \times S^1$, so that the second-factor projection $\phi : X \rightarrow S^1$ is then a submersion with fiber N . Finally, letting P be the connected sum of k copies of $\overline{\mathbb{CP}}_2$, we observe that the manifold $M = (N \times S^1) \# k\overline{\mathbb{CP}}_2$ of Lemma C is one of the 4-manifolds covered by Theorem 1. Since Lemma C implies Theorems A and B by Proposition 1, this provides one complete proof of our main results.

Nonetheless, by invoking Perelman's proof [6, 23] of Thurston's geometrization conjecture, we can improve Theorem 1 to yield the following sharp result:

Theorem 2. *Let N^3 be compact oriented connected 3-manifold, and let X^4 be a smooth compact oriented 4-manifold that admits a smooth submersion*

$\phi : X \rightarrow S^1$ with fiber N . Let P be any connected smooth compact oriented 4-manifold, and let $M = X \# P$. Then $\mathscr{Y}(N) \leq 0 \implies \mathscr{Y}(M) \leq 0$.

To prove this, we just need to first demonstrate the following:

Proposition 2. *If $\phi : Z \rightarrow N$ is a map of non-zero degree between compact oriented connected 3-manifolds, then $\mathscr{Y}(Z) > 0 \implies \mathscr{Y}(N) > 0$.*

Proof. Perelman's Ricci-flow proof of Thurston's uniformization conjecture implies [6, 23] that a closed oriented 3-manifold Z admits a metric of positive scalar curvature iff it is a connected sum of spherical spaces forms S^3/Γ_j and/or copies of $S^2 \times S^1$. Geometrization then goes on to tell us that a closed oriented 3-manifold N that does not admit metrics of positive scalar curvature must therefore be expressible as a connected sum $N = N_0 \# V$, where V is a $K(\pi, 1)$ and N_0 is some other 3-manifold. Consequently, if $\mathscr{Y}(N) \leq 0$, there is a degree-one map $\psi : N \rightarrow V$ to some compact oriented aspherical 3-manifold V obtained by collapsing $N_0 - B^3$ to a point.

Arguing by contradiction, we now assume that there exists some map $\phi : Z \rightarrow N$ of non-zero degree, where $\mathscr{Y}(Z) > 0$ but $\mathscr{Y}(N) \leq 0$. Letting $\psi : N \rightarrow V$ be a degree-one map to an aspherical manifold, we thus obtain a map $\Psi := \psi \circ \phi : Z \rightarrow V$ of non-zero degree, where V is a $K(\pi, 1)$ and

$$Z \approx (S^3/\Gamma_1) \# \cdots \# (S^3/\Gamma_k) \# \underbrace{(S^2 \times S^1) \# \cdots \# (S^2 \times S^1)}_{\ell}$$

for some $\ell \geq 0$. The latter may be restated as saying that $\coprod_{j=1}^k (S^3/\Gamma_j)$ can be obtained from Z by performing surgeries in dimension 2. Since every surgery can be realized by a cobordism, this means that there is a compact oriented 4-manifold-with-boundary W with $\partial W = \overline{Z} \sqcup \coprod_{j=1}^k (S^3/\Gamma_j)$, and such that W deform-retracts to a space obtained from Z by attaching 3-disks D^3 along their boundaries at $\max(\ell, k + \ell - 1)$ disjoint embedded 2-spheres in Z . Since $\pi_2(V) = 0$, we can therefore extend Ψ across these 3-disks, and thereby obtain a map $\widehat{\Psi} : W \rightarrow V$ with $\widehat{\Psi}|_Z = \Psi$. If we define $\Psi_j : S^3/\Gamma_j \rightarrow V$ to be the restriction of $\widehat{\Psi}$ to the corresponding component of ∂W , we thus have

$$-\Psi_*[Z] + \sum_{j=1}^k \Psi_{j*}[S^3/\Gamma_j] = \widehat{\Psi}_*[\partial W] = 0 \in H_3(V, \mathbb{Z}),$$

and pairing with the generator of $H^3(V, \mathbb{Z})$ therefore yields

$$(6) \quad \deg(\Psi) = \sum_{j=1}^k \deg(\Psi_j).$$

However, because $\pi_3(V) = 0$, any map $S^3 \rightarrow V$ is homotopic to a constant, and therefore has degree 0; and since mapping-degree is multiplicative under compositions, this in turn implies that every map $S^3/\Gamma_j \rightarrow V$ also has degree zero. Thus $\deg(\Psi_j) = 0$ for all j , and hence $\deg(\Psi) = 0$ by (6). Since this contradicts the fact that $\deg(\Psi) \neq 0$ by construction, our assumption was therefore false. Hence $\mathscr{Y}(Z) > 0 \implies \mathscr{Y}(N) > 0$ whenever there is a map $\phi : Z \rightarrow N$ of non-zero degree. $\square$

The proof of Theorem 2 is now identical to the proof of Theorem 1, except that, in the very last paragraph, Proposition 2 now supplants Lemmas 1 and 2. Our proof of Lemma C can then be rephrased in terms of Theorem 2 by simply observing that our proof of Proposition 2 also proves the following:

Proposition 3. *Let N be a compact oriented 3-manifold that admits a map $\phi : N \rightarrow V$ of non-zero degree to an aspherical manifold. Then $\mathscr{Y}(N) \leq 0$.*

This sharpens Lemma 2 by dropping the requirement that $b_1(V) \neq 0$. However, since recent results of Agol, Wise, and others [1] show that compact aspherical 3-manifolds always have finite-sheeted covers with $b_1 \neq 0$, the difference here is arguably more a matter of æsthetics than of substance. In any case, the pioneering papers of Schoen-Yau [37] and Gromov-Lawson [20], which proved strikingly similar results by entirely different methods, provide attractive alternatives for addressing the 3-dimensional aspects of our story.

Because the results of this section have been formulated in much greater generality than what we'd need to just prove Lemma C, they of course have many other applications. For example, we will later see, in §4 below, that they also imply interesting results concerning complex surfaces with b_1 odd and $\text{Kod} = -\infty$. For related results, including a non-Seiberg-Witten proof of the b_1 odd, $\text{Kod} = 0$ case of Theorems A and B, see [2, Theorem 4.8].

3. Proof via the Seiberg-Witten equations

In this section, we give an entirely different proof of Lemma C based on an indirect use of the Seiberg-Witten equations pioneered by Kronheimer [25].

Let M be a 4-manifold with a fixed spin^c structure $\mathbf{c}$. For any metric g on M , this gives us two rank-2 complex vector bundles $\mathbb{V}_\pm \rightarrow M$, along with formal isomorphisms

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2}$$

where $\mathbb{S}_\pm$ are the (locally-defined) spin bundles of (M, g) , and where

$$L = \det(\mathbb{V}_+) = \det(\mathbb{V}_-)$$

is called the determinant line bundle of the spin^c structure. Since

$$c_1(L) \equiv w_2(M) \pmod{2},$$

the square-root $L^{1/2}$, like the vector bundles $\mathbb{S}_\pm$, is not globally defined if $w_2(M) \neq 0$; however, the obstruction to consistently choosing the signs of the transition functions for $L^{1/2}$ exactly cancels the obstruction to consistently choosing signs for the transition functions of $\mathbb{S}_\pm$.

The *Seiberg-Witten equations* are equations for a pair (Φ, θ) , where Φ is a section of $\mathbb{V}_+$ and θ is a $\mathbf{U}(1)$ -connection on L . They are given by

$$(7) \quad \not{D}_\theta \Phi = 0$$

$$(8) \quad F_\theta^+ = -\frac{1}{2} \Phi \odot \bar{\Phi}$$

where $\not{D}_\theta : \Gamma(\mathbb{V}_+) \rightarrow \Gamma(\mathbb{V}_-)$ is the Dirac operator coupled to θ , and F_θ^+ is the self-dual part of the curvature of θ , while $\odot$ denotes the symmetric tensor product, and we have used the identification $\Lambda^+ \otimes \mathbb{C} = \odot^2 \mathbb{S}_+$ induced by Clifford multiplication. The Seiberg-Witten equations (7-8) then imply the Weitzenböck formula

$$(9) \quad 0 = 2\Delta|\Phi|^2 + 4|\nabla_\theta \Phi|^2 + s|\Phi|^2 + |\Phi|^4,$$

which leads to striking results relating curvature and differential topology.

On complex surfaces of Kähler type, Seiberg-Witten theory plays a crucial role in the calculation of Yamabe invariants. If a 4-manifold M with $b_+ > 1$ admits an almost-complex structure, Witten [46] defined an invariant that counts the expected number of gauge-equivalence classes of solutions of the equations with multiplicities, relative to the spin^c structure arising from an almost-complex structure. When this count is non-zero, the first Chern class $c_1(L)$ of the spin^c structure is then called a *basic class*.

One of Witten's most revolutionary discoveries was that if (M, J) is a compact Kähler surface with $b_+ > 1$, then $c_1(M, J)$ is a basic class. This in particular implies that no such M can admit metrics of positive scalar curvature.

However, Seiberg-Witten basic classes do not provide a workable method for proving Lemma [C](#). Indeed, a theorem of Biquard [[7](#) Théorème 8.1] shows, for example, that $S^1 \times N$ carries no basic classes if $N \rightarrow \Sigma$ is a non-trivial circle bundle over a hyperbolic Riemann surface. It might therefore be tempting to think that the Seiberg-Witten equations have no bearing at all on properly elliptic surfaces of non-Kähler type.

However, there are [[4](#), [33](#)] still many 4-manifolds without basic classes where, for specific spin^c structures, the Seiberg-Witten equations nonetheless admit solutions for every Riemannian metric. Such a state of affairs is usually described by using the following convenient definition, which was first introduced by Peter Kronheimer [[25](#) Definition 5]:

Definition 2. Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$. An element $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion}$, is called a **monopole class** of M iff there is some spin^c structure $\mathbf{c}$ on M with first Chern class

$$c_1(L) \equiv \mathbf{a} \pmod{\text{torsion}}$$

for which the Seiberg-Witten equations ([7](#)–[8](#)) have a solution for every Riemannian metric g on M .

Unfortunately, however, the expositional prominence accorded to this idea appears to have resulted in widespread misunderstanding of Kronheimer's paper [[25](#)], which ultimately never claimed to prove the existence of monopole classes on the 4-manifolds it studied. We will therefore need to carefully discuss Kronheimer's argument in order to highlight what it actually proves.

Let us now recall that an integral cohomology class $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion}$ is said to be *characteristic* if

$$\mathbf{a} \bullet \mathbf{b} \equiv \mathbf{b} \bullet \mathbf{b} \pmod{2}, \quad \forall \mathbf{b} \in H^2(M, \mathbb{Z})/\text{torsion},$$

where $\bullet$ denotes the intersection pairing. For any spin^c structure, the image $c_1^{\mathbb{R}}(L) \in H^2(M, \mathbb{Z})/\text{torsion} \subset H^2(M, \mathbb{R})$ of $c_1(L)$ is a characteristic element, and conversely, every characteristic element arises in this way from some spin^c structure on M .

When $\mathbf{a}$ is a monopole class, applying the Weitzenböck formula (9) to solutions of the Seiberg-Witten equations implies a scalar-curvature integral estimate involving $\mathbf{a}$ that must hold for every metric on the manifold. Without even mentioning the Seiberg-Witten equations, however, it is possible to simply axiomatize the property that this estimate holds for every possible metric. Doing so then leads to the following definition:

Definition 3. Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$. A characteristic integral cohomology class $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion} \subset H^2(M, \mathbb{R})$ will be called a **mock-monopole class** of M if every Riemannian metric g on M satisfies the inequality

$$(10) \quad \int_M s_-^2 d\mu_g \geq 32\pi^2[\mathbf{a}^+]^2,$$

where $s_-(x) := \min(s_g(x), 0) \forall x \in M$, and where $\mathbf{a}^+ \in H^2(M, \mathbb{R})$ is the orthogonal projection of $\mathbf{a}$, with respect to the intersection form $\bullet$, to the $b_+(M)$ -dimensional subspace $\mathcal{H}_g^+ \subset H^2(M, \mathbb{R})$ consisting of those deRham classes that are represented by self-dual harmonic 2-forms with respect to g .

We will see in a moment that every monopole class is also a mock-monopole class. However, Kronheimer's paper [25] develops a less direct way of using the Seiberg-Witten equations to show that certain 4-manifolds carry mock-monopole classes. Before discussing this further, however, let us first observe that Definition 3 has some pertinent immediate consequences.

Proposition 4. *Let M be a smooth compact oriented 4-manifold with $b_+ \geq 2$. If M carries a non-zero mock-monopole class, then $\mathcal{Y}(M) \leq 0$.*

Proof. Since the harmonicity of 2-forms on M^4 is unaffected by conformal changes of metric $g \rightsquigarrow \widehat{g} = u^2g$, and since the self-dual/anti-self-dual decomposition $\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-$ of 2-forms is also conformally invariant, the self-dual subspace $\mathcal{H}_g^+ = \mathcal{H}_{[g]}^+$ of $H^2(M, \mathbb{R})$ only depends on the conformal class $[g]$ of the metric g . Since the intersection form $\bullet$ is positive-definite on $\mathcal{H}_{[g]}^+$ for any conformal class $[g]$, we therefore have $[\mathbf{a}^+]^2 > 0$ whenever $\mathbf{a}^+ = (\mathbf{a}^+)_g \neq 0$. Thus, if a conformal class $[g]$ satisfies $(\mathbf{a}^+)_g \neq 0$, its Yamabe constant

$$Y(M, [g]) = \inf_{\widehat{g} \in [g]} \frac{\int_M \widehat{s} \widehat{d\mu}}{\sqrt{\int_M \widehat{d\mu}}}$$

must satisfy $Y(M, [g]) < 0$, because otherwise $[g]$ would contain a metric $\widehat{g}$ with $\widehat{s} \geq 0$, and hence with $\widehat{s}_- \equiv 0$, and this would then violate the key inequality (10) that must be satisfied by any mock-monopole class.

On the other hand, because we have $\mathbf{a} \neq 0$ by hypothesis, the set of metrics $\{g \mid (\mathbf{a}^+)_g \neq 0\}$ is *dense* in the C^2 topology, as a consequence of the fact [14] Proposition 4.3.14] that the period map

$$\begin{array}{ccc} \{C^2 \text{ metrics}\} & \longrightarrow & Gr_{b_+}^+(H^2(M, \mathbb{R})) \\ g & \longmapsto & \mathcal{H}_g^+ \end{array}$$

is everywhere transverse to the set of positive planes orthogonal to $\mathbf{a}$. But the Yamabe constant $Y(M, [g])$ is a continuous function of g in the C^2 topology [5] Proposition 4.31], so taking limits then shows that $Y(M, [g]) \leq 0$ for every metric g . Consequently, $\mathscr{Y}(M) = \sup_g Y(M, [g]) \leq 0$, as claimed. $\square$

Corollary 1. *Let X be a smooth compact oriented 4-manifold with $b_+ \geq 2$, and let $M = X \# k\overline{\mathbb{CP}}_2$ for some $k \geq 1$. If M admits a mock-monopole class, then neither M nor X can admit metrics of positive scalar curvature.*

Proof. Since the Mayer-Vietoris sequence gives us a canonical isomorphism $H^2(X \# k\overline{\mathbb{CP}}_2) = H^2(X) \oplus [H^2(\overline{\mathbb{CP}}_2)]^{\oplus k}$, we may take $\mathbf{E} \in H^2(M, \mathbb{Z})/\text{torsion}$ to be a generator of one $H^2(\overline{\mathbb{CP}}_2, \mathbb{Z})$ summand in $H^2(M, \mathbb{Z})/\text{torsion}$. We then have $\mathbf{E} \bullet \mathbf{E} = -1$. If $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion}$ is a mock-monopole class, the fact that $\mathbf{a}$ is characteristic then implies that

$$\mathbf{a} \bullet \mathbf{E} \equiv \mathbf{E} \bullet \mathbf{E} \equiv 1 \pmod{2}.$$

It therefore follows that $\mathbf{a} \neq 0$, and Proposition 4] therefore tells us that M cannot admit metrics of positive scalar curvature.

However, the Gromov-Lawson surgery lemma [19] implies that the connected sum of two positive-scalar-curvature 4-manifolds also admits metrics of positive scalar curvature. Since $\overline{\mathbb{CP}}_2$ admits metrics of positive scalar curvature, while $M = X \# k\overline{\mathbb{CP}}_2$ does not, it therefore follows that X cannot admit metrics of positive scalar curvature either. $\square$

Before we discuss Kronheimer's argument, let us first see why any monopole class is also a mock-monopole class. If M carries a monopole class $\mathbf{a}$, then there is a spin^c structure $\mathfrak{c}$ with $c_1^{\mathbb{R}}(L) = \mathbf{a}$ with respect to which there is some solution (Φ, θ) of the Seiberg-Witten equations (7)–(8) for each metric

g . The Weitzenböck formula (9) therefore tells us that

$$\begin{aligned} 0 &= 2\Delta|\Phi|^2 + 4|\nabla_\theta\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \\ &\geq 2\Delta|\Phi|^2 + (s_-)|\Phi|^2 + |\Phi|^4 \end{aligned}$$

and integration therefore tells us that

$$\int_M (-s_-)|\Phi|^2 d\mu_g \geq \int_M |\Phi|^4 d\mu_g.$$

Applying the Cauchy-Schwarz inequality, we thus have

$$\left(\int_M s_-^2 d\mu_g\right)^{1/2} \left(\int_M |\Phi|^4 d\mu_g\right)^{1/2} \geq \int_M |\Phi|^4 d\mu_g$$

and squaring therefore gives us

$$\int_M s_-^2 d\mu_g \geq \int_M |\Phi|^4 d\mu_g = 8 \int_M |F_\theta^+|^2 d\mu_g$$

where the last equality is an algebraic consequence of (8). Because iF_θ^+ differs from the harmonic representative of $2\pi(\mathbf{a}^+)_g$ by the self-dual part of an exact form, we also have

$$\int_M |F_\theta^+|^2 d\mu_g \geq 4\pi^2[(\mathbf{a}^+)_g]^2,$$

and putting these two inequalities together then shows that (10) is therefore satisfied for the given metric g . Since this argument applies equally well to every other metric, it therefore follows that $\mathbf{a}$ satisfies the definition of a mock-monopole class.

While the above argument is nearly standard in Seiberg-Witten theory, typical uses of it [25, 28] never even mention the function $s_- = \min(s_g, 0)$, and instead just use it to deduce lower bounds for the important Riemannian functional

$$\int s_g^2 d\mu_g \geq \int s_-^2 d\mu_g.$$

However, given the crucial role played by s_- in the proof of Proposition 4, our objectives require us to emphasize this subtle refinement of the conclusion.

With these ideas established, we now come to Kronheimer's construction. Let N be a compact connected oriented prime 3-manifold with $b_1 \geq 2$ that is equipped with a *taut foliation*. Prime means that N cannot be expressed as a non-trivial connected sum, and the assumption that $b_1(N) \geq 2$

implies, in particular, that N is not $S^2 \times S^1$. A taut foliation amounts to a Frobenius-integrable distribution $D \subset TN$ of oriented 2-planes for which there exists a smooth closed curve $S^1 \subset N$ that meets every integral surface of D transversely. We now set $X = N \times S^1$, and define an orientation-compatible almost-complex structure J' on X that, with respect to some arbitrary Riemannian metric h on N , sends $D^\perp \subset TN$ to TS^1 , while acting on D by $+90^\circ$ rotation. Deforming J' to be integrable near certain points and then blowing up now equips $M := X \# k\overline{\mathbb{CP}}_2$ with an almost-complex structure J in a specified manner. While there are many choices involved in this construction, the homotopy class of the resulting J is independent of these choices, so this construction equips M with a preferred spin^c structure. With one modest improvement, and using the terminology we have just introduced, the proof of [25, Proposition 8] then actually proves the following:

Proposition 5 (Kronheimer). *Let N be a compact oriented connected prime 3-manifold with $b_1(N) \geq 2$ that carries an integrable oriented distribution $D \subset TN$ of 2-planes that is tangent to a taut foliation. Set $X = N \times S^1$, and equip $M = X \# k\overline{\mathbb{CP}}_2$ with the almost-complex structure J described above. Then $\mathbf{a} := c_1^\mathbb{R}(M, J) \in H^2(M, \mathbb{Z})/\text{torsion}$ is a mock-monopole class.*

Let $X_\ell = N \times [0, \ell]$, set $M_\ell := X_\ell \# (k\ell)\overline{\mathbb{CP}}_2$, and let $\overleftrightarrow{M}_\ell$ be obtained from M_ℓ by identifying $N \times \{0\}$ with $N \times \{\ell\}$. Kronheimer begins by observing that $\overleftrightarrow{M}_\ell$ can be thought of as an ℓ -fold cover of M . Given a metric g on M , we can therefore pull it back to $\overleftrightarrow{M}_\ell$ and M_ℓ as a metric g_ℓ , and then consider the Seiberg-Witten equations (7-8) with respect to g_ℓ and the pulled-back spin^c structure. Because the tautness hypothesis implies that D can be deformed into a contact structure of either sign, Kronheimer shows that M_ℓ admits a symplectic form that is convex at both ends, allowing him to view it as the central region of an asymptotically conical almost-Kähler manifold obtained by adding conical ends that do not depend on ℓ . This allows him to predict the existence of solutions of the Seiberg-Witten equations on M_ℓ that are uniformly controlled in the regions near the boundary. By cutting these off and pasting, he then obtains pairs (Φ, θ) on $(\overleftrightarrow{M}_\ell, g_\ell)$ that satisfy some perturbation

$$\begin{aligned} \mathcal{D}_\theta \Phi &= \Omega \\ F_\theta^+ &= -\frac{1}{2} \Phi \odot \bar{\Phi} + i\eta \end{aligned}$$

of the Seiberg-Witten equations, where $\Omega \in \Gamma(\mathbb{V}_-)$ and $\eta \in \Gamma(\Lambda^+)$ are supported in $(N \times [0, \epsilon]) \cup (N \times [\ell - \epsilon, \ell])$ and satisfy uniform point-wise bounds that are independent of ℓ . By integrating the Weitzenböck formula

$$4|\not{D}_\theta \Phi|^2 = 4|\nabla_\theta \Phi|^2 + s|\Phi|^2 + |\Phi|^4 + 4\langle \eta, i\Phi \odot \bar{\Phi} \rangle + \text{divergence terms}$$

on $\overleftrightarrow{M}_\ell$, this yields

$$\int_{\overleftrightarrow{M}_\ell} \left[4|\Omega|^2 + (2\sqrt{2}|\eta| - s_-)|\Phi|^2 \right] d\mu_{g_\ell} \geq \int_{\overleftrightarrow{M}_\ell} |\Phi|^4 d\mu_{g_\ell}$$

and the Cauchy-Schwarz and triangle inequalities therefore yield

$$C + \left(\int_{\overleftrightarrow{M}_\ell} s_-^2 d\mu_{g_\ell} \right)^{1/2} \geq \left(8 \int_{\overleftrightarrow{M}_\ell} |\mathcal{F}^+|^2 d\mu_{g_\ell} \right)^{1/2}$$

where $C \geq \max(1, 4\|\Omega\|_2^2 + 2\sqrt{2}\|\eta\|_2)$ is a constant independent of ℓ , and where $\mathcal{F}^+$ is the pull-back to $\overleftrightarrow{M}_\ell$ of the self-dual part of the harmonic representative of $2\pi c_1^\mathbb{R}(M, J) = 2\pi \mathbf{a}$ with respect to g . Back down on (M, g) , however, this is equivalent to the statement that

$$\frac{C}{\sqrt{\ell}} + \left(\int_M s_-^2 d\mu_g \right)^{1/2} \geq (32\pi^2[\mathbf{a}^+]^2)^{1/2}$$

and since C is independent of ℓ , we therefore obtain the desired inequality (10) by taking the limit as $\ell \rightarrow \infty$. Since this works for any metric g on M , it therefore follows that $\mathbf{a} = c_1^\mathbb{R}(M, J)$ is a mock-monopole class, as claimed.

This provides a Seiberg-Witten proof of Lemma C. Indeed, notice that, by passing to covers, it suffices to consider the case when $N \rightarrow \Sigma$ is a circle bundle of Euler class +1 over a hyperbolic surface. Since a construction due to Milnor [32] endows $N \rightarrow \Sigma$ with a flat $\mathbf{SL}(2, \mathbb{R})$ -connection, N therefore admits a taut foliation; and since $b_1(N) = b_1(\Sigma) > 2$, Proposition 5 therefore applies. Thus $M = (N \times S^1) \# k\overline{\mathbb{CP}}_2$ carries a mock-monopole class for any $k \geq 0$, and Corollary 1 therefore asserts that neither $X = N \times S^1$ nor $M = X \# k\overline{\mathbb{CP}}_2$ can admit metrics of positive-scalar curvature, as claimed. Theorems A and B then once again follow by Proposition 1.

4. Pathological features of class VII

Our formulation of Theorems A and B has intentionally excluded surfaces with $\text{Kod} = -\infty$ and b_1 odd. These complex surfaces, which actually all have

$b_1 = 1$, are for historical reasons known as **surfaces of class VII**. The most familiar class-VII surfaces are the (primary) Hopf surfaces $(\mathbb{C}^2 - \{0\})/\mathbb{Z}$, which are diffeomorphic to $S^3 \times S^1$. These are already pathological from the standpoint of Theorem [B](#) because the results of Kobayashi [\[24\]](#) or Schoen [\[36\]](#) imply that a primary Hopf surface has Yamabe invariant $\mathscr{Y}(S^3 \times S^1) = \mathscr{Y}(S^4) = 8\sqrt{6}\pi$, while a theorem of Gursky-LeBrun [\[21\]](#) shows that its one-point blow-up has $\mathscr{Y}([S^1 \times S^3] \# \overline{\mathbb{CP}}_2) = \mathscr{Y}(\mathbb{CP}_2) = 12\sqrt{2}\pi$. The exclusion of class VII surfaces from Theorem [B](#) is therefore a matter of necessity.

However, surfaces of class VII must also be excluded from Theorem [A](#), because their Yamabe invariants are not always of the same sign. Indeed, while the Hopf surfaces discussed above have positive Yamabe invariant, a class of minimal class-VII surfaces discovered by Inoue [\[22\]](#) were shown by the first author [\[2\]](#) to have Yamabe invariant zero. We will call these examples Inoue-Bombieri surfaces, both because Inoue's paper credited Bombieri with their independent discovery, and to distinguish them from various other class-VII surfaces that also bear Inoue's name. As yet another application of Theorem [1](#), we now give a simplified proof of [\[2\]](#) Theorem 4.5]:

Theorem 3 (Albanese). *Let X be an Inoue-Bombieri surface, and let M be obtained from X by blowing up $k \geq 0$ points. Then $\mathscr{Y}(M) = 0$.*

Proof. Each Inoue-Bombieri surface X is the quotient of a half-space in $\mathbb{C}^2$ by a discrete group of affine transformations that preserves the half-space. Each one comes equipped [\[22\]](#) with a smooth submersion $\phi : X \rightarrow S^1$ whose fibers are aspherical 3-manifolds N with $b_1 \neq 0$; more specifically, N is always either the 3-torus $\mathbb{T}^3$ or a circle bundle $N \rightarrow \mathbb{T}^2$ over the 2-torus. Since each such N is expansive, Theorem [1](#) tells us that $\mathscr{Y}(M) \leq 0$. On the other hand, each such X admits an F -structure of positive rank, in the sense of Cheeger-Gromov [\[11\]](#), so each such X has $\mathscr{Y}(X) \geq 0$, and Kobayashi's connect-sum theorem [\[24\]](#) Theorem 2] therefore implies that $\mathscr{Y}(M) = \mathscr{Y}(X \# k\overline{\mathbb{CP}}_2) \geq 0$, too. We therefore conclude that $\mathscr{Y}(M) = 0$, as claimed. $\square$

Because class-VII surfaces inhabit a world where algebraic geometry holds so little sway, it is perhaps unsurprising that they actually turn out to violate many of our Kählerian expectations. Determining what sorts of new patterns actually hold in this realm is a problem that will merit further discussion in the next section.

5. Problems and perspectives

A recurrent theme of this article has been that Lemma [C](#) can actually be proved in many different ways, and that once this lemma is known, our main results, Theorems [A](#) and [B](#), then follow. In [§2](#), we gave a self-contained explanation of how the Schoen-Yau method works, and then showed why it implies our key results. However, a recent paper of Gromov [\[18\]](#) gives a systematic approach to the subject that subsumes many of the essential arguments used in [§2](#). Furthermore, an unpublished argument of Jian Wang shows that positive-scalar-curvature metrics cannot exist on 4-manifolds that admit maps of non-zero degree to aspherical 4-manifolds with $b_1 \neq 0$, and a recent paper of Chodosh, Li, and Liokumovich [\[12\]](#) improves this by dropping the $b_1 \neq 0$ hypothesis. Any of these approaches would provide alternative ways of proving Lemma [C](#) and Theorem [3](#).

While the Seiberg-Witten proof of Lemma [C](#) given in [§3](#) emphasizes the degree to which mock-monopole classes are quite sufficient for our purposes, it is unknown whether Proposition [5](#) might actually reflect the existence of a true monopole class on some high-degree cover of the manifold. Any result in this direction would certainly shed interesting new light on the subject.

Many intriguing questions about the Yamabe invariant remain open for complex surfaces of $\text{Kod} = -\infty$. For example, while the Yamabe invariant is always positive when such a complex surface has b_1 even, we still do not know whether blowing up a rational or ruled surface can ever change the precise value of its Yamabe invariant; all we know for sure is that the invariant must remain confined to a relatively narrow numerical range. For complex surfaces of class VII, the situation is even more daunting; because a full classification of these manifolds is lacking, we still cannot be absolutely certain that a class-VII surface can never have negative Yamabe invariant! Still, this seems rather unlikely. The global spherical shell conjecture [\[13\]](#) would imply that any class-VII surface is diffeomorphic to a blow-up of either a Hopf surface or an Inoue-Bombieri surface, and it is moreover definitively known [\[42\]](#) that this assertion does at least hold for surfaces with small b_2 . If the global spherical shell conjecture were true, Theorem [3](#) would thus immediately imply that any complex surface of $\text{Kod} = -\infty$ must necessarily have non-negative Yamabe invariant.

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SCHOOL OF COMPUTER AND MATHEMATICAL SCIENCES
UNIVERSITY OF ADELAIDE, ADELAIDE, SA 5005, AUSTRALIA
E-mail address: michael.albanese@adelaide.edu.au

DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY
STONY BROOK, NY 11794-3651, USA
E-mail address: claudemath@stonybrook.edu

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