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Constructing polylogarithms on higher-genus Riemann surfaces

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Abstract

An explicit construction is presented of homotopy-invariant iterated integrals on a Riemann surface of arbitrary genus in terms of a flat connection valued in a freely generated Lie algebra. The integration kernels consist of modular tensors, built from convolutions of the Arakelov Green function and its derivatives with holomorphic Abelian differentials, combined into a flat connection. Our construction thereby produces explicit formulas for polylogarithms as higher-genus modular tensors. This construction generalizes the elliptic polylogarithms of Brown-Levin, and prompts future investigations into the relation with the function spaces of higher-genus polylogarithms in the work of Enriquez-Zerbini.

Contents

1 Introduction

In a variety of research areas in theoretical physics, polylogarithms and related iterated integrals have become almost as widely used as elementary functions. In particular, perturbative computations in quantum field theory and string theory have benefitted significantly from the systematic investigation of iterated integrals on the sphere and the torus, namely on Riemann surfaces of genus zero and one. At genus zero, mathematical advances on multiple polylogarithms have become a driving force behind sophisticated loop calculations in high-energy physics and evaluations of higher-order effective interactions in the low-energy expansion of string theory. At genus one, the construction of elliptic polylogarithms in the mathematics literature has dramatically increased our computational reach in quantum field theory and string perturbation theory and spawned a vibrant collaboration between these two communities. Comprehensive overviews of the literature may be found in reviews and white papers, such as [1, 2, 3, 4] for quantum field theory and [5, 6, 7, 8, 9] for string theory.

A major impetus for the use of polylogarithms and their elliptic analogues is the fact that they span a space of functions which is closed under taking primitives. As a result, integration is rendered completely algorithmic. This property is at the source of the ubiquity of genus-zero polylogarithms [10, 11, 12] in the study of quantum-field-theory amplitudes [13, 14, 15, 16] and string-theory amplitudes [17, 18, 19]. Elliptic polylogarithms were introduced in [20, 21, 22], used to reformulate Feynman-integral calculations in [23, 24, 25, 26], and applied to one-loop string amplitudes in [27, 28].

The formulation of elliptic polylogarithms crucially hinges on the existence of suitable integration kernels, which were identified in [22] and naturally enter genus-one string amplitudes [29, 27, 30, 31]. For the torus, these kernels are usually expressed via Jacobi theta functions, obtained by expanding certain Kronecker-Eisenstein series, and combined into a flat connection. The flatness of the connection guarantees homotopy invariance of the iterated integrals generated by its path-ordered exponential. The translation of Kronecker-Eisenstein kernels from tori to elliptic curves was performed in [26].

Higher loop orders of scattering amplitudes in both quantum field theory and string theory involve functions and integrals on higher-genus Riemann surfaces, whose role in string theory dates back to the early days of the subject [32, 33, 34, 35]. In Feynman integrals, both hyperelliptic curves [36, 37, 38] and higher-dimensional geometric varieties, such as Calabi-Yau spaces, have recently been encountered [39, 40, 41, 42, 43]. However, a general and explicit construction of the functions necessary to describe Feynman integrals and string amplitudes beyond (elliptic) polylogarithms was still missing. For polylogarithms on higher-genus Riemann surfaces, proposals to characterize the function spaces and flat connections have been advanced in the mathematics literature by Enriquez [44] and Enriquez-Zerbini

[45, 46], but these have not yet led to tractable expressions for the individual polylogarithms necessary for physics applications. An investigation into the precise relation between these proposals and the construction presented here is relegated to future work.

In this paper, we shall eliminate this bottleneck for compact Riemann surfaces of arbitrary genus and present a generating series of homotopy-invariant iterated integrals that generalize the polylogarithms and their elliptic analogues to concrete expressions at arbitrary genera. These higher-genus polylogarithms are built out of modular tensors and can be organized to themselves enjoy tensorial modular transformation properties.

Our main result here is to provide an explicit proposal for the higher-genus generalization of the integration kernels and flat connection of Brown and Levin. The higher-genus integration kernels in this work share the logarithmic singularities of their genus-one counterparts and the Lie algebra structure of the formulation of Enriquez and Zerbini [45] (see also [47, 44] for earlier work). While the meromorphic higher-genus connections in the mathematics literature exhibit multi-valuedness [44], or poles of arbitrary order [45] in marked points, our construction features non-meromorphic kernels which reconcile single-valuedness with the presence of at most simple poles.

Instead of extending the Jacobi theta-function or elliptic-curve description of the Brown-Levin integration kernels used at genus one, our construction on Riemann surfaces of arbitrary genus is driven by the Arakelov Green function [48, 49] which, in turn, is built from the prime form [50] and Abelian integrals (for a recent account see [51]). We employ convolutions of Arakelov Green functions and their derivatives with holomorphic Abelian differentials to construct higher-genus analogues of the Kronecker-Eisenstein-type integration kernels that were crucial for elliptic polylogarithms. The differential properties of the Arakelov Green function then lead us to identify a flat connection which in turn yields infinite families of homotopy-invariant iterated integrals to be referred to as higher-genus polylogarithms.

The Arakelov Green function is by now widely used in string perturbation theory [52, 53, 51, 54, 55] and has stimulated the construction of tensor-valued functions on the Torelli-space of compact genus-h Riemann surfaces [56, 57, 58, 59]. Our integration kernels for higher-genus polylogarithms enjoy similar tensorial transformation properties under the modular group $Sp(2h,\mathbb{Z})$ of compact genus-h Riemann surfaces and are related to the vector-valued modular forms investigated by van der Geer and collaborators in [60, 61, 62].

Organization

In section 2, we review the construction of polylogarithms at genus zero following Goncharov and at genus one following Brown and Levin, while emphasizing the role played by flat connections. Section 3 provides a summary of the function theory on Riemann surfaces of arbitrary genus h that will be needed here, including the Arakelov Green function. We use these tools to build the $Sp(2h,\mathbb{Z})$ modular tensors needed for the explicit construction of a flat connection valued in a Lie algebra that is freely generated by $2h$ elements. In section 4, we use this connection to construct the promised higher-genus polylogarithms. We discuss their modular properties; provide examples at low order; discuss their meromorphic variants; give evidence for their closure under taking primitives; and present a proposal for higher-genus generalizations of elliptic associators. In section 5 we generalize the higher-genus connection to the case of multiple marked points on the surface. In section 6 we consider the behavior of the higher-genus flat connection under separating degenerations and recover the Brown-Levin connection at genus one from the degeneration of a genus-two surface. We conclude and discuss some open directions in section 7.

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2 Review of genus-zero and genus-one polylogarithms

The construction of homotopy-invariant iterated integrals on a surface of arbitrary genus, including genus zero and genus one, is based on the existence of a flat connection. We begin by reviewing this well-known construction. A flat connection $\mathcal{J}$ on a Riemann surface Σ, which takes values in a Lie algebra $\mathcal{L}$, is defined to satisfy the Maurer-Cartan equation,

$$d\mathcal{J} - \mathcal{J} \wedge \mathcal{J} = 0 \tag{2.1}$$

As a result, the differential equation $d\mathbf{\Gamma} = \mathcal{J}\mathbf{\Gamma}$ is integrable (and so is $d\mathbf{\Gamma}' = -\mathbf{\Gamma}'\mathcal{J}$). Its solution $\mathbf{\Gamma}$ takes values in the simply-connected Lie group associated with $\mathcal{L}$ and is given by the path-ordered exponential along an arbitrary open path $\mathcal{C}$ between points $z_0, z \in \Sigma$,

$$\mathbf{\Gamma}(\mathcal{C}) = \mathrm{P}\exp \int_{\mathcal{C}} \mathcal{J}(\cdot) = \mathrm{P}\exp \int_0^1 dt\, J(t) \tag{2.2}$$

We have parametrized the path $\mathcal{C}$ by $t \in [0,1]$ with $\mathcal{C}(0) = z_0$ and $\mathcal{C}(1) = z$ and set $\mathcal{J} = J(t)dt$. The path-ordered exponential is defined by placing $J(t)$ to the left of $J(t')$ for $t > t'$ following physics conventions. Its expansion in powers of $\mathcal{J}$ takes the form,

$$\mathrm{P}\exp \int_0^1 dt\, J(t) = 1 + \int_0^1 dt_1\, J(t_1) + \int_0^1 dt_1 \int_0^{t_1} dt_2\, J(t_1)J(t_2) + \cdots \tag{2.3}$$

Flatness of the connection $\mathcal{J}$ guarantees that $\mathbf{\Gamma}(\mathcal{C})$ is unchanged under continuous deformations of the path $\mathcal{C}$, so that $\mathbf{\Gamma}(\mathcal{C})$ depends only on the end-points z_0 and z. However, $\mathbf{\Gamma}(\mathcal{C})$ may be multiple-valued in z_0 and z as these points are taken around non-contractible cycles on Σ. Such iterated integrals will be referred to as *homotopy-invariant*.

Polylogarithms on surfaces Σ of arbitrary genus are obtained from the path-ordered exponential (2.3) by extracting the coefficients of independent words in the generators of $\mathcal{L}$. Homotopy invariance of $\mathbf{\Gamma}(\mathcal{C})$ implies that the resulting polylogarithms are functions of the endpoints z_0 and z and the homotopy class of the path $\mathcal{C}$, but do not depend on the specific path chosen within a homotopy class. In general, the polylogarithms are multiple valued in z and z_0 as these points are taken around a non-trivial homology cycle on Σ.

2.1 Genus zero

Multiple polylogarithms at genus zero are iterated integrals of rational forms $dz/(z-s)$ with $z, s \in \mathbb{C}$. They are defined recursively as follows [12],

$$G(s_1, s_2, \cdots, s_n; z) = \int_0^z \frac{dz_1}{z_1 - s_1} G(s_2, \cdots, s_n; z_1) \tag{2.4}$$

with initial value $G(\emptyset; z) = 1$ for $s_n \neq 0$. When $s_n = 0$, an endpoint divergence is *shuffle-regularized* preserving holomorphicity by setting $G(0; z) = \ln(z)$ (see for example [63, 2] for pedagogical accounts). The integer $n \geq 0$ is referred to as the *transcendental weight*. A generating series for the polylogarithms (2.4) may be constructed starting from the Knizhnik-Zamolodchikov (KZ) connection $\mathcal{J}_{\rm KZ}(z)$ for a Lie algebra $\mathcal{L}$ that is freely generated by elements $e_1, \cdots, e_m$ associated with the marked points $s_1, \cdots, s_m$,

$$\mathcal{J}_{\rm KZ}(z) = \sum_{i=1}^{m} \frac{dz}{z - s_i} \, e_i \tag{2.5}$$

Since $\mathcal{J}_{\rm KZ}(z)$ is a meromorphic $(1, 0)$ form in $z \in \hat{\mathbb{C}}$, it automatically satisfies the Maurer-Cartan equation (2.1) and is therefore a flat connection, away from the points s_i. The path-ordered exponential $\mathbf{\Gamma}_{\rm KZ}(z)$, produced by the connection $\mathcal{J}_{\rm KZ}(z)$ using (2.2) and (2.3), is *homotopy-invariant* by construction and depends only on the end-points. Choosing $z_0 = 0$ by translation invariance and $z_1 = z$, we may organize the expansion of the path-ordered exponential in powers of $\mathcal{J}_{\rm KZ}$ in terms of the generators $e_1, \cdots, e_m$ (of the universal enveloping algebra of $\mathcal{L}$),

$$\mathrm{P} \exp \int_0^z \mathcal{J}_{\rm KZ}(\cdot) = 1 + \sum_{\mathfrak{w}} \mathfrak{w} \, G(\mathfrak{w}; z) \tag{2.6}$$

The sum runs over all words $\mathfrak{w}$ with at least one letter, formed out of the alphabet $e_1, \cdots, e_m$, and we identify $G(e_{i_1}, e_{i_2}, \cdots, e_{i_n}; z) = G(s_{i_1}, s_{i_2}, \cdots, s_{i_n}; z)$ for $i_1, \cdots, i_n \in \{1, \cdots, m\}$. The construction confirms that every coefficient function $G(\mathfrak{w}; z)$ is a homotopy-invariant iterated integral. Without loss of generality we may set $(s_1, s_2) = (0, 1)$ by using the $SL(2, \mathbb{C})$ invariance on the sphere, as a result of which these coefficient functions $G(\mathfrak{w}; z)$ reduce to the standard genus-zero polylogarithms in $m-1$ variables [10, 11, 12].

2.2 Genus one

At genus zero, the coefficient of each Lie-algebra generator e_i in the connection (2.5) is a single-valued meromorphic $(1, 0)$-form with simple poles (as opposed to higher-order poles). On a compact Riemann surface of genus $h \geq 1$, however, it is not possible to maintain these properties simultaneously without introducing additional marked points. Instead, the available options are as follows.

1. a single-valued but non-meromorphic connection with at most simple poles;
2. a meromorphic but not single-valued connection with at most simple poles;
3. a meromorphic and single-valued connection with a single pole of higher order, or poles of lower order distributed over multiple marked points.

The Brown-Levin construction of elliptic polylogarithms via iterated integrals [22] follows option 1, and will be briefly reviewed below.[1] It will be generalized to higher genus in the next subsection following option 1. The constructions at genus one following options 1 and 2 will be related at the end of this section, while their relation with option 3 will be relegated to future work.

2.2.1 The Brown-Levin construction

A genus-one surface Σ with modulus τ in the Poincaré upper half plane may be represented by $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, and parametrized by local complex coordinates $z, \bar z$ subject to identifications $z \equiv z+1$ and $z \equiv z+\tau$. The Brown-Levin connection $\mathcal{J}_{\rm BL}(z|\tau)$ takes values in the Lie algebra $\mathcal{L}$ generated freely by elements a, b and is given as follows,

$$\mathcal{J}_{\rm BL}(z|\tau) = \frac{\pi}{{\rm Im}\,\tau} \big(dz - d\bar z\big)\, b + dz \left(a + \sum_{n=1}^\infty f^{(n)}(z|\tau)\, {\rm ad}_b^n(a) \right) \tag{2.7}$$

where ${\rm ad}_b(\cdot) = [b, \cdot]$. Flatness of the connection, namely $d\mathcal{J}_{\rm BL} - \mathcal{J}_{\rm BL} \wedge \mathcal{J}_{\rm BL} = 0$, requires the following relations between the coefficient functions $f^{(n)}(z|\tau)$,[2]

$$\partial_{\bar z} f^{(n)}(z|\tau) = - \frac{\pi}{{\rm Im}\,\tau}\, f^{(n-1)}(z|\tau) \qquad\qquad f^{(0)}(z|\tau) = 1 \tag{2.8}$$

The functions $f^{(n)}(z|\tau)$ may be constructed in different but equivalent ways. Following Brown and Levin, they are given by expanding the doubly-periodic Kronecker-Eisenstein series $\Omega(z,\alpha|\tau)$ in powers of an auxiliary parameter $\alpha \in \mathbb{C}$,

$$\Omega(z,\alpha|\tau) = \exp\left(2\pi i \alpha\, \frac{{\rm Im}\, z}{{\rm Im}\,\tau} \right) \frac{\vartheta_1'(0|\tau)\vartheta_1(z+\alpha|\tau)}{\vartheta_1(z|\tau)\vartheta_1(\alpha|\tau)} = \sum_{n=0}^\infty \alpha^{n-1} f^{(n)}(z|\tau) \tag{2.9}$$

The relations (2.8) immediately result from the following identity for $\Omega(z,\alpha|\tau)$ for $z \neq 0$,

$$\partial_{\bar z} \Omega(z,\alpha|\tau) = - \frac{\pi\, \alpha}{{\rm Im}\,\tau}\, \Omega(z,\alpha|\tau) \tag{2.10}$$

One may obtain the connection $\mathcal{J}_{\rm BL}$ in (2.7) by a formal substitution $\alpha \to {\rm ad}_b$ as follows,

$$\mathcal{J}_{\rm BL}(z|\tau) = \frac{\pi}{{\rm Im}\,\tau} \,(dz - d\bar z)\, b + dz\, {\rm ad}_b\, \Omega\big(z, {\rm ad}_b|\tau\big)\, a \tag{2.11}$$

The factor ${\rm ad}_b$ to the left of Ω ensures the cancellation of the pole that $\Omega(z,\alpha|\tau)$ has in α.

[1] An alternative construction of elliptic polylogarithms given in the same Brown-Levin reference [22] relies on certain averages of genus-zero polylogarithms which preserve meromorphicity. Throughout this work, we use the term *Brown-Levin (elliptic) polylogarithms* to refer to the non-meromorphic iterated integrals in [22] and not to the meromorphic functions obtained from the averaging procedure of the reference.

[2] As we shall see below, the relation for $n=1$ actually holds up to a δ-function, $\partial_{\bar z} f^{(1)}(z) = \pi\delta(z) - \pi/{\rm Im}\,\tau$, so that the corresponding relation (2.8) holds for $z \neq 0$, as does the flatness condition of $\mathcal{J}_{\rm BL}$. Throughout, we shall set $d^2 z = \frac{i}{2} dz \wedge d\bar z$ and normalize the δ-function by $\int_\Sigma d^2 z\, \delta(z) = 1$.

2.2.2 Alternative construction via convolutions of Green functions

An alternative construction of the functions $f^{(n)}(z|\tau)$, and the one that will generalize to higher genus, is in terms of the scalar Green function $g(z|\tau)$ on Σ, which is defined by,

$$\partial_{\bar z}\partial_z\, g(z|\tau) = -\pi\delta(z) + \frac{\pi}{\operatorname{Im}\tau} \qquad\qquad \int_\Sigma d^2z\, g(z|\tau) = 0 \tag{2.12}$$

and may expressed in terms of ϑ-functions and the Dedekind eta-function η as follows,

$$g(z|\tau) = -\ln\left|\frac{\vartheta_1(z|\tau)}{\eta(\tau)}\right|^2 - \pi\frac{(z-\bar z)^2}{2\operatorname{Im}\tau} \tag{2.13}$$

Furthermore, we define two-dimensional convolutions of g recursively as follows,

$$g_{n+1}(z|\tau) = \int_\Sigma \frac{d^2x}{\operatorname{Im}\tau}\, g(z-x|\tau)\, g_n(x|\tau) \qquad\qquad g_1(x|\tau) = g(x|\tau) \tag{2.14}$$

In terms of these convolutions $g_n(z|\tau)$ the integration kernels $f^{(n)}(z|\tau)$ are given by,

$$f^{(n)}(z|\tau) = -\partial_z^n g_n(z|\tau) \tag{2.15}$$

and may thus also be defined recursively by convolutions over Σ [31],

$$f^{(n)}(z|\tau) = -\int_\Sigma \frac{d^2x}{\operatorname{Im}\tau}\, f^{(1)}(z-x|\tau)\, f^{(n-1)}(x|\tau) \qquad\qquad n \geq 2 \tag{2.16}$$

We note that, in co-moving coordinates $u, v \in [0,1]$ with $z = u\tau+v$, the non-holomorphic prefactor in the definition (2.9) of $\Omega(z,\alpha|\tau)$ becomes $e^{2\pi i\alpha u}$ so that, for fixed u, v, the functions $f^{(n)}(u\tau+v|\tau)$ are holomorphic in the modulus τ.

2.2.3 Modular properties of the Brown-Levin construction

Under a modular transformation on the modulus τ, z, and α given by,

$$\tau \to \tilde\tau = \frac{A\tau+B}{C\tau+D} \qquad\qquad z \to \tilde z = \frac{z}{C\tau+D} \qquad\qquad \alpha \to \tilde\alpha = \frac{\alpha}{C\tau+D} \tag{2.17}$$

where $A, B, C, D \in \mathbb{Z}$ with $AD-BC = 1$, the Kronecker-Eisenstein series Ω and the functions $f^{(n)}$ in (2.9) transform as modular forms of weight $(1,0)$ and $(n,0)$, respectively,

$$\begin{aligned}\Omega(\tilde z, \tilde\alpha|\tilde\tau) &= (C\tau+D)\Omega(z,\alpha|\tau)\\ f^{(n)}(\tilde z|\tilde\tau) &= (C\tau+D)^n f^{(n)}(z|\tau)\end{aligned} \tag{2.18}$$

These transformation properties may be readily established by using (2.9) and the transformation properties of the Jacobi ϑ-function,

$$\vartheta_1(\tilde{z}, \tilde{\alpha}|\tilde{\tau}) = \varepsilon\,(C\tau + D)^{\frac{1}{2}} e^{i\pi C z^2/(C\tau+D)} \vartheta_1(z|\tau) \qquad\qquad \varepsilon^8 = 1 \tag{2.19}$$

or the modular invariance of the functions $g_n(z|\tau)$ along with the relation (2.15). The modular properties of the Brown-Levin connection and polylogarithms are most transparent by assigning the following transformation law to the generators a, b in (2.11),

$$a \to \tilde{a} = (C\tau + D)a + 2\pi i C b \qquad\qquad b \to \tilde{b} = \frac{b}{C\tau + D} \tag{2.20}$$

This choice renders the flat connection $\mathcal{J}_{\rm BL}$ modular invariant under (2.17). The extra contribution $2\pi i C b$ to $\tilde{a}$ in (2.20) is engineered to compensate the transformation of the first term in the expression (2.11) for the connection

$$\frac{\pi\, d\tilde{z}}{\mathrm{Im}\,\tilde{\tau}}\,\tilde{b} = \frac{C\bar{\tau} + D}{C\tau + D}\,\frac{\pi\, dz}{\mathrm{Im}\,\tau}\,b \tag{2.21}$$

2.2.4 Homotopy-invariant iterated integrals

Homotopy-invariant iterated integrals on a genus-one surface are constructed by expanding the path-ordered exponential in terms of words in the (rather frugal) alphabet a, b as follows,

$$\mathrm{P}\exp\int_0^z \mathcal{J}_{\rm BL}(\cdot|\tau) = 1 + \sum_{\mathfrak{w}} \mathfrak{w}\, \Gamma(\mathfrak{w}; z|\tau) \tag{2.22}$$

The Brown-Levin connection $\mathcal{J}_{\rm BL}$ can be found in (2.11), and the sum runs over all words $\mathfrak{w}$ with at least one letter, formed out of the alphabet a, b. The construction guarantees that every coefficient function $\Gamma(\mathfrak{w}; z|\tau)$ is a homotopy-invariant iterated integral. These functions were dubbed elliptic polylogarithms in [22].

In this construction, the requirement of doubly-periodicity introduces non-holomorphicity into the functions $f^{(n)}(z|\tau)$, into the connection $\mathcal{J}_{\rm BL}$, and into the elliptic polylogarithms in (2.22). Still, any $\bar{z}$-dependence of $\Gamma(\mathfrak{w}; z|\tau)$ occurs only via polynomials in $2\pi i(\mathrm{Im}\, z)/\mathrm{Im}\,\tau$, so that the key structure is carried essentially by meromorphic iterated integrals.

2.2.5 Meromorphic variant

Given the meromorphic counterparts of the doubly-periodic Kronecker-Eisenstein series $\Omega(z,\alpha|\tau)$ and its expansion coefficients $f^{(n)}(z|\tau)$ of (2.9),[3]

$$\frac{\vartheta_1'(0|\tau)\vartheta_1(z+\alpha|\tau)}{\vartheta_1(z|\tau)\vartheta_1(\alpha|\tau)} = \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z|\tau) \tag{2.23}$$

a meromorphic variant of the Brown-Levin polylogarithms (2.22) may be recursively defined as follows [26],

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; z|\tau\right) = \int_0^z dz_1\, g^{(n_1)}(z_1-a_1|\tau)\, \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_r \\ a_2 & \cdots & a_r \end{smallmatrix}; z_1|\tau\right) \tag{2.24}$$

with $\tilde{\Gamma}(\emptyset; z|\tau) = 1$ and $r, n_i \in \mathbb{N}_0$. These meromorphic elliptic polylogarithms coincide with the formulation via $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ a_1 & a_2 & \cdots & a_r \end{smallmatrix}; z|\tau\right)$ in [27] on the real line and exhibit a closer analogy to the recursive definition (2.4) at genus zero than the $\Gamma(\mathfrak{w}; z|\tau)$ in (2.22). However, the meromorphic integration kernels $g^{(n)}(z|\tau)$ in (2.23) such as

$$g^{(0)}(z|\tau) = 1 \qquad\qquad g^{(1)}(z|\tau) = \partial_z \ln \vartheta_1(z|\tau) = f^{(1)}(z|\tau) - 2\pi i \frac{\operatorname{Im} z}{\operatorname{Im} \tau} \tag{2.25}$$

which enter the construction (2.24) of $\tilde{\Gamma}$ are generically multiple-valued on the torus, and thus more properly live on the universal covering space, which is $\mathbb{C}$.

Note that the Brown-Levin polylogarithms (2.22) associated with words $\mathfrak{w} \to ab\cdots b$ reduce to a single integral over the meromorphic kernels (2.23), for instance

$$\Gamma(ab; z|\tau) = \int_0^z dt \left(2\pi i \frac{\operatorname{Im} t}{\operatorname{Im}\tau} - f^{(1)}(t|\tau)\right) = -\int_0^z dt\, g^{(1)}(t|\tau) = -\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z|\tau\right) \tag{2.26}$$

(see [27, 64] for the regularization of endpoint divergences) and more generally

$$\begin{aligned} \Gamma(a\underbrace{b\cdots b}_{n}; z|\tau) &= (-1)^n \int_0^z dt \sum_{j=0}^{n} \frac{1}{j!} \left(-2\pi i \frac{\operatorname{Im} t}{\operatorname{Im}\tau}\right)^j f^{(n-j)}(t|\tau) \\ &= (-1)^n \int_0^z dt\, g^{(n)}(t|\tau) = (-1)^n \tilde{\Gamma}\left(\begin{smallmatrix} n \\ 0 \end{smallmatrix}; z|\tau\right) \end{aligned} \tag{2.27}$$

These examples illustrate that the Brown-Levin polylogarithms in (2.22) are $\mathbb{Q}[2\pi i \frac{\operatorname{Im} z}{\operatorname{Im}\tau}]$-linear combinations of the meromorphic ones in (2.24).

[3] The meromorphic functions $g^{(n)}(z|\tau)$ in the expansion given in (2.23) are not to be confused with the real-analytic convolutions $g_n(z|\tau)$ of the scalar Green function on the torus defined in (2.14). Both notations have, by now, become standard for historical reasons and can be distinguished through the placement of $n \in \mathbb{N}_0$ in the superscript and in parenthesis in case of (2.23).

3 A flat connection at higher genus

This section is dedicated to the construction of a flat connection which generalizes the Brown-Levin flat connection $\mathcal{J}_{\rm BL}$ in (2.11) to higher genus. We begin by introducing some functions and forms on Riemann surfaces of arbitrary genus that will provide the key ingredients in our construction. Further background material on Riemann surfaces and their function theory, including ϑ-functions, may be found in [50, 65, 33].

3.1 Basics

The topology of a compact Riemann surface Σ without boundary is specified by its genus h. The homology group $H_1(\Sigma, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2h}$ and supports an anti-symmetric non-degenerate intersection pairing denoted by $\mathfrak{J}$. A canonical homology basis of cycles $\mathfrak{A}_I$ and $\mathfrak{B}_J$ with $I, J = 1, \cdots, h$ has symplectic intersection matrix $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = -\mathfrak{J}(\mathfrak{B}_J, \mathfrak{A}_I) = \delta_{IJ}$, and $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$, as illustrated in Figure 1 for genus two.

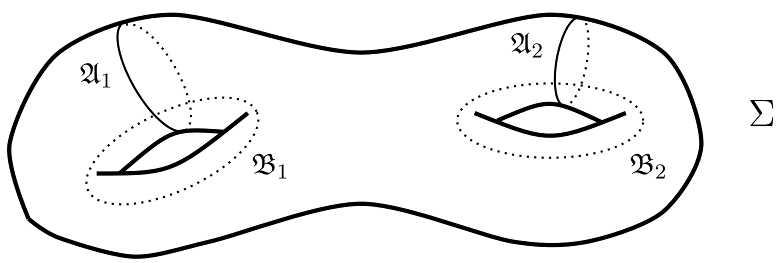

Figure 1: A choice of canonical homology basis on a compact genus-two Riemann surface Σ.

A canonical basis of holomorphic Abelian differentials $\boldsymbol{\omega}_I$ may be normalized on $\mathfrak{A}$-cycles,[4]

$$\oint_{\mathfrak{A}_I} \boldsymbol{\omega}_J = \delta_{IJ} \qquad\qquad \oint_{\mathfrak{B}_I} \boldsymbol{\omega}_J = \Omega_{IJ} \tag{3.1}$$

The complex variables Ω_{IJ} denote the components of the period matrix Ω of the surface Σ. By the Riemann relations, Ω is symmetric, and has positive definite imaginary part,

$$\Omega^t = \Omega \qquad\qquad Y = \text{Im}\,\Omega > 0 \tag{3.2}$$

[4]Throughout, differential forms are denoted in boldface, while their component functions in local complex coordinates $z, \bar{z}$ on Σ are denoted by the same letter in normal font, such as for example $\boldsymbol{\omega}_I = \omega_I(z)dz$. The coordinate volume form on Σ is $d^2z = \frac{i}{2}dz \wedge d\bar{z}$ and the δ-function is normalized by $\int_\Sigma d^2z\, \delta(z,w) = 1$ for any $w \in \Sigma$. Finally, repeated pairs of identical indices are to be summed following the Einstein convention so that, for example, we set $Y^{IJ}\boldsymbol{\omega}_J = \sum_{J=1}^h Y^{IJ}\boldsymbol{\omega}_J$.

The matrix $Y_{IJ} = \text{Im}\,\Omega_{IJ}$ and its inverse $Y^{IJ} = \big((\text{Im}\,\Omega)^{-1}\big)^{IJ}$ may be used to raise and lower indices as follows,

$$\boldsymbol{\omega}^I = Y^{IJ}\boldsymbol{\omega}_J \qquad \bar{\boldsymbol{\omega}}^I = Y^{IJ}\bar{\boldsymbol{\omega}}_J \qquad Y^{IK}Y_{KJ} = \delta^I_J \tag{3.3}$$

In particular, it will be useful to express the Riemann bilinear relations as follows,

$$\frac{i}{2}\int_\Sigma \boldsymbol{\omega}_I \wedge \bar{\boldsymbol{\omega}}^J = \delta_I^J \tag{3.4}$$

The choice of canonical $\mathfrak{A}$ and $\mathfrak{B}$-cycles is not unique: a new canonical basis $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ is obtained by applying a modular transformation $M \in Sp(2h,\mathbb{Z})$, such that M satisfies $M^t\mathfrak{J}M = \mathfrak{J}$, and the column matrices of cycles are transformed as follows,

$$\begin{pmatrix}\tilde{\mathfrak{B}}\\ \tilde{\mathfrak{A}}\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathfrak{B}\\ \mathfrak{A}\end{pmatrix} \qquad M = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \qquad \mathfrak{J} = \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix} \tag{3.5}$$

Under a modular transformation M, the row matrix $\boldsymbol{\omega}$ of holomorphic Abelian differentials $\boldsymbol{\omega}_I$, the period matrix Ω, and its imaginary part Y, transform as follows,

$$\begin{aligned}
\tilde{\boldsymbol{\omega}} &= \boldsymbol{\omega}\,(C\Omega + D)^{-1}\\
\tilde{\Omega} &= (A\Omega + B)(C\Omega + D)^{-1}\\
\tilde{Y} &= \big(\bar{\Omega}C^t + D^t\big)^{-1}\, Y\,(C\Omega + D)^{-1}
\end{aligned} \tag{3.6}$$

where we have denoted transformed quantities with a tilde as in the discussion of the genus-one case in section 2.2.3.

The moduli space of compact Riemann surfaces of genus h will be denoted by $\mathcal{M}_h$. The moduli space $\mathcal{M}_h$ for $h = 1, 2, 3$ may be identified with $\mathcal{H}_h/Sp(2h,\mathbb{Z})$ provided we remove from the Siegel upper half space $\mathcal{H}_h$ for $h = 2, 3$ all elements which correspond to disconnected surfaces, and take into account the effect of automorphisms including the involution on the hyper-elliptic locus for $h = 3$. For $h \geq 4$, the moduli space $\mathcal{M}_h$ is a complex co-dimension $\frac{1}{2}(h-2)(h-3)$ subspace of $\mathcal{H}_h/Sp(2h,\mathbb{Z})$ known as the Schottky locus.

3.2 The Arakelov Green function

The Arakelov Green function $\mathcal{G}(x, y|\Omega)$ on $\Sigma \times \Sigma$ generalizes the Green function $g(z|\tau)$ which was defined at genus one in (2.12), and is defined by,

$$\partial_{\bar{x}}\partial_x \mathcal{G}(x, y|\Omega) = -\pi\delta(x, y) + \pi\kappa(x) \qquad \int_\Sigma \boldsymbol{\kappa}(x)\mathcal{G}(x, y|\Omega) = 0 \tag{3.7}$$

where κ is given by the pull-back to Σ under the Abel map of the unique translation invariant Kähler form on the Jacobian variety $J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \Omega\mathbb{Z}^h)$, normalized to unit volume,[5]

$$\boldsymbol{\kappa} = \frac{i}{2h}\boldsymbol{\omega}_I \wedge \bar{\boldsymbol{\omega}}^I = \kappa(z)\, d^2z \qquad\qquad \int_\Sigma \boldsymbol{\kappa} = 1 \tag{3.8}$$

Here and throughout the rest of this work, we shall suppress the dependence on the period matrix Ω unless otherwise indicated. Both $\boldsymbol{\kappa}$ and $\mathcal{G}(x,y)$ are conformally invariant. An explicit formula for $\mathcal{G}(x,y)$ may be given in terms of the non-conformally invariant string Green function $G(x,y)$ as follows,

$$\mathcal{G}(x,y) = G(x,y) - \gamma(x) - \gamma(y) + \gamma_0 \tag{3.9}$$

where $\gamma(x)$ and γ_0 are given by,

$$\gamma(x) = \int_\Sigma \boldsymbol{\kappa}(z) G(x,z) \qquad\qquad \gamma_0 = \int_\Sigma \boldsymbol{\kappa}\gamma \tag{3.10}$$

The string Green function is given in terms of the prime form $E(x,y)$ by,[6]

$$G(x,y) = -\log|E(x,y)|^2 + 2\pi \left(\mathrm{Im}\int_y^x \boldsymbol{\omega}_I\right)\left(\mathrm{Im}\int_y^x \boldsymbol{\omega}^I\right) \tag{3.11}$$

see Appendix A for the definition of the prime form in terms of theta functions. The following double derivatives,

$$\begin{aligned}
\partial_x\partial_y \mathcal{G}(x,y) &= -\partial_x\partial_y \ln E(x,y) + \pi\, \omega_I(x)\, \omega^I(y) \\
\partial_x\partial_{\bar{y}} \mathcal{G}(x,y) &= \pi\, \delta(x,y) - \pi\, \omega_I(x)\, \bar{\omega}^I(y)
\end{aligned} \tag{3.12}$$

will prove useful in the sequel.

3.3 Convolution of Arakelov Green functions and modular tensors

Convolutions involving the Arakelov Green function and various integration measures were used to construct a variety of modular tensors in [56, 57, 58, 59]. Here we shall extend this library of modular tensors by including convolutions that involve not only Arakelov Green

[5] A recent account of the Arakelov Green function and its properties needed here may for instance be found in [51].

[6] Here, it is understood that the prime form, and thus the string Green function, are defined in a suitable fundamental domain for Σ in the universal covering space of Σ, and that the integrals in (3.10) are to be carried out in that same domain [51].

functions (as was the case in [58]) but also derivatives on Arakelov Green functions and holomorphic Abelian differentials. The building blocks of our flat connection and corresponding higher-genus polylogarithms will be modular tensors, and the resulting modular properties of the connection and the polylogarithms themselves will be discussed in section 3.5 and 4.4, respectively.

3.3.1 Definition of modular tensors

Modular tensors are defined on Torelli space, which is the moduli space of compact Riemann surfaces endowed with a choice of canonical homology basis of $\mathfrak{A}$ and $\mathfrak{B}$ cycles. The significance of modular tensors has been articulated in the work of Kawazumi [56, 57, 59] and two of the authors of the present paper in [58]. Modular tensors generalize modular forms at genus one by replacing the familiar automorphy factor $(C\tau + D)$ of $SL(2,\mathbb{Z})$ discussed in section 2.2.3 by an automorphy tensor Q and its inverse $R = Q^{-1}$,

$$\begin{aligned} Q &= Q(M,\Omega) = C\Omega + D \\ R &= R(M,\Omega) = (C\Omega + D)^{-1} \end{aligned} \tag{3.13}$$

for $M \in Sp(2h,\mathbb{Z})$ and the matrices C and D given in (3.5). The composition law for the automorphy tensors may be read off from the transformation properties of Ω given in (3.6),

$$Q(M_1 M_2, \Omega) = Q\big(M_1, (A_2\Omega + B_2)(C_2\Omega + D_2)^{-1}\big) Q(M_2, \Omega) \tag{3.14}$$

In (3.6) we already encountered the tensors $\boldsymbol{\omega}_I$, $\boldsymbol{\omega}^I$, Y_{IJ}, and its inverse Y^{IJ}. In the notation (3.13) of the automorphy factors Q and R, their transformation properties are given by,

$$\begin{aligned} \tilde{\boldsymbol{\omega}}_I &= \boldsymbol{\omega}_{I'} R^{I'}{}_I & \qquad \tilde{Y}_{IJ} &= Y_{I'J'}\, \bar{R}^{I'}{}_I\, R^{J'}{}_J \\ \tilde{\boldsymbol{\omega}}^J &= \bar{Q}^J{}_{J'}\, \boldsymbol{\omega}^{J'} & \qquad \tilde{Y}^{IJ} &= Q^I{}_{I'}\, \bar{Q}^J{}_{J'}\, Y^{I'J'} \end{aligned} \tag{3.15}$$

More generally, a modular tensor $\mathcal{T}$ of arbitrary rank transforms as follows,

$$\tilde{\mathcal{T}}^{I_1,\cdots,I_n;J_1,\cdots,J_{\bar{n}}}(\tilde{\Omega}) = Q^{I_1}{}_{I_1'} \cdots Q^{I_n}{}_{I_n'}\, \bar{Q}^{J_1}{}_{J_1'} \cdots \bar{Q}^{J_{\bar{n}}}{}_{J_{\bar{n}}'}\, \mathcal{T}^{I_1',\cdots,I_n';J_1',\cdots,J_{\bar{n}}'}(\Omega) \tag{3.16}$$

The tensors Y_{IJ} and Y^{IJ} may be used to lower and raise indices, respectively, and can be made to compensate any anti-holomorphic automorphy factor: instead of the $\bar{Q}^{J_i}{}_{J_i'}$ in (3.16), the tensor,

$$\mathcal{U}^{I_1,\cdots,I_n}_{J_1,\cdots,J_{\bar{n}}}(\Omega) = Y_{J_1K_1} \cdots Y_{J_{\bar{n}}K_{\bar{n}}}\, \mathcal{T}^{I_1,\cdots,I_n;K_1,\cdots,K_{\bar{n}}}(\Omega) \tag{3.17}$$

exclusively transforms with holomorphic automorphy factors $Q^{I_i}{}_{I_i'}$ and $R^{J_i'}{}_{J_i}$,

$$\tilde{\mathcal{U}}^{I_1,\cdots,I_n}_{J_1,\cdots,J_{\bar{n}}}(\tilde{\Omega}) = Q^{I_1}{}_{I_1'} \cdots Q^{I_n}{}_{I_n'} R^{J_1'}{}_{J_1} \cdots R^{J_{\bar{n}}'}{}_{J_{\bar{n}}}\, \mathcal{U}^{I_1',\cdots,I_n'}_{J_1',\cdots,J_{\bar{n}}'}(\Omega) \tag{3.18}$$

The above tensors may be reducible. Symmetrization, anti-symmetrization, and removal of the trace by contracting with factors of Y_{IJ} or δ_I^J may be used to extract irreducible tensors. For genus h, the anti-symmmetrization of h indices I (resp. J indices) produces a factor of $\det Q$ (resp. $\det \bar{Q}$).

3.3.2 The interchange lemma

The modular tensor defined by the following convolution,

$$\Phi^I{}_J(x) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar{\omega}^I(z) \omega_J(z) \tag{3.19}$$

was found to play a central role in the studies of more general modular tensors and their relations. In view of the second equation in (3.7), the tensor Φ is trace-less, and therefore vanishes identically at genus one. In particular, it enters into the *interchange lemma* which was stated and proved in [55, 58],

$$\partial_x \mathcal{G}(x,y)\, \omega_J(y) + \partial_y \mathcal{G}(x,y)\, \omega_J(x) - \partial_x \Phi^I{}_J(x)\, \omega_I(y) - \partial_y \Phi^I{}_J(y)\, \omega_I(x) = 0 \tag{3.20}$$

The genus-one version of this formula holds trivially by translation invariance on the torus and the vanishing of Φ. At higher genus, Φ may be viewed as compensating for the lack of translation invariance on higher-genus Riemann surfaces.

3.3.3 Higher convolutions of the Arakelov Green function

The fundamental building blocks for a flat connection on higher-genus Riemann surfaces are modular tensors defined recursively by convolutions on Σ as follows,

$$\begin{aligned} \Phi^{I_1 \cdots I_r}{}_J(x) &= \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar{\omega}^{I_1}(z)\, \partial_z \Phi^{I_2 \cdots I_r}{}_J(z) & \quad r \geq 2 \\ \mathcal{G}^{I_1 \cdots I_s}(x,y) &= \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar{\omega}^{I_1}(z)\, \partial_z \mathcal{G}^{I_2 \cdots I_s}(z,y) & \quad s \geq 1 \end{aligned} \tag{3.21}$$

where $\Phi^I{}_J(x)$ was defined in (3.19) and $\mathcal{G}^\emptyset(x,y) = \mathcal{G}(x,y)$. By construction, $\Phi^{I_1 \cdots I_r}{}_J(x)$ and $\mathcal{G}^{I_1 \cdots I_s}(x,y)$ are scalar functions of x, y and $Sp(2h,\mathbb{Z})$ tensors of rank $r{+}1$ and s, respectively, with the purely holomorphic transformation law in (3.18). The vanishing of the trace $\Phi^{I_1 \cdots I_r}{}_{I_r} = 0$ for arbitrary genus implies that Φ-tensors for arbitrary $r \geq 1$ vanish identically for genus one. Furthermore, at genus one, the derivatives of the tensor $\mathcal{G}^{I_1 \cdots I_s}$ for $I_1 = \cdots = I_s = 1$ equal the Kronecker-Eisenstein integration kernels $f^{(s+1)}$ given in (2.9) and (2.16),

$$\partial_x \mathcal{G}^{I_1 \cdots I_s}(x,y)\,\big|_{h=1} = -f^{(s+1)}(x{-}y|\tau) \tag{3.22}$$

From the differential relations satisfied by the Arakelov Green function in (3.7) and (3.12) we derive analogous differential relations satisfied by $\Phi^{I_1\cdots I_r}{}_J(x)$,

$$\begin{aligned}\partial_{\bar x}\partial_x\,\Phi^I{}_J(x) &= -\pi\,\bar\omega^I(x)\,\omega_J(x) + \pi\,\delta^I_J\,\kappa(x)\\ \partial_{\bar x}\partial_x\,\Phi^{I_1\cdots I_r}{}_J(x) &= -\pi\,\bar\omega^{I_1}(x)\,\partial_x\Phi^{I_2\cdots I_r}{}_J(x) \qquad\qquad r\geq 2\end{aligned} \tag{3.23}$$

and by $\mathcal{G}^{I_1\cdots I_s}(x,y)$ for $s\geq 1$,

$$\begin{aligned}\partial_{\bar x}\partial_x\,\mathcal{G}^{I_1\cdots I_s}(x,y) &= -\pi\,\bar\omega^{I_1}(x)\,\partial_x\mathcal{G}^{I_2\cdots I_s}(x,y)\\ \partial_{\bar y}\partial_x\,\mathcal{G}^{I_1\cdots I_s}(x,y) &= \pi\,\partial_x\mathcal{G}^{I_1\cdots I_{s-1}}(x,y)\,\bar\omega^{I_s}(y) - \pi\,\partial_x\Phi^{I_1\cdots I_s}{}_J(x)\,\bar\omega^J(y)\end{aligned} \tag{3.24}$$

These relations will be fundamental in the construction of the flat connection at higher genus.

3.3.4 Modular properties of convolutions

The convolutions $\mathcal{G}^{I_1\cdots I_s}(x,y)$ and $\Phi^{I_1\cdots I_r}{}_J(x)$ are modular tensors. Their transformation properties may be read off directly from those of $\boldsymbol{\omega}_J$ and $\bar{\boldsymbol{\omega}}^I$,

$$\tilde{\bar{\boldsymbol{\omega}}}^I(x) = Q^I{}_J\,\bar{\boldsymbol{\omega}}^J(x) \tag{3.25}$$

see (3.15), and provide the following transformation rules,

$$\begin{aligned}\tilde{\mathcal{G}}^{I_1\cdots I_s}(x,y) &= Q^{I_1}{}_{I_1'}\cdots Q^{I_s}{}_{I_s'}\mathcal{G}^{I_1'\cdots I_s'}(x,y)\\ \tilde\Phi^{I_1\cdots I_r}{}_J(x) &= Q^{I_1}{}_{I_1'}\cdots Q^{I_r}{}_{I_r'}\,\Phi^{I_1'\cdots I_r'}{}_{J'}(x)\,R^{J'}{}_J\end{aligned} \tag{3.26}$$

We note that all automorphy tensors Q,R in (3.13) are holomorphic on Torelli space. Thus we may view $\mathcal{G}^{I_1\cdots I_s}(x,y)$ and $\Phi^{I_1\cdots I_r}{}_J(x)$ as sections of holomorphic vector bundles over Torelli space, whose transition functions are given by the tensor Q and its inverse R.

3.4 Generating functions

For genus one, the functions $f^{(n)}(z|\tau)$ were obtained by expanding the Kronecker-Eisenstein series $\Omega(z,\alpha|\tau)$ in powers of the free parameter α. At higher genus, we may also assemble the families of modular tensors, defined in the previous subsection, into generating functions. To do so, we introduce a non-commutative algebra freely generated by B_I for $I=1,\cdots,h$, and that will soon be extended to a larger free algebra. We also fix an arbitrary auxiliary marked point p on the Riemann surface Σ. With the help of the generators B_I, we introduce the following generating functions,

$$
\begin{aligned}
\mathcal{H}(x,p;B) &= \partial_x \mathcal{G}(x,p) + \sum_{r=1}^{\infty} \partial_x \mathcal{G}^{I_1 I_2 \cdots I_r}(x,p) B_{I_1} B_{I_2} \cdots B_{I_r} \\
&= \partial_x \mathcal{G}(x,p) + \partial_x \mathcal{G}^{I_1}(x,p) B_{I_1} + \partial_x \mathcal{G}^{I_1 I_2}(x,p) B_{I_1} B_{I_2} + \cdots \\
\mathcal{H}_J(x;B) &= \omega_J(x) + \sum_{r=1}^{\infty} \partial_x \Phi^{I_1 I_2 \cdots I_r}{}_J(x) B_{I_1} B_{I_2} \cdots B_{I_r} \\
&= \omega_J(x) + \partial_x \Phi^{I_1}{}_J(x) B_{I_1} + \partial_x \Phi^{I_1 I_2}{}_J(x) B_{I_1} B_{I_2} + \cdots
\end{aligned} \tag{3.27}
$$

Note that the modular tensors $\mathcal{G}^{I_1 \cdots I_r}(x,p)$ and $\Phi^{I_1 \cdots I_r}{}_J(x)$ are not necessarily symmetric in their indices $I_1, \cdots, I_r$. This absence of symmetry is captured by the non-commutative nature of the algebra of the B_I. The differential relations (3.7), (3.23), and (3.24) on the components imply the following differential relations on the generating functions,

$$
\begin{aligned}
\partial_{\bar x} \mathcal{H}(x,p;B) &= \pi\, \kappa(x) - \pi \delta(x,p) - \pi \bar\omega^I(x) B_I\, \mathcal{H}(x,p;B) \\
\partial_{\bar p} \mathcal{H}(x,p;B) &= \pi \delta(x,p) + \pi \bar\omega^I(p) \big(\mathcal{H}(x,p;B) B_I - \mathcal{H}_I(x;B) \big) \\
\partial_{\bar x} \mathcal{H}_J(x;B) &= \pi\, \kappa(x) B_J - \pi\, \bar\omega^I(x) B_I\, \mathcal{H}_J(x;B)
\end{aligned} \tag{3.28}
$$

By forming the combination,

$$
\Psi_J(x,p;B) = \mathcal{H}_J(x;B) - \mathcal{H}(x,p;B) B_J \tag{3.29}
$$

the differential form κ in (3.8) is found to cancel between the $\bar x$-derivative of both terms in (3.28), and the result may be compactly written as follows,

$$
\begin{aligned}
\partial_{\bar x} \Psi_J(x,p;B) &= \pi \delta(x,p) B_J - \pi \bar\omega^I(x)\, B_I\, \Psi_J(x,p;B) \\
\partial_{\bar p} \Psi_J(x,p;B) &= -\pi \delta(x,p) B_J + \pi \bar\omega^I(p)\, \Psi_I(x,p;B)\, B_J
\end{aligned} \tag{3.30}
$$

The delta functions on the right-hand side signal the simple pole of $\partial_x \mathcal{G}(x,p) = -\frac{1}{x-p} + \mathcal{O}(1)$ whereas all the tensorial integration kernels $\partial_x \Phi^{I_1 \cdots I_r}{}_J(x)$ and $\partial_x \mathcal{G}^{I_1 \cdots I_r}(x,p)$ with $r \geq 1$ are regular on the entire surface. This generalizes the pole structure of the $f^{(n)}$ to arbitrary genus where $f^{(1)}(x{-}p|\tau) = \frac{1}{x-p} + \mathcal{O}(x{-}p)$ exhibits the only pole among the genus-one kernels.

To obtain tensorial modular transformations properties for the generating function (3.29), the modular transformations of its components must be accompanied by the following transformation properties for the algebra generators B_J,

$$
\begin{aligned}
\tilde B_J &= B_{J'} R^{J'}{}_J \\
\tilde{\mathcal{H}}_J(x;\tilde B) &= \mathcal{H}_{J'}(x;B) R^{J'}{}_J \\
\tilde\Psi_J(x,p;\tilde B) &= \Psi_{J'}(x,p;B) R^{J'}{}_J
\end{aligned} \tag{3.31}
$$

The generating function $\mathcal{H}(x,p;B)$ is then invariant.

3.5 The flat connection

We are now ready to assemble all the results of the previous sections into a flat and modular invariant connection.

To do so, we begin by extending the algebra generated by the elements B_I as follows. We introduce the Lie algebra $\mathcal{L}$ freely generated by elements a^I and b_I for $I = 1, \cdots, h$ and set $B_I = \mathrm{ad}_{b_I} = [b_I, \cdot]$. The algebra admits a dual grading counting independently the number of letters a and the number of letters b in each word, irrespective of the value of their indices. This algebra was already considered by Enriquez and Zerbini in their construction of Maurer-Cartan elements in [45], where the generators a^I and b_I correspond to generators of the fundamental group of the surface Σ with the point p removed. For a general reference on freely generated Lie algebras and their applications we refer to Reutenauer's book [66].

It remains to generalize the term proportional to $(dz - d\bar{z})b$ in the genus-one connection $\mathcal{J}_{\rm BL}$ in (2.11) to higher genus. Since the single b at genus one generalizes to a tensor b_I at higher genus, it might seem natural to generalize $dz - d\bar{z}$ to the closed differential $\boldsymbol{\omega}^I - \bar{\boldsymbol{\omega}}^I$. Actually, this choice does not lead to a flat connection, but promoting the ω^I part of this construction to $\mathcal{H}^I$ in (3.27) does the job. The result is the following theorem.

Theorem 3.1 *The connection $\mathcal{J}(x,p)$, on a Riemann surface Σ of arbitrary genus h with a marked point $p \in \Sigma$ and valued in the Lie algebra $\mathcal{L}$ freely generated by the $2h$ elements a^I, b_I with $I = 1, \cdots, h$, is given in terms of $\bar{\boldsymbol{\omega}}^I(x)$ and the generating functions $\mathcal{H}^I(x;B) = Y^{IJ}\,\mathcal{H}_J(x;B)$ and $\Psi_I(x,p;B)$ as follows,*

$$\mathcal{J}(x,p) = -\pi\, d\bar{x}\, \bar{\omega}^I(x)\, b_I + \pi\, dx\, \mathcal{H}^I(x;B)\, b_I + dx\, \Psi_I(x,p;B)\, a^I \tag{3.32}$$

where $B_I = \mathrm{ad}_{b_I} = [b_I, \cdot]$. The connection $\mathcal{J}(x,p)$ is flat and reproduces the Brown-Levin connection (2.11) at genus one.

To prove the theorem we begin by using the differential equations (3.28) and (3.30) satisfied by the generating functions $\mathcal{H}^I(x;B)$ and $\Psi_I(x,p;B)$, and readily establish the following results,

$$\begin{aligned} d_x\mathcal{J}(x,p) &= \pi d\bar{x} \wedge dx \Big\{ \delta(x,p)[b_I, a^I] - \pi\bar{\omega}^I(x)\, B_I\, \mathcal{H}^J(x;B)\, b_J \\ &\qquad\qquad - \bar{\omega}^I(x)\, B_I\, \Psi_J(x,p;B)\, a^J \Big\} \\ \mathcal{J}(x,p) \wedge \mathcal{J}(x,p) &= \pi d\bar{x} \wedge dx \Big(-\pi\, \bar{\omega}^I(x)\, B_I\, \mathcal{H}^J(x;B) b_J - \bar{\omega}^I(x)\, B_I\, \Psi_J(x,p;B)\, a^J \Big) \end{aligned} \tag{3.33}$$

The difference of the two lines shows that the connection is flat away from $x = p$,

$$d_x\mathcal{J}(x,p) - \mathcal{J}(x,p) \wedge \mathcal{J}(x,p) = \pi d\bar{x} \wedge dx\, \delta(x,p)\, [b_I, a^I] \tag{3.34}$$

To prove that the connection $\mathcal{J}(x,p)$ reduces to the non-holomorphic single-valued Brown-Levin connection, we specialize to the case of genus one and relabel $a^1 = a$ and $b_1 = b$. Since the tensor $\Phi^I{}_J$ of (3.19) and its higher-rank versions of (3.21) all vanish identically at genus one, the generating function $\mathcal{H}^1(x;B)$ reduces to,

$$\mathcal{H}^1(x;B)\Big|_{h=1} = \omega^1(x) = \frac{\omega_1(x)}{\mathrm{Im}\,\tau} \tag{3.35}$$

so that the first two terms in (3.32) combine to $\pi(dx - d\bar{x})b/\mathrm{Im}\,\tau$ thereby reproducing the contributions $\sim (\mathrm{Im}\,\tau)^{-1}$ to the non-meromorphic Brown-Levin connection of (2.11). The third term in (3.32) reproduces the Kronecker-Eisenstein series by (3.35) and (3.22),

$$\Psi_1(x,p;B)\Big|_{h=1} = \omega_1(x) - \mathcal{H}(x,p;B)B_1\Big|_{h=1} = \mathrm{ad}_b\,\Omega(x-p,\mathrm{ad}_b|\tau) \tag{3.36}$$

concluding the proof of Theorem 3.1.

The expression (3.32) for the flat connection is modular invariant for suitable $Sp(2h,\mathbb{Z})$ transformation rules of the generators a^I, b_I to be stated in the following theorem:

Theorem 3.2 *Under a modular transformation $M \in Sp(2h,\mathbb{Z})$, parametrized in (3.5), which acts on $\bar{\omega}^I$ as given in (3.25), on B_I, $\mathcal{H}_I$, and Ψ_I as given in (3.31), and on the Lie algebra generators a^I and b_I by,*

$$\begin{aligned} a^I \to \tilde{a}^I &= Q^I{}_J\, a^J + 2\pi i\, C^{IJ}\, b_J \\ b_I \to \tilde{b}_I &= b_J\, R^J{}_I \end{aligned} \tag{3.37}$$

the connection $\mathcal{J}(x,p)$ is invariant. In the basis $(\hat{a}^I, b_I)$ of generators of the Lie algebra $\mathcal{L}$,

$$\hat{a}^I = a^I + \pi Y^{IJ} b_J \tag{3.38}$$

subject to

$$\hat{a}^I \to \tilde{\hat{a}}^I = Q^I{}_J\, \hat{a}^J \tag{3.39}$$

the connection $\mathcal{J}(x,p)$ takes on a simplified form,

$$\mathcal{J}(x,p) = -\pi\, d\bar{x}\, \bar{\omega}^I(x)\, b_I + dx\, \Psi_I(x,p;B)\, \hat{a}^I \tag{3.40}$$

and is manifestly invariant under $Sp(2h,\mathbb{Z})$.

Proving modular invariance of $\mathcal{J}(x,p)$ is most transparently achieved by first carrying out the change of basis of (3.38) to the equivalent form (3.40) which immediately follows from the observation that $Y^{IJ}B_I b_J = 0$ so that $\Psi_I(x,p;B)Y^{IJ}b_J = \mathcal{H}^I(x;B)b_I$. The connection $\mathcal{J}(x,p)$, presented in the form of (3.40), is term-by-term invariant under the modular transformations of $\bar{\omega}^I$, B_I, Ψ_I, and a^I, b_I stated in the theorem: in both of $d\bar{x}\,\bar{\omega}^I(x)\,b_I$ and $dx\,\Psi_I(x,p;B)\,\hat{a}^I$, the respective ingredients transform with opposite automorphy factors as can be verified from (3.15), (3.31), (3.37) and (3.39). The modular invariance of $d\bar{x}\,\bar{\omega}^I(x)\,b_I$ and $dx\,\Psi_I(x,p;B)\,\hat{a}^I$ established in this way completes the proof of Theorem 3.2.

Finally, the connection $\mathcal{J}$ may be expanded in words with $r{+}1$ letters in the basis (a^I, b_I),

$$\begin{aligned}\mathcal{J}(x,p) = -\pi\, d\bar{x}\,\bar{\omega}^I(x) b_I + \pi\, dx\,\omega^I(x) b_I + \pi\, dx \sum_{r=1}^{\infty} \partial_x \Phi^{I_1\cdots I_r}{}_J(x)\, Y^{JK}\, B_{I_1}\cdots B_{I_r}\, b_K \\ +\; dx \sum_{r=1}^{\infty} \Big(\partial_x \Phi^{I_1\cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1\cdots I_{r-1}}(x,p)\delta^{I_r}_J \Big) B_{I_1}\cdots B_{I_r}\, a^J \end{aligned} \tag{3.41}$$

or, equivalently, in the basis $(\hat{a}^I, b_I)$,

$$\begin{aligned}\mathcal{J}(x,p) = \pi \Big(dx\,\omega^I(x) - d\bar{x}\,\bar{\omega}^I(x) \Big) b_I \\ + dx \sum_{r=1}^{\infty} \Big(\partial_x \Phi^{I_1\cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1\cdots I_{r-1}}(x,p)\delta^{I_r}_J \Big) B_{I_1}\cdots B_{I_r} \hat{a}^J \end{aligned} \tag{3.42}$$

where we abbreviate $B_I = \mathrm{ad}_{b_I}$. These expressions will be useful in the subsequent section to illustrate the expansion of higher-genus polylogarithms, and the simplified representation in (3.42) may be easily obtained from the change of basis using (3.38).

The higher-genus polylogarithms obtained by expanding the path-ordered exponential (2.2) of the connection $\mathcal{J}(x,p)$ in words of the generators (a^I, b_I) or $(\hat{a}^I, b_I)$ will automatically be homotopy invariant. Examples to low letter count will be given in the next section.

4 Higher-genus polylogarithms

In this section, the flat connection $\mathcal{J}$ assembled in the preceding section will be used to construct higher-genus polylogarithms. The flat connection $\mathcal{J}(x,p)$ of (3.32) is well-defined and single-valued on two copies of Σ but this property came at the cost of giving up meromorphicity in both of x and p. We shall outline a method to restore meromorphicity in the first variable x for certain combinations of integration kernels at the cost of giving up single-valuedness. Moreover, non-trivial evidence will be provided that the higher-genus polylogarithms in this section are closed under taking primitives. We close this section with a proposal for higher-genus analogues of elliptic associators.

4.1 Construction of higher-genus polylogarithms

The flat connection $\mathcal{J}(x,p)$ of Theorem 3.1 integrates to a homotopy-invariant path-ordered exponential $\mathbf{\Gamma}(x,y;p)$,

$$\mathbf{\Gamma}(x,y;p) = \mathrm{P}\exp\int_y^x \mathcal{J}(t,p) \tag{4.1}$$

Expanding the path-ordered exponential in terms of the generators of the Lie algebra $\mathcal{L}$ produces homotopy-invariant iterated integrals, as explained in the preamble of section 2,

$$\mathbf{\Gamma}(x,y;p) = 1 + \sum_{\mathfrak{w}} \mathfrak{w}\,\Gamma(\mathfrak{w};x,y;p) \tag{4.2}$$

Here, the sum over $\mathfrak{w}$ is over all words, containing at least one letter, made out of the alphabet of the Lie algebra generators introduced in section 3.5. In this way, the path-ordered exponential is the generating function for the iterated integrals $\Gamma(\mathfrak{w};x,y;p)$ and, for each word $\mathfrak{w}$, defines a polylogarithm $\Gamma(\mathfrak{w};x,y;p)$. Their iterated-integral definition via (4.1) and (4.2) implies the following shuffle property,

$$\Gamma(\mathfrak{w}_1;x,y;p)\cdot\Gamma(\mathfrak{w}_2;x,y;p) = \sum_{\mathfrak{w}\in\mathfrak{w}_1\sqcup\!\sqcup\,\mathfrak{w}_2} \Gamma(\mathfrak{w};x,y;p) \tag{4.3}$$

which holds in identical form for the polylogarithms (2.6) and (2.22) at genus zero and genus one. Following the standard shuffle product, the sum over $\mathfrak{w}\in\mathfrak{w}_1\sqcup\!\sqcup\,\mathfrak{w}_2$ includes all ordered sets obtained from $\mathfrak{w}_1\cup\mathfrak{w}_2$ that preserve the order within the individual words $\mathfrak{w}_1,\mathfrak{w}_2$.

In Theorem 3.1 and Theorem 3.2, the modular invariant connection $\mathcal{J}(x,p)$ was expressed in terms of two different bases for the same Lie algebra $\mathcal{L}$ in which $\mathcal{J}(x,p)$ takes its values. In the first basis (a^I,b_I) the relation with the Brown-Levin connection at genus one is manifest, while the second basis $(\hat{a}^I,b_I)$ leads to polylogarithms $\Gamma(\mathfrak{w};x,y;p)$ that transform as modular tensors by the $Sp(2h,\mathbb{Z})$-invariance of $\mathcal{J}(x,p)$ established in Theorem 3.2. The expansion in either basis will be of interest, and both will be pursued in the sequel.

In order to avoid cluttering, we compactly label the polylogarithms in the expansion of (4.2) through an ordered sequence of upper and lower indices encoding the accompanying words in either the basis (a^I, b_I) or the basis $(\hat{a}^I, b_I)$,

$$\begin{aligned}\Gamma_{\dots I \dots}{}^{\dots J \dots}(x,y;p) &= \Gamma(\dots a^I \dots b_J \dots ; x,y;p) \\ \hat{\Gamma}_{\dots I \dots}{}^{\dots J \dots}(x,y;p) &= \Gamma(\dots \hat{a}^I \dots b_J \dots ; x,y;p)\end{aligned} \tag{4.4}$$

For example, for words with at most two letters in the basis (a^I, b_I), the expansion (4.2) takes the following form,

$$\begin{aligned}\mathbf{\Gamma}(x,y;p) = 1 &+ a^I \Gamma_I(x,y;p) + b_I \Gamma^I(x,y;p) + a^I a^J \Gamma_{IJ}(x,y;p) \\ &+ b_I b_J \Gamma^{IJ}(x,y;p) + a^I b_J \Gamma_I{}^J(x,y;p) + b_I a^J \Gamma^I{}_J(x,y;p) + \cdots\end{aligned} \tag{4.5}$$

while in the basis $(\hat{a}^I, b_I)$ it is given by,

$$\begin{aligned}\mathbf{\Gamma}(x,y;p) = 1 &+ \hat{a}^I \hat{\Gamma}_I(x,y;p) + b_I \hat{\Gamma}^I(x,y;p) + \hat{a}^I \hat{a}^J \hat{\Gamma}_{IJ}(x,y;p) \\ &+ b_I b_J \hat{\Gamma}^{IJ}(x,y;p) + \hat{a}^I b_J \hat{\Gamma}_I{}^J(x,y;p) + b_I \hat{a}^J \hat{\Gamma}^I{}_J(x,y;p) + \cdots\end{aligned} \tag{4.6}$$

Identifying term by term in both expansions gives the relations $\Gamma_I = \hat{\Gamma}_I$ and $\Gamma_{IJ} = \hat{\Gamma}_{IJ}$ to be established in all generality in (4.10), as well as the following relations to this order,

$$\begin{aligned}\hat{\Gamma}^I &= \Gamma^I - \pi Y^{IJ}\,\Gamma_J \\ \hat{\Gamma}^I{}_J &= \Gamma^I{}_J - \pi Y^{IK}\,\Gamma_{KJ} \\ \hat{\Gamma}_I{}^J &= \Gamma_I{}^J - \pi\,\Gamma_{IK}\,Y^{KJ} \\ \hat{\Gamma}^{IJ} &= \Gamma^{IJ} - \pi Y^{IK}\,\Gamma_K{}^J - \pi\,\Gamma^I{}_K\,Y^{KJ} + \pi^2\,Y^{IK}\,\Gamma_{KL}\,Y^{LJ}\end{aligned} \tag{4.7}$$

We have suppressed the common arguments $(x,y;p)$ of the polylogarithms in order to avoid unnecessary clutter.

4.1.1 Tangential end-point regularization and specialization to genus one

Apart from their dependence on the endpoints x,y of the integration path in (4.1) and on the moduli of Σ, the higher-genus polylogarithms defined in (4.1) also depend on the marked point p that enters the connection $\mathcal{J}(t,p)$ in (3.32). Setting $p=x$ or $p=y$ leads to endpoint divergences caused by the simple pole of $\partial_t \mathcal{G}(t,p) = -\frac{1}{t-p} + \mathcal{O}(1)$ which we shall shuffle regularize using the procedure introduced in [27]. This may be done by shifting one or the other of the endpoints in the exponent of (4.1) by $|\varepsilon| \ll 1$ along a prescribed tangent vector [67, 68, 63, 2],

$$\int_y^{x-\varepsilon} \mathcal{J}(t,p) \text{ if } p=x \qquad \text{or} \qquad \int_{y+\varepsilon}^{x} \mathcal{J}(t,p) \text{ if } p=y \tag{4.8}$$

expanding the regularized integral in powers of ε, and defining the value of the integral as the term of order zero in the expansion, thus omitting divergent terms such as $\ln(2\pi i\varepsilon)$.

At genus one, the regularization procedure of [27] leads to the elliptic polylogarithms (2.22) of Brown-Levin which are obtained by setting $y = p = 0$ in (4.1) or (4.4),

$$\Gamma(\ldots a \ldots b \ldots; x) = \Gamma_{\ldots 1 \ldots}{}^{\ldots 1 \ldots}(x,0;0)\,\big|_{h=1} \tag{4.9}$$

4.2 The special case of polylogarithms for words without b_I

The polylogarithms associated with words $\mathfrak{w}$ that do not involve any of the letters b_I are the same in the bases (a^I, b_I) and $(\hat{a}^I, b_I)$ and are given by the following simple formula,

$$\Gamma_{I_1 I_2 \cdots I_r}(x,y;p) = \hat{\Gamma}_{I_1 I_2 \cdots I_r}(x,y;p) = \int_y^x \boldsymbol{\omega}_{I_1}(t_1) \int_y^{t_1} \boldsymbol{\omega}_{I_2}(t_2) \cdots \int_y^{t_{r-1}} \boldsymbol{\omega}_{I_r}(t_r) \tag{4.10}$$

which makes clear that these polylogarithms are actually independent of the marked point p and confirms explicitly that they are modular tensors of rank r. For the case $r = 1$, we recover the basic Abelian integrals. For $r = 2$, a particular combination of the $\mathfrak{A}$-cycle monodromy in x gives the Riemann vector [50] with base point z_0,

$$\Delta_I(z_0) = -\frac{1}{2} - \frac{1}{2}\Omega_{II} + \sum_{J \neq I} \oint_{\mathfrak{A}_J} \boldsymbol{\omega}_J(z) \int_{z_0}^{z} \boldsymbol{\omega}_I \tag{4.11}$$

For $r \geq 2$, the simple subclass (4.10) of polylogarithms generalizes Abelian integrals to modular tensors of rank r. They obey the differential equations,

$$\partial_x \Gamma_{I_1 I_2 \cdots I_r}(x,y;p) = \omega_{I_1}(x) \Gamma_{I_2 \cdots I_r}(x,y;p) \tag{4.12}$$

and the simplest instance of the shuffle property (4.3) reads,

$$\Gamma_I(x,y;p) \cdot \Gamma_J(x,y;p) = \Gamma_{IJ}(x,y;p) + \Gamma_{JI}(x,y;p) \tag{4.13}$$

Specializing to genus $h = 1$ they may be evaluated explicitly,

$$\Gamma_{\underbrace{11\ldots1}_{r}}(x,y;z)\,\big|_{h=1} = \frac{1}{r!}\,(x-y)^r \tag{4.14}$$

4.3 Low letter count polylogarithms in the basis (a^I, b_I)

Using the expansion (4.5) of the path-ordered exponential $\mathbf{\Gamma}(x,y;p)$ in powers of the connection combined with the expansion of the flat connection $\mathcal{J}(x,p)$ in (3.41),

$$\begin{aligned}\mathcal{J}(x,p) = \boldsymbol{\omega}_I a^I + \pi(\boldsymbol{\omega}^I - \bar{\boldsymbol{\omega}}^I)b_I + dx \Big(& \partial_x \Phi^I{}_J(x)[b_I, a^J] - \partial_x \mathcal{G}(x,p)[b_I, a^I] \\ & + \pi \partial_x \Phi^I{}_J(x) Y^{JK}[b_I, b_K] + \partial_x \Phi^{IJ}{}_K(x)\big[b_I,[b_J,a^K]\big] \\ & - \partial_x \mathcal{G}^I(x,p)\big[b_I,[b_J,a^J]\big] + \pi \partial_x \Phi^{IJ}{}_K(x) Y^{KL}\big[b_I,[b_J,b_L]\big] + \cdots \Big)\end{aligned} \tag{4.15}$$

in terms of words in the alphabet a^I, b_I, we construct polylogarithms for words that contain the letters b_I as well as a^I. For the single-letter word b_I, we obtain,

$$\Gamma^I(x,y;p) = \pi \int_y^x (\boldsymbol{\omega}^I - \bar{\boldsymbol{\omega}}^I) \tag{4.16}$$

which is independent of p but, as expected in the basis (a^I, b_I), not a modular tensor. For double-letter words with at least one letter b_I, we obtain,

$$\begin{aligned}\Gamma^{IJ}(x,y;p) &= \pi \int_y^x \Big(dt \, \big(\partial_t \Phi^I{}_K(t) Y^{KJ} - \partial_t \Phi^J{}_K(t) Y^{KI}\big) + \pi\big(\boldsymbol{\omega}^I(t) - \bar{\boldsymbol{\omega}}^I(t)\big) \int_y^t (\boldsymbol{\omega}^J - \bar{\boldsymbol{\omega}}^J) \Big) \\ \Gamma^J{}_I(x,y;p) &= \int_y^x \Big(dt \, \partial_t \Phi^J{}_I(t) - dt \, \partial_t \mathcal{G}(t,p) \delta^J_I + \pi\big(\boldsymbol{\omega}^J(t) - \bar{\boldsymbol{\omega}}^J(t)\big) \int_y^t \boldsymbol{\omega}_I \Big) \\ \Gamma_I{}^J(x,y;p) &= \int_y^x \Big(-dt \, \partial_t \Phi^J{}_I(t) + dt \, \partial_t \mathcal{G}(t,p) \delta^J_I + \pi \boldsymbol{\omega}_I(t) \int_y^t (\boldsymbol{\omega}^J - \bar{\boldsymbol{\omega}}^J) \Big)\end{aligned} \tag{4.17}$$

The entry Γ^{IJ} and the off-diagonal components of $\Gamma^J{}_I$ and $\Gamma_I{}^J$ are independent on p.

4.3.1 Simplified representations

The polylogarithms in (4.16) and (4.17) with upper indices admit simplified representations in terms of (4.10), their complex conjugates and contractions with Y^{IJ}. For words with a single letter b_I we have,

$$\Gamma^I(x,y;p) = \pi Y^{IJ}\big(\Gamma_J(x,y;p) - \overline{\Gamma_J(x,y;p)}\big) \tag{4.18}$$

while for two-letter words that contain at least one b_I, we have,

$$\Gamma_I{}^J(x,y;p) = \pi Y^{JK}\Gamma_{IK}(x,y;p) + \int_y^x dt\,\Big(-\partial_t\Phi^J{}_I(t) + \delta_I^J\partial_t\mathcal{G}(t,p) - \pi\omega_I(t)Y^{JK}\overline{\Gamma_K(t,y;p)}\Big)$$
$$\Gamma^I{}_J(x,y;p) = \pi Y^{IK}\big(\Gamma_{KJ}(x,y;p) - \Gamma_J(x,y;p)\overline{\Gamma_K(x,y;p)}\big) + \int_y^x dt\,\Big(\partial_t\Phi^I{}_J(t) - \delta_J^I\partial_t\mathcal{G}(t,p) + \pi\omega_J(t)Y^{IK}\overline{\Gamma_K(t,y;p)}\Big)$$
$$\Gamma^{IJ}(x,y;p) = \pi^2 Y^{IK}Y^{JL}\Big(\Gamma_{KL}(x,y;p) + \overline{\Gamma_{KL}(x,y;p)} - \overline{\Gamma_K(x,y;p)}\Gamma_L(x,y;p)\Big) + \pi\int_y^x dt\,\Big(\partial_t\Phi^I{}_K(t)Y^{KJ} - \partial_t\Phi^J{}_K(t)Y^{KI} + \pi\omega^J(t)Y^{IK}\overline{\Gamma_K(t,y;p)} - \pi\omega^I(t)Y^{JK}\overline{\Gamma_K(t,y;p)}\Big) \tag{4.19}$$

These expressions already illustrate the general fact that the indices on these polylogarithms cannot be simply raised or lowered by contraction with Y_{IJ} or its inverse. Indeed, $\Gamma^I(x,y;p)$ in (4.18) is not obtained by contracting $\Gamma_J(x,y;p)$ with Y^{IJ}.

Homotopy-invariance of the higher-genus polylogarithms in (4.18) and (4.19) can be seen directly from the fact that the integrands with respect to t entering $\Gamma_I{}^J(x,y;p), \Gamma^I{}_J(x,y;p)$ and $\Gamma^{IJ}(x,y;p)$ are meromorphic in t. The vanishing of the respective $\partial_{\bar t}$-derivatives can be checked via (3.7), (3.23) and (4.12). The shuffle relations $\Gamma^I\cdot\Gamma^J = \Gamma^{IJ}+\Gamma^{JI}$ and $\Gamma^I\cdot\Gamma_J = \Gamma^I{}_J + \Gamma_J{}^I$ in (4.3) are easily verified from the expressions in (4.18) and (4.19).

4.4 Low letter count polylogarithms in the basis $(\hat a^I, b_I)$

The polylogarithms $\hat\Gamma(x,y;p)$ in the basis $(\hat a^I, b_I)$ defined by the expansion (4.6) are modular tensors by the $Sp(2h,\mathbb{Z})$ invariance of the connection $\mathcal{J}(x,p)$ established in Theorem 3.2. For words involving only $\hat a^I$ letters and no b_I letters, the expressions were already given in (4.10). For the general case, it will be convenient to introduce the following expansion of the generating function $\Psi_I(x,p;B)$,

$$\Psi_J(x,p;B) = \omega_J(x) + \sum_{r=1}^\infty B_{I_1}\cdots B_{I_r}\, f^{I_1\cdots I_r}{}_J(x,p)$$
$$f^{I_1\cdots I_r}{}_J(x,p) = \partial_x\Phi^{I_1\cdots I_r}{}_J(x) - \partial_x\mathcal{G}^{I_1\cdots I_{r-1}}(x,p)\delta_J^{I_r} \tag{4.20}$$

Expanding the connection $\mathcal{J}(x,p)$ accordingly,

$$\mathcal{J}(x,p) = -\pi\bar{\boldsymbol{\omega}}^I(x)b_I + \Big(\boldsymbol{\omega}_I(x)\hat a^I + f^J{}_I(x,p)[b_J,\hat a^I] + f^{JK}{}_I(x,p)[b_J,[b_K,\hat a^I]] + \cdots\Big) \tag{4.21}$$

we compute the polylogarithms for a one- and two-letter words, starting with the coefficient of the letter b_I,

$$\hat{\Gamma}^I(x,y;p) = -\pi \int_y^x \bar{\boldsymbol{\omega}}^I = -\pi Y^{IK}\, \overline{\Gamma_K(x,y;p)} \tag{4.22}$$

This example illustrates that, also in the $(\hat{a}^I, b_I)$ basis, indices of polylogarithms $\hat{\Gamma}$ cannot be raised or lowered via Y^{IJ} or Y_{IJ}. For two-letter words that contain at least one letter b_I, the simplest examples (4.19) translate into,

$$\begin{aligned}
\hat{\Gamma}^{IJ}(x,y;p) &= \pi^2 \int_y^x \bar{\boldsymbol{\omega}}^I(t_1) \int_y^{t_1} \bar{\boldsymbol{\omega}}^J = \pi^2 Y^{IK} Y^{JL}\, \overline{\Gamma_{KL}(x,y;p)} \\
\hat{\Gamma}_I{}^J(x,y;p) &= -\int_y^x dt \left(f^J{}_I(t,p) + \pi\, \omega_I(t) \int_y^t \bar{\boldsymbol{\omega}}^J \right) \\
\hat{\Gamma}^I{}_J(x,y;p) &= \int_y^x dt \left(f^I{}_J(t,p) + \pi\, \omega_J(t) \int_y^t \bar{\boldsymbol{\omega}}^I \right) - \pi Y^{IK}\, \overline{\Gamma_K(x,y;p)}\, \Gamma_J(x,y;p)
\end{aligned} \tag{4.23}$$

under the conversion (4.7) between the (a^I, b_I) and $(\hat{a}^I, b_I)$ basis. These examples line up with the general transformation law of higher-genus polylogarithms $\hat{\Gamma}$ in the expansion (4.6) as modular tensors with holomorphic automorphy factors $R^{I'}{}_I$ and $Q^J{}_{J'}$ in (3.13),

$$\tilde{\hat{\Gamma}}_{\cdots I \cdots}{}^{\cdots J \cdots}(x,y;p) = \cdots R^{I'}{}_I \cdots Q^J{}_{J'} \cdots \hat{\Gamma}_{\cdots I' \cdots}{}^{\cdots J' \cdots}(x,y;p) \tag{4.24}$$

4.4.1 Genus-one illustration of the modular properties

We shall now illustrate how modularity is realized in the last two polylogarithms in equation (4.23) specialized to genus one. Upon specializing the expansion (4.6) and transformation law (4.24) to genus one, any elliptic polylogarithm $\hat{\Gamma}_{\cdots I \cdots}{}^{\cdots J \cdots}(x,y;p)|_{h=1}$ with m uppercase indices and n lowercase indices transforms as a modular form of $SL(2,\mathbb{Z})$ with weight $(m{-}n,0)$. In particular, the genus-one incarnation of the second example in (4.23),

$$\hat{\Gamma}_I{}^J(x,y;p)\,\big|_{h=1} = \tilde{\Gamma}\big(\tfrac{1}{p};y|\tau\big) - \tilde{\Gamma}\big(\tfrac{1}{p};x|\tau\big) + \frac{\pi}{\mathrm{Im}\,\tau}\Big[\tfrac{1}{2}(y{-}p)^2 - \tfrac{1}{2}(x{-}p)^2 + (\bar{y}{-}\bar{p})(x{-}y)\Big] \tag{4.25}$$

is modular invariant by this counting. Indeed, $SL(2,\mathbb{Z})$-invariance of the term $\sim \frac{(\bar{y}-\bar{p})(x-y)}{\mathrm{Im}\,\tau}$ is manifest whereas modular invariance of the leftover expression $\sim \frac{(y-p)^2}{2\mathrm{Im}\,\tau} - \frac{(x-p)^2}{2\mathrm{Im}\,\tau}$ and

$$\tilde{\Gamma}\big(\tfrac{1}{p};y|\tau\big) - \tilde{\Gamma}\big(\tfrac{1}{p};x|\tau\big) = \ln\vartheta(y{-}p|\tau) - \ln\vartheta(x{-}p|\tau) \tag{4.26}$$

relies on cancellations between the respective $SL(2,\mathbb{Z})$-transformations obtained from (2.19).

For the endpoint divergences of higher-genus polylogarithms $\hat{\Gamma}_{\cdots I \cdots}{}^{\cdots J \cdots}(x,y;p)$ at $p=x$ or $p=y$, the regularization prescription (4.8) necessitates the specification of a tangent vector. It remains to be investigated which choices of tangent vectors preserve the tensorial modular transformation of $\hat{\Gamma}_{\cdots I \cdots}{}^{\cdots J \cdots}(x,y;p)$ at $p \in \{x,y\}$.

4.5 Meromorphic variants

The first example of the higher-genus polylogarithms in (4.19) can be written as,

$$\begin{aligned}\Gamma_I{}^J(x,y;p) &= \Gamma(a^I b_J;x,y;p) \\ &= \int_y^x dt\,\Big(-\partial_t\Phi^J{}_I(t) + \delta_I^J\partial_t\mathcal{G}(t,p) + \pi\omega_I(t)Y^{JK}\big(\Gamma_K(t,y;p) - \overline{\Gamma_K(t,y;p)}\big)\Big)\end{aligned} \tag{4.27}$$

Upon specializing to genus $h=1$ and setting $p=y=0$, this reproduces the Brown-Levin polylogarithm $\Gamma(ab;p|\tau) = -\tilde\Gamma\big(\begin{smallmatrix}1\\0\end{smallmatrix};p|\tau\big)$ in (2.26), namely the integral over the meromorphic kernel $g^{(1)}(t|\tau)$ in (2.25). Accordingly, one may view the integrand with respect to t in the second line of (4.27) as a higher-genus uplift of the Kronecker-Eisenstein kernel $g^{(1)}(t|\tau)$,

$$g^J{}_I(t,y;p) = \partial_t\Phi^J{}_I(t) - \delta_I^J\partial_t\mathcal{G}(t,p) - 2\pi i\omega_I(t)Y^{JK}\,\mathrm{Im}\int_y^t \boldsymbol{\omega}_K \tag{4.28}$$

Indeed, the Laplace equations (3.7) and (3.23) of $\mathcal{G}(t,p)$ and $\Phi^J{}_I(t)$ readily imply meromorphicity in t,

$$\partial_{\bar t}g^J{}_I(t,y;p) = \pi\delta_I^J\delta(t,p) \tag{4.29}$$

verifying homotopy invariance of (4.27). However, (4.28) is not meromorphic in the endpoint y of the integration path or the second argument p of the flat connection. At genus $h=1$, setting $y\to p$ readily reduces $g^J{}_I(t,y;p)$ to the kernel $g^{(1)}(t-y|\tau)$ meromorphic in both t and y. Starting from genus $h\geq 2$, by contrast, (4.28) at $y=p$ does not yield a meromorphic function of two points t,y on Σ since

$$\partial_{\bar y}\big(g^J{}_I(t,y;p)\,\big|_{y=p}\big) = -\pi\delta_I^J\delta(t,y) + \pi\Big(\delta_I^J\omega_K(t)\bar\omega^K(y) - \omega_I(t)\bar\omega^J(y)\Big) \tag{4.30}$$

Corrections of $g^J{}_I(t,y;p)\,\big|_{y=p}$ by abelian integrals $\omega_I(t)\mathrm{Im}\int_y^t\boldsymbol{\omega}^J$ or $\delta_I^J\omega_K(t)\mathrm{Im}\int_y^t\boldsymbol{\omega}^K$ do not suffice to attain simultaneous meromorphicity in t and y. Still, one can add combinations of abelian integrals $\omega_I(t)\mathrm{Im}\int_p^y\boldsymbol{\omega}^J$ and $\delta_I^J\omega_K(t)\mathrm{Im}\int_p^y\boldsymbol{\omega}^K$ to render $g^J{}_I(t,y;p)$ meromorphic in both t and y at the cost of a separate dependence on a third marked point p.

One can similarly take the Brown-Levin polylogarithms $\Gamma(ab\cdots b;p|\tau) = (-1)^n\tilde\Gamma\big(\begin{smallmatrix}n\\0\end{smallmatrix};p|\tau\big)$ in (2.26) with $n\geq 2$ letters b as a starting point to motivate higher-genus analogues of the meromorphic kernels $g^{(n\geq 2)}$ in (2.23), e.g.

$$\begin{aligned}
{\Gamma_I}^{JK}(x,y;p) &= \Gamma(a^I b_J b_K;x,y;p) \\
&= \int_y^x dt \,\Big\{ \partial_t {\Phi^{KJ}}_I(t) - \delta_I^J \partial_t \mathcal{G}^K(t,p) - \pi \partial_t {\Phi^J}_I(t) Y^{KL}\big(\Gamma_L(t,y;p) - \overline{\Gamma_L(t,y;p)}\big) \\
&\qquad + \pi \delta_I^J \partial_t \mathcal{G}(t,z) Y^{KL}\big(\Gamma_L(t,y;p) - \overline{\Gamma_L(t,y;p)}\big) \\
&\qquad + \pi^2 \omega_I(t) Y^{JL} Y^{KM}\big(\Gamma_{LM}(t,y;p) - \overline{\Gamma_L(t,y;z)}\Gamma_M(t,y;p) + \overline{\Gamma_{LM}(t,y;p)}\big) \\
&\qquad + \pi \omega_I(t) \int_y^t dt' \big(\partial_{t'} {\Phi^J}_L(t') Y^{LK} - \pi \omega^J(t') Y^{KL} \overline{\Gamma_L(t',y;p)} - (J \leftrightarrow K)\big) \Big\}
\end{aligned} \tag{4.31}$$

Again, the integrand on the right-hand side is meromorphic in t and may be viewed as a higher-genus generalization ${g^{KJ}}_I$ of $g^{(2)}$. However, simultaneous meromorphicity in t and y cannot be attained without admitting dependences on additional marked points z. Hence, our construction does not suggest any straightforward generalizations of the meromorphic Kronecker-Eisenstein coefficients $g^{(n)}(t-y|\tau)$ to higher genus which meromorphically depend on two points t, y on the surface without any reference to additional points. Instead, the tensors $\partial_t {\Phi^{I_1\cdots I_r}}_J(t)$ and $\partial_t \mathcal{G}^{I_1\cdots I_s}(t,y)$ of section 3.3 naturally generalize the doubly-periodic Kronecker-Eisenstein kernels $f^{(n)}(t-y|\tau)$ to arbitrary genus.

The above examples motivate the study of gauge transformations $U(x,p)$, whose action on the connection is given by,

$$\tilde{\mathcal{J}}(x,p) = U(x,p)\mathcal{J}(x,p)U(x,p)^{-1} + \big(dU(x,p)\big)U(x,p)^{-1} \tag{4.32}$$

and which induce the following transformation on the path-ordered exponential,

$$\tilde{\mathbf{\Gamma}}(x,y;p) = U(x,p)\mathbf{\Gamma}(x,y;p)U(y,p)^{-1} \tag{4.33}$$

such that $\tilde{\mathbf{\Gamma}}(x,y;p)$ is meromorphic in x and y to all orders in a^I and b_J. Based on the generalized abelian integrals (4.10) and their complex conjugates, it is not difficult to construct a gauge transformation that implements meromorphicity in x and yields generating series of the elliptic polylogarithms (2.24) upon specialization to genus one. Refined choices of $U(x,p)$ that additionally preserve the vanishing of $\mathfrak{A}$-cycle monodromies of $\mathcal{J}(x,p)$ in x and make contact with the meromorphic connections in the work of Enriquez [44] and Enriquez-Zerbini [45, 46] are currently under investigation.

4.6 Closure under taking primitives

The closure of elliptic polylogarithms under taking primitives crucially hinges on translation invariance at genus one and the Fay identity among Kronecker-Eisenstein kernels [22, 27, 26].

We shall now present evidence that a similar closure holds for products of the higher-genus polylogarithms in (4.1) with the integration kernels in the connection (3.32). As we will see below, the two key mechanisms for closure under taking primitives are the interchange lemma (3.20) and Fay identities at higher genus.

4.6.1 Implications of the interchange lemma

Given that the path-ordered exponential in (4.1) only involves integrals over the first argument t of the connection $\mathcal{J}(t,p)$ in the integrand, the primitive of $\partial_t \mathcal{G}(t,p)\omega_I(p)$ with respect to p is not obvious from the definition of higher-genus polylogarithms. However, the interchange lemma (3.20) yields an alternative representation of $\partial_t \mathcal{G}(t,p)\omega_I(p)$ where its primitive with respect to p can be readily found in terms of higher-genus polylogarithms via (4.12) and (4.18),

$$\begin{aligned}\partial_t \mathcal{G}(t,p)\omega_J(p) &= \partial_p \Phi^I{}_J(p)\omega_I(t) + \partial_t \Phi^I{}_J(t)\omega_I(p) - \partial_p \mathcal{G}(p,t)\omega_J(t) \\ &= \partial_p \Big(-\Gamma_J{}^I(p,t;t)\omega_I(t) + \Gamma_I(p,t;t)\partial_t \Phi^I{}_J(t) \\ &\quad + \pi \omega_I(t) Y^{IK} \big[\Gamma_{JK}(p,t;t) - \Gamma_J(p,t;t)\overline{\Gamma_K(p,t;t)} \big] \Big)\end{aligned} \tag{4.34}$$

Similarly, convolutions of the interchange lemma (3.20) with Arakelov Green functions allow to rewrite higher-rank expressions $\partial_t \mathcal{G}^{I_1 I_2 \cdots I_s}(t,p)\omega_J(p)$ as (see section 5 of [69] for a proof),

$$\begin{aligned}\partial_t \mathcal{G}^{J_1 J_2 \cdots J_s}(t,p)\omega_I(p) &= (-1)^{s-1} \partial_p \mathcal{G}^{J_s \cdots J_2 J_1}(p,t)\omega_I(t) \\ &\quad + \Bigg\{ \sum_{i=1}^{s} (-1)^i \partial_t \Phi^{J_1 J_2 \cdots J_{s-i} K}{}_I(t) \partial_p \Phi^{J_s J_{s-1} \ldots J_{s-i+1}}{}_K(p) \\ &\qquad + \partial_t \Phi^{J_1 J_2 \cdots J_s K}{}_I(t)\omega_K(p) + (-1)^s \Big({}^{\;\;t \leftrightarrow p}_{J_1 J_2 \cdots J_s \leftrightarrow J_s \cdots J_2 J_1} \Big) \Bigg\}\end{aligned} \tag{4.35}$$

where the primitives of the right-hand side with respect to p are accessible from the higher-genus polylogarithms in (4.1) and their complex conjugates. The instruction to add $(-1)^s$ times the image under the simultaneous relabelling $t \leftrightarrow p$ and $J_1 J_2 \cdots J_s \leftrightarrow J_s \cdots J_2 J_1$ applies to both the second and the third line of (4.35).

4.6.2 Towards higher-genus Fay identities

In order to find the primitive of

$$\omega_I(1)\partial_2 \mathcal{G}(2,1)\partial_3 \mathcal{G}(3,1) = \omega_I(z_1)\partial_{z_2} \mathcal{G}(z_2,z_1)\partial_{z_3} \mathcal{G}(z_3,z_1) \tag{4.36}$$

with respect to z_1, the interchange lemmas (3.20) and (4.35) need to be augmented by higher-genus generalizations of the genus-one Fay identity,

$$\begin{aligned}\Omega(z_1,\alpha_1|\tau)\Omega(z_2,\alpha_2|\tau) &= \Omega(z_1,\alpha_1{+}\alpha_2|\tau)\Omega(z_2{-}z_1,\alpha_2|\tau)\\ &\quad + \Omega(z_2,\alpha_1{+}\alpha_2|\tau)\Omega(z_1{-}z_2,\alpha_1|\tau)\end{aligned} \tag{4.37}$$

of the Kronecker-Eisenstein series (2.9). The need for identities beyond the scope of interchange lemmas can be seen from the appearance of z_1 in two factors $\partial_2\mathcal{G}(2,1)$ and $\partial_3\mathcal{G}(3,1)$ of (4.36) which persists after trading $\partial_i\mathcal{G}(i,1)\omega_I(1)$ for $\partial_1\mathcal{G}(1,i)\omega_I(i)$ and Φ-tensors via (3.20). As a first example of higher-genus Fay identities, the expression (4.36) can be rewritten as,

$$\begin{aligned}\omega_I(1)\partial_2\mathcal{G}(2,1)\partial_3\mathcal{G}(3,1) = \Big\{&-\partial_1\mathcal{G}(1,3)\partial_2\mathcal{G}(2,3)\omega_I(3) - \omega_K(3)\partial_1\mathcal{G}^K(1,2)\omega_I(2)\\ &+\omega_K(3)\partial_1\Phi^{KL}{}_I(1)\omega_L(2) + \omega_K(3)\partial_1\Phi^K{}_L(1)\partial_2\Phi^L{}_I(2)\\ &+(z_2\leftrightarrow z_3)\Big\} + \omega_K(1)\partial_2\partial_3 V_I^K(2,3)\end{aligned} \tag{4.38}$$

such that each term on the right-hand side features only one z_1-dependent factor and can be readily integrated over z_1 via higher-genus polylogarithms (4.1). In the last line of (4.38), we encounter the convolution,

$$\begin{aligned}\partial_x\partial_y V_I^K(x,y) &= \int_\Sigma d^2z\,\partial_x\mathcal{G}(x,z)\bar\omega^K(z)\omega_I(z)\partial_y\mathcal{G}(y,z)\\ &= \partial_y\Phi^K{}_L(y)\partial_x\Phi^L{}_I(x) + \partial_y\Phi^{KL}{}_I(y)\omega_L(x) - \partial_y\mathcal{G}^K(y,x)\omega_I(x)\end{aligned} \tag{4.39}$$

which has been reduced to the integration kernels of (3.32) in passing to the second line. The derivation of (4.38) is based on the fact that the first three lines of the right-hand side have the same anti-holomorphic derivative with respect to z_1,z_2,z_3 as the left-hand side. Moreover, both sides of (4.38) vanish upon multiplication by $\bar\omega^P(1)\bar\omega^Q(2)\bar\omega^R(3)$ and integrating all of z_1,z_2,z_3 over the surface which excludes the addition of holomorphic terms.

At genus one, (4.38) reduces to the Fay identity,

$$f^{(1)}(z_1{-}z_2|\tau)f^{(1)}(z_2{-}z_3|\tau) + f^{(2)}(z_1{-}z_3|\tau) + \mathrm{cycl}(z_1,z_2,z_3) = 0 \tag{4.40}$$

at the $\alpha_1^0\alpha_2^0$ order of (4.37) through the identifications (3.22) as well as

$$\partial_x\partial_y V_I^K(x,y)\,\big|_{h=1} = f^{(2)}(x{-}y|\tau) \tag{4.41}$$

Similar Fay identities should reduce higher-rank analogues $\omega_I(1)\partial_2\mathcal{G}^{I_1\ldots I_s}(2,1)\partial_3\mathcal{G}^{J_1\ldots J_{s'}}(3,1)$ of (4.36) to convolutions of Arakelov Green functions whose primitives with respect to z_1 are determined by the differential equations of higher-genus polylogarithms. This is expected since products of convolutions of Arakelov Green functions span a rich function space which offers multiple distinct expressions with the same anti-holomorphic derivatives as $\omega_I(1)\partial_2\mathcal{G}^{I_1\cdots I_s}(2,1)\partial_3\mathcal{G}^{J_1\cdots J_{s'}}(3,1)$. By iteratively simplifying these alternative solutions to the differential equations with respect to $\bar{z}_i$ via interchange lemmas and lower-rank Fay identities, the z_1-dependence can be eventually arranged to occur through a single factor of either $\omega_I(1), \partial_1\Phi^{I_1\cdots I_r}{}_J(1)$ or $\partial_1\mathcal{G}^{I_1\cdots I_s}(1,i)$ in each term.

The detailed form of higher-rank Fay identities at arbitrary genus and their proof can be found in [69].

4.7 Higher-genus associators

The Drinfeld associator for genus zero evaluates monodromy properties of solutions to the Knizhnik-Zamolodchikov equation with a connection given in (2.5) [70, 71]. The associator was generalized to genus one in [72, 73, 74] using the meromorphic connection $\mathcal{J}_{\rm E}(z|\tau)$, and the associated Knizhnik-Zamolodchikov-Bernard equation,

$$\partial_z F(z|\tau) = \mathcal{J}_{\rm E}(z|\tau)F(z|\tau) \qquad\qquad \mathcal{J}_{\rm E}(z|\tau) = -dz\,\frac{\vartheta_1(z+{\rm ad}_b|\tau){\rm ad}_b}{\vartheta_1(z|\tau)\vartheta_1({\rm ad}_b|\tau)}(a) \tag{4.42}$$

Since the connection satisfies $\mathcal{J}_{\rm E}(z+1|\tau) = \mathcal{J}_{\rm E}(z|\tau)$ and $\mathcal{J}_{\rm E}(z+\tau|\tau) = e^{-2\pi i\,{\rm ad}_b}\mathcal{J}_{\rm E}(z|\tau)$ the functions $F(z|\tau), F(z+1|\tau)$ and $e^{2\pi ib}F(z+\tau|\tau)$ satisfy the same differential equation in z. The solutions are normalized in [73] by their behavior as $z\to 0$. As a result, their Wronskians are given as follows,[7]

$$\begin{aligned}\Phi_{\mathfrak{A}}(\tau) &= F(z|\tau)^{-1}F(z+1|\tau)\\ \Phi_{\mathfrak{B}}(\tau) &= F(z|\tau)^{-1}e^{2\pi ib}F(z+\tau|\tau)\end{aligned} \tag{4.43}$$

are independent of z, as the notation $\Phi_{\mathfrak{A}}(\tau)$ and $\Phi_{\mathfrak{B}}(\tau)$ indeed suggests, and are referred to as the elliptic associators introduced in [73]. Equivalently, the solution $F(z|\tau)$ may be expressed in terms of the path-ordered exponential of the connection $\mathcal{J}_{\rm E}$ in (4.42),

$$F(z|\tau) = \left({\rm P}\exp\int_{z_0}^{z}\mathcal{J}_{\rm E}(t|\tau)\right)F(z_0|\tau) \tag{4.44}$$

where t is the integration parameter and z_0 is a reference point chosen such that F satisfies the normalization as $z\to 0$ introduced in [73]. In terms of this formulation, we obtain the

[7] The associators $\Phi_{\mathfrak{A}}, \Phi_{\mathfrak{B}}$ and $\Phi_{\mathfrak{A}_I}, \Phi_{\mathfrak{B}_I}$ in this section are not to be confused with the modular tensors $\Phi^I{}_J(x)$ and $\Phi^{I_1\cdots I_r}{}_J(x)$ defined by (3.19) and (3.21).

following expressions for the elliptic associators,

$$\Phi_{\mathfrak{A}}(\tau) = F(z|\tau)^{-1} \left(\mathrm{P} \exp \int_z^{z+1} \mathcal{J}_{\mathrm{E}}(t|\tau) \right) F(z|\tau)$$
$$\Phi_{\mathfrak{B}}(\tau) = F(z|\tau)^{-1} e^{2\pi i b} \left(\mathrm{P} \exp \int_z^{z+\tau} \mathcal{J}_{\mathrm{E}}(t|\tau) \right) F(z|\tau) \tag{4.45}$$

Using the properties of the path-ordered exponential, it may be verified immediately that both expressions are independent of z. A parallel construction may be given in terms of the non-meromorphic but doubly periodic Brown-Levin connection $\mathcal{J}_{\mathrm{BL}}$ defined in (2.7), in which case the factor $e^{2\pi i b}$ should be omitted. It is this non-meromorphic version that we shall propose to generalize to higher genus.

Based on the polylogarithms that we have constructed in (4.1), we are led to a natural proposal for *higher-genus associators*, which generalize the elliptic associators reviewed above. They may be defined by evaluating the path-ordered exponential of (4.1) around the homology cycles of the Riemann surface. Those integrals, in turn, are generated by the path-ordered exponentials of (4.1) around the basis of homology cycles $\mathfrak{A}_I$ and $\mathfrak{B}_I$, and defined in close analogy with the construction given in (4.45). We introduce a slight variant of the generating function for higher-genus polylogarithms defined in (4.1),

$$F(x; p|\Omega) = \left(\mathrm{P} \exp \int_{x_0}^{x} \mathcal{J}(t; p|\Omega) \right) F(x_0; p|\Omega) = \mathbf{\Gamma}(x, x_0; p|\Omega) F(x_0; p|\Omega) \tag{4.46}$$

Here, we have exhibited the dependence on the moduli of the higher-genus Riemann surface Σ in terms of the period matrix Ω so that the presentation is as close as possible to the genus-one case (4.44), and retained the dependence on the point p explicitly. Since the connection $\mathcal{J}(t; p|\Omega)$ is single-valued on Σ, the functions $F(x; p|\Omega)$, $F(x + \mathfrak{A}_I; p|\Omega)$ and $F(x + \mathfrak{B}_I; p|\Omega)$ satisfy the same differential equation $d_x F(x; p|\Omega) = \mathcal{J}(x; p|\Omega) F(x; p|\Omega)$. As a result, the following combinations are independent of x,

$$\Phi_{\mathfrak{A}_I}(p|\Omega) = F(x; p|\Omega)^{-1} \left(\mathrm{P} \exp \int_x^{x+\mathfrak{A}_I} \mathcal{J}(t; p|\Omega) \right) F(x; p|\Omega)$$
$$\Phi_{\mathfrak{B}_I}(p|\Omega) = F(x; p|\Omega)^{-1} \left(\mathrm{P} \exp \int_x^{x+\mathfrak{B}_I} \mathcal{J}(t; p|\Omega) \right) F(x; p|\Omega) \tag{4.47}$$

A significant difference between the proposal for higher-genus associators made here and the elliptic associators of Enriquez is the dependence on the point p on Σ. Also, it remains to be investigated whether our proposal satisfies the general axioms enunciated for associators, or whether these axioms can be relaxed and generalized to the case of higher-genus polylogarithms. Moreover, it would be interesting to relate our proposal to the operad-theory approach to higher-genus associators in [75]. We shall return to these open questions in future work, see section 7.

5 Flat connection in the multiple variable case

In this section, we further generalize the construction of the flat connection and polylogarithms on higher-genus Riemann surfaces to an arbitrary number of marked points, thereby generalizing the multi-variable genus-one polylogarithms of Brown and Levin to higher genera. This generalized flat connection may be formulated entirely in terms of the modular tensors needed for the single-variable case, just as was the case for genus one.

The genus-one connection $\mathcal{J}_{\rm BL}$ in (2.7) may be generalized to depend on n additional marked points $z_1, \cdots, z_n$ on the torus, and is then given as follows by [22],

$$\begin{aligned}\mathcal{J}_{\rm BL}(z_1,\cdots,z_n;z|\tau) = & \frac{\pi}{\operatorname{Im}\tau}(dz-d\bar z)\, b + dz\ \operatorname{ad}_b \Omega(z,\operatorname{ad}_b|\tau)\, a \\ & + dz\sum_{i=1}^n \big(\Omega(z-z_i,\operatorname{ad}_b|\tau) - \Omega(z,\operatorname{ad}_b|\tau)\big)\, c_i \end{aligned} \tag{5.1}$$

We now have a Lie algebra $\mathcal{L}_n$ that is freely generated by the elements a, b of section 2.2 and additional elements $c_1, \cdots, c_n$ associated with the marked points $z_1, \cdots, z_n$. The generalization to higher genus is given by the following theorem.

Theorem 5.1 *The connection $\mathcal{J}_{\rm mv}$ is defined on a compact Riemann surface Σ of arbitrary genus h and with $n+1$ marked points $p, z_1, \cdots, z_n \in \Sigma$ as follows,*

$$\mathcal{J}_{\rm mv}(z_1,\cdots,z_n;x,p) = \mathcal{J}(x,p) + \mathcal{J}_n(z_1,\cdots,z_n;x,p) \tag{5.2}$$

The connection $\mathcal{J}(x,p)$ for the single-variable case was constructed in Theorem 3.1 and is repeated here for convenience, while the multi-variable addition $\mathcal{J}_n$ is given by,

$$\begin{aligned}\mathcal{J}(x,p) &= -\pi d\bar x\, \bar\omega^I(x)\, b_I + \pi dx\, \mathcal{H}^I(x;B)\, b_I + dx\, \Psi_I(x,p;B)\, a^I \\ \mathcal{J}_n(z_1,\cdots,z_n;x,p) &= -dx\sum_{i=1}^n \Big(\mathcal{H}(x,z_i;B) - \mathcal{H}(x,p;B)\Big)\, c_i \end{aligned} \tag{5.3}$$

The connection $\mathcal{J}_{\rm mv}$ takes values in a Lie algebra $\mathcal{L}_{\rm mv}$ that is freely generated by the elements a^I, b_I with $I = 1, \cdots, h$ as seen in section 3.5 and additional elements c_i with $i = 1, \cdots, n$ associated with the marked points z_i. The connection $\mathcal{J}_{\rm mv}$ is flat away from the points $p, z_1, \cdots, z_n$,

$$d\mathcal{J}_{\rm mv} - \mathcal{J}_{\rm mv}\wedge\mathcal{J}_{\rm mv} = \pi d\bar x\wedge dx\bigg(\delta(x,p)\Big(\big[b_I, a^I\big] - \sum_{i=1}^n c_i\Big) + \sum_{i=1}^n \delta(x,z_i)\, c_i\bigg) \tag{5.4}$$

and reduces to the multi-variable Brown-Levin connection (5.1) in the genus-one case. Alternatively, one may express the connection $\mathcal{J}(x,p)$ in terms of the basis $(\hat a^I, b_I)$ with tensorial *modular transformations. This leads to a modular invariant multi-variable connection (5.2) if the generators c_i are taken to be $Sp(2h, \mathbb{Z})$-invariant.*

To prove the theorem, we begin by showing that the connection $\mathcal{J}_{\rm mv}$ in (5.2) and (5.3) reduces to $\mathcal{J}_{\rm BL}$ for $h = 1$. Indeed, the differences between the generating series $\mathcal{H}$ in $\mathcal{J}_n$ reduce to differences between the Kronecker-Eisenstein series inside the sum in (5.1). In particular, by expanding the components, it is easy to see from (3.22) that we have,

$$\mathcal{H}(x, z_i; B) - \mathcal{H}(x, p; B) \Big|_{h=1} = -\Omega(x - z_i, \mathrm{ad}_b|\tau) + \Omega(x, \mathrm{ad}_b|\tau) \tag{5.5}$$

upon setting $p \to 0$ on the torus and identifying $a^1 = a$ as well as $b_1 = b$.

Furthermore, to prove flatness of $\mathcal{J}_{\rm mv}$ away from the points y, z_i, we use the flatness condition of the connection $\mathcal{J}$ in Theorem 3.1 to obtain the following relation,

$$d\mathcal{J}_{\rm mv} - \mathcal{J}_{\rm mv} \wedge \mathcal{J}_{\rm mv} = d\mathcal{J}_n - \mathcal{J} \wedge \mathcal{J}_n - \mathcal{J}_n \wedge \mathcal{J} + \pi d\bar{x} \wedge dx\, \delta(x, p)[b_I, a^I] \tag{5.6}$$

The remaining contributions may be worked out using (3.28) as follows,

$$\begin{aligned} d\mathcal{J}_n &= -d\bar{x} \wedge dx \sum_{i=1}^{n} \Big(\partial_{\bar{x}} \mathcal{H}(x, z_i; B) - \partial_{\bar{x}} \mathcal{H}(x, p; B) \Big)\, c_i \\ &= \pi d\bar{x} \wedge dx \sum_{i=1}^{n} \Big(\delta(x, z_i) - \delta(x, p) + \bar{\omega}^I(x) B_I \big(\mathcal{H}(x, z_i; B) - \mathcal{H}(x, p; B) \big) \Big)\, c_i \end{aligned} \tag{5.7}$$

Similarly, the wedge products may be worked out as follows,

$$\mathcal{J} \wedge \mathcal{J}_n + \mathcal{J}_n \wedge \mathcal{J} = \pi d\bar{x} \wedge dx\, \bar{\omega}^I(x) \Big[b_I, \sum_{i=1}^{n} \Big(\mathcal{H}(x, z_i; B) - \mathcal{H}(x, p; B) \Big)\, c_i \Big] \tag{5.8}$$

Flatness of $\mathcal{J}_{\rm mv}$ then follows in view of $B_I \mathcal{H}(x, p; B) c_i = [b_I, \mathcal{H}(x, p; B) c_i]$.

Finally, modular invariance of $\mathcal{J}(x, p), \mathcal{H}(x, z_i; B) - \mathcal{H}(x, p; B)$ and c_i readily carries over to the multi-variable connection $\mathcal{J}_{\rm mv}$ in (5.2) and (5.3).

Multi-variable generalizations of the higher-genus polylogarithms in section 4.1 may now be constructed from the flat connection $\mathcal{J}_{\rm mv}$ following the methods presented in section 4.

6 Separating degeneration

In this section, we shall study the behavior of the generating functions and flat connection at higher genus under separating degenerations of the genus-h Riemann surface Σ. To keep the discussion as simple and concrete as possible, we shall specialize here to the case of genus two degenerating to two genus-one surfaces connected by a long funnel. The generalization of this construction to the non-separating degeneration of higher-genus Riemann surfaces may be found in section 3 of [51], while the separating, non-separating, and tropical degenerations for genus two are presented in detail in [54].

6.1 The construction for genus two

A convenient parametrization of the neighborhood of the separating divisor is provided by the funnel construction given in Fay's book [50], and specifically for genus two in [54]. Here we shall give a simplified presentation that will suffice for the problem at hand. For genus two, the starting point of the construction of Σ is provided by the compact genus-one surfaces Σ_1 and Σ_2, to which we add punctures p_1 and p_2, respectively. Next, we introduce a system of local complex coordinates $(x_1, \bar{x}_1)$ and $(x_2, \bar{x}_2)$ on each surface, and denote the coordinates of the punctures simply by p_1 and p_2. We specify a disc $\mathfrak{D}_1$ centered at p_1 on Σ_1 and a disc $\mathfrak{D}_2$ centered at p_2 on Σ_2, as shown in Figure 2.

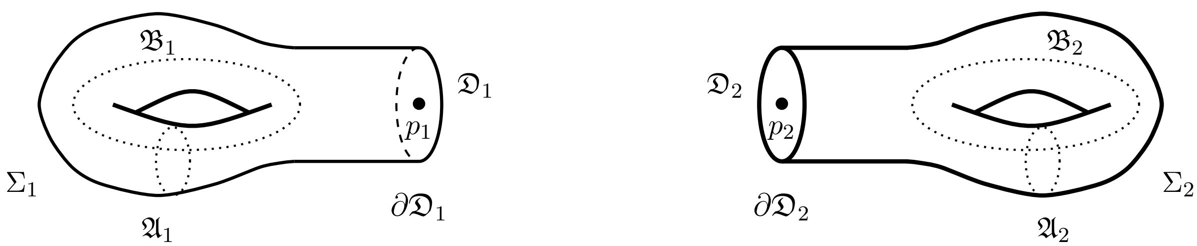

Figure 2: Funnel construction of a family of genus-two Riemann surfaces Σ near the separating divisor in terms of genus-one surfaces Σ_1 and Σ_2. The circles $\partial\mathfrak{D}_1$ and $\partial\mathfrak{D}_2$ are centered at the punctures p_1 and p_2 and bound the discs $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. The surface Σ is constructed from the surfaces $\Sigma_1 \setminus \mathfrak{D}_1$ and $\Sigma_2 \setminus \mathfrak{D}_2$ by identifying $\partial\mathfrak{D}_1$ and $\partial\mathfrak{D}_2$.

The genus-two surface Σ is obtained by identifying annuli surrounding $\partial\mathfrak{D}_1$ and $\partial\mathfrak{D}_2$ with respective local complex coordinates x_1 and x_2 via the relation (more complete discussions of the construction are given in [50, 54]),

$$(x_1 - p_1)(x_2 - p_2) = v_s \tag{6.1}$$

Here v_s is a complex parameter governing the separating degeneration (which is referred to as t in [50]) and is such that the separating degeneration corresponds to the limit $v_s \to 0$. Customarily, the curves $\partial\mathfrak{D}_1, \partial\mathfrak{D}_2$ are defined to be circles in the local complex coordinates on the surfaces but here instead we shall use a more intrinsic definition,

$$\begin{aligned}\partial\mathfrak{D}_1 &= \{x_1 \in \Sigma_1 \text{ such that } g(x_1 - p_1|\tau) = t_1\} \\ \partial\mathfrak{D}_2 &= \{x_2 \in \Sigma_2 \text{ such that } g(x_2 - p_2|\sigma) = t_2\}\end{aligned} \tag{6.2}$$

where the scalar Green function $g(z|\tau) = g_1(z|\tau)$ on the torus was defined in (2.12). The moduli τ and σ of the genus-one surfaces Σ_1 and Σ_2 in (6.2) are given by the diagonal entries of the period matrix Ω of the genus-two Riemann surface Σ which will be parametrized by,

$$\Omega = \begin{pmatrix} \tau & v \\ v & \sigma \end{pmatrix} \tag{6.3}$$

Note that, for sufficiently large values of t_1, t_2, each level-set $\partial\mathfrak{D}_1, \partial\mathfrak{D}_2$ in (6.2) is connected.

In the separating degeneration limit, the diagonal entries τ, σ are kept fixed as $v \to 0$. The relation between the off-diagonal entry v and v_s is linear and will be derived shortly in (6.10) below. The relation between the parameters t_1, t_2, and v_s is obtained as follows,

$$t_1 + t_2 = -\ln|2\pi v_s\, \eta(\tau)^2 \eta(\sigma)^2|^2 + \mathcal{O}(v_s^2) \tag{6.4}$$

Here, we have used the short-distance expansion of the scalar Green function on the torus,

$$g(z|\tau) = -\ln|2\pi z\, \eta(\tau)^2|^2 + \mathcal{O}(z^2) \tag{6.5}$$

to convert (6.1) into the expression above.

When performing integrals over the genus-two surface Σ, it will be convenient to decompose the integral into a sum of the contribution from $\Sigma_1 \setminus \mathfrak{D}_1$ plus the contribution from $\Sigma_2 \setminus \mathfrak{D}_2$, where the curves $\partial\mathfrak{D}_1, \partial\mathfrak{D}_2$ are defined such that $t_1 = t_2 = t$. Under these conditions, the Abelian differentials ω_1 and ω_2 remain uniformly bounded throughout Σ by a constant of order $\mathcal{O}(v_s^0)$, with corrections which are suppressed by powers of v_s.

Assuming that we set $t_1 = t_2$, we can evaluate the dependence on v of the radii of the coordinate discs $\mathfrak{D}_1, \mathfrak{D}_2$ in the limit where v is small. Using the asymptotic expression (6.5) for the Green functions, we have for $x_1 \in \mathfrak{D}_1$ and $x_2 \in \mathfrak{D}_2$,

$$|x_1 - p_1| = |x_2 - p_2| = |2\pi v_s \eta(\tau)^2 \eta(\sigma)^2|^{\frac{1}{2}} \tag{6.6}$$

Thus, in the limit where $v_s \to 0$, the coordinate areas of the coordinate discs tend to zero linearly in v_s and thus linearly in v, as we shall establish below.

6.2 Degeneration of Abelian differentials

We choose canonical homology bases $\mathfrak{A}_1, \mathfrak{B}_1 \subset \Sigma_1 \setminus \{p_1\}$ and $\mathfrak{A}_2, \mathfrak{B}_2 \subset \Sigma_2 \setminus \{p_2\}$ as in Figure 2, and extend those to a canonical homology basis for Σ. The genus-one holomorphic Abelian differentials $\hat{\omega}_1$ and $\hat{\omega}_2$ on Σ_1 and Σ_2, respectively, are normalized as follows,

$$\oint_{\mathfrak{A}_1} \hat{\omega}_1 = \oint_{\mathfrak{A}_2} \hat{\omega}_2 = 1 \qquad \oint_{\mathfrak{B}_1} \hat{\omega}_1 = \tau \qquad \oint_{\mathfrak{B}_2} \hat{\omega}_2 = \sigma \tag{6.7}$$

To construct holomorphic one-forms on the genus-two surface Σ with the period matrix Ω parametrized in (6.3) we extend $\hat{\omega}_1$ to a differential ω_1 on Σ and $\hat{\omega}_2$ to a differential ω_2 on Σ by using the identification (6.1). Choosing complex coordinates x_1, x_2 on Σ_1 and Σ_2 such that $\hat{\omega}_1 = dx_1$ and $\hat{\omega}_2 = dx_2$, we see that the differential dx_1 extends to $-v_s/(x_2-p_2)^2 dx_2$ in Σ_2 while the differential dx_2 extends to $-v_s/(x_1-p_1)^2 dx_1$ in Σ_1. Thus, the extensions are governed by meromorphic one-forms with a double pole. The meromorphic one-forms $\varpi(x_1, y_1|\tau)$ and $\varpi(x_2, y_2|\sigma)$ on Σ_1 and Σ_2, respectively, are normalized to have vanishing $\mathfrak{A}$-periods and a double pole of unit strength at $x_1 = y_1$ and $x_2 = y_2$. Their $\mathfrak{B}$-periods are given by the Riemann bilinear relations,

$$\oint_{\mathfrak{B}_1} \varpi(x_1, y_1|\tau) = 2\pi i \hat{\omega}_1(y_1) \qquad \oint_{\mathfrak{B}_2} \varpi(x_2, y_2|\sigma) = 2\pi i \hat{\omega}_2(y_2) \tag{6.8}$$

The holomorphic one-forms ω_1 and ω_2 on the genus-two surface Σ, canonically normalized on $\mathfrak{A}_1$ and $\mathfrak{A}_2$-cycles, are then given as follows,[8]

$$\begin{aligned} \omega_1(x) &\to \begin{cases} \hat{\omega}_1(x_1) & x_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \\ v\, \varpi(x_2, p_2|\sigma)/(2\pi i \hat{\omega}_2(p_2)) & x_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \end{cases} \\ \omega_2(x) &\to \begin{cases} v\, \varpi(x_1, p_1|\tau)/(2\pi i \hat{\omega}_1(p_1)) & x_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \\ \hat{\omega}_2(x_2) & x_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \end{cases} \end{aligned} \tag{6.9}$$

The parameter v_s is related to the off-diagonal entry v of the genus-two period matrix Ω of (6.3) by,

$$v = \oint_{\mathfrak{B}_1} \omega_2 = \oint_{\mathfrak{B}_2} \omega_1 = -2\pi i\, v_s\, \hat{\omega}_1(p_1)\, \hat{\omega}_2(p_2) \tag{6.10}$$

The expressions in (6.9) are valid up to corrections of order $\mathcal{O}(v^2)$ which have been omitted.

---

[8] Here and below, we shall use the notation x for a point on the genus-two surface Σ, and set it equal to x_1 when the point lies in the genus-one component $\Sigma_1 \setminus \mathfrak{D}_1$ and to x_2 when it lies in the component $\Sigma_2 \setminus \mathfrak{D}_2$.

6.3 Degeneration of the Arakelov Green function

The separating degeneration of the Arakelov Green function $\mathcal{G}$ on the funnel construction of genus-two surfaces Σ in Figure 2 is given by,

$$\mathcal{G}(x,y) \to \begin{cases} -\frac{1}{2}\ln|\hat{v}| + g(x_1-y_1|\tau) - \frac{1}{2}g(x_1-p_1|\tau) - \frac{1}{2}g(y_1-p_1|\tau) & x_1,y_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \\ -\frac{1}{2}\ln|\hat{v}| + g(x_2-y_2|\sigma) - \frac{1}{2}g(x_2-p_2|\sigma) - \frac{1}{2}g(y_2-p_2|\sigma) & x_2,y_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \\ +\frac{1}{2}\ln|\hat{v}| + \frac{1}{2}g(x_1-p_1|\tau) + \frac{1}{2}g(y_2-p_2|\sigma) & x_1 \in \Sigma_1 \setminus \mathfrak{D}_1,\ y_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \\ +\frac{1}{2}\ln|\hat{v}| + \frac{1}{2}g(x_2-p_2|\sigma) + \frac{1}{2}g(y_1-p_1|\tau) & x_2 \in \Sigma_2 \setminus \mathfrak{D}_2,\ y_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \end{cases} \tag{6.11}$$

where $\hat{v}$ is related to the entries v, τ, σ of the genus-two period matrix (6.3) by the Dedekind eta-function η,

$$\hat{v} = 2\pi\, v\, \eta(\tau)^2 \eta(\sigma)^2 \tag{6.12}$$

The combinations that will enter the flat connection and the generating functions of genus-two polylogarithms are,

$$\partial_x \mathcal{G}(x,y) \to \begin{cases} -f^{(1)}(x_1-y_1|\tau) + \frac{1}{2}f^{(1)}(x_1-p_1|\tau) & x_1,y_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \\ -f^{(1)}(x_2-y_2|\sigma) + \frac{1}{2}f^{(1)}(x_2-p_2|\sigma) & x_2,y_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \\ -\frac{1}{2}f^{(1)}(x_1-p_1|\tau) & x_1 \in \Sigma_1 \setminus \mathfrak{D}_1\,,\ y_2 \in \Sigma_2 \setminus \mathfrak{D}_2 \\ -\frac{1}{2}f^{(1)}(x_2-p_2|\sigma) & x_2 \in \Sigma_2 \setminus \mathfrak{D}_2\,,\ y_1 \in \Sigma_1 \setminus \mathfrak{D}_1 \end{cases} \tag{6.13}$$

Conveniently, the dependence on $\ln|\hat{v}|$ cancels out in these derivative combinations.

6.4 Degeneration of $\partial_x \mathcal{G}^{I_1\cdots I_s}(x,y)$

We begin with the separating degenerations of the convolutions $\partial_x \mathcal{G}^{I_1\cdots I_s}(x,y)$ of Arakelov Green functions defined in (3.21) that carry the dependence of the flat connection on the second marked point y,

$$\begin{aligned} \partial_x \mathcal{G}^{I_1\cdots I_s}(x,y) &= \int_\Sigma d^2z\, \partial_x \mathcal{G}(x,z) \overline{\omega}^{I_1}(z) \partial_z \mathcal{G}^{I_2\cdots I_s}(z,y) \\ &= \partial_x \mathcal{G}_{(1)}^{I_1\cdots I_s}(x,y) + \partial_x \mathcal{G}_{(2)}^{I_1\cdots I_s}(x,y) \end{aligned} \tag{6.14}$$

where the second equality is obtained by decomposing $\Sigma = \big(\Sigma_1 \setminus \mathfrak{D}_1\big) \cup \big(\Sigma_2 \setminus \mathfrak{D}_2\big)$ with,

$$\mathcal{G}_{(i)}^{I_1\cdots I_s}(x,y) = \int_{\Sigma_i \setminus \mathfrak{D}_i} d^2 z_i\, \mathcal{G}(x,z_i)\, \overline{\omega}^{I_1}(z_i)\, \partial_{z_i} \mathcal{G}^{I_2\cdots I_s}(z_i,y) \qquad i=1,2 \tag{6.15}$$

The asymptotics is given by the following lemma.

Lemma 6.1 *The functions $\partial_x \mathcal{G}^{I_1 \cdots I_s}(x,y)$ have the following separating degenerations, up to order $\mathcal{O}(v)$ which vanishes as $v \to 0$,*

$$
\begin{aligned}
\partial_{x_1} \mathcal{G}^{1_s}(x_1, y_1) &\to -f^{(s+1)}(x_1 - y_1|\tau) + \tfrac{1}{2} f^{(s+1)}(x_1 - p_1|\tau) \\
\partial_{x_1} \mathcal{G}^{1_s}(x_1, y_2) &\to -\tfrac{1}{2} f^{(s+1)}(x_1 - p_1|\tau) \\
\partial_{x_1} \mathcal{G}^{S}(x_1, y_i) &\to 0 \qquad\qquad \text{if } 2 \in S \qquad\qquad i = 1,2 \\
\\
\partial_{x_2} \mathcal{G}^{2_s}(x_2, y_2) &\to -f^{(s+1)}(x_2 - y_2|\sigma) + \tfrac{1}{2} f^{(s+1)}(x_2 - p_2|\sigma) \\
\partial_{x_2} \mathcal{G}^{2_s}(x_2, y_1) &\to -\tfrac{1}{2} f^{(s+1)}(x_2 - p_2|\sigma) \\
\partial_{x_2} \mathcal{G}^{S}(x_2, y_i) &\to 0 \qquad\qquad \text{if } 1 \in S \qquad\qquad i = 1,2
\end{aligned}
\tag{6.16}
$$

where $1_s = 1 \cdots 1$ with s entries and similarly for 2_s, and where S stands for an arbitrary array $I_1 \cdots I_s$ with each I_i taking values 1 or 2.

The proof proceeds by induction on s. For $s = 0$, only the cases on the first, second, fourth and fifth lines arise, and they are given by (6.13). We shall prove the validity of the formulas of (6.16) for $s = 1$ in Appendix B, as these cases involve some detailed analyses. The recursive definitions of (6.15) may then be used to show that the $v \to 0$ limit of any tensor $\partial_x \mathcal{G}^{I_1 \cdots I_s}(x,y)$ vanishes unless $I_1 = \cdots = I_s = 1$ or $I_1 = \cdots I_s = 2$. In these cases, it is readily shown that $\partial_x \mathcal{G}^{1_s}_{(2)}(x,y)$ and $\partial_x \mathcal{G}^{2_s}_{(1)}(x,y)$ have vanishing limits, while $\partial_x \mathcal{G}^{1_s}_{(1)}(x,y)$ and $\partial_x \mathcal{G}^{2_s}_{(2)}(x,y)$ produce the remaining contributions to (6.16).

6.5 Degeneration of $\partial_x \Phi^{I_1 \cdots I_s}{}_J(x)$

It will be convenient to rewrite the recursive definition (3.21) of the functions $\partial_x \Phi^{I_1 \cdots I_s}{}_J(x)$ in terms of $\partial_x \mathcal{G}^{I_1 \cdots I_{s-1}}(x,y)$ whose separating degeneration was already evaluated in the preceding subsection,

$$
\partial_x \Phi^{I_1 \cdots I_s}{}_J(x) = \int_\Sigma d^2 z \, \partial_x \mathcal{G}^{I_1 \cdots I_{s-1}}(x,z) \, \bar\omega^{I_s}(z) \omega_J(z) \tag{6.17}
$$

One may split up the integration into the contributions from the surfaces $\Sigma_1 \setminus \mathfrak{D}_1$ and $\Sigma_2 \setminus \mathfrak{D}_2$,

$$
\partial_x \Phi^{I_1 \cdots I_s}{}_J(x) = \big(\partial_x \Phi_{(1)}\big)^{I_1 \cdots I_s}{}_J(x) + \big(\partial_x \Phi_{(2)}\big)^{I_1 \cdots I_s}{}_J(x) \tag{6.18}
$$

where,

$$
\big(\Phi_{(i)}\big)^{I_1 \cdots I_s}{}_J(x) = \int_{\Sigma_i \setminus \mathfrak{D}_i} d^2 z_i \, \mathcal{G}^{I_1 \cdots I_{s-1}}(x, z_i) \, \bar\omega^{I_s}(z_i) \, \omega_J(z_i) \qquad i = 1,2 \tag{6.19}
$$

The analysis parallels the one carried out for $\partial_x \mathcal{G}^{I_1 \cdots I_s}(x,y)$, uses the results of Lemma 6.1, and produces the following lemma, stated without proof.

Lemma 6.2 *The separating degenerations of the functions $\partial_x \Phi^{I_1 \cdots I_s}{}_J(x)$ are given by,*

$$\begin{aligned} &\partial_{x_1} \Phi^{1_s}{}_1(x_1) \to \tfrac{1}{2} f^{(s)}(x_1 - p_1|\tau) && \partial_{x_1} \Phi^{1_{s-1}2}{}_2(x_1) \to -\tfrac{1}{2} f^{(s)}(x_1 - p_1|\tau) \\ &\partial_{x_2} \Phi^{2_s}{}_2(x_2) \to \tfrac{1}{2} f^{(s)}(x_2 - p_2|\sigma) && \partial_{x_2} \Phi^{2_{s-1}1}{}_1(x_2) \to -\tfrac{1}{2} f^{(s)}(x_2 - p_2|\sigma) \end{aligned} \tag{6.20}$$

where again $1_s = 1 \cdots 1$ with s entries and similarly for 2_s. All other components such as $\partial_{x_i} \Phi^{\cdots 1}{}_2(x_i), \partial_{x_i} \Phi^{\cdots 2}{}_1(x_i)$ with $i = 1, 2$ or $\partial_{x_2} \Phi^{1_s}{}_1(x_2)$ with $s \geq 2$ vanish. The degenerations of (6.20) are consistent with the traceless property $\Phi^{I_1 \cdots I_s}{}_{I_s} = 0$.

6.6 Degeneration of the flat connection

The separating degenerations of the modular tensors $\partial_x \mathcal{G}^{I_1 \cdots I_s}(x, y)$ and $\partial_x \Phi^{I_1 \cdots I_s}{}_J(x)$ for the special case of genus-two surfaces may now be used to evaluate the separating degeneration limit of the flat connections discussed in Theorem 3.1 and Theorem 5.1.

We begin by assembling the limits of the generating function $\mathcal{H}(x, y; B)$ in the first line of (3.27) with $B_I = \mathrm{ad}_{b_I}$,

$$\begin{aligned} \mathcal{H}(x_1, y_1) &\to \sum_{n=1}^{\infty} \Big(-f^{(n)}(x_1 - y_1|\tau) + \tfrac{1}{2} f^{(n)}(x_1 - p_1|\tau) \Big) B_1^{n-1} \\ \mathcal{H}(x_2, y_2) &\to \sum_{n=1}^{\infty} \Big(-f^{(n)}(x_2 - y_2|\sigma) + \tfrac{1}{2} f^{(n)}(x_2 - p_2|\sigma) \Big) B_2^{n-1} \\ \mathcal{H}(x_1, y_2) &\to -\frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_1 - p_1|\tau) B_1^{n-1} \\ \mathcal{H}(x_2, y_1) &\to -\frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_2 - p_2|\sigma) B_2^{n-1} \end{aligned} \tag{6.21}$$

as well as the components $I = 1, 2$ of $\mathcal{H}_I(x; B)$ in the second line of (3.27),

$$\begin{aligned} \mathcal{H}_1(x_1) &\to 1 + \frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_1 - p_1|\tau) B_1^n \\ \mathcal{H}_2(x_2) &\to 1 + \frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_2 - p_2|\sigma) B_2^n \\ \mathcal{H}_1(x_2) &\to -\frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_2 - p_2|\sigma) B_2^{n-1} B_1 \\ \mathcal{H}_2(x_1) &\to -\frac{1}{2} \sum_{n=1}^{\infty} f^{(n)}(x_1 - p_1|\tau) B_1^{n-1} B_2 \end{aligned} \tag{6.22}$$

Note that the rightmost generator in both of $\mathcal{H}_I(x_1)$ and $\mathcal{H}_I(x_2)$ is given by B_I in each term of (6.22). As a consequence, the contribution $\sim \mathcal{H}^I(x;B)\, b_I$ to the flat connection (3.32) at genus two reduces to $\omega^I(x) b_I$ under separating degenerations in view of $Y^{IJ} B_I b_J = 0$.

By the composition (3.29) of $\Psi_I(x,y)$, these results immediately imply the vanishing of two cases for its components and arguments,

$$\begin{aligned}
&\Psi_1(x_2,y_1) \to 0 \\
&\Psi_2(x_1,y_2) \to 0
\end{aligned} \tag{6.23}$$

Assembling the components $\Psi_I(x,y)$ with non-vanishing limits we find,

$$\begin{aligned}
\Psi_1(x_1,y_1) &\to 1 + \sum_{n=1}^{\infty} f^{(n)}(x_1-y_1|\tau) B_1^n = B_1\, \Omega(x_1-y_1, B_1|\tau) \\
\Psi_2(x_2,y_2) &\to 1 + \sum_{n=1}^{\infty} f^{(n)}(x_2-y_2|\sigma) B_2^n = B_2\, \Omega(x_2-y_2, B_2|\sigma) \\
\Psi_1(x_1,y_2) &\to 1 + \sum_{n=1}^{\infty} f^{(n)}(x_1-p_1|\tau) B_1^n = B_1\, \Omega(x_1-p_1, B_1|\tau) \\
\Psi_2(x_2,y_1) &\to 1 + \sum_{n=1}^{\infty} f^{(n)}(x_2-p_2|\sigma) B_2^n = B_2\, \Omega(x_2-p_2, B_2|\sigma) \\
\Psi_1(x_2,y_2) &\to \sum_{n=1}^{\infty} \Big(f^{(n)}(x_2-y_2|\sigma) - f^{(n)}(x_2-p_2|\sigma) \Big) B_2^{n-1} B_1 \\
&\qquad = \big(\Omega(x_2-y_2, B_2|\sigma) - \Omega(x_2-p_2, B_2|\sigma) \big) B_1 \\
\Psi_2(x_1,y_1) &\to \sum_{n=1}^{\infty} \Big(f^{(n)}(x_1-y_1|\tau) - f^{(n)}(x_1-p_1|\tau) \Big) B_1^{n-1} B_2 \\
&\qquad = \big(\Omega(x_1-y_1, B_1|\tau) - \Omega(x_1-p_1, B_1|\tau) \big) B_2
\end{aligned} \tag{6.24}$$

Our results may be summarized in the form of the following theorem.

Theorem 6.3 *To evaluate the separating degeneration of the flat connection in Theorem 3.1 at genus two, we may pick, without loss of generality, a point $y = y_1 \in \Sigma_1 \setminus \mathfrak{D}_1$. The components of the connection then enjoy the following asymptotics,*

$$\begin{aligned}
\mathcal{J}(x_1,y_1) &\to \frac{\pi}{\operatorname{Im} \tau} (dx_1 - d\bar{x}_1)\, b_1 + dx_1\, \mathrm{ad}_{b_1} \Omega\big(x_1-y_1, \mathrm{ad}_{b_1}|\tau\big)\, a^1 \\
&\qquad + dx_1 \Big(\Omega(x_1-y_1, \mathrm{ad}_{b_1}|\tau) - \Omega(x_1-p_1, \mathrm{ad}_{b_1}|\tau) \Big) [b_2, a^2] \\
\mathcal{J}(x_2,y_1) &\to \frac{\pi}{\operatorname{Im} \sigma} (dx_2 - d\bar{x}_2)\, b_2 + dx_2\, \mathrm{ad}_{b_2} \Omega\big(x_2-p_2, \mathrm{ad}_{b_2}|\sigma\big)\, a^2
\end{aligned} \tag{6.25}$$

The proof proceeds by inserting (6.22), (6.23) and (6.24) into the genus-two instance of the flat connection (3.32).

We conclude this section with a number of remarks.

1. Upon choosing $y_1 = p_1$ the second line in the expression (6.25) for $\mathcal{J}(x_1, y_1)$ cancels. With this assumption, the connection $\mathcal{J}(x, y)$ with $y = y_1 = p_1 = 0$ reduces to the genus-one Brown-Levin connection (2.11) on a torus Σ_1 when the point $x = x_1$ is on the surface $\Sigma_1 \setminus \{p_1\}$.
2. At distinct points $y_1 \neq p_1$ and $y_1 = 0$, in turn, the separating degeneration of $\mathcal{J}(x_1, y_1)$ in (6.25) yields the Brown-Levin connection (5.1) with one extra marked point $z_1 = p_1$, and free Lie algebra generator $c_1 = [a^2, b_2]$.
3. Finally, when the point $x = x_2$ is on the surface $\Sigma_2 \setminus \{p_2\}$, the genus-two connection $\mathcal{J}(x_2, y_1)$ reduces to the Brown-Levin connection on Σ_2 with a shift by p_2 in the Kronecker-Eisenstein series of (2.11).
4. The same methods determine the separating degeneration of the multi-variable addition $\mathcal{J}_n(z_1, \cdots, z_n; x, y)$ to the flat connection at genus two in (5.3). Once the extra punctures are distributed via $z_1, z_2, \cdots, z_m \in \Sigma_1 \setminus \mathfrak{D}_1$ and $z_{m+1}, \cdots, z_n \in \Sigma_2 \setminus \mathfrak{D}_2$, there are two inequivalent cases to consider: for $x = x_1$ and $y = y_1$ on the same surface $\Sigma_1 \setminus \mathfrak{D}_1$, we obtain the multi-variable additions (5.1) to the genus-one connection

$$\mathcal{J}_n(z_1, \cdots, z_n; x_1, y_1) \to dx_1 \sum_{i=1}^{m} \big(\Omega(x_1 - z_i, B_1|\tau) - \Omega(x_1 - y_1, B_1|\tau)\big)\, c_i \\ + dx_1 \sum_{i=m+1}^{n} \big(\Omega(x_1 - p_1, B_1|\tau) - \Omega(x_1 - y_1, B_1|\tau)\big)\, c_i \tag{6.26}$$

involving the generators $c_1, \cdots, c_n$ associated with the additional punctures $z_1, \cdots, z_n$ on both surfaces. If $x = x_1 \in \Sigma_1 \setminus \mathfrak{D}_1$ and $y = y_2 \in \Sigma_2 \setminus \mathfrak{D}_2$ are chosen to be on different surfaces, however, the generators $c_{m+1}, \cdots, c_n$ associated with the punctures $z_{m+1}, \cdots, z_n \in \Sigma_2 \setminus \mathfrak{D}_2$ on a different surface than x_1 are absent,

$$\mathcal{J}_n(z_1, \cdots, z_n; x_1, y_2) \to dx_1 \sum_{i=1}^{m} \big(\Omega(x_1 - z_i, B_1|\tau) - \Omega(x_1 - p_1, B_1|\tau)\big)\, c_i \tag{6.27}$$

7 Conclusions and further directions

In this work, we have presented an explicit construction of polylogarithms on compact Riemann surfaces of arbitrary genus. Generalizing the approach of Brown and Levin for the genus-one case, our construction relies on a flat connection whose path-ordered exponential plays the role of a generating series for higher-genus polylogarithms and manifests their homotopy-invariance. The flat connection takes values in the freely-generated Lie algebra introduced by Enriquez and Zerbini in [45] and is assembled from convolutions on higher-genus surfaces of Arakelov Green functions and their derivatives. Our construction furnishes the first explicit proposal for a "complete" set of integration kernels beyond genus one: the higher-genus polylogarithms constructed here are conjectured to close under taking primitives with respect to the points on the surface.

While our construction of higher-genus polylogarithms builds on the Brown-Levin connection at genus one, it also draws heavily on the structure of families of higher-genus modular tensors. We illustrate the importance of these tensorial building blocks and their properties in several examples of higher-genus polylogarithms, introduce a basis of polylogarithms which themselves transform as modular tensors and provide non-trivial evidence for their closure under taking primitives. Moreover, upon separating degeneration of the Riemann surface, our flat connection reduces to flat connections on the degeneration components. We illustrate this result here for the case of genus two, thereby paving the way for systematic investigations into the rich network of relations among the higher-genus polylogarithms that result from different types of degenerations.

It is expected that the higher-genus polylogarithms constructed in this work will find broad applications in theoretical physics, ranging from multi-loop amplitudes in string theory to Feynman integrals in quantum field theory and beyond. At the same time, our results offer a concrete forward leap towards a coherent theory of integration on arbitrary Riemann surfaces and should prove relevant to questions in number theory and algebraic geometry. Among the myriad of mathematical and physical open problems raised by our construction, the following questions readily qualify for tractable follow-up research:

(i) proving the conjecture advanced here that higher-genus polylogarithms close under taking primitives, based on the generalizations to all orders of the interchange lemma and Fay identities discussed in section 4.6 which are provided and proven in [69];

(ii) obtaining the separating and non-separating degenerations of the polylogarithms for arbitrary genera, exploiting the properties of the Arakelov Green function in [51, 54];

(iii) determining the differential relations with respect to moduli variations satisfied by higher-genus polylogarithms and their integration kernels using complex-structure deformation theory [53, 76];

(iv) identifying generalizations of the (non-holomorphic) higher-genus modular graph tensors in [56, 57, 58] that close under complex-structure variations and degenerations;

(v) exploring the properties of the higher-genus associators proposed in section 4.7, thereby generalizing the studies of elliptic associators introduced in [72, 73, 74];

(vi) re-formulation of higher-genus string amplitudes in terms of the integration kernels and polylogarithms constructed in this work, a program that was foreshadowed long ago by the cohomological analysis of chiral blocks for the case of genus two in [77] and implemented more recently at genus one in [27, 31, 78].

We plan to report progress on some of these topics in future work.

A Definiton of the prime form

Given that the Arakelov Green function is constructed from the prime form $E(x,y)$ in (3.9) and (3.11), we shall briefly review the definition of the prime form in this appendix. For this purpose, we introduce the Riemann ϑ-function of rank h,

$$\vartheta[\kappa](\zeta|\Omega) = \sum_{n\in\mathbb{Z}^h} e^{\pi i(n+\kappa')^t\Omega(n+\kappa')+2\pi i(n+\kappa')^t(\zeta+\kappa'')} \tag{A.1}$$

where $\zeta \in \mathbb{C}^h$ and where the characteristics $\kappa = [\kappa', \kappa'']$ comprises two h-component vectors $\kappa', \kappa'' \in \mathbb{C}^h$. Following the main text, we will henceforth suppress the dependence on the period matrix Ω.

The prime form at arbitrary genus h is built from a specialization of κ to odd half-characteristics or spin structures $\nu = [\nu', \nu'']$ with entries $\in \{0, \frac{1}{2}\}$ such that $4\nu' \cdot \nu''$ is odd [50]:

$$E(x,y) = \frac{\vartheta[\nu]\big(\int_y^x \boldsymbol{\omega}\big)}{h_\nu(x)h_\nu(y)} \tag{A.2}$$

By virtue of the holomorphic $(\frac{1}{2},0)$-forms $h_\nu(x)$ in the denominator subject to

$$h_\nu(x)^2 = \sum_{I=1}^{h} \boldsymbol{\omega}_I(x)\frac{\partial}{\partial \zeta_I}\vartheta[\nu](\zeta)\Big|_{\zeta=0} \tag{A.3}$$

the definition (A.2) of the prime form is independent on the choice of the odd spin structure ν. Since the ϑ-functions entering the prime form are odd under the flip $\zeta \to -\zeta$ in (A.1), the prime form is antisymmetric $E(y,x) = -E(x,y)$ under exchange of x, y and in particular exhibits the short-distance behaviour $E(x,y) = (x-y) + \mathcal{O}\big((x-y)^3\big)$ in local complex coordinates.

B Separating degeneration of $\partial_x \mathcal{G}^I(x,y)$

In this appendix, we derive the separating degeneration of the functions $\partial_x \mathcal{G}^I(x,y)$ for genus two with $I = 1, 2$. These derivations are essential components of Lemma 6.1. Actually, it will be convenient to derive first the asymptotic behavior in the separating degeneration of the modular tensor with a lowered index, defined by $\partial_x \mathcal{G}_I(x,y) = Y_{IJ}\, \partial_x \mathcal{G}^J(x,y)$, and then reconvert the result to $\partial_x \mathcal{G}^J(x,y)$.

B.1 Degeneration of $\partial_{x_1} \mathcal{G}_1(x_1, y_1)$ and $\partial_{x_2} \mathcal{G}_2(x_2, y_2)$

Following the decomposition of (6.14), we evaluate the contributions $\mathcal{G}_1^{(1)}(x,y)$ and $\mathcal{G}_2^{(2)}(x,y)$ by substituting the degeneration limits of the various factors in its integrand,

$$\mathcal{G}_1^{(1)}(x_1, y_1) = \int_{\Sigma_1 \setminus \mathfrak{D}_1} d^2 z_1 \Big[g(x_1 - z_1|\tau) - \tfrac{1}{2} g(x_1 - p_1|\tau) - \tfrac{1}{2} g(z_1 - p_1|\tau) - \tfrac{1}{2} \ln |\hat{v}| \Big] \\ \times \Big[\partial_{z_1} g(z_1 - y_1|\tau) - \tfrac{1}{2} \partial_{z_1} g(z_1 - p_1|\tau) \Big] \tag{B.1}$$

The integral may be extended to all of Σ_1 where it is absolutely convergent and differs from the original integral by a contribution that is proportional to the coordinate volume, which is of order $\mathcal{O}(v)$ in view of (6.6), and will be neglected. As an integral over Σ_1, the contributions of the z_1-independent terms inside the first bracket vanish in view of,

$$\int_{\Sigma_1} d^2 z_1\, \partial_{z_1} \Big(g(z_1 - y_1|\tau) \Big)^n = 0 \qquad\qquad n \geq 0 \tag{B.2}$$

Evaluating the remaining integrals using the concatenated Green functions g_n of (2.14), and taking the derivative in x_1, gives the following limit,

$$\partial_{x_1} \mathcal{G}_1^{(1)}(x_1, y_1) \to (\operatorname{Im} \tau) \Big(\partial_{x_1}^2 g_2(x_1 - y_1|\tau) - \tfrac{1}{2} \partial_{x_1}^2 g_2(x_1 - p_1|\tau) \Big) \tag{B.3}$$

The overall factor of $\operatorname{Im} \tau$ arises from the integrand in the definition of g_2 in (2.14).

We proceed analogously for $\partial_{x_1} \mathcal{G}_I^{(2)}(x_1, y_1)$,

$$\mathcal{G}_1^{(2)}(x_1, y_1) = \frac{1}{4} \overline{\left(\frac{v}{2\pi i \hat{\omega}_2(p_2)} \right)} \int_{\Sigma_2 \setminus \mathfrak{D}_2} d^2 z_2 \Big[g(x_1 - p_1|\tau) + g(z_2 - p_2|\sigma) + \ln |\hat{v}| \Big] \\ \times \overline{\partial_{z_2} \partial_{p_2} \ln \vartheta_1(z_2 - p_2|\sigma)}\, \partial_{z_2} g(z_2 - p_2|\sigma) \tag{B.4}$$

On the face of it, this contribution is automatically suppressed by a factor of v. However, the integral does not extend to a convergent integral over Σ_1, so part or all of the suppression might in principle be cancelled. To proceed, we extract the leading singularity near the puncture in terms of a contour integral over $\partial\mathfrak{D}_1$ which may be evaluated exactly in the limit where the size of the disc $\mathfrak{D}_1$ is small. First, we show that the combination,

$$\mathcal{I} = \int_{\Sigma_2\setminus\mathfrak{D}_2} d^2 z_2 \, \overline{\partial_{z_2}\partial_{p_2} \ln \vartheta_1(z_2 - p_2|\sigma)} \, \partial_{z_2} g(z_2 - p_2|\sigma) \tag{B.5}$$

remains bounded as $v \to 0$ so that its contribution to $\mathcal{G}_1^{(2)}(x_1, y_1)$ vanishes in this limit. To do so we write it as follows,

$$\begin{aligned} \mathcal{I} = &- \int_{\Sigma_2\setminus\mathfrak{D}_2} d^2 z_2 \, \overline{\partial_{p_2} \ln \vartheta_1(z_2 - p_2|\sigma)} \, \partial_{\bar z_2}\partial_{z_2} g(z_2 - p_2|\sigma) \\ &- \frac{i}{2} \int_{\Sigma_2\setminus\mathfrak{D}_2} d\Big[dz_2 \, \partial_{z_2} g(z_2 - p_2|\sigma) \overline{\partial_{p_2} \ln \vartheta_1(z_2 - p_2|\sigma)} \Big] \end{aligned} \tag{B.6}$$

Using $\partial_{\bar z_2}\partial_{z_2} g(z_2 - p_2|\sigma) = -\pi\delta(z_2, p_2) + \frac{\pi}{\mathrm{Im}\,\sigma}$, the fact that the support of the δ-functions is outside the domain of integration, and that the integral involving the constant term is convergent on Σ_1 and integrates to zero there, we see that the first line on the right vanishes. The remainder may be written as follows,

$$\mathcal{I} = -\frac{i}{2} \oint_{\partial\mathfrak{D}_2} dz_2 \, \big|\partial_{z_2} g(z_2 - p_2|\sigma)\big|^2 \tag{B.7}$$

In passing from the second line of (B.6) to (B.7) we have omitted a contribution proportional to $2\pi i \mathrm{Im}\,(z_2 - p_2)/\mathrm{Im}\,\sigma$, as the integral involving this term is of order $\mathcal{O}(v)$. The overall minus sign results from the fact that the orientation of the integrations over the boundary of $\Sigma_2 \setminus \mathfrak{D}_2$ is opposite to the ones over the boundary of $\mathfrak{D}_2$. The angular integration of the pole term vanishes, and the remaining contribution is bounded as $v \to 0$. We conclude that $\mathcal{G}_1^{(2)}(x_1, y_1) \to 0$ so that we obtain the following limit for $\partial_{x_1}\mathcal{G}_1(x_1, y_1)$ from (B.3) and, by swapping the role of the two surfaces,

$$\begin{aligned} \partial_{x_1}\mathcal{G}_1(x_1, y_1) &\to (\mathrm{Im}\,\tau)\Big(\partial_{x_1}^2 g_2(x_1 - y_1|\tau) - \tfrac{1}{2}\partial_{x_1}^2 g_2(x_1 - p_1|\tau) \Big) \\ \partial_{x_2}\mathcal{G}_2(x_2, y_2) &\to (\mathrm{Im}\,\sigma)\Big(\partial_{x_2}^2 g_2(x_2 - y_2|\sigma) - \tfrac{1}{2}\partial_{x_2}^2 g_2(x_2 - p_2|\sigma) \Big) \end{aligned} \tag{B.8}$$

where we have neglected all contributions that vanish as $v \to 0$.

B.2 Degeneration of $\partial_{x_1}\mathcal{G}_2(x_1, y_1)$ and $\partial_{x_2}\mathcal{G}_1(x_2, y_2)$

Following the corresponding analysis for these two cases, we find the limits,

$$\begin{aligned} \partial_{x_1}\mathcal{G}_2(x_1, y_1) &\to 0 \\ \partial_{x_2}\mathcal{G}_1(x_2, y_2) &\to 0 \end{aligned} \tag{B.9}$$

B.3 Degeneration of $\partial_{x_1}\mathcal{G}_1(x_1,y_2)$ and $\partial_{x_2}\mathcal{G}_2(x_2,y_1)$

Using once more the decomposition of (6.14), we have,

$$\mathcal{G}_1^{(1)}(x_1,y_2) = \frac{1}{2}\int_{\Sigma_1\setminus\mathfrak{D}_1} d^2z_1 \left[g(x_1-z_1|\tau) - \tfrac{1}{2}g(x_1-p_1|\tau) - \tfrac{1}{2}g(z_1-p_1|\tau) - \tfrac{1}{2}\ln|\hat{v}|\right] \times\partial_{z_1}g(z_1-p_1|\tau) \tag{B.10}$$

The integrals may be continued to absolutely convergent integrals on Σ_1. The z_1-independent terms inside the bracket integrate to zero, as does the term $g(z_1-p_1|\tau)$ so that,

$$\begin{aligned}\mathcal{G}_1^{(1)}(x_1,y_2) &= \frac{1}{2}\int_{\Sigma_1} d^2z_1\, g(x_1-z_1|\tau)\,\partial_{z_1}g(z_1-p_1|\tau)\\ &= \frac{1}{2}(\mathrm{Im}\,\tau)\partial_{x_1}g_2(x_1-p_1|\tau)\end{aligned} \tag{B.11}$$

The remaining contribution is given by,

$$\begin{aligned}\mathcal{G}_1^{(2)}(x_1,y_2) = \frac{1}{2}\overline{\left(\frac{v}{2\pi i\hat{\omega}_2(p_2)}\right)}\int_{\Sigma_2\setminus\mathfrak{D}_2} d^2z_2 \left[g(x_1-p_1|\tau)+g(z_2-p_2|\sigma)+\ln|\hat{v}|\right]&\\ \times\overline{\partial_{z_2}\partial_{p_2}\ln\vartheta_1(z_2-p_2|\sigma)}\left[\partial_{z_2}g(z_2-y_2|\sigma)-\tfrac{1}{2}\partial_{z_2}g(z_2-p_2|\sigma)\right]&\end{aligned} \tag{B.12}$$

The contribution of the second term in the bracket on the second line may be evaluated by the method used for $\mathcal{G}_1^{(2)}(x_1,y_1)$ and vanishes as $v\to 0$. In the remaining expression the z_2-independent terms in the first bracket integrate to a finite quantity as $v\to 0$, leaving contributions proportional to,

$$\mathcal{G}_1^{(2)}(x_1,y_2)\sim\bar{v}\int_{\Sigma_2\setminus\mathfrak{D}_2} d^2z_2\,\overline{\partial_{z_2}\partial_{p_2}\ln\vartheta_1(z_2-p_2|\sigma)}\,\partial_{z_2}g(z_2-y_2|\sigma)g(z_2-p_2|\sigma) \tag{B.13}$$

Isolating the leading singularity in the form of a contour integral,

$$\begin{aligned}\mathcal{G}_1^{(2)}(x_1,y_2)\sim -\bar{v}\int_{\Sigma_2\setminus\mathfrak{D}_2} d^2z_2\,\overline{\partial_{p_2}\ln\vartheta_1(z_2-p_2|\sigma)}\,\partial_{\bar{z}_2}\left[\partial_{z_2}g(z_2-y_2|\sigma)g(z_2-p_2|\sigma)\right]&\\ +\frac{i}{2}\bar{v}\oint_{\partial\mathfrak{D}_2} dz_2\,\partial_{z_2}g(z_2-y_2|\sigma)g(z_2-p_2|\sigma)\overline{\partial_{p_2}\ln\vartheta_1(z_2-p_2|\sigma)}&\end{aligned} \tag{B.14}$$

The angular integration of the second term cancels the pole and gives a finite result which vanishes as $v\to 0$. Applying $\partial_{\bar{z}_2}$ to the first factor in the bracket of the first line gives a $\delta(z_2,y_2)$ and a constant term, both of which give finite contributions to the integral leading to vanishing contributions to $\mathcal{G}_1^{(2)}(x_1,y_2)$ as $v\to 0$. The remaining integral is then,

$$\mathcal{G}_1^{(2)}(x_1,y_2)\sim -\bar{v}\int_{\Sigma_2\setminus\mathfrak{D}_2} d^2z_2\,\overline{\partial_{p_2}\ln\vartheta_1(z_2-p_2|\sigma)}\,\partial_{z_2}g(z_2-y_2|\sigma)\partial_{\bar{z}_2}g(z_2-p_2|\sigma) \tag{B.15}$$

The angular integration again kills the pole and we are left with a finite integral with a vanishing contribution as $v \to 0$. Hence, by the vanishing asymptotics of $\mathcal{G}_1^{(2)}(x_1, y_2)$ and the non-zero contribution (B.11) from $\mathcal{G}_1^{(1)}(x_1, y_2)$, we find the following asymptotics for $\partial_{x_1} \mathcal{G}_1(x_1, y_2)$ and, by swapping the two surfaces, we obtain,

$$\begin{aligned}
\partial_{x_1} \mathcal{G}_1(x_1, y_2) &\to \tfrac{1}{2}(\operatorname{Im}\tau)\, \partial_{x_1}^2 g_2(x_1 - p_1|\tau) \\
\partial_{x_2} \mathcal{G}_2(x_2, y_1) &\to \tfrac{1}{2}(\operatorname{Im}\sigma)\, \partial_{x_2}^2 g_2(x_2 - p_2|\sigma)
\end{aligned} \tag{B.16}$$

B.4 Degeneration of $\partial_x \mathcal{G}^I(x, y)$

Assembling the results obtained in (B.8), (B.9), and (B.16) for $\partial_x \mathcal{G}_J(x, y)$, we obtain those of $\mathcal{G}^I(x, y)$ by raising the index J with the help of Y^{IJ}. The off-diagonal elements of Y^{IJ} are proportional to v and do not contribute in the limit $v \to 0$. The role of the diagonal elements of Y^{IJ} is to cancel the prefactor $\operatorname{Im}\tau$ in (B.8) and $\operatorname{Im}\sigma$ in (B.16). The final result is the following table of limits,

$$\begin{aligned}
\partial_{x_1} \mathcal{G}^1(x_1, y_1) &\to \partial_{x_1}^2 g_2(x_1 - y_1|\tau) - \tfrac{1}{2}\partial_{x_1}^2 g_2(x_1 - p_1|\tau) \\
\partial_{x_1} \mathcal{G}^1(x_1, y_2) &\to \tfrac{1}{2}\partial_{x_1}^2 g_2(x_1 - p_1|\tau) \\
\partial_{x_1} \mathcal{G}^2(x_1, y_1) &\to 0 \\
\\
\partial_{x_2} \mathcal{G}^2(x_2, y_2) &\to \partial_{x_2}^2 g_2(x_2 - y_2|\sigma) - \tfrac{1}{2}\partial_{x_2}^2 g_2(x_2 - p_2|\sigma) \\
\partial_{x_2} \mathcal{G}^2(x_2, y_1) &\to \tfrac{1}{2}\partial_{x_2}^2 g_2(x_2 - p_2|\sigma) \\
\partial_{x_2} \mathcal{G}^1(x_2, y_2) &\to 0
\end{aligned} \tag{B.17}$$

Together with $\partial_x^2 g_2(x-p|\tau) = -f^{(2)}(x-p|\tau)$ as in (2.15), this completes the proof of Lemma 6.1 for the case $s = 1$.

References

[1] J. L. Bourjaily, *et al.* "Functions Beyond Multiple Polylogarithms for Precision Collider Physics," [arXiv:2203.07088 [hep-ph]].

[2] S. Abreu, R. Britto and C. Duhr, "The SAGEX review on scattering amplitudes Chapter 3: Mathematical structures in Feynman integrals," J. Phys. A **55** (2022) no.44, 443004 [arXiv:2203.13014].

[3] J. Blümlein and C. Schneider, "The SAGEX review on scattering amplitudes Chapter 4: Multi-loop Feynman integrals," J. Phys. A **55** (2022) no.44, 443005 [arXiv:2203.13015].

[4] S. Weinzierl, "Feynman Integrals, a Comprehensive Treatment for Students and Researchers," Springer Cham (2022), ISBN 978-3-030-99557-7.

[5] N. Berkovits, E. D'Hoker, M. B. Green, H. Johansson and O. Schlotterer, "Snowmass White Paper: String Perturbation Theory," [arXiv:2203.09099].

[6] D. Dorigoni, M. B. Green and C. Wen, "The SAGEX review on scattering amplitudes Chapter 10: Selected topics on modular covariance of type IIB string amplitudes and their supersymmetric Yang–Mills duals," J. Phys. A **55** (2022) no.44, 443011 [arXiv:2203.13021].

[7] E. D'Hoker and J. Kaidi, "Lectures on modular forms and strings," [arXiv:2208.07242].

[8] C. R. Mafra and O. Schlotterer, "Tree-level amplitudes from the pure spinor superstring," Phys. Rept. **1020** (2023), 1-162 [arXiv:2210.14241].

[9] E. D'Hoker and J. Kaidi, "Modular forms and string theory," Cambridge University Press (2024), ISBN 978-1-009-45753-8.

[10] A. B. Goncharov, "Geometry of Configurations, Polylogarithms, and Motivic Cohomology," Advances in Mathematics **114** (1995), 197-318.

[11] A. B. Goncharov, "Multiple polylogarithms, cyclotomy and modular complexes," Math. Res. Lett. **5** (1998), 497-516 [arXiv:1105.2076 [math.AG]].

[12] A. B. Goncharov, "Multiple polylogarithms and mixed Tate motives," [arXiv:math/0103059 [math.AG]].

[13] E. Remiddi and J. A. M. Vermaseren, "Harmonic polylogarithms," Int. J. Mod. Phys. A **15** (2000), 725-754 [arXiv:hep-ph/9905237].

[14] J. Vollinga and S. Weinzierl, "Numerical evaluation of multiple polylogarithms," Comput. Phys. Commun. **167** (2005), 177 [arXiv:hep-ph/0410259].

[15] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, "Classical Polylogarithms for Amplitudes and Wilson Loops," Phys. Rev. Lett. **105** (2010), 151605 [arXiv:1006.5703].

[16] C. Duhr, "Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes," JHEP **08** (2012), 043 [arXiv:1203.0454].

[17] J. Broedel, O. Schlotterer and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61** (2013), 812-870 [arXiv:1304.7267].

[18] O. Schlotterer and O. Schnetz, "Closed strings as single-valued open strings: A genus-zero derivation," J. Phys. A **52** (2019) no.4, 045401 [arXiv:1808.00713].

[19] P. Vanhove and F. Zerbini, "Single-valued hyperlogarithms, correlation functions and closed string amplitudes," Adv. Theor. Math. Phys. **26** (2022) no.2, 455-530 [arXiv:1812.03018].

[20] A. Beilinson and A. Levin, "The Elliptic Polylogarithm," in Proc. of Symp. in Pure Math. **55** (1994), 123-190.

[21] A. Levin and G. Racinet, "Towards multiple elliptic polylogarithms," [arXiv:math/0703237 [math.NT]].

[22] F. Brown and A. Levin, "Multiple Elliptic Polylogarithms," [arXiv:1110.6917 [math.NT]].

[23] S. Bloch and P. Vanhove, "The elliptic dilogarithm for the sunset graph," J. Number Theor. **148** (2015), 328-364 [arXiv:1309.5865].

[24] L. Adams and S. Weinzierl, "Feynman integrals and iterated integrals of modular forms," Commun. Num. Theor. Phys. **12** (2018), 193-251 [arXiv:1704.08895].

[25] J. Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C. G. Raab, C. S. Radu and C. Schneider, "Iterated Elliptic and Hypergeometric Integrals for Feynman Diagrams," J. Math. Phys. **59** (2018) no.6, 062305 [arXiv:1706.01299].

[26] J. Broedel, C. Duhr, F. Dulat and L. Tancredi, "Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism," JHEP **05** (2018), 093 [arXiv:1712.07089].

[27] J. Broedel, C. R. Mafra, N. Matthes and O. Schlotterer, "Elliptic multiple zeta values and one-loop superstring amplitudes," JHEP **07** (2015), 112 [arXiv:1412.5535].

[28] E. D'Hoker, M. B. Green, Ö. Gürdogan and P. Vanhove, "Modular Graph Functions," Commun. Num. Theor. Phys. **11** (2017), 165-218 [arXiv:1512.06779].

[29] L. Dolan and P. Goddard, "Current Algebra on the Torus," Commun. Math. Phys. **285** (2009), 219-264 [arXiv:0710.3743].

[30] A. G. Tsuchiya, "On new theta identities of fermion correlation functions on genus g Riemann surfaces," [arXiv:1710.00206].

[31] J. E. Gerken, A. Kleinschmidt and O. Schlotterer, "Heterotic-string amplitudes at one loop: modular graph forms and relations to open strings," JHEP **01** (2019), 052 [arXiv:1811.02548].

[32] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory Vol. 2: Loop Amplitudes, Anomalies and Phenomenology," Cambridge University Press (1988), ISBN 978-0-521-35753-1.

[33] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60** (1988), 917.

[34] J. Polchinski, "String Theory Vol II," Cambridge University Press (1997), ISBN 978-0-521-63304-8.

[35] E. Witten, "Superstring Perturbation Theory Revisited," [arXiv:1209.5461].

[36] R. Huang and Y. Zhang, "On Genera of Curves from High-loop Generalized Unitarity Cuts," JHEP **04** (2013), 080 [arXiv:1302.1023 [hep-ph]].

[37] A. Georgoudis and Y. Zhang, "Two-loop Integral Reduction from Elliptic and Hyperelliptic Curves," JHEP **12** (2015), 086 [arXiv:1507.06310].

[38] C. F. Doran, A. Harder, E. Pichon-Pharabod and P. Vanhove, "Motivic geometry of two-loop Feynman integrals," Quart. J. Math. Oxford Ser. **75** (2024) no.3, 901-967 [arXiv:2302.14840 [math.AG]].

[39] F. Brown and O. Schnetz, "A K3 in ϕ^4," Duke Math. J. **161** (2012) no.10, 1817-1862 [arXiv:1006.4064 [math.AG]].

[40] J. L. Bourjaily, A. J. McLeod, M. von Hippel and M. Wilhelm, "Bounded Collection of Feynman Integral Calabi-Yau Geometries," Phys. Rev. Lett. **122** (2019) no.3, 031601 [arXiv:1810.07689].

[41] C. Duhr, A. Klemm, F. Loebbert, C. Nega and F. Porkert, "Yangian-Invariant Fishnet Integrals in Two Dimensions as Volumes of Calabi-Yau Varieties," Phys. Rev. Lett. **130** (2023) no.4, 4 [arXiv:2209.05291].

[42] S. Pögel, X. Wang and S. Weinzierl, "Bananas of equal mass: any loop, any order in the dimensional regularisation parameter," JHEP **04** (2023), 117 [arXiv:2212.08908].

[43] C. Duhr, A. Klemm, C. Nega and L. Tancredi, "The ice cone family and iterated integrals for Calabi-Yau varieties," JHEP **02** (2023), 228 [arXiv:2212.09550].

[44] B. Enriquez, "Flat connections on configuration spaces and braid groups of surfaces," Advances in Mathematics **252** (2014), 204–226 [arXiv:1112.0864 [math.GT]].

[45] B. Enriquez and F. Zerbini, "Construction of Maurer-Cartan elements over configuration spaces of curves," [arXiv:2110.09341 [math.AG]].

[46] B. Enriquez and F. Zerbini, "Analogues of hyperlogarithm functions on affine complex curves," [arXiv:2212.03119 [math.AG]].

[47] D. Bernard, "On the Wess-Zumino-Witten models on Riemann surfaces," Nucl. Phys. B **309** (1988), 145-174.

[48] G. Faltings, "Calculus on Arithmetic Surfaces," Ann. Math. **119** (1984) 387.

[49] L. Alvarez-Gaume, G. W. Moore, P. C. Nelson, C. Vafa and J. b. Bost, "Bosonization in Arbitrary Genus," Phys. Lett. B **178**, 41-47 (1986).

[50] J. D. Fay, "Theta Functions on Riemann Surfaces," Lecture Notes in Math. **352** (1973).

[51] E. D'Hoker, M. B. Green and B. Pioline, "Higher genus modular graph functions, string invariants, and their exact asymptotics," Commun. Math. Phys. **366** (2019) 927-979 [arXiv:1712.06135].

[52] E. D'Hoker and M. B. Green, "Zhang-Kawazumi Invariants and Superstring Amplitudes," J. Number Theor. **144** (2014), 111-150 [arXiv:1308.4597].

[53] E. D'Hoker, M. B. Green, B. Pioline and R. Russo, "Matching the D^6R^4 interaction at two-loops," JHEP **01** (2015), 031 [arXiv:1405.6226].

[54] E. D'Hoker, M. B. Green and B. Pioline, "Asymptotics of the D^8R^4 genus-two string invariant," Commun. Num. Theor. Phys. **13** (2019), 351-462 [arXiv:1806.02691].

[55] E. D'Hoker, C. R. Mafra, B. Pioline and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part II. Low energy expansion and S-duality," JHEP **02** (2021), 139 [arXiv:2008.08687].

[56] N. Kawazumi, "Some tensor field on the Teichmüller space," Lecture at MCM2016, OIST (2016).

[57] N. Kawazumi, "Differential forms and functions on the moduli space of Riemann surfaces," in Séminaire Algèbre et topologie, Université de Strasbourg. Séminaire Algèbre et topologie, Université de Strasbourg (2017).

[58] E. D'Hoker and O. Schlotterer, "Identities among higher genus modular graph tensors," Commun. Num. Theor. Phys. **16** (2022) no.1, 35-74 [arXiv:2010.00924].

[59] N. Kawazumi, "A twisted invariant of a compact Riemann surface," [arXiv:2210.00532 [math.GT]].

[60] F. Cléry, G. van der Geer, S. Grushevsky, "Siegel modular forms of genus 2 and level 2," [arXiv:1306.6018 [math.AG]].

[61] F. Cléry, G. van der Geer, "Constructing vector-valued Siegel modular forms from scalar-valued Siegel modular forms," [arXiv:1409.7176 [math.AG]].

[62] G. van der Geer, "Siegel modular forms of degree two and three and invariant theory," [arXiv:2102.02245 [math.AG]].

[63] E. Panzer, "Feynman integrals and hyperlogarithms," [arXiv:1506.07243 [math-ph]].

[64] J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, "Elliptic symbol calculus: from elliptic polylogarithms to iterated integrals of Eisenstein series," JHEP **08** (2018), 014 [arXiv:1803.10256].

[65] D. Mumford, "Tata Lectures on Theta I," Progress in Mathematics **28** (1983), Birkhäuser Boston Inc., MA.

[66] C. Reutenauer, "Free Lie Algebras", Ocford University Press (1993).

[67] P. Deligne, "Le Groupe Fondamental de la Droite Projective Moins Trois Points," in Galois Groups Over, vol. 16, ed. by Y. Ihara, K. Ribet, J.P. Serre. Mathematical Sciences Research Institute Publications (Springer, New York, 1989), 79-297.

[68] F. Brown, "Motivic periods and the projective line minus three points," in Proceedings of the ICM 2014, vol. 2, ed. by S.Y. Jang Y.R. Kim, D.-W. Lee, I. Yie (2014), 295-318.

[69] E. D'Hoker and O. Schlotterer, "Fay identities for polylogarithms on higher-genus Riemann surfaces," [arXiv:2407.11476].

[70] V. Drinfeld, "Quasi Hopf algebras," Leningrad Math. **J. 1** (1989), 1419.

[71] V. Drinfeld, "On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)," Leningrad Math. **J. 2** (1991), 829.

[72] D. Calaque, B. Enriquez and P. Etingof, "Universal KZB equations: the elliptic case," Progr. Math. **269** (2009), 165-266. Birkhäuser Boston Inc., MA.

[73] B. Enriquez, "Elliptic associators," Selecta Math. (N.S.) **20** (2014), 491-584.

[74] R. Hain, "Notes on the universal elliptic KZB equation," [arXiv:1309.0580 [math.AG]].

[75] M. Gonzalez, "Surface Drinfeld Torsors I: Higher Genus Associators," [arXiv:2004.07303 [math.QA]].

[76] A. Basu, "Eigenvalue equation for genus two modular graphs," JHEP **02** (2019), 046 [arXiv:1812.00389].

[77] E. D'Hoker and D. H. Phong, "Two-Loop Superstrings. VII. Cohomology of Chiral Amplitudes," Nucl. Phys. B **804** (2008), 421-506 [arXiv:0711.4314].

[78] J. Broedel, N. Matthes, G. Richter and O. Schlotterer, "Twisted elliptic multiple zeta values and non-planar one-loop open-string amplitudes," J. Phys. A **51** (2018) no.28, 285401 [arXiv:1704.03449].