A Nominal Approach to Probabilistic Separation Logic

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ABSTRACT

Currently, there is a gap between the tools used by probability theorists and those used in formal reasoning about probabilistic programs. On the one hand, a probability theorist decomposes probabilistic state along the simple and natural product of probability spaces. On the other hand, recently developed probabilistic separation logics decompose state via relatively unfamiliar measure-theoretic constructions for computing unions of sigma-algebras and probability measures. We bridge the gap between these two perspectives by showing that these two methods of decomposition are equivalent up to a suitable equivalence of categories. Our main result is a probabilistic analog of the classic equivalence between the category of nominal sets and the Schanuel topos. Through this equivalence, we validate design decisions in prior work on probabilistic separation logic and create new connections to nominal-set-like models of probability.

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1 INTRODUCTION

Separation logic [39], now a standard tool for reasoning about programs with shared mutable state, grew out of Reynolds's Syntactic Control of Interference [38] — a substructural system for controlling the interaction of imperative program fragments. The basic ingredients for today's interpretations of separation logic connectives, present in the original model of Syntactic Control of Interference [32], can be seen as living in a category of functors known as the Schanuel topos, with noninterference defined in terms of the coproduct of finite sets. Over the years, this model has been reformulated to suit the needs of formal reasoning about imperative programs: modern models of separation logic live not in the Schanuel topos, but in categories more like **Set**, and separation is interpreted not by coproduct, but by algebraic structures such as partial commutative monoids (PCMs) [6, 19]. In particular, the now-standard model of separation logic in which separating conjunction splits stores into disjoint pieces is defined in terms of the partial function $\uplus$ sending a pair of disjoint stores to their union, giving rise to a PCM of stores. This shift in perspective is justified by a classic equivalence of categories:

Fact 1.1. The Schanuel topos **Sch** is equivalent to the category **Nom** of nominal sets, and the original coproduct-based model of separation in **Sch** corresponds to the standard union-based model in **Nom** across this equivalence.[1]

Today, there is a pressing need for syntactic control of *probabilistic* interference — that is, for establishing the *probabilistic independence* of program fragments. In response to this need, recent work has developed a number of probabilistic separation logics [2, 3, 5, 27], whose semantic models are given by PCMs made of probability-theoretic objects. Lilac [27] is a separation logic whose PCM-based model is particularly well-behaved: its notion of separation coincides with probabilistic independence [27, Lemma 2.5], and yields a frame rule identical to the standard one for store-based separation logics.

However, Lilac's PCM model does not match a probability theorist's intuition. One expects separation to be interpreted via a standard product of probability spaces [26], but Lilac interprets separation using *independent combination*: a partial binary operation on probability spaces constructed out of low-level set-theoretic operations on σ-algebras. Moreover, Lilac's model fixes up front an unconventional sample space — the space $[0,1]^\omega$ of infinite streams of real numbers in the interval, known as the Hilbert cube — and the soundness of Lilac's proof rules depends on various properties specific to it. These contrasts between Lilac's model and textbook probability raise a question: *how do we know Lilac provides a good notion of separation for probabilistic separation logic?*

We answer this question by showing Lilac's seemingly non-standard independent combination is in fact *equivalent* to a probability theorist's product-based intuition of state decomposition. Our result is a probabilistic analog of Fact 1.1: just as the coproduct model of separation corresponds to the now-standard model based on $\uplus$ across an equivalence between the Schanuel topos and **Nom**, the probability theorist's intuitive product-based model of independence corresponds to Lilac's independent-combination-based model across an equivalence between a category of *enhanced measurable*

[1] For a good reference documenting this equivalence, see Pitts [36, §6.3].

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sheaves and a category of *absolutely continuous sets* (Theorem 4.35). Our contributions are as follows:

- We introduce *absolutely continuous sets*: just as nominal sets are sets equipped with an action by permutations of names, absolutely continuous sets are sets equipped with a continuous action by measurable permutations of the Hilbert cube.
- We prove analogs of the equivalence **Sch** $\simeq$ **Nom** for both discrete and continuous probability (Theorems 3.18 and 4.34). In particular, we show that the category $\mathbf{Set}^{\ll}$ of absolutely continuous sets is equivalent to a topos **EMS** of *enhanced measurable sheaves*: a probabilistic analog of the Schanuel topos.
- We show that $\mathbf{Set}^{\ll}$ provides a natural background category for a fragment of Lilac. Theorem 4.35 then shows that, by transporting across the equivalence $\mathbf{Set}^{\ll} \simeq$ **EMS**, Lilac's model corresponds to a model in **EMS** where separation arises naturally from product of probability spaces via Day convolution [6, 14, 33].

2 THE NOMINAL SITUATION

Our main result is a probabilistic analog of Fact 1.1 (Theorem 4.35). To set the stage, we first make Fact 1.1 a precise mathematical statement (Proposition 2.18). We devote this section to describing the necessary pieces in this comfortable setting; the material in this section is standard, but we will deviate occasionally from the usual presentation in order to focus on the aspects that are most relevant to our eventual probabilistic counterpart.

At its core, Fact 1.1 states that two distinct approaches to modelling store-separation are equivalent. To illustrate this fact we will study a tiny separation logic consisting of propositions P, Q about integer-valued stores:

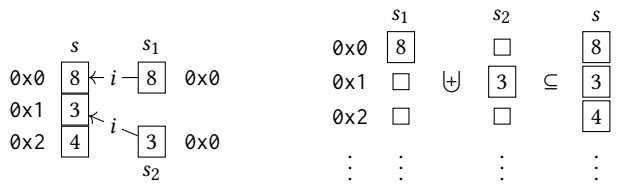
$$P, Q ::= x \mapsto i \mid \mathsf{True} \mid P * Q. \qquad \text{(TinySep)}$$

TinySep propositions are well-formed according to a judgment $\Gamma \vdash P$ defined as usual: a context Γ is a set of logical variables x, and $\Gamma \vdash P$ if Γ contains the variables used in P. Fact 1.1 asserts the equivalence of two different models for TinySep:

Model 1: separation as coproduct. In this model, a store consists of two components: (1) a *shape* L given as a finite set of available locations (i.e., memory addresses), and (2) a *valuation* $s : L \rightharpoonup \mathbb{Z}$, a partial function assigning values to a subset of the shape. An example is shown in Figure 1a; the store s has shape $\{\mathtt{0x0}, \mathtt{0x1}, \mathtt{0x2}\}$, and the valuation maps $\mathtt{0x0} \mapsto 8$ and so on. Under this model, the meaning of a proposition depends on the shape L: the interpretation of a proposition $\Gamma \vdash P$ has form $[\![\Gamma \vdash P]\!]_1^L : (\Gamma \to L) \to \mathcal{P}(L \rightharpoonup \mathbb{Z})$, associating each substitution $\gamma : \Gamma \to L$ to the set $[\![\Gamma \vdash P]\!]_1^L(\gamma)$ of L-shaped valuations satisfying P.

Under this interpretation, we define $s \in [\![\mathsf{True}]\!]_1^L(\gamma)$ always and $s \in [\![x \mapsto i]\!]_1^L(\gamma)$ if and only if $s(\gamma(x)) = i$. Separating conjunction is defined via the coproduct of store shapes: $P_1 * P_2$ holds of an L-shaped valuation s if and only if there are valuations s_1 of shape L_1 and s_2 of shape L_2, and an injective function $i : L_1 + L_2 \hookrightarrow L$ embedding the coproduct $L_1 + L_2$ into L such that s_1 satisfies P_1 and s_2 satisfies P_2 and s_1, s_2 embed into s along i. This situation is visualized in Figure 1a. For example,

$$s \in [\![(x \mapsto 8) * (y \mapsto 3)]\!]_1^{\{\mathtt{0x0},\mathtt{0x1},\mathtt{0x2}\}}(\{x \mapsto \mathtt{0x0}, y \mapsto \mathtt{0x1}\})$$



(a) Model 1: coproduct. **(b) Model 2: union.**

Figure 1: Visualizing separation in Model 1 and Model 2.

is witnessed by setting s_1 to the $\{\mathtt{0x0}\}$-shaped valuation $\{\mathtt{0x0} \mapsto 8\}$ and s_2 to the $\{\mathtt{0x0}\}$-shaped valuation $\{\mathtt{0x0} \mapsto 3\}$ and $i : \{\mathtt{0x0}\} + \{\mathtt{0x1}\} \hookrightarrow \{\mathtt{0x0}, \mathtt{0x1}\}$ to the injection defined by $i(\mathrm{inl}(\mathtt{0x0})) = \mathtt{0x0}$ and $i(\mathrm{inr}(\mathtt{0x0})) = \mathtt{0x1}$, where $\mathrm{inl} : L_1 \to L_1 + L_2$ and $\mathrm{inr} : L_2 \to L_1 + L_2$ are the coproduct injections.

Model 2: separation as union. In this model, one fixes upfront a "universal store shape" into which all store shapes can be embedded. Any countably-infinite set will do; we choose the natural numbers $\mathbb{N}$. A store is a partial function $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ defined on finitely many values of its domain, and a proposition $\Gamma \vdash P$ denotes a function $[\![\Gamma \vdash P]\!]_2 : (\Gamma \to \mathbb{N}) \to \mathcal{P}(\mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z})$. The interpretations of True and $x \mapsto i$ are as in the shape-indexed model: $s \in [\![\mathsf{True}]\!]_2(\gamma)$ always and $s \in [\![x \mapsto i]\!]_2(\gamma)$ if and only if $s(\gamma(x)) = i$. Separating conjunction is defined via union of stores: a store s is in $[\![P_1 * P_2]\!]_2(\gamma)$ if and only if there exist disjoint stores s_1 and s_2 with $s_1 \uplus s_2 \subseteq s$ such that s_1 is in $[\![P_1]\!]_2(\gamma)$ and s_2 is in $[\![P_2]\!]_2(\gamma)$. Figure 1b visualizes an example: $s \in [\![(x \mapsto 8) * (y \mapsto 3)]\!]_2\{x \mapsto \mathtt{0x0}, y \mapsto \mathtt{0x1}\}$ holds because s_1 and s_2 have a union contained in s and s_1 satisfies $x \mapsto 8$ and s_2 satisfies $y \mapsto 3$.

Relating the two models. Model 1 and Model 2 are equivalent by Fact 1.1. The equivalence is based on the following idea. Every store shape L can be encoded as a finite subset of $\mathbb{N}$ via a suitable pair of functions $\mathrm{enc}_L : L \to \mathbb{N}$ and $\mathrm{dec}_L : \mathbb{N} \rightharpoonup L$. Choosing an arbitrary such pair $(\mathrm{enc}_L, \mathrm{dec}_L)$ for every L allows translating Model 1 into Model 2 in a bijective way: an L-shaped store $s : L \rightharpoonup \mathbb{Z}$ corresponds to a finite partial function $s \circ \mathrm{dec}_L : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$, and a Model-1-substitution $\gamma : \Gamma \to L$ corresponds to a Model-2-substitution $\mathrm{enc}_L \circ \gamma : \Gamma \to \mathbb{N}$. Via these translations, it holds that $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ if and only if $s \circ \mathrm{dec}_L \in [\![\Gamma \vdash P]\!]_2(\mathrm{enc}_L \circ \gamma)$ for all L-shaped valuations s, propositions $\Gamma \vdash P$, and substitutions $\gamma : \Gamma \to L$, so both models induce the same notion of store-satisfaction.

This equivalence should seem plausible enough given how tiny TinySep is. What is remarkable about Fact 1.1 is that this equivalence continues to hold when the interpretations $[\![-]\!]_1^{(-)}$ and $[\![-]\!]_2$ are extended to include all the usual features of separation logic, including separating implication $-\!\!*$, the intuitionistic connectives $\wedge, \vee, \to, \mathsf{False}$, quantification at both first-order and higher type, quantification over propositions, predicates defined by structural recursion, and so on. In short, the semantic domains of Model 1 are equivalent in expressive power to those of Model 2.

Rather than laboriously verifying one by one that the standard interpretations of each of these features coincide, Fact 1.1 establishes a general result. The key is to place Model 1 and Model 2 into the context of suitable categories that bring out their essential structure. Model 1 naturally lives in a category **Sch** called the *Schanuel topos*: the interpretation $[\![\Gamma \vdash P]\!]_1^{(-)}$ of a proposition P defines a **Sch**-morphism from a **Sch**-object representing Γ-substitutions to a **Sch**-object representing store-predicates. Model 2 naturally lives in the category **Nom** of *nominal sets* [36]: the interpretation $[\![\Gamma \vdash P]\!]_2$ defines a **Nom**-morphism from a nominal set of Γ-substitutions to a nominal set of store-predicates. Having placed Model 1 and Model 2 into suitable background categories, Fact 1.1 follows from a classic theorem: the categories **Sch** and **Nom** are known to be equivalent [36, §6.3], and inspecting the proof of this equivalence shows that the functor **Sch** $\to$ **Nom** witnessing it sends Model 1 to Model 2 via the construction involving $(\mathrm{enc}_L, \mathrm{dec}_L)$.

The rest of this section is devoted to filling in the details of this category-theoretic setup. First we will describe how Model 1 lives in **Sch** and Model 2 lives in **Nom**. Then we will highlight the essential properties of this setup that make the equivalence **Sch** $\simeq$ **Nom** possible, and how Model 1 and Model 2 are instances of the same structure across this equivalence; Theorem 4.35 relies crucially on identifying analogous properties in the probabilistic setting.

2.1 Model 1 in the Schanuel topos

In this section we describe how Model 1 of Section 2 naturally lives in the Schanuel topos **Sch**. The benefit of this is that it makes the invariants maintained by $[\![-]\!]_1^{(-)}$ explicit: the category **Sch** is such that all constructions that make categorical sense — i.e., are well-defined as objects and morphisms of **Sch** — are forced to preserve all invariants. The invariants in this case are the following principles one intuitively expects to hold when reasoning about stores:

- *Extension*: propositions should continue to hold when new locations are introduced (such as when declaring a local variable or allocating a reference). More precisely, if $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ for some L-shaped valuation s and substitution $\gamma : \Gamma \to L$, and if L is a subset of some extended set of locations L', then it should hold that $s \in [\![\Gamma \vdash P]\!]_1^{L'}(\gamma)$, where we have implicitly coerced s into an L'-shaped valuation and γ into an L'-shaped substitution $\Gamma \to L'$ along the inclusion $L \subseteq L'$. [2]
- *Renaming*: propositions should be stable under renaming of locations. More precisely, if $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ for some L-shaped valuation s and substitution $\gamma : \Gamma \to L$, and if f is a bijective function $L \to L'$, then it should hold that $s \circ f^{-1} \in [\![\Gamma \vdash P]\!]_1^{L'}(f \circ \gamma)$.
- *Restriction*: the truth of a proposition should not depend on any unused locations. For example, suppose a proposition P holds of the $\{\ell_1, \ell_2\}$-shaped valuation $\{\ell_1 \mapsto 1\}$, which does not use the location ℓ_2. Then P should also hold of $\{\ell_1 \mapsto 1\}$ considered as an $\{\ell_1\}$-shaped valuation.

As basic principles of store-based reasoning, it is crucial that these invariants are preserved by the basic separation logic connectives: if P and Q satisfy Extension, Renaming, and Restriction, then their separating conjunction $P * Q$, separating implication $P -\!\!* Q$, conjunction $P \wedge Q$, and implication $P \to Q$ should as well.

A general strategy for preserving invariants like this is to work with **Set**-valued functors out of a category C capturing them. Such functors are very well-behaved: in particular, many subcategories of the functor category $[C^{\mathrm{op}}; \mathbf{Set}]$, called *categories of sheaves on* C, are automatically cartesian closed, and can be used to quickly obtain invariant-preserving interpretations of logical connectives. Placing Model 1 into the Schanuel topos **Sch** is an instance of this idea. The Schanuel topos is a particular subcategory of $[C^{\mathrm{op}}; \mathbf{Set}]$, where C is chosen so that functors $C^{\mathrm{op}} \to \mathbf{Set}$ capture Extension and Renaming, consisting only of functors that are *atomic sheaves* in order to capture Restriction. We build up to this model in steps.

2.1.1 The base category C. Essentially, Extension says propositions should be stable under subset-inclusions $L \subseteq L'$ and Renaming says they should be stable under bijections. These two invariants can be packaged into a *category of store shapes*:

Definition 2.1. Let **Shp** be the category whose objects are finite sets L and whose morphisms from L to M are functions $M \to L$ definable by composing subset-inclusions and bijections.

Note the direction $M \to L$ is the reverse of what one might expect; this is because we will consider contravariant functors on **Shp**. Intuitively, there is a morphism $M \to L$ if L is a "smaller" shape than M. Since every composite of subset-inclusions and bijections is an injective function, and every injective function is bijective onto its image, the category **Shp** has a simple abstract description:

Proposition 2.2. The category **Shp** is equal to $\mathbf{Inj}_{<\omega}^{\mathrm{op}}$, where $\mathbf{Inj}_{<\omega}$ is the category of injective functions between finite sets.

With **Shp** in hand, functors $\mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ (equivalently, functors $\mathbf{Inj}_{<\omega} \to \mathbf{Set}$) model Extension- and Renaming-invariant concepts. In particular, there is a functor modelling stores:

Definition 2.3 (Store functor)**.** The *store functor* $\mathrm{S} : \mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ is a functor that sends a finite set L to the set of all L-shaped valuations and a $\mathbf{Inj}_{<\omega}$-morphism $i : L \hookrightarrow M$ to a function coercing $\mathrm{S}(M)$ into $\mathrm{S}(L)$ defined by $\mathrm{S}(i)(L, s) = (M, s')$, where s' is the valuation $M \rightharpoonup \mathbb{Z}$ defined by $s'(m) = s(l)$ iff $m = i(l)$ for some l in L.

The action of S on **Shp**-morphisms captures the operations that we expect to be invariant under: if i is a subset inclusion $L \subseteq L'$, then $\mathrm{S}(i)$ coerces L-shaped stores into L'-shaped stores as in the description of Extension, and if f is a bijective function $L \to L'$, then $\mathrm{S}(f)$ sends an L-shaped valuation s to an L'-shaped valuation $s \circ f^{-1}$ as in the description of Renaming.

2.1.2 Using sheaves to capture Restriction. Recall the example used to illustrate Restriction: if a proposition holds of the $\{\ell_1, \ell_2\}$-shaped valuation $\{\ell_1 \mapsto 1\}$, then it should also hold of $\{\ell_1 \mapsto 1\}$ considered as an $\{\ell_1\}$-shaped valuation. We say that $\{\ell_1 \mapsto 1\} \in \mathrm{S}\{\ell_1, \ell_2\}$ *restricts to* $\{\ell_1 \mapsto 1\} \in \mathrm{S}\{\ell_1\}$ *along* i, where i is the subset-inclusion $\{\ell_1\} \subseteq \{\ell_1, \ell_2\}$. This is an instance of a more general property satisfied by the functor S:

Proposition 2.4. Let $i : L \hookrightarrow M$ be an injective function between finite sets L and M, and $s \in \mathrm{S}(M)$ an M-shaped valuation. If $\mathrm{dom}(s) \subseteq \mathrm{im}(i)$, then there exists a unique L-shaped valuation $s' \in \mathrm{S}(L)$, the *restriction of* s *along* i, such that $\mathrm{S}(i)(s') = s$.

[2] This makes the separation logic affine rather than linear; we will restrict our attention to affine separation logics in this paper, as Lilac is affine and our main goal is to obtain models for it.

Proposition 2.4 can be expressed more abstractly:

Definition 2.5. Let F be a functor $\mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ and $i : L \hookrightarrow M$ a $\mathbf{Inj}_{<\omega}$-morphism. An element y of $F(M)$ is *restrictable along* i if for all $\mathbf{Inj}_{<\omega}$-objects N and $\mathbf{Inj}_{<\omega}$-morphisms $j, k : M \to N$ with $j \circ i = k \circ i$ it holds that $F(j)(y) = F(k)(y)$.

Definition 2.6. A functor $F : \mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ *has a restriction operation* if for all $\mathbf{Inj}_{<\omega}$-morphisms $i : L \hookrightarrow M$ and elements y of $F(M)$ that are restrictable along i, there exists a unique $x \in F(L)$, called the *restriction of y along i*, such that $y = F(i)(x)$.

With these definitions in hand, one can show Proposition 2.4 is equivalent to S having a restriction operation. Functors with a restriction operation have a special name: they are called *atomic sheaves on* $\mathbf{Shp}$ [28, Lemma III.4.2]. The Schanuel topos $\mathbf{Sch}$ is the full subcategory of $[\mathbf{Shp}^{\mathrm{op}}; \mathbf{Set}]$ consisting of atomic sheaves.

In these new terms, Proposition 2.4 says S is an atomic sheaf on $\mathbf{Shp}$, and so an object of $\mathbf{Sch}$. Just as S captures the concept of stores as shape-indexed valuations, there are other atomic sheaves for each of the other concepts used to define Model 1:

Proposition 2.7. The following are objects of $\mathbf{Sch}$:

- The constant functor Prop sending every object of $\mathbf{Shp}$ to the set $\{\top, \bot\}$ and every morphism of $\mathbf{Shp}$ to the identity function.
- The functor Loc of locations, defined by $\mathrm{Loc}(L) = L$ on objects of $\mathbf{Shp}$ and $\mathrm{Loc}(i : L \hookrightarrow L')(l : L) = i(l)$ on $\mathbf{Inj}_{<\omega}$-morphisms $i : L \hookrightarrow L'$.
- The functor Loc^Γ of Γ-substitutions, which maps objects L to the set of all substitutions $L \to \Gamma$, and action on $\mathbf{Inj}_{<\omega}$-morphisms inherited pointwise from Loc.

With these sheaves in hand, one can show Model 1 lives in $\mathbf{Sch}$:

Proposition 2.8. If $\Gamma \vdash P$ then the L-indexed family of functions

$$\left(\llbracket \Gamma \vdash P \rrbracket_1^L : (\Gamma \to L) \to \mathcal{P}(L \rightharpoonup \mathbb{Z})\right)_{L \in \mathbf{Shp}}$$

is natural in L, so defines a morphism $\mathrm{Loc}^\Gamma \to \mathrm{Prop}^S$ in $\mathbf{Sch}$, where Prop^S is the exponential guaranteed to exist because $\mathbf{Sch}$ is cartesian closed by virtue of being a category of sheaves. Moreover, every morphism of this type satisfies Extension, Renaming, and Restriction.

2.2 Model 2 in nominal sets

We now turn to the other side of the equivalence given by Fact 1.1: the category of nominal sets $\mathbf{Nom}$, and how it naturally houses Model 2 of Section 2, in which separation is defined via union of finite partial functions on $\mathbb{N}$.

Just as $\mathbf{Sch}$ is a category capturing the invariants implicitly maintained by Model 1, $\mathbf{Nom}$ is a category capturing the invariants implicitly maintained by Model 2. In this case, the invariants are:

- *Permutation*: propositions should be stable under permuting locations. If $s \in \llbracket \Gamma \vdash P \rrbracket_2(\gamma)$ for some store $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ and substitution $\gamma : \Gamma \to \mathbb{N}$, and $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation of finitely-many natural numbers, then it should hold that $s \circ \pi \in \llbracket \Gamma \vdash P \rrbracket_2(\pi^{-1} \circ \gamma)$.
- *Finiteness*: more subtly, stores and substitutions can only mention finitely-many locations $n \in \mathbb{N}$; this models the fact that physical stores are necessarily finite, and ensures that one always has the ability to allocate fresh locations.

To capture Permutation, the objects of $\mathbf{Nom}$ are sets equipped with an action by a group of permutations to be invariant under. Specifically, let S_ω be the group of permutations of finitely-many natural numbers: elements of S_ω are bijective functions $\pi : \mathbb{N} \to \mathbb{N}$ such that there exists some $n \in \mathbb{N}$ with $\pi(m) = m$ for all $m \geq n$. An *S_ω-set* is a set X equipped with a right action by S_ω: a function $(\cdot) : X \times S_\omega \to X$ satisfying $x \cdot 1 = x$ and $x \cdot (\pi\sigma) = (x \cdot \pi) \cdot \sigma$ for all $x \in X$ and $\pi, \sigma \in S_\omega$. There is an S_ω-set $\overline{\mathrm{S}}$ of stores, whose group action says what it means to permute the locations in a store:[3]

Definition 2.9. Let $\overline{\mathrm{S}}$ be the S_ω-set of stores $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ with action $s \cdot \pi = s \circ \pi$.

A morphism of S_ω-sets $(X, \cdot_X) \to (Y, \cdot_Y)$ is an *equivariant function*: a function $f : X \to Y$ satisfying $f(x \cdot_X \pi) = f(x) \cdot_Y \pi$ for all $x \in X$ and $y \in Y$ and $\pi \in S_\omega$. This captures invariance under Permutation: S_ω-morphisms $\overline{\mathrm{S}} \to \overline{\mathrm{Prop}}$, where $\overline{\mathrm{Prop}}$ is the S_ω-set $\{\top, \bot\}$ with trivial action $p \cdot \pi = p$, are the permutation-invariant predicates on stores.

To capture Finiteness, the category $\mathbf{Nom}$ is a full subcategory of the category of S_ω-sets consisting of those S_ω-sets $(X, \cdot_X)$ in which every $x \in X$ only uses finitely many locations. The concept of "using" a location is made precise by looking at stabilizer subgroups: if $x \cdot \pi = x$ (i.e., π is in the stabilizer of x) then x can only "use" those locations fixed by π. An element x uses finitely many locations if its stabilizer is *open* for a suitable topology:

Definition 2.10 (Topology on S_ω)**.** A subset U of S_ω is *open* if for every π in U there exists a finite subset A of $\mathbb{N}$ such that $\pi \in \mathrm{Fix}\, A \subseteq U$, where $\mathrm{Fix}\, A$ is the subgroup of S_ω-permutations π that fix every element of A; i.e., $\pi(a) = a$ for all a in A.

Intuitively, a stabilizer subgroup $\mathrm{Stab}\, x$ is open if every π stabilizing x does so for some "finite reason" A: there is some subset A fixed by π such that any other permutation π' fixing A also stabilizes x. Nominal sets are S_ω-sets with open stabilizers [35, §6.2]:

Definition 2.11. A *nominal set* is a S_ω-set $(X, \cdot)$ such that for every x in X the stabilizer subgroup $\mathrm{Stab}\, x$ is open. $\mathbf{Nom}$ is the category of nominal sets and equivariant functions.

For example, $\overline{\mathrm{S}}$ is a nominal set: if s is a store with $s \circ \pi = s$, then π fixes the finite set $\mathrm{dom}(s)$, and moreover every permutation fixing $\mathrm{dom}(s)$ fixes s, so $\mathrm{dom}(s) \subseteq \mathrm{Stab}\, x$ and $\mathrm{Stab}\, s$ is open. There are nominal sets capturing each of the other concepts used in Model 2:

Proposition 2.12. The following are objects of $\mathbf{Nom}$:

- The S_ω-set $\overline{\mathrm{Prop}}$ of propositions
- The S_ω-set $\overline{\mathrm{Loc}}$ of locations $\mathbb{N}$ with action $x \cdot \pi = \pi^{-1}(x)$.
- The S_ω-set $\overline{\mathrm{Loc}}^\Gamma$ of Γ-substitutions $\gamma : \Gamma \to \mathbb{N}$ with action defined by $\gamma \cdot \pi = \pi^{-1} \circ \gamma$.

With these in hand, one can show Model 2 lives in $\mathbf{Nom}$:

Proposition 2.13. If $\Gamma \vdash P$ then the function $\llbracket \Gamma \vdash P \rrbracket_2$ is a morphism $\overline{\mathrm{Loc}}^\Gamma \to \overline{\mathrm{Prop}}^{\overline{\mathrm{S}}}$ in $\mathbf{Nom}$, and every morphism of this type satisfies Permutation and Finiteness.

[3] In general, we will overline objects of $\mathbf{Nom}$ to distinguish them from their $\mathbf{Sch}$-counterparts.

2.3 The equivalence

This section sketches the classic equivalence **Sch** ≃ **Nom** and how Models 1 and 2 correspond across it. We will not be concerned so much with the details of this particular equivalence, but rather with highlighting the key properties of **Sch** and **Nom** that make it possible — Theorem 4.35 relies on identifying analogous properties in the probabilistic setting.

In Section 2 we sketched the correspondence between Model 1 and Model 2, based on the idea that every store shape L can be encoded as a finite set of natural numbers via a pair of functions $(\mathrm{enc}_L, \mathrm{dec}_L)$. This idea also forms the basis for the equivalence **Sch** ≃ **Nom**. In the language of Section 2.1, every enc_L encodes the object L of **Shp** as a subset $\mathrm{im}(\mathrm{enc}_L)$ of $\mathbb{N}$. This encoding extends to morphisms of **Shp**: every **Shp**-morphism $M \to L$, equivalently an injective function $f : L \hookrightarrow M$, can be encoded as a permutation $\pi \in S_\omega$ that sends $\mathrm{im}(\mathrm{enc}_L)$ to $\mathrm{im}(\mathrm{enc}_M)$. More precisely,

Proposition 2.14 (Homogeneity [36, L1.14])**.** Let L, M be finite sets and enc_L and enc_M injective functions $L \hookrightarrow \mathbb{N}$ and $M \hookrightarrow \mathbb{N}$. For any injective function $i : L \hookrightarrow M$, there exists $\pi \in S_\omega$ such that $\pi \circ \mathrm{enc}_L = \mathrm{enc}_M \circ i$, making the following square commute:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\pi} & \mathbb{N} \\ {\scriptstyle \mathrm{enc}_L}\uparrow & & \uparrow{\scriptstyle \mathrm{enc}_M} \\ L & \xrightarrow[i]{} & M \end{array}$$

Furthermore, the relationships between encoded store shapes $\mathrm{im}(\mathrm{enc}_L)$ are faithfully captured by relationships between subgroups of S_ω:

Proposition 2.15 (Correspondence)**.** For any two store shapes L and M, it holds that $\mathrm{Fix}(\mathrm{im}(\mathrm{enc}_L)) \subseteq \mathrm{Fix}(\mathrm{im}(\mathrm{enc}_M))$ if and only if $\mathrm{im}(\mathrm{enc}_L) \supseteq \mathrm{im}(\mathrm{enc}_M)$.

Homogeneity and Correspondence together give the equivalence **Sch** ≃ **Nom**. For details, see MacLane and Moerdijk [28, Theorem III.9.2]. With this equivalence in hand, we are finally in a position to make Fact 1.1 precise. Abbreviating Loc^Γ as $[\![\Gamma]\!]$ and the exponential Prop^S as Pred, Proposition 2.8 shows that the Hom-set $\mathbf{Sch}([\![\Gamma]\!], \mathrm{Pred})$ serves as a semantic domain for Model-1 interpretations of TinySep propositions in context Γ. Analogously, Proposition 2.13 shows that $\mathbf{Nom}(\overline{[\![\Gamma]\!]}, \overline{\mathrm{Pred}})$ serves as a semantic domain for Model 2, where $\overline{[\![\Gamma]\!]}$ is $\overline{\mathrm{Loc}}^\Gamma$ and $\overline{\mathrm{Pred}}$ the exponential $\overline{\mathrm{Prop}}^{\overline{S}}$. The next proposition establishes that these semantic domains correspond across **Sch** ≃ **Nom**:

Proposition 2.16. Across the equivalence **Sch** ≃ **Nom**, the sheaf S corresponds to the nominal set $\overline{S}$, Prop to $\overline{\mathrm{Prop}}$, Loc to $\overline{\mathrm{Loc}}$, $[\![\Gamma]\!]$ to $\overline{[\![\Gamma]\!]}$, Pred to $\overline{\mathrm{Pred}}$, and $\mathbf{Sch}([\![\Gamma]\!], \mathrm{Pred})$ to $\mathbf{Nom}(\overline{[\![\Gamma]\!]}, \overline{\mathrm{Pred}})$.

It remains to show that Model 1 intepretations $[\![\Gamma \vdash P]\!]_1^{(-)}$ correspond to their Model 2 counterparts $[\![\Gamma \vdash P]\!]_2$. This is straightforward when P is True or $x \mapsto i$; the interesting case is the separating conjunction $P_1 * P_2$. One could show $[\![P_1 * P_2]\!]_1^{(-)}$ corresponds to $[\![P_1 * P_2]\!]_2$ by unwinding definitions and showing, via a careful calculation, that they correspond across the functor **Sch** → **Nom** witnessing the equivalence. But Fact 1.1 is far more general. The idea is to use the *internal language* of **Sch**: as a category of sheaves, any construction in higher-order logic can be interpreted in **Sch** [28, VI.7.1].

In this internal language, types denote sheaves and functions denote natural transformations, and Model 1's separating conjunction can be defined as $[\![\Gamma \vdash P_1 * P_2]\!]_1^{(-)} = [\![\Gamma \vdash P_1]\!]_1^{(-)} \circledast [\![\Gamma \vdash P_2]\!]_1^{(-)}$, where $\circledast$ is a special **Sch**-morphism denoting separating conjunction in the internal language of **Sch**. The meaning of $\circledast$ can be described by conveniently using the higher-order logic of **Sch**:

$$(\circledast) : \mathrm{Pred}^{[\![\Gamma]\!]} \times \mathrm{Pred}^{[\![\Gamma]\!]} \to \mathrm{Pred}^{[\![\Gamma]\!]}$$

$$(f_1 \circledast f_2)(\gamma : [\![\Gamma]\!])(s : \mathrm{S}) = \begin{pmatrix} \exists\, s_1\ s_2 : \mathrm{S}.\ s_1 \bullet s_2 \text{ defined} \wedge \\ s_1 \bullet s_2 \sqsubseteq s \wedge f_1\, \gamma\, s_1 \wedge f_2\, \gamma\, s_2 \end{pmatrix} \quad (1)$$

This definition is made of the following key ingredients:

- A symbol $\sqsubseteq$, which in the internal language looks like an ordering relation on stores, and externally denotes a suitable natural transformation $\mathrm{S} \times \mathrm{S} \to \mathrm{Prop}$.
- A symbol $\bullet$, which internally looks like a partial function combining stores, and externally denotes a natural transformation $\mathrm{S}^2_\perp \to \mathrm{S}$, where $\mathrm{S}^2_\perp$ is a subobject $i : \mathrm{S}^2_\perp \hookrightarrow \mathrm{S} \times \mathrm{S}$ of the sheaf $\mathrm{S} \times \mathrm{S}$ of pairs of stores carving out the domain on which $\bullet$ is defined.

The ordering $\sqsubseteq$ is the natural transformation $(\sqsubseteq) : \mathrm{S} \times \mathrm{S} \to \mathrm{Prop}$ defined by $(s_1 \sqsubseteq_L s_2) = \top$ if and only if s_1 is a subvaluation of s_2. The combining operation $\bullet$ is a natural transformation $\bullet : \mathrm{S}^2_\perp \to \mathrm{S}$. Its domain $\mathrm{S}^2_\perp$ is a sheaf defined in terms of the coproduct of finite sets. Each $\mathrm{S}^2_\perp(L)$ is a set consisting of pairs of L-shaped valuations that "factor through" a coproduct $L_1 + L_2$ along some $i : L_1 + L_2 \hookrightarrow L$:

$$\mathrm{S}^2_\perp(L) = \begin{array}{l} \{\, (\mathrm{S}(i \circ \mathrm{inl})(s_1), \mathrm{S}(i \circ \mathrm{inr})(s_2)) \\ \mid L_1, L_2 \in \mathbf{Shp}, s_1 \in \mathrm{S}(L_1), s_2 \in \mathrm{S}(L_2), i : L_1 + L_2 \hookrightarrow L \,\} \end{array}$$

The morphism $\bullet$ sends each pair $(\mathrm{S}(i \circ \mathrm{inl})(s_1), \mathrm{S}(i \circ \mathrm{inr})(s_2))$ of separated stores to the combined store $\mathrm{S}(i)[s_1, s_2]$, where the valuation $[s_1, s_2]$ of type $L_1 + L_2 \to \mathbb{Z}$ is the unique one defined by $[s_1, s_2] \circ \mathrm{inl} = s_1$ and $[s_1, s_2] \circ \mathrm{inr} = s_2$. Each $\mathrm{S}^2_\perp(L)$ is a subset of $(\mathrm{S} \times \mathrm{S})(L)$, and collecting the canonical subset-inclusions into an L-indexed family gives a monic natural transformation $i : \mathrm{S}^2_\perp \hookrightarrow \mathrm{S} \times \mathrm{S}$.

In the internal language, $\bullet$ looks like a partial function that is associative and commutative and monotone with respect to $\sqsubseteq$, with unit the natural transformation $\mathrm{emp} : 1 \to \mathrm{S}$ sending every store shape L to the empty valuation on L. Together, the tuple $(\sqsubseteq, \mathrm{S}^2_\perp, i, \bullet, \mathrm{emp})$ packages up the ingredients needed to model separation logic in **Sch** into a *resource monoid* internal to **Sch**:

Definition 2.17. A *resource monoid* [19] is a poset $(R, \sqsubseteq)$ with a least element $\perp$ and a monotone partial function $(\cdot) : R \times R \rightharpoonup R$ such that $(R, \cdot, \perp)$ forms a partial commutative monoid.[4]

We can similarly construct a resource monoid in **Nom**. There is an equivariant function $(\sqsubseteq) : \overline{\mathrm{S}} \times \overline{\mathrm{S}} \to \overline{\mathrm{Prop}}$ sending a pair (s_1, s_2) of finite partial functions on $\mathbb{N}$ to $\top$ if and only if $s_1 \subseteq s_2$, with least element $\overline{\mathrm{emp}}$ the empty finite partial function. There is a nominal set $\overline{\mathrm{S}}^2_\perp$ of separated stores: the set

$$\{(s_1, s_2) \mid s_1, s_2 \in \overline{\mathrm{S}}, \mathrm{dom}(s_1) \cap \mathrm{dom}(s_2) = \emptyset\}$$

of pairs of stores with disjoint domain, and pointwise action. Both the canonical inclusion $\bar{i} : \overline{\mathrm{S}}^2_\perp \hookrightarrow \overline{\mathrm{S}} \times \overline{\mathrm{S}}$ and the function $(\bar{\bullet}) : \overline{\mathrm{S}}^2_\perp \to \overline{\mathrm{S}}$ sending a pair (s_1, s_2) of disjoint stores to their union

[4] In this paper we are concerned with affine models of separation logic, and so consider an affine variant of the resource monoids defined in Galmiche et al. [19]. Our definition is closest in spirit to the affine PDMs sketched there.

$s_1 \uplus s_2$ are equivariant, hence morphisms in **Nom**. Finally, $\bar{\bullet}$ is monotone in $\bar{\imath}$ and has unit $\overline{\text{emp}}$ so $(\overline{\sqsubseteq}, \overline{\mathsf{S}^2_\perp}, \bar{\imath}, \bar{\bullet}, \overline{\text{emp}})$ forms a resource monoid internal to **Nom**, and reinterpreting Eq. (1) inside **Nom** with $(\overline{\sqsubseteq}, \overline{\mathsf{S}^2_\perp}, \bar{\imath}, \bar{\bullet}, \overline{\text{emp}})$ in place of $(\sqsubseteq, \mathsf{S}^2_\perp, i, \bullet, \text{emp})$ yields Model 2's separating conjunction. The following proposition, connecting the two resource monoids, makes Fact 1.1 precise:

Proposition 2.18. The resource monoid $(\sqsubseteq, \mathsf{S}^2_\perp, i, \bullet, \text{emp})$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathsf{S}^2_\perp}, \bar{\imath}, \bar{\bullet}, \overline{\text{emp}})$ across the equivalence **Sch** $\simeq$ **Nom**.

We are at last ready to appreciate the full power of this fact. First, it shows $[\![P_1 * P_2]\!]_1^{(-)}$ corresponds to $[\![P_1 * P_2]\!]_2$: both arise from the same internal-language definition, up to the replacement of types and function symbols following Propositions 2.16 and 2.18. Next, since the separating implication $-\!\!*$ and all intuitionistic connectives can be defined similarly using the internal language, they must correspond as well; this extends the equivalence of Models 1 and 2 from TinySep to all standard separation logic connectives. More generally, Fact 1.1 says that any construction in higher-order logic that only uses the types and functions of Propositions 2.16 and 2.18 corresponds across the equivalence **Sch** $\simeq$ **Nom**.

3 THE DISCRETE CASE

Theorem 4.35 imports quite a bit of measure theory in order to support continuous probability. To describe the key ideas, we temporarily set the measure theory aside by first presenting in detail a version of Theorem 4.35 adapted to discrete probability.

The structure of this section is completely analogous to Section 2. We first present two different probabilistic separation logics: one where separation is defined via the product of sample spaces, and a second based on Li et al. [27] where separation is defined via *independent combination*. Then, we will show how separation-as-product naturally lives in a category $\mathbf{EMS}_\mathrm{d}$ of *discrete enhanced measurable sheaves* analogous to the Schanuel topos, and how separation-as-independent-combination naturally lives in a category $\mathbf{Set}_\mathrm{d}^{\ll}$ of *discrete absolutely continuous sets*. Finally, we show these two categories equivalent, and that the two notions of separation correspond across this equivalence, giving an analog of Fact 1.1 suitable for discrete probability.

In Section 2 we considered a tiny separation logic TinySep for integer-valued stores. Analogously, we consider here a logic for integer-valued random variables:

$$P, Q ::= X \sim \mu \mid \mathsf{True} \mid P * Q. \qquad \text{(TinyProbSep)}$$

The proposition $X \sim \mu$ asserts that the logical variable X stands for an integer-valued random variable with probability mass function $\mu : \mathbb{Z} \to [0,1]$. As in TinySep, a proposition is well-formed in context Γ, written $\Gamma \vdash P$, if Γ contains the variables used by P. We shall establish the equivalence of two different models for TinyProbSep. In both cases, the basic idea is that a proposition denotes a predicate on *probability spaces* and logical variables denote *random variables*, just as a proposition in ordinary separation logic denotes a predicate on stores with logical variables denoting heap locations. The difference is in how these objects are represented:

Model 1: separation as product. In this model, a *probability space* consists of two components: (1) a nonempty countable set Ω called the *sample space*, and (2) a *probability space* $\mathcal{P}$ *on* Ω consisting of a pair $(\mathcal{F}, \mu)$ with $\mathcal{F}$ a σ-algebra on Ω and $\mu : \mathcal{F} \to [0,1]$ a probability measure. A *random variable on* Ω is a function $\Omega \to \mathbb{Z}$. We will write $\mathbb{P}(\Omega)$ and $\mathrm{RV}(\Omega)$ for the set of probability spaces and random variables on Ω respectively.

The meaning of a proposition depends on the underlying sample space: $\Gamma \vdash P$ denotes a map $[\![\Gamma \vdash P]\!]_1^\Omega : (\Gamma \to \mathrm{RV}(\Omega)) \to \mathscr{P}(\mathbb{P}(\Omega))$ associating each *random substitution* $G : \Gamma \to \mathrm{RV}(\Omega)$ to the set $[\![\Gamma \vdash P]\!]_1^\Omega(G)$ of probability spaces on Ω satisfying P.

Under this interpretation, we have $\mathcal{P} \in [\![\mathsf{True}]\!]_1^\Omega(G)$ for all probability spaces $\mathcal{P}$ on Ω, and $(\mathcal{F}, \mu) \in [\![X \sim \nu]\!]_1^\Omega(G)$ if and only if $G(X)$ is $\mathcal{F}$-measurable and has distribution ν; i.e., for all $i \in \mathbb{Z}$ it holds that $G(X)^{-1}(i) \in \mathcal{F}$ and $\mu(G(X)^{-1}(i)) = \nu(i)$. Separating conjunction is defined in terms of products of sample spaces. To make this precise, we need the following definitions:

Definition 3.1 (Pullback probability space)**.** Let X be a nonempty countable set, $(Y, \mathcal{G}, \nu)$ a countable probability space, and $f : X \twoheadrightarrow Y$ a surjective function. The *pullback of* $(\mathcal{G}, \nu)$ *along* f, written $f^{-1}(\mathcal{G}, \nu)$, is the probability space $(\mathcal{F}, \mu)$ on X defined by $\mathcal{F} = \{f^{-1}(G) \mid G \in \mathcal{G}\}$ and $\mu(f^{-1}(G)) = \nu(G)$. Note μ is well-defined because f surjective, so f^{-1} injective.

Definition 3.2 (Subspace)**.** Given two probability spaces $(\mathcal{F}, \mu)$ and $(\mathcal{G}, \nu)$ on Ω, say $(\mathcal{F}, \mu)$ is a *subspace* of $(\mathcal{G}, \nu)$, written $(\mathcal{F}, \mu) \sqsubseteq (\mathcal{G}, \nu)$, if $\mathcal{F} \subseteq \mathcal{G}$ and $\nu|_\mathcal{F} = \mu$.

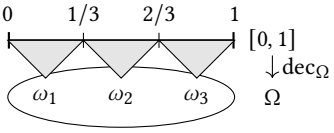
With these definitions, the separating conjunction $P_1 * P_2$ holds of a probability space $\mathcal{P}$ on Ω if and only if there exist probability spaces $\mathcal{P}_1$ on Ω_1 and $\mathcal{P}_2$ on Ω_2 and a surjective function $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$ such that $\mathcal{P}_1$ satisfies P_1 and $\mathcal{P}_2$ satisfies P_2 and $p^{-1}(\mathcal{P}_1 \otimes \mathcal{P}_2)$ is a subspace of $\mathcal{P}$, where $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the product probability space on $\Omega_1 \times \Omega_2$ whose measure is the product measure induced by the measures of $\mathcal{P}_1$ on Ω_1 and $\mathcal{P}_2$ on Ω_2 in the usual way.

For example, let Ω be the sample space $\{0,1\}^3$ of points $(x, y, z) \in \mathbb{R}^3$ with x, y, z all either 0 or 1. Let G be the random substitution of type $\{X, Y\} \to \mathrm{RV}(\Omega)$ where $G(X)$ is the random variable $(x, y, z) \mapsto x$ and $G(Y)$ is the random variable $(x, y, z) \mapsto y$. Let $(\mathcal{F}, \mu)$ be the uniform probability space on Ω, assigning each tuple (x, y, z) probability $1/8$. It holds that

$$(\mathcal{F}, \mu) \in [\![(X \sim \mathrm{Ber}(1/2)) * (Y \sim \mathrm{Ber}(1/2))]\!]_1^\Omega(G),$$

witnessed by setting p to the projection $\Omega \twoheadrightarrow \{0,1\} \times \{0,1\}$ defined by $p(x, y, z) = (x, y)$.

Model 2: separation as independent combination. In this model, one fixes upfront a single measurable space to serve as a "universal sample space" into which all discrete sample spaces can be embedded. Any standard Borel space will do; we choose the interval $[0,1]$. The idea is that, just as every finite store shape L can be encoded as a finite subset of $\mathbb{N}$ via an injective function $\mathrm{enc}_L : L \hookrightarrow \mathbb{N}$, every nonempty countable sample space Ω can be encoded as a countable partition of the interval via a random variable $\mathrm{dec}_\Omega : [0,1] \to \Omega$ with each $\mathrm{dec}_\Omega^{-1}(\omega)$ nonnegligible, visualized as:



(IntervalEncode)

This illustration gives one possible encoding of the three-point space $\{\omega_1, \omega_2, \omega_3\}$ as the partition $\{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$, generated by the random variable $\mathrm{dec}_\Omega : [0, 1] \to \{\omega_1, \omega_2, \omega_3\}$ taking value ω_1 on $[0, 1/3)$, ω_2 on $[1/3, 2/3]$, and ω_3 on $(2/3, 1]$.

With this fixed sample space in hand, the random variables $\Omega \to \mathbb{Z}$ of Model 1 can be encoded as Lebesgue-measurable functions $[0, 1] \to \mathbb{Z}$, quotiented by almost-everywhere equality. We will write $\overline{\mathrm{RV}}$ for the set of integer-valued random variables.

In order for our encoding of sample spaces as partitions generated by random variables to respect almost-everywhere equality of random variables, we must consider such partitions up to negligibility: for example, the partitions $\{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$ and $\{[0, 1/3], (1/3, 2/3), [2/3, 1]\}$ should be considered equivalent, as they arise from almost-everywhere-equal random variables. This idea motivates the following definition.

Definition 3.3. A *countable measurable partition of* $[0, 1]$ is a countable partition $\{A_i\}_{i \in I}$ with each A_i a Lebesgue-measurable and non-negligible subset of $[0, 1]$, quotiented by almost-everywhere equality: two partitions are almost-everywhere equal, written $\{A_i\}_{i \in I} =_{\text{a.s.}} \{B_j\}_{j \in J}$, if for all i in I there exists a unique j in J such that the symmetric difference $A_i \triangle B_j$ is Lebesgue-negligible.

Just as any L-shaped valuation can be encoded as a finite partial function on $\mathbb{N}$, any discrete probability space can be encoded as a countable measurable partition equipped with a measure:

Definition 3.4. A *countable measured partition of* $[0, 1]$ is a pair $(\{A_i\}_{i \in I}, \mu)$ with $\{A_i\}_{i \in I}$ a countable measurable partition of $[0, 1]$ and $\mu : \{A_i\}_{i \in I} \to [0, 1]$ a function satisfying $\sum_i \mu(A_i) = 1$. Two such partitions are equal if their measurable partitions are equal and their measures agree. Let $\overline{\mathbb{P}}$ be the set of countable measured partitions of $[0, 1]$.

Now that we have a way of encoding discrete probability spaces as countable measured partitions of $[0, 1]$, we can define a model of TinyProbSep purely in terms of countable measured partitions. A proposition $\Gamma \vdash P$ denotes a map $[\![\Gamma \vdash P]\!]_2 : (\Gamma \to \overline{\mathrm{RV}}) \to \mathscr{P}(\overline{\mathbb{P}})$ assigning each random substitution $G : \Gamma \to \overline{\mathrm{RV}}$ the set of countable measured partitions satisfying P. The interpretations of True and $X \sim \mu$ are as in Model 1: $\mathcal{P} \in [\![\mathsf{True}]\!]_2(G)$ for any countable measured partition $\mathcal{P}$, and $(\{A_i\}_{i \in I}, \mu) \in [\![X \sim \nu]\!]_2(G)$ if and only if for all $k \in \mathbb{Z}$ there exists $i \in I$ with $G(X)^{-1}(k) \triangle A_i$ negligible and $\mu(G(X)^{-1}(k)) = \nu(k)$. Separating conjunction is defined via *independent combination*, following Li et al. [27]:

Definition 3.5 (Discrete independent combination)**.** A countable measured partition $(\mathcal{A}, \mu)$ is an *independent combination* of $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$ if (1) the partition $\mathcal{A}$ is generated by the intersections $A_1 \cap A_2$ for A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$ and (2) $\mu(A_1 \cap A_2) = \mu_1(A_1)\mu_2(A_2)$ for all A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$. Independent combinations are unique if they exist [27, Lemma 2.3], defining a partial function $\bar{\bullet}$ with $(\mathcal{A}_1, \mu_1) \bar{\bullet} (\mathcal{A}_2, \mu_2) = (\mathcal{A}, \mu)$ if and only if $(\mathcal{A}, \mu)$ is the independent combination of $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$.

Definition 3.6 (Ordering on partitions)**.** For two countable measured partitions $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$, let $\overline{\sqsubseteq}$ be the ordering relation defined by $(\mathcal{A}, \mu) \overline{\sqsubseteq} (\mathcal{B}, \nu)$ if and only if the partition $\mathcal{A}$ is coarser than $\mathcal{B}$ and ν restricts to μ.

These definitions give an interpretation of separating conjunction: a countable measured partition $(\mathcal{A}, \mu)$ on $[0, 1]$ satisfies $P_1 * P_2$ with random substitution G, written $(\mathcal{A}, \mu) \in [\![P_1 * P_2]\!]_2(G)$, if and only if there exist $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$ independently combinable with $(\mathcal{A}_1, \mu_1) \bar{\bullet} (\mathcal{A}_2, \mu_2) \overline{\sqsubseteq} (\mathcal{A}, \mu)$ such that $(\mathcal{A}_1, \mu_1)$ is in $[\![P_1]\!]_2(G)$ and $(\mathcal{A}_2, \mu_2)$ is in $[\![P_2]\!]_2(G)$.

Relating the two models. We will show that Model 1 and Model 2 are equivalent. As shown in IntervalEncode, every nonempty countable sample space Ω can be encoded as a countable measured partition on $[0, 1]$ via a suitable random variable $\mathrm{dec}_\Omega : [0, 1] \to \Omega$. Choosing a dec_Ω for every Ω allows translating Model 1 into Model 2: a Model 1 random variable $X \in \mathrm{RV}(\Omega)$ corresponds to a Model 2 random variable $X \circ \mathrm{dec}_\Omega \in \overline{\mathrm{RV}}$, and probability spaces can be translated similarly. To extend this into an equivalence analogous to Fact 1.1, we repeat the recipe of Section 2: we will place Models 1 and 2 into suitable categories, show the categories equivalent, and show that the models correspond across this equivalence.

3.1 Discrete enhanced measurable sheaves

In Section 2.1 we saw how the Schanuel topos **Sch** captured the invariants maintained by Model 1 of TinySep. In this section we describe analogously how a category of *discrete enhanced measurable sheaves*, written $\mathbf{EMS}_\mathrm{d}$, captures the invariants maintained by Model 1 of TinyProbSep. Whereas the invariants of Section 2.1 were about extensions and restrictions of the store shape L, the invariants in our probabilistic setting are about extensions and restrictions of the sample space Ω, as observed by Simpson [44, 45]:

- *Extension*: propositions that hold in sample space Ω should continue to hold when Ω is extended to a larger sample space Ω' via a surjective function $p : \Omega' \twoheadrightarrow \Omega$. More precisely, if $(\mathcal{F}, \mu) \in [\![\Gamma \vdash P]\!]_1^\Omega(G)$ for some probability space $(\mathcal{F}, \mu)$ on Ω and random substitution $G : \Gamma \to \mathrm{RV}(\Omega)$, then it should hold that $p^{-1}(\mathcal{F}, \mu) \in [\![\Gamma \vdash P]\!]_1^{\Omega'}(G \cdot p)$, where $G \cdot p$ is the random substitution $(G \cdot p)(X) = G(X) \circ p$.[5]
- *Restriction*: the truth of a proposition should not depend on any unused samples. For example, let Ω be an arbitrary sample space. Suppose $G : \Gamma \to \mathrm{RV}(\Omega)$ sends every X in Γ to the constant random variable 0, so $G(X)(\omega) = 0$ for all ω, and let $(\mathcal{F}, \mu)$ be the minimal probability space on Ω where $\mathcal{F}$ is the minimal σ-algebra $\{\emptyset, \Omega\}$ and μ the minimal probability measure with $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Both G and $(\mathcal{F}, \mu)$ don't use any of the samples in Ω: every random variable $G(X)$ is a deterministic value, and μ only assigns probabilities to the deterministic events $\emptyset$ and Ω. Restriction says that if a proposition P holds in this situation, then it should hold of the one-point probability space on the one-point set with substitution sending every X in Γ to the constant random variable 0.

To capture these invariants, we replay the construction of the Schanuel topos: whereas the Schanuel topos is a category of atomic sheaves on the category **Shp** of store shapes, $\mathbf{EMS}_\mathrm{d}$ is a category of atomic sheaves on a category of discrete sample spaces.

[5] Note that this rolls the two invariants Extension and Renaming of Section 2.1 into one: it captures invariance under permutations of the underlying sample space in the case where p is a bijection.

First, we fix a base category capturing Extension: the category $\mathbf{Surj}_{\leq\omega}$ of nonempty countable sets and surjective functions. The idea is that an object Ω of $\mathbf{Surj}_{\leq\omega}$ is a countable sample space, and a morphism $p : \Omega' \twoheadrightarrow \Omega$ extends a sample space Ω to a larger space Ω' in which every sample ω in Ω is expanded to a set of samples $p^{-1}(\omega) \subseteq \Omega'$; surjectivity of p ensures that every $p^{-1}(\omega)$ is nonempty, so p never deletes any samples in Ω.

Functors $\mathbf{Surj}_{\leq\omega}^{\mathrm{op}} \to \mathbf{Set}$ model sample-space-dependent concepts. In particular, there is a functor modelling probability spaces: For a $\mathbf{Surj}_{\leq\omega}$-morphism $p : \Omega' \twoheadrightarrow \Omega$, setting $\mathbb{P}(p)$ to the function $\mathbb{P}(\Omega) \to \mathbb{P}(\Omega')$ that sends a probability space $\mathcal{P}$ on Ω to its pullback $p^{-1}\mathcal{P}$ makes $\mathbb{P}$ a functor $\mathbf{Surj}_{\leq\omega}^{\mathrm{op}} \to \mathbf{Set}$.

Next, we capture Restriction by cutting the functor category $[\mathbf{Surj}_{\leq\omega}^{\mathrm{op}}; \mathbf{Set}]$ down to a full subcategory of atomic sheaves. The notion of atomic sheaf is given by the notion of *atomic topology*, which exists for a given category if and only if the following *Ore property* holds [28, p.115]:

Definition 3.7. A category C has the *right Ore property* if for all $X \xrightarrow{f} Z \xleftarrow{g} Y$ there exists $X \xleftarrow{h} W \xrightarrow{k} Y$ such that $fh = gk$.

That $\mathbf{Surj}_{\leq\omega}$ satisfies this condition can be straightforwardly verified: any cospan can be completed to a commutative square by taking a pullback in **Set**. Thus the notion of atomic sheaf makes sense for $\mathbf{Surj}_{\leq\omega}$, a functor is an atomic sheaf if and only if it has a restriction operation in the sense of Definition 2.6, and the following definition makes sense:

Definition 3.8. Let $\mathbf{EMS}_{\mathrm{d}}$ be the full subcategory of the category $[\mathbf{Surj}_{\leq\omega}^{\mathrm{op}}; \mathbf{Set}]$ consisting of those functors that are atomic sheaves.

Just as $\mathbb{P}(\Omega)$ models the concept of probability spaces on Ω, there are other atomic sheaves corresponding to each of the other concepts used to define Model 1:

Proposition 3.9. The following are objects of $\mathbf{EMS}_{\mathrm{d}}$:

- The constant functor Prop of propositions sending every object of $\mathbf{Surj}_{\leq\omega}$ to the set $\{\top, \bot\}$ and every morphism of $\mathbf{Surj}_{\leq\omega}$ to the identity function.
- The functor RV of random variables, with action on morphisms defined by $\mathrm{RV}(p : \Omega' \twoheadrightarrow \Omega)(X : \mathrm{RV}(\Omega)) = (X \circ p : \mathrm{RV}(\Omega'))$.
- The functor RV^Γ of Γ-substitutions with $\mathrm{RV}^\Gamma(\Omega) = \Gamma \to \mathrm{RV}(\Omega)$ and action on morphisms defined by lifting RV pointwise.

With these sheaves in hand, one can show Model 1 lives in $\mathbf{EMS}_{\mathrm{d}}$:

Proposition 3.10. If $\Gamma \vdash P$ then the Ω-indexed family of functions

$$\left(\llbracket \Gamma \vdash P \rrbracket_1^\Omega : (\Gamma \to \mathrm{RV}(\Omega)) \to \mathcal{P}(\mathbb{P}(\Omega))\right)_{\Omega \in \mathbf{Surj}_{\leq\omega}}$$

is natural in Ω, so defines a morphism $\mathrm{RV}^\Gamma \to \mathrm{Prop}^{\mathbb{P}}$ in $\mathbf{EMS}_{\mathrm{d}}$, and every morphism of this type satisfies Extension and Restriction.

3.2 Discrete absolutely continuous sets

We now turn to Model 2 of TinyProbSep described in Section 3. Just as Model 2 of TinySep naturally lives in the category **Nom** of nominal sets, Model 2 of TinyProbSep naturally lives in a category $\mathbf{Set}_{\mathrm{d}}^{\ll}$ of *discrete absolutely continuous sets*. **Nom** captures two invariants held by Model 2 of TinySep: Permutation and Finiteness. Model 2 of TinyProbSep maintains analogous invariants:

- *Permutation*: propositions should be stable under permuting the sample space $[0, 1]$. More precisely, if $(\mathcal{A}, \mu) \in \llbracket \Gamma \vdash P \rrbracket_2(G)$ for some countable measured partition $(\mathcal{A}, \mu)$ and random substitution $G : \Gamma \to \overline{\mathrm{RV}}$, and if $\pi : [0, 1] \to [0, 1]$ is a measurable bijection, then it should hold that $(\mathcal{A}, \mu) \cdot \pi \in \llbracket \Gamma \vdash P \rrbracket_2(G \cdot \pi)$, where $(\mathcal{A}, \mu) \cdot \pi$ and $G \cdot \pi$ are the results of the permutation π acting on $(\mathcal{A}, \mu)$ and G.
- *Sparsity*: more subtly, the countable measured partitions $(\mathcal{A}, \mu)$ represent *countable* probability spaces only. This ensures $(\mathcal{A}, \mu)$ always leaves "enough room" in $[0, 1]$ for "fresh randomness": for any other discrete probability space, there exists an encoding of it as a countable measured partition $(\mathcal{B}, \nu)$ such that the discrete independent combination $(\mathcal{A}, \mu) \mathbin{\bar{\bullet}} (\mathcal{B}, \nu)$ is defined.

To capture Permutation, the objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$ are sets equipped with an action by a group of measurable automorphisms. Specifically, let $\mathrm{Aut}[0, 1]$ be the group of measurable maps $\pi : [0, 1] \to [0, 1]$ that are bijective mod almost-everywhere equality. The category of $\mathrm{Aut}[0, 1]$-sets is the category whose objects are sets X equipped with a right action by $\mathrm{Aut}[0, 1]$ and whose morphisms are equivariant functions. Just as there is a S_ω-set $\overline{\mathrm{S}}$ of stores, there is a $\mathrm{Aut}[0, 1]$-set $\overline{\mathbb{P}}$ of countable measured partitions on $[0, 1]$:

Definition 3.11. Let $\overline{\mathbb{P}}$ be the set of countable measured partitions on $[0, 1]$ with action $(\{A_i\}_{i\in I}, \mu) \cdot \pi = (\{\pi^{-1}(A_i)\}_{i\in I}, \mu \circ \pi)$.

Sparsity is captured by topologizing $\mathrm{Aut}[0, 1]$ via countable measurable partitions, so a stabilizer $\mathrm{Stab}\, x$ is open if every π stabilizing x does so for a "countable reason": there is a partition $\mathcal{A}$ fixed by π such that any other permutation fixing $\mathcal{A}$ also stabilizes x.

Definition 3.12 (Topology on $\mathrm{Aut}[0, 1]$)**.** A subset U of $\mathrm{Aut}[0, 1]$ is *open* if for every $\pi \in U$, there exists a countable measurable partition $\mathcal{A}$ of $[0, 1]$ such that $\pi \in \mathrm{Fix}\,\mathcal{A} \subseteq U$, where $\mathrm{Fix}\,\mathcal{A}$ is the subgroup of $\mathrm{Aut}[0, 1]$ consisting of those permutations π that fix every element of $\mathcal{A}$; i.e., $\pi(A) =_{\text{a.e.}} A$ for all $A \in \mathcal{A}$.

Definition 3.13. A *discrete absolutely continuous set* is a $\mathrm{Aut}[0, 1]$-set whose elements have open stabilizers. Let $\mathbf{Set}_{\mathrm{d}}^{\ll}$ be the category of discrete absolutely continuous sets and equivariant functions.

In addition to $\mathbb{P}$, there are objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$ corresponding to each of the other concepts used to define Model 2 of TinyProbSep:

Proposition 3.14. The following are objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$:

- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{Prop}} = \{\top, \bot\}$ with the trivial action.
- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{RV}}$ of random variables with action $X \cdot \pi = X \circ \pi$.
- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{RV}}^\Gamma$ of random Γ-substitutions $\Gamma \to \overline{\mathrm{RV}}$ with action defined by lifting $\overline{\mathrm{RV}}$ pointwise.

With these in hand, one can show Model 2 lives in $\mathbf{Set}_{\mathrm{d}}^{\ll}$:

Proposition 3.15. If $\Gamma \vdash P$ then the function $\llbracket \Gamma \vdash P \rrbracket_2$ is a morphism $\overline{\mathrm{RV}}^\Gamma \to \overline{\mathrm{Prop}}^{\overline{\mathbb{P}}}$ in $\mathbf{Set}_{\mathrm{d}}^{\ll}$, and every morphism of this type satisfies Permutation and Finiteness.

3.3 The equivalence of categories

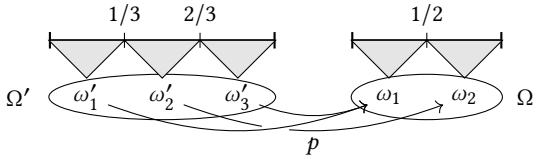
Having placed Models 1 and 2 of TinyProbSep described in Section 3 into the categories $\mathbf{EMS}_{\mathrm{d}}$ and $\mathbf{Set}_{\mathrm{d}}^{\ll}$ respectively, we describe in this section how $\mathbf{EMS}_{\mathrm{d}}$ are $\mathbf{Set}_{\mathrm{d}}^{\ll}$ equivalent, giving an analog

of **Sch** ≃ **Nom** for discrete probability. The key step is to establish probabilistic analogs of Homogeneity and Correspondence:

Lemma 3.16 (Homogeneity)**.** Let Ω, Ω' be nonempty countable sets and let dec_Ω and $\mathrm{dec}_{\Omega'}$ be measurable functions $[0,1] \to \Omega$ and $[0,1] \to \Omega'$ with $\mathrm{dec}_\Omega^{-1}(\omega)$ and $\mathrm{dec}_{\Omega'}^{-1}(\omega')$ nonnegligible for all $\omega \in \Omega$ and $\omega' \in \Omega'$. For any surjective function $f : \Omega' \twoheadrightarrow \Omega$, there exists $\pi \in \mathrm{Aut}[0,1]$ making the following square commute:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\pi} & [0,1] \\ {\scriptstyle \mathrm{dec}_{\Omega'}}\downarrow & & \downarrow{\scriptstyle \mathrm{dec}_\Omega} \\ \Omega' & \xrightarrow[p]{} & \Omega \end{array}$$

This lemma is particularly important, so we give some intuition about its proof. Consider the following visualization of a surjection p from $\Omega' = \{\omega'_1, \omega'_2, \omega'_3\}$ onto $\Omega = \{\omega_1, \omega_2\}$ and two decoding functions $\mathrm{dec}_{\Omega'}$ and dec_Ω visualized as in IntervalEncode:



Lemma 3.16 asserts that there exists π with $p \circ \mathrm{dec}_{\Omega'} = \mathrm{dec}_\Omega \circ \pi$. Indeed we can explicitly construct such a π for this example: let π be an automorphism that sends the interval $[0, 1/3]$ to $[0, 1/4]$, the interval $[2/3, 1]$ to $[1/4, 1/2]$, and finally $[1/3, 2/3]$ to $[1/2, 1]$. This construction generalizes nicely to any situation where the preimages $\mathrm{dec}_{\Omega'}^{-1}(\omega'_i)$ and $\mathrm{dec}_\Omega^{-1}(\omega_i)$ are all intervals. In the fully general case, these preimages can be arbitrary Lebesgue-measurable sets, but every such set is measurably isomorphic to an interval [17, 344J], so the general case reduces to the one sketched above.

Lemma 3.17 (Correspondence)**.** For any countable measurable partition $\{A_i\}_{i \in I}$ of $[0,1]$, let $\mathrm{Fix}\{A_i\}_{i\in I}$ be the subgroup of $\mathrm{Aut}[0,1]$ consisting of those automorphisms $\pi \in \mathrm{Aut}[0,1]$ fixing $\{A_i\}_{i\in I}$, so that $\pi(A_i) \triangle A_i$ negligible for all $i \in I$. For any two partitions $\mathcal{A}$ and $\mathcal{B}$, it holds that $\mathrm{Fix}\,\mathcal{A} \subseteq \mathrm{Fix}\,\mathcal{B}$ iff $\mathcal{A}$ is finer than $\mathcal{B}$.

Proof. If $\mathcal{A}$ is finer than $\mathcal{B}$ then certainly every π fixing $\mathcal{A}$ fixes $\mathcal{B}$. For the converse, suppose for contradiction that $\mathcal{A}$ is not finer than $\mathcal{B}$, so there is some $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ with $A \cap B_1$ and $A \cap B_2$ both nonnegligible. Pick an arbitrary π swapping $A \cap B_1$ with $A \cap B_2$; π fixes $\mathcal{A}$ but not $\mathcal{B}$, contradicting $\mathrm{Fix}\,\mathcal{A} \subseteq \mathrm{Fix}\,\mathcal{B}$. □

The equivalence follows from these lemmas:

Theorem 3.18. $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

For details, see the appendix: Theorem 3.18 follows from a specialization of Theorem C.33 using Lemmas 3.16 and 3.17 to satisfy the preconditions. This equivalence of categories extends to an equivalence of Models 1 and 2 of TinyProbSep. The argument is as in Section 2.3: we package Models 1 and 2 into resource monoids in $\mathbf{EMS}_\mathrm{d}$ and $\mathbf{Set}_\mathrm{d}^{\ll}$ respectively, and then show they correspond across the equivalence $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

To construct the resource monoid packaging Model 1 into $\mathbf{EMS}_\mathrm{d}$, we make use of a general recipe for constructing models of separation logic via the *Day convolution* [6, 14, 33]. The Day convolution is a general construction lifting a monoidal structure on a base category C to a monoidal structure on $[C^{\mathrm{op}}; \mathbf{Set}]$, see Day [13]. The resource monoid $(\sqsubseteq, S^2_\perp, i, \bullet)$ in **Sch** described in Section 2.3 can be constructed using the Day convolution: the base category **Shp** has a monoidal product given by coproduct of finite sets, and Day convolution lifts this to a monoidal product $\otimes$ on $[\mathbf{Shp}^{\mathrm{op}}; \mathbf{Set}]$; applying the Day convolution to the sheaf S gives the functor $S \otimes S$, which one can show is naturally isomorphic to $S^2_\perp$; the operations $\sqsubseteq, i, \bullet$ can then be defined straightforwardly.

To apply this recipe for discrete probability, we replace **Shp** with $\mathbf{Surj}_{\leq\omega}$ and coproduct + of finite sets with product $\times$ of sample spaces. This makes $(\mathbf{Surj}_{\leq\omega}, \times, 1)$ a monoidal category, where the unit 1 is the one-point sample space. Via the Day convolution, $\times$ lifts to a monoidal product $\otimes$ on $[\mathbf{Surj}^{\mathrm{op}}_{\leq\omega}; \mathbf{Set}]$. Just as $S \otimes S$ is isomorphic to the functor $S^2_\perp$ modelling separated stores, the Day convolution $\mathbb{P} \otimes \mathbb{P}$ is isomorphic to a sheaf of probability spaces that can be rendered independent with a suitable joint measure:

Proposition 3.19. The functor $\mathbb{P} \otimes \mathbb{P}$ is isomorphic to an atomic sheaf $\mathbb{P}^2_\perp$ sending each $\Omega \in \mathbf{Surj}_{\leq\omega}$ to the set

$$\{\, ((\pi_1 \circ p)^{-1}(\mathcal{P}_1), (\pi_2 \circ p)^{-1}(\mathcal{P}_2)) \mid \Omega_1, \Omega_2 \in \mathbf{Surj}_{\leq\omega}, \mathcal{P}_1 \in \mathbb{P}(\Omega_1), \mathcal{P}_2 \in \mathbb{P}(\Omega_2), p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2 \,\}$$

of pairs of probability spaces on Ω that "factor through" a product $\Omega_1 \times \Omega_2$ along some projection $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$.

The resource monoid operations can be defined as follows. First, the subspace ordering $\sqsubseteq$ forms a natural transformation $(\sqsubseteq) : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$. Next, there is natural transformation $(\bullet) : \mathbb{P}^2_\perp \to \mathbb{P}$ sending a pair $((\pi_1 \circ p)^{-1}(\mathcal{P}_1), (\pi_2 \circ p)^{-1}(\mathcal{P}_2))$ of probability spaces that factor through some $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$ to $p^{-1}(\mathcal{P}_1 \otimes \mathcal{P}_2)$, where $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the usual product probability space on $\Omega_1 \times \Omega_2$. Each $\mathbb{P}^2_\perp(\Omega)$ is a subset of $(\mathbb{P}\times\mathbb{P})(\Omega)$, and collecting the canonical subset-inclusions into an Ω-indexed family forms a natural transformation $i : \mathbb{P}^2_\perp \hookrightarrow \mathbb{P} \times \mathbb{P}$. Finally, $\bullet$ is associative and commutative and monotone with respect to $\sqsubseteq$, and has unit the natural transformation $\mathrm{emp} : 1 \to \mathbb{P}$ sending a sample space Ω to the trivial probability space $(\Omega, \{\emptyset, \Omega\}, \mu)$ with $\mu(\Omega) = 1$.

Proposition 3.20. $(\sqsubseteq, \mathbb{P}^2_\perp, i, \bullet, \mathrm{emp})$ is a resource monoid in $\mathbf{EMS}_\mathrm{d}$.

Model 2 can be packaged into a resource monoid in $\mathbf{Set}_\mathrm{d}^{\ll}$ analogously. Let $\overline{\mathbb{P}}^2_\perp$ be the discrete absolutely continuous set $\{(\mathcal{P}_1, \mathcal{P}_2) \mid \mathcal{P}_1, \mathcal{P}_2 \in \overline{\mathbb{P}}, \mathcal{P}_1 \mathbin{\overline{\bullet}} \mathcal{P}_2 \text{ defined}\}$ of pairs of independently combinable countable measured partitions of $[0,1]$ with pointwise group action. This is a subset of $\overline{\mathbb{P}} \times \overline{\mathbb{P}}$; both the canonical inclusion map $\overline{i}$ and the function $\overline{\bullet} : \overline{\mathbb{P}}^2_\perp \to \overline{\mathbb{P}}$ sending a pair $(\mathcal{P}_1, \mathcal{P}_2)$ of independently combinable probability spaces on $[0,1]$ to their independent combination $\mathcal{P}_1 \mathbin{\overline{\bullet}} \mathcal{P}_2$ are equivariant. Finally, the ordering relation $\overline{\sqsubseteq}$ on probability spaces on $[0,1]$ is equivariant, so defines a morphism $(\overline{\sqsubseteq}) : \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \overline{\mathrm{Prop}}$, and this ordering relation has as least element $\overline{\mathrm{emp}}$ the measured partition containing a single component with probability 1.

The following theorem establishes that $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \overline{i}, \overline{\bullet})$ is a resource monoid together with an analog of Fact 1.1 for discrete probability:

Theorem 3.21. The resource monoid $(\sqsubseteq, \mathbb{P}^2_\perp, i, \bullet, \mathrm{emp})$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \overline{i}, \overline{\bullet}, \overline{\mathrm{emp}})$ across the equivalence $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

4 THE CONTINUOUS CASE

In this section we generalize Section 3 from discrete to continuous probability: $\mathbf{EMS}_\mathrm{d}$ becomes a category $\mathbf{EMS}$ of *enhanced measurable sheaves*, and $\mathbf{Set}_\mathrm{d}^{\ll}$ becomes a category $\mathbf{Set}^{\ll}$ of *absolutely continuous sets*. Due to the amount of measure theory required, we stick to stating the key definitions and lemmas; the full details can be found in the appendix.

4.1 Enhanced measurable sheaves

The first step in generalizing $\mathbf{EMS}_\mathrm{d}$ to $\mathbf{EMS}$ is to replace the base category $\mathbf{Surj}_{\leq\omega}$ of discrete sample spaces with a base category of continuous sample spaces.

The starting point for this generalization is the following observation. Let $\mathbf{Prob}^+_{\leq\omega}$ be the category whose objects are countable probability spaces $(\Omega, \mu : \Omega \to [0,1])$ with $\mu(\omega) > 0$ for all $\omega \in \Omega$, and whose morphisms $(\Omega, \mu) \to (\Omega', \mu')$ are measure-preserving maps $f : \Omega \to \Omega'$; i.e., $\sum_{f(\omega)=\omega'} \mu(\omega) = \mu'(\omega')$ for all $\omega' \in \Omega'$. There is a functor $\mathrm{U}_\mathrm{d} : \mathbf{Prob}^+_{\leq\omega} \to \mathbf{Surj}_{\leq\omega}$ that forgets the measures μ: measure-preserving maps f between probability spaces with strictly positive measure are surjective because $f^{-1}(y)$ must be nonempty for all $y \in \mathrm{cod}(f)$. The category $\mathbf{Surj}_{\leq\omega}$ is the image of $\mathbf{Prob}^+_{\leq\omega}$ under U_d: every nonempty countable set Ω can be equipped with a strictly positive probability measure, and for every surjective function $p : \Omega' \twoheadrightarrow \Omega$, there exist strictly positive probability measures μ' on Ω' and μ on Ω making p a measure-preserving map $(\Omega', \mu') \to (\Omega, \mu)$. Thus $\mathbf{Surj}_{\leq\omega}$ can be thought of as a category of probability spaces where one has forgotten all measures.

To generalize this situation from discrete to continuous probability, we replace the category $\mathbf{Prob}^+_{\leq\omega}$ of countable probability spaces with a category of continuous probability spaces:

Definition 4.1 (The category $\mathbf{Prob}_\mathrm{std}$)**.** Let $\mathbf{Prob}_\mathrm{std}$ be the category of *standard probability spaces* [40] and measure-preserving maps quotiented by almost-everywhere equality.

Then, we replace the functor $\mathrm{U}_\mathrm{d} : \mathbf{Prob}^+_{\leq\omega} \to \mathbf{Surj}_{\leq\omega}$ with a functor U that forgets continuous probability measures. The idea behind this forgetting process is as follows. Given a probability space $(X, \mathcal{F}, \mu)$, one can forget everything about the measure μ except for which subsets are negligible, leaving behind an *enhanced measurable space* $(X, \mathcal{F}, \mathcal{N})$, where $\mathcal{N}$ is the σ-ideal of μ-negligible sets [34, Definition 4.4]. Given a measure-preserving map $[f] : (X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$ quotiented by almost-everywhere equality where μ has negligibles $\mathcal{N}$ and ν has negligibles $\mathcal{M}$, one can forget everything about $[f]$ measure-preserving except for the fact that $\nu(G) = 0$ iff $\mu(f^{-1}(G)) = 0$, leaving behind an equivalence class $[f]$ with $f^{-1}(G) \in \mathcal{N}$ iff $G \in \mathcal{M}$ for all $G \in \mathcal{G}$. This motivates the following definitions.

Definition 4.2 (The category $\mathbf{EMS}_\mathrm{std}$)**.** A *standard enhanced measurable space* is tuple $(X, \mathcal{F}, \mathcal{N})$ for which there exists a measure μ making $(X, \mathcal{F}, \mu)$ a standard probability space with negligibles $\mathcal{N}$. Given enhanced measurable spaces $(X, \mathcal{F}, \mathcal{N})$ and $(Y, \mathcal{G}, \mathcal{M})$, a measurable map $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is *negligible-preserving and reflecting* if $f^{-1}(G) \in \mathcal{N}$ iff $G \in \mathcal{M}$ for all $G \in \mathcal{G}$; two such maps f, f' are *almost-everywhere equal* if $f^{-1}(G) \triangle f'^{-1}(G') \in \mathcal{N}$ for all $G, G' \in \mathcal{G}$ with $G \triangle G' \in \mathcal{M}$. Let $\mathbf{EMS}_\mathrm{std}$ be the category of standard enhanced measurable spaces and negligible-preserving-and-reflecting maps quotiented by almost-everywhere equality.

Proposition 4.3. Let $\mathrm{U} : \mathbf{Prob}_\mathrm{std} \to \mathbf{EMS}_\mathrm{std}$ be the functor that sends probability spaces $(X, \mathcal{F}, \mu)$ with negligibles $\mathcal{N}$ to enhanced measurable spaces $(X, \mathcal{F}, \mathcal{N})$. This functor is surjective on objects, and any morphism of standard enhanced measurable spaces arises from a measure-preserving map equipping those spaces with standard probability measures.

Then, just as $\mathbf{EMS}_\mathrm{d}$ is the category of atomic sheaves on $\mathbf{Surj}_{\leq\omega}$, $\mathbf{EMS}$ is the category of atomic sheaves on $\mathbf{EMS}_\mathrm{std}$:

Proposition 4.4. $\mathbf{EMS}_\mathrm{std}$ has the right Ore property.

Definition 4.5. Let $\mathbf{EMS}$ be the full subcategory of $[\mathbf{EMS}_\mathrm{std}^{\mathrm{op}}; \mathbf{Set}]$ consisting of atomic sheaves. Objects of $\mathbf{EMS}$ will be called *enhanced measurable sheaves*.

Inside $\mathbf{EMS}$, there are continuous analogs of the discrete enhanced measurable sheaves RV of random variables and $\mathbb{P}$ of discrete probability spaces. The continuous analog of RV models A-valued random variables for A Polish, following Simpson [44, 45]:

Definition 4.6. For any measurable space $(A, \mathcal{G})$ arising from a Polish space, the *sheaf of random variables* is:

$$\mathrm{RV}_A(\Omega, \mathcal{F}, \mathcal{N}) = \{\text{measurable maps } (\Omega, \mathcal{F}) \to (A, \mathcal{G})\} \,/ =_{\mathcal{N}\text{-a.e.}}$$
$$\mathrm{RV}_A(p : \Omega' \to \Omega)([X] : \mathrm{RV}_A(\Omega)) : \mathrm{RV}_A(\Omega') = [X \circ p]$$

For proof that RV_A is indeed a sheaf, see the appendix. Next, to generalize $\mathbb{P}$ from discrete to continuous probability, we make use of the following observation: every discrete probability space $(\Omega, \mathcal{F}, \mu)$ arises via pullback from a surjection $X : \Omega \twoheadrightarrow A$ in which the set A is equipped with a probability mass function $\nu : A \to [0,1]$, by setting $\mathcal{F} := \{X^{-1}(a) \mid a \in A\}$ and $\mu(X^{-1}(a)) = \nu(a)$. Thus, discrete probability spaces $(\Omega, \mathcal{F}, \mu)$ can be represented by $\mathbf{Surj}_{\leq\omega}$ morphisms $\Omega \to \mathrm{U}_\mathrm{d}(A, \mu)$ for $(A, \mu) \in \mathbf{Prob}^+_{\leq\omega}$, where U_d is the functor $\mathbf{Surj}_{\leq\omega} \to \mathbf{Prob}^+_{\leq\omega}$ that forgets measures. This motivates the following generalization to the continuous setting.

Definition 4.7. The *sheaf of probability spaces* is

$$\mathbb{P} := \mathrm{colim}_{A : \mathrm{Core}(\mathbf{Prob}_\mathrm{std})} \text{⅄}(\mathrm{U}A),$$

where ⅄ is the Yoneda embedding, $\mathrm{Core}(\mathbf{Prob}_\mathrm{std})$ is the subcategory of $\mathbf{Prob}_\mathrm{std}$-isomorphisms, and the colimit is taken in presheaves. (See the appendix for proof that $\mathbb{P}$ is indeed a sheaf.) Concretely, the presheaf $\mathbb{P}$ sends $(\Omega, \mathcal{F}, \mathcal{N}) : \mathbf{EMS}_\mathrm{std}$ to the set of pairs $((A, \mathcal{G}, \mu), X)$ where $(A, \mathcal{G}, \mu) : \mathbf{Prob}_\mathrm{std}$ and X is a $\mathbf{EMS}_\mathrm{std}$-map from $(\Omega, \mathcal{F}, \mathcal{N})$ to $\mathrm{U}(A, \mathcal{G}, \mu)$, quotiented by $((A, \mathcal{G}, \mu), X) \sim ((A', \mathcal{G}', \mu'), X')$ iff there is a $\mathbf{Prob}_\mathrm{std}$-iso $i : (A, \mathcal{G}, \mu) \to (A', \mathcal{G}', \mu')$ with $X' = U(i)\, X$. The action on morphisms is given by precomposition.

Using RV and $\mathbb{P}$, we generalize the resource monoid of Theorem 3.21 to a resource monoid of continuous probability spaces. The monoidal category $(\mathbf{Surj}_{\leq\omega}, \times, 1)$ of discrete sample spaces becomes a monoidal category $(\mathbf{EMS}_\mathrm{std}, \otimes, 1)$ of continuous sample spaces, with monoidal product $\otimes$ inherited from the usual tensor product $\otimes_{\mathbf{Prob}_\mathrm{std}}$ of standard probability spaces:

Definition 4.8. Given two standard enhanced measurable spaces X, Y, their *tensor product* $X \otimes Y$ is defined to be $\mathrm{U}(X' \otimes_{\mathbf{Prob}_\mathrm{std}} Y')$, where X' and Y' are arbitrary standard probability spaces with $\mathrm{U}(X') = X$ and $\mathrm{U}(Y') = Y$.

This is well-defined — the choice of X', Y' does not matter — and extends to a bifunctor on $\mathbf{EMS}_{\mathrm{std}}$ making $(\mathbf{EMS}_{\mathrm{std}}, \otimes, 1)$ a symmetric monoidal category with unit the one-point space 1. For details, see the appendix. Lifting $\otimes$ to $[\mathbf{EMS}_{\mathrm{std}}^{\mathrm{op}}; \mathbf{Set}]$ via the Day convolution yields a resource monoid in $\mathbf{EMS}$:

Lemma 4.9. The Day convolution $\mathbb{P} \otimes \mathbb{P}$ is a sheaf, and there is a monic map of sheaves $i : \mathbb{P} \otimes \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$.

Lemma 4.10. There is a map of sheaves $\sqsubseteq : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$, where Prop is the constant sheaf at $\{\top, \bot\}$, and a map of sheaves $\mathrm{emp} : 1 \to \mathbb{P}$, making $(\mathbb{P}, \mathrm{emp})$ a poset in $\mathbf{EMS}$ with least element emp.

Lemma 4.11. There is a map of sheaves $\bullet : \mathbb{P} \otimes \mathbb{P} \to \mathbb{P}$, monotone with respect to $\sqsubseteq$, such that $(\mathbb{P}, \bullet, \mathrm{emp})$ is a partial commutative monoid in $\mathbf{EMS}$.

Theorem 4.12. $(\sqsubseteq, \mathbb{P}^2_\bot, i, \bullet, \mathrm{emp})$ is a resource monoid in $\mathbf{EMS}$.

For details, see the appendix. While the colimit presentation of $\mathbb{P}$ makes it easier to check for sheafhood and to construct the above resource monoid, it is difficult to work with in the concrete calculations to follow. To address this, we show $\mathbb{P}$ equivalent to a sheaf of continuous probability spaces that arise via pullback along $\mathbf{EMS}_{\mathrm{std}}$-maps. To do this, we must take care to define pullback in a way that respects the negligible ideals contained in $\mathbf{EMS}_{\mathrm{std}}$-objects.

Definition 4.13. For $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ and $(Y, \mathcal{G}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$ and $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$, the *enhanced pullback of* $(Y, \mathcal{G}, \mu)$ *along* f, written $f^*(\mathcal{G}, \mu)$, is the pair $(f^*\mathcal{G}, f^*\mu)$ defined by

$$f^*\mathcal{G} = \{f^{-1}(G) \triangle N \mid G \in \mathcal{G}, N' \in \mathcal{N}\}$$
$$f^*\mu(f^{-1}(G) \triangle N) = \mu(G) \text{ for all } G \in \mathcal{G}, N \in \mathcal{N}$$

Enhanced pullback makes $(X, f^*\mathcal{G}, f^*\mu)$ a probability space with negligibles $\mathcal{N}$ and f a measure-preserving map $(X, f^*\mathcal{G}, f^*\mu) \to (Y, \mathcal{G}, \mu)$.

Definition 4.14. A *probability space on* $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ is a pair $(\mathcal{G}, \mu)$ with $\mathcal{N} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and μ a probability measure with negligibles $\mathcal{N}$. Call such a pair *standardizable* if $(X, \mathcal{G}, \mu)$ arises via enhanced pullback along a map $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$ for some $(Y, \mathcal{G}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$.

With these definitions in hand, the colimit $\mathbb{P}$ is equivalent to a sheaf of standardizable probability spaces, with action on morphisms given by enhanced pullback:

Lemma 4.15. $\mathbb{P}$ is equivalent to the following sheaf:

$$\hat{\mathbb{P}}(\Omega) = \{(\mathcal{G}, \mu) \mid (\mathcal{G}, \mu) \text{ standardizable on } \Omega\}$$
$$\hat{\mathbb{P}}(f : \Omega' \to \Omega)(\mathcal{G}, \mu) = f^*(\mathcal{G}, \mu)$$

Moreover, the Day convolution $\mathbb{P} \otimes \mathbb{P}$ corresponds to a sheaf of independently combinable probability spaces:

Lemma 4.16. $\mathbb{P} \otimes \mathbb{P}$ is equivalent to the following sheaf $\mathbb{P}^2_\bot$:

$$\mathbb{P}^2_\bot(\Omega) = \left\{ ((\mathcal{G}, \mu), (\mathcal{H}, \nu)) \;\middle|\; \begin{array}{l} (\mathcal{G}, \mu) \text{ and } (\mathcal{H}, \nu) \text{ standardizable} \\ \text{and independently combinable} \end{array} \right\}$$
$$\mathbb{P}^2_\bot(f : \Omega' \to \Omega)((\mathcal{G}, \mu), (\mathcal{H}, \nu)) = (f^*(\mathcal{G}, \mu), f^*(\mathcal{H}, \nu))$$

Via these equivalences, the resource monoid in Theorem 4.12 parallels its discrete analog (Proposition 3.20). Across $\mathbb{P} \cong \hat{\mathbb{P}}$, the ordering $\sqsubseteq$ corresponds to the generalization of Definition 3.6 from countable measured partitions to standardizable probability spaces. Across $\mathbb{P} \otimes \mathbb{P} \cong \mathbb{P}^2_\bot$, the monic map i corresponds to the canonical inclusion $\mathbb{P}^2_\bot \hookrightarrow \mathbb{P} \times \mathbb{P}$, and the combining operation $\bullet$ corresponds to the map $\mathbb{P}^2_\bot \to \mathbb{P}$ that sends independently-combinable pairs of standardizable probability spaces to their independent combination. For details, see the appendix.

4.2 Absolutely continuous sets

Finding a continuous analog to $\mathbf{Set}^{\ll}_{\mathrm{d}}$ boils down to showing continuous analogs of Lemmas 3.16 and 3.17. In the discrete setting, these lemmas hold because every discrete probability space can be encoded as a measured partition that leaves enough room in the sample space $[0, 1]$ for fresh randomness. To create a continuous analog, we fix an enormous sample space following Li et al. [27]:

Definition 4.17. The Hilbert cube $\mathbb{I}^\omega$ is the standard enhanced measurable space $([0, 1]^\omega, \mathcal{F}, \mathcal{N})$ of infinite sequences in the interval $[0, 1]$. The σ-algebra $\mathcal{F}$ and negligibles $\mathcal{N}$ are those of the usual Lebesgue measure on $[0, 1]^\omega$.

Then, to ensure that there is always enough room left over in $\mathbb{I}^\omega$ for fresh randomness, we encode all probability spaces using only finitely many dimensions at a time:

Definition 4.18. A standardizable probability space $(\mathcal{G}, \mu)$ on $\mathbb{I}^\omega$ has *finite footprint* if it arises by enhanced pullback along a map $\mathbb{I}^\omega \to X$ that factors through $\mathrm{proj}_{1..n}$ for some n, where $\mathrm{proj}_{1..n}$ is the canonical projection $\mathbb{I}^\omega \to [0, 1]^n$.

Analogously, the group $\mathrm{Aut}[0, 1]$ of $\mathbf{Set}^{\ll}_{\mathrm{d}}$ becomes a group of finite-dimensional permutations of the Hilbert cube:

Definition 4.19. A $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi : \mathbb{I}^\omega \to \mathbb{I}^\omega$ has *finite width* if it is of the form $\pi' \times 1_{\mathbb{I}^\omega}$ for some $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi' : [0, 1]^n \to [0, 1]^n$. Let $\mathbf{G}^{\ll}$ be the subgroup of $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}} \mathbb{I}^\omega$ consisting only of those automorphisms with finite width.

Then, the topology on $\mathrm{Aut}[0, 1]$ generated by countable measurable partitions becomes a topology on $\mathbf{G}^{\ll}$ generated by standardizable sub-σ-algebras with finite footprint:

Definition 4.20 (Topology on $\mathbf{G}^{\ll}$)**.** A subgroup U of $\mathbf{G}^{\ll}$ is *open* if for every π in U there exists $(\mathcal{F}, \mu)$ with finite footprint such that $\pi \in \mathrm{Fix}\,\mathcal{F} \subseteq U$, where $\mathrm{Fix}\,\mathcal{F}$ is the subgroup of those π in $\mathbf{G}^{\ll}$ with $\pi(F) =_{\mathrm{a.e.}} F$ for all $F \in \mathcal{F}$.

Definition 4.21. $\mathbf{Set}^{\ll}$ is the category of $\mathbf{G}^{\ll}$-sets with open stabilizers and equivariant functions between them; objects of $\mathbf{Set}^{\ll}$ will be called *absolutely continuous sets*.

There are absolutely continuous sets analogous to the sheaves RV_A of random variables and $\mathbb{P}$ of standardizable probability spaces:

Definition 4.22. For A a Polish space, a random variable $X : \mathbb{I}^\omega \to A$ has *finite footprint* if it factors through $\mathrm{proj}_{1..n}$ for some n. Let $\overline{\mathrm{RV}}_A$ be the set of random variables with finite footprint. This forms an absolutely continuous set, with action $X \cdot \pi = X \circ \pi$.

Definition 4.23. Let $\overline{\mathbb{P}}$ be the set of standardizable probability spaces on $\mathbb{I}^\omega$ with finite footprint. This forms an absolutely continuous set, with action $(\mathcal{F}, \mu) \cdot \pi = \pi^*(\mathcal{F}, \mu)$.

These yield a resource monoid in $\mathbf{Set}^{\ll}$.

Theorem 4.24. $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{\imath}, \bar{\bullet}, \overline{\mathrm{emp}})$ is a $\mathbf{Set}^{\ll}$ resource monoid, where

- $\overline{\sqsubseteq}: \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \overline{\mathrm{Prop}}$ is the map that sends $((\mathcal{G}, \mu), (\mathcal{H}, \nu))$ to $\top$ iff $\mathcal{G} \subseteq \mathcal{H}$ and $\nu|_{\mathcal{G}} = \mu$, where $\overline{\mathrm{Prop}}$ is the two-element set with trivial action.
- $\overline{\mathbb{P}}^2_\perp$ is the set of pairs $((\mathcal{G}, \mu), (\mathcal{H}, \nu)) \in \overline{\mathbb{P}} \times \overline{\mathbb{P}}$ for which $(\mathcal{G}, \mu)$ and $(\mathcal{H}, \nu)$ are independently combinable.
- $\bar{\imath}$ is the inclusion $\overline{\mathbb{P}}^2_\perp \hookrightarrow \overline{\mathbb{P}} \times \overline{\mathbb{P}}$.
- $\bar{\bullet} : \overline{\mathbb{P}}^2_\perp \to \overline{\mathbb{P}}$ is the map that sends independently-combinable pairs to their independent combination.
- $\overline{\mathrm{emp}} : 1 \to \overline{\mathbb{P}}$ is the constant map at the probability space f^*1 on $\mathbb{I}^\omega$ arising from enhanced pullback along the unique $\mathbf{EMS}_{\mathrm{std}}$-map $\mathbb{I}^\omega \to \mathrm{U}(1)$ into the one-point probability space 1.

4.3 The equivalence

By choosing $\mathbb{I}^\omega$ as underlying sample space and topologizing $\mathrm{Aut}\,\mathbb{I}^\omega$ to permit only objects that use finitely-many dimensions of $\mathbb{I}^\omega$ at a time, we obtain continuous analogs of Homogeneity and Correspondence. This relies crucially on both the finiteness of footprints and the inclusion of negligible ideals in the base category $\mathbf{EMS}_{\mathrm{std}}$. Negligible ideals allow passing to *measure algebra* [17, 321A]:

Definition 4.25. A *measure algebra* is a tuple $(\mathfrak{A}, \overline{\mu})$ consisting of a complete Boolean algebra $\mathfrak{A}$ and a function $\overline{\mu} : \mathfrak{A} \to [0, 1]$ such that (1) $\overline{\mu}(A) > 0$ for $A \neq \bot$ and (2) $\overline{\mu}$ is countably additive in the sense that $\overline{\mu}(\bigvee_i A_i) = \sum_i \overline{\mu}(A_i)$ for all countable families $\{A_i\}_{i \in I}$ with $A_i \wedge A_j = \bot$ for all $i \neq j$. A *measure algebra homomorphism* from $(\mathfrak{A}, \overline{\mu})$ to $(\mathfrak{B}, \overline{\nu})$ is a complete Boolean algebra homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, measure-preserving in the sense that $\overline{\nu}(f(A)) = \overline{\mu}(A)$ for all $A \in \mathfrak{A}$.

Every $(X, \mathcal{F}, \mu)$ in $\mathbf{Prob}_{\mathrm{std}}$ yields a measure algebra $(\mathcal{F}/\mu, \overline{\mu})$, where $\mathcal{F}/\mu$ is the complete Boolean algebra of events $F \in \mathcal{F}$ mod $F \sim F'$ iff $\mu(F \triangle F') = 0$, and $\overline{\mu}([F]) = \mu(F)$ [17, 321H]. Every measure-preserving map f from $(X, \mathcal{F}, \mu)$ to $(Y, \mathcal{G}, \nu)$ defines a homomorphism f^* from $(\mathcal{G}/\nu, \overline{\nu})$ to $(\mathcal{F}/\mu, \overline{\mu})$ sending $[G] \in \mathcal{G}/\nu$ to $[f^{-1}(G)] \in \mathcal{F}/\mu$ [17, 324M]. This gives a duality:

Definition 4.26. A *standard probability algebra* is a measure algebra $(\mathfrak{A}, \overline{\mu})$ arising from a standard probability space as described above. Let $\mathbf{ProbAlg}_{\mathrm{std}}$ be the category of standard probability algebras and measure algebra homomorphisms between them.

Lemma 4.27. $\mathbf{Prob}_{\mathrm{std}} \simeq \mathbf{ProbAlg}_{\mathrm{std}}^{\mathrm{op}}$.

A similar duality holds also for $\mathbf{EMS}_{\mathrm{std}}$:

Definition 4.28. A *standard measurable algebra* is a complete Boolean algebra $\mathfrak{A}$ arising from a standard probability space; i.e. $\mathfrak{A}$ is isomorphic to a Boolean algebra $\mathcal{F}/\mu$ for some $(X, \mathcal{F}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$. Let $\mathbf{MbleAlg}_{\mathrm{std}}$ be the category of standard measurable algebras and *injective* complete boolean algebra homomorphisms.

Lemma 4.29. $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbf{MbleAlg}_{\mathrm{std}}^{\mathrm{op}}$.

Lemmas 4.27 and 4.29 allow importing the extensive technical development of measure algebras from Fremlin [17]. In particular, the algebraic perspective reveals that the finite-footprint property from Section 4.2 is a means of producing *relatively-atomless* subalgebras:

Definition 4.30 (Fremlin [17, 331A])**.** Let $\mathfrak{A}$ be a complete Boolean algebra and $\mathfrak{B} \subseteq \mathfrak{A}$ a subalgebra. An element $a \in \mathfrak{A}$ is a $\mathfrak{B}$-*relative atom* of $\mathfrak{A}$ if the principal ideal generated by a in $\mathfrak{A}$ is $\{a \cap b \mid b \in \mathfrak{B}\}$. The algebra $\mathfrak{A}$ is $\mathfrak{B}$-*relatively atomless* if it has no $\mathfrak{B}$-relative atoms.

Theorem 4.31. Let $\mathfrak{A}$ be the measurable algebra of $\mathbb{I}^\omega$. For any $(\mathcal{G}, \mu)$ with finite footprint, $\mathfrak{A}$ is $\mathcal{G}/\mu$-relatively atomless.

Relative-atomlessness is key to obtaining continuous analogs of Homogeneity and Correspondence, which hold specifically for the case where subalgebras are relatively atomless:

Lemma 4.32 (Homogeneity)**.** For $\mathfrak{A}$ a standard measurable algebra and subalgebras $\mathfrak{B}, \mathfrak{C} \subseteq \mathfrak{A}$ that render it relatively-atomless, and a $\mathbf{MbleAlg}_{\mathrm{std}}$-morphism $f : \mathfrak{B} \hookrightarrow \mathfrak{C}$, there exists a complete Boolean algebra automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ with $\pi(b) = f(b)$ for all $b \in \mathfrak{B}$.

Lemma 4.33 (Correspondence)**.** Let $\mathfrak{A}$ be a standard measurable algebra. For any subalgebra $\mathfrak{C} \subseteq \mathfrak{A}$, let $\mathrm{Fix}\,\mathfrak{C}$ be the group of $\mathfrak{A}$-automorphisms fixing every c in $\mathfrak{C}$. If $\mathfrak{A}$ is $\mathfrak{C}$-relatively atomless then $\mathrm{Fix}\,\mathfrak{C} \subseteq \mathrm{Fix}\,\mathfrak{D}$ iff $\mathfrak{D} \subseteq \mathfrak{C}$.

These yield a continuous analog of Theorem 3.18:

Theorem 4.34. $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$.

Finally, a careful calculation across this equivalence shows that the resource monoids in Theorems 4.12 and 4.24 indeed correspond, yielding an analog of Fact 1.1 for continuous probability:

Theorem 4.35. Across $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$, the sheaf $\mathbb{P}$ corresponds to $\overline{\mathbb{P}}$, the sheaf RV_A corresponds to $\overline{\mathrm{RV}}_A$, and the resource monoid $(\sqsubseteq, \mathbb{P} \otimes \mathbb{P}, i, \bullet, \mathrm{emp})$ in $\mathbf{EMS}$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{\imath}, \bar{\bullet}, \overline{\mathrm{emp}})$ in $\mathbf{Set}^{\ll}$.

5 DISCUSSION & RELATED WORK

Atomic sheaves for probability. Tao [49] defines probabilistic notions as those invariant under extension of the sample space. Along these lines, Simpson [45] constructs a topos of atomic sheaves on a category of probability spaces and measure-preserving maps; in it, he presents a sheaf of random variables and an extension of the Giry monad [20] to sheaves, and shows how concepts such as independence and expectation can be internalized [44, 46].

Simpson's topos is similar to our $\mathbf{EMS}$, but our base category $\mathbf{EMS}_{\mathrm{std}}$ omits measures and its maps are quotiented by almost-everywhere equality; we instead model measures explicitly via the sheaf $\mathbb{P}$. As we have focused on separation logic, we have not investigated whether the Giry monad extends to $\mathbf{EMS}$ and the probabilistic concepts that can be expressed internally; this would make interesting future work. Simpson [45] mentions a resemblance to nominal sets, but does not extensively develop the notion to the best of our knowledge.

Simpson's topos also serves as a model of Atomic Sheaf Logic [47], a recently-developed logic axiomatizing the interaction between conditional independence and a notion of *atomic equivalence*, which

in the probabilistic setting denotes equidistribution of random variables, with potential applications to developing proof-relevant probabilistic separation logics; it would be interesting to explore whether our topos admits analogous constructions.

Categorical probability. There are numerous categorical formulations of probability. Fritz [18] develops probability theory purely synthetically by axiomatizing equational properties known to hold for Markov kernels. Jackson [24], building on Breitsprecher [9], gives an alternative sheaf-theoretic model of probability by taking sheaves on a single measurable space rather than a category of measurable spaces; we speculate that there could be a relationship between this model and ours similar to the relationship between petit and gros topoi of sheaves on topological spaces [28].

Quasi-Borel spaces. The category QBS of quasi-Borel spaces [22] is a richly developed model of higher-order probability. Whereas QBS has been used extensively to model higher-order probabilistic languages [1, 42, 43, 50], our goal in constructing $\mathbf{EMS}$ and $\mathbf{Set}^{\ll}$ has been focused on refining models of probabilistic separation logic. Structurally, QBS and $\mathbf{EMS}$ are quite different: QBS is a well-pointed quasi-topos while $\mathbf{EMS}$ is a non-well-pointed topos. However, as remarked in Heunen et al. [22, Prop. 34], QBS is related to particular presheaves on the category of standard measurable spaces. This suggests connections to $\mathbf{EMS}$, since it is a category of sheaves on $\mathbf{EMS}_{\mathrm{std}}$, but there is a gap between these two settings: $\mathbf{EMS}_{\mathrm{std}}$-morphisms are quotiented by almost-everywhere equality whereas maps of standard measurable spaces are not. We leave elucidating the relationship between our setting and QBS to future work.

General representation theorems. The equivalence $\mathbf{Sch} \simeq \mathbf{Nom}$ can be obtained via a *Fraïssé limit* [23, §7.1], a recipe for making universal objects (e.g., $\mathbb{N}$) capable of representing a class of models (e.g., finite sets). More generally, there is a long line of results giving groupoid-based representations of categories [7, 8, 10, 15, 25, 29], with a history going back to Grothendieck [16, 21]. Caramello [11] is particularly relevant, as it gives conditions closely resembling Lemmas 3.16 and 3.17 under which categories of atomic sheaves are equivalent to categories of continuous $\mathrm{Aut}(u)$-sets for suitable objects u. We are currently investigating whether Theorems 3.18 and 4.34 can be obtained via these general results, with an eye towards generalizing beyond probability to the quantum setting.

Probabilistic separation logic. PSL [5] is the first separation logic whose separating conjunction models independence, by splitting random substitutions; it has since been extended to support conditional independence [2] and negative dependence [3], and to the quantum setting [51]. In contrast to PSL and its extensions, Lilac [27] has an alternative model of separation, via independent combination. Lilac's model is complicated: independent combination is an intricate measure-theoretic operation, an intricate proof is required to show it forms a monoid, and many side conditions on this monoid are needed for soundness of Lilac's proof rules.

Theorem 4.35 simplifies and clarifies Lilac's model. It shows that independent combination arises naturally from the well-known tensor product of standard probability spaces; that independent combination forms a monoid then follows from the fact that tensor product is monoidal. The resource monoid in Theorem 4.35 replaces the side conditions on Lilac's monoid with the single notion of standardizability — a condition well-motivated by the intuition that probability spaces should arise via pullback along $\mathbf{EMS}_{\mathrm{std}}$-maps.

Theorem 4.35 also improves on the model in Li et al. [27] in multiple ways. Quotienting by negligiblity yields a model invariant under almost-everywhere equality, whereas the model in Li et al. [27] must manually track σ-ideals of negligible sets. Interpreting propositions as equivariant maps implies our model is invariant under finite-width permutations of $\mathbb{I}^{\omega}$. Finally, using the internal language of $\mathbf{EMS}$, one can interpret quantification over propositions, allowing to generalize Lilac to a higher-order logic; in the future, we would like to explore whether this higher-order generalization can be used to specify properties of higher-order programs.

An aspect of Lilac not captured by our model is its *conditioning modality*, interpreted by disintegration [12]. This is difficult to capture in our model because $\mathbf{EMS}_{\mathrm{std}}$-objects come with a fixed collection of negligible sets, whereas disintegration can change which sets are negligible.

Probability and name generation. Recent work has identified connections between probability theory and name generation: Staton et al. [48] provides a semantics for a probabilistic language that treats random variables as dynamically-allocated read-only names, and Sabok et al. [41] show that QBS can be used to characterize observational equivalence of stateful imperative programs by interpreting dynamic allocation as probabilistic sampling. The resemblance between our probabilistic Theorem 4.35 and the store-based Fact 1.1 provides further evidence along these lines.

Nominal sets. Many constructions exist in $\mathbf{Nom}$ beyond its ability to capture permutation-invariance: *freshness quantification* [30] captures the informal convention of picking fresh names [4], a *name abstraction* [36, §4] type former gives a uniform treatment of binding, and *nominal restriction sets* [36, §9.1] models languages with locally generated names [31, 37]. It would be interesting to explore whether analogous constructions can be carried out in $\mathbf{Set}^{\ll}$, to obtain analogous treatments of the informal convention of picking fresh sample spaces [17, §27] and to provide models of probabilistic languages with locally generated random variables.

6 CONCLUSION

We unify two different approaches to separating probabilistic state: the usual product of probability spaces and independent combination. To do this, we show that separation-as-product lives in a category $\mathbf{EMS}$ of enhanced measurable sheaves, that separation-as-independent-combination lives in a category $\mathbf{Set}^{\ll}$ of absolutely continuous sets, and that these two notions of separation correspond across an equivalence $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$. This validates the use of independent combination in probabilistic separation logic [27], clarifies independent combination's relationship with traditional formulations of independence, and suggests improvements to existing models. Finally, as a probabilistic analog of $\mathbf{Nom}$, the category $\mathbf{Set}^{\ll}$ creates new probabilistic interpretations of nominal concepts, which we hope will create more opportunities for using nominal techniques in probability.

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